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QUANTUM VACUUM DYNAMICS: EXPLORING THE IMPACTS OF ACCELERATION, ROTATION AND BOUNDARIES IN INTERACTING QUANTUM FIELD THEORY

PhD Candidate
MARCO PASINI

Referees
Dr. VINCENZO VITAGLIANO
Dr. MAXIM CHERNODUB

Supervisor
Prof. ANDREA VACCHI
Co-Supervisor
Prof. STEFANO ANSOLDI

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MARCO PASINI
☞ marco.pasini@uniud.it

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SUPERVISORS:
Andrea Vacchi, *Università degli Studi di Udine*
Stefano Ansoldi, *Università degli Studi di Udine*

ABSTRACT

Quantum fluctuations and zero-point energy are cornerstones of quantum field theory, representing the ever-present quantum corrections to classical fields. These phenomena, which embody the intrinsic properties of the quantum vacuum, play a profound role in shaping the physical behaviour of a given system. This thesis investigates how these quantum contributions behave under three specific influences: acceleration, rotation, and boundaries, particularly for self-interacting fields. Motivated by unresolved issues or discrepancies in the literature, this work seeks to shed light on the complex nuances of this subject.

Specifically, the first problem addressed focuses on the phenomenon of symmetry breaking and the possible associated restoration induced by acceleration via the Unruh effect. Fundamental physical principles are challenged in this matter. For a broken symmetry to be restored by a change in the reference frame, it implies that the concept of scalar quantities breaks down at the quantum level. Conversely, if symmetry breaking persists, the interpretation of the Unruh effect as a genuine thermodynamic phenomenon comes into question. Given the lack of consensus on symmetry restoration, this work examines the various methods for calculating quantum corrections and identifies how the differing renormalization prescriptions lead to contrasting outcomes across the existing literature.

The second investigated issue examines the quantum vacuum energy in non-relativistic (Schrödinger) quantum field theories. While these theories are typically not influenced by zero-point energy, it is shown that such contributions cannot be neglected when, in addition to Dirichlet boundary conditions, rotations and interactions are considered. In this framework, the calculation of the unperturbed spectrum reveals a non-vanishing quantum vacuum energy, along with a Casimir-like force that includes both a repulsive contribution from rotation and an attractive contribution from interactions. These results are further corroborated by examining the corresponding relativistic theory, ensuring consistency with the non-relativistic limit. The novel contributions to the interacting quantum vacuum, combined with the potential for experimental verification through systems such as Bose-Einstein condensates, provide a compelling motivation for further exploration of these phenomena.

PUBLICATIONS

This thesis is based on the following publications:

[1] Domenico Giuseppe Salluce, Marco Pasini, Antonino Flachi, Antonio Pittelli, and Stefano Ansoldi. «Symmetry restoration and uniformly accelerated observers in Minkowski spacetime.» In: *Journal of High Energy Physics* 2024.5 (2024), pp. 1–18.

[2] Matthew Edmonds, Antonino Flachi, and Marco Pasini. «Quantum vacuum effects in nonrelativistic quantum field theory.» In: *Physical Review D* 108.12 (2023), p. L121702.

ACRONYMS

BEC	Bose-Einstein Condensate
EFT	Effective Field Theory
EOM	Equation of Motion
EPR	Einstein-Podolsky-Rosen
GR	General Relativity
KMS	Kubo-Martin-Schwinger
KG	Klein-Gordon
KGE	Klein-Gordon Equation
LHY	Lee-Huang-Yang
ODE	Ordinary Differential Equation
PDE	Partial Differential Equation
QFT	Quantum Field Theory
QVE	Quantum Vacuum Energy
SSB	Spontaneous Symmetry Breaking
UV	Ultraviolet
VEV	Vacuum Expectation Value
ZPE	Zero-point Energy
1PI	One-particle Irreducible

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INTRODUCTION

This thesis focuses on the calculation and physical interpretation of quantum corrections, which arise from quantum fluctuations and vacuum energy. These corrections, acting upon the classical field behavior, can profoundly influence the system's dynamics and properties. As demonstrated in this work, their nature and magnitude can depend on a variety of factors, including temperature, acceleration, rotation, interactions, and boundary conditions. These factors not only shape the form and strength of the quantum corrections, but also govern phenomena such as symmetry breaking and restoration, phase transitions, and the stability of vacuum configurations. For instance, temperature can induce thermal quantum fluctuations that restore a broken symmetry, while boundary conditions can constrain the field, modifying its zero-point energy. Similarly, acceleration, as described by the Unruh effect, and rotation can significantly alter the vacuum structure and energy spectrum.
By systematically exploring these dependencies, this thesis aims to uncover how quantum corrections affect the field dynamics through the influence on external or intrinsic parameters. This analysis provides new insights into the behavior of quantum fields across diverse conditions, with implications ranging from condensed matter systems, cosmology, and quantum gravity.

The natural framework for incorporating these quantum effects is the effective action approach to field theories. The effective action is also known as the quantum-corrected version of the classical action, and determines the quantum equations of motion governing the field's dynamics. It can be expanded in loops, which inherently contain all quantum effects and provide the complete corrections to the field's behavior. The contribution of each successive loop in the expansion is expected to decrease, with higher-order loops providing progressively smaller corrections. For this reason, the first loop contribution is often regarded as the most significant, as it contains the majority of the quantum corrections. Consequently, it is possible to define the one-loop effective action as the sum of its classical and (first) quantum components, i.e.,

$$\Gamma_{1\text{-loop}}[\phi] = S[\phi] + \hbar\Gamma^{(1)}[\phi] \,. \tag{0.1}$$

The explicit form of $\Gamma^{(1)}$ can be expressed in various ways, depending on the context of the calculation. For the purposes of this thesis, $\Gamma^{(1)}$ is presented in three distinct forms:

- $\Gamma^{(1)} \sim V_{\text{eff}}$, in terms of the effective potential.
- $\Gamma^{(1)} \sim \langle \phi^2 \rangle$, in terms of the so-called 'vacuum polarization', which is calculated by considering the two-point Green's function $G(x, x')$ in the limit $x' \to x$.
- $\Gamma^{(1)} \sim \sum_n \omega_n$, in terms of the zero-point energy, defined as the sum over all possible eigenvalues of quantum field fluctuations.

The one-loop correction $\Gamma^{(1)}$, is inherently divergent and requires renormalization to yield finite and physically meaningful quantities. As extensively discussed in this work, the chosen renormalization procedure plays a crucial role in determining these finite results.

This thesis focuses on the study of scalar fields, which, despite their simplicity compared to more complex field theories, capture the most significant features of the phenomena under examination.

The manuscript is structured into two chapters, each based on separate publications resulting from my doctoral research. Each chapter addresses a distinct aspect of quantum corrections. The first one involves the (possible) quantum corrections derived by the Unruh effect, a thermal-like behaviour that the vacuum manifests for accelerated observers, in the case of a system affected by a spontaneous symmetry breaking (SSB). The second one discusses the significance of quantum vacuum energy in interacting, non-relativistic quantum field theories in the presence of boundary conditions and rotations.

Symmetry restoration for uniformly accelerated observers

Spontaneous symmetry breaking occurs when a symmetry transformation leaves the action invariant but fails to preserve the vacuum state of the system. In the context of quantum field theory, this means that the field configuration invariant under the symmetry transformation does not correspond to the vacuum expectation value (VEV) of the field. Since the action and the equations of motion remain symmetric, the symmetry transformations map a set of degenerate vacuum configurations, all equally valid as candidates for the system's ground state. The VEV of the field breaks the symmetry by selecting a specific vacuum state from the set of degenerate configurations. If the (internal)

symmetry is spontaneously broken at the classical level, these configurations are determined by the minima of the classical potential. Thus, their behaviour is encapsulated in the classical contribution $S(\phi)$ to the effective action (0.1).
Remarkably, quantum corrections can, in principle, modify the classical behavior significantly, sometimes acting as more than just subleading contributions and effectively restoring the symmetry that was initially broken. For instance, thermal quantum fluctuations are a prominent example of quantum corrections that can lead to the restoration of a broken symmetry [1].

On the other hand, since the late 1970s, it has been widely recognized that the canonical quantization of fields in Riemannian spacetime is intrinsically non-unique, introducing ambiguities in the definition of a universal vacuum state. This challenge is evident even in Minkowski spacetime, where the standard Minkowski vacuum is no longer viewed as the true ground state by a non-inertial observer. Specifically, when described in the context of uniformly accelerated coordinates, the Minkowski vacuum appears to contain radiation rather than being empty. This phenomenon can, in principle, be observed by an accelerating detector, which would measure thermal radiation characterized by a temperature given by the equation:

$$T_U = \frac{a}{2\pi}, \tag{0.2}$$

commonly referred to as the Unruh temperature. This phenomenon, known as the Unruh effect, reveals that the temperature of an accelerated system in vacuum increases significantly due to the interactions with quantum fluctuations. The discovery of the Unruh effect led physicists to explore its physical significance, giving rise to numerous questions about its nature, including one particularly fundamental issue:

Can a spontaneously broken symmetry undergo a phase transition to an unbroken phase due to the influence of Unruh radiation, arising from acceleration?

By naively extending the established analogy between acceleration and temperature, one might conclude that, similar to the finite-temperature case, a critical acceleration could trigger a transition from a broken symmetry phase to a restored one. Expanding on this intuitive idea, comparable to the temperature melting of a condensate, this inquiry becomes crucial in unraveling the intricate relationship between quantum field theory and general relativity.
Well-established physical principles are challenged in this matter, including the invariance of scalar quantities under changes in reference frame. Specifically, for symmetry to be restored, the scalar quantities

represented by the VEV of the field and the vacuum polarization must be affected by the Unruh temperature and, therefore, by the choice of reference frame. Conversely, if symmetry persists regardless of the acceleration, this would significantly call into question the nature of the Unruh temperature as a genuine thermodynamic phenomenon. The literature on this topic can be categorized into three main perspectives:

- those who claim that the symmetry phase remains unchanged regardless of the observer's motion;
- those who argue that symmetry restoration occurs at sufficiently high acceleration;
- those who assert that symmetry breaking is enhanced for uniformly accelerated observers.

The first point of view has been primarily supported by seminal works of W.G. Unruh and N. Weiss [2], C.T. Hill [3], and D.N. Page [4], among others. In contrast, the idea of symmetry restoration has gained traction in recent years, with numerous contributions exploring the phenomenon across various field theories, including bosonic and fermionic models [5–10]. The symmetry breaking strengthening standpoint has been considered in a few recent studies [11, 12]. However, its underlying mechanisms were already hinted at by P. Candelas and D.J. Raine [13] in their study of Green's function behavior in incomplete manifolds, despite not explicitly addressing the issue of symmetry breaking.

This thesis provides a detailed examination of the various approaches, concluding that the differing outcomes ultimately arise from the choice of inequivalent renormalization schemes or from selecting different Green's functions as the appropriate physical quantity. As one can easily imagine, quantum corrections are significantly influenced by these choices, leading to distinct physical predictions depending on the chosen framework.

Quantum vacuum effects in non-relativistic quantum field theories

One-loop corrections to the effective potential are sometimes expressed through the expression of zero-point energy:

$$E = \frac{1}{2}\sum_n \omega_n, \tag{0.3}$$

a divergent quantity in quantum field theory that requires renormalization to yield finite physical results. Historically, this term was often disregarded as a trivial constant energy shift, with its relevance primarily recognized in relation to gravitational or cosmological problems.

This perspective changed with the introduction of boundary conditions at finite distances, capable of discretizing the modes of quantum fluctuations within the bounded region, thereby altering both the quantum vacuum and the zero-point energy. These developments led to the Casimir prescription for the calculation of quantum vacuum energy:

$$E_{\text{vac}} = E_B - E_{NB}, \tag{0.4}$$

where E_B and E_{NB} represent the zero-point energy in the presence and absence of boundaries, respectively. Notably, this reduces to the normal-ordering result when the boundaries are taken to infinity. The resulting shift in vacuum energy produces the well-known Casimir effect [14].

Building on this foundation, modifications to the dispersion relation, as seen in relativistic field theories, have been shown to influence vacuum energy. However, non-relativistic quantum field theories, such as Schrödinger field theory, were traditionally excluded from such considerations due to their peculiar quantization prescription, which inherently lacks a canonical zero-point energy. However, when viewed as effective theories derived from relativistic fields, the inclusion of zero-point energy becomes natural. Despite this, the computation of quantum vacuum energy for non-relativistic free fields in bounded regions yields a vanishing result.

In this thesis, we extend these ideas to interacting and rotating systems, showing that non-relativistic quantum field theories exhibit non-vanishing vacuum energy under such conditions. Specifically, we consider a 1+1 dimensional non-relativistic interacting quantum field theory in a rotating ring with Dirichlet boundary conditions imposed. For the first time, we derive the unperturbed solutions to the equations of motion and the corresponding spectrum, which are then applied to calculate the quantum vacuum energy. Two regularization methods were employed: zeta-function regularization and a novel scheme combining a 'window function' approach, i.e.,

$$E_B = \lim_{\ell_c \to 0} \sum_{n=1}^{\infty} \omega_n e^{-\omega_n \ell_c}, \tag{0.5}$$

with the Jacobi theta function. Both methods yield consistent results, with the latter providing a more physically meaningful interpretation in scenarios with physical cut-off scales ℓ_c. Our findings show that the vacuum energy and associated forces include contributions from both interactions and rotations. Importantly, these results are validated by replicating the calculations in the associated relativistic field theory, with agreement achieved in the non-relativistic limit.

An intriguing connection emerges between this quantum field theoretical framework and ultra-cold atom systems, particularly Bose-Einstein condensates (BEC) confined in optical tweezers (see more in 2.3.6). This provides a promising connection between theoretical studies of quantum vacuum energy, experimental investigations of ultra-cold atom systems, and potential cosmological analogues, such as those involving vacuum energy phenomena.

1 SYMMETRY RESTORATION FOR UNIFORMLY ACCELERATED OBSERVERS

1.1 PRELIMINARIES

In this short introduction, we give the basic definitions and concepts to understand the state of the art of the literature concerning the problem of spontaneously broken symmetry in Rindler space and the results of our article. This summary is far from a complete review of the topic discussed, which is extensively covered in the referenced literature [15, 16]. This small outline includes an introduction to:

- Rindler space and Unruh effect,
- Klein-Gordon field and Rindler-Fulling quantization,
- Effective action method and effective potential,
- Finite temperature symmetry restoration.

1.1.1 *Rindler space and Unruh effect*

1.1.1.1 *Rindler coordinates and space*

Since we are interested in the physics experienced by an uniformly accelerated observer with proper acceleration a, it is useful to describe Minkowski space in a suitable system of coordinates, which are called Rindler coordinates. They are defined, with respect to the usual Cartesian Minkowski coordinates, as follow

$$\begin{cases} t = \rho \sinh(a\tau) \\ x = \rho \cosh(a\tau) \\ y = y, \quad z = z, \end{cases} \tag{1.1}$$

with $\rho \in (-\infty, \infty)$, $\tau \in (-\infty, \infty)$ and assuming the accelerated motion to be in the x direction. These coordinates, for fixed y, z, and a specific constant value of ρ, define a hyperbola that represents the worldline of a uniformly accelerated observer with proper acceleration ρ^{-1}. Varying ρ generates a one-parameter family of such hyperbolae, each corresponding to a different observer experiencing a distinct proper acceleration. These hyperbolae also share the same asymptotes, which are the lines $x = \pm t$ and divide the Minkowski space into 4 different regions, as shown in Figure 1.1. The left region, or left wedge, is associated with negative values of ρ and is called the left Rindler wedge, the right region to positive ρ and is called the right Rindler wedge.

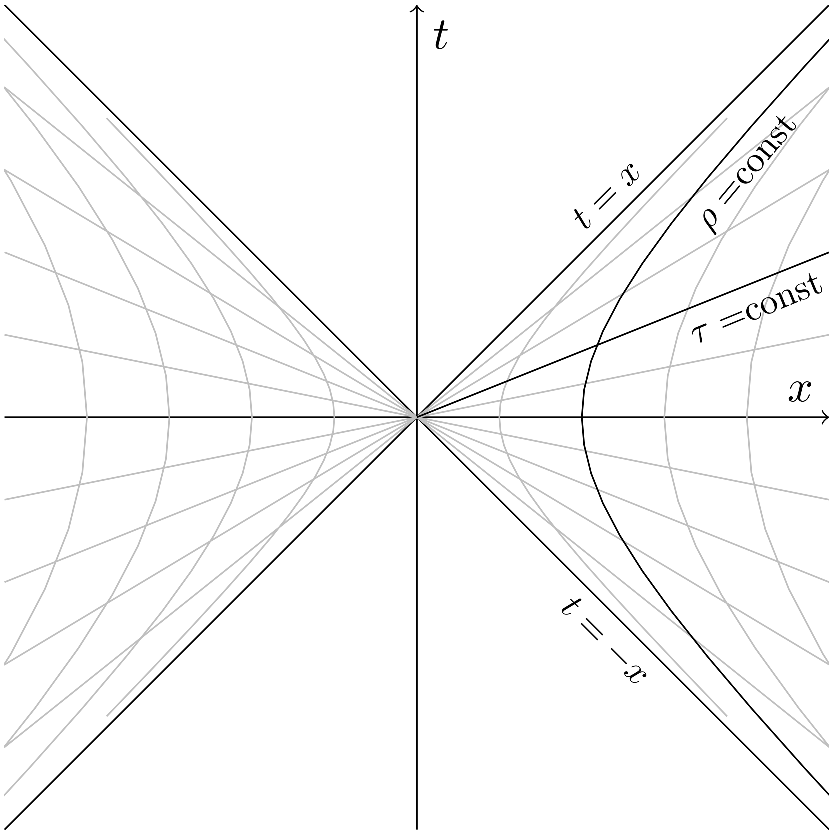

Figure 1.1: Rindler wedges in the plane (t, x). Rindler coordinates are defined in the regions R and L: time coordinates $\tau = const$ are straight lines passing through the origin, and space coordinates $\rho = const$ are branches of hyperbolae, corresponding to the worldlines of different Rindler observers.

Minkowski line element in Rindler coordinates become

$$ds^2 = a^2\rho^2 d\tau^2 - d\rho^2 - dy^2 - dz^2. \tag{1.2}$$

The proper time experienced by an observer at fixed ρ, y, z is

$$ds = a\rho\, d\tau, \tag{1.3}$$

and, for an observer at $\rho = 1/a$, we can identify the proper time with the τ coordinate. This is the reason why the Rindler coordinates are said to be adapted to the motion of a uniformly accelerated observer. Consequently, in the context of quantum field theory, it is possible to define a Rindler Hamiltonian that generates translation in the proper time τ for an observer at $\rho = 1/a$ accelerating with the proper acceleration a.

We call the Rindler space the open submanifold of the Minkowski space that is covered by the Rindler coordinates for $\rho \geq 0$, i.e. the right Rindler wedge. It has some peculiar features:

- *Incomplete manifold*. Uniformly accelerated observers are confined to a submanifold of the (maximally extended) Minkowski space. Their motion starts and ends at infinity, without crossing the light cone determined by the asymptotes, which is the boundary of such region, and it is confined in the right wedge. The same arguments apply both to the left and the right wedges, but not to the future and past wedges, which are not mapped by Rindler coordinates.

- *Presence of horizons*. The light-cone boundary at $t = x$ marks a region from which an accelerating observer cannot access any information. It is effectively an horizon for him/her.

- *Causal structure*. An accelerated observer in the right wedge can hypotetically access information from all of its own and past wedges. He/her can send information to the future wedge (without being able to receive any) and is causally disconnected from the left wedge.

An interesting similarity emerges from the previously described causal structure and the causal structure of the maximally extended Schwarzschild spacetime. Despite the different topological nature of these spacetimes, it is possible in fact to relate, respectively, left/right, past and future wedges to the causally disconnected region outside black hole horizon, white hole, and black hole.

1.1.1.2 *Symmetries and Killing vectors*

A symmetry of a generic spacetime is a transformation that leaves the metric invariant and, by extension, that preserves the way distances are calculated, i.e. an isometry. From a more practical perspective, a symmetry of spacetime can be described by a vector field ξ along which the Lie derivative of the metric is vanishing, i.e.

$$\mathcal{L}_\xi \, g_{\mu\nu} = 0. \tag{1.4}$$

The Lie derivative of the metric can also be described in terms of covariant derivatives which, for a generic (k, l)-type tensor T, is defined as

$$\begin{aligned}\nabla_\rho T^{\mu_1 \dots \mu_k}{}_{\nu_1 \dots \nu_l} = \partial_\rho T^{\mu_1 \dots \mu_k}{}_{\nu_1 \dots \nu_l} + \sum_{i=1}^{k} \Gamma^{\mu_i}_{\rho\sigma} T^{\mu_1 \dots \sigma \dots \mu_k}{}_{\nu_1 \dots \nu_l} \\ - \sum_{j=1}^{l} \Gamma^{\sigma}_{\rho\nu_j} T^{\mu_1 \dots \mu_k}{}_{\nu_1 \dots \sigma \dots \nu_l},\end{aligned} \tag{1.5}$$

where the Γs are the connection coefficents. If the metric is compatible with connection, like in ordinary GR, we have $\nabla g = 0$ and equation (1.4) can be written as

$$\begin{aligned}\mathcal{L}_\xi g_{\mu\nu} &= \xi^\sigma \cancel{\nabla_\sigma g_{\mu\nu}} + \nabla_\mu \xi^\sigma g_{\sigma\nu} + \nabla_\nu \xi^\sigma g_{\mu\sigma} \\ &= \nabla_\mu \xi_\nu + \nabla_\nu \xi_\mu = 0.\end{aligned} \tag{1.6}$$

This equation is also called the Killing equation, and any vector field that satisfies the Killing equation is called a Killing vector field. To each different Killing vector field corresponds a symmetry of spacetime. Symmetries of spacetime, however, can not be arbitrarily many, with the maximum number of symmetries being

$$N_{max} = d + \frac{d\,(d-1)}{2} = \frac{d\,(d+1)}{2}, \tag{1.7}$$

with d the spacetime dimension. It corresponds to the sum of d translational symmetries and $d(d-1)/2$ rotational/boost symmetries. Spacetimes that possess the maximum number of symmetries are called maximally symmetric and are characterized by a constant curvature.
Naturally, Minkowski space is maximally symmetric, and the same can be said for Rindler space (even if only locally), being a Minkowski subspace. However, as we will see below, depending on the specific set of coordinates adopted and their physical interpretation, the nature of symmetries can change.
First, let us take a look at Minkowski space-time symmetries. Translational symmetries can be easily found by considering the Killing vector field

$$k = k^\mu \partial_\mu \tag{1.8}$$

with basis $(\partial_t, \partial_{x_1}, \partial_{x_2}, \partial_{x_3})$, and the vector components $k^\mu = \delta^\mu_{\sigma^*}$ with σ^* the specific component along which we operate the translation. A sufficient condition for a vector field k to be a Killing vector field is that the metric is independent of the coordinate $\sigma^\star$. In this case, equation

(1.6) is satisfied, provided we consider the unique, torsion-free, metric-compatible Levi-Civita connection, namely,

$$\Gamma^{\alpha}_{\mu\nu} = \frac{1}{2} g^{\alpha\sigma} \left(\partial_\mu g_{\nu\sigma} + \partial_\nu g_{\sigma\mu} - \partial_\sigma g_{\mu\nu} \right) . \tag{1.9}$$

Minkowski spacetime, having a coordinate-independent metric, is translational invariant along any of the possible directions t, x_1, x_2, x_3.
This condition is sufficient, but clearly not necessary since we only have d spacetime coordinates but the number of possible symmetries is higher (1.7). One possible approach to find other symmetries of spacetime is to perform a change of coordinates and check if the metric is independent with respect to some of the new coordinates. For example, if we consider the change in coordinates (1.1), the new metric (1.2) is independent of τ, which we interpreted as the proper time for a uniformly accelerated observer with proper acceleration a. This means that there is an isometry in time τ translations and ∂_τ is a Killing vector in this new set of coordinates. Using (1.1) it is possible to express it in Minkowski coordinates as

$$\frac{\partial}{\partial \tau} = a \left(x \frac{\partial}{\partial t} + t \frac{\partial}{\partial x} \right) , \tag{1.10}$$

in other words, a times what is known to be the Minkowki boost generator.
At a quantum field theory level, it is then possible to identify the physical Hamiltonian operator of an accelerated observer, i.e. the Rindler Hamiltonian, with the Minkowski boost operator (modulo the proportionality constant a).

1.1.2 *Rindler-Fulling quantization and Unruh effect*

In this chapter, we focus on exploring the interplay between spontaneous symmetry breaking and accelerated reference frames. To narrow the scope, we consider a real scalar interacting quantum field theory, as SSB can occur across a wide variety of field theories. Despite its simplicity, this model captures most of the novel and physically significant effects introduced by the non-inertial nature of accelerated reference frames. As a foundational step, we adopt the Rindler-Fulling quantization framework for the free Klein-Gordon field and subsequently establish its connection to the standard Minkowski quantization used for inertial observers. Finally, we conclude by deriving the Unruh effect.

1.1.2.1 *Rindler-Fulling quantization*

The free (minimally coupled) Klein-Gordon Lagrangian in a generic spacetime, can be written as

$$\mathcal{L} = \frac{\sqrt{-g}}{2}\left[g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi - m^2\phi^2\right] \tag{1.11}$$

and the associated equation of motion is

$$\left[\frac{1}{\sqrt{-g}}\partial_\mu\left(g^{\mu\nu}\sqrt{-g}\partial_\nu\right) + m^2\right]\phi = 0. \tag{1.12}$$

In Rindler coordinates (1.1), the Klein-Gordon partial differential equation (PDE) (1.12) takes the form

$$\left(\frac{1}{a^2\rho^2}\partial_\tau^2 - \frac{1}{\rho}\partial_\rho\left(\rho\,\partial_\rho\right) - \partial_y^2 - \partial_z^2 + m^2\right)\phi = 0, \tag{1.13}$$

which solutions can be obtained by separation of variables. They correspond to plane wave solutions in τ, y, z coordinates

$$\phi_{\Omega,\vec{k}_\perp}\left(\tau,\rho,\vec{x}_\perp\right) = Ne^{-i\Omega\tau}e^{i\vec{k}_\perp\vec{x}_\perp}\tilde{\phi}_{\Omega,\vec{k}_\perp}\left(\rho\right) \tag{1.14}$$

with $\vec{x}_\perp = (y,z)$, $\vec{k}_\perp = \left(k_y,k_z\right)$ and N an arbitrary constant. The remaining ordinary differential equation (ODE)

$$\left(\partial_\rho^2 + \frac{1}{\rho}\partial_\rho + \frac{\Omega^2}{a^2\rho^2} - \mu_{k_\perp}^2\right)\tilde{\phi}_{\Omega,\vec{k}_\perp}\left(\rho\right) = 0 \tag{1.15}$$

with $\mu_{k_\perp} = \sqrt{\vec{k}_\perp^2 + m^2}$, is the well-known modified Bessel equation, which closed-form solutions [17] are the modified Bessel functions. Solutions (1.14) can then be written as

$$\phi_{\Omega,\vec{k}_\perp}\left(\tau,\rho,\vec{x}_\perp\right) = N_k e^{-i\Omega\tau}e^{i\vec{k}_\perp\vec{x}_\perp}K_{i\Omega/a}\left(\mu_{k_\perp}\rho\right). \tag{1.16}$$

At this point, a few remarks can help guide the quantization of the field.
Since we are not operating within the Lorentz transformation framework, the 'eigenvalues' k_ρ of the kinetic operator $\partial_\rho^2 + \rho^{-1}\partial_\rho$ depends on the ρ coordinate, such as

$$k_\rho = \sqrt{\frac{\Omega^2}{a^2\rho^2} + \vec{k}_\perp^2 + m^2} \tag{1.17}$$

with Ω, k_y, k_z, m constants in ρ, as implicitly considered in the plane wave decomposition assumption (1.14). The coordinate dependence in k_ρ is a direct reflection of the fact that ρ translations are not an isometry of the spacetime since the metric depends on it. Consequently, on-shell, k_ρ is not a constant of motion and is not a suitable candidate as a quantum number that describes the state of the system. For this reason, in the field expansion, it is preferable to integrate over (Ω, k_y, k_z) instead of (k_x, k_y, k_z) to obtain something of the form

$$\phi(x) = \int_0^\infty d\Omega \int d^2k_\perp \left[b_k f_k(x) + b_k^\dagger f_k^\star(x) \right], \tag{1.18}$$

with $b_k, b_k^\dagger$ annihilation and creation operator, which we later discuss in more detail, and $f_k(x)$ are the modes (1.16). The orthogonality and normalization coefficient of the modes are determined by the Klein-Gordon inner product, defined like

$$(f_1, f_2) \equiv i \int_{\Sigma_t} d\Sigma \sqrt{-g} n^\nu \left(f_1^\star(x) \overleftrightarrow{\partial_\nu} f_2(x) \right) \tag{1.19}$$

with Σ_t a spacelike hypersurface, $d\Sigma$ its element, n^ν its future-directed unit normal and n the dimensions of the spacetime.
Klein-Gordon inner product has some remarkable properties:

- is independent on the particular foliation of the spacetime. Any choice of Cauchy hypersurface leads to the same result;
- is time independent for a Klein-Gordon field, i.e., $d/dt\,(f_1, f_2) = 0$;
- together with the standard conjugate symmetry of scalar products in $\mathbb{C}$ it also has an additional useful property, that we will utilize in the following

$$(f_1, f_2)^\star = (f_2, f_1), \qquad (f_1^\star, f_2^\star) = -(f_2, f_1). \tag{1.20}$$

It is possible to prove, using Klein-Gordon inner product, that the basis of solutions constructed from (1.16) is indeed the orthogonal one, and can be opportunely normalized. However, it is not complete because of the missing modes pertaining to the left Rindler wedge. Ultimately, the complete, normalized basis is constituted by the elements

$$f^{(\sigma)}_{\Omega,\vec{k}_\perp}(\tau, \rho, \vec{x}_\perp) = \Theta(\sigma\rho) \frac{1}{2\pi^2} \sqrt{\frac{1}{a} \sinh\left(\pi \frac{\Omega}{a}\right)} e^{-i\sigma\Omega\tau} e^{i\vec{k}_\perp \vec{x}_\perp} K_{i\Omega/a}\left(\mu_{k_\perp} \sigma\rho\right), \tag{1.21}$$

with Θ the Heaviside step function and $\sigma = \{-, +\}$ labeling the left and right wedge respectively.
Importantly, because of the existence of a time-like Killing vector field

∂_τ, we can discern between positive and negative frequencies, as is customary for inertial observers, i.e.,

$$\begin{aligned}\frac{\partial}{\partial\tau} f^{(\sigma)}_{\Omega,\vec{k}_\perp}(\tau,\rho,\vec{x}_\perp) &= -i\sigma\Omega f^{(\sigma)}_{\Omega,\vec{k}_\perp}(\tau,\rho,\vec{x}_\perp)\\ \frac{\partial}{\partial\tau} f^{(\sigma)\star}_{\Omega,\vec{k}_\perp}(\tau,\rho,\vec{x}_\perp) &= i\sigma\Omega f^{(\sigma)\star}_{\Omega,\vec{k}_\perp}(\tau,\rho,\vec{x}_\perp)\,.\end{aligned} \tag{1.22}$$

It is peculiar that, in the left wedge, the positive frequency modes appear with the opposite sign, as the proper time has the opposite time ordering with respect to the usual Minkowski one. This is merely a consequence of the choice of coordinates. Specifically, because in the left Rindler wedge, the accelerations $\rho^{-1} = const$ of the uniformly accelerated observers are considered to be negative, while a remains positive. As a consequence, the definition of proper time in (1.3) also has the opposite sign.
Bringing everything together, we arrive at the Rindler-Fulling quantization

$$\phi(x) = \sum_\sigma \int_0^\infty d\Omega \int d^2k_\perp \left[b^{(\sigma)}_k f^{(\sigma)}_k(x) + b^{(\sigma)\dagger}_k f^{(\sigma)\star}_k(x)\right], \tag{1.23}$$

with $k = \{\Omega, \vec{k}_\perp\}$ and $b^{(\sigma)}_k, b^{(\sigma)\dagger}_k$ the annihilation and creation operator which define the Rindler vacua $|0\rangle_{R(\sigma)}$ in the following way

$$b^{(\sigma)}_k |0\rangle_{R(\sigma)} = 0 \qquad \forall\sigma, k. \tag{1.24}$$

In this sense, the Hilbert space of the whole (left wedge + right wedge) Rindler spacetime can be seen as a bipartite system described by the tensorial product of the Hilbert spaces in the two wedges. The states in the Hilbert spaces can be constructed from the vacuum via creation operators and, for a single mode k, they are

$$|n_k, m_k\rangle_R = |n_k\rangle_{R(-)} \otimes |m_k\rangle_{R(+)} = \frac{\left(b^{(-)\dagger}\right)^n}{\sqrt{n!}} \frac{\left(b^{(+)\dagger}\right)^m}{\sqrt{m!}} |0,0\rangle_R\,. \tag{1.25}$$

From this point onward, to simplify the notation, we will denote $|0\rangle_{R(-)}, |0\rangle_{R(+)}$ by just $|0\rangle_R$, since the action of the creation and annihilation operators of a given wedge in the vacuum is unambiguous. Rindler-Fulling operators $b^{(\sigma)\dagger}_k, b^{(\sigma)}_k$ satisfy the canonical commutation relations, which retain the same structure as in Minkowski quantization, apart from a Kronecker delta in the σ variable,

$$\begin{aligned}\left[b^{(\sigma)}_k, b^{(\sigma')\dagger}_{k'}\right] &= \delta_{\sigma\sigma'}\delta(\Omega-\Omega')\,\delta^{(2)}\left(\vec{k}_\perp - \vec{k}'_\perp\right)\\ \left[b^{(\sigma)}_k, b^{(\sigma')}_{k'}\right] &= \left[b^{(\sigma)\dagger}_k, b^{(\sigma')\dagger}_{k'}\right] = 0\end{aligned} \tag{1.26}$$

The Hamiltonian of the system can be constructed from the stress-energy tensor associated to Lagrangian (1.11), which, as in General Relativity, takes the standard form

$$\begin{aligned} \mathcal{T}_{\mu\nu} &\equiv 2\sqrt{-g}\frac{\delta\mathcal{L}}{\delta g^{\mu\nu}} \\ &= \partial_\mu\phi\partial_\nu\phi - \frac{1}{2}g_{\mu\nu}\left(\partial^\alpha\phi\partial_\alpha\phi - m^2\phi\right). \end{aligned} \tag{1.27}$$

If we consider a Killing vector field K and the conserved stress-energy tensor $T_{\mu\nu}$, it is easy to prove that the quantity $K^\mu T_{\mu\nu}$ is a conserved vector. For this reason, the definition of the Hamiltonian

$$H \equiv \int_{\Sigma_t} d\Sigma\sqrt{-g}n^\nu K^\mu T_{\mu\nu} \tag{1.28}$$

is independent on the particular choice of the Cauchy hypersurface Σ_t. Choosing hypersurfaces at constant Rindler time τ and the Killing vector field $K = \partial_\tau$, it is possible to rewrite the Hamiltonian as

$$\begin{aligned} H &= \int d^{n-1}x\,\sqrt{-g}g^{\tau\tau}T_{\tau\tau} \\ &= \int_{-\infty}^{\infty}\frac{d\rho}{|a\rho|}\int dx_\perp\left[\frac{1}{2}(\partial_\tau\phi)^2 + \frac{a^2\rho^2}{2}\left((\partial_\rho\phi)^2 + (\partial_{x_\perp}\phi)^2 + m^2\phi^2\right)\right], \end{aligned} \tag{1.29}$$

also referred to as Rindler Hamiltonian.
Considering quantum fields, in the second quantization formalism, the Hamiltonian operator H_R is constructed by substituting the field quantization (1.23) into the Rindler Hamiltonian. Thus, we obtain, upon integration of the modes over the spatial coordinates,

$$H_R = H_R^{(+)} - H_R^{(-)}, \tag{1.30}$$

with

$$H_R^{(\sigma)} = \int_0^\infty d\Omega\int d^2k_\perp\,\frac{\Omega}{2}\left(b_k^{(\sigma)\dagger}b_k^{(\sigma)} + b_k^{(\sigma)}b_k^{(\sigma)\dagger}\right) \tag{1.31}$$

the Hamiltonian operator in the left, $\sigma = -$, and right, $\sigma = +$, wedges. We immediately recognize the same structure of the ordinary Minkowski Hamiltonian operator (aside from the wedge identification σ), with $b_k^{(\sigma)\dagger}b_k^{(\sigma)}$ the usual number operator, which counts the number of particles with mode $\left(\Omega,\vec{k}_\perp\right)$ in the wedge (σ).

1.1.2.2 *Unruh effect*

An important distinction between the Rindler-Fulling Hilbert/Fock space and the (inertial) Minkowski Hilbert/Fock space arises from the absence of a unitary transformation connecting the two. In other words, they carry (unitarily) inequivalent representations of the Poincarè group, which define a different particle content in each respective Hilbert space. This behavior should come with little surprise since no Poincaré transformation can relate an inertial and a uniformly accelerated observer. This difference in particle interpretation between an inertial observer and a uniformly accelerated one gives rise to the well-known Unruh effect, which we will briefly explore in this subsection.

Let us now focus on the phenomenon of positive/negative frequency mixing that takes place when we try to expand Rindler modes $f^{(\sigma)}_{\Omega,\vec{k}_\perp}$ (1.21), in terms of Minkowski ones, which are

$$u_{\omega,\vec{k}} = \frac{1}{\sqrt{(2\pi)^3\, 2\omega_k}} e^{-i\left(\omega_k t - \vec{k}\vec{x}\right)}. \tag{1.32}$$

Here, we consider the standard field quantization for an inertial observer in Minkowski spacetime, which is given by

$$\phi\,(x) = \sum_{\sigma} \int_0^\infty d^3k \left[a_k u_{\omega,\vec{k}} + a_k^\dagger u^\star_{\omega,\vec{k}}\right], \tag{1.33}$$

where a_k and $a_k^\dagger$ are the usual annihilation and creation operators. The vacuum state $|0\rangle_M$ is defined by $a_k\,|0\rangle_M$ for all $\vec{k}$. The operators satisfy the canonical commutation relations

$$\begin{aligned} \left[a_k, a^\dagger_{k'}\right] &= \delta^{(3)}\left(\vec{k} - \vec{k}'\right) \\ \left[a_k, a_{k'}\right] &= \left[a^\dagger_k, a^\dagger_{k'}\right] = 0. \end{aligned} \tag{1.34}$$

As we shall see in the following, this difference in perception of quantum field modes between an inertial and a uniformly accelerated observer gives rise to new, unexpected interpretations of some well established concepts in "inertial" quantum field theory, such as vacuum and particles.
Given the two aforementioned bases of eigenfunctions, the elements of one basis can be represented in terms of the elements of the other as follows

$$\begin{aligned} u_{\omega,\vec{k}'} &= \left(f^{(\sigma)}_{\Omega,\vec{k}_\perp}, u_{\omega,\vec{k}'}\right) f^{(\sigma)}_{\Omega,\vec{k}_\perp} + \left(f^{(\sigma)\star}_{\Omega,\vec{k}_\perp}, u_{\omega,\vec{k}'}\right) f^{(\sigma)\star}_{\Omega,\vec{k}_\perp} \\ u^\star_{\omega,\vec{k}'} &= \left(f^{(\sigma)}_{\Omega,\vec{k}_\perp}, u^\star_{\omega,\vec{k}'}\right) f^{(\sigma)}_{\Omega,\vec{k}_\perp} + \left(f^{(\sigma)\star}_{\Omega,\vec{k}_\perp}, u^\star_{\omega,\vec{k}'}\right) f^{(\sigma)\star}_{\Omega,\vec{k}_\perp}. \end{aligned} \tag{1.35}$$

The frequency mixing is given by the non-vanishing coefficients in the terms relating positive frequencies modes with negative frequencies ones, which are expected to disappear for orthogonal modes. These coefficients, also known as Bogoliubov coefficients, play a pivotal role in quantum field theory in curved spacetime and non-inertial reference frames, as well as in condensed matter physics. All of Bogoliubov coefficients in (1.35) can be directly calculated from the definition of the Klein-Gordon inner product given in (1.19), even though it is not the only possible way to evaluate them. All possible combinations of inner products between Minkowski and Rindler modes can be derived from just two independent cases, for instance

$$\begin{aligned}\left(f^{(\sigma)}_{\Omega,\vec{k}_\perp}, u_{\omega,\vec{k}}\right) &= \frac{1}{\sqrt{2\pi\omega}}\frac{1}{\sqrt{2\sinh\left(\pi\Omega/a\right)}}\delta^{(2)}\left(\vec{k}_\perp - \vec{k}'_\perp\right) e^{\frac{\pi}{2}\frac{\Omega}{a}}\left(\frac{\omega_{k'}+k'_1}{\omega_{k'}-k'_1}\right)^{i\sigma\frac{\Omega}{2a}}\\ \left(f^{(\sigma)}_{\Omega,\vec{k}_\perp}, u^\star_{\omega,\vec{k}}\right) &= \frac{1}{\sqrt{2\pi\omega}}\frac{1}{\sqrt{2\sinh\left(\pi\Omega/a\right)}}\delta^{(2)}\left(\vec{k}_\perp + \vec{k}'_\perp\right) e^{-\frac{\pi}{2}\frac{\Omega}{a}}\left(\frac{\omega_{k'}+k'_1}{\omega_{k'}-k'_1}\right)^{i\sigma\frac{\Omega}{2a}};\end{aligned} \tag{1.36}$$

the other ones can be obtained using properties (1.20). Inserting relations (1.35) in the Rindler-Fulling field quantization (1.23), we obtain the relation between creation and annihilation operators. It is useful for this purpose to redefine the Minkowski creation and annihilation operator by defining

$$d^{(\sigma)}_{\Omega,\vec{k}_\perp} = \int_{-\infty}^{\infty} dk_1 \frac{1}{\sqrt{2\pi\omega_k}}\left(\frac{\omega_{k'}+k'_1}{\omega_{k'}-k'_1}\right)^{i\sigma\frac{\Omega}{2a}} a_{k1,\vec{k}_\perp}. \tag{1.37}$$

An important property of the operators $d^{(\sigma)}, d^{(\sigma)\dagger}$ is that, being a rescaled versions of the Minkowski annihilation and creation operators, they act on the Minkowski vacuum in the same manner as the $a, a^\dagger$, i.e.,

$$d^{(\sigma)}_{\Omega,\vec{k}_\perp}\left|0\right\rangle_M = 0 \qquad \forall\sigma,\Omega,\vec{k}_\perp. \tag{1.38}$$

It can be proven [15] that they satisfy the usual commutation relations

$$\begin{aligned}\left[d^{(\sigma)}_{\Omega,\vec{k}_\perp}, d^{(\sigma')\dagger}_{\Omega',\vec{k}'_\perp}\right] &= \delta_{\sigma\sigma'}\delta\left(\Omega-\Omega'\right)\delta^{(2)}\left(\vec{k}_\perp - \vec{k'}_\perp\right)\\ \left[d^{(\sigma)}_{\Omega,\vec{k}_\perp}, d^{(\sigma')}_{\Omega',\vec{k}'_\perp}\right] &= \left[d^{(\sigma)\dagger}_{\Omega,\vec{k}_\perp}, d^{(\sigma')\dagger}_{\Omega',\vec{k}'_\perp}\right] = 0.\end{aligned} \tag{1.39}$$

Therefore, the relation between Rindler and Minkowski creation and annihilation operators is given by

$$\begin{aligned} b^{(\sigma)}_{\Omega,\vec{k}_\perp} &= \frac{e^{\frac{\pi}{2}\frac{\Omega}{a}}}{\sqrt{2\sinh\left(\pi\Omega/a\right)}} d^{(\sigma)}_{\Omega,\vec{k}_\perp} + \frac{e^{-\frac{\pi}{2}\frac{\Omega}{a}}}{\sqrt{2\sinh\left(\pi\Omega/a\right)}} d^{(-\sigma)\dagger}_{\Omega,-\vec{k}_\perp} \\ b^{(\sigma)\dagger}_{\Omega,\vec{k}_\perp} &= \frac{e^{\frac{\pi}{2}\frac{\Omega}{a}}}{\sqrt{2\sinh\left(\pi\Omega/a\right)}} d^{(\sigma)\dagger}_{\Omega,\vec{k}_\perp} + \frac{e^{-\frac{\pi}{2}\frac{\Omega}{a}}}{\sqrt{2\sinh\left(\pi\Omega/a\right)}} d^{(-\sigma)}_{\Omega,-\vec{k}_\perp} . \end{aligned} \tag{1.40}$$

One of the most well-established and notable features of the Unruh effect is the inequivalence of the Minkowski and Rindler vacua, which we are going to display. We consider for example, in the right wedge, the Rindler number operator $N^{(+)}_R = b^{(+)\dagger}_{\Omega,\vec{k}_\perp} b^{(+)}_{\Omega,\vec{k}_\perp}$ and the Minkowski number operator $N^{(+)}_M = d^{(+)\dagger}_{\Omega,\vec{k}_\perp} d^{(+)}_{\Omega,\vec{k}_\perp}$, and we calculate their expectation value with respect to Minkowski vacuum

$$\begin{aligned} &{}_M\langle 0|\, N^{(+)}_M \,|0\rangle_M = 0 \\ &{}_M\langle 0|\, N^{(+)}_R \,|0\rangle_M = \frac{e^{\pi\frac{\Omega}{a}}}{2\sinh\left(\pi\Omega/a\right)}\, {}_M\langle 0|\, d^{(+)}_{\Omega,-\vec{k}_\perp} d^{(+)\dagger}_{\Omega,-\vec{k}_\perp} \,|0\rangle_M = \frac{1}{e^{2\pi\Omega/a}-1} . \end{aligned} \tag{1.41}$$

Minkowski and Rindler observers do not agree on the particle content of the Minkoski vacuum. The first one sees the vacuum as empty. The second one sees the vacuum as populated by particles which, surprisingly enough, manifest a thermal behaviour displaying a Bose-Einstein-like distribution at Unruh temperature $T_U = a/(2\pi)$. This phenomenon is one of the most notorious embodiments of the Unruh effect, which takes place whenever we are considering observables (in both Minkowski and Rindler representations) with support in only one of the two Rindler wedges, i.e., $\mathbb{I}^{(-)} \otimes O^{(+)}$ or $O^{(-)} \otimes \mathbb{I}^{(+)}$. These seem to be the natural kind of observables for an uniformly accelerated observer, which cannot measure or experience anything from the opposite wedge he/she is living in. This observation is corroborated by the form of the modes (1.21) which support lies in only one of the two possible wedges. Building on this observation, several authors (e.g. [2],[18]) have shown that the Unruh effect can be expressed in a different way than in (1.41) and can be directly observed through the computation of the expectation values of the aforementioned observables. Without delving in the specifics of calculations (which can be found in [15]), starting from the relations (1.40), it is possible to express Minkowski vacuum in terms of Rindler states which, for a single mode, can be written as

$$
\begin{aligned}
|0\rangle_M &= \sqrt{1-e^{-2\pi\frac{\Omega}{a}}}\exp\left(e^{-\pi\frac{\Omega}{a}} b^{(+)\dagger}_{\Omega,\vec{k}_\perp} b^{(-)\dagger}_{\Omega,-\vec{k}_\perp}\right)|0\rangle_R \\
&= \sqrt{1-e^{-2\pi\frac{\Omega}{a}}}\sum_n e^{-n\pi\frac{\Omega}{a}}|n,n\rangle_R\,,
\end{aligned} \tag{1.42}
$$

where in the last step we have expanded the exponential in order to apply the creation operators to the vacuum. It is readily observed that the relation between the two vacua is non-unitary, and Minkowski vacuum, in terms of Rindler Fock space, appears as a set of pairwise particles (one couple for every possible mode) correlated between the two different wedges. This kind of correlation is the Einstein-Podolsky-Rosen (EPR) type. Ultimately, Minkowski vacuum ends up to be a statistical mixture of Rindler states and, once again, what is vacuum to an inertial observer does not appear as empty space for an uniformly accelerated observer. The expectation value of a generic operator which support is the right Rindler wedge is

$$
\begin{aligned}
&{}_M\langle 0|\mathbb{I}^{(-)}\otimes O^{(+)}|0\rangle_M = \\
&\quad = \left(1-e^{-2\pi\frac{\Omega}{a}}\right)\sum_{n,m}{}_R\langle m,m|\,e^{-m\pi\frac{\Omega}{a}}\left(\mathbb{I}^{(-)}\otimes O^{(+)}\right)e^{-n\pi\frac{\Omega}{a}}|n,n\rangle_R \\
&\quad = \left(1-e^{-2\pi\frac{\Omega}{a}}\right)\sum_n e^{-2n\pi\frac{\Omega}{a}}{}_{R^{(+)}}\langle n|\,O^{(+)}|n\rangle_{R^{(+)}}\,.
\end{aligned} \tag{1.43}
$$

In the second step we have traced out the degrees of freedom in the left wedge as a consequence of the specific kind of observables we are considering. The result can be reformulated in a more formal way by considering the normal ordered Rindler Hamiltonian

$$
:H_R^{(+)}:=\int_0^\infty d\Omega\int d^2k_\perp\,\Omega\, b_k^{(+)\dagger}b_k^{(+)} \tag{1.44}
$$

which simplifies to $:H_R^{(+)}:=\Omega/a\, b_k^{(+)\dagger}b_k^{(+)}$ for a single mode. Specifically, it can be written as follow

$$
{}_M\langle 0|\,\mathbb{I}^{(-)}\otimes O^{(+)}|0\rangle_M = \mathrm{Tr}_{(+)}\left(\rho^{(+)}O^{(+)}\right), \tag{1.45}
$$

with $\rho^{(+)}$ the density matrix of Rindler states in the right wedge

$$
\rho^{(+)} = \frac{\exp\left(-\frac{2\pi}{a}:H_R^{(+)}:\right)}{\mathrm{Tr}_{(+)}\left[\exp\left(-\frac{2\pi}{a}:H_R^{(+)}:\right)\right]}. \tag{1.46}
$$

Once again, identifying the Unruh temperature to $T_U = a/(2\pi)$ we obtain the canonical density matrix for a system at finite temperature T_U; what appears to be empty space to an inertial observer, is a thermal bath of Rindler quanta to a uniformly accelerated observer. This is due to the disruption of the EPR correlation between the two wedges when we trace out one of them. To an experienced reader, equation (1.45), which goes by the name of thermalization theorem, is non other than the KMS condition in disguise [19–21]. The first to unveil this mechanism was G.L. Sewell [18], who, by utilizing the Bisognano-Wichmann theorem [22], established a rigorous connection between quantum field theory in curved spacetime and thermofield dynamics.

This result will be of crucial importance in the following of this chapter and will serve as a key point in the interpretation of the issues discussed in the published article.

1.1.3 *Spontaneous symmetry breaking and restoration: a path integral approach*

The concept of symmetries has played a pivotal role in physics for more than a century. Symmetry principles often serve us as a compass, guiding our reasoning and analyses in the pursuit of understanding the physical world, influencing the structure of physical laws and the behavior of systems. Symmetries in physics can take many different forms; there are spacetime symmetries, as we briefly encountered in the previous subsection, internal symmetries, discrete symmetries, dynamical symmetries, topological symmetry, and so on. However, these symmetries are not always realized exactly and sometimes can be approximate or broken. We are interested in spontaneously broken symmetries, that is, symmetries of the theory that are not realized as symmetry transformations for the vacuum and for the physical states.

To determine whether the symmetry condition (i.e., the phase) of a given theory is broken or unbroken, several approaches are available. The approaches of particular interest to us are the study of the VEV of the fields and the analysis of the minima of the (effective) potential.
In practical terms, an (internal) symmetry is said to be in a broken phase when the field configuration that is preserved under the symmetry transformation $\mathcal{T}$, which leaves the Lagrangian unchanged, differs from the VEV of the field. That is,

$$\mathcal{T}(\mathcal{L}) = \mathcal{L} \quad \wedge \quad \mathcal{T}\left(\phi_{sym}\right) = \phi_{sym} \quad \wedge \quad \phi_{sym} \neq \langle 0|\, \phi\, |0\rangle \, . \qquad (1.47)$$

Since, for most of the symmetries of interests, the potential is still invariant under the action of such symmetry, the difference between ϕ_{sym} and $\langle 0|\, \phi\, |0\rangle$ suggests the presence of multiple minima in the potential. The vacuum is then said to be degenerate, and so, non-unique. Different types of symmetries lead to distinct forms of vacuum degeneracy, and the set of field configurations that correspond to the vacuum states can be parametrized by their field VEVs, forming what is known as the moduli space.
As we already anticipated, in this chapter, we will analyze a real scalar field with a ϕ^4 potential, which, for negative values of the m^2 parameter, provides one of the simplest cases of SSB. In this scenario, the broken symmetry is the discrete $\mathbb{Z}_2$ internal symmetry, characterized by the transformation $\phi \to -\phi$, which results in two distinct, disconnected vacua. As the symmetry is discrete, the theory does not exhibit the presence of Goldstone bosons, which are characteristic of continuously broken symmetries. Nevertheless, the key physical insights and features relevant to our study of SSB in this 'toy model' can be generalized to more sophisticated and complex theories.

Since we are interested in the shape of the potential, let us first specify the Lagrangian of the system

$$\mathcal{L} = \frac{1}{2} g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi - \frac{m^2}{2} \phi^2 - \frac{\lambda}{4!} \phi^4 \tag{1.48}$$

so we can trivially retrieve the classical potential

$$V = \frac{m^2}{2} \phi^2 + \frac{\lambda}{4!} \phi^4 \tag{1.49}$$

(we have omitted the overall multiplicative term $\sqrt{-g}$ from the Lagrangian definition to simplify the following calculations, we will introduce it directly in the definition of the volume element in the action). If we allow the mass-squared parameter m^2 to take on negative values, SSB occurs for real field configurations. The vacuum is degenerate and is realized, at tree level ($O(\hbar^0)$), for the field configurations

$$\phi_{cl}^{\pm} = \pm \sqrt{-\frac{6m^2}{\lambda}}. \tag{1.50}$$

As expected, according to the nature of the $\mathbb{Z}_2$ symmetry, the two vacua are mapped to each other as $\phi \to -\phi$.
In the following, we will explore the possibility of a phase transition of the system, with the symmetry, originally broken, shifting to an unbroken phase.
In practice, the phase state and the position of the minima are determined by the VEV

$$\Phi(x) = \langle 0 | \phi(x) | 0 \rangle , \tag{1.51}$$

which is also referred to as 'order parameter' and represents the quantum corrected version of ϕ_{cl}. The state of the $\mathbb{Z}_2$ symmetry is defined as

$$\begin{aligned} \Phi(x) &\neq 0 \qquad \text{broken phase} \\ \Phi(x) &= 0 \qquad \text{unbroken phase.} \end{aligned} \tag{1.52}$$

Alternatively, expressed in words, if the minima of the effective potential, due to quantum and/or statistical effects, are shifted to such an extent that they converge to the origin, forming a single absolute minimum, then the symmetry is restored.
Many factors can modify the classical values of the field at the minima, including curvature, temperature, and others. Adopting a path integral formalism, we define a prescription to evaluate the effect of one-loop quantum corrections and analyze the resulting modified shape of the potential and shift of the position of its minima.

1.1.3.1 *Effective action, background field method, one-loop corrections and equations of motion*

Let us first define the generating functional $Z[J]$ for Green's functions in the presence of an external current J,

$$Z[J] = \int \mathcal{D}\phi \, e^{i\left(S[\phi]+\int d^4x\sqrt{-g}J(x)\phi(x)\right)}. \tag{1.53}$$

As the name implies, any n-point Green's function can be derived from it, as these functions correspond to the coefficients of its functional expansion around $J = 0$, i.e.,

$$G^{(n)}(x_1,\ldots,x_n) = \left(\frac{-i}{\sqrt{-g}}\right)^n \frac{\delta^n Z[J]}{\delta J(x_1)\cdots\delta J(x_n)}\bigg|_{J=0}. \tag{1.54}$$

However, since we are interested in calculating the loop expansion (specifically at first order), we are primarily concerned with a smaller subset of the full set of Green's functions generated by (1.54). These are the connected Green's function and one-particle irreducible (1PI) diagrams. For this reason, it is convenient to introduce another functional, known as the generating functional of connected Green's function, denoted by $W[J]$, and defined as

$$W[J] = -i\ln\left(Z[J]\right), \tag{1.55}$$

from which connected Green's functions can be generated in the same manner as with $Z[J]$,

$$\begin{aligned} G_c^{(n)}(x_1,\ldots,x_n) &= \left(\frac{-i}{\sqrt{-g}}\right)^n \frac{\delta^n W[J]}{\delta J(x_1)\cdots\delta J(x_n)}\bigg|_{J=0} \\ &= \langle 0|\, T\left(\phi(x_1)\ldots\phi(x_n)\right)|0\rangle_c \end{aligned} \tag{1.56}$$

with T the time-ordering operator.
More generally, in the path integral formalism, the expectation value of a composite operator $O[\phi]$ can be written as

$$\langle 0|\, O[\phi]\,|0\rangle_J = Z^{-1}[J]\int \mathcal{D}\phi\, O[\phi]\; e^{i\left(S[\phi]+\int d^4x\sqrt{-g}J(x)\phi(x)\right)}. \tag{1.57}$$

In this way, it is possible to determine the expectation value of the field

$$\Phi_J(x) \equiv \frac{-i}{\sqrt{-g}}\frac{\delta W[J]}{\delta J(x)} = \langle 0|\,\phi(x)\,|0\rangle_J, \tag{1.58}$$

which we previously referred to as order parameter.
Lastly, we want to define the Effective action $\Gamma[\Phi]$, which is mathematically the Legendre transform of $W[J]$. In order to do so, the functional

dependence of Φ on J, given by (1.58), must be inverted to express J as a functional of Φ. This allows us to define the effective action as

$$\Gamma[\Phi] = W[J] - \int d^4x \sqrt{-g}\, J\, \Phi. \tag{1.59}$$

The effective action, which generates 1PI diagrams in the same way as Z and W do, has a profound physical interpretation. It represents a quantum-corrected version of the action, with all quantum corrections taken into account. The equation of motion for the field Φ is derived following the same approach as in the classical case, i.e.,

$$\begin{cases} \left.\dfrac{1}{\sqrt{-g}}\dfrac{\delta S}{\delta \phi}\right|_{\phi_0} = -J & \text{classical equation of motion} \quad (1.60a) \\ \left.\dfrac{1}{\sqrt{-g}}\dfrac{\delta \Gamma}{\delta \phi}\right|_{\Phi} = -J & \text{quantum equation of motion.} \quad (1.60b) \end{cases}$$

For this reason, Φ represents the quantum-corrected value, with respect to ϕ_0, of the field VEV.
Most of the times, the quantum equations of motion are difficult to solve exactly, and a perturbative approach is invoked instead. Perturbation is realized through a loop expansion, expressed in powers of $\hbar$, of the effective action

$$\Gamma[\Phi] = S[\Phi] + \hbar\Gamma^{(1)}[\Phi] + O\left(\hbar^2\right). \tag{1.61}$$

The inclusion of additional, higher order loops leads to a more accurate determination of the value of the VEV Φ. For the purpose of this thesis, as is frequently considered adequate in this type of problems, we limit our analysis to a one-loop expansion.
Therefore, we adopt the approach commonly referred to as the background field method.
Firstly, we decompose the field ϕ in two quantitatively distinct contributions, as

$$\phi = \phi_B + \tilde{\phi}, \tag{1.62}$$

where ϕ_B represents the 'background field', a non-dynamical but arbitrary field, and $\tilde{\phi}$ represents quantum fluctuations upon the background. When we say that ϕ_B is non-dynamical, it means that it is not integrated over in the path integral; that is the role of quantum fluctuations $\tilde{\phi}$. For this reason, there are no ϕ_B particles in loops and no external $\tilde{\phi}$ fields. For the actual calculation we make use of a slightly different definiton of effective action, since we are interested in the case where the external current J is vanishing, i.e.,

$$\Gamma[\phi_b] = -i \ln\left(\int_{1PI} \mathcal{D}\tilde{\phi}\, e^{iS[\phi_b+\tilde{\phi}]}\right) \tag{1.63}$$

Subsequently, considering the theory described by (1.48), we expand the Lagrangian density inside the classical action, and obtain

$$\begin{aligned}\sqrt{-g}\mathcal{L}[\phi_B+\tilde{\phi}] &= \sqrt{-g}\mathcal{L}\left[\phi_B\right] + \cancel{\partial_\mu\left(\sqrt{-g}\tilde{\phi}\partial^\mu\phi_B\right)} \\ &+ \sqrt{-g}\left(\frac{1}{2}\partial_\mu\tilde{\phi}\partial^\mu\tilde{\phi} - \frac{m^2}{2}\tilde{\phi}^2 - \frac{\lambda}{24}\tilde{\phi}^4 - \frac{\lambda}{4}\tilde{\phi}^2\phi_B^2 - \frac{\lambda}{6}\tilde{\phi}^3\phi_B\right) \\ &- \sqrt{-g}\tilde{\phi}\left(\partial_\mu\partial^\mu\phi_B + m^2\phi_B + \frac{\lambda}{6}\phi_B^3\right).\end{aligned} \tag{1.64}$$

The total derivative terms coming from the Lagrangian density have been neglected. It is also possible to ignore the linear term in quantum fluctuations, as it can only enter one-particle reducible graphs and never 1PI amplitudes. Consequently, it follows that

$$\begin{aligned}&\mathcal{L}[\phi_B+\tilde{\phi}] = \\ &= \mathcal{L}\left[\phi_B\right] + \frac{1}{2}\partial_\mu\tilde{\phi}\partial^\mu\tilde{\phi} - \frac{1}{2}\tilde{\phi}^2\left(m^2 + \frac{\lambda}{2}\phi_B^2\right) - \frac{\lambda}{6}\phi_B\tilde{\phi}^3 - \frac{\lambda}{24}\tilde{\phi}^4.\end{aligned} \tag{1.65}$$

Inserting this expansion inside the effective action (1.63), we obtain

$$\begin{aligned}&\Gamma\left[\phi_B\right] = S\left[\phi_B\right] - \\ &i\ln\left(\int\mathcal{D}_{\tilde{\phi}}\exp\left[i\int d^4x\sqrt{-g}\left(\frac{1}{2}\partial^\mu\tilde{\phi}\partial_\mu\tilde{\phi} - \frac{1}{2}\tilde{\phi}^2\left(m^2+\frac{\lambda}{2}\phi_B^2\right) - \frac{\lambda}{6}\phi_B\tilde{\phi}^3 - \frac{\lambda}{24}\tilde{\phi}^4\right)\right]\right).\end{aligned} \tag{1.66}$$

So far, we have not performed any approximation in the computation of the effective action action; thus, in equation (1.66), it remains in its exact form.

We are now interested in the evaluation of the first loop contribution. To achieve this, we must disregard all cubic and quartic terms in the fluctuations present in the exponential, as they only contribute starting from the second loop correction. In this way, we obtain the one-loop effective action

$$\begin{aligned}&\Gamma\left[\phi_B\right] \approx S\left[\phi_B\right] - \\ &i\ln\left(\int\mathcal{D}_{\tilde{\phi}}\exp\left[i\int d^4x\sqrt{-g}\left(\frac{1}{2}\partial^\mu\tilde{\phi}\partial_\mu\tilde{\phi} - \frac{1}{2}\tilde{\phi}^2\left(m^2+\frac{\lambda}{2}\phi_B^2\right)\right)\right]\right).\end{aligned} \tag{1.67}$$

We are now interested in deriving the quantum equation of motion (1.60b) for the field ϕ_b and vanishing source. It can be written as

$$\left.\frac{\delta\Gamma}{\delta\phi_B}\right|_\Phi = \left.\frac{\delta S}{\delta\phi_B}\right|_\Phi - \frac{\lambda}{2}\Phi\left\langle 0\right|\tilde{\phi}^2\left|0\right\rangle = 0 \tag{1.68}$$

or, equivalently,

$$-\partial_\mu\left(\sqrt{-g}\partial^\mu\Phi\right)-\Phi\left(\frac{\lambda}{6}\Phi^2+m^2+\frac{1}{2}\lambda\left\langle 0\right|\tilde{\phi}^2\left|0\right\rangle\right)=0, \tag{1.69}$$

where we have used (1.57) to define $\left\langle 0\right|\tilde{\phi}^2\left|0\right\rangle$ (also $\langle\tilde{\phi}^2\rangle$ for brevity). We must now ensure that the (background) solutions of (1.69) correspond to the minima of the effective action Γ. To verify this, we need to confirm that the second derivative is positive at this stationary configurations, i.e.,

$$\frac{\delta^2\Gamma}{\delta\Phi^2}\geq 0 \tag{1.70}$$

which, in our case, translates to the positive semi-definiteness of the operator

$$A\left[\Phi\right]=-\partial_\mu\left(\sqrt{-g}\partial^\mu\right)-\left(\frac{\lambda}{2}\Phi^2+m^2+\frac{1}{2}\lambda\langle\tilde{\phi}^2\rangle\right)-\cancel{\frac{1}{2}\lambda\Phi\frac{\delta\langle\tilde{\phi}^2\rangle}{\delta\Phi}}. \tag{1.71}$$

We have neglected the last term, as the derivative of $\langle\tilde{\phi}^2\rangle$ with respect to Φ, once applied the definition in (1.57), is of order $O\left(\lambda\right)$. Consequently, the neglected term exhibits an overall dependence on λ of order $O\left(\lambda^2\right)$. Since we expect the coupling constant to be small, $\lambda\ll 1$, this term is treated as a higher-order correction.

Unfortunately, the stationary background configurations solving (1.69) correspond to saddle points, as the kinetic operator $-\partial_\mu\left(\sqrt{-g}\partial^\mu\right)$ is not positive-definite due to the indefinite nature of the Minkowski metric. Specifically, when considering different directions in the space of fluctuations, some directions lead to an increase of the action, while others lead to a decrease, depending on whether the fluctuations are predominantly along the time direction or the spatial directions. As a result, when analyzing the operator $A[\Phi]$, the true minima of the theory manifest as saddle points. For this purpose, it is convenient to express the one-loop effective action (1.67) in Euclidean coordinates (see also Appendix A), by considering

$$t=-it_{\mathrm{E}},\qquad \partial_t=i\partial_{t_{\mathrm{E}}},\qquad d^4x\sqrt{-g}=-i\,d^4x_{\mathrm{E}}\sqrt{g_{\mathrm{E}}}, \tag{1.72}$$

the associated Euclidean Action

$$S_{\mathrm{E}}=\int d^4x_{\mathrm{E}}\sqrt{g_{\mathrm{E}}}\left[\frac{1}{2}g_{\mathrm{E}}^{\mu\nu}\partial_{\mathrm{E}\mu}\phi\partial_{\mathrm{E}\nu}\phi+\frac{m^2}{2}\phi^2+\frac{\lambda}{4!}\phi^4\right], \tag{1.73}$$

and by redefining the generating functional W as

$$W_{\mathrm{E}}[J]=-\ln\left(Z_{\mathrm{E}}\left[J\right]\right). \tag{1.74}$$

Thus, by repeating the same procedure as before, we obtain the Euclidean one-loop effective action

$$\Gamma_{\mathrm{E}}[\phi_B] \approx S_{\mathrm{E}}[\phi_B] - \ln\left(\int \mathcal{D}_{\tilde{\phi}} \exp\left[-\int d^4x\sqrt{g_{\mathrm{E}}}\left(\frac{1}{2}\partial^{\mu}_{\mathrm{E}}\tilde{\phi}\partial_{\mathrm{E}\mu}\tilde{\phi} + \frac{1}{2}\tilde{\phi}^2\left(m^2 + \frac{\lambda}{2}\phi_B^2\right)\right)\right]\right), \tag{1.75}$$

with the new quantum equation of motion

$$-\partial_{\mathrm{E}\mu}\left(\sqrt{g_{\mathrm{E}}}\partial^{\mu}_{\mathrm{E}}\Phi\right) + \Phi\left(\frac{\lambda}{6}\Phi^2 + m^2 + \frac{1}{2}\lambda\langle 0|\,\tilde{\phi}^2\,|0\rangle\right) = 0, \tag{1.76}$$

which remarkably presents the same background solutions as the Minkowski counterpart, and the second variation of the effective action

$$A[\Phi] \equiv \frac{\delta^2\Gamma_{\mathrm{E}}}{\delta\Phi^2} = -\partial_{\mathrm{E}\mu}\left(\sqrt{g_{\mathrm{E}}}\partial^{\mu}_{\mathrm{E}}\right) + \left(\frac{\lambda}{2}\Phi^2 + m^2 + \frac{1}{2}\lambda\langle\tilde{\phi}^2\rangle\right) \geq 0. \tag{1.77}$$

The latter equation will be the focus in the analysis of the minima and maxima of the theory.

Broadly speaking, for an arbitrary (potentially curved) spacetime, this is the extent to which we can explore and, to say something more about the background field Φ, one has to specify the particular spacetime of interest. Furthermore, equation (1.69) is not as easy to solve as it seems in a curved spacetime. In flat space, the full group of translational symmetries can be employed, granting Φ to be constant, and transforming (1.69) in an algebraic equation rather than a differential one. However, even in particularly simple curved spacetimes like Schwarzschild's, the number of translational symmetries is not maximal and, as a result, Φ is generally not constant. Consequently, to determine Φ from (1.69), one have to solve the full differential equation, which depends on $\langle\tilde{\phi}^2\rangle$. This expectation value, in turn, depends on Φ via the definition in (1.57). In such cases, Φ and $\langle\tilde{\phi}^2\rangle$ must be computed iteratively, often using numerical methods, until convergence is achieved.
From all of this analysis, the significance of the quantity $\langle\tilde{\phi}^2\rangle$, also known as vacuum polarization (in a broad sense), has emerged. Vacuum polarization is divergent and requires a regularization procedure for its computation, adding further complexity to the whole process. This quantity will be our primary focus in the following discussion.
We will briefly outline the calculation of Φ in flat Minkowski space at finite temperature, as this will be useful for subsequent discussions.

1.1.3.2 *Symmetry breaking, restoration and effective potential*

Using the formalism just introduced, it is important to precisely define under which conditions the system is in a broken or unbroken phase of symmetry. Starting from the effective action and the associated equation of motion, it is possible to define the state of the $\mathbb{Z}_2$ symmetry of theory (1.48) as follows

$$\begin{aligned} \left.\frac{\delta\Gamma}{\delta\phi}\right|_{\Phi\neq 0} &= 0 \qquad \text{broken phase} \\ \left.\frac{\delta\Gamma}{\delta\phi}\right|_{\Phi=0} &= 0 \qquad \text{unbroken phase.} \end{aligned} \tag{1.78}$$

It can also be said that the field configuration satisfing the quantum equation of motion and representing the true vacuum of the theory, should be vanishing for the symmetry to be unbroken. We are therefore interested in exploring the mechanism underlying the system's symmetry states and phase transitions. We initially study the case of the symmetry breaking and restoration by temperature in a flat Minkowski spacetime, using equations of motion (and derivative) (1.69, 1.71). Later on, the effective potential will be defined as an alternative framework to explore these kind of problems.

Symmetry breaking and restoration in Minkoski spacetime

Let us consider flat Minkowski spacetime. We recall the previously derived quantum equation of motion along with its derivative

$$\begin{cases} -\partial_\mu\left(\sqrt{-g}\partial^\mu\Phi\right) - \Phi\left(\dfrac{\lambda}{6}\Phi^2 + m^2 + \dfrac{1}{2}\lambda\langle\tilde{\phi}^2\rangle\right) = 0, & \text{(1.79a)} \\ A\left[\Phi\right] \equiv -\partial_{\mathrm{E}\mu}\left(\sqrt{g_{\mathrm{E}}}\partial^\mu_{\mathrm{E}}\right) + \left(\dfrac{\lambda}{2}\Phi^2 + m^2 + \dfrac{1}{2}\lambda\langle\tilde{\phi}^2\rangle\right) \geq 0. & \text{(1.79b)} \end{cases}$$

Firstly, we are interested in the solution of (1.79a), i.e. the field values at the stationary points. In a flat Minkowski space, endowed with the full set of translational symmetries, we can consider Φ to be constant, and, as a consequence, the kinetic term in (1.79a) vanishes. We are then left with

$$\Phi\left(\frac{\lambda}{6}\Phi^2 + m^2 + \frac{1}{2}\lambda\langle\tilde{\phi}^2\rangle\right) = 0, \tag{1.80}$$

which has solutions

$$\begin{cases} \Phi = 0 \\ \Phi = \pm\sqrt{-6\dfrac{m^2}{\lambda} - 3\langle\tilde{\phi}^2\rangle}. \end{cases} \tag{1.81}$$

Not every stationary point is a minimum, so we need to check that condition (1.79b) is satisfied. Before proceeding, we acknowledge that the spectrum of the operator $-\partial_{\mathrm{E}\mu}(\sqrt{g}\partial^{\mu}_{\mathrm{E}})$ is positive and bounded from below by zero. For this reason, we consider zero as the reference value for the Euclidean Laplacian operator, ensuring that condition (1.79b) is satisfied for every possible eigenvalue. The positivity conditions for A then becomes

$$\begin{cases} A\,[0] \geq 0 \quad \Longrightarrow \quad -m^2 \leq \frac{1}{2}\lambda\langle\tilde{\phi}^2\rangle \\ A\left[\pm\sqrt{-6\frac{m^2}{\lambda} - 3\langle\tilde{\phi}^2\rangle}\right] \geq 0 \quad \Longrightarrow \quad -m^2 \geq \frac{1}{2}\lambda\langle\tilde{\phi}^2\rangle. \end{cases} \tag{1.82}$$

(Recall that m^2 must be negative for symmetry breaking to occur, cfr. (1.1.3)).

This means that, regardless of the cause of the quantum fluctuations, the field configurations $\Phi = \pm\sqrt{-6m^2/\lambda - 3\langle\tilde{\phi}^2\rangle}$ represent the true minima of the (effective) potential as long as $\langle\tilde{\phi}^2\rangle \leq -2m^2/\lambda$, while $\Phi = 0$ is a local maximum. Thus, we still find the theory to be in a broken phase.

With the growth in magnitude of $\langle\tilde{\phi}^2\rangle$, the minima shifts towards the origin at $\Phi = 0$ that, for $\langle\tilde{\phi}^2\rangle \geq -2m^2/\lambda$, become the absolute minimum of the potential and the symmetry is restored.

As a double check, we immediately notice that in the broken phase, by neglecting the contribution from quantum fluctuation given by $\langle\tilde{\phi}^2\rangle$, the minima are found to be $\Phi = \pm\sqrt{-6\frac{m^2}{\lambda}}$ and we immediately retrieve the classical value obtained in (1.50).

So far, we have not discussed the nature of $\langle\tilde{\phi}^2\rangle$; let us now explore this aspect by examining the case of quantum fluctuations induced by temperature.

In the finite-temperature scenario, Φ exhibits a temperature dependence that is embedded in the $\langle\tilde{\phi}^2\rangle$ term, which typically accounts for all quantum and statistical corrections to the classical minima solutions. This calculation has been performed by several authors (e.g. [23]) that obtained, in the limit $m/T \to 0$,

$$\langle\tilde{\phi}^2\rangle = \frac{T^2}{12}. \tag{1.83}$$

Using the conditions derived in (1.82), we can determine a critical temperature above which the system transitions into an unbroken phase

$$T_c^2 = -24\frac{m^2}{\lambda}. \tag{1.84}$$

1.1.3.3 *Effective potential and symmetry restoration*

An alternative method that is often invoked in the study of quantum correction is the calculation of the effective potential. Similarly to the effective action, the effective potential represents the quantum-corrected version of the classical potential, whose minima determine the possible vacua of the field.
The effective potential can be derived directly from the effective action, as

$$\Gamma\left[\Phi\right] = \int d^4x\sqrt{-g}\left[-V\left(\Phi\right) + \frac{1}{2}Z\left(\Phi\right)\partial_\mu\Phi\partial^\mu\Phi + (\text{h.o. derivative terms})\right]. \tag{1.85}$$

$V\left(\Phi\right)$ is the effective potential, which includes both the tree-level terms present in the Lagrangian and the loop contributions with external fields having vanishing momenta. The calculation of the effective potential in the general scenario of a non-constant Φ must also account for the derivative terms in (1.85). However, if we limit ourselt to translationally invariant spacetimes (as the inner region of Rindler space), the relation (1.85) symply reduces to

$$\Gamma\left[\Phi\right] = -\int d^4x\,\sqrt{-g}\,V\left(\Phi\right). \tag{1.86}$$

and the symmetry phases in (1.78) can be written as

$$\begin{cases} \left.\dfrac{\partial V}{\partial \Phi}\right|_{\Phi\neq 0} = 0 & \text{broken phase} \qquad (1.87\text{a}) \\ \left.\dfrac{\partial V}{\partial \Phi}\right|_{\Phi= 0} = 0 & \text{unbroken phase.} \qquad (1.87\text{b}) \end{cases}$$

Let us examine more closely the spontaneously broken ϕ^4 theory at finite temperature discussed before and, more specifically, the associated one-loop correction to the potential.

It is useful to separate the different contributions to the potential as follows [1],

$V^{(0)}$	Tree-level (classical) potential
$V_0^{(1)}$	Zero-temperature one-loop effective potential
$V_0 = V^{(0)} + V_0^{(1)}$	Zero-temperature total effective potential
$V_\beta^{(1)}$	Finite-temperature one-loop effective potential
$V^{(1)} = V_0^{(1)} + V_\beta^{(1)}$	Total one-loop effective potential
$V = V_0 + V_\beta^{(1)}$	Finite-temperature total effective potential

The field configuration that is invariant under the symmetry transformation is trivially $\Phi_{sym} = 0$ and potential $V\left(\Phi_{sym}\right) = 0$. However, starting from a broken phase, this corresponds to a local maximum and cannot represent the true vacuum of the theory. For the symmetry to be restored, the quantum fluctuation should be strong enough to shift the true broken minima towards $\Phi = 0$, that becomes the new unique absolute minimum. This can be summarized by requiring the second derivative of the effective potential being positive al $\Phi = 0$, ie.,

$$\begin{cases} \left.\dfrac{\partial^2 V}{\partial\Phi\partial\Phi}\right|_{\Phi=0} \leq 0 & \text{broken phase} & (1.88a) \\ \left.\dfrac{\partial^2 V}{\partial\Phi\partial\Phi}\right|_{\Phi=0} \geq 0 & \text{unbroken phase.} & (1.88b) \end{cases}$$

We now present the conditions under which quantum fluctuations induce symmetry restoration in the effective potential. We start by imposing (1.88b), which can be written as

$$0 \leq \frac{\partial}{\partial\Phi}\left(\frac{\partial V}{\partial\Phi}\right)\bigg|_{\Phi=0} = \frac{\partial}{\partial\Phi}\left(2\Phi\frac{\partial V}{\partial\Phi^2}\right)\bigg|_{\Phi=0} = 2\,\frac{\partial V}{\partial\Phi^2}\bigg|_{\Phi=0} \tag{1.89}$$

and, together with the usual definition of mass parameter in quantum field theory, which is

$$m^2 = \frac{\partial^2 V_0}{\partial\Phi\partial\Phi}\bigg|_{\Phi=0} = 2\,\frac{\partial V_0}{\partial\Phi^2}\bigg|_{\Phi=0} \tag{1.90}$$

we obtain

$$\left.\frac{\partial V_\beta^{(1)}}{\partial \Phi^2}\right|_{\Phi=0} \geq -\frac{m^2}{2}. \tag{1.91}$$

We have generally described quantum corrections to the effective potential $V_\beta^{(1)}$ as originating from finite-temperature contributions. However, we have not yet considered any thermal processes in this derivation. In fact, the prescription above is valid for any kind of quantum contribution that acts from first loop corrections and that can be detached from the always present one-loop term $V_0^{(1)}$.

Let us specialize the calculation to the case of ϕ^4 theory at finite temperature. Starting from the one-loop effective action (1.67) (and replacing $\phi_B \to \Phi$), we can perform the integration over the quantum fluctuation, that can be reformulated as

$$\int \mathcal{D}_{\tilde{\phi}} \exp\left[-i\int d^4x \frac{1}{2}\tilde{\phi}\left[\sqrt{-g}\left(\Box + M_\Phi^2\right)\right]\tilde{\phi}\right], \tag{1.92}$$

where we have used the shortcut $\Box \equiv (-g)^{-\frac{1}{2}}\partial_\mu\sqrt{-g}\partial^\mu$ and $M_\Phi^2 = m^2 + \frac{\lambda}{2}\Phi^2$. After performing the Wick rotation $t \to -it_{\mathrm{E}}$ (see also Appendix A) this becomes a standard Gaussian integral over the field $\tilde{\phi}$, and has solution

$$\begin{gathered}\mathcal{N}\det^{-\frac{1}{2}}\left[\sqrt{-g}\left(\Box + \left(m^2 + \frac{\lambda}{2}\Phi^2\right)\right)\right] \\ = \\ \mathcal{N}\exp\left[-\frac{1}{2}\mathrm{Tr}\ln\left(\sqrt{-g}\left(\Box + M_\Phi^2\right)\right)\right].\end{gathered} \tag{1.93}$$

The resulting effective action is

$$\Gamma[\Phi] = S[\Phi] + \underbrace{\frac{i}{2}\mathrm{Tr}\ln\left[\sqrt{-g}\left(\Box + M_\Phi^2\right)\right]}_{\Gamma^{(1)}} \cancel{-i\ln(\mathcal{N})} \tag{1.94}$$

where we have neglected the last term since potentials can be always redefined up to a constant. In this decomposition we can immediately recognize the one-loop total contribution to the effective action $\Gamma^{(1)}$. The operator $\sqrt{-g}\left(\Box + M_\Phi^2\right)$ is nothing but the inverse propagator $\mathcal{G}^{-1}\left(M_\Phi^2\right)$, which act on the Hilbert space spanned by positions and momenta eigenvectors; its trace needs to be calculated accordingly.

After establishing all the definitions in a general context, the remainder of this subsection will focus on the case of flat spacetime, using Minkowski coordinates.

Since the inverse propagator is diagonal in momentum space, it is convenient to calculate the trace over such states. The general matrix element is given by

$$\begin{aligned} \langle p'|\, \mathcal{G}^{-1}\left(M_\Phi^2\right)|p\rangle &= \delta\left(p'-p\right)\mathcal{G}^{-1}\left(p, M_\Phi^2\right) \\ \langle p'|\ln\left(\mathcal{G}^{-1}\left(M_\Phi^2\right)\right)|p\rangle &= \delta\left(p'-p\right)\ln\left(\mathcal{G}^{-1}\left(p, M_\Phi^2\right)\right) \end{aligned} \tag{1.95}$$

so that $\Gamma^{(1)}$ can be written as

$$\Gamma^{(1)} = \frac{i}{2}\mathrm{Tr}\ln\left(\mathcal{G}^{-1}\left(M_\Phi^2\right)\right) = \frac{i}{2}\int d^4x \int \frac{d^4p}{(2\pi)^4}\ln\left(-p^2+M_\Phi^2\right), \tag{1.96}$$

where we have used the identity $\delta^4(p)|_{p=0} = (2\pi)^{-4}\int d^4x$.
Applying definition (1.86), we obtain the one-loop effective potential $V^{(1)}$

$$V^{(1)} = -\frac{i}{2}\int\frac{d^4p}{(2\pi)^4}\ln\left(-p^2+M_\Phi^2\right) = \frac{1}{2}\int\frac{d^4p_{\mathrm{E}}}{(2\pi)^4}\ln\left(p_{\mathrm{E}}^2+M_\Phi^2\right) \tag{1.97}$$

where in the last equation we have performed the Wick rotation $p^0 \to ip^0_{\mathrm{E}}$.
We now consider the system to be thermalized at an inverse temperature β and, as explained in Appendix A, the energy spectrum becomes discretized and expressed in terms of the Matsubara frequencies.
The effective potential then becomes

$$V^{(1)} = \frac{1}{2\beta}\sum_n\int\frac{d^3p}{(2\pi)^3}\ln\left(\frac{4\pi^2n^2}{\beta^2} + \underbrace{\vec{p}^{\,2}+M_\Phi^2}_{E_M^2}\right). \tag{1.98}$$

The sum over Matsubara frequencies can be performed exactly, by considering

$$\begin{aligned} S\left(E_M\right) &= \sum_n \ln\left(\frac{4\pi^2n^2}{\beta^2}+E_M^2\right) \\ S'\left(E_M\right) &= \sum_n \frac{2E_M}{4\pi n^2/\beta^2+E_M^2} = 2\beta\left(\frac{1}{2}+\frac{1}{e^{\beta E_M}-1}\right), \end{aligned} \tag{1.99}$$

where we have used the renowned summation of the mathematical series

$$\sum_{n=-\infty}^{\infty}\frac{y}{y^2+n^2} = \pi\coth\left(\pi y\right) = \pi\left(1+\frac{2}{e^{2\pi y}-1}\right). \tag{1.100}$$

Integrating back, we obtain

$$S\left(E_M\right) = 2\beta\left(\frac{E_M}{2} + \frac{1}{\beta}\ln\left(1 - e^{-\beta E_M}\right)\right) + \cancel{const} \tag{1.101}$$

where we have neglected the constant term by a redefinition of the potential.
The resulting one-loop effective potential $V^{(1)}$ at finite temperature can be then decomposed as $V^{(1)} = V_0^{(1)} + V_\beta^{(1)}$, with

$$\begin{cases} V_0^{(1)} = \dfrac{1}{2}\displaystyle\int \frac{d^3p}{(2\pi)^3}\, E_M & (1.102a) \\ V_\beta^{(1)} = \dfrac{1}{2\pi^2\beta^4}\displaystyle\int_0^\infty dp\, p^2 \ln\left(1 - e^{-\left(p^2+\beta^2 M_\Phi^2\right)^{1/2}}\right) & (1.102b) \end{cases}$$

where in (1.102b) definition we have adopted spherical coordinates, integrating over $|\vec{p}| = p$.
Typically, the divergent part of the one-loop correction is completely contained in the omnipresent term $V_0^{(1)}$, which is usually removed through a suitable renormalization prescription. However, as will be discussed in the following chapter, this is not always the case, particularly when the system is subject to boundary conditions. In such instances, a finite contribution may remain, and $V_0^{(1)}$ may play a pivotal role in the calculation of one-loop corrections.
Focusing on the conditions for symmetry restoration, as outlined in equation (1.91), we proceed to compute $V_\beta^{(1)}$. However, this is only feasible through a series expansion at high temperatures, that is, as $\beta \to 0$ and $M\beta \to 0$, a customary assumption in the study of such phenomena. The physical interpretation of this limit is that the temperature energy scale vastly exceeds the rest mass. As shown in [1], $V_\beta^{(1)}$ can be expanded as

$$V_\beta^{(1)} = -\frac{\pi^2}{90\beta^4} + \frac{M^2}{24\beta^2} + \frac{c}{64\pi^2}M^4 - \frac{1}{12\pi}\frac{M^3}{\beta} - \frac{1}{64\pi^2}M^4\ln\left(M^2\beta^2\right) + O\left(M^6\beta^2\right) \tag{1.103}$$

with $c \approx 5.41$. In the symmetry breaking scenario, we are interested only in physically acceptable terms that do not become complex for negative M^2. Only the first three terms of (1.103) satisfy this condition. Taking into account only the leading contribution of the first two, we focus on the computation of the symmetry restoration condition (1.91), that becomes

$$\frac{\lambda}{48\beta^2} \geq -\frac{m^2}{2} \quad \Rightarrow \quad T_c^2 = -24\frac{m^2}{\lambda}, \tag{1.104}$$

in complete agreement with (1.84).
Schematically, the behaviour of the effective potential under the influence of temperature or other sources of quantum corrections (in the case of a second-order phase transition) is represented in the figure below.

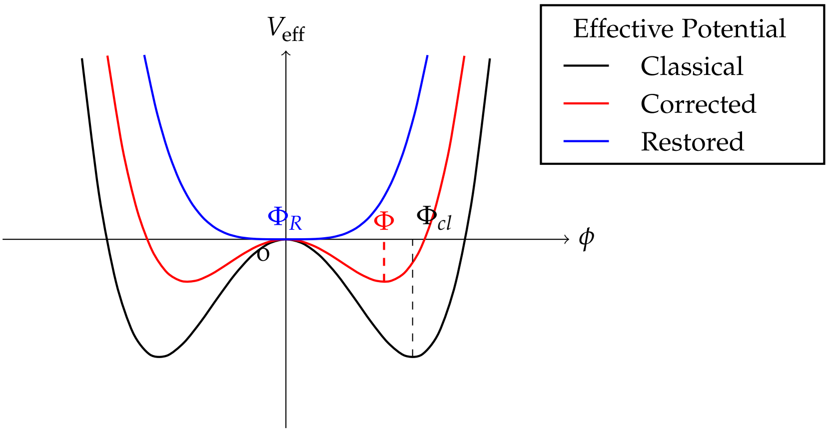

Figure 1.2: One-loop effective potential of a second order phase transition. The black line represents the effective potential for vanishing quantum corrections ($\langle\phi^2\rangle_{\text{ren}} = 0$) which is equivalent to the classical potential.
The red line represents the quantum corrected potential
($0 < \langle\phi^2\rangle_{\text{ren}} < -2m^2/\lambda$).
The blue line represents the restored potential ($\langle\phi^2\rangle_{\text{ren}} = -2m^2/\lambda$).

1.2 STATE OF THE ART

In this section, we provide an overview of the current state of research on symmetry breaking and phase transitions in uniformly accelerated reference frames. Surprisingly, the literature remains divided, with seemingly contradictory contributions on the topic. These works can be broadly classified into three groups: those opposing symmetry restoration, those supporting it, and those suggesting acceleration enhances symmetry breaking.

From the 'symmetry phase preservation' group, we summarize three seminal papers. The first, by W.G. Unruh and N. Weiss [2], examines the influence of acceleration on n-point Green's functions in interacting field theories and directly addresses the possibility of acceleration-induced symmetry restoration, which the authors deem impossible. Next, we consider the work of C.T. Hill [3, 24], who computes $\langle\phi^2\rangle_{\text{ren}}$ with respect to both Minkowski and Unruh vacuum and comments on which physical vacuum polarization should be adopted in the computation of quantum correction in the context of symmetry breaking in accelerated frame. Finally, we review a contribution from gravitational physics by D.N. Page [4], who evaluates $\langle\phi^2\rangle_{\text{ren}}$ in the broader framework of Einstein metrics, deriving an exact, physically meaningful expression for vacuum polarization in maximally symmetric spacetimes.

Regarding the 'symmetry phase restoration' group, the extensive literature spans across many different theories [5–10], including both bosonic and fermionic models, making a comprehensive review impractical. However, despite technical differences, the core methodologies are conceptually similar. To illustrate the reasoning underlying these works, we focus on the real, ϕ^4 self-interacting, spontaneously broken framework (1.48) as a representative model.

For the 'symmetry breaking enhancement' group, we first highlight key aspects of P. Candelas' early work [13], particularly his analysis of the structure of the two-point Green's function in incomplete manifolds and its renormalization in the coincidence limit. We then present the more recent contributions of S. Benic and K. Fukushima [11], and M. Chernodub [12], who directly investigate symmetry breaking in accelerating frames and derive a 'cooling' effect in such settings.

1.2.1 *Symmety phase persistence standpoint*

W.G Unruh and N. Weiss contribution

The most interesting result, obtained in this article, is the path-integral generalization of the thermalization theorem (1.45) to self-interacting field theories. Although the article also treats the fermionic case, we limit our presentation to scalar fields, giving a brief adapted version of such a proof. In the conclusions, the authors also discussed the role of this theorem for theories characterized by multiple vacua, inferring the impossibility of a phase transition as a function of acceleration alone.

In order to prove the thermalization theorem in a path-integral approach, we need to prove that:

1. The partition function, for vanishing external current J, at finite temperature $1/\beta_U = a/2\pi$ in the right Rindler wedge is equal to the (Euclidean) partition function in Minkowski coordinates, i.e.,

$$Z^R_{\beta_U} = Z_E; \tag{1.105}$$

2. The finite temperature $1/\beta_U = a/2\pi$, n-points Green's function in the right Rindler wedge is equal to the (Euclidean) n-point Green's function in Minkowski coordinate, i.e.,

$$\begin{aligned} &{}_M\langle 0|\left(\phi\left(t_1,\vec{x}_1\right),\cdots,\phi\left(t_n,\vec{x}_n\right)\right)_t|0\rangle_M = \\ &\mathrm{Tr}\left[e^{-\beta_U H^R}\left(\phi\left(\tau_1,\rho_1,\vec{x}_{1\perp}\right),\cdots,\phi\left(\tau_n,\rho_n,\vec{x}_{n\perp}\right)\right)_\tau\right]/\mathrm{Tr}\left[e^{-\beta_U H^R}\right], \end{aligned} \tag{1.106}$$

with $(\cdots)_t$ and $(\cdots)_\tau$ respectively the Minkowski and Rindler time ordering operator.

Through this proof, we only deal with coordinates in the Wick-rotated spacetime (see also Appendix A). Rindler coordinates transformation (1.1) then becomes

$$\begin{cases} t_{\mathrm{E}} = \rho\sin\left(a\tau_{\mathrm{E}}\right) \\ x = \rho\cos\left(a\tau_{\mathrm{E}}\right) \\ y = y, \qquad\qquad z = z. \end{cases} \tag{1.107}$$

and the associated Euclidean metric can be written as

$$d^2s_{\mathrm{E}} = a^2\rho^2 d\tau_{\mathrm{E}}^2 + d\rho^2 + dy^2 + dz^2, \qquad\qquad \sqrt{g^R_{\mathrm{E}}} = a\rho. \tag{1.108}$$

The classical action of the theory, written in terms of the right wedge Rindler coordinates $(\tau_E, \rho, \vec{x}_\perp)$, assumes the form

$$S^R_{\text{E}} = i \int d\tau_{\text{E}}\, d\rho\, d^2x_\perp \sqrt{g^R_{\text{E}}}\mathcal{L}^R_{\text{E}} \tag{1.109}$$

with

$$\sqrt{g^R_{\text{E}}}\mathcal{L}^R_{\text{E}} = \frac{1}{2}\frac{1}{a\rho}\left(\frac{\partial\phi}{\partial\tau_{\text{E}}}\right)^2 + a\rho\left(\left(\frac{\partial\phi}{\partial\rho}\right)^2 + \frac{1}{2}\left(\nabla_\perp\phi\right)^2 + V(\phi)\right). \tag{1.110}$$

As shown in detail in [25] and summarized in A, we know the general expression of the finite temperature $1/\beta$ partition function, which, expressed in (Euclidean) Rindler coordinates, can be written as

$$Z^R_\beta = \text{Tr}\left[e^{-\beta H^R}\right] = N \int_{\phi(0)=\phi(\beta)} \mathcal{D}\phi \exp\left[-\int_0^\beta d\tau_{\text{E}} \int_{\rho>0} d\rho\, d^2x_\perp \sqrt{g^R_{\text{E}}}\mathcal{L}^R_{\text{E}}\right]. \tag{1.111}$$

Notice that, until now, the inverse temperature $1/\beta$ has been completely arbitrary.

To perform the change in integration variables (1.107), we need to carefully account for the transformation of the integration domain. Specifically, since we are performing a polar coordinate transformation, we have to avoid double-covering the coordinate space (t_E, x, y, z). To achieve this, $a\tau$ must be treated as an angular coordinate, imposing a constraint on the inverse temperature β, namely $\beta \leq 2\pi/a$. The radial coordinate $\rho \in (0, \infty)$ already spans the correct domain as a radial-type variable. As a result, for $\beta = 2\pi/a$, the right Rindler wedge is mapped to the entire domain of the Euclidean Minkowski space, $\mathbb{R}^4$. The periodic boundary condition $\phi(\tau = 0) = \phi(\tau = \beta)$ becomes a 'consistency condition' since, being τ an angular-like variable, it ensures that after a 2π rotation, the field remains single-valued and no discontinuity arises in $\mathbb{R}^4$. The finite temperature partition function (1.111) for $\beta_U = 2\pi/a$, can then be written as

$$Z^R_\beta = N \int \mathcal{D}\phi \exp\left[-\int dx^4_{\text{E}} \mathcal{L}_{\text{E}}\right] = Z_{\text{E}} \tag{1.112}$$

with $d^4x_{\text{E}} = a\rho\, d\tau_{\text{E}}\, d\rho\, d\vec{x}_\perp$ the Euclidean volume element, and $\mathcal{L}_{\text{E}}$ the Euclidean Minkowski Lagrangian

$$\begin{aligned}\mathcal{L}_{\text{E}} &= \frac{1}{2}\left(\frac{\partial\phi}{\partial t_{\text{E}}}\right)^2 + \frac{1}{2}\left(\nabla\phi\right)^2 + V(\phi) = \\ \mathcal{L}^R_{\text{E}} &= \frac{1}{2}\frac{1}{a^2\rho^2}\left(\frac{\partial\phi}{\partial\tau_{\text{E}}}\right)^2 + \left(\frac{\partial\phi}{\partial\rho}\right)^2 + \frac{1}{2}\left(\nabla_\perp\phi\right)^2 + V(\phi)\,.\end{aligned} \tag{1.113}$$

The first statement (1.105) is then proved.
The proof of (1.106) follows analogously, as the scalar field ϕ is independent of the choice of coordinates and thus takes the same value at identical spacetime points. Therefore, we can immediately say

$$\begin{aligned} &N \int \mathcal{D}\phi\, \phi\left(t_{\mathrm{E}\,1}, \vec{x}_1\right) \cdots \phi\left(t_{\mathrm{E}\,n}, \vec{x}_n\right) \exp\left[-\int dx_{\mathrm{E}}^4 \mathcal{L}_{\mathrm{E}}\right] = \\ =&N \int_{\phi(0)=\phi(\beta)} \mathcal{D}\phi\, \phi\left(\tau_{\mathrm{E}\,1}, \rho_1, \vec{x}_{1\perp}\right) \cdots \phi\left(\tau_{\mathrm{E}\,n}, \rho_n, \vec{x}_{n\perp}\right) \cdot \\ &\cdot \exp\left[-\int_0^{\beta_U} d\tau_{\mathrm{E}} \int_{\rho>0} d\rho\, d^2x_\perp \sqrt{g_{\mathrm{E}}^R} \mathcal{L}_{\mathrm{E}}^R\right] \end{aligned} \tag{1.114}$$

or equivalently, by noting that the time ordering $(\cdots)_t$ and $(\cdots)_\tau$ are equivalent in the right Rindler wedge (in the left wedge they are opposite, cfr. 1.1.2.1),

$$\begin{aligned} &{}_M\langle 0| \left(\phi\left(t_{\mathrm{E}\,1}, \vec{x}_1\right), \cdots, \phi\left(t_{\mathrm{E}\,n}, \vec{x}_n\right)\right)_t |0\rangle_M = \\ &\mathrm{Tr}\left[e^{-\beta_U H^R} \left(\phi\left(\tau_{\mathrm{E}\,1}, \rho_1, \vec{x}_{1\perp}\right), \cdots, \phi\left(\tau_{\mathrm{E}\,n}, \rho_n, \vec{x}_{n\perp}\right)\right)_\tau\right] / \mathrm{Tr}\left[e^{-\beta_U H^R}\right]. \end{aligned} \tag{1.115}$$

Transforming the Euclidean times back to Minkowski and Rindler times completes our proof.

In the conclusion of the paper, the authors discussed how this result could be applied to a theory that exhibits SSB in an inertial frame. They considered the specific potential $V(\phi) = m^2/2 + \lambda\phi^4/4!$ with $m^2 < 0$, and concluded that, at tree-level, no phase transition can occur as a function of acceleration, despite the Hilbert space being populated with Unruh thermal radiation. In this theory, the VEV of a field at zero temperature in an inertial frame is given by

$${}_M\langle \pm| \phi(x) |\pm\rangle_M = \pm\sqrt{\frac{-6m^2}{\lambda}} \neq 0 \tag{1.116}$$

where $|+\rangle, |-\rangle$ are the two degenerate vacua.
Applying the result (1.106), which in this case simplifies to

$$\mathrm{Tr}\left[e^{-\beta_U H^R} \phi\right] = {}_M\langle \pm| \phi |\pm\rangle_M \neq 0 \tag{1.117}$$

it is evident that the expectation value of the field at finite Unruh temperature $\beta = 2\pi/a$ is bounded to remain identical Minkowski VEV. Consequently, at least at tree-level, the broken symmetry cannot be restored through acceleration.
Lastly, the authors discussed the possible renormalization corrections

to the result when moving beyond the tree-level approximation and performing the calculation of higher-order Feynman graphs.

> "Finally, we note that the results of this paper should not be affected by renormalization of the theory. The easiest way to see this is to note that the result holds in any number of spatial dimensions d. Thus any Feynman graph regulated using dimensional regularization in the Rindler system at temperature $T = a/2\pi$ will be equal to the corresponding regulated $T = 0$ graph in Minkowski space. Since all counterterms are Lorentz invariant, the renormalized graphs should be equal as well."

With this statement, they ruled out the possibility of novel contributions arising from loop corrections when adopting Lorentz-invariant renormalization schemes.

As we have discussed in section 1.1.3.2, the one-loop quantum corrected position of the minima in the spontaneously broken ϕ^4 theory is given by

$$\Phi = \pm\sqrt{-6\frac{m^2}{\lambda} - 3\langle\phi^2\rangle}, \tag{1.118}$$

where the quantum correction is fully encapsulated in the vacuum polarization $\langle\phi^2\rangle$. This contribution, defined in 1.1.3.1, is nothing other than the (renormalized) two-point Green's function $G(x, x')$ in the coincidence limit $x' \to x$. In their seminal paper, W.G. Unruh and N. Weiss showed that, under a suitable renormalization prescription, no additional corrections are expected from the calculation of $\langle\phi^2\rangle$ for accelerated observers that do not also appear for inertial observers. In this scenario, symmetry restoration is deemed impossible.

C.T. Hill contribution

Three years later, a direct calculation of the renormalized $\langle\phi^2\rangle$ was performed by C.T. Hill, with the aim to observe whether a scalar ϕ^4 broken symmetry theory, i.e.,

$$\mathcal{L} = \frac{1}{2}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi - \frac{m^2}{2}\phi^2 - \frac{\lambda}{4!}\phi^4, \tag{1.119}$$

with $m^2 < 0$, could be restored due to Unruh effect. In the introduction, he commented on the following:

> "Simple heuristic arguments suggest that the answer must be no. In the ground state we have the order parameter, $\Phi = \langle 0|\,\phi\,|0\rangle$ which is a scalar and must transform into itself in the Rindler coordinate system. [...] These arguments

> raise paradoxes however. For example, if an accelerating observer truly observes radiation of temperature T_H, then he must find thermal corrections to operator matrix elements, e.g. ϕ^2 must display a thermal correction of $T_H^2/12$ which conflicts with the covariance requirement that all observers must agree that it have the same value. The energy density, which for a spontaneously broken theory is essentially the effective potential, must be the same for all observers up to covariant transformation factors. How then does the accelerating observer obtain the usual thermal corrections to this quantity which would normally lead to symmetry restoration?"

The prescription adopted for its calculation was the usual finite-temperature renormalization prescription, i.e.

$$\mathrm{Tr}\left[\rho_\beta O\right] - \langle 0| O |0\rangle , \tag{1.120}$$

with ρ_β the finite-temperature density matrix and O a generic observable. Since the Minkowski vacuum state appears as populated by a thermal distribution of Rindler quanta to an accelerated observer, it is straightforward to generalize (1.120) to this context

$$\mathrm{Tr}\left[\rho_{\beta_U} O\right] - {}_R\langle 0| O |0\rangle_R = {}_M\langle 0| O |0\rangle_M - {}_R\langle 0| O |0\rangle_R , \tag{1.121}$$

where we have used the thermalization theorem, to relate the expression on the left-hand side to that on the right-hand side.
The author proceed to the calculation of the two-point Green's function with respect to Minkowski and Rindler vacua. He obtains respectively

$$\begin{cases} {}_M\langle 0| \phi\left(x+\frac{1}{2}\epsilon\right)\phi\left(x-\frac{1}{2}\epsilon\right)|0\rangle_M = \Phi^2 + \dfrac{M_\Phi}{4\pi^2\epsilon} & \text{(1.122a)} \\ {}_R\langle 0| \phi\left(x+\frac{1}{2}\epsilon\right)\phi\left(x-\frac{1}{2}\epsilon\right)|0\rangle_R = \Phi^2 + \dfrac{M_\Phi}{4\pi^2\epsilon} - \dfrac{1}{48\pi^2\rho^2}, & \text{(1.122b)} \end{cases}$$

where $M_\Phi^2 = m^2 + \lambda\Phi^2/2$, ρ is the ordinary Rindler coordinate (1.1), and ϵ is the typical separation (spacelike in this case) in the Feynman propagator. Using (1.120), the renormalized vacuum polarization for an uniformly accelerated observer with acceleration $a = 1/\rho$, simply becomes

$${}_M\langle 0| \phi^2 |0\rangle_M - {}_R\langle 0| \phi^2 |0\rangle_R = \frac{a^2}{48\pi^2} = \frac{T_U^2}{12}. \tag{1.123}$$

This result appears striking at first glance because, contrary to the result of W.G Unruh and N. Weiss, it implies that, at one-loop, there is in fact an acceleration-dependent quantum correction that can shift the position of the classical minima, potentially restoring the broken symmetry. Moreover, this outcome is fully consistent with the standard finite-temperature contribution (1.83). However, as the author insightfully observed, the finite temperature-like contribution does not originate from the supposedly 'hot' term ${}_M\langle 0|\,\phi^2\,|0\rangle_M$. Instead, it arises from the Rindler ground state value, which is suppressed by an amount of $-\frac{1}{12}T^2$. He ultimately suggested that the thermally correct value for an accelerated observer should be (1.122a) and that thermal-like terms should be added to eliminate the fictitious term derived earlier. In this way, general invariance of $\langle\phi^2\rangle$ is ensured, and consequently, symmetry restoration cannot occur.

D.N Page contribution

From the gravitational scenario, an important contribution comes from the contribution of D.N. Page. He calculated the quantum correcting term $\langle\phi^2\rangle_{\text{ren}}$ for a real, massless field, in the case of Einstein spacetimes, which are defined by the relation $R_{\mu\nu} = \Lambda\, g_{\mu\nu}$. In other words, we consider spacetimes where the metric tensor is proportional to the Ricci tensor, with the proportionality constant given by the cosmological constant Λ.

Summarizing the fundamental steps, the author proceeded to:

- express the original metric in terms of an 'optical metric' $g_{\mu\nu}$, i.e.,

$$g_{\mu\nu} = \Omega^{-2}\bar{g}_{\mu\nu} \tag{1.124}$$

  where Ω^{-2} is a space-dependent conformal factor, to make the optical metric component $g_{tt} = -1$ (given the opposite convention of the metric signature in the article with respect to this thesis);

- calculate the optical thermal propagators, in the Gaussian path-integral approximation of Bekenstein and Parker [26], obtaining

$$G_{\text{Gauss}}\left(\tau,\vec{x};0,\vec{x}'\right) = \frac{T}{4\pi}\frac{\Delta^{\frac{1}{2}}\sinh\left(2\pi Tr\right)}{r\left(\cosh\left(2\pi Tr\right)-\cos\left(2\pi T\tau\right)\right)}, \tag{1.125}$$

  with $r = \sqrt{2^{(3)}\sigma}$ the spatial distance between the two points in the optical metric, and Δ the three-dimensional version of the Van-Vleck determinant

$$\Delta\left(\vec{x},\vec{x}'\right) = \sqrt{g\left(\vec{x}\right)}\det\left(-\frac{\partial^{2\,(3)}\sigma\left(\vec{x},\vec{x}'\right)}{\partial x^i\partial x^j}\right)\sqrt{g\left(\vec{x}'\right)}; \tag{1.126}$$

- subtract the counterterms given by the (covariant) point-separation method derived by Wald [27] (one can also employ directly the Christensen counterterms (1.176)) and take the coincidence limit;
- write the vacuum polarization in terms of the original coordinates, and get

$$\langle \phi^2 \rangle_{\text{Gauss}} = \frac{1}{48\pi^2} \left(4\pi^2 T^2 \Omega^{-2} - \Omega^{-4} \nabla^\alpha \Omega \nabla_\alpha \Omega - \Lambda \right). \quad (1.127)$$

Remarkably, given that $\Omega = \sqrt{\bar{g}_{00}}$, it is possible to recognize that the first term in parenthesis can be written as a function of the local Tolman–Ehrenfest temperature

$$T_{\text{loc}} = T\sqrt{\bar{g}_{00}}. \quad (1.128)$$

The second term can be written as a function of the Unruh acceleration temperature $T_U = a/(2\pi)$, by using the definition of the four-acceleration modulus

$$\begin{aligned} a^2 = a^\mu a_\mu &= \bar{g}^{\mu\nu} \partial_\nu \left(\ln \left(\sqrt{\bar{g}_{00}} \right) \right) \partial_\mu \left(\ln \left(\sqrt{\bar{g}_{00}} \right) \right) = \\ &= \Omega^{-4} \bar{g}^{\mu\nu} \partial_\mu \Omega \partial_\nu \Omega. \end{aligned} \quad (1.129)$$

Finally we get an approximate expression for $\langle \phi^2 \rangle_{\text{ren}}$, which reads

$$\langle \phi^2 \rangle_{\text{Gauss}} = \frac{1}{12} \left(T_{\text{loc}}^2 - T_{\text{acc}}^2 \right) - \frac{1}{48\pi^2} \Lambda. \quad (1.130)$$

In flat space and de Sitter spacetime, this approximation becomes exact. Eq. (1.130) has a deep physical interpretation, and shows that the vacuum polarization comprises two fundamental contribution: a temperature-like contribution and a curvature contribution.

Since we are interested in ordinary flat space, we can set $\Lambda = 0$ and analyze $\langle \phi^2 \rangle_{\text{Gauss}}$ accordingly. In Minkowski spacetime for inertial observers, at finite temperature T, we have $T_{\text{loc}} = T$ and $T_{\text{acc}} = 0$, and we retrieve the standard result (1.83).
The author also pointed out that the expression of $\langle \phi^2 \rangle_{\text{ren}}$ at $T = 0$, coming from (1.130), remains invariant whether computed in Minkowski or Rindler coordinates and vanishes in both cases. However, if for inertial observers $T_{\text{loc}} = T_{\text{acc}} = 0$, then for uniformly accelerating observers, $T_{\text{loc}} = T_{\text{acc}} = a/(2\pi)$ holds. This equivalence relies on the implicit assumption that the lowest possible temperature T_{loc} an accelerated observer can experience is the Unruh temperature. Consequently, the appropriate temperature T in (1.128) has a lower bound given by $T_{\text{loc}} \geq T_{acc}$.
As we will see in the following, this assumption is not (implicitly or explicitly) universally accepted in the literature.

1.2.2 *Symmetry phase restoration standpoint*

Considering the spontaneously broken theory (1.119), we adopt the effective action formalism, discussed in 1.1.3.1 and 1.1.3.3, to study the one-loop correction to the effective potential and to evaluate the condition of symmetry restoration as a function of acceleration.
From (1.94), we know the one-loop effective action takes the form

$$\Gamma\left[\Phi\right] = S\left[\Phi\right] + \frac{i}{2}\mathrm{Tr}\ln\left[\partial_\mu\left(\sqrt{-g}g^{\mu\nu}\partial_\nu\right) + \sqrt{-g}M_\Phi^2\right] \tag{1.131}$$

with $M_\Phi^2 = m^2 + \lambda\Phi^2/2$. It is possible to reformulate the one-loop correction, eliminating the presence of the logarithmic function (often troublesome for calculation purposes). Using the complete orthonormal basis of position states, for whom $\langle x|x'\rangle = \delta(x-x')$ with projector operator $\mathbb{1} = \int dx\,|x\rangle\langle x|$, it is possible to simplify the 1-loop contribution by evaluating the trace of the differential operator as follows

$$\begin{aligned}&\mathrm{Tr}\ln[\partial_\mu(\sqrt{-g}g^{\mu\nu}\partial_\nu) + \sqrt{-g}M_\Phi^2]\\ &= \int d^4x' \int_0^{M_\Phi^2} dq\frac{d}{dq}\left(\langle x'|\ln\left[\partial_\mu\sqrt{-g}g^{\mu\nu}\partial_\nu + \sqrt{-g}q\right]|x'\rangle\right) + \\ &\int d^4x'\,\langle x'|\ln\left[\partial_\mu\sqrt{-g}g^{\mu\nu}\partial_\nu\right]|x'\rangle\,,\end{aligned} \tag{1.132}$$

or equivalently, using the fact that
$G(x,x',M_\Phi) = \langle x|\left[\partial_\mu\sqrt{-g}g^{\mu\nu}\partial_\nu + \sqrt{-g}M_\Phi^2\right]^{-1}|x'\rangle$, equation (1.132) takes the form

$$\begin{aligned}&\mathrm{Tr}\ln[\partial_\mu(\sqrt{-g}g^{\mu\nu}\partial_\nu) + \sqrt{-g}M_\Phi^2]\\ &= \int d^4x'\sqrt{-g}\int_0^{M_\Phi^2} dq\,\lim_{x\to x'} G(x,x',q) + \int d^4x'\ln\left[\partial_\mu\sqrt{-g}g^{\mu\nu}\partial_\nu\right].\end{aligned} \tag{1.133}$$

In the RHS of (1.133), the integral of the logarithm is a purely kinetic term, therefore independent of Φ^2. Since we are ultimately interested in applying the symmetry restoration condition (1.91) which involves differentiation over Φ^2, this term can be safely neglected. From the definition of the effective potential (1.86), i.e.,

$$\Gamma\left(\Phi\right) = -\int d^4x\sqrt{-g}\,V\left(\Phi\right), \tag{1.134}$$

and substituting (1.133) in (1.131), we get

$$V(\Phi) = \frac{1}{2}m^2\Phi^2 + \frac{\lambda}{4!}\Phi^4 - \frac{i}{2}\int_0^{M_\Phi^2} dq^2\,\lim_{x\to x'} G\left(x,x',q\right). \tag{1.135}$$

Imposing the symmetry restoration condition (1.91), we obtain

$$\left.\frac{\partial V}{\partial \Phi^2}\right|_{\Phi=0} = \frac{m^2}{2} - i\frac{\lambda}{4} \underbrace{\lim_{x\to x'} G\left(x,\, x', m^2\right)}_{i\langle\phi^2\rangle} \geq 0, \tag{1.136}$$

where $G\left(x,\, x', m^2\right)$ is a free propagator, independent of Φ.
We can observe that the lower bound of this condition corresponds to the critical point for the phase transition and it is, in fact, the relation (1.118) for Φ in disguise. At this stage, it remains to identify the Green's function and determine how to renormalize it in the coincidence limit.

Observation 1.
In the calculation of the one-loop correction that follows, we will work with propagators $\langle 0|\, \phi(x)\phi(x')\, |0\rangle$, assuming the fields already ordered in time ($\tau > \tau'$). Under this condition, Feynman propagators reduce to Wightman functions. As a consequence, all previous results, including the thermalization theorem, remain valid when the time-ordering operator is omitted in this restricted regime.

Most of the authors identify the propagator for the accelerated observer, as the thermal Green's function

$$G_\beta\left(x,\, x',\, m\right) = i\,\mathrm{Tr}\left[e^{-\beta_U H^{\mathcal{R}}} \phi(x)\, \phi(x')\right] / \mathrm{Tr}\left[e^{-\beta_U H^{\mathcal{R}}}\right] \tag{1.137}$$

with fields $\phi(x)$, $\phi(x')$ expressed in terms of Rindler coordinates (1.1).

Observation 2.
Applying the results of thermalization theorem (1.106),

$$\mathrm{Tr}\left[e^{-\beta_U H^{\mathcal{R}}} \phi(x)\, \phi(x')\right] / \mathrm{Tr}\left[e^{-\beta_U H^{\mathcal{R}}}\right] = {}_M\langle 0|\, \phi(x)\phi(x')\, |0\rangle_M \tag{1.138}$$

such a Green's function is indistinguishable from the usual propagator in Minkowski space. As we argued before, the choice of coordinate system, whether Minkowski or Rindler, should not affect the scalar field ϕ, as its value at any given spacetime point is coordinate-independent. Consequently, one might expect no differences to arise in the calculation of the symmetry phase for an accelerated observer compared to an inertial one. However, as we will see, the situation is more subtle than it initially appears.

Subsequently, many authors approached the calculation of the LHS of (1.138), adopting the Matsubara frequencies formalism at finite temperature $1/\beta_U$, and imposing the coincidence limit adapted to the worldline of a uniformly accelerated observer with proper acceleration a, i.e.,

$$\begin{cases} \tau' = \tau \\ \rho' = \rho = \dfrac{1}{a} \\ \vec{x}'_\perp = \vec{x}_\perp . \end{cases} \tag{1.139}$$

In their calculations, they often implicitly considered the high acceleration (or Unruh temperature) limit, i.e. $m/a \to 0$, which we already encountered expressed in terms of temperature in 1.1.3.3.
As a double-check of the thermalization theorem, as we show in detail in (B), it is possible to calculate the RHS of (1.138) instead of the LHS, by using Rindler-Fulling quantization of the field ϕ (1.23) together with Bogoliubov transformations (1.40). The result of the two different calculations turn out to be the same, and can be written as

$$\begin{aligned} \left.\frac{\partial V}{\partial \Phi^2}\right|_{\Phi=0} &= \frac{m^2}{2} + \frac{\lambda}{16\pi^2}\int_0^\infty d\Omega\,\Omega \coth\left(\frac{\pi\Omega}{a}\right) \\ &= \frac{m^2}{2} + \frac{\lambda}{16\pi^2}\int_0^\infty d\Omega\,\Omega\left(1 + \frac{2}{e^{\frac{2\pi\Omega}{a}} - 1}\right) . \end{aligned} \tag{1.140}$$

The integral over the frequencies Ω is manifestly divergent and should be properly renormalized. For this purpose, some authors chose to renormalize the mass parameter $m^2 \to m^2 + \delta m^2$, others adopt physically similar techniques, to remove completely the quadratic divergence term in the integral. As a result, they all obtain the condition for symmetry restoration

$$\frac{m^2}{2} + \frac{\lambda}{16\pi^2}\int_0^\infty d\Omega\,\frac{2\Omega}{e^{\frac{2\pi\Omega}{a}} - 1} \geq 0, \tag{1.141}$$

(or its analogue, depending on the specific theory under consideration). Computing the integral in Ω, it is possible to obtain the critical value for the accelaration, that is

$$T_{Uc}^2 = \frac{a_c^2}{4\pi^2} = -\frac{24m^2}{\lambda}, \tag{1.142}$$

in perfect agreement with the finite temperature results (1.84, 1.104). These renormalization approaches can broadly be classified into what we refer to in this thesis as 'temperature-like' renormalization schemes. A critical analysis of this issue is presented in the following section.
Once more, as observed in the C.T. Hill review, we obtained a non-vanishing, acceleration-dependent correction at one-loop. Hence, for sufficiently high acceleration, the symmetry can be restored.

The results obtained from the calculation of the quantum corrections at one-loop, both in the case of the finite-temperature propagator in an ensemble of Rindler quanta and in the case of the Minkowski propagator using Rindler-Fulling quantization, are the same. Therefore, we can conclude that the thermalization theorem holds true and is not responsible for the discrepancy observed in the literature. Instead, this contradiction can be attributed to the specific renormalization procedure adopted. In the following section, we will present our original work, which focused on studying the different choices of renormalization prescriptions, their results, and how they relate to each other.

1.2.3 *Symmetry breaking enhancement standpoint*

P. Candelas and D.J. Raine contribution

Candelas and Raine focused on the study of the (free) Green's functions on incomplete manifolds, a category that includes, among others, the case of Rindler space, which we recall to be a submanifold of the maximally extended Minkowski space. We will call with M the incomplete manifold, with M_0 its analytical extension and with L the hyperbolic operator on this manifold (Klein-Gordon operator in our specific case). A Green's function G for L in M satisfies the relation

$$LG\left(x,x'\right)=\delta\left(x-x'\right). \tag{1.143}$$

The authors emphasize the fact that Green's functions are global objects, sensible to manifold properties and boundary conditions. Consequently, there is no reason to believe that G will be also a Green's function on the larger manifold M_0. In general, there will be singularities in $M0-M$. Suppose that, when x lies in M_0-M, we can express (1.143) as

$$\rho\left(x,x'\right)=LG\left(x,x'\right)-\delta\left(x-x'\right), \tag{1.144}$$

with the function ρ quantifying the extent to which the Green's function G fails to satisfy the usual Green's equation in the larger manifold. Since L possesses an inverse G_0 in M_0, we can rewrite G using (1.144) as

$$G\left(x,x'\right)=G_0\left(x,x'\right)-\int dy\, G_0\left(x,y\right)\rho\left(y,x'\right). \tag{1.145}$$

In our specific case, this translate to a relation between the Rindler Green's function (LHS) and the usual Minkowski Green's function.
Firstly, the authors obtained the Rindler Green's function, that can be

obtained by using the Rindler-Fulling quantization (1.23) restricted to the right wedge, and get

$$G\left(x,x'\right) = {}_R\langle 0|\,\phi\left(x\right)\phi\left(x'\right)|0\rangle_R =$$
$$\frac{i}{4\pi^4 a}\int_0^\infty d\Omega\int d^2k_\perp \sinh\left(\pi\frac{\Omega}{a}\right)e^{-i\left(\Omega(\tau-\tau')-\vec{k}_\perp(\vec{x}_\perp-\vec{x}'_\perp)\right)}K_{i\frac{\Omega}{a}}\left(\mu_k\rho\right)K_{i\frac{\Omega}{a}}\left(\mu_k\rho'\right), \tag{1.146}$$

(see also B).

The authors proceeded to algebraically manipulate the Rindler propagator, rewriting it in the form (1.145), i.e.,

$$G\left(x,x'\right) = G_0\left(\sqrt{2\sigma}\right) - \int_\infty^\infty \frac{d\lambda}{\lambda^2+\pi^2}G_0\left(\gamma\left(\lambda\right)\right), \tag{1.147}$$

with

$$G_0\left(\alpha\right) = \frac{i}{2\pi}\left(\frac{m}{2\pi\alpha}\right)^{(n-2)/2}K_{(n-2)/2}\left(m\alpha\right), \tag{1.148}$$

n being the number of spacetime dimensions, $\sqrt{2\sigma}$ the geodesic distance between x and x', and

$$\gamma = \sqrt{\rho^2+\rho'^2+2\rho\rho'\cosh\left(\lambda+a\left(\tau'-\tau\right)\right)+\left(\vec{x}_\perp-\vec{x}'_\perp\right)}. \tag{1.149}$$

We recognize that, for $n=4$, $G_0\left(\sqrt{2\sigma}\right)$ is the usual Minkowski propagator (1.173) in four dimensions. It can be proven that Eq. (1.147), in the coincidence limit $x\to x'$, the first term on the RHS of (1.147) contains all the divergences, while the second term is finite.

The authors then proceeded to calculate, in $n=2$ spacetime dimensions, the renormalized stress energy tensor $\langle T_{\mu\nu}\rangle$. This can be derived directly from the renormalized expression of the Rindler Green's function, which the authors evaluated by subtracting the usual counterterm in Minkowski spacetime $G_0\left(\sqrt{2\sigma}\right)$ from (1.147). In analogy to (1.27), the two-dimensional VEV of the stress-energy tensor can be written as

$$\langle T_{\mu\nu}\rangle = -i\lim_{x\to x'}\left(\frac{\partial}{\partial x^\mu}\frac{\partial}{\partial x'^\nu}-\frac{1}{2}g_{\mu\nu}\left(g^{\alpha\alpha'}\frac{\partial}{\partial x^\alpha}\frac{\partial}{\partial x'^\alpha}-m^2\right)\right)G\left(x,x'\right). \tag{1.150}$$

Using the integral representation of the modified Bessel function of the second kind, i.e.,

$$K_\nu\left(z\right) = \frac{1}{2}\left(\frac{1}{2}z\right)^\nu\int_0^\infty\frac{dt}{t^{\nu+1}}e^{-t-\frac{z^2}{4t}}, \tag{1.151}$$

and (1.147) for $n = 2$, it is now possible to evaluate the components of $\langle T_{\mu\nu} \rangle$

$$\begin{aligned} \langle T_\tau{}^\tau \rangle &= -\langle T_\rho{}^\rho \rangle = \frac{1}{8\pi} \int_{-\infty}^{\infty} \frac{d\lambda}{\lambda^2 + \pi^2} \int_0^\infty \frac{ds}{s} \left(\frac{\gamma^2}{4s^2} - m^2 \right) e^{-m^2 s - \frac{\gamma^2}{4s}} \\ \langle T_\tau{}^\rho \rangle &= \langle T_\rho{}^\tau \rangle = 0. \end{aligned} \tag{1.152}$$

The $\langle T_\tau{}^\tau \rangle$ and $\langle T_\rho{}^\rho \rangle$ components can be evaluated in the limit of vanishing mass $m \to 0$, and, using relation (1.172), one gets

$$\langle T_\tau{}^\tau \rangle = -\langle T_\rho{}^\rho \rangle = \frac{1}{24\pi\rho^2}. \tag{1.153}$$

By choosing a specific worldline $\rho = a^{-1}$, we get acceleration-dependent diagonal components which, surprisingly, do not vanish and lead to an acceleration-induced conformal anomaly.

Since our primary interest concerns the calculation of the Green's function (1.147), for $n = 4$ and in the coincidence limit, we adopt the same procedure as the authors and subtract once again $G_0\left(\sqrt{2\sigma}\right)$ from it, and obtain

$$G_{\text{reg}}\left(x, x'\right) = -\frac{im}{4\pi^2} \int_{-\infty}^{\infty} \frac{d\lambda}{\lambda^2 + \pi^2} \frac{K_1\left(m\gamma\left(\lambda\right)\right)}{\gamma\left(\lambda\right)}. \tag{1.154}$$

Using the integral representation (1.151), its coincidence limit is given by

$$\lim_{x \to x'} G_{\text{reg}}\left(x, x'\right) = -\frac{i}{16\pi^2} \int_{\infty}^{\infty} \frac{d\lambda}{\lambda^2 + \pi^2} \int_0^\infty \frac{ds}{s^2} e^{-m^2 s - \frac{\gamma^2}{4s}}, \tag{1.155}$$

and in the vanishing mass regime, we obtain

$$\begin{aligned} \lim_{x \to x'} G_{\text{reg}}\left(x, x'\right) &= -\frac{i}{8\pi^2\rho^2} \int_{\infty}^{\infty} \frac{d\lambda}{\lambda^2 + \pi^2} \frac{1}{1 + \cosh\left(\lambda\right)} \\ &= -\frac{i}{48\pi^2\rho^2}, \end{aligned} \tag{1.156}$$

where we have used (1.172) again.
Choosing the specific worldline $\rho = a^{-1}$, we obtain an acceleration-dependent vacuum polarization

$$\langle \phi^2 \rangle_{\text{ren}} = -\frac{a^2}{48\pi^2}. \tag{1.157}$$

Interestingly, according to this prescription, the acceleration-induced vacuum polarization exhibits the opposite behavior to the symmetry phase restoration condition 1.2.2. Specifically, applying the symmetry restoration criterion (1.136) and substituting (1.157), we obtain

$$\frac{m^2}{2} - \frac{a^2}{48\pi^2} \geq 0, \tag{1.158}$$

which is never realized, since $m^2 < 0$. Hence, symmetry restoration is never achieved, and acceleration effectively cools the system.
We will show that this result, along with the conformal anomaly (1.153), directly follows from the chosen physical Green's function $G(x, x')$, which is different from the Green's function considered in the previous perspectives (symmetry phase preservation and restoration standpoints).
Candelas and Raine were likely among the first, but surely not the last, to consider (1.146) as the correct physical quantity to calculate vacuum polarization, stress-energy tensor, and potentially other observables, as we will discuss below.

S. Benić, K. Fukushima and M. Chernodub contribution

We are now going to briefly discuss the key points of two different contributions who agree with the cooling effects induced by acceleration on the system.

First, we will discuss Benić and Fukushima paper. Among several notheworthy comments regarding the two-point Green's function $G(x, x')$ and its coincidence limit, the following are particularly relevant to our analysis:

- "The observer defines the operators we should put in the expectation value".
  In other words, the operator whose expectation value we want to calculate should be expressed according to the specific description of the observer measuring it. This was the case when we first introduced the Unruh effect in (1.41), where we expressed the number operator in terms of the uniformly accelerated observer's creation and annihilation operators, associated to the Rindler-Fulling quantization.

- The expectation value of an operator expressed solely in terms of field operators is independent on the particular field quantization (i.e., the specific definition of creation and annihilation operators). Operationally, this occurs because, once the states on which we calculate the expectation value are fixed, the differences between the various definitions of creation and annihilation operators for distinct observers are compensated by the Bogoliubov transformations employed to make them act on such states.

- The expectation value of an operator that cannot be expressed solely in terms of field operators but is also expressed in terms of the energy dispersion relation is dependent on the peculiar observer who is measuring it.
  This is the case again of (1.41), where we found a non-vanishing VEV for the Rindler number operator $N_R^{(+)} = b^{(+)\dagger}_{\Omega,\vec{k}_\perp} b^{(+)}_{\Omega,\vec{k}_\perp}$, contrary to the Minkowski number operator. The authors showed that this arises from the impossibility of expressing the creation and annihilation operators solely in terms of fields, as incorporating the energy dispersion relation is also necessary.

- What may play a role in the VEV of operators composed exclusively by fields are the vacuum state considered and possibly, in case of divergent quantities, the renormalization procedure adopted.

They then proceeded to calculate the quantum corrections in the form of $\langle \phi^2 \rangle$. The particular and explicit choice they made was to consider as the physical quantity of interest the two-point Green's function ${}_R\langle 0|\, \phi^2 \,|0\rangle_R$. Meanwhile, the renormalization was carried out by removing the usual divergent terms of the Minkowski propagator in the coincidence limit. Overall, they obtained a renormalized vacuum polarization given by

$$\begin{aligned} \langle \phi^2 \rangle_{\text{ren}} &= \lim_{x \to x'} \left({}_R\langle 0|\, \phi\,(x)\, \phi\,(x')\, |0\rangle_R - {}_M\langle 0|\, \phi\,(x)\, \phi\,(x')\, |0\rangle_M \right) \\ &= -\frac{T_U^2}{12}. \end{aligned} \tag{1.159}$$

This is exactly the result Candelas and Raine would have obtained in their prescription. The temperature-like contribution has the opposite sign compared to the conventional thermal contribution, obtained in the symmetry restoration standpoint (1.2.2), leading once again to a strengthening of the broken symmetry phase.
Result (1.159) comes with little surprise if one compares it to (1.123) since, operationally speaking, the definition of $\langle \phi^2 \rangle_{\text{ren}}$ is the opposite.

Chernodub contribution

Coming from a different angle, also Chernodub contributed to this issue, asserting that acceleration has a 'cooling' effect on the phase of the system.
Firstly, a preliminary distinction of two different physical scenarios has been made:

- Minkowski vacuum as perceived by an accelerating observer. There are no physical objects accelerating with the observer;
- An accelerating physical system and a co-moving observer moving with the same acceleration.

The author manifests his interest in studying the second scenario, considering it the physical one in which the system can genuinely experience the effects of acceleration. He proceeded by calculating the quantum correction to the condensate as a function of temperature and acceleration. His computation led to the final expression for $\langle\phi^2\rangle_{\text{ren}}$, which reads

$$\langle\phi^2\rangle_{\text{ren}} = \frac{1}{12}\left(T^2 - \frac{a^2}{4\pi^2}\right). \tag{1.160}$$

This result is in perfect agreement with that of D.N. Page (1.130), but its interpretation turns out to be entirely different. In this case, the author considers the system to be in its ground state at $T = 0$ whereas in 1.2.1, the ground state was considered to be thermalized at the Unruh temperature, $T_{\text{loc}} = T_U = a/(2\pi)$. This fundamentally shifts the perspective on the Unruh effect for a uniformly accelerated system. To obtain a vanishing ϕ_{ren}, the system must be heated from $T = 0$ to $T = T_U$, as acceleration effectively lowers the system 'effective' temperature by $a/(2\pi)$. As a consequence, the critical temperature required for the system to undergo a phase transition increases with acceleration.
Notice that, in the case $T = 0$, Eq. (1.160) is in perfect agreement with both Candelas and Raine (1.157) and Benić and Fukushima (1.159) results.

1.3 ORIGINS OF DIVERGENT RESULTS AND THEIR EXPLANATIONS

In this section, we present our published work [28], supplemented with additional comments and calculations to examine the differing perspectives and results in the literature regarding symmetry breaking in accelerated frames, as discussed in 1.2. Practically, this reduces to studying the renormalized vacuum polarization $\langle\phi^2\rangle_{\text{ren}}$. However, due to the subtleties involved and the different definitions used in its calculation, the literature does not provide a unanimous conclusion on the behavior of the broken symmetry phase under such a change of reference frame.
Our analysis has shown that the three distinct outcomes - symmetry restoration, persistence of symmetry breaking, and an 'enhancement' of symmetry breaking — fundamentally arise from two key factors:

- Inequivalent renormalization schemes;
- Different definitions of the true vacuum state of the system;

We will analyze these aspects in detail.

Ultimately, only one of these three perspectives can be realized in Nature. However, to date, we lack physical conditions that would allow us to discriminate between them and identify the correct one.

In this section, since the symmetry restoration condition (1.141) involves analyzing the Green's function at the field configuration $\Phi = 0$, we will focus on studying free (in the sense of $\Phi = 0$) propagators.

1.3.1 *Symmetry Restoration and Breaking Persistence: Inequivalent Renormalization Schemes*

We will now focus on the study of the two scenarios in which the phase of the symmetry is either restored or preserved. These two cases are characterized by the fact that the physical quantity used to calculate the vacuum polarization is the coincidence limit of the propagator ${}_M\langle 0|\, \phi(x)\phi(x')\, |0\rangle_M$ calculated with respect to Minkowski vacuum.
Subsequently, this divergent quantity must be appropriately renormalized.
The most commonly used renormalization schemes in the literature can be divided into two main categories:

1. *Frame-dependent renormalization scheme.*
   As we have mentioned before, many authors performed this kind of renormalization approach, in the attempt of renormalize

their theory. The result of this choice leads directly to a frame-dependent result and, therefore, to a change in the VEV of the field and, possibly, a restoration of the broken symmetry. At first glance, this renormalization scheme appears, given the analogy acceleration-temperature, as the natural one. Given its similarity with the finite temperature case, seen in detailed in 1.1.3.3, it will also be referred to as temperature-like renormalization scheme from now on.

2. *Covariant renormalization scheme.*
   A covariant renormalization scheme is a method to renormalize quantum field theories in a way that preserves the invariance of the theory under general coordinate transformations, ensuring that the physical predictions remain independent of the reference frame and respect general covariance. What we expect from such procedure is an invariant VEV of the field, which behaves properly as a scalar quantity, for both inertial and uniformly accelerated observer, prohibiting the theory to undergo a phase transition by changing reference frame. This approach is more commonly applied in curved spacetimes, although several authors have applied it to flat spacetime in the context of accelerated observers.

We will then summarize the different results obtained and propose a possible solution to this otherwise irreconcilable contradiction.

1.3.1.1 *Frame-dependent renormalization scheme*

The renormalization scheme employed by many proponents of symmetry restoration is formally equivalent to the finite-temperature prescription introduced in (1.1.3.3) and utilized in C.T. Hill's seminal work (1.121), expressed as

$$\langle \phi^2 \rangle_{\text{ren}} = \text{Tr}\left[\rho_{\beta_U} O\right] - {}_R\langle 0| O |0\rangle_R = {}_M\langle 0| O |0\rangle_M - {}_R\langle 0| O |0\rangle_R \,. \tag{1.161}$$

This scheme is inherently frame-dependent.
Observers subjected to different accelerations do not share the same Rindler vacuum state $|0\rangle_R$. Similarly to the relationship between inertial and accelerated observers, there are no unitary transformations connecting the different vacuum states. Consequently, the particle content differs between the frames. Although the structure of the renormalization scheme (1.161) remains unchanged, the variation in the Rindler vacuum state for each observer with different acceleration alters both the calculation of $\langle \phi^2 \rangle_{\text{ren}}$ and its resulting value. As a consequence, the vacuum polarization $\langle \phi^2 \rangle_{\text{ren}}$ can no longer be regarded as an invariant scalar quantity.

With this understanding, we proceed with the calculation of (1.161). The two Green's functions, evaluated in the coincidence and high-acceleration limits with respect to both the Minkowski and Rindler vacuum states, are given by

$$\begin{cases} {}_M\langle 0|\,\phi^2\,|0\rangle_M = \dfrac{1}{(2\pi)^2}\displaystyle\int_0^\infty d\Omega\,\Omega\left(1+\dfrac{2}{e^{\frac{2\pi\Omega}{a}}-1}\right) & (1.162a) \\ {}_R\langle 0|\,\phi^2\,|0\rangle_R = \dfrac{1}{(2\pi)^2}\displaystyle\int_0^\infty d\Omega\,\Omega. & (1.162b) \end{cases}$$

The difference between the two gives

$$\langle\phi^2\rangle_{\text{ren}} = \frac{1}{(2\pi)^2}\int_0^\infty d\Omega\,\Omega\frac{2}{e^{\frac{2\pi\Omega}{a}}-1} = \frac{a^2}{48\pi^2} = \frac{T_U^2}{12} \tag{1.163}$$

in perfect agreement with the 'renormalized mass' result (1.141). Inspecting the two terms, it seems that the temperature-like contribution comes from (1.162a) through the acceleration-dependent term in the integral, which is not removed during renormalization. This assumption is also supported by the interpretation of the Minkowski vacuum as populated by a thermal spectrum of Rindler quanta. It appears natural to associate (1.162b) to the zero-temperature contribution and (1.162a) to the finite temperature contribution, analogous to the decomposition of terms in the finite-temperature effective potential, as seen in (1.102a) and (1.102b).
However, we will show that this is not true. Surprisingly enough, the term Ω contains a hidden dependence on acceleration. By removing it through renormalization, we artificially shift the temperature-like contribution from ${}_R\langle 0|\,\phi^2\,|0\rangle_R$ to ${}_M\langle 0|\,\phi^2\,|0\rangle_M$. We will prove this statement in more detail in the following.

Frame-dependent counterterms in position space

As shown in Appendix (B), it is possible to represent the propagators, calculated with respect to both Minkowski and Rindler vacua, in Rindler coordinates as

$$\begin{gathered} {}_M\langle 0|\,\phi(x)\phi(x')\,|0\rangle_M = \\ \frac{1}{4\pi^4}\int_0^\infty d\Omega\int d^2k_\perp \cosh\left(\frac{\Omega}{a}\left(\pi - ia\left(\tau-\tau'\right)\right)\right)e^{i\vec{k}_\perp\left(\vec{x}_\perp-\vec{x}'_\perp\right)}K_{i\frac{\Omega}{a}}\left(\mu_k\rho\right)K_{i\frac{\Omega}{a}}\left(\mu_k\rho'\right) \\ {}_R\langle 0|\,\phi(x)\phi(x')\,|0\rangle_R = \\ \frac{1}{4\pi^4}\int_0^\infty d\Omega\int d^2k_\perp \sinh\left(\pi\frac{\Omega}{a}\right)e^{-i\Omega(\tau-\tau')}e^{i\vec{k}_\perp\left(\vec{x}_\perp-\vec{x}'_\perp\right)}K_{i\frac{\Omega}{a}}\left(\mu_k\rho\right)K_{i\frac{\Omega}{a}}\left(\mu_k\rho'\right). \end{gathered} \tag{1.164}$$

Using the relation

$$\sinh\left(\pi\nu\right)e^{-\nu\psi} = \cosh\left(\nu\left(\psi-\pi\right)\right) - \cosh\left(\nu\psi\right)e^{-\pi\nu} \tag{1.165}$$

it is possible to express one propagator in terms of the other, i.e.,

$$\begin{aligned} &{}_R\langle 0|\,\phi(x)\phi(x')\,|0\rangle_R = {}_M\langle 0|\,\phi(x)\phi(x')\,|0\rangle_M - \\ &\frac{1}{4\pi^4}\int_0^\infty d\Omega \int d^2k_\perp \cosh\left(\Omega\left(\tau-\tau'\right)\right)e^{-\pi\frac{\Omega}{a}}e^{i\vec{k}_\perp\left(\vec{x}_\perp-\vec{x}'_\perp\right)}K_{i\frac{\Omega}{a}}\left(\mu_k\rho\right)K_{i\frac{\Omega}{a}}\left(\mu_k\rho'\right). \end{aligned} \tag{1.166}$$

The second term on the RHS has been calculated independently by P. Candelas and D.J. Raine [13] and by B. Linet [29]. Overall, we can express (1.166) as

$$\begin{aligned} &{}_R\langle 0|\,\phi(x)\phi(x')\,|0\rangle_R = {}_M\langle 0|\,\phi(x)\phi(x')\,|0\rangle_M - \\ &\frac{m}{8\pi^3}\int_{-\infty}^\infty du\, F\left(u, ia\tau - ia\tau'\right)\frac{K_1\left(m\,R_4(u)\right)}{R_4(u)} \end{aligned} \tag{1.167}$$

with

$$\begin{aligned} R_4(u) &= \sqrt{2\rho\rho'\cosh\left(u\right)+\rho^2+\rho'^2+\left(\vec{x}_\perp-\vec{x}'_\perp\right)^2}, \\ F\left(u,\psi\right) &= -\frac{\pi+\psi}{\left(\pi+\psi\right)^2+u^2}+\frac{\psi-\pi}{\left(\psi-\pi\right)^2+u^2}. \end{aligned} \tag{1.168}$$

Using the relation $\sqrt{1+\cosh(x)} = \sqrt{2}\cosh(x/2)$ and the coincidence limit (1.139), we obtain

$$ {}_R\langle 0|\,\phi^2\,|0\rangle_R = {}_M\langle 0|\,\phi^2\,|0\rangle_M - i\frac{ma}{4\pi^2}\int_{-\infty}^\infty \frac{du}{\pi^2+u^2}\frac{K_1\left(2\frac{m}{a}\cosh\left(\frac{u}{2}\right)\right)}{\sqrt{2\left(1+\cosh(u)\right)}}. \tag{1.169}$$

To proceed with our analysis and explicitly calculate the integral, it is necessary to apply once again the high acceleration limit, $m/a \ll 1$. In Thus, it is possible to expand the modified Bessel function as

$$K_1\left(2\frac{m}{a}\cosh\left(\frac{u}{2}\right)\right) = \frac{a}{2m}\cosh\left(\frac{u}{2}\right) + O\left(\frac{m}{a}\right) \tag{1.170}$$

and neglect terms of order $O\left(\frac{m}{a}\right)$. The Rindler vacuum polarization (1.169) then becomes

$${}_R\langle 0|\,\phi^2\,|0\rangle_R = {}_M\langle 0|\,\phi^2\,|0\rangle_M - \frac{a^2}{8\pi^2}\int_{-\infty}^\infty \frac{du}{\pi^2+u^2}\frac{1}{1+\cosh\left(u\right)}. \tag{1.171}$$

The remaining integral can be solved exactly and gives

$$\int_{-\infty}^\infty \frac{du}{\pi^2+u^2}\frac{1}{1+\cosh\left(u\right)} = \frac{1}{6}. \tag{1.172}$$

It is also possible to express ${}_M\langle 0|\,\phi(x)\phi(x')\,|0\rangle_M$ using the well-known expression for the massive scalar propagator in Minkowski spacetime (calculated with respect to the Minkowski vacuum)

$$G_M\left(x,x'\right) = i\,{}_M\langle 0|\,\phi(x)\phi(x')\,|0\rangle_M = i\frac{m}{4\pi^2\sqrt{2\sigma}}K_1\left(m\sqrt{2\sigma}\right) \quad (1.173)$$

with $(2\sigma)^{\frac{1}{2}}$ the geodesic distance between x and x' We have consequently derived the final expressions for both ${}_M\langle 0|\,\phi^2\,|0\rangle_M$ and ${}_R\langle 0|\,\phi^2\,|0\rangle_R$, which are now given by

$$\begin{cases} {}_M\langle 0|\,\phi(x)\phi(x')\,|0\rangle_M = \dfrac{m}{4\pi^2\sqrt{2\sigma}}K_1\left(m\sqrt{2\sigma}\right) & (1.174a) \\ {}_R\langle 0|\,\phi^2\,|0\rangle_R = {}_M\langle 0|\,\phi^2\,|0\rangle_M - \dfrac{a^2}{48\pi^2} = {}_M\langle 0|\,\phi^2\,|0\rangle_M - \dfrac{T_U^2}{12}, & (1.174b) \end{cases}$$

in complete agreement with Hill's result (1.122a,1.122b), and with our previous result (1.163).

The propagator ${}_M\langle 0|\,\phi(x)\phi(x')\,|0\rangle_M$ in (1.174a) is expressed solely in terms of invariant quantities, such as the geodesic distance. There is no notion of acceleration or temperature within it. We can clearly express it in a coordinate system where it seems to depend on these parameters. However, given its covariant nature, the dependence is merely apparent.

The reason why we performed this calculation, despite the result being already known, is to illustrate explicitly what Hill previously suggested:

- the thermal-like behaviour originates from the Rindler vacuum expectation value, not Minkowski vacuum;
- Rindler vacuum is 'thermally depressed' with respect to Minkowski vacuum, it is not the Minkowski vacuum that is thermally excited;
- being the Rindler vacuum expectation value the counter-term, the temperature-like behaviour arises from our specific choice of renormalization scheme, and is not an intrinsic property of the system.

The reader might dismiss these points, arguing that as long as there is a temperature difference, the physical implications and the result remain unchanged. However, it is crucial to track the origins of the quantum corrections. If these depend on a particular choice of renormalization scheme we made, they can also be tossed away whenever a different prescription is adopted. In our case, the specific contribution from acceleration/temperature arises from the counterterms in ${}_R\langle 0|\,\phi^2\,|0\rangle_R$, for a given $|0\rangle_R$. Changing the observer's acceleration alters the associated Rindler vacuum, and consequently, the value of the counterterms $T_U/12$. If we adopt a different, generic renormalization scheme that is

entirely independent of acceleration or temperature, the thermal-like behavior of the system vanishes.
Conversely, in the 'pure' case of finite temperature in Minkowski spacetime, the thermal behavior of $\langle\phi^2\rangle_{\text{ren}}$ persists under any possible renormalization scheme, except those where the counterterms include a temperature-dependent contribution capable of altering or nullifying the thermal effects. In this context, it is customary, and supported by both theoretical and experimental evidence, to fix the form of the counterterms once and evaluate the thermal contributions based on this fixed choice.
For these reasons, although the two scenarios share many similarities, they may not be treated on the same footing.

1.3.1.2 *Covariant renormalization scheme*

In the early studies of quantum field theory in curved spacetimes, one of the primary goals was to develop a theory that, like General Relativity, preserves general covariance and ensures invariance under changes of coordinates. Despite the emergence of novel quantum phenomena, such as Hawking radiation and the conformal trace anomaly, the tensorial and covariant structures of field equations and observables remained the guiding principle for constructing a consistent theoretical framework.
In this context, the development of a covariant renormalization method was essential to ensure that vacuum expectation values are calculated consistently in all reference frames. One of the most renowned methods is the covariant point-splitting method. An explicit expression for the counterterms of the two-point Green's function in a scalar field theory of the type

$$S\left[\phi\right] = \int d^4x \, \frac{\sqrt{-g}}{2} \left(\partial_\mu\phi\partial^\mu\phi - m^2\phi^2 - \xi R\phi^2\right) \tag{1.175}$$

in an arbitrary curved spacetimes, was derived by S.M. Christensen [30], and can be written as

$$G_{\text{CT}}\left(x, x'\right) = i\left(\frac{1}{8\pi^2\sigma} + \frac{m^2 + \left(\xi - \frac{1}{6}\right)}{8\pi^2}\left[\gamma_E + \frac{1}{2}\ln\left(\frac{m^2\sigma}{2}\right)\right] - \frac{m^2}{16\pi^2} + \frac{1}{96\pi^2}R_{\alpha\beta}\frac{\sigma^\alpha\sigma^\beta}{\sigma}\right), \tag{1.176}$$

with γ_E the Euler-Mascheroni constant. In Minkowski space, which is globally flat, the Ricci tensor and scalar vanish and the counterterms simplify to

$$G_{CT}\left(x,x'\right) = i\left(\frac{1}{8\pi^2\sigma} + \frac{m^2}{8\pi^2}\left[\gamma_E + \frac{1}{2}\ln\left(\frac{m^2\sigma}{2}\right)\right] - \frac{m^2}{16\pi^2}\right). \quad (1.177)$$

What we usually require from a free theory in flat spacetime for inertial observers, is that quantum corrections at on-loop vanish, as we have discussed at the end of section 1.1.3.3. Thus, if we consider the Minkowski propagator (1.173), and we expand it for small geodesic distances, i.e. $\sigma \to 0^+$, we get

$$\begin{aligned} &i_M\langle 0|\,\phi\left(x\right)\phi\left(x'\right)|0\rangle_M \\ &\overset{x\to x'}{=} \\ &i\left(\frac{1}{8\pi^2\sigma} + \frac{m^2}{8\pi^2}\left[\gamma_E + \frac{1}{2}\ln\left(\frac{m^2\sigma}{2}\right)\right] - \frac{m^2}{16\pi^2} + O\left(\sigma\right)\right), \end{aligned} \quad (1.178)$$

which, neglecting $O(\sigma)$ terms, is exactly (1.177). In this sense, (1.176) can be regarded as the natural covariant extension of Minkowski counterterms to curved spacetimes.
This reinforces the well-established idea that, in a covariant renormalization scheme for flat spacetime, the counterterms are determined by $i\,{}_M\langle 0|\,\phi(x)\phi(x')\,|0\rangle_M$.
No acceleration-dependent or frame-dependent terms appear in (1.177), which make this renormalization procedure inequivalent from the one employed in 1.3.1.1.
Taking this into account, we observe that, by definition, the vacuum polarization $\langle\phi^2\rangle_{\text{ren}}$ is null

$$\begin{aligned} \langle\phi^2\rangle_{\text{ren}} &= \text{Tr}\left[e^{-\beta_U H^{\mathcal{R}}}\phi(x)\,\phi(x')\right]/\text{Tr}\left[e^{-\beta_U H^{\mathcal{R}}}\right] - {}_M\langle 0|\,\phi(x)\phi(x')\,|0\rangle_M = \\ &= {}_M\langle 0|\,\phi(x)\phi(x')\,|0\rangle_M - {}_M\langle 0|\,\phi(x)\phi(x')\,|0\rangle_M = 0. \end{aligned} \quad (1.179)$$

As a consequence, the symmetry restoration condition (1.136) becomes

$$\left.\frac{\partial V}{\partial\Phi^2}\right|_{\Phi=0} = \frac{m^2}{2} \geq 0, \quad (1.180)$$

which is never satisfied for the spontaneously broken theory with $m^2 < 0$.
Thus, symmetry restoration cannot occur.

Covariant counterterms in momentum space

Let us derive the appropriate counterterm we would have considered in renormalizing (1.140) if we had chosen to adopt a covariant renormalization scheme instead.
We start by expressing the geodesic distance in terms of Rindler coordinates (1.1)

$$\begin{aligned}\sqrt{2\sigma} &= \sqrt{(t-t')^2-(x-x')^2-\left(\vec{x}_\perp-\vec{x}'_\perp\right)^2} \\ &= \sqrt{2\rho\rho'\cosh\left(a\left(\tau-\tau'\right)\right)-\rho^2-\rho'^2-\left(\vec{x}_\perp-\vec{x}'_\perp\right)^2}\end{aligned} \tag{1.181}$$

and we apply the incomplete coincidence limit (1.139)

$$\begin{aligned}\tau' &= 0 \\ \rho' &= \rho = \frac{1}{a} \\ \vec{x}'_\perp &= \vec{x}_\perp.\end{aligned} \tag{1.182}$$

The resulting geodesic distance can be written as

$$\sqrt{2\sigma_\tau} = \frac{1}{a}\sqrt{2\left(\cosh\left(a\left(\tau-\tau'\right)\right)-1\right)} = \frac{2}{a}\sinh\left(\frac{a\tau}{2}\right), \tag{1.183}$$

where we have used the relation $\cosh(x)-1 = 2\sinh^2(x/2)$.
Inserting (1.183) in the free Minkowski propagator (1.173), we obtain

$$G_M\left(\tau\right) = i\frac{ma}{8\pi^2\sinh a\frac{\tau}{2}}K_1\left(2\frac{m}{a}\sinh\left(a\frac{\tau}{2}\right)\right). \tag{1.184}$$

Using the high acceleration limit $m/a \ll 1$, it is possible to expand the modified Bessel function as

$$K_1\left(2\frac{m}{a}\sinh\left(a\frac{\tau}{2}\right)\right) = \frac{a}{2m\sinh\left(a\frac{\tau}{2}\right)} + O\left(\frac{m}{a}\right), \tag{1.185}$$

and, neglecting terms of order $O\left(m/a\right)$, we get

$$G_M\left(\tau\right) = i\frac{a^2}{16\pi^2\sinh^2\left(a\frac{\tau}{2}\right)}, \tag{1.186}$$

which is exactly the massless Minkowski propagator in the (incomplete) coincidence limit (1.182) expressed in Rindler coordinates [31].

Ultimately, we are interested in the momentum representation of this propagator, thus, we calculate its Fourier transform, i.e.

$$\tilde{G}_M(\Omega) = \frac{a^2}{16\pi^2}\frac{i}{\sqrt{2\pi}}\int_{-\infty}^{\infty} d\tau \underbrace{\frac{e^{-i\Omega\tau}}{\sinh^2\left(a\frac{\tau}{2}\right)}}_{f_\Omega(\tau)}. \tag{1.187}$$

To evaluate this integral, we need to complexify the variable $\tau \to \tau_c$ and perform the integration in the complex plane. A suitable integration contour is the closed lower semicircle γ of radius R, with a small semi-circle of radius ϵ turning around τ (see figure 1.3).

The integrand has poles at $\tau_c = i2n\pi/a$ with $n \in \mathbb{Z}$ so, considering

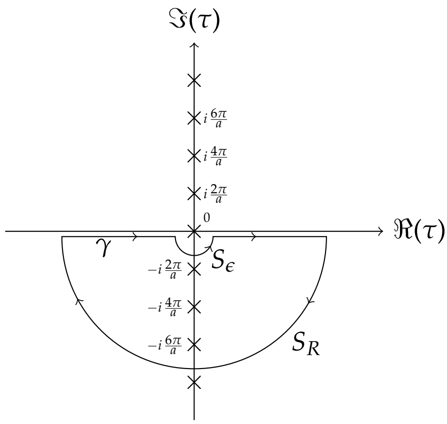

Figure 1.3: $\Omega > 0$ case. Integration contour and poles of $f_\Omega(\tau)$ in the τ complex-plane.

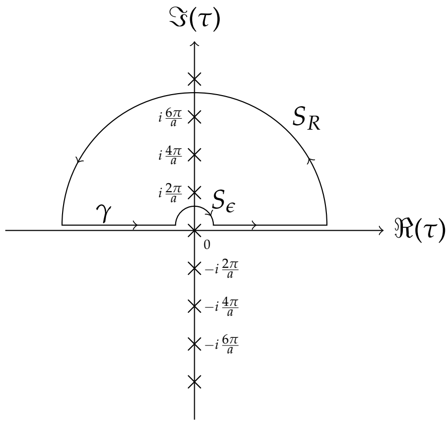

Figure 1.4: $\Omega < 0$ case. Integration contour and poles of $f_\Omega(\tau)$ in the τ complex-plane.

the lower half-plane, we are interested only in the contribution of the $n \leq 0$ poles. Thus, we obtain

$$\begin{aligned} 2\pi i \sum_{n=1}^{N} \text{Res}\left(f_\Omega(\tau_c), -i\frac{2\pi n}{a}\right) &= \oint_\gamma d\tau_c\, f_\Omega(\tau_c) \\ &= \int_{-R}^{R} d\tau\, f_\Omega(\tau) + \int_{S_R} d\tau_c\, f_\Omega(\tau_c) + \int_{S_\epsilon} d\tau_c\, f_\Omega(\tau_c), \end{aligned} \tag{1.188}$$

where N is the number of poles inside the semi-circle. Taking the limit $R \to \infty$ and $\epsilon \to 0^+$, we realize that the contribution from the integration along S_R vanishes.

Note that γ is the correct integration contour only for positive frequencies $\Omega > 0$, ensuring that the integration along S_R vanishes rather than diverging. For negative values of Ω, the appropriate contour should

instead be the semicircle in the upper half-plane (see figure 1.4). This will be enforced as a constraint in the frequency domain in the final form of the transformed propagator.
We can now evaluate the integral along the real τ-axis.

$$\begin{aligned}\int_{-\infty}^{\infty} d\tau\, f_\Omega(\tau) &= 2\pi i \sum_{n=1}^{\infty} \mathrm{Res}\left(f_\Omega(\tau_c), -i\frac{2\pi n}{a}\right) + i\pi \mathrm{Res}\left(f_\Omega(\tau_c), 0\right) \\ &= \frac{8\pi}{a^2}\Omega \sum_{n=0}^{\infty} e^{-\frac{2\pi n}{a}} + \frac{4\pi}{a^2}\Omega\end{aligned} \tag{1.189}$$

Performing the geometric series, we finally obtain

$$\int_{-\infty}^{\infty} d\tau\, f_\Omega(\tau) = \frac{4\pi}{a^2}\Omega\left(1 + \frac{2}{e^{\frac{2\pi\Omega}{a}} - 1}\right). \tag{1.190}$$

We can express the Fourier transform of the propagator for positive Ω frequencies as

$$\tilde{G}_M(\Omega) = i\frac{1}{4\pi}\frac{\Theta(\Omega)}{\sqrt{2\pi}}\Omega\left(1 + \frac{2}{e^{\frac{2\pi\Omega}{a}} - 1}\right). \tag{1.191}$$

where the Heaviside Theta function $\Theta(\Omega)$ is used to impose the frequency constraint discussed earlier.
Repeating the same procedure for $\Omega < 0$, by closing the contour integration with the same circle in the higher half-plane, we can express the Fourier transform of the propagator for negative Ω frequencies as

$$\tilde{G}_M(\Omega) = -i\frac{1}{4\pi}\frac{\Theta(-\Omega)}{\sqrt{2\pi}}\Omega\left(1 + \frac{2}{e^{\frac{-2\pi\Omega}{a}} - 1}\right). \tag{1.192}$$

Combining the two contributions and performing an inverse Fourier transform, we obtain

$$\begin{aligned}&G_M(\tau) = \\ &i\frac{1}{8\pi^2}\int_{-\infty}^{\infty} d\Omega\, e^{i\Omega\tau}\left[\Theta(\Omega)\,\Omega\left(1 + \frac{2}{e^{\frac{2\pi\Omega}{a}} - 1}\right) - \Theta(\Omega)\,\Omega\left(1 + \frac{2}{e^{\frac{2\pi\Omega}{a}} - 1}\right)\right] \\ &= i\frac{1}{4\pi^2}\int_0^{\infty} d\Omega\,\Omega\left(1 + \frac{2}{e^{\frac{2\pi\Omega}{a}} - 1}\right)\cos(\Omega\tau)\end{aligned} \tag{1.193}$$

Lastly, we complete the coincidence limit (1.182) by letting $\tau \to 0^+$, i.e.

$$\lim_{\tau\to 0^+} G_M(\tau) = i\frac{1}{(2\pi)^2}\int_0^{\infty} d\Omega\,\Omega\left(1 + \frac{2}{e^{\frac{2\pi\Omega}{a}} - 1}\right). \tag{1.194}$$

We can immediately recognize that the resulting covariant form of the counterterms (1.194) cancels completely with (1.162a). Thus, the symmetry restoration condition reverts to (1.180), which can never be realized for $m^2 < 0$.

1.3.2 *Symmetry Breaking Enhancement: Different Physical Vacuum States*

Contrary to the previous cases of symmetry restoration and preservation, the symmetry breaking enhancement scenario emerges when the true physical vacuum of an accelerating system is taken to be the Rindler vacuum instead of the Minkowski vacuum. All VEVs needs to be computed accordingly, including propagators and vacuum polarization. This choice was made explicit by Benić and Fukushima in (1.159) and by Candelas and Raine, whereas it is more subtly implied in the Chernodub's work. However, the equivalence of their approaches can be inferred from both their arguments and their results.
In the cases of Candelas-Raine and Benić-Fukushima results, it is clear that the physical quantity relevant for the calculation of the vacuum polarization is the Rindler propagator ${}_R\langle 0|\, \phi(x)\phi(x')\, |0\rangle_R$. Since it shares the same UV-structure with Minkowski propagator, they subtracted ${}_M\langle 0|\, \phi(x)\phi(x')\, |0\rangle_M$ from it, to obtain a meaningful results upon taking the coincidence limit $x \to x'$.
For Chernodub, this choice is less evident but can be identified by comparing his result with that of D.N. Page. Specifically, starting from

$$\langle \phi^2 \rangle_{\text{ren}} = \frac{1}{12}\left(T^2 - \frac{a^2}{4\pi^2}\right), \tag{1.195}$$

the fact that Chernodub considers $T = 0$ as the ground state temperature while Page considers $T = a/(2\pi)$ implies that they are implicitly considering different vacuum states in their vacuum polarization calculations. Since Chernodub's vacuum is thermally depressed by an amount $a/(2\pi)$ relative to Page's, we can directly deduce that the former adopts the Rindler vacuum as the true vacuum for the accelerated system, whereas the latter assumes the Minkowski vacuum as the correct choice.
We can then conclude that all the symmetry breaking enhancement approaches agree in the same definition of ϕ_{ren} at zero temperature, which is

$$\langle \phi^2 \rangle_{\text{ren}} = \lim_{x \to x'} \left({}_R\langle 0|\, \phi\left(x\right) \phi\left(x'\right) |0\rangle_R - {}_M\langle 0|\, \phi\left(x\right) \phi\left(x'\right) |0\rangle_M \right) \tag{1.196}$$

Building on the analysis conducted in the previous discussion of renormalization schemes, we can further examine (1.196) and highlight the following points:

- The renormalization scheme adopted is covariant;
- The result is frame-dependent.

The covariance of the renormalization scheme is ensured by considering ${}_M\langle 0|\,\phi(x)\phi(x')\,|0\rangle_M$ as the 'universal' counter-term, which remains independent of the specific reference frame.
The frame-dependent result is induced by the specific Rindler vacuum under consideration. As discussed before, different uniformly accelerated observers identify distinct, unitarily inequivalent Rindler vacua. Consequently, also the calculation of vacuum polarization will vary depending on the specific observer performing the measurement.
A dynamical concept of vacuum emerges in this picture, where the true physical grounds state changes accordingly to the peculiar accelerating observer's frame.

The physical implications of considering Rindler vacuum as the true physical ground state of an accelerating system however, extend beyond the calculation of vacuum polarization the calculation of vacuum polarization, spanning across the evaluation of the VEV of any observable. In this context, Unruh effect needs to be reinterpreted accordingly. While it holds true that Minkowski vacuum is perceived as a thermal state, it does not represent a physically significant state for the accelerated observer (except for quantities that require renormalization). For instance, let us consider the VEV of the Rindler number operator $N_R^{(+)} = b^{(+)\dagger}_{\Omega,\vec{k}_\perp} b^{(+)}_{\Omega,\vec{k}_\perp}$ performed in (1.41). In this case, it is necessary to use the Rindler vacuum rather than the Minkowski vacuum, and the expectation value trivially vanishes,

$$ {}_R\langle 0|\,N_R^{(+)}\,|0\rangle_R = {}_R\langle 0|\,b^{(+)\dagger}_{\Omega,\vec{k}_\perp} b^{(+)}_{\Omega,\vec{k}_\perp}\,|0\rangle_R = 0. \tag{1.197}$$

Therefore, the accelerating observer detects no thermal spectrum from the vacuum.
Moreover, the thermalization theorem (Eq. 1.106), given by

$$ \begin{aligned} &{}_M\langle 0|\left(\phi\left(t_1,\vec{x}_1\right),\cdots,\phi\left(t_n,\vec{x}_n\right)\right)_t|0\rangle_M = \\ &\mathrm{Tr}\left[e^{-\beta_U H^R}\left(\phi\left(\tau_1,\rho_1,\vec{x}_{1\perp}\right),\cdots,\phi\left(\tau_n,\rho_n,\vec{x}_{n\perp}\right)\right)_\tau\right]/\mathrm{Tr}\left[e^{-\beta_U H^R}\right], \end{aligned} \tag{1.198}$$

changes interpretation. The accelerating system is no longer intrinsically thermalized at the Unruh temperature. Instead, it must be heated from $T = 0$ to $T_U = a/(2\pi)$ for the accelerated to measure the same

Green's functions as the Minkowski observer in his vacuum state.

In conclusion, the physical implications of considering the Rindler vacuum as the true ground state for the accelerating observer are profound. These implications extend beyond the calculation of vacuum polarization or the stress-energy tensor, requiring a fundamentally different interpretation of quantum field theory for the accelerating observer as a whole.

1.3.3 *Outlook*

Our work demonstrates that the evident contradictions in the literature ultimately arise from different choices of (inequivalent) renormalization schemes or differing definitions of the 'true' vacuum state for an accelerating observer, leading to conflicting results. Specifically, as we have seen in sections 1.1.3.2 and 1.2.2, the symmetry restoration condition at one-loop can be expressed equivalently in terms of the VEV of the field or studying the derivatives of the effective potential, i.e,

$$\begin{aligned} \Phi &= \pm\sqrt{-\frac{6m^2}{\lambda} - 3\langle\phi^2\rangle} = 0 \\ \left.\frac{\partial V_{\text{eff}}}{\partial\Phi^2}\right|_{\Phi=0} &= \frac{m^2}{2} + \frac{\lambda}{4}\langle\phi^2\rangle = 0. \end{aligned} \tag{1.199}$$

Both quantities heavily depend on $\langle\phi^2\rangle$ and the way its calculation is performed. The three major different prescriptions adopted in the literature to compute and renormalize this divergent quantity have been shown to be inequivalent. In the following, we briefly summarize the results obtained.

Symmetry phase restoration:
Minkowski 'true' vacuum and frame-dependent renormalization

1. Both quantities Φ and $\langle\phi^2\rangle_{\text{ren}}$ lose their scalar behaviour under general coordinates transformations.
   In this prescription, a spontaneously broken symmetry can, in principle, be restored due to a sufficiently high acceleration.

2. The calculation of the renormalized $\langle\phi^2\rangle_{\text{ren}}$ can be written as

$$\begin{aligned}\langle\phi^2\rangle_{\text{ren}} &= \text{Tr}\left[e^{-\beta_U H^{\mathcal{R}}}\phi^2\right]/\text{Tr}\left[e^{-\beta_U H^{\mathcal{R}}}\right] - \underbrace{{}_R\langle 0|\,\phi^2\,|0\rangle_R}_{\text{counterterm}} \\ &= {}_M\langle 0|\,\phi^2\,|0\rangle_M - {}_R\langle 0|\,\phi^2\,|0\rangle_R = \frac{a^2}{48\pi^2} = \frac{T_U^2}{12}.\end{aligned} \tag{1.200}$$

3. The finite Unruh temperature contribution originates directly from the counterterms, rather than the 'thermal Rindler propagator'.

4. Counterterms can be expressed as follows

$${}_R\langle 0|\,\phi^2\,|0\rangle_R = {}_M\langle 0|\,\phi^2\,|0\rangle_M - \frac{a^2}{48\pi^2} = \frac{1}{4\pi^2}\int_0^\infty d\Omega\,\Omega = -\frac{2}{\lambda}\delta m^2. \tag{1.201}$$

Symmetry phase preservation:
Minkowski 'true' vacuum and covariant renormalization

1. Both quantities Φ and $\langle\phi^2\rangle_{\text{ren}}$ preserve their scalar behaviour under general coordinates transformations.
   As a result, a spontaneously broken symmetry can never be restored as a function of acceleration.

2. The calculation of the renormalized $\langle\phi^2\rangle_{\text{ren}}$ can be written as

$$\begin{aligned}\langle\phi^2\rangle_{\text{ren}} &= \text{Tr}\left[e^{-\beta_U H^{\mathcal{R}}}\phi^2\right]/\text{Tr}\left[e^{-\beta_U H^{\mathcal{R}}}\right] - \underbrace{{}_M\langle 0|\,\phi^2\,|0\rangle_M}_{\text{counterterm}} \\ &= {}_M\langle 0|\,\phi^2\,|0\rangle_M - {}_M\langle 0|\,\phi^2\,|0\rangle_M = 0,\end{aligned} \tag{1.202}$$

   which vanish by definition.

3. The finite Unruh temperature contribution is absent.

4. Counterterms can be expressed as follows

$${}_M\langle 0|\,\phi^2\,|0\rangle_M = \frac{1}{4\pi^2}\int_0^\infty d\Omega\,\Omega\left(1 + \frac{2}{e^{\frac{2\pi\Omega}{a}} - 1}\right) = -\frac{2}{\lambda}\delta m^2. \tag{1.203}$$

Possible schemes reconciliation with a covariant outcome

The only possible way, in principle, to reconcile the two schemes, is by introducing a new framework where the renormalized mass is not bounded to be a constant anymore, but becomes a general function of acceleration, i.e.,

$$m \to m_R(a) + \delta m^2. \tag{1.204}$$

In this way, the symmetry restoration condition becomes

$$\left.\frac{\partial V_{\text{eff}}}{\partial \Phi^2}\right|_{\Phi=0} = \frac{m_R(a)^2}{2} + \frac{\lambda}{4}\langle \phi^2 \rangle_{\text{ren}} = 0. \tag{1.205}$$

In order for the derivative of the effective potential to be invariant under coordinates transformation, we have two possibilities:

- the calculation of $\langle \phi^2 \rangle$ is performed covariantly. Thus, the renormalized mass $m_R(a)$ is a constant so that the combination of the two terms is constant in any reference frames.
- the calculation of $\langle \phi^2 \rangle_{\text{ren}}$ yields an acceleration-dependent result. Thus, the renormalized mass $m_R(a)$ is a definite function of acceleration so that the combination of the two terms is constant in any reference frames.

While it is true that a less intricate way to obtain an invariant result is to adopt a covariant renormalization scheme, it is possible to also change the definition of the renormalized parameters, such as mass, in order to reconcile the two schemes. It is important to emphasize that this remains an empirical prescription, not derived from the fundamental principles of QFT, and should be regarded as such.

Symmetry breaking enhancement:
Rindler 'true' vacuum and covariant renormalization

1. Both quantities Φ and $\langle \phi^2 \rangle_{\text{ren}}$ lose their scalar behaviour under general coordinates transformations.
   In this prescription, the spontaneous symmetry breaking of the theory is enhanced, with an increase in the condensate.

2. The calculation of the renormalized $\langle \phi^2 \rangle_{\text{ren}}$ can be written as

$$\langle \phi^2 \rangle_{\text{ren}} = {}_R\langle 0|\, \phi^2\, |0\rangle_R - {}_M\langle 0|\, \phi^2\, |0\rangle_M = -\frac{a^2}{48\pi^2} = -\frac{T_U^2}{12}. \tag{1.206}$$

3. The finite, negative Unruh temperature contribution originates from the choice of Rindler vacuum as the 'true' physical one,

which varies dynamically for each accelerated observer.

4. The vacuum polarization divergences are removed by using the usual, covariant, Minkowski term ${}_M\langle 0|\, \phi^2 \,|0\rangle_M$.

2 QUANTUM VACUUM EFFECTS IN NON-RELATIVISTIC QUANTUM FIELD THEORIES

2.1 PRELIMINARIES

The Casimir effect can be understood as a byproduct of the perturbation of quantum fluctuations in a space where boundary conditions are imposed at finite distances. In practice, these boundary conditions are realized by using a setup that interacts with the field and imposes physical constraints, such as conducting plates for the electromagnetic field. As a consequence, the fluctuation spectrum in the region between the plates becomes discretized.
Heuristically, in the absence of boundary conditions, vacuum fluctuations within a given region of space are considered to be in equilibrium with fluctuations in the surrounding regions. However, when boundary conditions are introduced, this equilibrium is disrupted. By removing certain fluctuations from the confined volume, the number of permitted quantum modes inside the plates becomes significantly smaller than outside. The resulting energy difference between the inside and the outside gives rise to a force, known as the Casimir force.

Operatively, the zero-point energy (ZPE) (in the presence of Dirichlet boundary conditions) is defined as the sum over all energy eigenstates of the field, i.e.,

$$E = \frac{1}{2}\sum_{n=1}^{\infty} \omega_n, \tag{2.1}$$

which is clearly divergent. To make this quantity physically meaningful, it must be renormalized by removing the divergent terms. The finite quantity that emerges from this process is commonly referred to as the 'quantum vacuum energy' (QVE).
In this introduction, we will present the general framework for understanding QVE within the context of quantum corrections, along with the foundational knowledge required to treat and calculate it for free scalar fields subject to boundary conditions. Building on these fundamental concepts and insights gained in this introduction, we will explore, in the next section, the case of interacting fields, which is the central focus of my Ph.D. work.

2.1.1 *Vacuum energy as quantum correction*

In the following, we will show how the QVE can be interpreted as the one-loop quantum correction in the effective action.
As we have seen in 1.1.3.3, the one-loop effective action, in flat spacetimes and unbroken symmetry regime $m^2 > 0$, $\Phi = 0$, can be expressed as

$$\Gamma[\phi] = S[\phi] + \frac{i}{2}\mathrm{Tr}\ln\left[\Box + m^2\right] \tag{2.2}$$

and we have proceeded to calculate the finite temperature effective potential, through the Matsubara frequency formalism. The one-loop correction was then split into a zero-temperature contribution $V_0^{(1)}$ (1.102a) and a finite-temperature contribution $V_\beta^{(1)}$ (1.102b). Setting the limit of zero temperature $1/\beta \to 0$, the thermal contribution vanishes and the total one-loop contribution is given by the effective potential

$$V_0^{(1)} = \frac{1}{2}\int \frac{d^3k}{(2\pi)^3}\,\omega_k, \qquad \text{with } \omega_k \text{ energy eigenvalues} \tag{2.3}$$

Note that the same result could have been achieved without employing the finite temperature approach, by considering the effective potential (1.97) expressed in Minkowski momenta, with the usual $i\epsilon$ prescription, i.e,

$$V^{(1)} = -\frac{i}{2}\int \frac{d^4k}{(2\pi)^4}\,\ln\left(-k_0^2 + \omega_k^2 - i\epsilon\right) \tag{2.4}$$

and integrating k_0 in the complex plane, discarding an infinite constant. As previously mentioned, it is standard practice to renormalize out this term so that one-loop quantum corrections for a free field, without boundary conditions imposed, vanish. The same prescription was adopted in the previous chapter, the covariant renormalization prescription had the effect of canceling out completely the one-loop correction.

Typically, the renormalization prescription is fixed, and any factor that alters or modifies the energy dispersion relation (such as potentials, boundary conditions, or other external influences) and the ZPE, will lead to a non-vanishing vacuum energy. This, in turn, contributes to the one-loop correction of the effective action.
This is the case of the boundary conditions we are imposing on the system. The energy dispersion relation is modified, as frequencies and momenta become discretized. By reversing the thermodynamic limit, the one-loop correction can be expressed in terms of the vacuum energy.

Considering:

- theories where dispersion relations are invariant under parity transformation (as in both relativistic and non-relativistic settings):

$$\frac{1}{2\,(2\pi)^3}\iiint_{-\infty}^{\infty} dk_x dk_y dk_z\,\omega_k = \frac{1}{2\pi^3}\iiint_0^{\infty} dk_x dk_y dk_z\,\omega_k, \tag{2.5}$$

- Dirichlet boundary conditions $\phi(0) = \phi(L_{x,y,z}) = 0$, with $L_{x,y,z}$ distances between the plates, and

$$k_n = \pi n/L, \qquad d_k \to \pi/L, \qquad \omega_k \to \omega_n, \tag{2.6}$$

- ω_k translationally invariant,

it is possible to express the one-loop effective action as

$$\begin{aligned}\Gamma^{(1)} &= -\int d^4x V_{\text{eff}} = -V^{(4)}\frac{1}{2V^{(3)}}\sum_{n=1}^{\infty}\omega_n \\ &= -T\sum_{n_x,n_y,n_z=1}^{\infty}\frac{\omega_n}{2}\end{aligned} \tag{2.7}$$

where $V^{(4)}$ and $V^{(3)} = L^3$ are respectively the spacetime and the space volume within the plates and T the time interval.

The ZPE in (2.7) is still divergent and require renormalization.
In this context, it is useful to compare the one-loop correction the the effective action $\Gamma^{(1)}$ both in the case of continuous and discrete spectra.
The relations (2.3) and (2.7) can be rewritten as

$$\begin{cases}\Gamma^{(1)}_{NB} = -\dfrac{V^4}{2}\displaystyle\int\frac{d^3k}{(2\pi)^3}\,\omega_k, & \text{without boundary conditions}\\ \Gamma^{(1)}_{B} = -\dfrac{T}{2}\displaystyle\sum_{n_x,n_y,n_z=1}^{\infty}\omega_n, & \text{with boundary conditions.}\end{cases} \tag{2.8}$$

To compare the vacuum energy in the two setups, one needs to consider the following relation

$$\frac{V^3}{(2\pi)^3}\frac{1}{2}\int d^3k\,\omega_k \iff \frac{1}{2}\sum_{n_x,n_y,n_z=1}^{\infty}\omega_n \tag{2.9}$$

to ensure that calculation is performed consistently, also from a dimensional point of view.

In the following, we will show some of the most prominent renormalization techniques.

2.1.1.1 *Regularization and renormalization prescriptions*

In order to regularize and subsequently renormalize the divergent ZPE, various methods can be employed. Below, we outline a few of these approaches, with a focus on some of the most well-established techniques that we intend to utilize throughout this thesis:

Casimir's renormalization definition

Casimir's idea to renormalize the ZPE follows the logic we presented earlier. The renormalized QVE E_{vac} is given by the difference between the ZPE in the presence, E_B, and in the absence, E_{NB}, of boundaries,

$$E_{\text{vac}} = E_B - E_{NB}. \tag{2.10}$$

Such a definition is compatible with the normal ordering prescription and with the vanishing of the VEV of the Hamiltonian in the non-interacting vacuum (i.e., no boundaries) and gives a calculable recipe of the QVE in response to changes in external conditions.

To apply this definition in practice, several different methods can be taken. We will present some of the most renowned techniques, while adopting a very introductory approach. In fact, the purpose of this introduction is to provide the reader with the essential tools to understand the upcoming sections, where the Ph.D. results are presented. For more advanced insights into the topics, the reader can consult the references [32][33].

Regulator-based approaches

Introducing a regulator parameter allows the separation of the divergent component of the ZPE from its finite part. The divergent component then needs to be renormalized by virue of (2.10) or another suitable renormalization prescription. In the following we present the two regulator-based techniques that we will utilize in this thesis

- *Frequency-dependent 'window function'.*
  A regulator of the form $\exp(-\epsilon f(\omega))$ is introduced into the sum

or integral of the ZPE to ensure the result is finite for any $\epsilon > 0$. In 1+1 dimensions, this can be schematically written as:

$$\begin{aligned} E_B &= \frac{1}{2}\sum_n \omega_n e^{-\epsilon f(\omega_n)} \\ E_{NB} &= \frac{L}{2}\int \frac{dk}{2\pi}\omega_k e^{-\epsilon f(\omega_k)} \end{aligned} \tag{2.11}$$

Any reasonable choice of the function f should not affect the final result. However, the specific choice $f(\omega) = \omega_\infty$, where ω_∞ represents the leading large-frequency asymptotics of the spectrum, is particularly natural, as it allows ϵ to be interpreted as a length-scale cutoff. In this context, we will use ϵ and ℓ_c interchangeably. ℓ_c determines how high-frequency modes are suppressed and, in the limit $\ell_c \to 0$, the zero point energies (2.11) return to their unregularized expression.
The general procedure we will follow is to calculate the ZPE using its new definition, expand the result for small ϵ, and apply Casimir renormalization (2.10). The 'window function technique' can be useful even in the presence of real physical cut-off scale associated with a minimal length scale (e.g., the interatomic separation scale).

- *Zeta-function regularization*.
  Zeta-function regularization is a mathematical technique used to assign finite values to otherwise divergent sums. The fundamental concept is to express these sums in terms of the Riemann zeta function

  $$\zeta_R(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}, \tag{2.12}$$

  or its generalizations. The regularization technique involves finding a representation of the zeta function that converges in a specific region of the complex-s plane. This is followed by analytically continuing the function to the physical value for $s \to 0$, in a region where the original series did not converge. The uniqueness property of analytic continuation ensures that the obtained result is unique as well.
  Practically, this is achieved by generalizing the ZPE (2.1) to a complex-valued function

  $$E(s) = \frac{1}{2}\sum_{n=1}^{\infty} \omega_n \left(\frac{\omega_n}{\mu}\right)^{-s} \tag{2.13}$$

  where $s \in \mathbb{C}$ is a complex-valued regularization parameter and μ is a renormalization scale with dimension of energy. $E(s)$ is then analytically continued to regions of the complex plane where it can be evaluated in the limit $s \to 0$. In flat space with flat boundaries, this limit provide a finite result which is independent of the renormalization scale. In a more general scenario it may diverge and a renormalization prescription is required.

Regulator-independent approach - Abel-Plana formula

Although originally conceived as a method for expressing mathematical series in terms of integrals, the Abel-Plana formula adapts surprisingly well to the calculation of Casimir energy. We can represent it in different ways [34], but the most used frequently used form in physical applications is

$$\sum_{n=0}^{\infty} f(n) - \int_0^{\infty} dx\, f(x) = \frac{1}{2} f(0) + i \int_0^{\infty} dt\, \frac{f(it) - f(-it)}{e^{2\pi t} - 1}, \tag{2.14}$$

where $f(z)$ is a meromorphic function whose poles do not lie on the real axis. Additionally, the following growth condition must be satisfied

$$\lim_{y\to\infty} e^{-2\pi|y|} \left| f(x+iy) \right| = 0, \tag{2.15}$$

uniformly for x within any finite interval. It indeed allows us to calculate the QVE directly from the definition given in (2.10).

2.1.2 *Casimir effect in 1+1 dimensions*

We now examine the classic case of the Casimir effect, using it as a simplified model to apply the concepts and techniques discussed up to this point.
The Casimir effect, in its original formulation, was studied within the framework of quantum electrodynamics. It involved imposing boundary conditions on the electromagnetic field, where the field vanishes on the surfaces of two parallel, perfectly conducting metallic plates, effectively introducing Dirichlet boundary conditions. In the Coulomb gauge, where $\vec{\nabla} \cdot \vec{A}$, the electromagnetic potential A^{μ} has two independent components, which correspond to the two polarization states of light. These can each be treated as independent real massless scalar fields, simplifying the problem while requiring a factor of two in the final result to account for both polarizations.
We now turn to the simplified case of a free real massless scalar field in 1+1 dimensions, which captures the essential features of the Casimir effect. The Lagrangian density and corresponding equation of motion are given by:

$$\mathcal{L} = \int d^2x \frac{1}{2}\, \partial_{\mu}\phi \partial^{\mu}\phi, \qquad \partial_{\mu}\partial^{\mu}\phi = 0. \tag{2.16}$$

Given the Dirichlet boundary conditions

$$\phi(0) = \phi(L) = 0, \tag{2.17}$$

with L the distance between the plates, we obtain two different mode decomposition. The first one in the presence of plates, with discrete energy eigenvalues ω_n, the other without the plates, with continuous energy eigenvalues ω_k, i.e.

$$\begin{aligned} u_{NB}(k) &= \frac{e^{i\omega_k t}e^{-ikx}}{2\sqrt{\pi\omega_k}} && \text{with } \omega_k = |k| \\ u_B(k) &= \frac{e^{-i\omega_n t}\sin(k_n x)}{\sqrt{\omega_n L}} && \text{with } \omega_n = |k_n| = \frac{\pi n}{L}, \quad n \in \mathbb{Z}/\{0\}. \end{aligned} \tag{2.18}$$

The corresponding quantization of the field can be written as

$$\begin{aligned} \phi_{NB}(x) &= \int_{-\infty}^{\infty} \frac{dk}{2\sqrt{\pi\omega_k}} \left(a_k e^{i\omega_k t} e^{-ikx} + a_k^\dagger e^{i\omega_k t} e^{ikx} \right) \\ \phi_B(x) &= \sum_{n=1}^{\infty} \frac{1}{\sqrt{\omega_n L}} \left(a_n e^{-i\omega_n t} + a_n^\dagger e^{i\omega_n t} \right) \sin(k_n x). \end{aligned} \tag{2.19}$$

We are now interested in computing the Casimir energy of the system. To perform the calculation we define the window function $f(\omega) = \exp(-\omega\ell_c)$, compute the vacuum energy E_B inside the region $0 < x < L$, and expand the result for small ℓ_c,.i.e.,

$$\begin{aligned} E_B(\ell_c) &= \frac{1}{2}\sum_{n=1}^{\infty} \frac{\pi n}{L} e^{-\ell_c \frac{\pi n}{L}} = -\frac{\pi}{L}\frac{\partial}{\partial \ell_c}\left(\sum_{n=1}^{\infty} e^{-\ell_c \frac{\pi n}{L}} \right) = -\frac{\pi}{L}\frac{\partial}{\partial \ell_c}\left(\frac{1}{e^{\frac{\pi \ell_c}{L}} - 1} \right) \\ &= \underbrace{\frac{L}{2\pi\ell_c^2}}_{\text{divergent for } \ell_c \to 0} - \frac{\pi}{24L} + O\left(\ell_c^2\right). \end{aligned} \tag{2.20}$$

The same procedure is then applied to calculate the vacuum energy E_{NB} in the region $0 < x < L$, taking into account the relation (2.9) in 1+1 dimensions

$$\begin{aligned} E_{NB}(\ell_c) &= \frac{L}{2}\int_{-\infty}^{\infty} \frac{dk}{2\pi} |k| e^{-|k|\ell_c} = -L\frac{\partial}{\partial \ell_c}\left(\int_0^{\infty} \frac{dk}{2\pi} e^{-k\ell_c} \right) = -\frac{L}{2\pi}\frac{\partial}{\partial \ell_c}\left(\frac{1}{\ell_c} \right) \\ &= \underbrace{\frac{L}{2\pi\ell_c^2}}_{\text{divergent for } \ell_c \to 0}. \end{aligned} \tag{2.21}$$

Using definition (2.10), it is possible to renormalize the vacuum energy, observing that the divergent contributions in the regulator ℓ_c in (2.20) and in (2.21) cancel out exactly

$$E_{\text{vac}} = E(0) = -\frac{\pi}{24L}. \tag{2.22}$$

This is also called the 'Casimir energy'.
This vacuum energy could also be calculated considering the zeta-function regularization. In fact,

$$E_{\text{vac}} = \lim_{s\to 0}\frac{\mu^s}{2}\sum_{n=1}^{\infty}\left(\frac{\pi n}{L}\right)^{1-s} = \lim_{s\to 0}\frac{\mu^s}{2}\left(\frac{\pi}{L}\right)^{1-s}\zeta_R\left(-1+s\right) = -\frac{\pi}{24L} \tag{2.23}$$

(where $\zeta_R(-1) = -1/12$).
The difference in QVE produces a force that pulls the plates close together, i.e,

$$F \equiv -\frac{\partial}{\partial L}E_{\text{vac}} = \frac{\pi}{24L^2}, \tag{2.24}$$

this force is also called 'Casimir force'.

2.1.3 *The issue of non-relativistic vacuum energy*

Before discussing QVE in non-relativistic quantum field theories, it is crucial to determine whether such a field can actually possess ZPE. Non-relativistic and relativistic quantum field theories differ significantly. Non-relativistic theories lack the concepts of antiparticles, intrinsic vacuum fluctuations, and the associated ZPE peculiar to relativistic frameworks, despite complying with the Heisenberg uncertainty principle. Therefore, it seems that QVE, as understood in relativistic contexts, cannot be meaningfully defined in a non-relativistic system.
More specifically, the non-relativistic quantum field theory can be described in terms of the Schrödinger action

$$S = \int dtd^3x\left[\frac{i}{2}\left(\Psi^\dagger\dot{\Psi} - \dot{\Psi}^\dagger\Psi\right) - \frac{1}{2m}\left|\nabla\Psi\right|^2 - V\left|\Psi\right|^2\right] \tag{2.25}$$

with the associated Hamiltonian

$$H = \int d^3x\Psi^\dagger\left(-\frac{1}{2m}\nabla^2 + V\right)\Psi. \tag{2.26}$$

The usual field quantization can be expressed as

$$\Psi\left(t,\vec{x}\right) = \sum_n a_n e^{-i\omega_n t} f_n\left(\vec{x}\right), \tag{2.27}$$

with creation and annihilation operators $a_n^\dagger, a_n$ that satisfy the canonical equal-time commutation relations

$$\begin{aligned}
&\left[\Psi\left(t,\vec{x}\right),\Psi^\dagger\left(t,\vec{x}'\right)\right]=\delta\left(\vec{x}-\vec{x}'\right)\\
&\left[\Psi\left(t,\vec{x}\right),\Psi\left(t,\vec{x}'\right)\right]=\left[\Psi^\dagger\left(t,\vec{x}\right),\Psi^\dagger\left(t,\vec{x}'\right)\right]=0\\
&\left[a_n,a_{n'}^\dagger\right]=\delta_{n,n'}\\
&\left[a_n,a_{n'}\right]=\left[a_n^\dagger,a_{n'}^\dagger\right]=0
\end{aligned}\tag{2.28}$$

and $\{f_n\left(\vec{x}\right)\}_n$ a complete orthonormal set of eigenfunctions.
We immediately notice that the field is expanded only in terms of positive frequency modes, lacking the negative frequency modes associated with antiparticles, which are present in the relativistic case.
As a result, the Hamiltonian operator can be written in terms of creation and annihilation operator as

$$H=\sum_n \omega_n a_n^\dagger a_n \tag{2.29}$$

without the characteristic ZPE contribution.
This is why, canonically, it is assumed that non-relativistic quantum fields do not possess QVE.

Nonetheless, the Schrödinger theory can be regarded as an effective field theory that emerges as the non-relativistic limit of a relativistic scalar field theory.
Consider the well-known Hamiltonian of a relativistic complex scalar field, expressed in terms of creation and annihilation operators:

$$H=\sum_n \omega_n\left(a_n^\dagger a_n+b_n b_n^\dagger\right)=\sum_n \omega_n\left(a_n^\dagger a_n+b_n^\dagger b_n+1\right)\tag{2.30}$$

where $a_n, a_n^\dagger, b_n, b_n^\dagger$, annihilation and creation operators of particles and antiparticles, respectively. These operators satisfy the canonical commutation relations

$$\begin{aligned}
&\left[a_n,a_{n'}^\dagger\right]=\delta_{n,n'}\\
&\left[a_n,a_{n'}\right]=\left[a_n^\dagger,a_{n'}^\dagger\right]=0\\
&\left[b_n,b_{n'}^\dagger\right]=\delta_{n,n'}\\
&\left[b_n,b_{n'}\right]=\left[b_n^\dagger,b_{n'}^\dagger\right]=0.
\end{aligned}\tag{2.31}$$

We can regard the non-relativistic limit as a theory which is free from antiparticles. As a consequence, we can consider the Fock space of this theory as restricted to the states $|\psi\rangle$ satisfying $b_n^\dagger b_n |\psi\rangle = 0$. Moreover, the non-relativistic eigenvalues ω_n^{NR} are obtained as an approximation of the relativistic eigenvalues ω_n^R in the regime of small momenta $k/m \ll 1$. Expanding ω_n^R, we find

$$\omega_n^R = \sqrt{k_n^2 + m^2} \approx m + \frac{k_n^2}{2m} = m + \omega_n^{NR}, \tag{2.32}$$

Neglecting the rest mass term, the resulting Hamiltonian reduces to

$$H \approx \sum_n \omega_n \left(a_n^\dagger a_n + 1 \right), \tag{2.33}$$

which reproduce the correct limit (2.29) with the addition of the ZPE.

In summary, the Schrödinger field theory behaves as an effective field theory (EFT), which inherently limits its ability to predict phenomena specific of the higher-energy relativistic theory, such as quantum fluctuations or ZPE. However, these phenomena remain an intrinsic part Nature, even though an effective field theory may lack the tools to predict them. A more deeper understanding emerges from the UV-complete theory, which shows that some of these effects may persist at the low-energy scale where the effective theory operate, highlighting the importance of incorporating them from the UV perspective. Specifically, if a UV-complete theory predicts corrections or novel phenomena in the low-energy limit that differ from those of the corresponding EFT, the predictions of the UV-complete theory are generally more reliable, provided the corrections lie within the regime of validity of both theories.

Hence, by adopting a 'hybrid' approach (2.33) that incorporates ZPE into the Schrödinger field theory, it becomes possible to investigate phenomena involving QVE within the framework of non-relativistic quantum field theory as well. This will be our approach in the calculation of the QVE of a non-relativistic interacting system in the presence of boundaries, 2.2. This result will be then endorsed by the calculation of the associated relativistic system in the low energy limit 2.3.42.3.5.

2.1.4 *Non-relativistic quantum vacuum energy in 1+1 dimensions: non-interacting case*

Considering the 'hybrid' Hamiltonian (2.33), we investigate whether a non-relativistic theory in presence of Dirichlet boundary conditions can acquire a non-vanishing QVE. We will use the analysis conducted in the guided example in (2.1.2) and renormalize the ZPE in the same manner. As a regularization prescription, we will again use the window function, with $f(\omega) = \omega^{NR}$, and verify the result with zeta-function regularization. We have previously shown that the non-relativistic dispersion relation takes the form $\omega_k^{NR} = k^2/2m$. When boundary conditions are imposed, the momentum becomes discrete, i.e. $p = \pi n/L$, and the corresponding energy eigenvalues are $\omega_n^{NR} = \pi^2 n^2/2mL^2$. Therefore, the ZPEs of the system, with and without boundary conditions, in the region $x \in [0, L]$ is

$$E_B(\ell_c) = \frac{1}{2}\sum_{n=1}^{\infty} \frac{\pi^2 n^2}{2mL^2} e^{-\ell_c \frac{\pi^2 n^2}{2mL^2}} \tag{2.34}$$

$$E_{NB}(\ell_c) = \frac{L}{2}\int_{-\infty}^{\infty} \frac{dk}{2\pi}\frac{k^2}{2m} e^{-\frac{k^2}{2m}\ell_c}. \tag{2.35}$$

For the calculation of E_B it is useful to introduce the Jacobi thetanull function $\theta(x)$ (also expressed as the Jacobi $\theta_3(0, e^{\pi x})$ function), which is defined as

$$\theta(x) = \sum_{n=-\infty}^{\infty} e^{-\pi x n^2} = 1 + 2\sum_{n=1}^{\infty} e^{-\pi x n^2}. \tag{2.36}$$

The θ function possesses a particularly convenient modular transformation property:

$$\theta(x) = \frac{1}{\sqrt{x}}\theta\left(\frac{1}{x}\right), \tag{2.37}$$

which enables us to express, as $x \to 0$, an infinite, divergent sum of exponentials, each tending individually to 1, as the sum of a single divergent term and an infinite series of exponentially vanishing terms. In practice:

$$\theta(x) = \sqrt{\frac{1}{x}} + O\left(\frac{e^{-\frac{\pi}{x}}}{\sqrt{x}}\right). \tag{2.38}$$

Going back to our case, we can proceed the calculation of E_B (2.34) as

$$\begin{aligned} E_B(\ell_c) &= -\frac{1}{2}\frac{\partial}{\partial \ell_c}\left(\sum_{n=1}^{\infty} e^{-\ell_c \frac{\pi^2 n^2}{2mL^2}}\right) = -\frac{1}{4}\frac{\partial}{\partial \ell_c}\left(\sqrt{\frac{2mL^2}{\pi \ell_c}} - 1 + O\left(\frac{e^{-\frac{2mL^2}{\ell_c}}}{\sqrt{\ell_c}}\right)\right) \\ &= \frac{\sqrt{m}L}{4\sqrt{2\pi}\ell_c^{\frac{3}{2}}} + O\left(\frac{e^{-\frac{2mL^2}{\ell_c}}}{\ell_c^{\frac{3}{2}}}\right) \end{aligned} \tag{2.39}$$

where we have used (2.38) and (2.38) with $x = \pi \ell_c / 2mL^2$.
The computation of the ZPE E_{NB} (2.35) gives

$$\begin{aligned} E_{NB}\left(\ell_c\right) &= -\frac{L}{4\pi}\frac{\partial}{\partial \ell_c}\left(\int_{-\infty}^{\infty} e^{-\frac{\ell_c}{2m}\ell_c}\right) = \frac{L}{4\pi}\frac{\partial}{\partial \ell_c}\left(\sqrt{\frac{2\pi m}{\ell_c}}\right) \\ &= \frac{\sqrt{m}L}{4\sqrt{2\pi}\ell_c^{\frac{3}{2}}}. \end{aligned} \tag{2.40}$$

Lastly, by applying the renormalization prescription (2.10) we obtain

$$E_{\text{vac}} \approx 0. \tag{2.41}$$

The same result could have been obtained more directly by using zeta-regularization, i.e.,

$$E_{\text{vac}} = \lim_{s\to 0}\frac{\mu^s}{2}\sum_{n=1}^{\infty}\left(\frac{\pi^2 n^2}{2mL^2}\right)^{1-s} = \lim_{s\to 0}\frac{\mu^s}{2}\left(\frac{\pi^2}{2mL^2}\right)^{1-s}\zeta_R\left(-2+s\right) = 0. \tag{2.42}$$

It appears that, either due to the absence of ZPE in a 'pure' Schrödinger quantum field theory or through direct calculation in a 'hybrid' approach, the QVE is bound to vanish.
As we will see in the next section, this is neither the case for an interacting theory nor for a rotating system, where the symmetry of the system is spontaneously broken by Dirichlet boundary conditions.

2.2 QUANTUM VACUUM ENERGY IN NON-RELATIVISTIC, SELF-INTERACTING, QFT

We consider a system of nonrelativistic interacting bosons described by a complex Schrödinger quantum field

$$\Phi = \frac{(\phi_1 + i\phi_2)}{\sqrt{2}}, \quad \phi_1, \phi_2 \in \mathbb{R}. \tag{2.43}$$

confined to a one-dimensional ring of radius R rotating with a constant angular velocity Ω. We assume that the periodicity of the ring is broken externally by a barrier, which is modeled by imposing Dirichlet boundary conditions at a single point on the ring in the corotating.
The system in the fixed laboratory frame can be described by the Lagrangian density

$$\mathcal{L} = \frac{i}{2}\left(\Phi^\star\partial_t\Phi - \Phi\partial_t\Phi^\star\right) - \frac{1}{2m}\partial_x\Phi^\star\partial_x\Phi - \frac{\lambda}{4}\left(\Phi^\star\Phi\right)^2, \tag{2.44}$$

or, expressed in angular coordinates by the change of variable $x = R\varphi$,

$$\mathcal{L} = \frac{i}{2}\left(\Phi^\star\dot{\Phi} - \Phi\dot{\Phi}^\star\right) - \frac{1}{2mR^2}\Phi^{\star\prime}\Phi' - \frac{\lambda}{4}\left(\Phi^\star\Phi\right)^2, \qquad (2.45)$$

where $0 \leq \varphi \leq 2\pi$, $\dot{} = \partial_t$ and $' = \partial_\varphi$.
We now consider the ring spinning at a non-relativistic angular velocity $\Omega R \ll 1$ with respect to the laboratory frame. Indicating with (t_0, φ_0) the laboratory coordinates, we perform a change of coordinates to the co-rotating frame ones (t, φ), i.e.,

$$t = t_0, \qquad \varphi = \varphi_0 + \Omega t_0. \qquad (2.46)$$

This change in frame of reference considerably simplifies the problem since Dirichlet boundary conditions become time-independent in the co-rotating frame

$$\Phi\left(t_0, \Omega t_0\right) = \Phi\left(t_0, 2\pi + \Omega t_0\right) = 0 \quad \rightarrow \quad \Phi\left(t, 0\right) = \Phi\left(t, 2\pi\right) = 0. \qquad (2.47)$$

With the corresponding change in derivatives

$$\frac{\partial}{\partial t_0} = \frac{\partial}{\partial t} + \Omega\frac{\partial}{\partial \varphi}, \qquad \frac{\partial}{\partial \varphi_0} = \frac{\partial}{\partial \varphi}. \qquad (2.48)$$

the Lagrangian density becomes

$$\mathcal{L} = \frac{i}{2}\left(\Phi^\star\dot{\Phi} - \Phi\dot{\Phi}^\star\right) + \frac{i}{2}\Omega\left(\Phi^\star\Phi' - \Phi\Phi^{\star\prime}\right) - \frac{1}{2mR^2}\Phi^{\star\prime}\Phi' - \frac{\lambda}{4}\left(\Phi^\star\Phi\right)^2. \qquad (2.49)$$

2.2.1 *Equations of Motions*

We are now interested in the study of the equation of motion (EOM) of the field Φ. They can be derived by solving the Euler-Lagrange equations in terms of the field $\Phi^\star$, i.e.,

$$\frac{\partial\mathcal{L}}{\partial\Phi^\star} - \partial_\mu\frac{\partial\mathcal{L}}{\partial\,\partial_\mu\Phi^\star} = 0. \qquad (2.50)$$

Performing the calculation with Lagrangian density (2.49), we obtain

$$i\dot{\Phi} + i\Omega\Phi' + \frac{1}{2mR^2}\Phi'' - \frac{\lambda}{2}\left|\Phi\right|^2\Phi = 0. \qquad (2.51)$$

This partial differential equation (PDE) represents a rotating version of the Gross-Pitaevskii equation with the external trapping potential set to zero. The normal mode decomposition of (2.51) can be performed by seeking stationary solutions of the form

$$\Phi\left(t,\varphi\right)=e^{-i\omega_p t}f_p\left(\varphi\right), \tag{2.52}$$

thus, our PDE reduces to an ordinary differential equation (ODE) of the form

$$\frac{1}{2mR^2}f_p''+i\Omega f_p'-\left(\frac{\lambda}{2}\left|f_p\right|^2-\omega_p\right)f_p=0. \tag{2.53}$$

The persistent modulus term prevents the EOMs for Φ, or f_p, from completely decoupling from $\Phi^\star$, or $f_p^\star$. Nevertheless, we can take advantage of this behavior to redefine the complex-valued function f_p in terms of its modulus and phase, as

$$f_p\left(\varphi\right)=\rho\left(\varphi\right)e^{i\alpha\left(\varphi\right)}, \qquad \text{with } \rho\left(\varphi\right),\alpha\left(\varphi\right)\in\mathbb{R}. \tag{2.54}$$

Substituting (2.54) in (2.53) we get

$$\begin{aligned}&\frac{1}{2mR^2}\rho''+\left(\omega_p-\frac{\alpha'^2}{2mR^2}-\Omega\alpha'-\gamma\frac{m}{2}R^2\Omega^2\right)\rho-\frac{\lambda}{2}\rho^3+\\&+i\left[\frac{1}{2mR^2}\alpha''\rho+\frac{1}{mR^2}\alpha'\rho'+\Omega\rho'\right]=0.\end{aligned} \tag{2.55}$$

This equation can be separated in its real and imaginary part, and solved as two different, coupled differential equations. In conclusion, our goal is to determine analytic solutions for the system of ODEs:

$$\begin{cases}\frac{1}{2mR^2}\rho''+\left(\omega_p-\frac{\alpha'^2}{2mR^2}-\Omega\alpha'-\gamma\frac{m}{2}R^2\Omega^2\right)\rho-\frac{\lambda}{2}\rho^3=0 & \text{(2.56a)}\\ \frac{1}{2mR^2}\alpha''\rho+\frac{1}{mR^2}\alpha'\rho'+\Omega\rho'=0. & \text{(2.56b)}\end{cases}$$

Section 2.2.2 will focus entirely on the search for analytic solutions for $\rho\left(\varphi\right)$ and $\alpha\left(\varphi\right)$. Readers primarily interested in these solutions may proceed directly to the results in (2.2.3).

2.2.2 *Methodology for Solving the ODEs system*

Firstly, we note that equation (2.56b) depends only on the (angular) derivatives of α. Therefore, it is convenient to initially solve it by redefining $\alpha'\left(\varphi\right)=\beta\left(\varphi\right)$. This allows the equation to be rewritten as

$$\beta'\rho+2\beta\rho'+2mR^2\Omega\rho'=0. \tag{2.57}$$

By separating the variables and integrating, the solution for β in terms of ρ can be found by solving

$$\int \frac{d\beta}{2\beta + 2mR^2\Omega} = -\int \frac{d\rho}{\rho}, \tag{2.58}$$

and can be expressed as

$$\alpha' = \beta = \frac{C}{\rho^2} - mR^2\Omega \tag{2.59}$$

with C the constant of integration. Inserting the result (2.59) in (2.56a) we obtain

$$\frac{1}{2mR^2}\rho'' - \frac{C^2}{2mR^2}\frac{1}{\rho^3} + \left(\omega_p + \frac{m}{2}R^2\Omega^2 - \gamma\frac{m}{2}R^2\Omega^2\right)\rho - \frac{\lambda}{2}\rho^3 = 0. \tag{2.60}$$

To simplify the notation for the forthcoming calculation, we will express equation (2.60) as

$$\rho'' + F_1\frac{1}{\rho^3} + F_2\rho + F_3\rho^3 = 0 \tag{2.61}$$

with

$$\begin{aligned} F_1 &= -C^2 \\ F_2 \equiv \epsilon_p &= 2mR^2\left(\omega_p + \frac{m}{2}R^2\Omega^2\right). \\ F_3 &= -\lambda mR^2 \end{aligned} \tag{2.62}$$

Multiplying (2.61) by $2\rho'$ we can rewrite it as a total derivative of φ, i.e.,

$$\begin{gathered} 2\rho'\rho'' + 2F_1\frac{\rho'}{\rho^3} + 2F_2\rho'\rho + 2F_3\rho'\rho^3 \\ = \\ \left(\rho'^2\right)' - F_1\left(\frac{1}{\rho^2}\right)' + F_2\left(\rho^2\right)' + \frac{F_3}{2}\left(\rho^4\right)' = 0, \end{gathered} \tag{2.63}$$

which is possible to integrate and obtain

$$\rho'^2 - F_1\frac{1}{\rho^2} + F_2\rho^2 + \frac{F_3}{2}\rho^4 + H = 0, \tag{2.64}$$

where H is the integration constant.
Finally, multiplying both sides of (2.64) for ρ^2 and changing variables

$$\begin{cases} s = \rho^2 \\ \dfrac{s'}{2} = \rho\rho' \end{cases} \tag{2.65}$$

gives

$$\left(s'\right)^2 = 4F_1 - 4Hs - 4F_2 s^2 - 2F_3 s^3. \tag{2.66}$$

The above equation corresponds to the differential equation of an undamped quadratic anharmonic oscillator, whose canonical form can be obtained by differentiating with respect to φ and dividing by $2s'$,

$$s'' = -2H - 4F_2 s - 3F_3 s^2. \tag{2.67}$$

Following [35], it is possible to rewrite the RHS polynomial of (2.66) in terms of its roots $\alpha_1, \alpha_2, \alpha_3$, as

$$\begin{aligned} \left(s'\right)^2 &= d\left(s-\alpha_1\right)\left(s-\alpha_2\right)\left(s-\alpha_3\right) \\ &= -d\alpha_1\alpha_2\alpha_3 + d\left(\alpha_1\alpha_2 + \alpha_2\alpha_3 + \alpha_3\alpha_1\right)s - \\ &\quad -d\left(\alpha_1+\alpha_2+\alpha_3\right)s^2 + ds^3. \end{aligned} \tag{2.68}$$

By comparing the coefficients term by term with our specific system (2.66), we obtain the relations between the roots and the coefficients F_1, F_2 and F_3:

$$\begin{cases} d = -2F_3 \\ \alpha_1 + \alpha_2 + \alpha_3 = -2\dfrac{F_2}{F_3} \\ \alpha_1\alpha_2 + \alpha_2\alpha_3 + \alpha_3\alpha_1 = 2\dfrac{H}{F_3} \\ \alpha_1\alpha_2\alpha_3 = 2\dfrac{F_1}{F_3}. \end{cases} \tag{2.69}$$

The advantage of expressing (2.66) as (2.68) is that the latter is one of the standard non-linear ODE, whose solutions can be expressed in terms of Jacobi elliptic sine function as

$$s\left(\varphi\right) = \alpha_3 - \left(\alpha_3 - \alpha_2\right)\mathrm{sn}^2\left(q\varphi, k\right) \tag{2.70}$$

with elliptic modulus k and q given by

$$k = \sqrt{\frac{\alpha_3 - \alpha_2}{\alpha_3 - \alpha_1}}, \qquad q = \sqrt{\frac{F_3}{2}\left(\alpha_3 - \alpha_1\right)}. \tag{2.71}$$

Ignoring this may lead to misunderstandings and incorrect outcomes. Re-expressing the result in terms of the variable ρ, by using (2.65), we obtain:

$$\rho(\varphi) = \sqrt{\alpha_3 - (\alpha_3 - \alpha_2)\,\mathrm{sn}^2(q\varphi, k)}. \tag{2.72}$$

We have formally solved the initial system of ODEs (2.56a, 2.56b). However, the result is expressed in terms of the parameters $\alpha_1, \alpha_2, \alpha_3$ which exhibit an intricate dependence on the physical parameters of interest m, R, Ω, λ and on the integration constants C, H. Given the algebraic complexity of the problem, expressing the solutions (2.72) as functions of these physically meaningful parameters and deriving the energy eigenvalues ω_p, is a challenging task. However, part of this issue can be alleviated by imposing the Dirichlet boundary conditions (2.47) in the co-rotating frame, i.e.

$$\Phi(t, 0) = \Phi(t, 2\pi R) = 0 \tag{2.73}$$

which implies

$$\Phi(t, 0) = e^{-i\omega_p t}\rho(0)\, e^{i\alpha(0)} \quad \Longrightarrow \quad \rho(0) = 0. \tag{2.74}$$

Exploiting the fact that

$$\mathrm{sn}(0, k) = 0, \qquad \forall k \in \mathbb{R} \tag{2.75}$$

we can constrain the parameter space, and obtain

$$\rho(0) = \sqrt{\alpha_3 - (\alpha_3 - \alpha_2)\,\mathrm{sn}^2(0, k)} = 0 \quad \Longrightarrow \quad \alpha_3 = 0. \tag{2.76}$$

This simplifies the solution to

$$\rho(\varphi) = \sqrt{\alpha_2}\,\mathrm{sn}\left(\sqrt{-\frac{F_3}{2}\alpha_1}\varphi, \sqrt{\frac{\alpha_2}{\alpha_1}}\right). \tag{2.77}$$

Interestingly, the condition $\alpha_3 = 0$, along with the last equation in the system (2.69) and the definition (2.62), determines the value of the integration constant C to be zero, ie $C = 0$.
At this point, it is more convenient to restart from (2.61), rewriting it for $F_1 = 0$

$$\rho'' + F_2\rho + F_3\rho^3 = 0. \tag{2.78}$$

From (2.77), we know the specific form of the solutions, which we express more generally as

$$\rho(\varphi) = A\,\mathrm{sn}(q\varphi, k) \tag{2.79}$$

and substitute into (2.78) to find the expressions for A, q, and k in terms of F_2 and F_3. Given the non-linearity of the differential equation, we cannot neglect the coefficient A as we would in the linear case. In fact, it plays an important role in determining the elliptic modulus. The expression for the second derivative of the Jacobi sine can be expressed as

$$\frac{d^2}{dz^2}\mathrm{sn}(z,k) = -\left(1+k^2\right)\mathrm{sn}(z,k) + 2k^2\mathrm{sn}^3(z,k)\,. \tag{2.80}$$

Using this relation and the definition (2.79) in the ODE (2.78), we obtain

$$2k^2q^2\mathrm{sn}^3(q\varphi,k) - \left(1+k^2\right)q^2\mathrm{sn}(q\varphi,k) = -F_2\,\mathrm{sn}(q\varphi,k) - F_3A^2\,\mathrm{sn}^3(q\varphi,k)\,. \tag{2.81}$$

For the equation to hold, we compare and match the coefficients of sn and sn^3, and get

$$\begin{cases} F_3 = -\dfrac{2k^2q^2}{A^2} & (2.82a) \\ F_2 = \epsilon_p = \left(1+k^2\right)q^2. & (2.82b) \end{cases}$$

Among the Dirichlet boundary conditions (2.73), we have not yet utilized the condition at $\varphi = 2\pi$. This condition will provide the quantization of the solutions and the energy eigenvalues. Given the periodic property of the Jacobi sine function,

$$\mathrm{sn}(2nK(m) + 2ilK(1-m), m) = 0, \qquad \forall n, l \in \mathbb{Z}, \tag{2.83}$$

with $K(m)$ the complete elliptic integral of the second kind, we obtain the quantization condition

$$q = \frac{n}{\pi}K(k)\,. \tag{2.84}$$

Combining all the results and solving (2.82a) and (2.82b) for q and k, we obtain the system of equations

$$\begin{cases} q_n^2 = \dfrac{A_n^2F_3 + 2F_2}{2} & (2.85a) \\ k_n^2 = -\dfrac{A_n^2F_3}{A_n^2F_3 + 2F_2} & (2.85b) \\ q_n = \dfrac{n}{\pi}K(k_n)\,. & (2.85c) \end{cases}$$

The normalization coefficients A_n are computed using the non-relativistic normalization condition

$$\langle \phi | \phi \rangle = \int_V dV \phi^\star(x)\, \phi(x) = 1. \tag{2.86}$$

Before proceeding with the calculation, let us first introduce some useful properties of Jacobi and elliptic functions

$$\begin{cases} \mathrm{dn}^2(x,m) + m^2 \mathrm{sn}^2 = 1 & \text{(2.87a)} \\ \mathcal{E}(x,k) = \int_0^x \mathrm{dn}^2(t,k)\, dt & \text{(2.87b)} \\ \mathcal{E}(nK(k),k) = nE(k) & \text{(2.87c)} \end{cases}$$

where $E(k)$ is the complete elliptic integral of the first kind, $\mathcal{E}(k)$ is the Jacobi epsilon function, and $dn(x,m)$ is the delta amplitude. Property (2.87c) is derived from a combination of quasi-addition and quasi-periodic formulas. The normalization coefficient A_n is given by

$$\begin{aligned} 1 &= A_n^2 R \int_0^{2\pi} \mathrm{sn}^2(q_n \varphi, k_n)\, d\varphi \\ &= 2\pi \frac{A_n^2 R}{k_n^2} - \frac{A_n^2 R}{k_n^2} \int_0^{2\pi} \mathrm{dn}^2(q_n \varphi, k_n)\, d\varphi \\ &= \frac{A_n^2 R}{k_n^2} \left(2\pi - \frac{1}{q_n} \mathcal{E}(2\pi q_n, k_n) \right) \end{aligned} \tag{2.88}$$

where in the first step, we used relation (2.87a), and in the second step, definition (2.87b). The integral can be further simplified as follows:

$$\begin{aligned} 1 &= \frac{A_n^2 R}{k_n^2} \left(2\pi - \frac{1}{q_n} \mathcal{E}(2\pi q_n, k_n) \right) \\ &= \frac{A_n^2 R}{k_n^2} \left(2\pi - \frac{\pi}{nK(k_n)} \mathcal{E}(2nK(k_n), k_n) \right) \\ &= \frac{2\pi A_n^2 R}{k_n^2} \left(1 - \frac{E(k_n)}{K(k_n)} \right) \end{aligned} \tag{2.89}$$

where in the first step, we used the quantization relation (2.85c), and in the second step, definition (2.87c). The final expression for the normalization coefficient A_n is given by

$$A_n^2 = \frac{k_n^2}{2\pi R \left(1 - \frac{E(k_n)}{K(k_n)} \right)}. \tag{2.90}$$

Using relations (2.85a), (2.85b), and (2.85c), it takes only a few steps to arrive at the defining relation of the elliptic modulus, which can be written as:

$$-\frac{F_3}{R}\frac{\pi}{4n^2} = K(k_n)\left(K(k_n) - E(k_n)\right). \tag{2.91}$$

We have successfully expressed q_n and A_n as functions of physical parameters and the elliptic modulus k_n. However, despite k being a function of only physical parameters and the quantum number n, the relation (2.91) is not invertible. Therefore, k_n and ultimately ω_n cannot be expressed in a closed form. Nevertheless, it is still possible to further study their behavior in regimes of no interaction and large n. We dedicate subsections 2.2.4 and 2.3.1 to this purpose, respectively.
Summarizing the results of the quantization procedure, from (2.85c), (2.90), (2.82b), and (2.91), we can write the system of equations that characterize k_n and ω_n solely in terms of physical parameters and the quantum number n

$$\begin{cases} \epsilon_n = \dfrac{n^2}{\pi^2}\left(1 + k_n^2\right) K^2(k_n) & \text{(2.92a)} \\ -\dfrac{F_3}{R}\dfrac{\pi}{4n^2} = K(k_n)\left(K(k_n) - E(k_n)\right) & \text{(2.92b)} \end{cases}$$

To conclude our analysis, we need to determine the phase $\alpha(\varphi)$, with the constraint $C = 0$. By integrating (2.59), we obtain

$$\alpha(\varphi) = -mR^2\Omega\varphi + \Xi \tag{2.93}$$

with Ξ constant of integration.

2.2.3 *Complete solution of PDE system*

To summarize the results obtained in (2.2.2), we can write the full analytical solution to the original PDE (2.51) as follows:

$$\Phi(t,\varphi) = \sqrt{\frac{k_n^2}{2\pi R\left(1 - \frac{E(k_n)}{K(k_n)}\right)}}\, e^{-i\omega_n t} e^{-i\left(mR^2\Omega\varphi - \Xi\right)} \mathrm{sn}\left(\frac{n}{\pi}K(k_n)\,\varphi, k_n\right) \tag{2.94}$$

where $\exp(i\Xi)$ is a constant phase, reflecting the $U(1)$ global symmetry of the Lagrangian, which leaves the equations of motion unaffected. The elliptic modulus k_n and the energy eigenvalues ω_n cannot be expressed in closed form and are determined by the system of equations

$$\begin{cases} \omega_n = \dfrac{n^2}{2\pi^2 m R^2}\left(1+k_n^2\right) K^2\left(k_n\right) - \dfrac{m}{2}R^2\Omega^2 & \text{(2.95a)} \\ \lambda m R \dfrac{\pi}{4n^2} = K\left(k_n\right)\left(K\left(k_n\right) - E\left(k_n\right)\right). & \text{(2.95b)} \end{cases}$$

This limitation does not prevent us from studying their asymptotic behaviors for $\lambda \to 0$ and $n \to \infty$. These two regimes will be investigated in the following sections due to their crucial consequences in the study of the non-interacting limit of the theory and for the regularization and renormalization of the QVE.

2.2.4 *Non-interacting limit*

We investigate whether it is possible to retrieve the non-interacting results by considering the non-interacting limit of our own results, as expressed in 2.2.3. By taking the limit $\lambda \to 0$, equation (2.95b) becomes:

$$K\left(k_n\right)\left(K\left(k_n\right) - E\left(k_n\right)\right) = 0. \tag{2.96}$$

From the basic properties of complete elliptic integrals, we know that $K(m)$ is always positive for all $m \in [0,1)$, and $K(m) = E(m)$ if and only if $m = 0$. It follows that k_n vanishes in the limit $\lambda \to 0$. This new condition affects the form of eigenfunctions and eigenvalues. Specifically, from the basic properties of the Jacobi sn function, we know that $\mathrm{sn}(x,0) = \sin(x)$, and equation (2.79) becomes:

$$\mathrm{sn}\left(x,0\right) = \sin\left(x\right) \tag{2.97}$$

and Eq.(2.79) becomes

$$\rho\left(\varphi\right) = A_n \sin\left(q_n\varphi\right). \tag{2.98}$$

Furthermore, knowing that $K(0) = \pi/2$, the relation for the energy eigenstates (2.92a) becomes

$$\omega_n = \frac{n^2}{8mR^2} - \frac{m}{2}R^2\Omega^2. \tag{2.99}$$

To compute the non-interacting limit of the normalization coefficient A_n, it is necessary to set aside the quantization of the elliptic modulus by expressing it as a continuous variable $k_n \to k$. Therefore, taking the limit $k \to 0$, we obtain:

$$\begin{aligned} A^2 &= \lim_{k\to 0} \frac{k^2}{2\pi R \left(1 - \frac{E(k)}{K(k)}\right)} \\ &= \frac{1}{\pi R} \end{aligned} \tag{2.100}$$

Summarizing, the non-interacting eigenfunction is:

$$\begin{cases} \Phi(t,\varphi) = \sqrt{\dfrac{1}{\pi R}}\, e^{-i\omega_n t} e^{-i\left(mR^2\Omega\varphi - \Xi\right)} \sin\left(\dfrac{n}{2}\varphi\right). \\ \omega_n = \dfrac{n^2}{4}\dfrac{1}{2mR^2} - \dfrac{m}{2}R^2\Omega^2 \end{cases} \tag{2.101}$$

which is perfectly compatible with the standard, non-interacting result. Proving that the results obtained in the interacting case are consistent in the non-interacting limit gives us confidence in the reliability of our outcomes.

2.3 ASYMPTOTIC BEHAVIOURS OF k_n AND ω_n AND VACUUM ENERGY COMPUTATION

2.3.1 *Asymptotic behaviour of k_n*

To be able to renormalize the ZPE, we need to regularize the sum over the energy eigenstates. We previously expressed these as functions of k_n, but we could not write them in closed form. This requires examining the large-n behavior of ω_n, which in turn depends on the asymptotic behavior of k_n.
We are particularly interested in examining the relation that defines the elliptic moduli (2.95b), i.e.,

$$\frac{\pi}{4}\frac{\lambda m R}{n^2} = K(k_n)\left(K(k_n) - E(k_n)\right) \tag{2.102}$$

or, expanding the RHS,

$$\frac{\pi}{4}\frac{\lambda m R}{n^2} = \frac{\pi^2}{4}\left(c_1 k_n^2 + c_2 k_n^4 + \dots\right). \tag{2.103}$$

We point out that the function $K(k_n)\left(K(k_n) - E(k_n)\right)$ is continuous in $k \in [0,1)$ and has image $[0,\infty)$ within the same interval. Consequently, for every positive m, λ, R, there is always a value $k \in [0,1)$ that satisfies (2.102). This property is significant because it ensures that the solutions obtained in 2.2.3 are reliable even for large λ and not only in the

perturbative regime $\lambda \ll 1$. Moreover, it allows us to extract the large-n behavior of k_n. We finally observe that for $n \to \infty$ the LHS tends to zero, thus also $k_n \to 0$ in the RHS.
Collecting $c_1 k_n^2$ from the RHS of (2.103), we obtain

$$\frac{\lambda m R}{\pi n^2} = c_1 k_n^2 \left(1 + \frac{c_2}{c_1} k_n^2 + \frac{c_3}{c_1} k_n^4 + \dots\right). \tag{2.104}$$

Since it is possible to prove that the coefficients $\{c_l\}_l$ are a monotonically decreasing sequence and, therefore, $c_l/c_1 < 1 \;\; \forall l > 1$, we can see that

$$\begin{aligned} \frac{\lambda m R}{\pi n^2} &= c_1 k_n^2 \left(1 + \frac{c_2}{c_1} k_n^2 + \dots\right) < c_1 k_n^2 \left(1 + k_n^2 + \dots\right) \leq \\ &\leq c_1 k_n^2 \sum_{j=0}^{\infty} k_n^{2j} = c_1 k_n^2 \frac{1}{1 - k_n^2} \xrightarrow{n \to \infty} c_1 k_n^2, \end{aligned} \tag{2.105}$$

where we have used the fact that $k_n^2 \in (-1, 1)$.
This means that, as $n \to \infty$, terms of order $O\left(k_n^4\right)$ do not contribute to the sum. Therefore, for large n,

$$k_n^2 \approx \frac{2\lambda m R}{\pi} \frac{1}{n^2}, \tag{2.106}$$

where we have used the fact that $c_1 = 1/2$. It is evident from (2.106) that as the quantum number n increases, the asymptotic elliptic modulus decreases rapidly. Consequently, the system increasingly resembles a non-interacting one. This behavior can be attributed to the dominance of the kinetic energy contribution over the attractive interaction at large n.

2.3.2 *Asymptotic behaviour of ω_n*

We are now interested in studying the behavior of ω_n for large n, as this will be crucial for the renormalization of the QVE. We begin with the energy eigenvalues equation (2.95a), i.e.,

$$\omega_n = \frac{n^2}{2\pi^2 m R^2} \left(1 + k_n^2\right) K^2\left(k_n\right) - \frac{m}{2} R^2 \Omega^2 \tag{2.107}$$

we can extract the asymptotic behaviour of ω_n by expanding $K^2\left(k_n\right)$

$$\omega_n = \frac{n^2}{2\pi^2 m R^2} \left(\frac{\pi^2}{4} + \frac{3\pi^2}{8} k_n^2\right) - \frac{m}{2} R^2 \Omega^2 + O\left(k_n^4\right), \tag{2.108}$$

and substituting the asymptotic behaviour of k_n (2.106), we obtain

$$\omega_n^{(a)} = \frac{n^2}{8mR^2} + \frac{3}{8}\frac{\lambda}{\pi R} - \frac{m}{2}R^2\Omega^2 + O\left(n^{-2}\right). \tag{2.109}$$

Terms of order $O\left(n^{-2}\right)$ will be neglected in the definition of $\omega^{(a)}$ since they are not divergent in the summation over n when calculating the QVE. We will call $\omega_n^{(a)}$ the spectral asymptotics of the energy eigenvalues. We can immediately observe that, in the non-interacting and non-rotating limit, the energy eigenvalues exhibit the correct behavior of the free theory (with $R = L/2\pi$).

2.3.3 *Asymptotic quantum vacuum energy*

We are now ready to synthesize all the calculations performed so far and approach the calculation of the QVE. We will utilize the information obtained about the asymptotic behavior of the spectrum to regularize and then renormalize the ZPE, similar to what we did in the free case in 2.1.4.

2.3.3.1 *ζ regularization: Chowla-Selberg formula*

We start by considering the definition of ZPE,

$$E = \frac{1}{2}\sum_{n=1}^{\infty}\omega_n^{\text{ex}} \tag{2.110}$$

where ω_n^{ex} are the exact energy eigenvalues of the theory, derived directly from (2.95a, 2.95b) as solutions.
As discussed in (2.1.1.1), the ZPE can be generalized to a complex-valued function of a complex variable s, namely:

$$E(s) = \frac{1}{2\mu^{-s}}\sum_{n=1}^{\infty}\left(\omega_n^{\text{ex}}\right)^{-s}, \tag{2.111}$$

where the physical energy is obtained in the limit $s \to -1$. Subsequently, we add and subtract the series for the spectral asymptotic $\omega_n^{(a)}$, defined in (2.109), and obtain

$$E(s) = \frac{1}{2\mu^{-s}}\sum_{n=1}^{\infty}\left(\omega_n^{\text{ex}}\right)^{-s} - \frac{1}{2\mu^{-s}}\sum_{n=1}^{\infty}\left(\omega_n^{(a)}\right)^{-s} + \frac{1}{2\mu^{-s}}\sum_{n=1}^{\infty}\left(\omega_n^{(a)}\right)^{-s}. \tag{2.112}$$

By initially considering s in a region of the complex plane where the three series converge, i.e. $\Re(s) > 1/2$ for $\omega_n \propto n^2$, we can recast them as

$$E(s) = \frac{1}{2\mu^{-s}} \sum_{n=1}^{\infty} \left((\omega_n^{\text{ex}})^{-s} - \left(\omega_n^{(a)}\right)^{-s} \right) + \frac{1}{2\mu^{-s}} \sum_{n=1}^{\infty} \left(\omega_n^{(a)}\right)^{-s} . \quad (2.113)$$

If we now consider the limit $s \to -1$, we observe that the series on the left remains convergent since $\omega_n^{ex} - \omega_n^{(a)} \sim O\left(n^2\right)$, whereas the series on the right diverges. Consequently, by applying the limit exclusively to the first series, the generalized ZPE can be split into two parts, as follows:

$$E(s) = \Delta + E^{(a)}(s) \quad (2.114)$$

with

$$\Delta = \frac{1}{2} \sum_{n=1}^{\infty} \left(\omega_n^{\text{ex}} - \omega_n^{(a)} \right) , \quad (2.115)$$

$$E^{(a)}(s) = \frac{1}{2\mu^{-s}} \sum_{n=1}^{\infty} \left(\omega_n^{(a)}\right)^{-s} . \quad (2.116)$$

The explicit calculation of Δ must be computed numerically. $E^{(a)}(s)$, on the other hand, diverges and needs to be regularized, but is much more treatable than the original series (2.110).
We are now interested in regularizing the divergent part of the vacuum energy, i.e. $E^{(a)}(s)$. We begin by generalizing this ZPE to the complexed-valued function

$$\begin{aligned} E^{(a)}(s) &= \frac{\mu^s}{2} \sum_{n=1}^{\infty} \omega_n^{-s} \\ &= \frac{(8\mu m R)^s}{2} \sum_{n=1}^{\infty} \left(n^2 + \rho^2\right)^{-s} , \end{aligned} \quad (2.117)$$

with

$$\rho^2 = \frac{3m\lambda R}{\pi} - 4m^2 R^4 \Omega^2 . \quad (2.118)$$

It is now possible to analitically continue $E^{(a)}(s)$ to the physical value for $s \to -1$. For this purpose, we define the Chowla-Selberg formula, which can be derived using zeta-regularization techniques, and that is

$$S(s,\rho) \equiv \sum_{n=1}^{\infty} \left(n^2+\rho^2\right)^{-s}$$
$$= -\frac{\rho^{-2s}}{2} + \frac{\sqrt{\pi}}{2}\frac{\Gamma(s-1/2)}{\Gamma(s)}\rho^{1-2s} + \frac{2\pi^s}{\Gamma(s)}\rho^{-s+1/2}\sum_{l=1}^{\infty} l^{s-1/2}K_{s-1/2}(2\pi l\rho)\,. \tag{2.119}$$

We are now interested in the convergence properties of $S(s,\rho)$ and its result, in the limit $s \to -1$. By noting that

$$\lim_{s\to -1}\frac{1}{\Gamma(s)} = 0 \tag{2.120}$$

we can immediatelly conclude that the second term on the RHS of (2.119) is trivially zero. The third term requires some analysis. While it is true that the $\Gamma(s)$ factor in the dominator diverges, we must ensure that its series does not diverge as well, to avoid indeterminate situations. Considering the parity property of the order of the modified Bessel function K and its behaviour for large arguments , i.e.,

$$K_\mu(x) = K_{-\mu}(x), \qquad\qquad K_\mu(x) \approx \sqrt{\frac{\pi}{2x}}e^{-x} \tag{2.121}$$

we can conclude that the series converges for all $x > 0$. Thus, under this assumption, the third term in the RHS of (2.119) also vanishes. We then remain with the finite result

$$S(-1,\rho) = -\frac{\rho^2}{2}, \tag{2.122}$$

and the regularized expression for $E^{(a)}$ becomes

$$E^{(a)}_{\text{reg}} = E^{(a)}(-1) = -\frac{3}{32}\frac{\lambda}{\pi R} + \frac{m}{8}R^2\Omega^2. \tag{2.123}$$

2.3.3.2 *Window function regularization*

Our interest now is to confirm (2.123) by using the window function regularization. Specifically, we adopt the function

$$f_w(\ell_c) \equiv \exp\left(-\ell_c\omega_\infty\right), \tag{2.124}$$

with

$$\begin{aligned} \omega_{n\,\infty} &= \frac{n^2}{8mR^2} && \text{with boundary conditions} \\ \omega_{p\,\infty} &= \frac{p^2}{2m} && \text{without boundary conditions} \end{aligned} \tag{2.125}$$

We choose this particular ω_∞, favoring it over the full spectral asymptotics (2.109), since it captures the same large-n leading behaviour without unnecessarily complicating the calculations.
The asymptotic ZPE (2.116) can be written as

$$\begin{aligned} E_B^{(a)}(\ell_c) &= \frac{1}{2}\sum \omega_n^{(a)} f_w(\ell_c) \\ &= \frac{1}{2}\sum_{n=1}^{\infty} \frac{n^2}{8mR^2} e^{-\ell_c \frac{n^2}{8mR^2}} + \sum_{n=1}^{\infty} \left(\frac{3}{8}\frac{\lambda}{\pi R} - \frac{m}{2}R^2\Omega^2\right) e^{-\ell_c \frac{n^2}{8mR^2}}. \end{aligned} \tag{2.126}$$

We could have split the first series into two, because all the series represented in (2.126) converge to a finite value for all $\ell_c > 0$.
Performing the calculation by using the θ function (2.36), we can express $E_B^{(a)}$ as

$$E_B^{(a)} = -\frac{1}{4}\frac{\partial}{\partial \ell_c}\left(\theta\left(\frac{\ell_c}{\pi\eta^2}\right)\right) + \frac{\rho^2}{4\eta^2}\theta\left(\frac{\ell_c}{\pi\eta^2}\right) - \frac{\rho^2}{4\eta^2} \tag{2.127}$$

with

$$\begin{aligned} \eta^2 &= 8mR^2 \\ \rho^2 &= \frac{3m\lambda R}{\pi} - 4m^2R^4\Omega^2. \end{aligned} \tag{2.128}$$

Using the modular transformation (2.38), we obtain

$$E_B^{(a)} = \frac{\sqrt{\pi}}{8}\eta\ell_c^{-\frac{3}{2}} + \sqrt{\pi}\frac{\rho^2}{4\eta}\ell_c^{-\frac{1}{2}} - \frac{\rho^2}{4\eta^2} + O\left(\frac{e^{-\frac{\pi^2\eta^2}{\ell_c}}}{\ell_c^{\frac{1}{2}}}\right) \tag{2.129}$$

We can recognize that, upon substitutions (2.128) and $L = 2\pi R$, the first term is equal to the non-interacting one, calculated in (2.39). The second and third one, comes from the contribution of interactions and rotations.
To renormalize the ZPE, we need to subtract to $E_B^{(a)}$ the contribution of

the ZPE in the same region, without Dirichlet boundary conditions, that is

$$\begin{aligned} E_{NB}^{(a)} &= \frac{L}{2}\int_{-\infty}^{\infty}\frac{dp}{2m}\omega_p^{(a)}e^{-\ell_c\frac{p^2}{2m}} = \pi R\int_{-\infty}^{\infty}\frac{dp}{2m}\left(\frac{p^2}{2m}+\frac{\rho^2}{\eta^2}\right)e^{-\ell_c\frac{p^2}{2m}} \\ &= \frac{\sqrt{\pi}}{8}\eta\ell_c^{-\frac{3}{2}} + \sqrt{\pi}\frac{\rho^2}{4\eta}\ell_c^{-\frac{1}{2}}. \end{aligned} \tag{2.130}$$

We recognize that the divergent terms for $\ell_c \to 0$ of (2.129) and (2.130) cancel completely in the renormalization procedure $E_B^{(a)} - E_{NB}^{(a)}$. Therefore, the QVE is

$$E_{\text{reg}}^{(a)} = -\frac{3}{32}\frac{\lambda}{\pi R} + \frac{m}{8}R^2\Omega^2 + O\left(\frac{e^{-\frac{\pi^2\eta^2}{\ell_c}}}{\ell_c^{\frac{1}{2}}}\right) \tag{2.131}$$

in perfect agreement with (2.123) when $\ell_c \to 0$.

This choice of regularization has the advantage that the regulator-independent term of (2.129) corresponds to the fully resummed result, while the cut-off contributions are exponentially small and encoded in the last term of (2.131).

2.3.3.3 *Quantum vacuum energy in laboratory frame: results and comments*

Up to this point, we have only considered the asymptotic term of the QVE, which is calculated in the co-rotating reference frame. In this section, we will address the QVE in its entirety, calculating it in the laboratory reference frame. We will show its characteristics and peculiarities, relying on the results of the numerical calculations.

As we have seen in (2.114), the total QVE, in the co-rotating frame (E_{rot}), can be expressed as $E_{\text{rot}} = E_{\text{rot}}^{(a)} + \Delta$. To find the energy in the laboratory frame E_s, we use the relation

$$E_s - E_{\text{rot}} = \Omega L, \tag{2.132}$$

where

$$L = -\frac{\partial}{\partial\Omega}E_{\text{rot}} \tag{2.133}$$

represents the angular momentum. This gives us:

$$E_s = \Delta - \Omega\frac{\partial\Delta}{\partial\Omega} - \frac{3\lambda}{32\pi R} - \frac{mR^2\Omega^2}{8}. \tag{2.134}$$

Next, we derive the force F_{lab} as:

$$F_s = -\frac{\partial E_s}{\partial R} = -\frac{\partial \Delta}{\partial R} + \Omega \frac{\partial^2 \Delta}{\partial \Omega \partial R} - \frac{3\lambda}{32\pi R^2} + \frac{mR\Omega^2}{4} \tag{2.135}$$

If we ignore the contributions from Δ, we notice:

- *Interacting-attracting term:* a term proportional to $-\lambda/R$, which vanishes as λ approaches zero and scales inversely with the ring size. This term represents an attractive "Casimir-like" force.
- *Rotating-repulsive term:* a rotating, $E_I = \frac{1}{2}\mathcal{I}_R\Omega^2$, proportional to the ring moment of inertia, with $\mathcal{I}_R = mR^2$.
- *Critical radius:* the force vanishes at the critical radius

$$R_{\text{crit}} \approx \sqrt{\frac{3\lambda}{8\pi m\Omega^2}}, \tag{2.136}$$

  with its sign changing from negative (attractive) for $R < R_{\text{crit}}$ to positive (repulsive) for $R > R_{\text{crit}}$. R_{crit} is an unstable equilibrium point.
  (The symbol "$\approx$" in the formula indicates that we have ignored the contribution of Δ)

Furthermore, the energy vanishing behavior for $\lambda \to 0$ and $\Omega \to 0$ aligns with our expectation that QVE should disappear in absence of interactions and rotation. The angular velocity appears quadratically, consistent with the symmetry of our model with respect to $\Omega \leftrightarrow -\Omega$. Remarkably, the force scaling with the ring size changes with the angular velocity: it scales linearly in the fast rotation regime and as the inverse square of the ring size for slow rotation.

In the numerical calculations, we have restored units of $\hbar$, with l denoting a generic length scale.

Figure 2.1 illustrates the QVE (panels a and c) as a function of radius R and rotation strength Ω, respectively. The lower panels (b and d) show the corresponding force associated with each data-set from (a) and (c). The gray-shaded region indicates the region where the force is repulsive.

Figure 2.2 presents heat maps of Eq. (2.136) in the (R, Ω) and (R, λ) parameter spaces. In panel (a), the interaction strength is $\lambda ml/\hbar^2 = 10$, while, in panel (b), the rotation strength is $\Omega ml^2/\hbar = 5$. The solid blue lines in both panels mark the border between the repulsive regime and the causality limit defined by $\Omega Rml/\hbar = 1$. The red dashed line indicates where the force changes sign, derived from Eq. (2.136). The red data point in each panel corresponds to the point (R_0, Ω_0) in (a) and (R_0, λ_0) in (b) where the critical radius (2.136) and the causality limit coincide, with

$$(R_0, \Omega_0) = \left(\frac{3}{8\pi} \frac{\lambda m l^2}{\hbar^2}, \frac{8\pi}{3} \frac{\hbar^2}{m^2 l^3} \frac{1}{\lambda} \right). \qquad (2.137)$$

This point in 2.2 shows the maximum rotation strength where repulsive solutions are obtained. We expect our model (2.49) to support a causal repulsive force in the region between $\Omega_c < \Omega < R^{-1}(ml/\hbar)$ and $R > R_0$. Similarly, for panel (b), the causal repulsive regime is defined between $0 < \lambda < \lambda_c$ and $0 < R < R_0$. We can perform an analysis similar to that in 2.2 panel (b) for the (R, Ω) parameter space with constant λ.

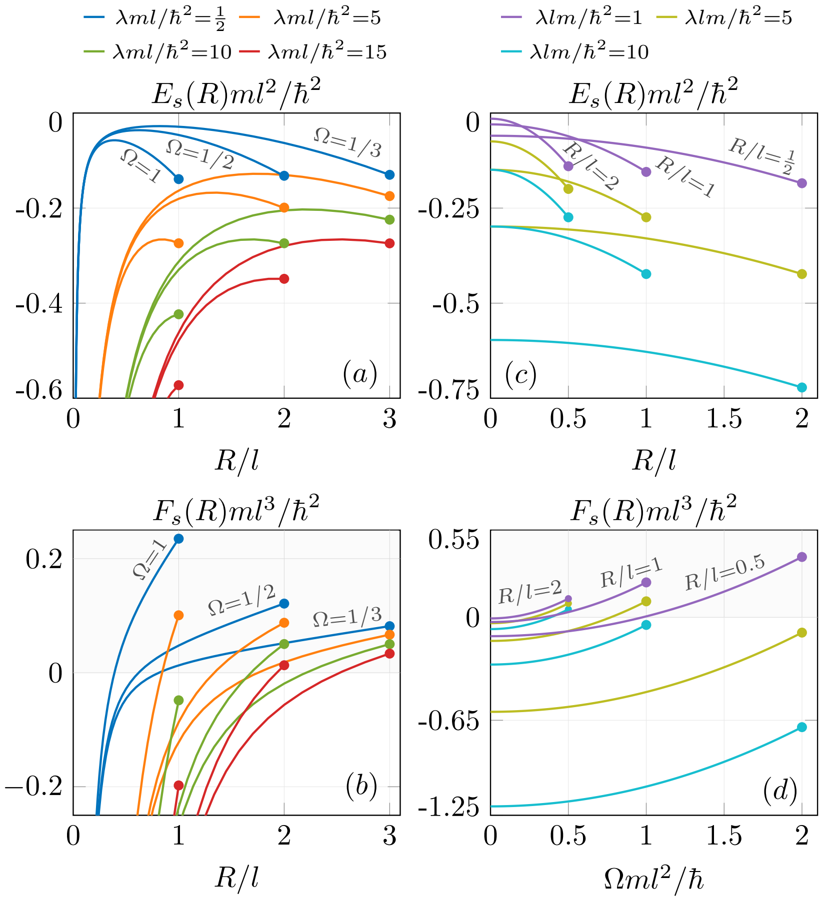

Figure 2.1: Quantum vacuum energy and force. Panels (a) and (c) show the QVE, Eq. (2.134), for the same λ (color groups) and varying Ω (panel (a)) or R (panel (c)) (see text labels). Panels (b) and (d) show the corresponding force, Eq. (2.135), for each data-set. The light gray shading indicates the parameter regions where the force changes sign from attractive to repulsive. Corresponding vertical axis labels are displayed above each panel.

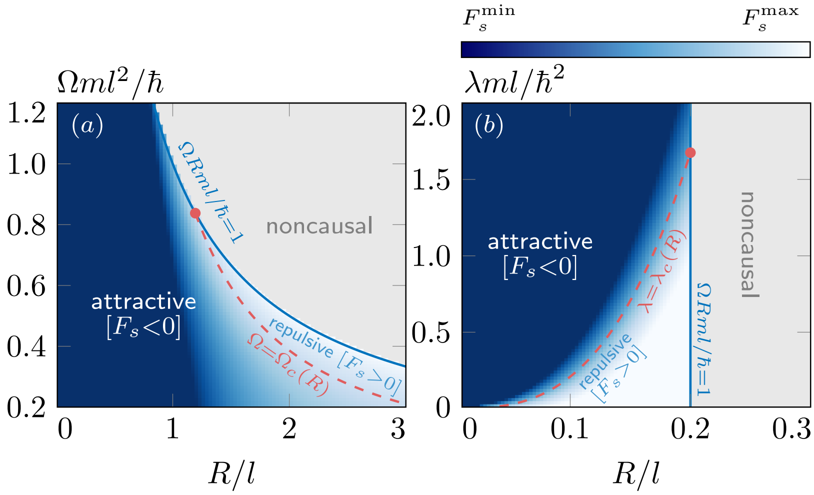

Figure 2.2: Quantum fluctuation induced-force heat maps. Panel (a) shows Eq. (2.135) in the limit $\Delta = 0$ in the (Ω, R) parameter space for fixed $\lambda ml/\hbar^2 = 10$. The solid blue line indicates the border between the repulsive and noncausal regions, while the dashed red line indicates the point at which the force changes sign. Panel (b) shows the magnitude of the force in the (λ, R) parameter space for fixed $\Omega ml^2/\hbar = 5$.

2.3.4 *Relativistic case*

We are now interested in determining whether the derivation of analytical, non-perturbative solutions to EOM, and the calculation of the QVE in Schrödinger quantum field theory (2.44), can be generalized to a relativistic complex scalar quantum field theory of the same type.

Let us preface by stating that this work is not yet fully completed and, therefore, has not been published. However, the results obtained so far are remarkable. We have managed to calculate, in a way similar to the non-relativistic case, the solutions to EOM, the asymptotic spectrum, and the non-relativistic limit of the theory. It will be particularly interesting to analyze the latter, which justifies the emergence of QVE even in non-relativistic systems to which the approach described in 2.1.3 has been applied.

We start by considering a relativistic complex scalar quantum field theory in a 1+1 dimensional rotating ring with Dirichlet boundary conditions imposed. The Lagrangian density describing the system, expressed in angular coordinates, is

$$\mathcal{L} = \frac{1}{2}\partial_{t_0}\Phi^\star\partial_{t_0}\Phi - \frac{1}{2R^2}\partial_{\varphi_0}\Phi^\star\partial_{\varphi_0}\Phi - \frac{m^2}{2}\Phi^\star\Phi - \frac{\lambda}{4}\left(\Phi^\star\Phi\right)^2. \tag{2.138}$$

The coordinate transformation from the laboratory frame to the co-rotating frame, for a non-relativistic rotation where $\beta \equiv R\Omega \ll 1$, can be described again by:

$$t = t_0, \qquad \varphi = \varphi_0 + \Omega t_0, \tag{2.139}$$

where (t_0, φ_0) are the coordinates associated to the laboratory frame and (t, φ) the coordinates associated to the co-rotating frame. They corresponding relation for partial derivatives is described by:

$$\frac{\partial}{\partial t_0} = \frac{\partial}{\partial t} + \Omega\frac{\partial}{\partial\varphi}, \qquad \frac{\partial}{\partial\varphi_0} = \frac{\partial}{\partial\varphi}. \tag{2.140}$$

In the co-rotating frame, Dirichlet boundary conditions become time independent and can be written as:

$$\Phi\left(t_0, \Omega t_0\right) = \Phi\left(t_0, 2\pi + \Omega t_0\right) = 0 \quad \rightarrow \quad \Phi\left(t, 0\right) = \Phi\left(t, 2\pi\right) = 0. \tag{2.141}$$

The lagrangian density then becomes

$$\mathcal{L} = \frac{1}{2}\dot\Phi^\star\dot\Phi + \frac{\Omega}{2}\dot\Phi^\star\Phi' + \frac{\Omega}{2}\Phi^{\star\prime}\dot\Phi - \frac{1}{2}\frac{1-\beta^2}{R^2}\Phi^{\star\prime}\Phi' - \frac{m^2}{2}\Phi^\star\Phi - \frac{\lambda}{4}\left(\Phi^\star\Phi\right)^2, \tag{2.142}$$

with $\dot{} = \partial_t$ and $' = \partial_\varphi$.

2.3.4.1 *Equation of Motion*

Regarding the EOMs, the Euler-Lagrange equation for the field Φ is given by

$$\frac{\partial\mathcal{L}}{\partial\Phi^\star} - \partial_\mu\frac{\partial\mathcal{L}}{\partial\,\partial_\mu\Phi^\star} = 0, \tag{2.143}$$

and gives us

$$\frac{1}{2}\ddot\Phi + \Omega\dot\Phi' - \frac{1}{2}\frac{1-\beta^2}{R^2}\Phi'' + \frac{m^2}{2}\Phi + \frac{\lambda}{2}\left|\Phi\right|^2\Phi = 0. \tag{2.144}$$

Seeking again stationary solutions of the form

$$\Phi\left(t,\varphi\right)=e^{-i\omega_{p}t}f_{p}\left(\varphi\right) \tag{2.145}$$

then equation of motion (2.144) becomes

$$-\frac{\omega_p^2}{2}f_p-i\omega_p\Omega f_p'-\frac{1}{2}\frac{1-\beta^2}{R^2}f_p''+\frac{m^2}{2}f_p+\frac{\lambda}{2}\left|f_p\right|^2f_p=0, \tag{2.146}$$

where we have multiplyied by $e^{i\omega_p t}$ both LHS and RHS. EOM (2.146) can be written equivalently as

$$f_p''+2i\frac{\omega_p\beta R}{1-\beta^2}f_p'+\frac{\omega_p^2R^2}{1-\beta^2}f_p-\frac{m^2R^2}{1-\beta^2}f_p-\frac{\lambda R^2}{1-\beta^2}\left|f_p\right|^2f_p=0. \tag{2.147}$$

Representing $f_p\left(\varphi\right)$ by its modulus and phase, as

$$f_p\left(\varphi\right)=\rho\left(\varphi\right)e^{i\alpha\left(\varphi\right)},\qquad\text{with }\rho\left(\varphi\right),\alpha\left(\varphi\right)\in\mathbb{R}, \tag{2.148}$$

EOM (2.147) becomes

$$\begin{aligned}&\rho''+2i\rho'\alpha'+i\rho\alpha''-\rho\alpha'^2+2i\frac{\omega_p\beta R}{1-\beta^2}\rho'-2\frac{\omega_p\beta R}{1-\beta^2}\rho\alpha'+\\&+\frac{\omega_p^2R^2}{1-\beta^2}\rho-\frac{m^2R^2}{1-\beta^2}\rho-\frac{\lambda R^2}{1-\beta^2}\rho^3=0.\end{aligned} \tag{2.149}$$

Analogously to the non-relativistic case, EOM (2.149) can be divided into its real and imaginary part and regarded as a system of two coupled differential equations

$$\begin{cases}\rho''+\left[-\alpha'^2-2\frac{\omega_p\beta R}{1-\beta^2}\alpha'+\left(\omega_p^2-m^2\right)\frac{R^2}{1-\beta^2}\right]\rho-\frac{\lambda R^2}{1-\beta^2}\rho^3=0\\ \rho\alpha''+2\rho'\alpha'+2\frac{\omega_p\beta R}{1-\beta^2}\rho'=0.\end{cases}$$

(2.150a)

(2.150b)

2.3.4.2 *Methodology for solving the ODEs system*

Even in the relativistic case, the system of coupled ODE (2.150a,2.150b) depends only on derivatives of $\alpha(\varphi)$. We begin by solving ODE (2.150b), and, substituting $\Delta(\varphi) \equiv \alpha'(\varphi)$, we get

$$\rho\Delta' + 2\rho'\Delta + 2\frac{\omega_p \beta R}{1-\beta^2}\rho' = 0. \tag{2.151}$$

Separating variables, we can solve for $\Delta(\varphi)$ and get

$$\int \frac{d\Delta}{2\Delta + 2\frac{\omega_p \beta R}{1-\beta^2}} = -\int \frac{d\rho}{\rho}, \tag{2.152}$$

which has solutions

$$\alpha' \equiv \Delta = \frac{C}{\rho^2} - \frac{\omega_p \beta R}{1-\beta^2}. \tag{2.153}$$

Substituting (2.153) in (2.150a) we obtain

$$\rho'' - \frac{C}{\rho^3} + \left(\frac{\omega_p^2 R^2}{(1-\beta^2)^2} - \frac{m^2 R^2}{1-\beta^2}\right)\rho - \frac{\lambda R^2}{1-\beta^2}\rho^3 = 0, \tag{2.154}$$

ODE (2.154) takes the same form of its non-relativistic counterpart (2.60). Thus, we can write it as

$$\rho'' + \frac{G_1}{\rho^3} + G_2\rho + G_3\rho^3 = 0 \tag{2.155}$$

with

$$\begin{cases} G_1 = -C & (2.156a) \\ G_2 \equiv \varepsilon_p^2 = \dfrac{\omega_p^2 R^2}{(1-\beta^2)^2} - \dfrac{m^2 R^2}{1-\beta^2} & (2.156b) \\ G_3 = -\dfrac{\lambda R^2}{1-\beta^2} & (2.156c) \end{cases}$$

and obtain the same ODE as (2.61) with different coefficients.
Since ODE (2.155) has the same form as (2.61), we can repeat the calculation steps from (2.61) to equations (2.85a,2.85b,2.85c), just substituting $F_1 \to G_1, F_2 \to G_2, F_3 \to G_3, \epsilon_p \to \varepsilon_p^2$.

Summarizing the results of the omitted steps, we obtain a solution of the form

$$\Phi\left(t,\varphi\right) = A_n\, e^{-i\omega_n t} e^{-i\left(\frac{\omega_n \beta R}{1-\beta^2}\varphi - \Xi\right)} \operatorname{sn}\left(q_n\varphi, k_n\right) \tag{2.157}$$

with quantization conditions

$$\begin{cases} q_n^2 = \dfrac{A_n^2 G_3 + 2G_2}{2} & (2.158a) \\ k_n^2 = -\dfrac{A_n^2 G_3}{A_n^2 G_3 + 2G_2} & (2.158b) \\ q_n = \dfrac{n}{\pi} K\left(k_n\right). & (2.158c) \end{cases}$$

or, expressed in terms of G_2 and G_3,

$$\begin{cases} G_3 = -\dfrac{2k_n^2 q_n^2}{A_n^2} & (2.159a) \\ G_2 = \epsilon_p^2 = \left(1 + k_n^2\right) q_n^2. & (2.159b) \end{cases}$$

To determine the normalization constant A_n, we need to fix the normalization condition

$$\langle\Phi|\Phi\rangle = \int_V dV \rho\left(x\right) = 1. \tag{2.160}$$

However, the probability density in a relativistic field theory is fundamentally different from the non-relativistic counterpart (cfr. 2.163 and 2.163). In fact, it can be written as

$$\rho\left(x\right) = i\left(\Phi^\star\partial_0\Phi - \Phi\partial_0\Phi^\star\right), \tag{2.161}$$

and, together with (2.160), it defines the Klein-Gordon scalar product. For stationary solutions, the probability density becomes

$$\rho\left(x\right) = 2\omega_n\Phi^\star\Phi \tag{2.162}$$

and the normalization condition (2.160) can be written as

$$2\omega_n \int_V dV\Phi^\star\Phi = 1. \tag{2.163}$$

We recognize that this calculation is essentially equivalent to the one performed from (2.88) to (2.90), apart from the multiplicative factor

$2\omega_n$. Therefore we can immediately conclude that the normalization coefficient is

$$A_n^2 = \frac{k_n^2}{4\omega_n \pi R \left(1 - \frac{E(k_n)}{K(k_n)}\right)}. \tag{2.164}$$

Complete solutions and quantization conditions

In conclusion, the full solution to the original PDE equation (2.144), can be written as

$$\Phi\left(t,\varphi\right) = \sqrt{\frac{k_n^2}{4\omega_n \pi R \left(1 - \frac{E(k_n)}{K(k_n)}\right)}}\, e^{-i\omega_n t} e^{-i\left(\frac{\omega_n \beta R}{1-\beta^2}\varphi - \Xi\right)} \operatorname{sn}\left(q_n \varphi, k_n\right). \tag{2.165}$$

Analogously to the non-relativistic case, the solution comprises an arbitrary constant phase, due to the $U(1)$ symmetry of the theory.
The elliptic modulus k_n and the energy eigenvalues ω_n cannot be expressed in closed forms, as they are determined by the non invertible equations

$$\begin{cases} \omega_n^2 = \dfrac{n^2}{\pi^2}\left(1 + k_n^2\right) K^2\left(k_n\right) \dfrac{\left(1-\beta^2\right)^2}{R^2} + m^2\left(1 - \beta^2\right) & \text{(2.166a)} \\ \dfrac{\pi}{8\omega_n n^2} \dfrac{\lambda R}{1-\beta^2} = K\left(k_n\right)\left(K\left(k_n\right) - E\left(k_n\right)\right). & \text{(2.166b)} \end{cases}$$

We can immediately identify a formal difference compared to the non-relativistic case: Eq. (2.166b), which provides the values or asymptotic behavior of the elliptic moduli k_n, cannot be determined independently of ω_n. Special attention will be given to deriving the large-n behavior of k_n.

2.3.4.3 *Non-interacting limit*

Once more, we investigate whether it is possible to retrieve the non-interacting results by considering the non-interacting limit $\lambda \to 0$. Guided by previous calculations, we know that the interacting behaviour of the solutions is embodied in the elliptic coefficients k_n, and in ω_n through k_n.
Thus, considering Eq. (2.166b) and the fact that $K(k_n)\left(K(k_n) - E(k_n)\right) \iff k_n = 0$, we can deduce that $\lambda = 0$ implies $k_n = 0$. Consequently, it is

possible to derive the non-interacting limit of the various terms that compose the complete solution, i.e.

$$\begin{aligned} K(k_n) &\xrightarrow{k_n \to 0} \frac{\pi}{2} \\ \mathrm{sn}\left(\frac{n}{\pi}K(k_n)\varphi, k_n\right) &\xrightarrow{k_n \to 0} \sin\left(\frac{n}{2}\varphi\right) \\ \omega_n(k_n) &\xrightarrow{k_n \to 0} \frac{n^2}{4R^2}\left(1-\beta^2\right)^2 + m^2\left(1-\beta^2\right) \\ A_n = \frac{k_n^2}{4\omega_n \pi R\left(1-\frac{E(k_n)}{K(k_n)}\right)} &\xrightarrow{k_n \to 0} A^2 = \frac{1}{2\omega_n \pi R}, \end{aligned} \tag{2.167}$$

recovering the correct relativistic, non-interacting quantities. Eventually, the full solution in the non-interacting limit becomes

$$\Phi(t,\varphi) = \sqrt{\frac{1}{2\omega_n \pi R}} e^{-i\omega_n t} e^{-i\left(\frac{\omega_n \beta R}{1-\beta^2}\varphi - \Xi\right)} \sin\left(\frac{n}{2}\varphi\right). \tag{2.168}$$

2.3.4.4 *Asymptotic behaviour of k_n*

Starting from (2.166b), we have

$$\frac{\pi}{8\omega_n n^2}\frac{\lambda R}{1-\beta^2} = K(k_n)\left(K(k_n) - E(k_n)\right) \tag{2.169}$$

The procedure used in (2.3.1) cannot be directly applied here due to the presence of the new ω_n term on the LHS. To proceed, we must first multiply both sides by ω_n, square them, and substitute (2.166a), leading to

$$\frac{\pi^2}{64n^4}\frac{\lambda^2 R^2}{(1-\beta^2)^2} = K^2(k_n)\left(K(k_n) - E(k_n)\right)^2 \left(\frac{n^2}{\pi^2}\left(1+k_n^2\right)K^2(k_n)\frac{(1-\beta^2)^2}{R^2} + m^2\left(1-\beta^2\right)\right). \tag{2.170}$$

Since the RHS exhibits a behavior similar to $K(k_n)(K(k_n) - E(k_n))$ under the arguments made in (2.3.1), we can expand it to the lowest order in k_n and evaluate the asymptotic behavior of the elliptic moduli. This leads to

$$\frac{\pi^2}{64n^4}\frac{\lambda^2 R^2}{(1-\beta^2)^2} = \frac{\pi^4\left(1-\beta^2\right)\left(n^2\left(1-\beta^2\right) + 4m^2R^2\right)}{256R^2}k_n^4. \tag{2.171}$$

Equation (2.171) can be simplified, yielding

$$k_n^2 \approx \frac{2R\lambda}{n^2\pi\left(1-\beta^2\right)\sqrt{\frac{n^2}{4R^2}\left(1-\beta^2\right)^2+m^2\left(1-\beta^2\right)}}, \tag{2.172}$$

where we recognize that the term inside the square root corresponds to the non-interacting energy eigenvalue derived in (2.167).

2.3.4.5 *Behaviour of energy eingenvalues ω_n*

We are now able to compute the asymptotic behaviour of the energy eigenvalues. Analogously to what we did in the non-relativistic case, we start by considering Eq. (2.166a), i.e.,

$$\omega_n^2 = \frac{n^2}{\pi^2}\left(1+k_n^2\right)K^2\left(k_n\right)\frac{\left(1-\beta^2\right)^2}{R^2} + m^2\left(1-\beta^2\right) \tag{2.173}$$

and expand the $K^2\left(k_n\right)$ term for small k_n. The leading behaviour of the energy eigenvalues then becomes

$$\omega_n^2 = m^2\left(1-\beta^2\right) + \frac{n^2}{4R^2}\left(1-\beta^2\right)^2 + \frac{3n^2}{8R^2}\left(1-\beta^2\right)^2 k_n^2 + O\left(k_n^4\right), \tag{2.174}$$

and, substituting the asymptotics of k_n (2.172), we obtain

$$\begin{aligned}\omega_n^{(a)\,2} = m^2\left(1-\beta^2\right) + \frac{n^2}{4R^2}\left(1-\beta^2\right)^2 \\ - \frac{3\left(1-\beta^2\right)\lambda}{8\pi R\sqrt{\frac{n^2}{4R^2}\left(1-\beta^2\right)^2+m^2\left(1-\beta^2\right)}} + O\left(n^{-4}\right).\end{aligned} \tag{2.175}$$

Terms of order $O\left(n^{-4}\right)$ will be neglected from the definition of $\omega_n^{(a)\,2}$ because, upon taking the square root, they do not contribute divergently to the summation over n in the calculation of the QVE. As in the non-relativistic case, we observe that in the limit $\lambda \to 0$, $\omega_n^{(a)\,2}$ reduces to the exact energy eigenvalues expected in a free relativistic theory.

2.3.5 *Non-relativistic limit*

We now address the issue discussed in 2.1.3 to verify whether, in the non-interacting limit, the relativistic solutions for the equations of motion and energy eigenvalues reduce to their non-relativistic counterparts. If this is confirmed, the Schrödinger quantum field theory and the relativistic quantum field theory become indistinguishable, except for the inclusion of the ZPE in the latter. Based on the arguments in 2.1.3, the relativistic theory should be preferred in the low-energy regime, provided that QVE corrections remain within the energy scales where the Schrödinger field theory is valid.

To explore the low-energy limit of the relativistic theory, we examine the large mass $m \gg 1$ and slow rotation $\beta \ll 1$. Expanding the asymptotic values of the energy eigenstates (2.175), we find

$$\omega_n^R = m + \frac{n^2}{8mR^2} - \frac{3\lambda}{16m^2\pi R} - \frac{m}{2}\beta^2 - \cancel{\frac{3n^2\beta^2}{16mR^2}} + O\left(m^{-3}\right) + O\left(\beta^3\right) \tag{2.176}$$

where we have omitted the labels (a) in favour of R (relativistic) and NR (non-relativistic) for clarity. Additionally, the term $3n^2\beta^2/16mR^2$ is neglected as it is subleading, combining corrections of order $O\left(m^{-1}\right) \cdot O\left(\beta^2\right)$.
The non-relativistic limit of the energy eigenvalues ω_n^R, aside from the rest mass term, matches the non-relativistic counterpart (2.109), except for a factor $1/2m^2$ in the interaction term. This discrepancy is expected because the mass dimensions and definitions of the coupling constants differ between the relativistic and non-relativistic theories. In the non-relativistic theory, the coupling constant λ_{NR} is dimensionless $[\lambda_{NR}] = 0$, whereas in the relativistic theory, λ_R has mass dimension $[\lambda_R] = 2$ (see more in C). Hence, we can see that

$$\omega_n^{R\to NR} = \cancel{m} + \frac{n^2}{8mR^2} - \frac{3\lambda_{NR}}{8\pi R} - \frac{m}{2}R^2\Omega^2, \tag{2.177}$$

perfectly reproducing the non-relativistic value of the energy eigenvalues.

Lastly, we examine the solutions in both theories.
Using (2.165), we focus on the elliptic moduli k_n of the two theories. Since these cannot be expressed in closed form, direct comparison is not possible. However, if their defining equations (2.95b) and (2.166b) are equivalent in the non-relativistic limit, their solutions k_n will also be identical. Substituting the non-relativistic limit

$$\frac{1}{\omega_n^R} = \frac{1}{m} + O\left(m^{-3}\right) \tag{2.178}$$

into (2.166b), we recover (2.95b), with the redefinition $\lambda_R = 2m^2\lambda_{NR}$. Thus, the moduli are equivalent in the non-relativistic limit.

Given this equivalence, we can also compare the normalization constants and phases of the solutions:

$$A_n^R = \sqrt{\frac{k_n^2}{4\omega_n \pi R \left(1 - \frac{E(k_n)}{K(k_n)}\right)}} \xrightarrow[m\to\infty]{\beta\to 0} \sqrt{\frac{1}{2m}} A_n^{NR} = \sqrt{\frac{k_n^2}{2\pi m R \left(1 - \frac{E(k_n)}{K(k_n)}\right)}}$$
$$e^{-i\left(\frac{\omega_n \beta R}{1-\beta^2}\varphi\right)} \xrightarrow[m\to\infty]{\beta\to 0} e^{-imt} e^{-i\left(mR^2\Omega\varphi\right)}. \tag{2.179}$$

The differences in the normalization factor $\frac{1}{\sqrt{2m}}$ and the phase term $\exp(-imt)$ are due to the normalization conventions and the rest mass phase (see more in C).

In conclusion, the non-relativistic limit of the relativistic theory fully aligns with the non-relativistic theory.

2.3.6 *Outlook*

In this chapter, we have obtained several original results for the theories described by (2.49) and (2.142) in the presence of Dirichlet boundary conditions in a 1+1 dimensional ring:

- Analytical, non-perturbative solutions of the relativistic and non-relativistic EOMs

$$\Phi^{NR}(t,\varphi) = \sqrt{\frac{k_n^2}{2\pi R \left(1 - \frac{E(k_n)}{K(k_n)}\right)}} e^{-i\omega_n t} e^{-i\left(mR^2\Omega\varphi - \Xi\right)} \mathrm{sn}\left(\frac{n}{\pi} K(k_n)\,\varphi, k_n\right)$$
$$\Phi^{R}(t,\varphi) = \sqrt{\frac{k_n^2}{4\omega_n \pi R \left(1 - \frac{E(k_n)}{K(k_n)}\right)}} e^{-i\omega_n t} e^{-i\left(\frac{\omega_n \beta R}{1-\beta^2}\varphi - \Xi\right)} \mathrm{sn}\left(\frac{n}{\pi} K(k_n)\,\varphi, k_n\right). \tag{2.180}$$

  with exact energy eigenvalues, determined by

$$\begin{cases} \omega_n^{NR} = \dfrac{n^2}{2\pi^2 m R^2}\left(1 + k_n^2\right) K^2(k_n) - \dfrac{m}{2} R^2 \Omega^2 \\ \lambda_{NR} m R \dfrac{\pi}{4n^2} = K(k_n)\left(K(k_n) - E(k_n)\right). \end{cases} \tag{2.181}$$

  and

$$\begin{cases} \omega_n^{R\,2} = \dfrac{n^2}{\pi^2}\left(1 + k_n^2\right) K^2(k_n) \dfrac{\left(1-\beta^2\right)^2}{R^2} + m^2\left(1 - \beta^2\right) \\ \dfrac{\pi}{8\omega_n n^2} \dfrac{\lambda_R R}{1-\beta^2} = K(k_n)\left(K(k_n) - E(k_n)\right); \end{cases} \tag{2.182}$$

- A non-vanishing vacuum energy in a non-relativistic theory

$$E = -\frac{3}{32}\frac{\lambda}{\pi R} + \frac{m}{8}R^2\Omega^2 + \frac{1}{2}\sum_{n=1}^{\infty}\left(\omega_n^{\text{ex}} - \omega_n^{(a)}\right) \tag{2.183}$$

  with

$$\omega_n^{(a)} = \frac{n^2}{8mR^2} + \frac{3}{8}\frac{\lambda}{\pi R} - \frac{m}{2}R^2\Omega^2 \tag{2.184}$$

  (results expressed in the co-rotating frame). This was achieved using an original regularization approach combining a 'window function' method with Jacobi theta functions.

- A Casimir-like force with attractive or repulsive behavior depending on the physical parameters.

- A strong motivation for including a ZPE term even in non-relativistic systems.

CONCLUSIONS

COMPENDIUM OF THIS THESIS

This thesis explores the intricate dynamics of QVE and symmetry breaking in interacting QFTs under the influence of acceleration, rotation, and boundaries. Specifically, it addresses two fundamental problems: symmetry restoration in accelerated frames and QVE in non-relativistic field theories.

The first part investigates the phenomenon of symmetry breaking and its possible restoration due to acceleration, mediated by the Unruh effect. The conflicting results in the literature have been shown to arise either from differences in renormalization prescriptions or from different choices of the vacuum state.

Advocates of symmetry restoration at high accelerations compare the 'Unruh-thermalized' (divergent) observables of an accelerated observer to their zero-temperature counterparts, which are computed using the Rindler vacuum. However, the Rindler vacuum is frame-dependent, varying with the observer's acceleration and exhibiting different particle content and thermal properties across distinct accelerated frames. Our analysis demonstrated that applying a frame-dependent renormalization in Rindler frequencies is formally equivalent to a finite-temperature renormalization, but with a critical distinction. In a purely thermal scenario, the counterterm is fixed, while the system's physical temperature varies. Conversely, in the accelerated case, the temperature of the Minkowski vacuum remains fixed, and the counter-term incorporates the temperature dependence. This dependence changes with the observer's state of motion, specifically the magnitude of their acceleration. Subtracting this counterterm yields a positive temperature dependence, analogous to the thermal scenario. However, while the results align, the origins of the thermal-like contributions differ: finite temperature in Minkowski space arises intrinsically, while for accelerated observers, it reflects a vacuum dynamics shaped by their specific state of motion.

Proponents of the persistence of symmetry breaking argue that the Minkowski vacuum corresponds to the lowest possible temperature for any observer, whether inertial or accelerating. In this framework, the choice of counterterm is unique, even though its representation may vary across different coordinate systems. Given the result of the thermalization theorem (1.106), the invariance of scalar quantities such as ϕ^2 or the VEV of the field is inherently maintained. We also showed

that this perspective aligns with the conventional renormalization prescription employed in the broader framework of quantum field theory in curved spacetime.
As a result, while inertial and non-inertial observers perceive different particle contents in the quantum vacuum, they ultimately arrive at the same observed value for the VEV of the field or $\langle \phi^2 \rangle$. The inertial observer interprets these quantities as intrinsic properties of the Minkowski vacuum, whereas the non-inertial observer attributes them to the thermal radiation generated by acceleration.
Advocates of symmetry breaking enhancement suggest, explicitly or implicitly, that the true physical vacuum is the Rindler vacuum. Consequently, they argue that the physical ground state remains at zero temperature for any observer, whether inertial or accelerated. In analogy with the symmetry restoration scenario, adopting the Rindler vacuum in calculations renders the results frame-dependent, as the Rindler vacuum is a dynamical entity that varies between different accelerating observers. However, significant conceptual differences arise between the two approaches, leading to fundamentally opposite outcomes.
If the Rindler vacuum is considered the true physical vacuum, and divergent quantities are regularized using (covariant) Minkowski counterterms, the resulting expression remains frame-dependent. Computationally, this is the opposite of the symmetry restoration approach, as it leads to an increase in the condensate with growing acceleration. Assuming the Rindler vacuum as the true ground state also introduces a novel interpretation of QFT for accelerated observers: observables (such as the Green's function) must be heated and thermalized at the Unruh temperature to reproduce those of a Minkowski observer in their vacuum state. This follows from the fact that the lowest temperature is now zero, rather than the Unruh temperature.

It is evident that these differing outcomes cannot be reconciled as they stand. A suitable criterion is therefore needed to determine which one truly represents Nature's behavior.

The second part of this work focuses on the study of QVE in non-relativistic interacting quantum field theories with Dirichlet boundary conditions imposed on a rotating 1+1 dimensional ring. This investigation led to several significant results, advancing our understanding of quantum vacuum dynamics in constrained and rotating systems.

First, we derived exact, non-perturbative solutions to the EOMs. These solutions are refinedly expressed in terms of Jacobi elliptic functions and elliptic integrals, capturing the intricate interplay between rotation, boundary conditions, and field interactions. Additionally, we provided the relations necessary to compute the energy eigenvalues numerically and derived a closed-form expression for the asymptotic behavior of the spectrum at large quantum numbers. These results not only establish the groundwork for analyzing such systems, but also underscore the precision of the mathematical framework employed.

Using these unperturbed solutions and incorporating a manually introduced zero-point energy term, we demonstrated that the QVE does not vanish in the non-relativistic case. Importantly, when analyzing the relativistic analog of the same system, we found that the results reduce to those of the purely non-relativistic case in the appropriate low-energy limit. In fact, in this limit, the two theories should be physically indistinguishable. This agreement between the two frameworks, despite their differing mathematical origins, provides strong theoretical justification for including ZPE contributions in the Schrödinger theory.

The computed QVE and the resulting force, expressed in the laboratory frame, exhibit a dual contribution: an attractive component arising from interactions and a repulsive centrifugal term induced by rotation. This interplay highlights the unique behavior of vacuum energy in non-relativistic systems, especially under rotational and boundary constraints.

The prospect of experimentally observing such phenomena in condensed matter systems, particularly using BEC confined in optical traps, offers an exciting avenue for future research. This connection between theoretical predictions and experimental feasibility motivates further exploration of quantum vacuum dynamics in both relativistic and non-relativistic domains.

In conclusion, the significant results obtained in this thesis, ranging from the nuanced role of renormalization in symmetry breaking under acceleration to the demonstration of nonvanishing vacuum energy in rotating systems, underscore the profound richness of quantum vacuum dynamics.

IMPLICATIONS AND FUTURE DIRECTIONS

In the first part of this thesis, we explored the various scenarios and outcomes discussed in the literature concerning the behaviour of a system exhibiting a spontaneous symmetry breaking, as perceived by a uniformly accelerated observer in flat spacetime. Despite being the simplest non-inertial reference frame, this setup has been sufficient to divide the community into distinct schools of thought for decades. We specifically highlighted the differences among these approaches, uncovering their explicit and implicit assumptions, lines of reasoning, and consequences. However, this theoretical playground is only the first stepping stone toward a complete understanding of non-inertial reference frames in more complicated scenarios, such as gravitational fields. As an immediate generalization, consider a static observer in Schwarzschild or Reissner–Nordström spacetimes, subject to a constant acceleration. As noted by P. Candelas and K.W. Howard [36], based on D.N. Page result [4], the vacuum polarization with respect to Hartle-Hawking vacuum in Schwarzschild spacetime (calculated using a covariant renormalization scheme), can be expressed as the sum of two contributions

$$\langle \phi^2 \rangle_{\text{ren}} = \frac{1}{12(8\pi M)^2} \frac{1-(2M/r)^4}{1-2M/r} + \frac{\Delta(r)}{(8\pi M)^2} \tag{2.185}$$

or, more generally,

$$\langle \phi^2 \rangle_{\text{ren}} = (T_{\text{loc}}^2 - T_{\text{acc}}^2) + \frac{\Delta(r)}{(8\pi M)^2}. \tag{2.186}$$

The second term, $\Delta(r)$, typically requires numerical evaluation but is generally much smaller than the first contribution, and will thus be neglected in the following discussion. Once again, the pivotal role is played by the local temperature T_{loc}, and the acceleration temperature T_{acc}. As explored in this thesis, it is reasonable to infer, even in gravitational contexts, that T_{loc} is associated to the temperature of the (assumed) true physical vacuum, whereas T_{acc} relates to the temperature reference intrinsic to the renormalization procedure. A different physical choice for either T_{loc} or T_{acc}, for instance selecting a different physical vacuum or adopting a different renormalization scheme, would alter the gravitational results in a manner closely comparable to the accelerated case in flat spacetime. This clearly illustrates that the study of the latter is crucial for inferring the correct behavior of scalar quantities and symmetry phases in gravitational scenarios as well.

As discussed throughout this thesis, the general standpoint within the physics community concerning the calculation of $\langle\phi^2\rangle$, and thus the symmetry phase of the system in this setup, is to adopt a covariant renormalization approach and to identify the physical vacuum with a state incorporating the black hole's black-body radiation (implying a minimum, non-vanishing vacuum temperature). This is the one and only only approach that preserves the general covariance of the theory, ensuring that scalar quantities remain invariant across reference frames. However, if general covariance is not regarded as an indispensable requirement, alternative possibilities emerge, such as those discussed in this work.

The problem of the persistence, restoration, or enhancement of spontaneous symmetry breaking in non-inertial reference frames becomes even more intriguing when considered within the context of the Standard Model and its electroweak symmetry breaking. Although the Standard Model is more complex than the ϕ^4 theory analyzed here, mostly due to the presence of Yukawa interactions with fermion fields, it is plausible to expect that the qualitative behavior of the symmetry mechanism under changes in reference frame remains similar. If so, profound implications arise: massive particles in the Standard Model might appear equally massive, more massive, less massive, or even massless to observers in different frames, depending on the actual behavior selected by Nature among the competing possibilities discussed. Such frame-dependent mass behavior would dramatically affect the dynamics of particles and the physical laws experienced by accelerated observers.

In conclusion, it is imperative to clarify how scalar quantities and symmetry phases behave for observers in non-inertial reference frames. Additional criteria, possibly even stronger than general covariance, must be established to consistently favor one viewpoint, ruling out the others, and to definitively settle this long-standing issue.

In the second part of the thesis, we investigated the QVE in a non-relativistic, ϕ^4 interacting QFT. This problem is particularly intriguing, not only due to its relevance in both QFT and condensed matter physics within a non-relativistic framework, but also for the potential cosmological implications emerging from its relativistic counterpart. Our analysis of the non-relativistic case demonstrates that interactions and/or rotations, when combined with boundaries, can significantly alter the structure of the quantum vacuum and its energy in an observable, Casimir-like fashion. This is yet another of the few known instances where QVE emerges as an essential ingredient for the accurate description of non-relativistic systems. Notably, the first such instance appeared in the 1957 work by Lee, Huang, and Yang (LHY) [37, 38], where the ground state energy of a dilute Bose gas was shown to receive crucial interaction-induced corrections from quantum fluctuations; a result that has since been confirmed experimentally. Our result can be seen as a Casimir analog of the LHY effect, arising in a boundary-sensitive context where interactions similarly induce a significant shift in vacuum energy. Moreover, the non-perturbative approach we have adopted enables us to further expand our analysis from a weakly coupled to a strongly coupled regime of the interaction strength.

From an experimental standpoint, a promising connection arises between our quantum field theoretical framework and the domain of ultra-cold atomic systems, suggesting potential experimental feasibility. A particularly illustrative example is provided by the experiment described in [39], where a ^{23}Na BEC is confined within a ring of radius $R \sim 20\mu$m. Within a quasi-1D approximation, the effective interaction strength λ can be related to the s-wave scattering length a_s via the expression $\lambda = g/(\pi l^2)$, with $g = 4\pi\hbar^2 a_s/m$ characterizing the atomic interaction. For ^{23}Na, taking $a_s = 50a_0$ and choosing $l \sim 2\ \mu$m, ensuring the transverse confinement is much smaller than the ring radius, and assuming $N = 2 \times 10^3$ atoms, we find a dimensionless interaction strength $\lambda ml/\hbar^2 \sim 4a_s N/l \sim 10$, which aligns closely with the parameters used in Fig. 2.2.

A similar estimate can be made for the dimensionless Casimir-like force F_s, based on Eq. (2.135). The resulting expression is:

$$F_s \frac{ml^3}{\hbar^2} = -\frac{3}{8\pi} \frac{a_s N l}{R^2} + \frac{m^2 l^3}{\hbar^2} \frac{R\Omega^2}{4}. \tag{2.187}$$

By adopting a rotation frequency of $\Omega \sim 2\pi \times 25$Hz, as realized in [40], we obtain a dimensionless force $F_s ml^3/\hbar^2 \sim 0.1$. Although small, this value could still be within reach of experimental observation. Under these conditions, a ring of radius $R \sim 20\ \mu$m would fall within the causally repulsive regime of Fig.2.2, and the scaled angular velocity $\Omega ml^2/\hbar \sim 0.2$ suggests that even lower rotation speeds could be advantageous.

While our theoretical model employs Dirichlet boundary conditions, similar constraints could be experimentally implemented using weak links, as demonstrated in the ring-shaped BEC experiments [41, 42]. These considerations reinforce the potential relevance of our theoretical results to near-future experimental platforms.

The physical system investigated in this study may find applications in the field of atomtronics [43], providing an additional opportunity to probe the fundamental aspects of quantum vacuum physics. Exploring analogous setups involving fermionic systems [44] or more complex, multiply connected geometries [45] opens up new directions to examine the phenomena discussed here in less explored contexts.

Lastly, the study of QVE in interacting relativistic systems with boundaries could potentially offer novel insights into cosmology, particularly regarding the cosmological constant problem.
From a study by M. Bordag [46], which considered a ϕ^4-interacting relativistic QFT, it appears possible to obtain a constant, non-vanishing, interaction-induced vacuum energy density in flat spacetime. This result holds in the limit where the boundaries are taken to infinity. Such a term could provide a new contribution to the cosmological constant. To illustrate the relevance of this idea, consider the general relativistic action for a minimally coupled scalar field

$$S = \frac{1}{16\pi G_N}\int d^4x\sqrt{-g}R + \int d^4x\sqrt{-g}\left[\frac{1}{2}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi - \rho_{\Lambda\text{vac}} - V(\phi)\right], \tag{2.188}$$

where

$$\rho_{\Lambda\text{vac}} = \frac{\Lambda}{8\pi G_N}, \tag{2.189}$$

and $V(\phi)$ is a Higgs-type potential that undergoes spontaneous symmetry breaking (SSB)

$$V(\phi) = \frac{1}{2}m^2\phi^2 + \frac{1}{4!}\lambda\phi^4, \qquad (m^2 < 0, \lambda > 0). \tag{2.190}$$

The classical minima of this potential and their associated energy densities are

$$\langle\phi\rangle = \pm\sqrt{-6m^2/\lambda} \simeq 246\,\text{GeV}, \qquad \langle V(\phi)\rangle = -\frac{3}{2}\frac{m^4}{\lambda} \simeq -10^8\,\text{GeV}^4. \tag{2.191}$$

The stress-energy tensor of the scalar field reads

$$T^\phi\mu\nu = g\mu\nu\rho_{\Lambda\text{vac}} + \partial_\mu\phi\partial_\nu\phi - \frac{1}{2}g_{\mu\nu}\partial_\sigma\phi\partial^\sigma\phi + g_{\mu\nu}V(\phi), \tag{2.192}$$

which expectation value in the vacuum, given the vanishing contribution of kinetic terms, becomes

$$\langle T^\phi_{\mu\nu}\rangle = g_{\mu\nu}\left(\rho_{\Lambda\text{vac}} + \underbrace{\langle V(\phi)\rangle}_{\rho_{\Lambda\text{ind}}}\right). \tag{2.193}$$

The total contribution to the physical cosmological constant is then

$$\rho_{\text{ph}} = \rho_{\Lambda\text{vac}} + \rho_{\Lambda\text{ind}} \approx 10^{-47}\,\text{GeV}^4. \tag{2.194}$$

However, given the classical value of $\rho_{\Lambda\text{ind}}$ in Eq. (2.191), we observe a striking mismatch in magnitude

$$\left|\frac{\rho_{\Lambda\text{ind}}}{\rho_{\Lambda\text{ph}}}\right| \approx O(10^{55}). \tag{2.195}$$

This means that $\rho_{\Lambda\text{vac}}$ must be fine-tuned to cancel $\rho_{\Lambda\text{ind}}$ with extraordinary precision, up to its 55^{th} decimal place. This is the well-known cosmological constant fine-tuning problem.

The entire calculation so far has been performed at the classical level. However, a more accurate and complete treatment requires taking into account quantum corrections to the effective potential (see [47] for a comprehensive discussion).

As discussed in this thesis (see 2.1.1), the first quantum correction to the effective potential can be expressed in terms of the QVE. Therefore, by generalizing the relativistic interacting model (without rotation) to the $3+1$-dimensional case, and assuming a negative mass-squared parameter, it is conceivable that a constant QVE term, scaling as $\sim m^4$, survives even as the boundaries are taken to infinity. If such a contribution exists, it could represent a new mechanism to offset part of the large vacuum expectation value of the energy density $\rho_{\Lambda\text{ind}}$. This could potentially mitigate the severity of the fine-tuning problem.

Much remains to be said and taken into account in this procedure, such as the role of renormalization in shaping vacuum energy contributions and the physical interpretation of imposing boundary conditions at infinity (e.g. cosmological horizons). Although this possibility remains highly speculative, it is definitely intriguing to study further the implication of interactions to QVE in this cosmological context. Even were it unable to resolve the fine-tuning problem, it nonetheless provides a novel avenue for examining how interactions shape vacuum energy dynamics.

APPENDIX

A FINITE TEMPERATURE FIELD THEORY

In this Appendix section we introduce the basic knowledge and formalism of finite temperature field theory.
First of all, let us consider a QFT described by the Hamiltonian density $\mathcal{H}(\phi, \pi)$, where $\phi(t, \vec{x})$ is the field operator defined in Heisenberg picture and $\pi(t, x)$ its conjugate momentum. Since we are interested in the transition amplitude between an initial state $\phi_0(0, \vec{x})$, and a final state $\phi_1(t_1, \vec{x})$, we can use the Feynman functional formula in Hamiltonian form, and obtain

$$\begin{aligned} \langle \phi_1(t_1) | \phi_0(0) \rangle &= \langle \phi_1(0) | e^{-iHt_1} | \phi_0(0) \rangle \\ &= N \int \mathcal{D}\phi \mathcal{D}\pi \exp \left[i \int_0^{t_1} dt \int d^3x \left(\pi \dot{\phi} - \mathcal{H}(\phi, \pi) \right) \right], \end{aligned} \tag{A.1}$$

where H is the Hamiltonian of the theory, N is a normalization factor and $\dot{}$ is the time derivative. The integration over ϕ runs over all possible field configurations that starts at $\phi_0(0, \vec{x})$ and go to $\phi_1(t_1, \vec{x})$, while the integration over π is unrestricted.
We now perform a Wick rotation, by considering the shift from real time coordinate to imaginary time coordinate, i.e. $t = -it_{\text{E}}$, also known as Euclidean time. The quantity of interest affected by this coordinate change are

$$\begin{cases} t & \rightarrow \quad -it_{\text{E}} \\ t_1 & \rightarrow \quad -i\beta \\ \dfrac{\partial}{\partial t} & \rightarrow \quad i\dfrac{\partial}{\partial t_{\text{E}}} \\ p_0 & \rightarrow \quad ip_{0_{\text{E}}} \\ \eta = \text{diag}(+,-,-,-) & \rightarrow \quad -\delta = -\text{diag}(+,+,+,+). \end{cases} \tag{A.2}$$

From this moment on, $\dot{}$ will represent the derivative with respect to the Euclidean time t_{E}.
The transition amplitude then becomes

$$\langle \phi_1(t_1) | \phi_0(0) \rangle = N \int \mathcal{D}\phi \mathcal{D}\pi \exp \left[\int_0^{\beta} dt \int d^3x \left(i\pi\dot{\phi} - \mathcal{H}(\phi, \pi) \right) \right]. \tag{A.3}$$

In QFT in general, and especially in thermal systems, we are interested in the calculation of the partition function

$$Z = \mathrm{Tr}\left(e^{-\beta H}\right) = \sum_{\phi} \langle\phi| e^{-\beta H} |\phi\rangle \tag{A.4}$$

which, in practice, corresponds to a sum over all transition amplitudes from an initial state $|\phi\rangle$ back into itself after an imaginary-time evolution of duration $t_{\mathrm{E}} = \beta$. This is equivalent to requiring the field configurations to be periodic in imaginary time with period β. What we have done is to compactify the Euclidean time dimension on a circle of circumference β. As a result, the energy spectrum becomes discretized, as we will see in the following. Moreover, in this scenario, β represents the inverse temperature of the system.
The partition function can be expressed as

$$Z = N \int \mathcal{D}\pi \int_{\phi(0)=\phi(\beta)} \mathcal{D}\phi \exp\left[\int_0^{\beta} dt_{\mathrm{E}} \int d^3x \left(i\pi\dot{\phi} - \mathcal{H}(\phi,\pi)\right)\right], \tag{A.5}$$

and, considering Hamiltonian densities of the form

$$\mathcal{H} = \frac{1}{2}\left(\pi^2 + (\nabla\phi)^2\right) + V(\phi), \tag{A.6}$$

we can rewrite the integral inside the exponential as

$$\begin{aligned}&\int_0^{\beta} dt_{\mathrm{E}} \int d^3x \left(i\pi\dot{\phi} - \mathcal{H}(\phi,\pi)\right)\\ &= \int_0^{\beta} dt_{\mathrm{E}} \int d^3x \left(-\frac{1}{2}(\pi - i\dot{\phi})^2 - \frac{1}{2}\dot{\phi}^2 - \frac{1}{2}(\nabla\phi)^2 - V(\phi)\right),\end{aligned} \tag{A.7}$$

by completing the square in the momentum field. Performing the substitution $\pi - i\dot{\phi} \to \pi$ does not change the integral measure over the momenta configurations and enables us to perform the Gaussian integration over them. We finally obtain

$$Z = \tilde{N} \int_{\phi(0)=\phi(\beta)} \mathcal{D}\phi \exp\left[-\int_0^{\beta} dt_{\mathrm{E}} \int d^3x \mathcal{L}_{\mathrm{E}}(\phi,\dot{\phi})\right], \tag{A.8}$$

with $\mathcal{L}_{\mathrm{E}}$ the Euclidean Lagrangian density

$$\mathcal{L}_{\mathrm{E}} = \frac{1}{2}\dot{\phi}^2 + \frac{1}{2}(\nabla\phi)^2 + \frac{m^2}{2}\phi^2, \tag{A.9}$$

and $\tilde{N}$ a new normalization constant given by the π integration. It is possible now to expand the field ϕ in Fourier series and, given its periodicity property in the Euclidean time coordinate, we can write

$$\phi\left(t_{\mathrm{E}}, \vec{x}\right) = \frac{1}{\beta} \sum_{n=-\infty}^{\infty} \int \frac{d^3k}{(2\pi)^3} e^{i\vec{k}\cdot\vec{x}} e^{i\omega_n t_{\mathrm{E}}} \phi_n(\vec{k}), \tag{A.10}$$

with

$$\omega_n = \frac{2\pi n}{\beta} \tag{A.11}$$

which take the name of Matsubara frequencies.
The orthonormality relation for the 'Matsubara modes' are given by

$$\int_0^\beta dt_{\mathrm{E}} e^{i\left(\omega_n - \omega_{n'}\right)t_{\mathrm{E}}} = \beta\, \delta_{n,n'}, \tag{A.12}$$

that shows the conservation of discrete energy and a factor β at each vertex.
Finally, the Feynman rules for the thermal field are equivalent to the zero-temperature ones, with the replacements

$$\begin{cases} k_{0_{\mathrm{E}}} & \rightarrow \quad i\omega_n \\ \int \frac{dk_0}{2\pi} & \rightarrow \quad \frac{1}{\beta} \sum_{n=-\infty}^{\infty} \\ (2\pi)^4 \delta^4 \left(k_1 + k_2 + \cdots\right) & \rightarrow \quad -i\,(2\pi)^3 \delta_{\omega_{n_1}+\omega_{n_2}+\cdots}\, \delta^3 \left(\vec{k}_1 + \vec{k}_2 + \cdots\right). \end{cases} \tag{A.13}$$

B PROPAGATORS IN MINKOWSKI AND RINDLER VACUA

We are interested in obtaining the expressions of the free propagators, starting from the Rindler-Fulling quantization (1.23), evaluated with respect to Minkowski and Rindler vacua.
The Rindler-Fulling quantization, restricted to the right Rindler wedge, reads

$$\phi^{(+)}(x) = \int_0^\infty d\Omega \int d^2k_\perp \left[b_k^{(+)} f_k^{(+)}(x) + b_k^{(+)\dagger} f_k^{(+)\star}(x) \right], \tag{B.1}$$

with

$$f^{(+)}_{\Omega,\vec{k}_\perp}(x) = \frac{1}{2\pi^2}\sqrt{\frac{1}{a}\sinh\left(\pi\frac{\Omega}{a}\right)} e^{-i\Omega\tau} e^{i\vec{k}_\perp \vec{x}_\perp} K_{i\Omega/a}\left(\mu_{k_\perp}\rho\right), \tag{B.2}$$

and $\mu_{k_\perp} = \sqrt{\vec{k}_\perp^2 + m^2}$.
In order to evaluate how the fields act on the Minkowski vacuum, it is necessary to transform the creation and annihilation operators $b_k^\dagger, b_k$ in terms of $d_k^\dagger, d_k$ ones. To achieve this, we invoke the Bogoliubov transformations (1.40), i.e.,

$$\begin{aligned} b^{(+)}_{\Omega,\vec{k}_\perp} &= \frac{e^{\frac{\pi}{2}\frac{\Omega}{a}}}{\sqrt{2\sinh(\pi\Omega/a)}} d^{(+)}_{\Omega,\vec{k}_\perp} + \frac{e^{-\frac{\pi}{2}\frac{\Omega}{a}}}{\sqrt{2\sinh(\pi\Omega/a)}} d^{(-)\dagger}_{\Omega,-\vec{k}_\perp} \\ b^{(+)\dagger}_{\Omega,\vec{k}_\perp} &= \frac{e^{\frac{\pi}{2}\frac{\Omega}{a}}}{\sqrt{2\sinh(\pi\Omega/a)}} d^{(+)\dagger}_{\Omega,\vec{k}_\perp} + \frac{e^{-\frac{\pi}{2}\frac{\Omega}{a}}}{\sqrt{2\sinh(\pi\Omega/a)}} d^{(-)}_{\Omega,-\vec{k}_\perp} . \end{aligned} \tag{B.3}$$

Taking into account that $d_k^{(\sigma)}|0\rangle_M = 0$, we obtain

$$\phi^{(+)}(x)|0\rangle_M = \int_0^\infty d\Omega \int d^2k_\perp \left[N(\Omega)^{\frac{1}{2}} f_k^{(+)}(x) d^{(-)\dagger}_{\Omega,-\vec{k}_\perp} + (1+N(\Omega))^{\frac{1}{2}} f_k^{(+)\star}(x) d^{(+)\dagger}_{\Omega,\vec{k}_\perp} \right] |0\rangle_M , \tag{B.4}$$

with $N(\Omega) = \left(e^{2\pi\Omega/a} - 1\right)^{-1}$.
Finally, employing the canonical commutation relations (1.39)

$$\begin{aligned} \left[d^{(\sigma)}_{\Omega,\vec{k}_\perp}, d^{(\sigma')\dagger}_{\Omega',\vec{k}'_\perp} \right] &= \delta_{\sigma\sigma'}\delta\left(\Omega - \Omega'\right)\delta\left(\vec{k}_\perp - \vec{k'}_\perp\right) \\ \left[d^{(\sigma)}_{\Omega,\vec{k}_\perp}, d^{(\sigma')}_{\Omega',\vec{k}'_\perp} \right] &= \left[d^{(\sigma)\dagger}_{\Omega,\vec{k}_\perp}, d^{(\sigma')\dagger}_{\Omega',\vec{k}'_\perp} \right] = 0, \end{aligned} \tag{B.5}$$

we get

$$G^M\left(x,x'\right) = i_M\langle 0|\,\phi^{(+)}\left(x\right)\phi^{(+)}\left(x'\right)|0\rangle_M$$
$$= i\int_0^\infty d\Omega\int d^2k_\perp\left[\left(1+N\left(\Omega\right)\right)g_k\left(x,x'\right)+N\left(\Omega\right)g_k^\star\left(x,x'\right)\right], \tag{B.6}$$

with

$$g_k\left(x,x'\right) = \frac{1}{4\pi^4 a}\sinh\left(\pi\frac{\Omega}{a}\right)e^{-i\Omega(\tau-\tau')}e^{i\vec{k}_\perp\left(\vec{x}_\perp-\vec{x}'_\perp\right)}K_{i\Omega/a}\left(\mu_{k_\perp}\rho\right)K_{i\Omega/a}\left(\mu_{k_\perp}\rho'\right)$$
$$= e^{-i\Omega(\tau-\tau')}h_k\left(x,x'\right), \tag{B.7}$$

where we have used the parity property of the modified Bessel function $K_\mu(x)$ with respect to its order μ.
Expression (B.6) can be further simplified by performing the change of integration variables $k_2 \to -k_2$ and $k_3 \to -k_3$ to the term

$$\int d^2k_\perp g_k^\star\left(x,x'\right). \tag{B.8}$$

Under this transformation, $\mu_{k_\perp}, \vec{k}_\perp\left(\vec{x}_\perp-\vec{x}'_\perp\right)$ and $d\vec{k}_\perp^2$ remain unchanged, and the overall sign of the integral is preserved after interchanging the integration extrema. Thus, the propagator becomes

$$G^M\left(x,x'\right) =$$
$$= i\int_0^\infty d\Omega\int d^2k_\perp\left[\left(1+N\left(\Omega\right)\right)e^{-i\Omega(\tau-\tau')}+N\left(\Omega\right)e^{i\Omega(\tau-\tau')}\right]h_k\left(x,x'\right), \tag{B.9}$$

and, after simple algebraic manipulations of the term in squared brackets, i.e.,

$$\begin{aligned}\left(1+N\left(\Omega\right)\right)e^{-i\Omega(\tau-\tau')}+N\left(\Omega\right)e^{i\Omega(\tau-\tau')} &=\\ = e^{\pi\frac{\Omega}{a}}\frac{e^{\pi\frac{\Omega}{a}}e^{-i\Omega(\tau-\tau')}+e^{-\pi\frac{\Omega}{a}}e^{i\Omega(\tau-\tau')}}{e^{2\pi\frac{\Omega}{a}}-1} &=\\ = \frac{\cosh\left[\frac{\Omega}{a}\left(\pi-ia\left(\tau-\tau'\right)\right)\right]}{\sinh\left(\pi\frac{\Omega}{a}\right)}\end{aligned} \tag{B.10}$$

we can write the final expression for the free Minkowski propagator as

$$G^M\left(x,x'\right) =$$
$$\frac{i}{4\pi^4 a}\int_0^\infty d\Omega\int d^2k_\perp\cosh\left[\frac{\Omega}{a}\left(\pi-ia\left(\tau-\tau'\right)\right)\right]e^{i\vec{k}_\perp\left(\vec{x}_\perp-\vec{x}'_\perp\right)}K_{i\frac{\Omega}{a}}\left(\mu_k\rho\right)K_{i\frac{\Omega}{a}}\left(\mu_k\rho'\right). \tag{B.11}$$

A comparable yet simpler procedure can be used to calculate the propagator with respect to the Rindler vacuum. Without requiring the use of Bogoliubov transformations, and employing the canonical commutation relations (1.26),

$$\begin{aligned} \left[b_k^{(\sigma)}, b_{k'}^{(\sigma')\dagger}\right] &= \delta_{\sigma\sigma'}\delta\left(\Omega-\Omega'\right)\delta\left(\vec{k}_\perp-\vec{k'}_\perp\right) \\ \left[b_k^{(\sigma)}, b_{k'}^{(\sigma')}\right] &= \left[b_k^{(\sigma)\dagger}, b_{k'}^{(\sigma')\dagger}\right] = 0 \end{aligned} \tag{B.12}$$

together with the fact that $b_k^{(\sigma)}\left|0\right\rangle_R = 0$, one can readily conclude that

$$G^R\left(x,x'\right) = i_R\langle 0|\,\phi^{(+)}\left(x\right)\phi^{(+)}\left(x'\right)|0\rangle_R = \frac{i}{4\pi^4 a}\int_0^\infty d\Omega\int d^2k_\perp \sinh\left(\pi\frac{\Omega}{a}\right)e^{-i\Omega(\tau-\tau')}e^{i\vec{k}_\perp\left(\vec{x}_\perp-\vec{x}'_\perp\right)}K_{i\frac{\Omega}{a}}\left(\mu_k\rho\right)K_{i\frac{\Omega}{a}}\left(\mu_k\rho'\right). \tag{B.13}$$

COINCIDENCE LIMITS

At this stage, the coincidence limit of these propagators can be evaluated. By considering

$$\begin{cases} \tau' = \tau \\ \rho' = \rho = \dfrac{1}{a} \\ \vec{x}'_\perp = \vec{x}_\perp, \end{cases} \tag{B.14}$$

the Green's functions in the coincidence limit simply become

$${}_M\langle 0|\,\phi^{(+)}\left(x\right)\phi^{(+)}\left(x'\right)|0\rangle_M = \frac{1}{4\pi^4 a}\int_0^\infty d\Omega\int d^2k_\perp \cosh\left(\pi\frac{\Omega}{a}\right)K^2_{i\frac{\Omega}{a}}\left(\frac{\mu_k}{a}\right), \tag{B.15}$$

and

$${}_R\langle 0|\,\phi^{(+)}\left(x\right)\phi^{(+)}\left(x'\right)|0\rangle_R = \frac{1}{4\pi^4 a}\int_0^\infty d\Omega\int d^2k_\perp \sinh\left(\pi\frac{\Omega}{a}\right)K^2_{i\frac{\Omega}{a}}\left(\frac{\mu_k}{a}\right). \tag{B.16}$$

Integration over the transverse momenta k_2 and k_3 can now be performed. To simplify the process, the $k_2 - k_3$ plane is mapped to polar coordinates, followed by a change of variables $\sqrt{\vec{k}_{\perp}^{2}} \equiv k \to u = \sqrt{k^2+m^2}/a$, i.e.,

$$\begin{aligned}\int d^2k_{\perp} K^2_{i\frac{\Omega}{a}}\left(\frac{\sqrt{\vec{k}_{\perp}^{2}+m^2}}{a}\right) &= 2\pi \int_0^{\infty} dk\, k K^2_{i\frac{\Omega}{a}}\left(\frac{\sqrt{k^2+m^2}}{a}\right) \\ &= 2\pi a^2 \int_{\frac{m}{a}}^{\infty} du\, u K^2_{i\frac{\Omega}{a}}\left(u\right).\end{aligned} \tag{B.17}$$

The last integration is particularly challenging to evaluate in its current form. However, we can simplify it by considering the high-acceleration limit, $m/a \ll 1$, which is the Unruh analogue of the high-temperature limit frequently used in symmetry restoration calculations [1]. Under this approximation, the integral in Eq. (B.17) reduces to

$$\int d^2k_{\perp} K^2_{i\frac{\Omega}{a}}\left(\frac{\sqrt{\vec{k}_{\perp}^{2}+m^2}}{a}\right) \approx 2\pi a^2 \int_0^{\infty} du\, u K^2_{i\frac{\Omega}{a}}\left(u\right) \tag{B.18}$$

which can be computed using the known result [68]:

$$\int_0^{\infty} dx\, x K_{\nu}\left(ax\right) K_{\nu}\left(bx\right) = \frac{\pi \left(ab\right)^{-\nu}\left(a^{2\nu}-b^{2\nu}\right)}{2\sin\left(\pi\nu\right)\left(a^2-b^2\right)}. \tag{B.19}$$

To evaluate this expression, we take the double limit $a \to b \to 1$, yielding

$$\lim_{b\to 1}\left(\lim_{a\to b} \frac{\pi \left(ab\right)^{-\nu}\left(a^{2\nu}-b^{2\nu}\right)}{2\sin\left(\pi\nu\right)\left(a^2-b^2\right)}\right) = \frac{\pi\nu}{2\sin\left(\pi\nu\right)}. \tag{B.20}$$

Consequently, the integration over the transverse momenta simplifies to a closed form

$$\int d^2k_{\perp} K^2_{i\frac{\Omega}{a}}\left(\frac{\sqrt{\vec{k}_{\perp}^{2}+m^2}}{a}\right) \approx \frac{\pi^2 \Omega a}{\sinh\left(\pi\frac{\Omega}{a}\right)}. \tag{B.21}$$

With this result, the propagators in the coincidence limit become

$${}_M\langle 0|\,\phi^2\,|0\rangle_M = \frac{1}{\left(2\pi\right)^2}\int_0^{\infty} d\Omega\, \Omega\left(1+\frac{2}{e^{\frac{2\pi\Omega}{a}}-1}\right) \tag{B.22}$$

and

$${}_R\langle 0|\,\phi^2\,|0\rangle_R = \frac{1}{\left(2\pi\right)^2}\int_0^{\infty} d\Omega\, \Omega. \tag{B.23}$$

C NON-RELATIVISTIC LIMIT: INTERACTING KLEIN-GORDON EQUATION

To effectively compare solutions of the relativistic KGE with those of the Schrödinger equation, along with their associated energy eigenstates, it is useful to derive the non-relativistic limit of the Klein-Gordon equation. This approach allows for a direct comparison with the Schrödinger equation, shedding light on the differences in the definitions of fields and coupling constants between the two theories.
First, we note that the mass dimensions of these quantities differ fundamentally between the two theories. Specifically, in the 1+1-dimensional case:

Relativistic	$[\Phi_R] = 0$	$[\lambda_R] = 2$
Non-relativistic	$[\Phi_{NR}] = \frac{1}{2}$	$[\lambda_{NR}] = 0$

Keeping this in mind, we proceed with the non-relativistic limit and verify that the resulting expressions remain consistent with these dimensional considerations.
Starting from (2.144), the generic ϕ^4 KGE for the non-rotating field Φ_R can be expressed as:

$$\frac{1}{2}\ddot{\Phi}_R - \frac{1}{2}\Phi_R'' + \frac{m^2}{2}\Phi_R + \frac{\lambda_R}{2}\left|\Phi_R\right|^2\Phi_R = 0, \tag{C.1}$$

where $\dot{}$ denotes the time derivative, and $'$ represents the spatial derivative.
Given that we are seeking stationary solutions of the form

$$\Phi_R = \frac{1}{\sqrt{2\omega_R}}e^{-i\omega_R t}f(x), \tag{C.2}$$

where the factor $(2\omega_R)^{-\frac{1}{2}}$ comes from the KG inner product normalization (2.163), the KGE equation becomes

$$-\omega_R^2 f(x) - f(x)'' + m^2\Phi_R + \frac{\lambda_R}{2\omega_R}\left|f(x)\right|^2 f(x) = 0, \tag{C.3}$$

where we have simplified an overall phase $\exp(-i\omega_R t)$ and a factor $2(2\omega_R)^{-\frac{1}{2}}$.

Performing the non-relativistic limit requires an expansion in the regime of large m. Considering

$$\begin{aligned} \omega_R &= m + \frac{p^2}{2m} + O\left(m^{-3}\right) = m + \omega_{NR} + O\left(m^{-3}\right), \\ \frac{1}{\omega_R} &= \frac{1}{m} + O\left(m^{-3}\right), \end{aligned} \tag{C.4}$$

and neglecting terms of order $O\left(m^{-2}\right)$, we can write the KGE equation as

$$-\left(\cancel{m^2} + 2m\omega_{NR} + \omega_{NR}^2\right) f\left(x\right) - f''\left(x\right) + \cancel{m^2 f\left(x\right)} + \frac{\lambda_R}{2m}\left|f\left(x\right)\right|^2 f\left(x\right) = 0, \tag{C.5}$$

or equivalently,

$$-\left(\omega_{NR} + \frac{\omega_{NR}^2}{2m}\right) f\left(x\right) - \frac{1}{2m} f''\left(x\right) + \frac{\lambda_R}{4m^2}\left|f\left(x\right)\right|^2 f\left(x\right) = 0. \tag{C.6}$$

The term $\omega_R^2/2m$ is of order $O\left(m^{-3}\right)$ and can therefore also be neglected. The final expression of the non-relativistic limit of the KGE is then given by

$$-\omega_{NR} f\left(x\right) - \frac{1}{2m} f''\left(x\right) + \frac{\lambda_R}{4m^2}\left|f\left(x\right)\right|^2 f\left(x\right) = 0. \tag{C.7}$$

At this point, starting from the Schrödinger equation

$$\dot{\Phi}_{NR} - \frac{1}{2m}\Phi''_{NR} + \frac{\lambda_{NR}}{2}\left|\Phi_{NR}\right|^2 \Phi_{NR} = 0, \tag{C.8}$$

and requiring the solutions to be stationary again, i.e.,

$$\Phi_{NR} = e^{-i\omega_{NR}t} f\left(x\right), \tag{C.9}$$

we obtain

$$-\omega_{NR} f\left(x\right) - \frac{1}{2m} f''\left(x\right) + \frac{\lambda_{NR}}{2}\left|f\left(x\right)\right|^2 f\left(x\right) = 0. \tag{C.10}$$

We immediately notice that equations (C.7) and (C.10) are identical up to a redefinition of the coupling constants. Specifically, we can find the relation

$$\lambda_R = 2m^2 \lambda_{NR}. \tag{C.11}$$

We conclude by noting that, using (C.4), the non-relativistic limit of the field Φ_R is given by

$$\Phi_R \xrightarrow{m\to\infty} \frac{1}{\sqrt{2m}} e^{-imt}\Phi_{NR}. \tag{C.12}$$

We notice that both the mass dimensions of Φs (C.12) and λs (C.11) are compatible with the observations previously made.

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