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Landau–Lifshitz and Weinberg Energy Distributions for the Static Regular Simpson–Visser Space-Time Geometry

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Abstract: The symmetric Landau–Lifshitz and Weinberg energy–momentum complexes are utilized in order to determine the energy distribution in a four-dimensional, static and spherically symmetric regular Simpson–Visser space-time geometry. For different values of the metric parameter a , the static Simpson–Visser space-time geometry corresponds to the Schwarzschild black hole solution, to a regular black hole solution with a one-way spacelike throat, to a one-way wormhole solution with an extremal null throat, or to a traversable Morris–Thorne wormhole solution. Both symmetric prescriptions yield a zero momentum, while the energy distributions calculated have an expression dependent on the mass m , the radial coordinate r , and the metric parameter a . Some special limiting cases of the results derived are considered, while a possible astrophysical application to questions of gravitational lensing is indicated.

Keywords: Simpson–Visser static solution; energy–momentum localization; symmetric Landau–Lifshitz energy–momentum complex; symmetric Weinberg energy–momentum complex



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1. Introduction

The energy–momentum localization of the gravitational field belongs to the oldest but still unresolved problems in classical general relativity. Indeed, given any space-time geometry, we are lacking a proper definition for the energy density of the gravitational field. In 1915, Einstein was the first to attack the problem (see [1,2]) by introducing a so called energy–momentum complex, i.e., a pseudo-tensorial quantity for which there exists a local conservation law. Since then, Einstein’s attempt was followed by a number of various but similar pseudo-tensorial prescriptions of which we notice the complexes of Landau and Lifshitz [3], Papapetrou [4], Bergmann and Thomson [5], Møller [6], and Weinberg [7]. An essential common ingredient of these pseudotensorial definitions, except Møller’s pseudo-tensorial definition, is their dependence on the coordinate system used for their application.

Indeed, Cartesian or quasi-Cartesian coordinates are required, while for the implementation of the Møller energy–momentum complex any coordinate system can be employed, given any four-dimensional space-time. This coordinate dependence has raised a lot of criticism [8,9]. Nevertheless, an abundance of physically reasonable and persuasive results [10–24] obtained by using different localization prescriptions for the gravitational field of various $(d + 1)$ -dimensional gravitational backgrounds, where $d = 1, 2, 3$, has led to the revival as well as rehabilitation of the energy–momentum complexes in the past decades. At this point, a significant result must be stressed according to which the application of

different energy–momentum complexes yields the same energy–momentum distribution for any metric belonging to the Kerr–Schild class but also for more general metrics [25–27]. The Landau–Lifshitz and Weinberg definitions, also yielded several physically meaningful results for various space-time geometries [23,28,29]. Being symmetric energy–momentum complexes, both the Landau–Lifshitz and Weinberg energy–momentum complexes are shown to be suitable tools for defining the angular momentum.

Besides the energy–momentum complexes, other approaches have also been applied to the problem of energy–momentum localization. Two of them stand out: the method of super-energy tensors [30–32] and the quasi-local mass approach [33–36]. In fact, results derived by the Einstein, Landau–Lifshitz, Papapetrou, Bergmann–Thomson, Weinberg, and Møller energy–momentum complexes agree with those obtained by the application of the quasi-local mass approach. Worth noticing are the investigations based on the notion of a quasi-local energy–momentum with respect to a closed 2-surface bounding a 3-volume in space-time. Within this framework, the main role is played by the concept of the Wang–Yau quasi-local energy [37]. Another remarkable attempt to revive the use of energy–momentum complexes is associated the so called covariant Hamiltonian approach. In this context, a Minkowski reference geometry is isometrically matched on the 2-surface boundary and it is found that quasi-local superpotentials stemming from various energy–momentum complexes linearly conform with the Freud superpotential, thus yielding the same quasi-local energy for any closed 2-surface (see, e.g., [38]).

Before concluding this part, it should be indicated that the effort to avoid the coordinate dependence of the energy–momentum complexes has opened the way to the development of alternative calculation techniques within the framework of the Teleparallel Equivalent of General Relativity and several outcomes showing a resemblance to results obtained by the general-relativistic energy–momentum localization methods have been produced (see, e.g., [39–47]).

This paper is organized as follows: We introduce the geometry of the static, regular Simpson–Visser space-time and present the line element of the metric in Section 2. Section 3 consists of the definitions and the basic properties of the symmetric Landau–Lifshitz complex and the general analytic formulae for the calculation of the energy and momentum distributions. We present the calculated superpotentials, energy, and momenta together with the graph showing the behavior of the energy with the radial coordinate r , near the origin, for various values of the parameter of the metric a . In Section 4, the symmetric Weinberg energy–momentum complex is presented, together with the computed superpotentials, energy, and momenta and the graph of the energy distribution near the origin. Section 5 Concluding Remarks encompasses a discussion of the obtained results including some comments on their conjectured applicability in astrophysical contexts. Further, focusing on some rather interesting limiting values of the metric parameter a , and of the radial coordinate r , we give the corresponding energies. Geometrized units ($c = G = 1$) have been utilized and the metric signature is $(+, -, -, -)$. In both symmetric prescriptions we have used Schwarzschild Cartesian coordinates (t, x, y, z) , while Greek indices run from 0 to 3 and Latin indices range from 1 to 3.

2. The Simpson–Visser Gravitational Background

This section is devoted to the description of the Simpson–Visser gravitational background. The Simpson–Visser space-time has been shown to play an important role in various studies dedicated to the understanding of the gravitational lensing of light rays reflected by a photon sphere of black holes and wormholes [48–50]. The Simpson–Visser metric is characterized by a parameter $a > 0$ which is responsible for the regularization of the central singularity and the ADM mass $m \geq 0$. Depending on the values of the positive metric parameter $a > 0$ the static Simpson–Visser solution can describe: (a) a Schwarzschild metric in the case $a = 0$ and $m \neq 0$, (b) a non-singular black hole metric for $a < 2m$, and in this case the singularity is replaced by a bounce to a different universe, e.g., a “black-bounce” or a “hidden wormhole”. The solution describes a regular

black hole space-time which does not belong to the traditional family of regular black hole solutions, while it yields the ordinary Schwarzschild solution as a special case, (c) a one-way traversable wormhole metric with a null throat in the case $a = 2m$, (d) a two-way traversable wormhole metric space-time of the Morris–Thorne type when the mathematical condition $a > 2m$ is met, and (e) an Ellis–Bronnikov wormhole metric for $a \neq 0$ and $m = 0$.

Regarding the massless Ellis–Bronnikov wormhole it is worth noticing that it is an exact solution with zero ADM mass but having a positive “Wheelerian mass” (the latter was introduced by J.A. Wheeler for his geon, see, e.g., [51]) that is considered responsible for the gravitational reaction of this wormhole. Indeed, massless wormholes of this kind can have as a source (rather exotic) fields, such as a ghost scalar field (which has a negative energy density) or a dilaton field. These fields can assign a non-zero mass to the wormhole thus imparting to the latter the ability to exhibit a strong lensing behavior (see, e.g., [52,53]).

Regular black holes and traversable wormholes have been the subject of various studies of an apparent mathematical as well as astrophysical interest. Despite this purely theoretical motivation connected with a unified handling of such objects that has led to the Simpson–Visser solution and has stimulated our study of its energy, the formulation, in this context, of associated phenomenological models is conceivable in the near future.

The Simpson–Visser static and spherically symmetric gravitational background is described by the line element

$$ds^2 = B(r)dt^2 - A(r)dr^2 - (r^2 + a^2)(d\theta^2 + \sin^2\theta d\phi^2), \quad (1)$$

where $B(r) = f(r)$, $A(r) = \frac{1}{f(r)}$, while the metric function is given by

$$f(r) = 1 - \frac{2m}{(r^2 + a^2)^{1/2}}. \quad (2)$$

Note that the r coordinate can take positive as well as negative real values $r \in (-\infty, +\infty)$. In this paper, as we consider a classical spherically symmetric black hole, we are physically allowed to restrict our calculations to only positive-definite values of the r coordinate, starting with the value $r = 0$ at the center of the black hole.

Further, in their interesting paper Mazza, Franzin, and Liberati [54] have elaborated a novel family of rotating black hole mimickers and developed a proposal for a spinning generalisation of the Simpson–Visser space-time metric that can be used for comparisons with future observational data on strong-field gravitational lensing [55]. To develop this spinning generalization of the Simpson–Visser metric they used the Newman–Janis procedure. The rotating Simpson–Visser metric reduces to the Simpson–Visser metric in the case of a vanishing value of the parameter $l = 0$ and to the Kerr metric for $a = 0$.

3. Energy-Momentum Distribution of the Simpson–Visser Space-Time in the Landau–Lifshitz Prescription

In this section, we present the symmetric Landau–Lifshitz energy–momentum complex and the general analytic formulae for the calculation of the energy and momentum distributions.

The Landau–Lifshitz energy–momentum complex is given by [3]

$$L^{\mu\nu} = \frac{1}{16\pi} S^{\mu\nu\rho\sigma}_{,\rho\sigma}. \quad (3)$$

The Landau–Lifshitz superpotentials are

$$S^{\mu\nu\rho\sigma} = -g(g^{\mu\nu}g^{\rho\sigma} - g^{\mu\rho}g^{\nu\sigma}). \quad (4)$$

Note that L^{00} and L^{0i} components represent the energy and the momentum densities, respectively. For the Landau–Lifshitz pseudotensorial definition the local conservation law holds

$$L^{\mu\nu}_{,\nu} = 0. \quad (5)$$

By integrating $L^{\mu\nu}$ over the 3-space, one obtains the expressions for the energy and momentum:

$$P^\mu = \iiint L^{\mu 0} dx^1 dx^2 dx^3. \quad (6)$$

Using Gauss' theorem one obtains

$$P^\mu = \frac{1}{16\pi} \iint S_{,\sigma}^{\mu 0 i \sigma} n_i dS = \frac{1}{16\pi} \iint U^{\mu 0 i} n_i dS. \quad (7)$$

In the Landau–Lifshitz prescription, the calculations have to be made using the line element (1) which is transformed into Schwarzschild Cartesian coordinates with the aid of the coordinate transformation $x = r \sin \theta \cos \phi$, $y = r \sin \theta \sin \phi$, $z = r \cos \theta$. Using these coordinate transformations the metric (1) becomes

$$ds^2 = B(r)dt^2 - (dx^2 + dy^2 + dz^2) - \frac{A(r) - 1}{r^2} (xdx + ydy + zdz)^2. \quad (8)$$

The non-vanishing components of the the Landau–Lifshitz superpotentials are given by

$$U^{001} = x \frac{r^2 + a^2}{r^3} \left[\frac{1}{1 - 2m(r^2 + a^2)^{-1/2}} + \frac{a^2 - r^2}{r^2} \right], \quad (9)$$

$$U^{002} = y \frac{r^2 + a^2}{r^3} \left[\frac{1}{1 - 2m(r^2 + a^2)^{-1/2}} + \frac{a^2 - r^2}{r^2} \right], \quad (10)$$

$$U^{003} = z \frac{r^2 + a^2}{r^3} \left[\frac{1}{1 - 2m(r^2 + a^2)^{-1/2}} + \frac{a^2 - r^2}{r^2} \right]. \quad (11)$$

Using (9)–(11) and setting $\mu = 0$ in (7), we obtain the energy

$$E_{LL} = \frac{r^2 + a^2}{2r} \left[\frac{1}{1 - 2m(r^2 + a^2)^{-1/2}} + \frac{a^2 - r^2}{r^2} \right]. \quad (12)$$

Taking into account that $U^{\mu 0 i} = 0$ for $\mu \neq 0$ and applying Equation (7) we obtain that all the momentum components vanish:

$$P_x = P_y = P_z = 0. \quad (13)$$

Figure 1 shows the Landau–Lifshitz energy distribution given by (12) as a function of the radial coordinate r near the origin for four values of the metric parameter a and $m = 1$. For the case $a = 0$, the energy is very low and almost coincides with the r -axis.

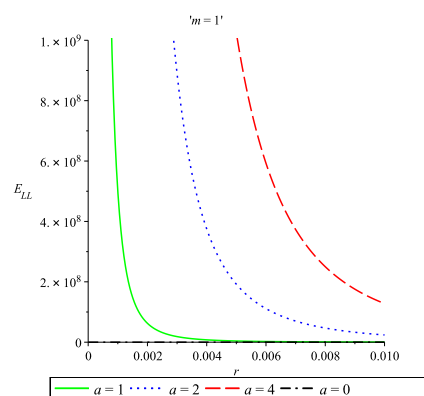


Figure 1. Landau–Lifshitz energy distribution vs. the radial coordinate r near the origin.

4. Energy-Momentum Distribution of the Simpson–Visser Space-Time in the Weinberg Prescription

This section is devoted to the introduction of the the symmetric Weinberg energy–momentum complex, together with the computed superpotentials, energy, and momenta.

The Weinberg energy–momentum complex [7] is given by

$$W^{\mu\nu} = \frac{1}{16\pi} D^{\lambda\mu\nu}_{,\lambda}. \quad (14)$$

The corresponding superpotentials are

$$D^{\lambda\mu\nu} = \frac{\partial h^\kappa_\kappa}{\partial x_\lambda} \eta^{\mu\nu} - \frac{\partial h^\kappa_\kappa}{\partial x_\mu} \eta^{\lambda\nu} - \frac{\partial h^{\kappa\lambda}}{\partial x^\kappa} \eta^{\mu\nu} + \frac{\partial h^{\kappa\mu}}{\partial x^\kappa} \eta^{\lambda\nu} + \frac{\partial h^{\lambda\nu}}{\partial x_\mu} - \frac{\partial h^{\mu\nu}}{\partial x_\lambda}, \quad (15)$$

with

$$h_{\mu\nu} = g_{\mu\nu} - \eta_{\mu\nu}. \quad (16)$$

The W^{00} and W^{0i} components are the energy and the momentum densities, respectively. In the Weinberg prescription the local conservation law is respected:

$$W^{\mu\nu}_{,\nu} = 0. \quad (17)$$

The expression for the energy–momentum is obtained by integrating $W^{\mu\nu}$ over the 3-space

$$P^\mu = \iiint W^{\mu 0} dx^1 dx^2 dx^3. \quad (18)$$

with the aid of Gauss' theorem and integrating over the surface of a sphere of radius r , the energy–momentum distribution has the expression:

$$P^\mu = \frac{1}{16\pi} \iint D^{i0\mu} n_i dS. \quad (19)$$

In the case of the Weinberg prescription, in order to compute the energy–momentum the metric given by (1) has also to be converted into Schwarzschild Cartesian coordinates using the coordinate transformation $x = r \sin \theta \cos \phi$, $y = r \sin \theta \sin \phi$, $z = r \cos \theta$, thus the metric is given by (8).

The nonvanishing superpotential components are as follows:

$$D^{100} = \frac{x}{r} \left[\frac{1}{1 - 2m(r^2 + a^2)^{-1/2}} + \frac{a^2 - r^2}{r^2} \right], \quad (20)$$

$$D^{200} = \frac{y}{r} \left[\frac{1}{1 - 2m(r^2 + a^2)^{-1/2}} + \frac{a^2 - r^2}{r^2} \right], \quad (21)$$

$$D^{300} = \frac{z}{r} \left[\frac{1}{1 - 2m(r^2 + a^2)^{-1/2}} + \frac{a^2 - r^2}{r^2} \right]. \quad (22)$$

Substituting these expressions into (19), the expression for the energy distribution inside a 2-sphere of radius r is given by

$$E_W = \frac{r}{2} \left[\frac{1}{1 - 2m(r^2 + a^2)^{-1/2}} + \frac{a^2 - r^2}{r^2} \right]. \quad (23)$$

The vanishing of the spatial components of the superpotential leads to the vanishing of all momentum components, and we have:

$$P_x = P_y = P_z = 0. \quad (24)$$

Figure 2, exhibits the behavior of the Weinberg energy distribution given by (23) as a function of the radial coordinate r near the origin, for four values of the metric parameter a and $m = 1$. For the case $a = 0$, the energy is very low and almost coincides with the r -axis.

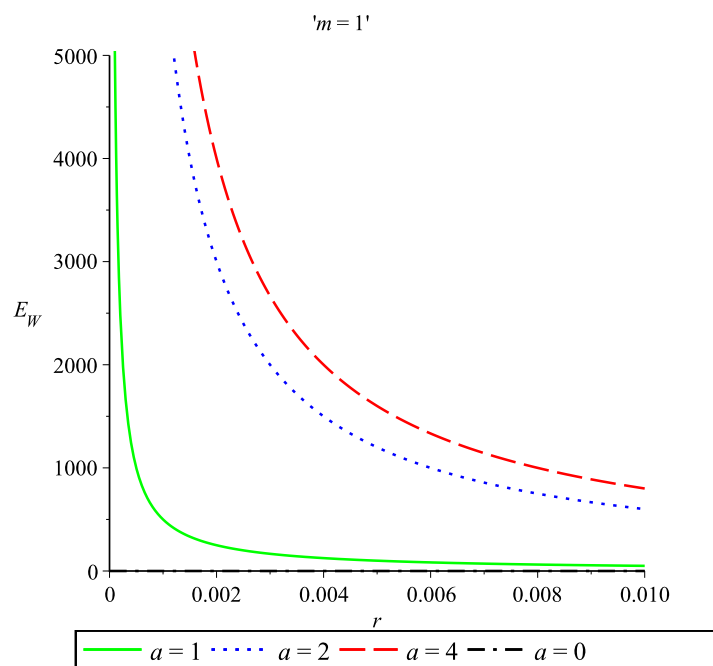


Figure 2. Weinberg energy distribution vs. the radial coordinate r near the origin.

5. Concluding Remarks

In this work, we have investigated the energy–momentum distribution for the static Simpson–Visser space-time. The Landau–Lifshitz and Weinberg prescriptions have been employed for the calculations of energy distribution and momenta. We found that both prescriptions yield the same result regarding the momentum components, namely that all the momenta vanish. In fact, the momenta also vanish using the Einstein and Møller energy–momentum complexes, as shown in [56]. This means that the specific metric does not “allow” the existence of any momentum density. It is not a result depending on the choice of the energy–momentum complex and certainly it is not always expected though it is often obtained. The expressions of the energy distribution are well-defined and physically meaningful presenting a dependence on the mass m , the metric parameter a , and on the radial coordinate r .

In Table 1, we present the limiting behavior of the energy for $r \rightarrow \infty$ and $r \rightarrow 0$, and in the particular cases $a = 0$ and $m \neq 0$, $a = r$ and $m \neq 0$, and $a \neq 0$ and $m = 0$.

Table 1. Landau–Lifshitz energy E_{LL} and Weinberg energy E_W for limiting values of r and specific values of a and m .

Energy	$r \rightarrow \infty$	$r \rightarrow 0$	$a = 0, m \neq 0$	$a = r, m \neq 0$	$a \neq 0, m = 0$
E_{LL}	m	∞	$\frac{m}{1 - \frac{2m}{r}}$	$\frac{r}{1 - 2m/\sqrt{2}r}$	$\frac{a^2}{2r^3}(r^2 + a^2)$
E_W	m	∞	$\frac{m}{1 - \frac{2m}{r}}$	$\frac{r}{2[1 - 2m/\sqrt{2}r]}$	$\frac{a^2}{2r}$

We notice that the behavior of the energy distribution for the Simpson–Visser gravitational background is a very interesting one and is strongly affected by the value of the metric parameter a . For $a = 0$ and $m \neq 0$ the Simpson–Visser solution corresponds to the Schwarzschild metric and the expression of the energy distribution in both Landau–Lifshitz

and Weinberg pseudotensorial prescriptions is equal to $\frac{m}{1-\frac{2m}{r}}$. This expression of the energy distribution is in perfect agreement with the result obtained by Virbhadra for the energy distribution of the Schwarzschild metric [26]. Figure 3 shows the energy in this case for $m = 1$.

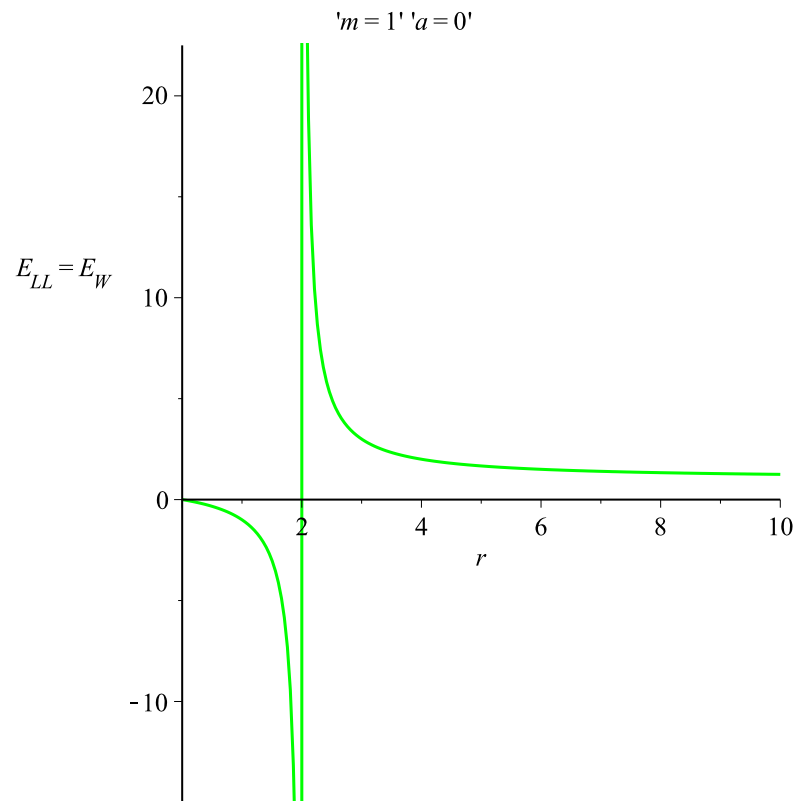


Figure 3. Energy distribution ($E_{LL} = E_W$) vs. r for $m = 1$ and $a = 0$.

In the cases $a = 0$, $m \neq 0$ (Schwarzschild black hole metric) and $a < 2m$ (which corresponds to the regular black hole metric), the Landau–Lifshitz E_{LL} and Weinberg E_W energy distributions exhibit a discontinuity for $r = (-a^2 + 4m^2)^{1/2}$ whose values depend on the values of the mass m and the metric parameter a . The Landau–Lifshitz E_{LL} and Weinberg E_W energy distributions are positive and monotonically decreasing near the origin until they vanish to become negative, then exhibiting a discontinuity, and after this point they become positive and are decreasing again as r increases and finally they reach the value of the ADM mass m for $r \rightarrow \infty$. In Figure 4, both energy functions (Landau–Lifshitz in red and Weinberg in green) are presented for $a = 1$ and $m = 1$. The aforementioned discontinuity which appears relatively close to the origin is a feature of the particular space-time metric and it does not cause any particular problem to our current interpretation of the results obtained.

In the case of a one-way traversable wormhole metric with a null throat characterized by $a = 2m$, the Landau–Lifshitz E_{LL} and Weinberg E_W energy distributions take only positive values and decrease from a maximum value to the ADM mass m . The two-way traversable wormhole metric space-time of the Morris–Thorne type obtained for $a > 2m$ presents also only positive values for the Landau–Lifshitz E_{LL} and Weinberg E_W energy distributions, that in this case are also decreasing functions from a maximum value to the ADM mass m .

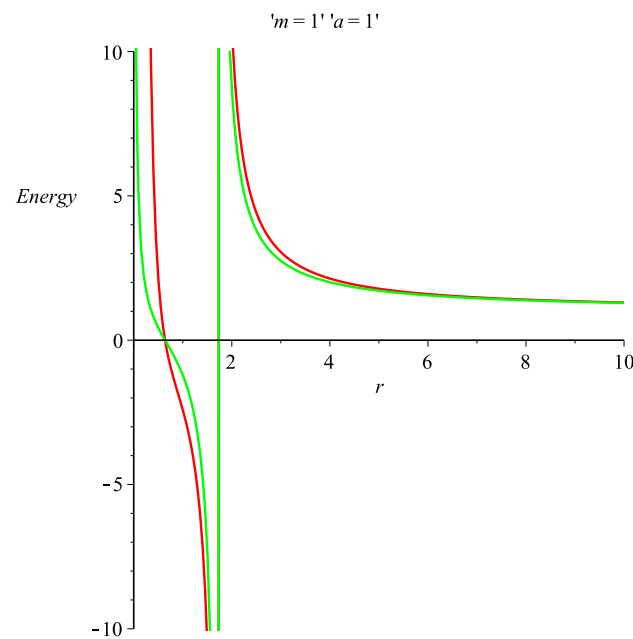


Figure 4. Energy distribution vs. r for $m = 1$ and $a = 1$ (E_{LL} in red and E_W in green).

Note that for $a = r$ and $m \neq 0$ one obtains $E_{LL} = \frac{r}{1-2m/\sqrt{2}r} = 2E_W$. In the particular case of the Ellis–Bronnikov wormhole metric with $a \neq 0$ and $m = 0$ the calculations show that $E_{LL} = \frac{a^2}{2r^3}(r^2 + a^2) > 0$ and $E_W = \frac{a^2}{2r} > 0$.

In Figures 5 and 6, respectively, we plot the energy distributions in the Landau–Lifshitz and Weinberg prescriptions in the particular case of the Ellis–Bronnikov wormhole metric for three different values of the metric parameter $a \neq 0$ and $m = 0$. The Landau–Lifshitz E_{LL} energy takes only positive values and is equal to $\frac{E_W}{r^2}(r^2 + a^2)$. Note that in this case the Weinberg E_W energy is equal to the Einstein energy E_E that we obtained in [56], being always positive and equal to $a^2/2r$.

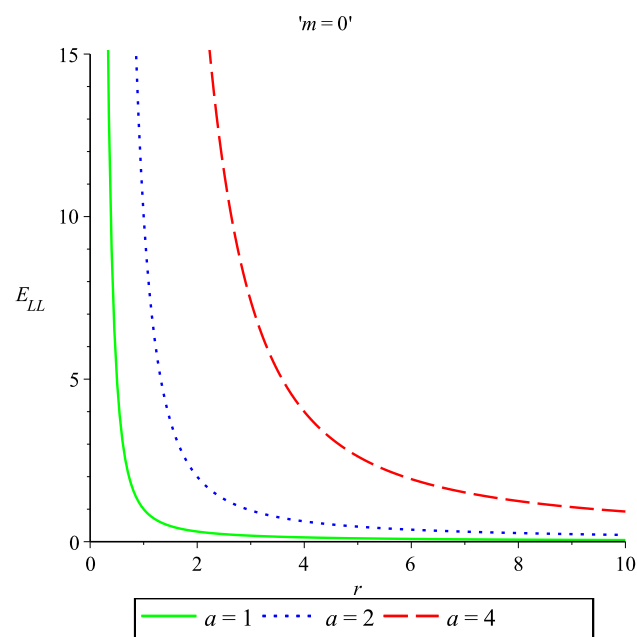


Figure 5. E_{LL} vs. r for $a \neq 0$ and $m = 0$.

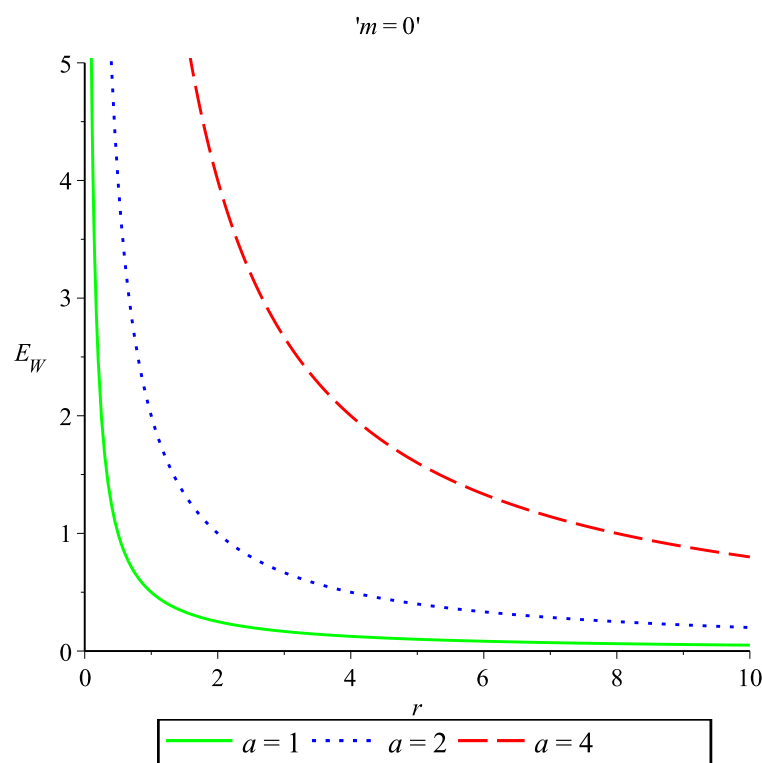


Figure 6. E_W vs. r for $a \neq 0$ and $m = 0$.

The results obtained by applying the Landau–Lifshitz and Weinberg prescriptions come to support the use of energy–momentum complexes for the determination of the energy of a four-dimensional gravitational background. In an astrophysical context, a positive energy region could possibly serve as a convergent gravitational lens, while a negative energy region could play the role of a divergent gravitational lens [57]. Overall, the Simpson–Visser metric and its possible generalizations appear to provide a rather advantageous point of view for an enhanced understanding of strong-field gravitational lensing of light reflected by a photon sphere of black holes and wormholes [55,58,59]. However, one can remark that, in the application to gravitational lensing of light rays in a Simpson–Visser space-time, for very large values of a the corresponding photon sphere can become very large. Thus, from observations in usual gravitational lensing, for example in cases of micro-lensing, a possible restriction can be put on the value of a in the study of the photon sphere stability and deflection angle issues (see, e.g., [60]).

In general, for all tested values of the pair (a, m) in the cases $a = 0$ and $m \neq 0$ and $a < 2m$, there appears a small region of negativity for the energy near the origin. Indeed, we have found that both the Landau–Lifshitz and Weinberg prescriptions exhibit a discontinuity when $r = (-a^2 + 4m^2)^{1/2}$ thus leading to some difficulty in order to provide a physically meaningful interpretation of the energy in specific regions near the origin of the space-time geometry considered. Thus, for example, for $a < 2m$, corresponding to the space-time geometry exterior to a regular black hole, we see that, for the pair of values $a = 1, m = 1$, the discontinuity appears at $r = 1.73$. Overall, the behavior of the energy starts decreasing from positive values, then it becomes negative until it comes to a point of discontinuity, and after this point it decreases again from positive values. For the special case $a = 0$, the discontinuity point is slightly shifted to a larger value of r (i.e., to $r = 2$), while for $m = 0$ there is no discontinuity but only a monotonically decreasing positive energy. Further, in the cases $a = 2m$ and $a > 2m$, that describe a one-way traversable wormhole metric and a two-way traversable wormhole metric space-time of the Morris–Thorne type, respectively, both Landau–Lifshitz and Weinberg prescriptions give only positive values for the energy distribution. Figure 7 shows both energies as a

function of r in the case $a = 2$ m, while Figure 8 shows an example of the case $a > 2$ m. In both figures the Landau-Lifshitz energy is plotted in red while the Weinberg energy is plotted in green.

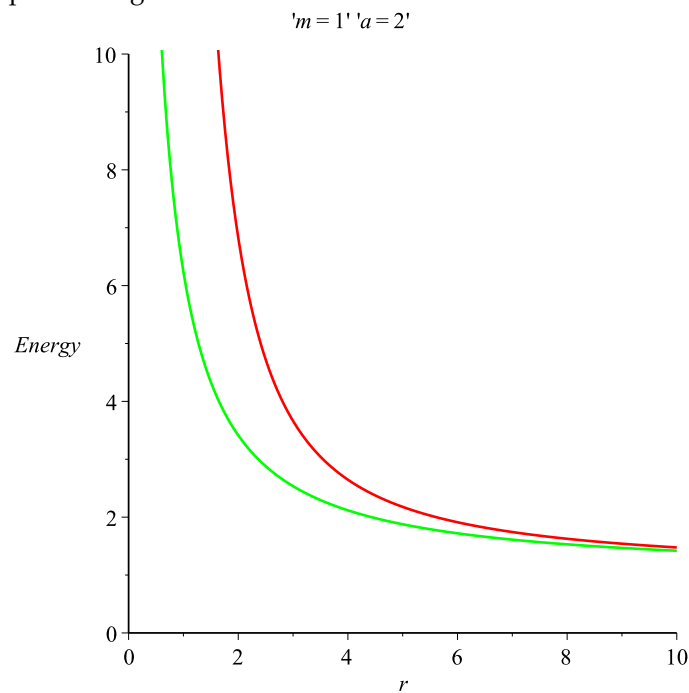


Figure 7. Energy distribution vs. r for $a = 2$ m (E_{LL} in red and E_W in green).

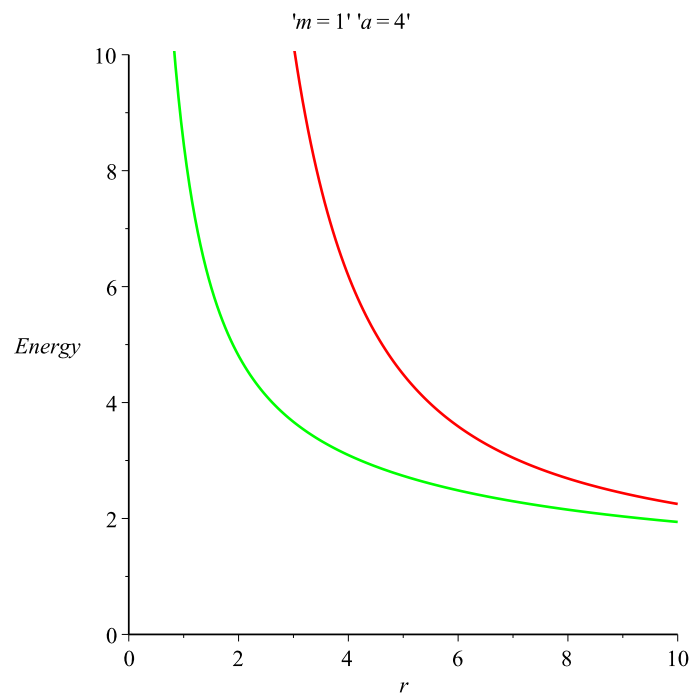


Figure 8. Energy distribution vs. r for $a > 2$ m (E_{LL} in red and E_W in green).

To our knowledge there is still no direct way of measuring the energy density of the gravitational field for a given space-time. The superpotentials in the Landau-Lifshitz and Weinberg prescriptions are constructed entirely from the space-time metric. Hence, a possible way of determining the energy for a given space-time would rely on the space-time curvature obtained from astronomical observations (see, e.g., Pirani's thought experiment based on the notion of geodesic deviation [61]). Indeed, Pirani's idea is simply illus-

trated in https://www.esa.int/ESA_Multimedia/Images/2015/09/Measuring_spacetime_curvature (accessed on 21 April 2022). In a more recent context, the space-time curvature could be determined by combining cosmic microwave background measurements with galaxy clustering data (see, e.g., [62] for a nice review). Another possible way of determining the geometry of space-time based on quantum measurements is suggested very recently in [63]. By using the curvature thus obtained and after coping with the relevant differential-geometric difficulties since this subtle question touches upon the non-trivial “equivalence problem” of metrics (see, e.g., [64]), and possibly by applying the Cartan–Karlhede algorithm [65], it could be theoretically possible to determine the metric and from it obtain the energy–momentum of the gravitational field.

As a future work, we consider to apply the other two notable energy–momentum complexes, namely those of Bergmann–Thomson and Papapetrou, for the Simpson–Visser metric and find out if there appear regions of space-time that can, due to their energy, act as regions of strong convergent (in the case of positive energy) or divergent (in the case of negative energy) gravitational lensing (as an example see our references [57,60,66]). Such space-time regions would be associated, for example, with supermassive black holes or supermassive exotic objects consisting of dark matter/energy or even strongly naked singularities (see, e.g., [67]).

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