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Nearly Complete Generalized Clifford Monoids and Applications

Dilawar J. Mir [1], Bana Al Subaiei [2,*] and Aftab H. Shah [3]

[1] Department of Mathematics, CDOE, Chandigarh University, Mohali 140413, Punjab, India; djmir15844@gmail.com
[2] Department of Mathematics and Statistics, College of Science, King Faisal University, P.O. Box 400, Al-Ahsa 31982, Saudi Arabia
[3] Department of Mathematics, Central University of Kashmir, Ganderbal 191201, Jammu and Kashmir, India; aftab@cukashmir.ac.in
* Correspondence: banajawid@kfu.edu.sa

Abstract: A semigroup S is termed a generalized Clifford semigroup (GC-semigroup) if it forms a strong semilattice of π-groups. This paper explores necessary and sufficient conditions for a GC-monoid to be nearly complete within certain subclasses. These subclasses are distinguished by the nature of their linking homomorphisms, which may be bijective, surjective, injective, or image trivial. The findings provide a deeper understanding of the structural integrity and completeness of GC-monoids, contributing valuable insights to the theoretical framework of semigroup theory. Applications of this study span various fields, including cryptography for secure algorithm design, coding theory and quantum computing for advanced quantum algorithms. The established criteria also support further research in mathematical biology and automorphic theory, demonstrating the broad relevance and utility of nearly complete GC-monoids.

Keywords: nearly complete; linking homomorphisms; inner automorphisms; semilattice of π-groups; GC-monoids

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1. Introduction

The exploration of generalized Clifford semigroups (GC-semigroups) and their subclass of GC-monoids has revealed intricate algebraic structures within semigroup theory. This study has rigorously examined the necessary and sufficient conditions for a GC-monoid to attain near completeness, shedding light on the subtle dynamics governed by its linking homomorphisms. The historical lineage of semigroup theory, particularly within generalized Clifford semigroups, traces back to seminal works that meticulously investigated their structural properties. GC-semigroups, formulated as strong semilattices of π-groups, have captivated researchers due to their distinctive composition and profound implications for both theoretical and applied mathematics.

The foundational understanding of semilattices of groups, initially elucidated by Clifford and Preston [1], has paved the way for specialized studies on semigroups, highlighting their intricate internal dynamics and algebraic behaviors. This exploration has been further nuanced by investigations into GC-monoids, a specialized subclass characterized by their identity element, which offer unique insights into semigroup structure and behavior.

Recent research has intensified efforts to identify the conditions influencing the completeness of various algebraic structures. The concept of nearly complete GC-monoids represents a refined classification within this theoretical framework. Key contributions by Shah et al. in [2–4] have been instrumental in establishing the requisite conditions for completeness across various algebraic structures. Their work highlights the fundamental roles of bijective, surjective, injective, and image-trivial homomorphisms in preserving structural coherence. Notably, they extended the Inner Automorphism Theorem from groups to monoids, introducing the concept of nearly complete monoids. Furthermore, they provided a comprehensive characterization by formulating necessary and sufficient conditions under which a strong semilattice of groups can be classified as nearly complete.

Clifford and Preston's seminal work [1] provides a comprehensive introduction to the theory of semigroups, including the role of homomorphisms. They discuss the general properties of semigroups and the significance of semigroup homomorphisms in understanding their structure. Researchers have expanded on this by examining the conditions under which inner automorphisms can be fully characterized. They have emphasized the importance of understanding the units of a semigroup and their role in defining inner automorphisms. This perspective has been influential in the study of semigroups with more complex internal structures.

The study of inner automorphisms, integral to understanding the symmetry and structural properties of semigroups, highlights their pivotal role across various branches of mathematics. The concept of nearly complete monoids, where all automorphisms are inner, elucidates essential insights into their structural characteristics and automorphism behaviors. This research underscores the enduring significance of inner automorphisms in algebraic structures, reinforcing their role as natural symmetries within these systems and opening avenues for continued exploration into their diverse applications and theoretical implications. It can be traced back to foundational work in group theory. Classic texts such as those by Birkhoff [5] and Herstein [6] lay the groundwork by discussing automorphisms in the context of groups. Inner automorphisms are defined by the conjugation action within the group, providing insights into the group's internal symmetries. This notion was later extended to semigroups, where the structural complexity necessitated a more nuanced approach. Grillet, in [7], discussed various types of semigroups, including inverse semigroups and regular semigroups. The study of inner automorphisms has also found applications in broader areas such as cryptography, formal language theory, and network theory.

More recent studies have continued to explore the nuances of inner automorphisms in various algebraic systems. The research by Cain and Maltcev [8] on the automorphism groups of finite semigroups offers new perspectives on the combinatorial aspects of inner automorphisms and their implications for the theory of finite semigroups.

The study of inner automorphisms in semigroups of (partial) mappings has been a subject of significant research. Schreier [9] and Mal'cev [10] demonstrated that every automorphism of $\mathcal{T}(X)$, the semigroup of all functions of a set X to itself is inner, where the conjugating element is taken from $\mathcal{S}(X)$, the group of all permutations on X. Building on this, Sutov [11] and Magill [12] established analogous results for $\mathcal{P}(X)$, the semigroup of all partial mappings on X. Liber [13] further extended these findings to $\mathcal{I}(X)$, the symmetric inverse semigroup of all partial injective functions on X.

Additional contributions to this area include the works of Gluskin [14] and Symons [15], who provided further examples and insights. Sullivan [16] and Levi [17] expanded the scope of these results by exploring transformation semigroups that are closed under conjugation by permutations, known as $\mathcal{S}(X)$-normal semigroups. Recently, Mir et al. [18] and Al Subaiei [19] studied $\mathcal{P}_{\mathcal{M}}(X)$, the posemigroup of all partial monotone transformations on a poset X. Mir et al. [18] extended Sullivan's results to $\mathcal{P}_{\mathcal{M}}(X)$. Additionally, Mir and Alali, in [20], investigated the automorphisms of a semigroup S of centralizers of idempotent transformations with restricted range. However, the study of inner automorphism groups, in general, monoids has received relatively less attention. Araújo et al. [21,22]

made notable contributions by formulating general theorems on inner automorphisms and by developing an algorithm to compute inner automorphisms within specific classes of semigroups. This underscores the need for further research in this field.

This paper focuses on the inner automorphisms of GC-semigroups. For a GC-semigroup S, there exists a semilattice $\mathcal{X}$, a family of π-groups U_a, and structure homomorphisms $\phi_{a,b}\colon U_a \to U_b$ for each $a \geq b$, where $S = \bigcup_{a\in\mathcal{X}} U_a$, and the linking homomorphisms define the multiplication in S. We investigate the necessary and sufficient conditions for a GC-monoid to be nearly complete within specific subclasses. These subclasses are characterized based on the properties of their linking homomorphisms, which may be bijective, surjective, injective, or image-trivial. By examining these conditions, this study provides a detailed characterization of nearly complete GC-monoids, offering new insights into their algebraic structure.

The structure of this paper is outlined as follows: In Section 2, we review relevant background information and previous work on GC-semigroups and GC-monoids. Section 3 introduces the subclasses of GC-monoids under consideration and presents the main results, including necessary and sufficient conditions for nearly completeness. In Section 3.5, we discuss the implications of these findings and potential applications in various fields. Finally, Section 4 concludes this paper by summarizing our contributions and providing suggestions for future research.

2. Preliminaries

Some notations and several important results that will be needed in our work are presented in this section. The reader can be referred to [23,24] for basic information on semigroup theory.

A *semigroup* is an algebraic structure comprising a set S along with an associative binary operation. When a semigroup possesses an identity element, it is referred to as a *monoid*. If S lacks an identity element, one can adjoin an identity $1 \notin S$ and extend the multiplication of S to the set $S \cup \{1\}$, denoted as S^1, such that $1^2 = 1$ and $1s = s1 = s$ for all $s \in S$. Notably, if S is already a monoid, then $S^1 = S$.

Given two semigroups S and U, a function $\Phi : S \to U$ is called a *homomorphism* if it preserves the semigroup operation, i.e., $\Phi(st) = \Phi(s)\Phi(t)$ for all $s, t \in S$. A *homomorphism* from a semigroup to itself is referred to as an *endomorphism*. If a homomorphism (or endomorphism) is bijective, it is called an *isomorphism* (or *automorphism*, respectively). The collection of all automorphisms of S, denoted by $\mathrm{Aut}(S)$, constitutes a group under the operation of function composition. The identity automorphism, which maps each element of S to itself, is denoted by id_S. Additionally, in the case of a monoid, every automorphism must preserve the identity element.

An element $s \in S$ is said to be an *idempotent* if it satisfies the equation $s^2 = s$. The collection of all idempotents in S is denoted by $E(S)$.

An element $s \in S$ is called *regular* if there exists an element $x \in S$ such that $sxs = s$. If every element of S is regular, then semigroup S is said to be *regular*.

Every group is a regular semigroup. However, the class of regular semigroups is significantly broader than the class of groups. One of the well-known examples of a regular semigroup that is not a group is the full transformation semigroup $\mathcal{T}(X)$ on a non-empty set X.

For an element $s \in S$, we say that $s' \in S$ is an *inverse* of s if it satisfies the following conditions: $s = ssis$, and $s' = s'ss'$. A semigroup S is called an *inverse semigroup* if every element of S has a unique inverse. Equivalently, a regular semigroup in which idempotents commute is an inverse semigroup.

A *Clifford semigroup* is a special type of inverse semigroup where all idempotents are central, meaning they commute with every element of S.

An element s of a semigroup S is called π-regular if there exists a positive integer $n \in \mathbb{N}$ such that s^n is regular. Moreover, the semigroup S itself is said to be π-regular if every element of S is π-regular.

A semigroup S is called a π-group if there exists a subgroup G^S of S, which is also an ideal subgroup in S, such that for every $s \in S$, there exists a positive integer $n \in \mathbb{N}$ satisfying $s^n \in G^S$. Note that every group is a π-group, but not conversely.

In the study of non-regular semigroups, π-regular semigroups form a significant subclass, as they generalize the concept of regular semigroups. The set of all regular elements in a semigroup S is denoted by R^S. We can express S as the union $S = R^S \cup N^S$, where $N^S = S \setminus R^S$ represents the set of non-regular elements of S. Notably, N^S constitutes a partial semigroup, meaning that for $x, y \in N^S$, if the product xy is defined, then $xy \in N^S$.

Let $\mathcal{X}$ be a semilattice, and let $E(S)$ denote the set of all idempotents of a semigroup S. We construct a semigroup S with $E(S) \cong \mathcal{X}$ as follows:

For each $a \in \mathcal{X}$, let U_a be a π-group, and assume that $U_a \cap U_b = \emptyset$ for $a \neq b$. For every pair $a, b \in \mathcal{X}$ with $a \geq b$, let $\phi_{a,b} \colon U_a \to U_b$ be a homomorphism that satisfies the following properties:

(i) $\phi_{a,a} = \mathrm{id}_{U_a}$ for all $a \in \mathcal{X}$.
(ii) For any $a, b, c \in \mathcal{X}$ with $a \geq b \geq c$, we have $\phi_{b,c} \circ \phi_{a,b} = \phi_{a,c}$.

Define $S = \bigcup_{a \in \mathcal{X}} U_a$ and introduce a multiplication $*$ on S as follows: for $s \in U_a$ and $t \in U_b$,

$$s * t = \phi_{a,a\wedge b}(s)\phi_{b,a\wedge b}(t).$$

Here, $a \wedge b$ denotes the greatest lower bound of a and b in the semilattice $\mathcal{X}$.

The set S forms a semigroup, denoted by $S = [\mathcal{X}; U_a, \phi_{a,b}]$, which is referred to as a *strong semilattice of π-groups*. Such semigroups are also known as *generalized Clifford semigroups (GC-semigroups)*. The homomorphisms $\phi_{a,b}$ are termed as *linking homomorphisms*, $\mathcal{X}$ is called the *linking semilattice of* S, and the π-groups U_a are referred to as the *components of* S. Note that if every component of $S = [\mathcal{X}; U_a, \phi_{a,b}]$ is a group, then S forms a Clifford semigroup.

We first start with the following lemma:

Lemma 1. *Let S be a π-group and $s \in R^S$. Then, $G^S = R^S$.*

Proof. Let $e \in S$ denote the unique idempotent element of S. For any $s \in R^S$, it follows that $s = se \in G^S$, since G^S is an ideal of S. Consequently, we have $R^S \subseteq G^S$. Furthermore, it is clear that $G^S \subseteq R^S$. Therefore, in a π-group, we conclude that $G^S = R^S$. □

Lemma 2 ([4], Lemma 2.1)**.** *Let $S = [\mathcal{X}; U_a, \phi_{a,b}]$ and $W = [\mathcal{Y}; V_a, \xi_{a,b}]$ be two GC-semigroups. Suppose that Θ is a homomorphism from S to W. Then, the following properties hold:*

(i) The restriction $\Theta|_{E(S)}$ is a homomorphism of semilattices.
(ii) If $G \subseteq S$ is a group, then there exists $a \in \mathcal{X}$ such that $G \subseteq U_a$.
(iii) For each $a \in \mathcal{X}$, the restriction $\Theta|_{U_a}$ is a π-group homomorphism from U_a to V_d, where $\Theta(e_a) = e_d$.

Let $\phi_{a,b}$ be a homomorphism between two π-groups U_a and U_b. Then, we define the kernel of $\phi_{a,b}$, denoted by $\mathcal{K}(\phi_{a,b})$, the set $\{s \in U_a : \phi_{a,b}(s) = e_b\}$, and the image of $\phi_{a,b}$ is denoted by $\Delta(\phi_{a,b})$.

The connection between the images and the kernels associated with the linking isomorphisms is an essential aspect of understanding their structure and properties. This relationship is formally stated in the following result. As the proof is straightforward, it is omitted for brevity.

Corollary 1. *Let $S = [\mathcal{X}; U_a, \phi_{a,b}]$ and $T = [\mathcal{Y}; V_a, \xi_{a,b}]$ be two GC-semigroups, and let Θ be an isomorphism from S to T. If $a, b \in \mathcal{X}$ with $a \geq b$, then the following hold:*

(i) $\Theta_b\big(\Delta(\phi_{a,b})\big) = \Delta(\xi_{\lambda(a),\lambda(b)})$;

(ii) $\Theta_a\big(\mathcal{K}(\phi_{a,b})\big) = \mathcal{K}(\xi_{\lambda(a),\lambda(b)})$.

3. The Inner Automorphisms of GC-Semigroups

Let S be a monoid with identity element 1, and let $\mathcal{M}_S$ denote the set of elements $h \in S$ such that there exists $k \in S$ satisfying $hk = kh = 1$. This set $\mathcal{M}_S$ is referred to as the *group of units*.

According to [25], an endomorphism ψ of S is termed *inner* if there exist elements $u, v \in S$ such that $\psi(x) = uxv$ for all $x \in S$. In ([25], Theorem 1), ψ is an automorphism of S if and only if S is a monoid with identity 1 and $uv = vu = 1$, where $u \in \mathcal{M}_S$ and $v = u^{-1}$. We denote such an automorphism by ψ_u^S.

An automorphism ψ of a monoid S is said to be an *inner* automorphism if there exists an element $u \in \mathcal{M}_S$, such that $\psi(x) = uxu^{-1}$ for all $x \in S$.

The collection of all inner automorphisms of a semigroup S, commonly denoted as $\mathrm{Inn}(S)$, constitutes a subgroup of the automorphism group $\mathrm{Aut}(S)$. Consequently, for a monoid S, we can write the following:

$$\mathrm{Inn}(S) = \{\psi_h^S : h \in \mathcal{M}_S\}.$$

Moreover, if S is a group, this definition coincides with the standard definition of the inner automorphisms of a group.

An automorphism of a semigroup S is called *outer* if it is not an inner automorphism. The set of all such outer automorphisms of S, which may potentially be an empty set, is denoted by $\mathrm{Out}(S)$. Specifically, for any automorphism $\varphi \in \mathrm{Aut}(S)$, if there does not exist an element $u \in S$ such that $\varphi(x) = uxu^{-1}$ for all $x \in S$, then φ is classified as an outer automorphism. Thus, the set $\mathrm{Out}(S)$ encapsulates the automorphisms that are fundamentally distinct from inner automorphisms.

The set of all elements in a semigroup S that commute with every other element in S is referred to as the *center* of the semigroup. This set is denoted by Z_S, and it consists of those elements $z \in S$ such that $zs = sz$ for every $s \in S$.

Formally, the center of a semigroup S is defined as follows:

$$Z_S = \{z \in S \mid zs = sz \text{ for all } s \in S\}.$$

The center of a semigroup S plays a crucial role in analyzing its structure, as it comprises elements that exhibit commutative behavior with all other elements under the multiplication operation in S.

The following result generalizes a classical theorem from group theory, which asserts that the inner automorphism group of a group G, denoted by $\mathrm{Inn}(G)$, is isomorphic to the quotient $\frac{G}{Z_G}$, where Z_G is the center of the group G. This generalization extends the concept of inner automorphisms to the setting of monoids, providing an analogous result for monoids. This result establishes a correspondence between the inner automorphism group and the quotient structure in the context of monoids, similar to the well-known result in group theory.

Theorem 1 ([2], Theorem 3.2)**.** *Let S be a monoid and $\mathcal{M}_S$ be the group of units of S. Then, $Inn(S) \cong \frac{\mathcal{M}_S}{(Z_S \cap \mathcal{M}_S)}$.*

Note that unlike in the case of groups, the above theorem does not imply that S is commutative if and only if $\mathrm{Inn}(S)$ is the trivial group. Therefore, it is important to investigate $\mathrm{Inn}(S)$ even when S is commutative.

The following result provides a characterization of the inner automorphisms of a monoid by relating them to the inner automorphisms of its group of units. Specifically, it establishes a connection between the automorphism structure of the monoid and the automorphism structure of its invertible elements, offering a deeper insight into the behavior of inner automorphisms within the monoid.

Theorem 2 ([2], Theorem 3.3)**.** *If S is a monoid, then the map $Inn(S) \to Inn(\mathcal{M}_S)$ defined by $\phi_h^S \mapsto \phi_h^{\mathcal{M}_S}$ for each $h \in \mathcal{M}_S$ is an isomorphism if and only if $Z_{\mathcal{M}_S} = Z_S \cap \mathcal{M}_S$.*

Lemma 3. *Let $S = [\mathcal{X}; U_a, \phi_{a,b}]$ be a strong semilattice of semigroups. If S is a monoid with identity element 1, then $1 \in U_\sigma$ if and only if σ is the maximum element of $\mathcal{X}$. In this case, we have $\mathcal{M}_S \subseteq U_\sigma$.*

Proof. Let 1 be the identity element of the semigroup S, such that $1 \in S_\sigma$. For any $s \in S$, assume that $s \in U_b$ for some $b \neq \sigma$. This implies that $s1 = s = 1s$. In other words, $S_b \subseteq U_{\sigma b}$, which leads to the conclusion that $b \leq \sigma$ for all $b \in \mathcal{X}$. Therefore, σ is the maximum element of $\mathcal{X}$.

Conversely, suppose that σ is the maximum element of $\mathcal{X}$. Now, assume that $1 \in U_b$ for some $b \neq \sigma$. Let $x \in U_\sigma$. Since $1x = x = x1$, we obtain $b\sigma = \sigma$, which contradicts the assumption that σ is the maximum element of $\mathcal{X}$. Hence, we must have $b = \sigma$.

Now, consider any $s \in \mathcal{M}_S$. This implies that the inverse of s, denoted as s^{-1}, exists. We also have $1 = ss^{-1} \in U_\sigma$. If $s \in U_a$ and $s^{-1} \in U_b$, then we know that $ss^{-1} \in U_{ab}$. Therefore, we must have $a = b = \sigma$; otherwise, we would reach a contradiction to the maximality of σ. Hence, we conclude that $\mathcal{M}_S \subseteq U_\sigma$. □

Corollary 2. *If each U_a is a π-group in Lemma 3, then $\mathcal{M}_S = U_\sigma$.*

Proof. Let $s \in U_\sigma$. There exists some $a' \in \mathcal{X}$ such that $s = s_{a'} \in U_{a'}$, so we have $s_{a'} s_{a'}^{-1} = e_\sigma \in U_\sigma$. This implies that $a' = \sigma$; otherwise, we obtain a contradiction for σ to be a maximum element of $\mathcal{X}$. Therefore, we have $\mathcal{M}_S = U_\sigma$. □

Note that if S is a π-group, then $\mathcal{M}_S = G^S$, where G^S is the subgroup of S.

A group G is said to be *complete* if $\mathrm{Inn}(G) = \mathrm{Aut}(G)$, and its center Z_G is trivial. We extend this concept to monoids as follows:

A monoid S is called *nearly complete* if every automorphism of S is inner and complete if, in addition, the intersection of the center Z_S and the group of units U_S is trivial.

The following lemma provides a straightforward method for identifying the elements that belong to both the set Z_S and the set $\mathcal{M}_S$, i.e., the intersection $Z_S \cap \mathcal{M}_S$.

Lemma 4. *Let $S = [\mathcal{X}; U_a, \phi_{a,b}]$ be a GC-monoid, where σ denotes the maximum element of the set $\mathcal{X}$. Then, the intersection of Z_S and $\mathcal{M}_S$, denoted by $Z_S \cap \mathcal{M}_S$, consists precisely of the elements of $Z_{\mathcal{M}_S}$, where the linking homomorphisms preserve the property of being in the center. Specifically, we have the following characterization:*

$$Z_S \cap \mathcal{M}_S = \{g \in Z_{\mathcal{M}_S} : \phi_{\sigma,a}(h) \in Z_{u_a} \text{ for each } a \in \mathcal{X}\}.$$

Proof. By applying Lemma 2, we deduce that $\mathcal{M}_S = U_\sigma$. Consequently, if $h \in Z_{\mathcal{M}_S}$ and $s \in U_a$, the commutation relation $hs = sh$ holds if and only if the equation

$$\phi_{\sigma,a}(h) \cdot s = s \cdot \phi_{\sigma,a}(h)$$

is satisfied. This completes the proof of the result. □

Given the diverse nature of the linking homomorphisms in the strong semilattices of π-groups, a comprehensive description of all nearly complete GC-monoids can be intricate. To address this complexity, we have investigated several constraints on the nature of these linking homomorphisms that have enabled us to characterize nearly complete GC-monoids effectively.

It is crucial to emphasize that the property of being nearly complete does not always carry over from GC-semigroups to their corresponding linking semilattices. This distinction highlights that the structure and properties of a semigroup may not necessarily preserve certain characteristics when examined through the lens of its associated semilattice. For a concrete illustration of this phenomenon, readers are encouraged to consult ([2], Example 4.1).

We now extend the property of $\mathcal{X}$ having a trivial automorphism group as follows: Consider GC-semigroups, denoted by $S = [\mathcal{X}; U_a, \phi_{a,b}]$. We define the automorphism group of $\mathcal{X}$ with respect to S, denoted by $\mathrm{Aut}(\mathcal{X})_S$, as the set of all automorphisms λ of $\mathcal{X}$ such that $\lambda = \Theta_{\mathcal{X}}$, for some $\Theta_{\mathcal{X}} \in \mathrm{Aut}(S)$.

If the automorphism group $\mathrm{Aut}(\mathcal{X})_S$ is trivial, meaning that the only element of $\mathrm{Aut}(\mathcal{X})_S$ is the identity automorphism, this condition implies that no non-trivial automorphisms exist that preserve the structure of $\mathcal{X}$ in relation to the semilattice S.

Now, we have the corollary that immediately follows from Lemma 4.

Corollary 3. *If S is a GC-monoid that is nearly complete, then the automorphism group of $\mathcal{X}$, related to S, is trivial.*

The inner automorphisms of these monoids are restricted to the inner automorphisms of their respective components. Additionally, these restrictions impose certain conditions on the conjugating elements, ensuring that the conjugation is limited within the structure of the individual components. More precisely, if an element x of the monoid is conjugated by an element y, the automorphism induced by this conjugation must respect the decomposition of the monoid into its components, thereby preserving the internal structure of each component.

Lemma 5. *Let S be a GC-monoid, and let $h \in \mathcal{M}_S$. For each element $a \in \mathcal{X}$, the inner automorphism ψ_h^S on S restricts to an inner automorphism $\psi_x^{U_a}$ on U_a, where $x = \phi_{\sigma,a}(h) \in \mathcal{M}_{U_a}$.*

Proof. For $a \in U_a$, we have

$$\psi_h^U(a) = hah^{-1} = \phi_{\sigma,a}(h) \cdot a \cdot \phi_{\sigma,a}(h^{-1}) = \phi_{\sigma,a}(h) \cdot a \cdot (\phi_{\sigma,a}(h))^{-1} = \psi^{U_a}_{\phi_{\sigma,a}(h)}(a),$$

which shows that ψ_h^S restricts to the inner automorphism $\psi^{U_a}_{\phi_{\sigma,a}(h)}$ of U_a, yielding the first result. □

3.1. The Bijective Case

In this section, we characterize the inner automorphisms of GC-monoids where all linking homomorphisms are bijective. The simplicity of our findings is derived from the following two key results:

Lemma 6 ([4], Lemma 2.6)**.** *Let S be a GC-monoid where all the linking homomorphisms are bijective. Then, for any $a \in \mathcal{X}$, we have $S \cong \mathcal{X} \times U_a$.*

Theorem 3. *Let $\mathcal{X}$ be a semilattice and H be a π-group. Then, $Aut(\mathcal{X} \times H) \cong Aut(\mathcal{X}) \times Aut(H)$.*

Corollary 4. *Let $S = \mathcal{X} \times H$, where $\mathcal{X}$ is a semilattice and H is a π-group. Then, an automorphism $\psi \in \mathrm{Aut}(S)$ is inner if and only if it can be expressed as $\psi(a,t) = (a, \psi_h^H(t))$ for some $h \in \mathcal{M}_H$.*

Proof. Let σ be the greatest element of $\mathcal{X}$, so that $\mathcal{M}_S = \{(\sigma, u) : u \in \mathcal{M}_H\}$. Then, for any $k = (\sigma, k) \in \mathcal{M}_S$ and $(a,t) \in S$, we have

$$\psi_k^S(a,t) = (\sigma,k)(a,t)(\sigma,k)^{-1} = (\sigma,k)(a,t)(\sigma,k^{-1}) = (a, ktk^{-1}).$$

Hence, $\psi_k^S = (\mathrm{id}_{\mathcal{X}}, \psi_k^G)$.

Conversely, suppose that $\psi(a,t) = (a, \psi_k^G(t))$ for some $k \in \mathcal{M}_H$. By Theorem 3, we know that ψ is an automorphism of S. Now, we only need to show that ψ is inner. To this end, for any $(a,t) \in S$, we have

$$\psi(a,t) = (a, \psi_k^H(t)) = (a, ktk^{-1}) = (a,k)(a,t)(a,k^{-1}) = (a,k)(a,t)(a,k^{-1}).$$

Thus, $\psi(t) = ktk^{-1}$ for all $t \in S$. This completes the proof. □

The subsequent result follows directly from Theorem 3 and the preceding corollary. By applying the conclusions drawn from these two established results, we can immediately derive the desired outcome.

Corollary 5. *Let $S = \mathcal{X} \times H$ for some semilattice monoid $\mathcal{X}$ and a π-group H. Then, S is nearly complete if and only if both $\mathcal{X}$ and H are so.*

3.2. The Surjective Case

In this section, we focus on the scenario where all the linking homomorphisms are surjective, referring to them as *surjective GC-semigroups*. This generalization is particularly motivated by the favorable behavior of central units in such structures:

Lemma 7. *If S is a surjective GC-monoid with $\mathcal{M}_S = U_\sigma$, then we have the equality $Z_S \cap \mathcal{M}_S = Z_{\mathcal{M}_S}$. In particular, the map $Inn(S) \longrightarrow Inn(\mathcal{M}_S)$, $\phi_h^S \mapsto \phi_h^{\mathcal{M}_S}$, for each $h \in \mathcal{M}_S$, is an isomorphism.*

Proof. Let $S = [\mathcal{X}; U_a, \phi_{a,b}]$ and let σ denote the greatest element of $\mathcal{X}$. Suppose $u \in Z_{\mathcal{M}_S} = Z_{U_\sigma}$ and $v \in U_a$ for some $a \in \mathcal{X}$. Since $\phi_{\sigma,a}$ is surjective, there exists $w \in U_\sigma$ such that $\phi_{\sigma,a}(w) = v$. Therefore, we have

$$uv = \phi_{\sigma,a}(u) \cdot v = \phi_{\sigma,a}(u) \cdot \phi_{\sigma,a}(w) = \phi_{\sigma,a}(uw) = \phi_{\sigma,a}(wu) = \phi_{\sigma,a}(w) \cdot \phi_{\sigma,a}(u) = vu,$$

where the fourth equality follows from the fact that u is central in U_σ. Thus, we conclude that $Z_{\mathcal{M}_S} \subseteq Z_S \cap \mathcal{M}_S$, and the reverse inclusion is immediate.

By Theorem 2, it follows that $\mathrm{Inn}(S)$ is isomorphic to $\mathrm{Inn}(\mathcal{M}_S)$. □

We now proceed to demonstrate that the automorphisms of surjective GC-monoids can be uniquely determined by the combined automorphisms of two distinct structural components: the linking semilattice and the group of units. Specifically, we will show how the intrinsic properties and symmetries of these components are intertwined to define the automorphism group of the entire GC-monoid. This relationship highlights the critical role played by the linking semilattice in the structural integrity of the monoid, while the group of units further enriches its algebraic properties.

Theorem 4. *Let $S = [\mathcal{X}; U_a, \phi_{a,b}]$ be a surjective GC-monoid with $\mathcal{M}_S = U_\sigma$. Let $\Theta_\sigma \in \mathrm{Aut}(U_\sigma)$ and $\lambda \in \mathrm{Aut}(\mathcal{X})$. Then, the following statements are equivalent:*

(i) The automorphism Θ_σ extends to an automorphism Θ of S such that $\Theta_\mathcal{X} = \lambda$.

(ii) For each $a \in \mathcal{X}$, we have
$$\Theta_\sigma(\mathcal{K}(\phi_{\sigma,a})) = \mathcal{K}(\phi_{\sigma,\lambda(a)}).$$

(iii) For each $a \in \mathcal{X}$, the map
$$\phi_{\sigma,\lambda(a)} \circ \Theta_\sigma \circ \phi_{\sigma,a}^{-1}$$
is an isomorphism from U_a to $U_{\lambda(a)}$.

Moreover, in this case, the automorphism Θ, which extends Θ_σ, possesses the property
$$\Theta_a = \phi_{\sigma,\lambda(a)} \circ \Theta_\sigma \circ \phi_{\sigma,a}^{-1}, \quad \forall a \in \mathcal{X}.$$

Proof. (i) $\Rightarrow$ (ii). Suppose that the automorphism Θ_σ extends to an automorphism Θ of S such that $\Theta_\mathcal{X} = \lambda$. Then, by Corollary 1, it follows that for each $a \in \mathcal{X}$, we have
$$\Theta_\sigma\big(\mathcal{K}(\phi_{\sigma,a})\big) = \mathcal{K}(\phi_{\lambda(\sigma),\lambda(a)}) = \mathcal{K}(\phi_{\sigma,\lambda(a)}).$$

(ii) $\Rightarrow$ (iii) We proceed to prove the implication by demonstrating that the map $\Theta_a = \phi_{\sigma,\lambda(a)}\Theta_\sigma\phi_{\sigma,a}^{-1}$ is well defined and injective. To this end, let us assume $k_1, k_2 \in U_a$. Suppose further that $\phi_{\sigma,a}(h_i) = k_i$, for $i = 1,2$, where $h_1, h_2 \in U_\sigma$ (the preimage set under $\phi_{\sigma,a}$). Then,
$$\begin{aligned}
\Theta_a(k_1) = \Theta_a(k_2) &\Leftrightarrow \phi_{\sigma,\lambda(a)}\Theta_\sigma(h_1) = \phi_{\sigma,\lambda(a)}\Theta_\sigma(h_2) \\
&\Leftrightarrow \phi_{\sigma,\lambda(a)}\Theta_\sigma(h_1)\Theta_\sigma(h_2^{-1}) = \phi_{\sigma,\lambda(a)}\Theta_\sigma(h_2)\Theta_\sigma(h_2^{-1}) \ \ (\text{as } h_2 \in \mathcal{M}_S = U_\sigma) \\
&\Leftrightarrow \phi_{\sigma,\lambda(a)}\Theta_\sigma(h_1h_2^{-1}) = \phi_{\sigma,\lambda(a)}\Theta_\sigma(h_2h_2^{-1}) \\
&\Leftrightarrow \phi_{\sigma,\lambda(a)}\Theta_\sigma(h_1h_2^{-1}) = \phi_{\sigma,\lambda(a)}\Theta_\sigma(1) \\
&\Leftrightarrow \phi_{\sigma,\lambda(a)}\Theta_\sigma(h_1h_2^{-1}) = \phi_{\sigma,\lambda(a)}(1) \\
&\Leftrightarrow \phi_{\sigma,\lambda(a)}\Theta_\sigma(h_1h_2^{-1}) = e_{\lambda(a)} \\
&\Leftrightarrow \Theta_\sigma(h_1h_2^{-1}) \in \mathcal{K}(\phi_{\sigma,\lambda(a)}) \ \ (\text{as } h_2 \in \mathcal{M}_S = U_\sigma) \\
&\Leftrightarrow h_1h_2^{-1} \in \mathcal{K}(\phi_{\sigma,a}) \ \ (\text{ by property (ii)}) \\
&\Leftrightarrow k_1 = k_2.
\end{aligned}$$

Hence, the map Θ_a is well defined and injective. To establish surjectivity, we note that Θ_σ is a surjective homomorphism by definition, and the linking homomorphisms involved are also surjective. Since Θ_a is composed of Θ_σ and these surjective linking homomorphisms, it follows that Θ_a is a surjective homomorphism as well.

(iii) $\Rightarrow$ (i). Let $a \in \mathcal{X}$ be an arbitrary element, and consider the map $\phi_{\sigma,\lambda(a)} \circ \Theta_\sigma \circ \phi_{\sigma,a}^{-1} : U_a \to U_{\lambda(a)}$, which is an isomorphism. By ([26], Theorem 2), it follows that $\phi_{\sigma,\lambda(a)} \circ \Theta_\sigma \circ \phi_{\sigma,a}^{-1} = \Theta_\alpha$. Equivalently, we have $\phi_{\sigma,\lambda(a)} \circ \Theta_\sigma = \Theta_\alpha \circ \phi_{\sigma,a}$. Thus, by ([4], Theorem 2.4), the proof is complete. Moreover, the concluding statement follows directly from ([26], Theorem 2). $\square$

It follows that for each pair $(\lambda, \psi) \in \mathrm{Aut}(\mathcal{X}) \times \mathrm{Aut}(\mathcal{M}_S)$, there exists at most one automorphism of S that extends ψ and simultaneously corresponds to the linking semilattice automorphism λ. This uniqueness ensures a well-defined relationship between the automorphisms of S, $\mathcal{M}_S$, and $\mathcal{X}$.

From Corollary 3, we observe that in nearly complete GC-semigroups, where the underlying semilattice possesses a trivial automorphism group relative to S, the structural behavior of these semigroups becomes significantly simplified. Specifically, if we consider all linking homomorphisms to be surjective, Theorem 4 provides a clear criterion for determining when such configurations arise.

By analyzing the interplay between the automorphisms of $\mathcal{M}_S$ and the kernels of the linking homomorphisms, we can gain a deeper insight into the structural dynamics of the semigroup S. This approach not only aids in classifying the automorphisms but also reveals how the properties of $\mathcal{M}_S$ influence the overall automorphism group of S under specific conditions, such as the surjectivity of the linking homomorphisms.

Corollary 6. *Let $S = [\mathcal{X}; U_a, \phi_{a,b}]$ be a surjective GC-monoid with $\mathcal{M}_S = U_\sigma$. Then, the automorphism group of $\mathcal{X}$ with respect to S is trivial if and only if, for every non-identity element $\lambda \in \mathrm{Aut}(\mathcal{X})$ and every automorphism ψ of $\mathcal{M}_S = U_\sigma$, there exists an element $a \in \mathcal{X}$ such that $\psi\big(\ker \phi_{\sigma,a}\big) \neq \ker \phi_{\sigma,\lambda(a)}$.*

Theorem 5. *Let S be a surjective GC-monoid. Then, S is nearly complete if and only if $Aut(\mathcal{X})_S$ is trivial, and every outer automorphism of $\mathcal{M}_S$ fails to preserve the kernel of at least one linking homomorphism.*

Proof. Let the semilattice $\mathcal{X}$ have the greatest element, σ, such that $\mathcal{M}_S = U_\sigma$.

Suppose that S is nearly complete, so $\mathrm{Aut}(\mathcal{X})_S$ is trivial by Corollary 3. Let $\Theta_\sigma \in \mathrm{Out}(U_\sigma)$. By Lemma 5, Θ_σ cannot be extended to an automorphism of S. Thus, the result follows directly from Theorem 4, where λ is taken as $\mathrm{id}_{\mathcal{X}}$.

Conversely, let $\Psi = \bigcup_{a\in\mathcal{X}} \Psi_a$ be an automorphism of S. If Ψ_σ is outer, it fails to preserve the kernels of all linking homomorphisms, contradicting Theorem 4. Thus, Ψ_σ must be inner, say $\psi_h^{U_\sigma}$. For any $u \in U_a$, we have

$$\Psi(u) = \Psi_a(u) = \phi_{\sigma,a}\Psi_\sigma\phi_{\sigma,a}^{-1}(u) = \phi_{\sigma,a}(h \cdot \phi_{\sigma,a}^{-1}(u) \cdot h^{-1}) = \phi_{\sigma,a}(h) \cdot u \cdot \phi_{\sigma,a}(h^{-1}) = huh^{-1}.$$

Hence, $\Psi = \psi_h^S$ is inner. □

Corollary 7. *Let S be a surjective GC-monoid with $\mathcal{M}_S = U_\sigma$. If the semilattices $\mathcal{X}$ and $\mathcal{M}_S$ are nearly complete, then S is also nearly complete.*

The converse of above corollary is not true and is illustrated below.

Example 1. *Let $\mathcal{X} = \{0, 1, a, b\}$ denote the diamond semilattice, where 1 is the greatest element, 0 is the least element, and $a \wedge b = 0$. The semilattice $\mathcal{X}$ possesses a single non-trivial automorphism, which swaps the elements a and b.*

Next, consider the set $U_a = \{x, y, z\}$ with the following Cayley table:

	x	y	z
x	x	x	x
y	x	z	x
z	x	x	x

It is evident that U_a is a π-group and that $Reg(U_a) = \{x\}$. Let $U_1 \cong U_a$ and $U_b \cong U_0$ be trivial π-groups. Define the monoid $S = U_0 \cup U_1 \cup U_a \cup U_b$, where $\phi_{1,a}$ is an isomorphism, and all other linking homomorphisms have trivial images. Since $U_a \ncong U_b$, it follows that for every

$\Psi \in Aut(S)$, the corresponding semilattice automorphism is trivial. Furthermore, each component of S has a trivial automorphism group, which implies that $Aut(S)$ is trivial. In particular, S is a nearly complete surjective GC-monoid, whereas $\mathcal{X}$ is not nearly complete.

It is important to note that both Theorem 5 and its corollary fail to hold if surjectivity is omitted, as demonstrated below.

Example 2. *Let $\mathcal{X} = \{a, b\}$ be a semilattice with $b \geq a$. Let $U_b = \{1\}$ and U_a be any π-group. Then, let $S = U_b \cup U_a$ be the GC-semigroup with the linking homomorphism $\varphi_{b,a}$, which is injective. Since it is clear that $Z_S \cap U_b = Z_{U_b}$, by Theorem 2, we obtain $Inn(S) = Inn(U_b)$. Therefore, S is not nearly complete if $|Aut(U_a)| > 1$.*

3.3. The Injective Case

In this section, we examine the case where all the linking homomorphisms in a GC-semigroup are injective. A GC-semigroup S is said to be an *injective GC-semigroup* if all of its linking homomorphisms are injective. This property imposes certain structural constraints on the semigroup, which we aim to explore further.

To begin, we investigate the scenario where the linking semilattice associated with S has a least element. This condition is crucial, as it introduces additional structure to the semigroup. Specifically, we focus on the component of S that corresponds to the minimum element of the linking semilattice. This component plays a pivotal role in the study of injective GC-semigroups, much in the same way that the maximal component, often referred to as the *group of units*, is of significant importance in the case of surjective linking homomorphisms.

The analysis of the minimum component reveals key insights into the structure and behavior of injective GC-semigroups, as it provides a foundation for understanding how a semigroup behaves under injectivity constraints. In particular, the interaction between this minimum component and the overall semigroup structure highlights the distinctive features of injective GC-semigroups.

Lemma 8. *Let $S = [\mathcal{X}; U_a, \phi_{a,b}]$ be an injective GC-monoid, where $\mathcal{X}$ has a least element denoted by r. In this context, we can express the intersection of Z_S and $\mathcal{M}_S$ as $Z_S \cap \mathcal{M}_S = \phi_{\sigma,r}^{-1}(Z_{U_r})$, where σ represents the maximum element of $\mathcal{X}$. This result leads to the conclusion that the inner automorphism group of S, denoted as $Inn(S)$, is isomorphic to the quotient of $\mathcal{M}_S$ by $\phi_{\sigma,r}^{-1}(Z_{U_r})$:*

$$Inn(S) \cong \frac{\mathcal{M}_S}{\phi_{\sigma,r}^{-1}(Z_{U_r})}.$$

Proof. Recall that $\mathcal{M}_S = U_\sigma$. If $g \in Z_S \cap \mathcal{M}_S$, then, for every $x \in U_r$, we have

$$xh = hx \implies x \cdot \phi_{\sigma,r}(h) = \phi_{\sigma,r}(h) \cdot x \implies h \in \phi_{\sigma,r}^{-1}(Z_{U_r}).$$

Hence, $Z_S \cap \mathcal{M}_S \subseteq \phi_{\sigma,r}^{-1}(Z_{U_r})$.

Conversely, let $k \in \mathcal{M}_S$ such that $\phi_{\sigma,r}(k) = y \in Z_{U_r}$. For any $a \in \mathcal{X}$ and $s \in U_a$, we have $ks \in U_a$, and since the linking homomorphisms are transitive, it follows that

$$\begin{aligned}\phi_{a,r}(ks) &= \phi_{a,r}(\phi_{\sigma,a}(k) \cdot s) = \phi_{\sigma,r}(k) \cdot \phi_{a,r}(s) = y \cdot \phi_{a,r}(s) \\ &= \phi_{a,r}(s) \cdot y = \phi_{a,r}(s) \cdot \phi_{\sigma,r}(k) = \phi_{a,r}(s \cdot \phi_{\sigma,a}(k)) = \phi_{a,r}(sk),\end{aligned}$$

where the fourth equality holds because y is central in U_r. Thus, $sk = ks$, since $\phi_{a,r}$ is injective. Therefore, $k \in Z_S \cap \mathcal{M}_S$.

The final result follows immediately from Theorem 1. $\square$

Theorem 6. *Let S be an injective GC-monoid in which $\mathcal{X}$ has a least element r. Suppose $\Theta_r \in \mathrm{Aut}(U_r)$ and $\lambda \in \mathrm{Aut}(\mathcal{X})$. Then, the following conditions are equivalent:*

(i) Θ_r *extends to an automorphism of S with the corresponding underlying semilattice automorphism λ.*

(ii) $\Theta_r(\Delta(\phi_{a,r})) = \Delta(\phi_{\lambda(a),r})$ *for each* $a \in \mathcal{X}$.

(iii) *The map* $\phi^{-1}_{\lambda(a),r}\Theta_r\phi_{a,r}$ *for each* $a \in \mathcal{X}$ *is a bijective homomorphism from* U_a *to* $U_{\lambda(a)}$.

Moreover, in this case, the automorphism Θ, which extends Θ_r, possesses the property $\Theta_a = \phi^{-1}_{\lambda(a),r}\Theta_r\phi_{a,r}$, *for each* $a \in \mathcal{X}$.

Proof. (i) ⇒ (ii). Suppose that the automorphism Θ_r extends to an automorphism Θ of S such that $\Theta_{\mathcal{X}} = \lambda$. Then, by Corollary 1, it follows that for each $a \in \mathcal{X}$, we have

$$\Theta_r\big(\Delta(\phi_{a,r})\big) = \Delta(\phi_{\lambda(a),\lambda(r)}) = \Delta(\phi_{\lambda(a),r}).$$

(ii) ⇒ (iii) Note that the composition is possible because $\Theta_r(\Delta(\phi_{a,r})) = \Delta(\phi_{\lambda(a),r})$ for each $a \in \mathcal{X}$, as guaranteed by Condition (ii). The map is a well-defined and one-to-one homomorphism since both Θ_r and the linking homomorphisms are injective.

Now, let $x \in U_{\lambda(a)}$. Define $y = \Theta_r^{-1}(\phi_{\lambda(a),r}(x))$. By Condition (ii), it follows that $y \in \Delta(\phi_{a,r})$, which means there exists some $x' \in U_a$ such that $\phi_{a,r}(x') = y$.

We now compute the following:

$$\phi^{-1}_{\lambda(a),r} \circ \Theta_r \circ \phi_{a,r}(x') = \phi^{-1}_{\lambda(a),r}\big(\Theta_r(y)\big) = \phi^{-1}_{\lambda(a),r}\big(\phi_{\lambda(a),r}(x)\big) = x.$$

Thus, the map is surjective. Combining injectivity and surjectivity, we conclude that the map is a bijective homomorphism.

(iii)⇒ (i) and the final statement follow on immediately from ([26], Theorem 2). □

Corollary 8. *Let S be an injective GC-monoid in which $\mathcal{X}$ has a least element denoted by r. Then,* $\mathrm{Aut}(\mathcal{X})_S$ *is trivial if and only if for every non-identity automorphism* $\lambda \in \mathrm{Aut}(\mathcal{X})$ *and every automorphism θ of the set U_r, there exists an element $a \in \mathcal{X}$ such that* $\theta(\Delta(\phi_{a,r})) \neq \Delta(\phi_{\lambda(a),r})$.

Theorem 7. *Let S be an injective GC-monoid in which $\mathcal{X}$ has a least element r. Then, S is nearly complete if and only if* $\mathrm{Aut}(\mathcal{X})_S$ *is trivial and every automorphism*

$$\theta \in Out(U_r) \cup \{\phi_h^{S_r} \text{ such that } h \cdot \phi_{\sigma,r}(h^{-1}) \notin Z_{U_r} \text{ for all } h \in U_\sigma\}$$

does not preserve the image of some linking homomorphism, i.e., $\theta(\Delta(\phi_{\sigma,r})) \neq \Delta(\phi_{\sigma,r})$.

Proof. Let the semilattice $\mathcal{X}$ have a greatest element σ, so that $\mathcal{M}_S = U_\sigma$.

To prove the forward direction, we appeal to Lemma 5, as in the proof of Theorem 5. Observe that

$$\{\phi_h^{S_r} \mid g \cdot \phi_{\sigma,r}(h^{-1}) \notin Z_{U_r} \text{ for all } h \in U_\sigma\}$$

is precisely the set of inner automorphisms of U_r that are not restrictions of the inner automorphisms of S.

Conversely, let $\Phi = \bigcup_{a\in\mathcal{X}} \Phi_a$ be an automorphism of S. Then, for some $x \in \Delta(\phi_{\sigma,r})$, we have $\Phi_r = \psi_x^{U_r}$; otherwise, by hypothesis, Φ we would fail to preserve the images of all the linking homomorphisms. This would contradict Theorem 6.

Let $x = \phi_{\sigma,r}(h)$ for some $h \in U_\sigma$. For any $a \in \mathcal{X}$ and any $y \in U_a$, Theorem 6 gives

$$\Phi_a(y) = \phi_{a,r}^{-1}\Theta_r\phi_{a,r}(y) = \phi_{a,r}^{-1}\phi_x^{U_r}\phi_{a,r}(y) = \phi_{a,r}^{-1}\left(x \cdot \phi_{a,r}(y) \cdot x^{-1}\right)$$
$$= \phi_{a,r}^{-1}\left(\phi_{\sigma,r}(h) \cdot \phi_{a,r}(y) \cdot \phi_{\sigma,r}(h^{-1})\right).$$

Now, since $\phi_{a,r}\phi_{\sigma,a} = \phi_{\sigma,r}$ and the linking homomorphisms are injective, it follows that $\phi_{a,r}^{-1}\phi_{\sigma,r} = \phi_{\sigma,a}$. Substituting this relation back, we find

$$\Phi_a(y) = \phi_{\sigma,a}(h) \cdot \phi_{a,r}^{-1}\phi_{a,r}(y) \cdot \phi_{\sigma,a}(h^{-1}) = \phi_{\sigma,a}(h) \cdot y \cdot \phi_{\sigma,a}(h^{-1}).$$

Hence, $\Phi_a(y) = hyh^{-1}$ for all $y \in U_a$ and $h \in U_\sigma$. Consequently, $\Phi = \psi_h^S$ is an inner automorphism, and we conclude that S is nearly complete. □

3.4. The Image-Trivial Case

Examining the automorphisms of the semilattice of a GC-monoid does not yield insights into the nearly complete structure of a GC-monoid. Furthermore, it is natural to explore how elements $\mathrm{Aut}(\mathcal{M}_S)$ and the π-group U_r for the smallest r (if it exists) interact with the kernels and images of the linking homomorphisms.

To tackle this, we investigate the class of GC-monoids in which every linking homomorphism has a trivial image. We refer to such GC-monoids as *image-trivial*. In this context, the automorphisms can be constructed in a straightforward manner.

The study of the automorphisms of the semilattice associated with a GC-monoid does not yield significant insights into the nearly complete structure of the GC-monoid itself. This observation motivates a natural question: How do the automorphisms of the group of units of a GC-monoid, and, if it exists, the π-group S_r corresponding to the minimum r, interact with the kernels and images of the linking homomorphisms?

To explore this interplay, we focus on a specific subclass of GC-monoids, which is characterized by the property that every linking homomorphism has a trivial image. We designate such GC-monoids as *image-trivial*. This restriction simplifies the construction of their automorphisms, allowing for a more direct analysis.

The following result highlights the nature of automorphisms in image-trivial GC-monoids. The proof is detailed in ([4], Corollary 2.7).

Corollary 9. *Let $S = [\mathcal{X}; U_a, \phi_{a,b}]$ be an image-trivial GC-monoid, and let $\lambda \in Aut(\mathcal{X})$ such that $U_a \cong U_{\lambda(a)}$ for every $a \in \mathcal{X}$. For any collection of isomorphisms $\Theta_a\colon U_a \to U_{\lambda(a)}$, the map $\Theta = \bigcup_{a\in\mathcal{X}} \Theta_a$ is an automorphism of S. Moreover, if Θ is any automorphism of S, there exists an automorphism $\lambda \in Aut(\mathcal{X})$ and a corresponding collection of isomorphisms $\{\Theta_a\colon U_a \to U_{\lambda(a)}\}$ such that $\Theta = \bigcup_{a\in\mathcal{X}} \Theta_a$.*

Theorem 8. *Let $S = [\mathcal{X}; U_a, \phi_{a,b}]$ be an image-trivial GC-monoid, where $\mathcal{M}_S = U_\sigma$, and let $S' = S \setminus U_\sigma$. Then, the following holds:*

$$Z_{U_\sigma} = Z_S \cap U_\sigma, \quad \textit{and} \quad \mathrm{Inn}(S) = \{\psi_h^{U_\sigma} \cup \mathrm{id}_{S'} : h \in U_\sigma\} \cong \mathrm{Inn}(U_\sigma).$$

Moreover, S is nearly complete if and only if the following conditions are satisfied:

(i) For each non-identity automorphism λ of $\mathcal{X}$, there exists $a \in \mathcal{X}$ such that $U_a \ncong U_{\lambda(a)}$.
(ii) The component U_σ is nearly complete.
(iii) $\mathrm{Aut}(U_a) = \{\mathrm{id}_{U_a}\}$ for $a \neq \sigma$.

Proof. Let $k \in Z_{U_\sigma}$ and $x \in U_a$. Then,

$$kx = \phi_{\sigma,a}(k) \cdot x = e_a x = x e_a = x \cdot \phi_{\sigma,a}(k) = xk,$$

and, thus, $Z_{U_\sigma} = Z_S \cap U_\sigma$. Hence, $\mathrm{Inn}(S) \cong \mathrm{Inn}(U_\sigma)$ by Theorem 2.

Moreover, if $\psi_h^S \in \mathrm{Inn}(S)$, then for $u \in U_a$, a similar calculation yields $\psi_h^S(u) = huh^{-1} = u$, which implies that ψ_h^S acts as the identity on S'.

Let S be nearly complete. By Corollary 9, any automorphism of $\mathcal{X}$ that preserves the isomorphism types of the components of S can be extended to an automorphism of S, thereby proving Statement (i).

Moreover, every automorphism of a connected component can be extended to an automorphism of S, where the underlying semilattice automorphism is the identity. This leads to the conclusion of Statements (ii) and (iii).

Conversely, suppose that Statements (i)–(iii) hold, and let $\Theta = [\lambda, \Theta_a] \in \mathrm{Aut}(S)$. Then, λ is the identity in accordance to (1), $\Theta_\sigma = \psi_k^{U_\sigma}$ for some $k \in S_\sigma$ in accordance to (ii), and $\Theta_a = \mathrm{id}_{U_a}$ for $a < \sigma$ in accordance to (iii). Hence, $\Theta = \psi_k^S$. □

Corollary 10. *There exists a GC-monoid S and a set $\mathcal{X}$ with a minimum element r such that*

(i) *S is not nearly complete;*
(ii) *Every automorphism $\lambda \in Aut(\mathcal{X})$ is trivial;*
(iii) *Every outer automorphism of U_σ does not preserve the kernel of some linking homomorphism;*
(iv) *Every automorphism in*

$$Out(U_r) \cup \left\{ \psi_h^{S_r} \mid g \cdot \phi_{\sigma,r}(h^{-1}) \notin Z_{U_r}, \forall h \in U_\sigma \right\}$$

does not preserve the image of some linking homomorphism.

Proof. Let $\mathcal{X} = \{\sigma, a, r\}$ be a set where the elements satisfy the relation $\sigma > a > r$. Define the following subsets:

$$U_\sigma = \{e_\sigma\}, \quad U_r = \{e_r\},$$

where both U_σ and U_r are singleton sets corresponding to trivial π-groups. Additionally, let U_a be any π-group with a non-trivial automorphism group.

Now, consider the GC-monoid S defined as $S = U_\sigma \cup U_a \cup U_r$. This monoid is *image-trivial*. By invoking the previous theorem, S is shown to be *not nearly complete*, and the automorphism group of $\mathcal{X}$ is trivial. Consequently, Properties (i) and (ii) hold.

Furthermore, since U_σ and U_r are trivial π-groups, they satisfy the required properties for triviality. Thus, Properties $(iiii)$ and (iv) follow immediately. □

3.5. Applications

Let $\mathcal{X} = \{0, a, b, c, 1\}$ denote a semilattice where the partial order is defined by $1 \geq x \geq 0$ for all $x \in \mathcal{X}$. In this semilattice, the elements a, b, and c form a subset $\{a, b, c\}$ of pairwise incomparable elements. Additionally, the meet (greatest lower bound) of any two elements in $\{a, b, c\}$ is 0.

Now, consider the structure $S = [\mathcal{X}, U_a, \phi_{a,b}]$, which represents a strong semilattice of π-groups, where structure morphisms are assumed to be bijective.

According to Corollary 4.2 of [3], the automorphism group of the semilattice $\mathcal{X}$, denoted as $\mathrm{Aut}(\mathcal{X})$, has a cardinality of $3! = 6$. Explicitly, we write the following:

$$\mathrm{Aut}(\mathcal{X}) = \{\lambda_1, \lambda_2, \ldots, \lambda_6\}.$$

Let Θ represent an arbitrary automorphism of the structure S, i.e., $\Theta \in \mathrm{Aut}(S)$. For every such Θ, we can establish a correspondence as follows:

$$\Theta \longleftrightarrow \{\lambda_i; \{\Theta_a\}_{a\in\mathcal{X}}\}, \quad \text{where} \quad i = 1, 2, \ldots, 6. \tag{1}$$

Here, $\lambda_i \in \mathrm{Aut}(\mathcal{X})$ and $\{\Theta_a\}_{a\in\mathcal{X}}$ denote the collection of morphisms induced by Θ at each $a \in \mathcal{X}$. This correspondence highlights the interplay between the automorphisms of the semilattice $\mathcal{X}$ and the automorphisms of the strong semilattice structure S.

Consider the product of elements in $\mathrm{Aut}(S)$. Let $\varphi, \mu \in \mathrm{Aut}(S)$ and $s \in S$. Then, there exists an element $a \in \mathcal{X}$ such that $s \in U_a$. For instance, if $a = b$, then $s \in U_b$.

Now, consider $\lambda_4 \in \mathrm{Aut}(\mathcal{X})$ associated with φ, satisfying the following:

$$\lambda_4(b) = c, \quad \lambda_4(c) = a.$$

Since $\varphi\mu \in \mathrm{Aut}(S)$, it follows from Equation (1) that the action of $\mu\varphi$ on s is given by

$$\mu\varphi(s) = \mu(\varphi_b(s)) = \mu_{\lambda_4(b)}(\varphi_b(s)) = \mu_c(\varphi_b(s)).$$

Now, consider a group isomorphism Θ_b, defined as $\Theta_b : U_b \to U_{\lambda_4(b)}$. We express Θ_b in terms of other mappings by Theorem 4 as $\Theta_b = \phi_{a,c}\Theta_c\phi_{c,b}^{-1}$. This relation shows that Θ_b is entirely determined by Θ_c. Furthermore, by Theorem 6, any group isomorphism of U_a (for $a \in \mathcal{X}$) can ultimately be determined by Θ_0, where 0 is the least element in the set $\mathcal{X}$.

3.5.1. Expressing Θ_a in Terms of Θ_0

Consider any isomorphism Θ_1 associated with an automorphism Θ of S, along with a function λ_j for some $j \in \{1, 2, \ldots, 6\}$. Assume that λ_j satisfies

$$\lambda_j(a) = c, \quad \lambda_j(0) = 0.$$

Under these assumptions, by Theorem 4, Θ_a can be explicitly expressed in terms of Θ_0 as

$$\Theta_a = \phi^{-1}_{\lambda_j(a),0}\Theta_0\phi_{a,0} = \phi^{-1}_{c,0}\Theta_0\phi_{a,0}.$$

This formula establishes that Θ_0 plays a fundamental role in determining any component isomorphism Θ_a associated with the automorphism Θ of S.

3.5.2. Expressing Θ_c in Terms of Θ_1

Next, we demonstrate that any component isomorphism of U_a (for $a \in \mathcal{X}$) can be uniquely determined by the isomorphism Θ_1, where 1 is the maximum element in $\mathcal{X}$.

To illustrate this, consider Θ_c to be associated with an automorphism Θ of S, and let λ_k be a permutation defined for some $k \in \{1, 2, \ldots, 6\}$ such that

$$\lambda_k(c) = b, \quad \lambda_k(1) = 1.$$

By Theorem 4, we express Θ_c in terms of Θ_1 as follows:

$$\Theta_c = \phi_{\lambda_k(1),\lambda_k(c)}\Theta_1\phi^{-1}_{1,c} = \phi_{1,b}\Theta_1\phi^{-1}_{1,c}.$$

This expression indicates that the behavior of Θ_c is fully governed by Θ_1, establishing its role as a fundamental isomorphism.

3.5.3. Expressing Θ_b in Terms of Θ_c

Finally, by Theorem 6, any component isomorphism of U_a (where $a \in \mathcal{X}$) can be explicitly determined through the mappings of Θ_b.

To illustrate this, let $a = b$ and $b = c$, with $b \wedge c = 0$. Suppose that λ_p satisfies

$$\lambda_p(b) = a, \quad \lambda_p(c) = b, \quad \lambda_p(0) = 0.$$

Then, Θ_b is given by

$$\Theta_b = \phi^{-1}_{\lambda_p(b),\lambda_p(0)} \phi_{\lambda_p(c),\lambda_p(0)} \Theta_c \phi^{-1}_{c,0} \phi_{b,0}.$$

Next, we substitute the following mappings:

$$\Theta_b = \phi^{-1}_{a,0} \phi_{b,0} \Theta_c \phi^{-1}_{c,0} \phi_{a,0}.$$

Thus, Θ_c serves as a determining factor for any π-group isomorphism associated with Θ.

Thus, it is evident that Θ_c serves as a determining factor for any π-group isomorphism associated with the automorphism Θ of S. This result highlights the dependency of the component isomorphisms on specific mappings, reinforcing the structural coherence governed by Θ.

4. Conclusions

This paper investigated the conditions under which generalized Clifford monoids (GC-monoids) can be nearly complete, focusing on specific subclasses characterized by the nature of their linking homomorphisms: bijective, surjective, injective, or image-trivial. By establishing the necessary and sufficient conditions for near completeness within these subclasses, we have contributed to a deeper understanding of GC-monoids and their structural properties.

Our findings reveal that the linking homomorphisms play a crucial role in determining the near completeness of GC-monoids. This insight not only enriches the theoretical framework of GC-semigroups but also paves the way for practical applications in various fields, such as cryptography, coding theory, formal language theory, network theory, mathematical biology, and quantum computing. The conditions identified in this study can be utilized to design more efficient algorithms, model complex systems, and optimize network structures.

Furthermore, this study enhances automorphic theory by providing a better understanding of inner automorphisms in algebraic structures. The results also open new avenues for research in the semigroup theory, encouraging further exploration and development.

In summary, this paper provided a comprehensive characterization of nearly complete GC-monoids and offered valuable criteria for both theoretical analyses and practical applications. Future research may build upon these findings to explore more complex subclasses and their implications, contributing to the continued advancement of the semigroup theory and its applications.

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