Analytical Solution of Dirac Equation for q-Deformed Hyperbolic Manning-Rosen Potential in D Dimensions using SUSY QM and its Thermodynamics'Application

Analytical Solution of Dirac Equation for q-Deformed Hyperbolic Manning-Rosen Potential in D Dimensions using SUSY QM and its Thermodynamics'Application

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Abstract. The Dirac equation of q-deformed hyperbolic Manning Rosen potential in D dimension was solved by using Supersymmetric Quantum Mechanics (SUSY QM). The D dimensional relativistic energy spectra were obtained by using SUSY QM and shape invariant properties and D dimensional wave functions of q-deformed hyperbolic Manning Rosen potential were obtained by using the SUSY raising and lowering operators. In the non-relativistic limit, the relativistic energy spectra for exact spin symmetry case reduced into non-relativistic energy spectra and so for the wave functions. In the classical regime, the partition function, the vibrational specific heat, and the vibrational mean energy of some diatomic molecules were calculated from the non-relativistic energy spectra with the help of error function and imaginary error function.

1. Introduction

One of the important tasks of relativistic quantum mechanic is finding an accurate exact solution of Dirac equation for a certain potential. The bound state solutions of Dirac equation for some potentials, central and non-central, have been investigated by some authors [1-6]using NU method, SUSY QM method [7-11], and Romanovski polynomial method [12-16]. It is known that for very limited potential, three dimensional radial Dirac equation is exactly solvable only for s-wave ($l = 0$). However, the three dimensional radial Dirac equation for the spherically symmetric potentials can not be solved analytically for $l \neq 0$ states because of the centrifugal term $\sim r^{-2}$[17-19]. the Schrodinger equation can only be solved approximately for different suitable approximation scheme. One of the suitable approximation scheme is conventionally proposed by Greene and Aldrich [23-20]. Furthermore, the extension in higher dimensionalspaces for some physical problems is very important in some area. The multidimensional non-relativistic andrelativistic physical systems have been investigatedby many authors, such asring-shaped pseudoharmonic potential[21], isotropic harmonic oscillator plus inversequadratic potential [22], Pseudoharmonic potential[23], Kratzer-Fues potential[24-25],hydrogen atom [26],modified Poschl-Teller potential[27], linierly energy dependent quadratic potential [28], trigonometric scarf potential [29], ring-shaped Kratzer potential [30].

In this paper we will attempt to solve the Dirac equation for a charged particle moving in a field governed by hyperbolic Manning Rosen potential [31] using supersymmetric quantum mechanic (SUSY QM) with idea of shape invariance. SUSY QM method is developed based on Witten proposal [32] and the idea of shape invariant potentialis proposed by Gendenshtein [33]. SUSY QM is a

powerful tool to determine energy spectrum and wave function of a class of shape invariant potentials. The relativistic energy spectrum obtained by using the idea of shape invariance and the wave functions are achieved by using lowering and raising SUSY operator. Some of hyperbolic and trigonometric potentials are exactly solvable within the approximation of centrifugal term and their bound state solutions have been reported in the previous papers[10-11]. In the non-relativistic limit, the relativistic energy equation reduces the non-relativistic energy equation that will be applied to study the thermal properties including vibrational mean energy U, and specific heat C[34,35]. This paper is organized as follows. Brief review of SUSY quantum mechanics is presented in section 2, solution of Dirac equations are presented in section 3 and conclusion is presented in section 4.

2. Review of Supersymmetric Quantum Mechanics Approach Using Operator

2.1. Supersymmetry Quantum Mechanics (SUSY QM)

According to the definition proposed by Witten, in a supersymmetry quantum system there are super charge operators Q_i which commute with the Hamiltonian H_{ss} [32] and given as

$$\left[Q_i, H_{ss}\right] = 0 \text{ with, } i = 1, 2, 3, \ldots \text{N} \tag{1}$$

and they obey to anti commutation algebra

$$\left\{Q_i, Q_j\right\} = \delta_{ij} H_{ss} \tag{2}$$

with H_{ss} is called supersymmetric Hamiltonian. Witten proposed that the SUSY QM is the one dimensional model of SUSY field theory and he stated that the simplest SUSY QM system has N=2 [32] where

$$Q_1 = \left(1/\sqrt{2}\right)\left(\sigma_1\left(p/\sqrt{2m}\right) + \sigma_2\varphi(x)\right) \text{ and } Q_2 = \left(1/\sqrt{2}\right)\left(\sigma_2\left(p/\sqrt{2m}\right) + \sigma_1\varphi(x)\right) \tag{3}$$

where σ_i are the usual Pauli spin matrices, $p = -i\hbar\left(\partial/\partial x\right)$ is the usual momentum operator, and $\phi(x)$ is superpotential. By inserting equation (2) into equation (1) we get,

$$H_{ss} = \begin{pmatrix} -\frac{\hbar^2}{2m}\frac{d^2}{dx^2} + \frac{\hbar}{\sqrt{2m}}\frac{d\phi(x)}{dx} + \phi^2(x) & 0 \\ 0 & -\frac{\hbar^2}{2m}\frac{d^2}{dx^2} - \frac{\hbar}{\sqrt{2m}}\frac{d\phi(x)}{dx} + \phi^2(x) \end{pmatrix} = \begin{pmatrix} H_+ & 0 \\ 0 & H_- \end{pmatrix} \tag{4}$$

Here H_- and H_+, are supersymmetry partner of the Hamiltonian, $V_-(x)$ and $V_+(x)$ are the supersymmetry partner potential. To simplify the determination of the energy spectrum and the wave functions, the new operators are introduced given as

$$A^+ = -\frac{\hbar}{\sqrt{2m}}\frac{d}{dx} + \phi(x) \text{ and } A = \frac{\hbar}{\sqrt{2m}}\frac{d}{dx} + \phi(x) \tag{5}$$

A^+ as raising operator, and A as lowering operator. By inserting equation (5) into equation (4) we get

$$H_-(x) = A^+A, \text{ and } H_+(x) = AA^+ \tag{6}$$

It is always possible to factorize the usual Hamiltonian as

$$H = H_- + E_0 = -\frac{\hbar^2}{2m}\frac{d^2}{dx^2} + V_-(x;a_0) + E_0 \tag{7}$$

From equations (4) and (7) we get,

$$V(x) = V_-(x;a_0) + E_0 = \phi^2(x;a_0) - \frac{\hbar}{\sqrt{2m}}\phi'(x;a_0) + E_0 \tag{8}$$

where $V(x)$ is the effective potential, while $\phi(x)$ is determined hypothetically from equation (8) based on the shape of effective potential from the associated system.

2.2. Shape Invariance

The supersymmetry only gives the relationshipbetween the eigenvalues and eigenfunctions of the two Hamiltonian partners but does not yield the actual spectrum [36]. The energy spectrum is obtained by applying the shape invariant condition proposed by Gendenshtein [33]. If the pair of supersymmetric partner potentials $V_{\pm}(x)$ are similar in shape and differ only in the parameters, then they are said to be shape invariant. More specifically, if $V_{\pm}(x,a_0)$ satisfy the requirement that

$$V_{+}(x;a_j) = V_{-}(x;a_{j+1}) + R(a_{j+1}) \tag{9}$$

with

$$V_{+}(x;a_j) = \phi^2(x;a_j) + \frac{\hbar}{\sqrt{2m}}\phi'(x;a_j) \tag{10}$$

$$V_{-}(x;a_{j+1}) = \phi^2(x;a_{j+1}) - \frac{\hbar}{\sqrt{2m}}\phi'(x;a_{j+1}) \tag{11}$$

where *j = 0,1,2,..*, and *a* is a parameter in our original potential, V-, whose ground state energy is zero, $a_j = f_j(a_0)$ where $f_j$is a function applied j times, the remainder $R(a_j)$ is *a'*s dependence.

The energy eigenvalue of the Hamiltonian H_{-} is given by [33]

$$E_n^{(-)} = \sum_{k=1}^{n} R(a_k) \tag{12}$$

and by using equations (7) and (12) we get the total energy spectra,

$$E_n = E_n^{(-)} + E_0 \tag{13}$$

Based on the characteristics of lowering operator, the ground state wave function is obtained from condition that,

$$A\psi_0^{(-)} = 0 \tag{14}$$

The n^{th} level of the the wavefunctions are obtained by applying raising operator operated to the lower wave function [37], given as

$$\psi_n^{(-)}(x;a_0) \approx A^{+}(x;a_0)A^{+}(x.a_1)....A^{+}(x,a_{n-1})\psi_0^{(-)}(x;a_n) \tag{15}$$

The potential partners $V_{\pm}(x,a_0)$ and the SUSY operators, A^{+} and A are obtained from equations (4), (5), and (8), and the energy spectrum from equations (8), and (12), the wave function obtained from equations (14) and (15).

3. Solution of Dirac Equation in D Dimension

The Dirac equation with the scalar potential $S(\vec{r})$ and magnitude of vector potential $V(\vec{r})$ is given as [38]

$$\left\{\vec{\alpha}.\vec{p} + \beta(M + S(\vec{r}))\right\}\psi(\vec{r}) = \left\{E - V(\vec{r})\right\}\psi(\vec{r}) \tag{16}$$

where *M* is the relativistic mass of the particle, *E* is the total relativistic energy, and $\vec{p}$ is the three-dimensional momentum operator, $-i\nabla$

$$\vec{\alpha} = \begin{pmatrix} 0 & \vec{\sigma} \\ \vec{\sigma} & 0 \end{pmatrix}, \text{ and } \beta = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix} \tag{17}$$

with $\vec{\sigma}$ are the three-dimensional Pauli matrices and I is the 2×2 identity matrix. The potential in equation (16) is spherically symmetric potential, and we have taken $\hbar = 1$, *c=1*. The Dirac equation expressed in equation (16) is invariant under spatial inversion and therefore its eigen states have definite parity. By writing the spinor in D dimension [39-40]as

$$\psi(\vec{r}) = \begin{pmatrix} \zeta(\vec{r}) \\ \Omega(\vec{r}) \end{pmatrix} = \frac{1}{r^{\frac{D-1}{2}}} \begin{pmatrix} F_{nK}(r)Y_{jm}^{l}(\theta,\varphi) \\ iG_{nK}(r)Y_{jm}^{\tilde{l}}(\theta,\varphi) \end{pmatrix} \tag{18}$$

If we insert equations (17) and (18) into equation (16) and use matrices multiplication, we achieve

$$\vec{\sigma}.\vec{p}\Omega(\vec{r}) = \{-M - S(\vec{r}) + E - V(\vec{r})\}\zeta(\vec{r}) \tag{19}$$

$$\vec{\sigma}.\vec{p}\zeta(\vec{r}) = \{M + S(\vec{r}) + E - V(\vec{r}\})\Omega(\vec{r}) \tag{20}$$

In the exact spin symmetric case, when the scalar potential is equal to the magnitude of vector potential $S(\vec{r}) = V(\vec{r})$, then from equations (19) and (20) we have

$$\vec{\sigma}.\vec{p}\frac{\vec{\sigma}.\vec{p}}{M+E}\zeta(\vec{r}) = \{-M - 2V(\vec{r}) + E\}\zeta(\vec{r}) \tag{21}$$

By applying the Pauli matrices, it is simply shown that if $(\vec{\sigma}.\vec{p})(\vec{\sigma}.\vec{p}) = p^2$, then equation (21) becomes

$$p^2 + 2V(\vec{r})(M+E)\zeta(\vec{r}) = (E^2 - M^2)\zeta(\vec{r}) \tag{22}$$

Since in D dimensions [39-40], $p^2 = -\Delta_D = -\nabla_D^2$ where

$$\nabla_D^2 = r^{1-D}\frac{\partial}{\partial r}\left\{r^{D-1}\frac{\partial}{\partial r}\right\} - \frac{L_D^2}{r^2}; L_D^2 Y_{lm} = l(l+D-2)Y_{lm} \tag{23}$$

then we get the Dirac equation in D dimension for the case of exact spin symmetry by inserting equations (18) and (23) into equation (22) as

$$\frac{\partial^2 F_{nK}(r)}{\partial r^2} - \frac{(l+\frac{D-1}{2})(l+\frac{D-3}{2})}{r^2}F_{nK}(r) - (2V(\vec{r})(M+E))F_{nK}(r) = -(E^2 - M^2)F_{nK}(r) \tag{24}$$

By setting $V \to (1/2)V$ in equation (24) then the Dirac equation in equation (24) reducesinto one dimensional Schrodinger type equation. Forthe similar vector and scalar potentials which is given as hyperbolic Manning-Rosen potential, $V(r) = \left(t^2\left(\eta(\eta-1)/\sinh_q^2 tr\right) - 2tv\coth_q tr\right)$, equation (24) becomes

$$\frac{\partial^2 F_{nK}(r)}{\partial r^2} - \frac{(l+\frac{D-1}{2})(l+\frac{D-3}{2})}{r^2}F_{nK}(r) - \left(t^2\frac{\eta(\eta-1)}{\sinh_q^2 tr} - 2tv\coth_q tr\right)(M+E)F_{nK}(r) = -(E^2 - M^2)F_{nK}(r) \tag{25}$$

where $0 \le tr \le \pi/2$, $\eta > 1$, $v > 0$, and in this case, $t > 0$, the parameter has to control the width of the hyperbolic Manning-Rosen potential. In order to solve theradial Dirac equation in equation (25), we use the approximation value for the centrifugal term as in Greene and Aldirch, and in Ikdhair [13,23], $\frac{1}{r^2} \cong t^2\left(d_0 + \frac{1}{\sinh_q^2 tr}\right)$, for $tr << 1$ and $d_0 = 1/12$. In the centrifugal approximation scheme, equation (25) becomes

$$\frac{\partial^2 F_{nK}(r)}{\partial r^2} - \left(\frac{t^2\eta(\eta-1)(M+E) + (l+\frac{D-1}{2})(l+\frac{D-3}{2})t^2}{\sinh_q^2 tr} - 2tv(M+E)\coth_q tr\right)F_{nK}(r)$$
$$= \left\{(l+\frac{D-1}{2})(l+\frac{D-3}{2})t^2 d_0 - (E^2 - M^2)\right\}F_{nK}(r) \tag{26}$$

By setting

$$\eta(\eta-1)(M+E) + (l+\frac{D-1}{2})(l+\frac{D-3}{2}) = \eta'(\eta'-1)\ v(M+E) = v' \tag{27}$$

$$\left\{(l+\frac{D-1}{2})(l+\frac{D-3}{2})t^2 d_0-\left(E^2-M^2\right)\right\}=-E' \tag{28}$$

in equation (26) then it becomes

$$\frac{\partial^2 F_{nK}(r)}{\partial r^2}-\left(\frac{\eta'(\eta'-1)t^2}{\sinh_q^2 tr}-2tv'\coth_q tr\right)F_{nK}(r)=-E'F_{nK}(r) \tag{29}$$

and the effective potential in equation (29) is given as

$$V=\left(\frac{\eta'(\eta'-1)t^2}{\sinh_q^2 tr}-2tv'\coth_q tr\right) \tag{30}$$

Equation (29) is solved using SUSY QM and by introducing thehypothetical super-potential as in [32-33]

$$\phi(r)=ta\coth_q tr+\frac{b}{a} \tag{31}$$

By inserting equations (30) and (31) into equation (10) and by taking the ground state energy as ε_0 we get

$$\frac{t^2a^2q}{\sinh_q^2 tr}+t^2a^2+2tb\coth_q tr+\frac{b^2}{a^2}+\left(\frac{t^2qa}{\sinh_q^2 tr}\right)=\frac{t^2\{\eta'(\eta'-1)\}}{\sinh_q^2 tr}-2tv'\coth_q tr-\varepsilon_0 \tag{32}$$

From equation (32) we have

$$a^2+a=\frac{\eta'(\eta'-1)}{q},\ \ b=-v',\left(t^2a^2+\frac{b^2}{a^2}\right)=-\varepsilon_0 \tag{33}$$

and thus from all expressions in equation (33) we get the values of *a*,*b*, and ε_0 that have physical meaning as,

$$a=-1/2+\sqrt{\left\{\frac{\eta'(\eta'-1)}{q}+\frac{1}{4}\right\}}\ ;\ b=-v' \tag{34}$$

$$\varepsilon_0=-\left(t^2a^2+\frac{b^2}{a^2}\right) \tag{35}$$

where equation (35) is the ground state relativistic energy equation of the system. By using equations (30), (33) and (34), the super-potential is obtained, given as

$$\phi(r)=ta\coth_q tr+\frac{b}{a} \tag{36}$$

By inserting equation (36) into equatons (5) and (6) we get the super-partner potentials as

$$V_-(r,a_0)=\frac{t^2qa(a+1)}{\sinh_q^2 tr}+2tb\coth_q tr+t^2a^2+\frac{b^2}{a^2} \tag{37}$$

$$V_+(r,a_0)=\frac{t^2qa(a-1)}{\sinh_q^2 tr}+2tb\coth_q tr+t^2a^2+\frac{b^2}{a^2} \tag{38}$$

For $a\to a-1$

$$V_-(r,a_1)=\frac{t^2\{(a-1)(a)\}}{\sinh_q^2 tr}-2tv'\coth_q tr+t^2(a-1)^2+\frac{b^2}{(a-1)^2} \tag{39}$$

By comparing the coefficient of the variables in equations (37) and (38) we obtain the mapping parameters $a_0,a_1,\ldots,a_n$ given as: $a_0=a,a_1=a-1,....,a_n=a-n$ (40)

By using equations (37) and (40) we have

$$V_{-}(r,a_2)=\frac{t^2\{(a-2)(a-1)\}}{\sinh^2 tr}-2tv'\coth tr+t^2(a-2)^2+\frac{b^2}{(a-2)^2} \tag{41}$$

From equations (39) and (41) we can see that $V_{+}(r,a_0)$ have the same function form as $V_{-}(r,a_1)$ and by using the shape invariance condition in equation (11),we get

$$R(a_1)=V_{+}(r;a_0)-V_{-}(r;a_1)=t^2a^2+\frac{b^2}{a^2}-\left(t^2(a-1)^2+\frac{b^2}{(a-1)^2}\right) \tag{42}$$

Using generalizations of equation (42) and by using the mapping condition equation (40) we obtain

$$R(a_n)=V_{+}(r;a_{n-1})-V_{-}(r;a_n)=t^2(a-(n-1))^2+\frac{b^2}{(a-(n-1))^2}-\left(t^2(a-n)^2+\frac{b^2}{(a-n)^2}\right) \tag{43}$$

Using equations (14), (15), (42) and (43) we get

$$\varepsilon_n^{(-)}=t^2a^2+\frac{b^2}{a^2}-\left(t^2(a-n)^2+\frac{b^2}{(a-n)^2}\right) \tag{44}$$

that gives

$$\varepsilon_n=E'=-\left(t^2(a-n)^2+\frac{b^2}{(a-n)^2}\right) \tag{45}$$

$$a=-1/2+\sqrt{\left\{\frac{\eta'(\eta'-1)}{q}+1/4\right\}}=-1/2+\sqrt{\left\{\frac{(\eta'-1/2)^2}{q}+1/4-1/(4q)\right\}} \tag{46}$$

By inserting equation (27) into equation (46) we get

$$a=-1/2+\sqrt{\left\{\frac{\eta(\eta-1)(M+E)+(l+\frac{D-1}{2})(l+\frac{D-3}{2})}{q}+1/4\right\}};b=-v'=-v(M+E) \tag{47}$$

By inserting equations (28)and (47) into equation (45) we obtain the relativistic energy equation given as

$$(E^2-M^2)=\left\{(l+\frac{D-1}{2})(l+\frac{D-3}{2})t^2d_0-\left(t^2(a-n)^2+\frac{b^2}{(a-n)^2}\right)\right\} \tag{48}$$

with *a* and *b* are expressed in equation (47).

The relativistic energy spectrum in Table 1 is obtained numerically from the relativistic energy equation in equation (48) with the help of the Math-Lab software application.

Table 1. Relativistic energy spectra ($d_0=\frac{1}{12}, t=0.8, n=1, l=2, M=10\ fm^{-1}, \eta=2, v=1.5$)

Parameter	$E(fm^{-1})$				
	$q=0.1$	$q=0.2$	$q=0.3$	$q=0.4$	$q=0.5$
$D=2$	-13.88	-13.30	-12.72	-12.16	-11.69
$D=3$	-17.22	-16.40	-15.52	-14.65	-13.84
$D=4$	-21.54	-20.46	-19.24	-17.99	-16.81
$D=5$	-26.85	-25.47	-23.88	-22.22	-20.60
$D=6$	-33.15	-31.45	-29.44	-27.30	-25.18

In the non-relativistic limit, the relativistic energy reduces to non-relativistic energy as follows since $(E+M)\to 2\mu$, μ is the non-relativistic mass, and $(E-M)\to E_{NR}$, E_{NR} is the non-relativistic energy then we have

$$E^2 - M^2 = (E+M)(E-M) = 2\mu E_{NR} \tag{49}$$

and equation (47) becomes

$$a_{NR} = -1/2 + \sqrt{\left\{\frac{\eta(\eta-1)2\mu + (l+\frac{D-1}{2})(l+\frac{D-3}{2})}{q} + 1/4\right\}} \tag{50}$$

The relativistic energy equation in equation (48) for special case, $D = 3$, reduces the the non-relativistic energy given as

$$E_{NR} = \left\{\frac{(l+1)(l)t^2 d_0}{2\mu} - \frac{1}{2\mu}\left(\begin{array}{l} t^2\left(-1/2+\sqrt{\left\{\frac{\eta(\eta-1)2\mu+(l+1)l}{q}+1/4\right\}}-n\right)^2 \\ +\frac{4\mu^2 v^2}{\left(-1/2+\sqrt{\left\{\frac{\eta(\eta-1)2\mu+(l+1)l}{q}+1/4\right\}}-n\right)^2}\end{array}\right)\right\} \tag{51}$$

which is in agreement with the energy of the hyperbolic Manning-Rosen potential obtained using other methods [41], and for any dimensions, D, the non-relativistic energy is given by

$$E_{NR} = \frac{1}{2\mu}\left\{(l+\frac{D-1}{2})(l+\frac{D-3}{2})t^2 d_0 - \left(t^2(a_{NR}-n)^2 + \frac{v'^2}{(a_{NR}-n)^2}\right)\right\} \tag{52}$$

with a_{NR} is expressed in equation (50).

For small value of d_0, $(l+\frac{D-1}{2})(l+\frac{D-3}{2})t^2 d_0 \approx 0$, therefore the non-relativistic energy is approximated by:

$$E_{NR} = -\frac{1}{2\mu}\left(t^2(a_{NR}-n)^2 + \frac{v'^2}{(a_{NR}-n)^2}\right) \tag{53}$$

By manipulating equations (7), (16), (17), and (41) we obtain the relativistic ground state and first excited state wave function as follows. By inserting equations (41) and (7) into equation (16), we get the radial ground state wave function as

$$\left(\frac{d}{dr} + (ta\coth_q tr - \frac{v'}{a})F_0 = 0 \to F_0 = (\sinh_q tr)^{-a} e^{\frac{v'}{a}r}\right. \tag{54}$$

The first exited state of wave function is obtained by

$$F_1 = \left(-\frac{d}{dr} + ta\coth_q tr - \frac{v'}{a}\right)F_0 \to F_1 = 2\left[at(\coth_q tr) - \left(\frac{v'}{a}\right)\right](\sinh_q tr)^{-a} e^{\frac{v}{a}r} \tag{55}$$

and the second exited state wave function is

$$F_2 = 2\left\{\left[at^2 q(\operatorname{csch}^2 tr)\right] + 2\left[at(\coth tr) - \left(\frac{v'}{a}\right)\right]^2 (\sinh\ tr)^{-a} e^{\frac{v}{a}r}\right\} \tag{56}$$

and for the higher levels of the relativistic wavefunctionscan be obtained using SUSY operators.

3.1. Thermodynamical Properties

In classical regimes [42], the vibrational partition function, vibrational mean energy, and specific heat are obtained from the non-relativistic energy equation in equation (53).The vibrational partition function is defined as

$$Z(\zeta,\beta)=\sum_{n=0}^{\zeta}e^{-\beta E_{nl}},\beta=\frac{1}{kT} \tag{57}$$

k is Boltzman constant, E_{nl} is non-relativistic energy spectrum of the system. The non-relativistic energy of the system in equation (53) for special case when v is very small then the non-relativistic energy in equation reduces to

$$E_{NR}=-(1/2\mu)\left(t^2\left(n-\varsigma\right)^2\right) \tag{58}$$

and therefore the vibrational partition function in equation (57) reduces to

$$Z(\zeta,\beta)=\sum_{n=0}^{\zeta}e^{-\beta E_{nl}}=\sum_{n=0}^{\zeta}e^{\beta\frac{1}{2\mu}\left(t^2(n-\varsigma)^2\right)}=\sum_{n=0}^{\zeta}e^{\left\{\frac{(n-\varsigma)}{\delta}\right\}^2}, \tag{59}$$

By setting

$$\frac{\beta t^2}{2\mu}=\frac{1}{\delta^2};\frac{n-\zeta}{\delta}=y;\varsigma=a_{NR}=-1/2+\sqrt{\left\{\frac{\eta(\eta-1)2\mu+(l+\frac{D-1}{2})(l+\frac{D-3}{2})}{q}+1/4\right\}} \tag{60}$$

in equation (59) and in the classical regime when the temperature, *T*, is high enough, causes the value of ς is high , and β is small then equation (59) could be written into integral form as

$$Z(\varsigma,\beta)=\sum_{n=0}^{\zeta/\delta}e^{y^2}=\int^{\zeta/\delta}e^{y^2}dy=\left(\sqrt{\pi}/2\right)erfi\left(\zeta/\delta\right)=i\left(\sqrt{\pi}/2\right)erf\left(i\zeta/\delta\right) \tag{61}$$

In this section, the thermodynamics properties will be expressed in terms of two mathemathicalfunctions: the Dawson function and the imaginary error function. The Dawson function or Dawson integral (named for John M. Dawson) is denoted as [43]

$$F(x)=e^{-x^2}\int_0^x e^{y^2}dy=\frac{\sqrt{\pi}e^{-x^2}erfi\left(x\right)}{2} \tag{62}$$

and the imaginary error function is defined by

$$erfi(x)=ierf\left(ix\right) \tag{63}$$

where erf is the error function given as

$$erf\left(x\right)=\frac{2}{\sqrt{\pi}}\int_0^x e^{-t^2}dt \tag{64}$$

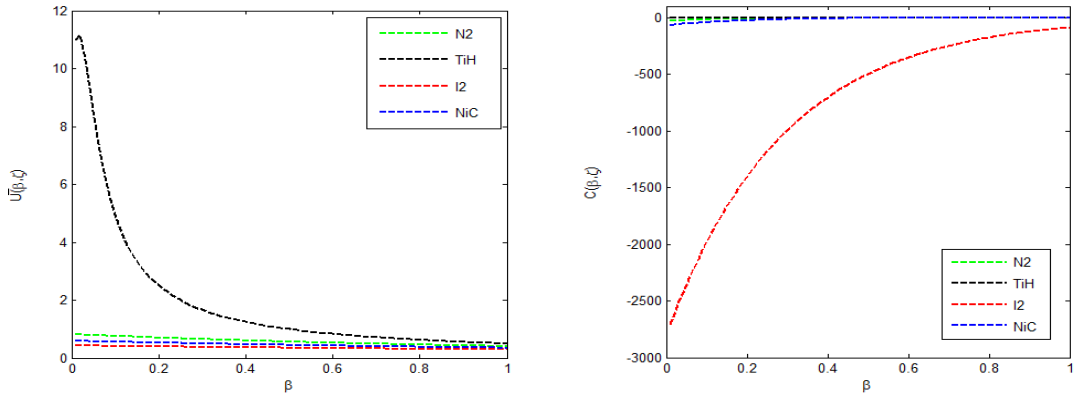
By applying equations (58,61-64) the vibrational specific heat and the mean energy are obtained.The vibrational mean energy is

$$U(\beta,\zeta)=-\frac{\partial}{\partial\beta}\ln Z(\zeta,\beta)==-\left(\left(i\frac{t\varsigma}{2\sqrt{2\mu\beta}}e^{\frac{t^2\varsigma^2\beta}{2\mu}}\right)\Big/erf\left(i\frac{t\sqrt{\beta}\varsigma}{\sqrt{2\mu}}\right)\right)=-\left(\left(\frac{t\varsigma\sqrt{\pi}}{4\sqrt{2\mu\beta}}\right)\Big/DawsonF\left(\frac{t\sqrt{\beta}\varsigma}{\sqrt{2\mu}}\right)\right) \tag{65}$$

The vibrational specific heat is

$$C(\beta,\zeta) = -\frac{\partial}{\partial T}U(\zeta,\beta) == k\beta^2 \frac{t\varsigma}{2\sqrt{2\mu\beta}} \frac{\sqrt{\pi}}{2} \left\{ \frac{\left(-\frac{1}{2\beta} + \frac{t^2\varsigma^2}{2\mu}\right)}{Dawson\,F\left(\frac{t\sqrt{\beta}\varsigma}{\sqrt{2\mu}}\right)} - \frac{\left(\frac{t\varsigma}{2\sqrt{2\mu\beta}}\right)}{\left[Dawson\,F\left(\frac{t\sqrt{\beta}\varsigma}{\sqrt{2\mu}}\right)\right]^2} \right\} \quad (66)$$

The vibrational mean energy and the vibrational specific heat of the system of diatomic molecules governed by hyperbolic potential are calculated numerically using equations (65) and (66) with the help of Math-Lab software, which is shown in figure 1 below,



(a) (b)

Figure 1. Graph of (a) mean energy $U(\beta,\zeta)$ as a function of β, (b)vibrational specific heat $C(\beta,\zeta)$ as a function of β (for D=3)

From figure 1(a) and 1(b) we see that vibrational mean energywhose is governed by q-deformed hyperbolic Manning-Rosen potential are positives and specific heat for system are negatives. The negative specific heat may occur at the astronomical objects [44], at the glass transitions [45] and refers to previous research[46,47].

4. Conclusion

The relativistic and non-relativistic energy equations for q-deformed the hyperbolic Manning-Rosen potential are obtainable by using SUSY quantum mechanics. The q-deformed hyperbolic potential is used to describe the behavior of diatomic molecules. In the non-relativistic limit, the relativistic energy equation reduces to the non-relativistic energy. By using the imaginary error function and the Dawson function, the vibrational partition function, specific heat and the mean energy are derived from the non-relativistic energy equation in the classical regime.

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