



Effects of order α^3 on the determination of the Pauli form factor F_2 of the τ -lepton

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Abstract We introduce optimal observables to measure the Pauli form factor F_2 of the τ -lepton in the pair-production process $e^-e^+ \rightarrow \tau^-\tau^+$ from the intensity distribution of the decay products. The spin-density matrix for the production process is calculated in QED up to order α^3 including virtual photon-loops and soft bremsstrahlung, as well as the γZ^0 interference. We find that the decay channel $(\rho^- \nu_\tau) \times (\rho^+ \bar{\nu}_\tau)$ yields the best resolution for $\text{Re}F_2(s)$ and $\text{Im}F_2(s)$ due to its high branching fraction. We also study the bias that is introduced in the determination of F_2 , if the production spin-density matrix is taken in tree-level (one-photon exchange) approximation.

1 Introduction

In the light of the very accurate results for the anomalous magnetic moment $a_\mu = (g_\mu - 2)/2$ of the muon presented recently in Ref. [1], a measurement of the analogous quantity for the τ -lepton would clearly be of great interest. In quantum electrodynamics (QED) the coupling vertex of the photon to a τ -lepton pair is of the form $e \bar{u}_{\tau^-} \Gamma^\mu v_{\tau^+}$, with \bar{u}_{τ^-} and v_{τ^+} free Dirac-spinors and the composed Dirac-matrix:

$$\Gamma^\mu = F_1(s)\gamma^\mu + \frac{i}{2m_\tau}F_2(s)\sigma^{\mu\nu}q_\nu, \quad \sigma^{\mu\nu} = \frac{i}{2}[\gamma^\mu, \gamma^\nu], \quad (1)$$

where q^μ is the four-momentum carried by the virtual photon and $s = q^\mu q_\mu$. Here, one has introduced the so-called Dirac form factor $F_1(s)$, normalized to $F_1(0) = 1$, and the so-called Pauli form factor $F_2(s)$, that gives the anomalous magnetic moment $F_2(0) = a_\tau = (g_\tau - 2)/2$.

One way to access this coupling is the production of τ -lepton pairs at e^-e^+ -colliders:

$$e^- + e^+ \rightarrow \tau^- + \tau^+. \quad (2)$$

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In such a process, a non-zero real part $\text{Re}F_2(s)$ leads to a modification of the cross section, while a non-zero imaginary part $\text{Im}F_2(s)$ even alters the spin-correlation between the produced τ^- and τ^+ [2]. However, due to the very short life-time (of less than a picosecond), τ -leptons typically decay before they can reach any detector and therefore only their spin correlation can be accessed through the angular distribution of the decay products. At the same time, one should keep in mind, that in τ -pair production only the kinematical region $s > 4m_\tau^2$ is accessible, and thus one does not measure the anomalous magnetic moment a_τ directly.

While a similar measurement aiming at the electric dipole moment d_τ has been performed [3], no such analysis was done for a_τ because of expected contaminations of the measurement by higher order QED effects. Since the $\gamma\tau^-\tau^+$ coupling that incorporates the electric dipole moment d_τ stems from a CP-odd higher dimensional operator, these QED corrections have been assumed to be of no relevance for the pertinent CP-odd observable. It has been argued in the past that a measurement of $F_2(s)$ in τ -pair production events may be meaningful and feasible, e.g. due to suppression of the contaminations at $\sqrt{s} = m_{\Upsilon(4S)} = 10.58$ GeV. For this reason we want to determine in this work the actual size of these contaminations by comparing results from leading order QED with next-to-leading order QED calculations, and relate these corrections to the resolution that can be achieved in an extraction of $F_2(s)$.

Our paper is organized as follows. In Sect. 2 we introduce our analysis method employing the so-called optimal observables. In Sect. 3 we derive analytical expressions for the spin-density matrix of τ -pair production including virtual photon-loops and soft bremsstrahlung up to order α^3 , and we estimate the effect of these next-to-leading order corrections in Sect. 4. Finally, the conclusions of our study are given in Sect. 5. Appendix A collects the basic loop-functions that are encountered in the evaluation of the two-photon exchange box diagrams, and in appendix B we present our results based

on optimal observables concerning the measurement of the electric dipole moment d_τ . Furthermore, in appendix C we examine the effect of varying the infrared cutoff λ for undetected soft photons.

2 Pauli form factor $F_2(s)$ and optimal observables

In order to access the Pauli form factor $F_2(s)$, we start from the full intensity distribution \mathcal{I} for $\tau^- \tau^+$ pair production, including their weak decays. It involves four sums over helicities and takes the form:

$$\mathcal{I} = \sum_{\lambda_\mp, \lambda_\mp^{(0)}} \chi_{\lambda_- \lambda_+, \lambda'_- \lambda'_+}^{\text{prod}} \cdot \chi_{\lambda_-, \lambda'_-}^- \cdot \chi_{\lambda_+, \lambda'_+}^+, \quad (3)$$

where $\chi_{\lambda_-, \lambda'_-}^-$ and $\chi_{\lambda_+, \lambda'_+}^+$ stand for the spin-density matrices of the respective weak τ^\mp -decays, and $\chi_{\lambda_- \lambda_+, \lambda'_- \lambda'_+}^{\text{prod}}$ denotes the spin-density matrix of $\tau^- \tau^+$ pair-production in $e^- e^+$ annihilation, which we will give in analytical form up to order α^3 (with $\alpha = 1/137.036$) in Sect. 3. The spin-density matrices $\chi_{\lambda_\mp, \lambda'_\mp}^\mp$ for the weak decays arise from tree-level amplitudes involving W^\mp exchange in the case of leptonic decays, and for hadronic τ^\mp -decays they can be found in Refs. [4, 5]. Since the Pauli form factor F_2 is small, one can expand the intensity distribution in terms of F_2 up to linear order:

$$\mathcal{I} = \mathcal{I}_0 + \text{Re} F_2 \mathcal{I}_{\text{Re}} + \text{Im} F_2 \mathcal{I}_{\text{Im}}. \quad (4)$$

Using this expansion, we define the following optimal observables [6]:

$$\mathcal{O}_{\text{Re,Im}}^{\text{opt}} = \frac{\mathcal{I}_{\text{Re,Im}}}{\mathcal{I}_0}, \quad (5)$$

which are exclusively sensitive to contributions from F_2 . For a given set \mathcal{S} of data, the value of F_2 is determined through the expectation values $\langle \mathcal{O}_{\text{Re,Im}}^{\text{opt}} \rangle_{\mathcal{S}}$ of the optimal observables. We generate Monte Carlo data sets in order to predict the dependence of the expectation values of the optimal observables on F_2 . In contrast to previous analyses [3, 7], we do not generate data sets at various working points for F_2 and then perform a linear fit through these working points. Instead, we generate a single data set \mathcal{S}_0 , using the value $F_2 = 0$ and determine the dependence of the expectation values using a reweighting approach. For this purpose, we construct a weight $w(F_2)$ for every event that reweights the data set, such that it would correspond to a non-zero value $F_2 \neq 0$:

$$w(F_2) = (\mathcal{I}_0 + \text{Re} F_2 \mathcal{I}_{\text{Re}} + \text{Im} F_2 \mathcal{I}_{\text{Im}} + |F_2|^2 \mathcal{I}_{|F_2|^2}) / \mathcal{I}_0, \quad (6)$$

where quadratic terms have to be kept, since these weights will enter non-linearly. Having these weights, we can predict the expectation value for an optimal observable with $F_2 \neq 0$ using the formula:

$$\langle \mathcal{O}_{\text{Re,Im}}^{\text{opt}} \rangle_{F_2 \neq 0} = \langle w(F_2) \mathcal{O}_{\text{Re,Im}}^{\text{opt}} \rangle_{F_2=0} / \langle w(F_2) \rangle_{F_2=0}, \quad (7)$$

which results in two coupled quadratic equations that can be solved for $\text{Re} F_2$ and $\text{Im} F_2$. Note that since F_2 is small, one takes the solution with the smallest magnitude $|F_2|$ as the correct one.

2.1 Reconstruction effects

The method described so far relies on the calculation of the intensity \mathcal{I} from the kinematics of the decay products in an event. However, in reality this kinematics is not fully accessible since at least one neutrino escapes for every τ -decay. This results in a two-fold kinematical ambiguity if one single neutrino is produced in each weak decay, corresponding to semi-leptonic channels. Moreover, in leptonic decays there remains an ambiguity in three unknown kinematical variables, the invariant mass $m_{\nu_\tau \bar{\nu}_\ell}$ and the two angles describing the $\nu_\tau \bar{\nu}_\ell$ system.

In order to account for this problem, we use the integrated intensity $\bar{\mathcal{I}}$, where all unknown kinematical degrees of freedom are integrated over and thus all ambiguities are averaged out. Using an expansion of $\bar{\mathcal{I}}$ analogous to Eq. (4), we define integrated optimal observables $\bar{\mathcal{O}}_{\text{Re,Im}}^{\text{opt}}$ according to Eq. (5), by substituting \mathcal{I} with $\bar{\mathcal{I}}$ and the same for its components $\mathcal{I}_{0,\text{Re,Im}}$.

Using these averages, we can extract $\text{Re} F_2$ and $\text{Im} F_2$ in an approach similar to Eq. (7) by setting:

$$\langle \bar{\mathcal{O}}_{\text{Re,Im}}^{\text{opt}} \rangle_{F_2 \neq 0} = \langle w(F_2) \bar{\mathcal{O}}_{\text{Re,Im}}^{\text{opt}} \rangle_{F_2=0} / \langle w(F_2) \rangle_{F_2=0}, \quad (8)$$

where the calculation of the weights $w(F_2)$ has to be performed according to Eq. (6), using the not integrated and not averaged components $\mathcal{I}_{0,\text{Re,Im}}$. These weights can in fact be calculated, since the data set \mathcal{S}_0 is generated by Monte Carlo, and the kinematically not reconstructible information of events is available. Moreover, detector effects can be taken into account in a similar way by calculating $\bar{\mathcal{O}}_{\text{Re,Im}}^{\text{opt}}$ from a data set \mathcal{S}_0 with detector simulation included, while using the known generator information in the calculation of the weights $w(F_2)$.

3 Spin-density matrix for τ -lepton pair production in QED to one-loop order

So far we have treated the extraction of the form factor F_2 by splitting the intensity distribution \mathcal{I} into a part proportional to F_2 , and a standard model (one-photon exchange) part. However, the spin-density matrix for the pair production process $e^-(\vec{k}) + e^+(-\vec{k}) \rightarrow \tau^-(\vec{p}) + \tau^+(-\vec{p})$ in the center of mass frame is composed of several contributions up to order α^3 in QED:

$$\chi^{\text{prod}} = \frac{\alpha^2 \sqrt{s-4}}{16m_\tau^2 s^{5/2}} \left\{ \mathcal{M}_{\text{tree}} + \mathcal{M}_{e\text{-vert}} + \mathcal{M}_{\text{vacpol}} + \mathcal{M}_{\tau\text{-vert}} + \mathcal{M}_{\text{box}} + \mathcal{M}_{\text{tree}} \delta_{\text{soft}} \right\}, \tag{9}$$

where the prefactor is adjusted to the differential cross section $d\sigma/d\Omega_{\text{cm}}$. For notational simplicity, we omit from now on the four helicity indices $\lambda_+ \lambda_- , \lambda'_+ \lambda'_-$ on the spin-density matrix χ^{prod} .

In the following we will present explicit expressions for the six contributions $\mathcal{M}_{\text{tree}}, \dots, \delta_{\text{soft}}$ appearing in Eq. (9) as functions of the dimensionless Mandelstam variables s, t and u , which satisfy the constraint $s + t + u = 2$. With this convenient choice of variables¹ the initial and final momenta are given in units of the τ -lepton mass $m_\tau = 1776.86 \text{ MeV}$ by $|\vec{k}| = \sqrt{s} m_\tau/2$ and $|\vec{p}| = \sqrt{s-4} m_\tau/2$. The invariant squared momentum-transfers, t and u , are expressed in terms of s and the cosine of the scattering angle, $\cos \theta = \hat{p} \cdot \hat{k}$, as:

$$t = \frac{1}{2} \left(2 - s + \sqrt{s^2 - 4s} \cos \theta \right), \quad 2 - s < t, u < 0, \\ u = \frac{1}{2} \left(2 - s - \sqrt{s^2 - 4s} \cos \theta \right). \tag{10}$$

Using these definitions and arranging the prefactor as in Eq. (9), one obtains from the tree-level one-photon exchange diagram:

$$\mathcal{M}_{\text{tree}} = \left[s + 4 + (s - 4) \cos^2 \theta \right] \mathbf{1} + (s - 4) \sin^2 \theta \vec{\sigma}_1 \cdot \vec{\sigma}_2^* \\ - 2 \left[s - 4 + (\sqrt{s} - 2)^2 \cos^2 \theta \right] \vec{\sigma}_1 \cdot \hat{p} \vec{\sigma}_2^* \cdot \hat{p} \\ + 2(s - 2\sqrt{s}) \cos \theta \left[\vec{\sigma}_1 \cdot \hat{p} \vec{\sigma}_2^* \cdot \hat{k} + \vec{\sigma}_1 \cdot \hat{k} \vec{\sigma}_2^* \cdot \hat{p} \right] \\ - 2s \vec{\sigma}_1 \cdot \hat{k} \vec{\sigma}_2^* \cdot \hat{k}, \tag{11}$$

where spin-space 1 belongs to the τ^- ($\vec{\sigma}_1 = \vec{\sigma}_{\lambda_+ \lambda'_-}$) and spin-space 2 belongs to the τ^+ ($\vec{\sigma}_2 = \vec{\sigma}_{\lambda_- \lambda'_+}$), with the same quantization axis for helicity eigenstates. The unit-operator $\mathbf{1}$ and the products of hermitian spin-operators $O_1 O_2$ get sandwiched with two-component spin wave functions in the way: $(\chi_1^\dagger O_1 \chi_1)(\chi_2^\dagger O_2 \chi_2^*) = (\chi_1^\dagger O_1 \chi_1)(\chi_2^\dagger O_2^* \chi_2) = (\chi_1^\dagger O_1 \chi_1)(\chi_2^\dagger O_2^* \chi_2)$. The first arrangement of χ -states results from multiplying the original expression (obtained from reducing Dirac-spinors and Dirac-matrices in the diagrammatic amplitude times its complex-conjugated) from the right with the spin-exchange operator $(\mathbf{1} + \vec{\sigma}_1 \cdot \vec{\sigma}_2)/2$, and the second or third explain the occurrence of the transposed or complex-conjugated Pauli spin-matrix $\vec{\sigma}_2^t = \vec{\sigma}_2^*$. Note that $\vec{\sigma}_2^*$ and $-\vec{\sigma}_2$ are unitarily equivalent due to the relation: $(i\sigma_{2y})\vec{\sigma}_2^*(i\sigma_{2y})^{-1} = -\vec{\sigma}_2$. In this context it is worth to stress that our result for $\mathcal{M}_{\text{tree}}$ in Eq. (11) agrees with Ref. [8] after the substitution $\vec{\sigma}_2^* \rightarrow -\vec{\sigma}_2$.

¹ Note that only in Sect. 3 and appendix A the Mandelstam variable s is dimensionless.

The first next-to-leading order correction of order α comes from the electronic vertex correction by a photon-loop. Its contribution to the spin-density matrix reads:

$$\mathcal{M}_{e\text{-vert}} = 2\mathcal{M}_{\text{tree}} \text{Re} F_1^e(s) \\ = \mathcal{M}_{\text{tree}} \frac{\alpha}{\pi} \left\{ (2\xi_{\text{IR}} + \ln r) \left(1 - \ln \frac{s}{r} \right) - \frac{1}{2} \ln^2 \frac{s}{r} + \frac{3}{2} \ln \frac{s}{r} + \frac{2\pi^2}{3} - 2 \right\}, \tag{12}$$

where only those terms that do not vanish in the limit $r \rightarrow 0$ have been kept, with the very small squared mass ratio $r = (m_e/m_\tau)^2 = 8.271 \cdot 10^{-8}$. The infrared divergence due to the vanishing photon mass is treated in dimensional regularization and handled by the pole-term:

$$\xi_{\text{IR}} = \frac{1}{d-4} + \frac{1}{2} (\gamma_E - \ln 4\pi) + \ln \frac{m_\tau}{\mu}, \tag{13}$$

with d the number of space-time dimensions and μ an arbitrary mass scale. Note that the contribution of the Pauli form factor $F_2^e(s)$ of the electron vanishes in the limit $r \rightarrow 0$.

The next radiative correction arises from vacuum polarization in the one-photon exchange:

$$\mathcal{M}_{\text{vacpol}} = 2\mathcal{M}_{\text{tree}} \Delta\alpha(\sqrt{s} m_t), \tag{14}$$

with $\Delta\alpha(E_{\text{cm}}) = \text{Re}\Pi(E_{\text{cm}})$ taken from empirical determinations that include leptonic and hadronic contributions [9]. Ignoring sharp bottomonium resonances, we take an average value of about $\Delta\alpha(10 \text{ GeV}) \simeq 3.5\%$. We note as an aside that the unique vacuum polarization due to charged leptons (e^\mp, μ^\mp, τ^\mp) reads:

$$\text{Re}\Pi(\sqrt{s} m_\tau)^{\text{lept}} = \frac{2\alpha}{3\pi} \left\{ \frac{s+2}{s^{3/2}} \sqrt{s-4} \ln \frac{\sqrt{s} + \sqrt{s-4}}{2} - \frac{5}{2} - \frac{2}{s} + \ln s + \ln \frac{m_\tau^2}{m_e m_\mu} \right\}, \tag{15}$$

and this part amounts to 2.1% at 10 GeV. Here, the contributions from electrons and muons are well approximated by leading logarithms plus constants.

The next radiative correction that one has to take into account is the vertex correction at the τ -lepton photon interaction. The corresponding contribution to the spin-density matrix reads:

$$\mathcal{M}_{\tau\text{-vert}} = 2\mathcal{M}_{\text{tree}} \text{Re} \left[F_1^\tau(s) + F_2^\tau(s) \right] \\ + (s-4) \left\{ \text{Im} F_2^\tau(s) \sqrt{s} \cos \theta (\vec{\sigma}_1 - \vec{\sigma}_2^*) \cdot (\hat{p} \times \hat{k}) + \text{Re} F_2^\tau(s) \left[2 \sin^2 \theta (\mathbf{1} - \vec{\sigma}_1 \cdot \vec{\sigma}_2^*) + 2 \left(2 + (\sqrt{s} - 2) \cos^2 \theta \right) \vec{\sigma}_1 \cdot \hat{p} \vec{\sigma}_2^* \cdot \hat{p} - \sqrt{s} \cos \theta \left(\vec{\sigma}_1 \cdot \hat{p} \vec{\sigma}_2^* \cdot \hat{k} + \vec{\sigma}_1 \cdot \hat{k} \vec{\sigma}_2^* \cdot \hat{p} \right) \right] \right\}, \tag{16}$$

and we recognize here the first correction that is not purely proportional to $\mathcal{M}_{\text{tree}}$. The photon-loop form factors $F_1^\tau(s)$ and $F_2^\tau(s)$ of the τ -lepton in the time-like region $s > 4$ are given by the expressions:

$$\begin{aligned} \text{Re}[F_1^\tau(s) + F_2^\tau(s)] &= \frac{\alpha}{\pi} \left\{ \xi_{\text{IR}} \left(1 + \frac{4-2s}{\sqrt{s^2-4s}} \ln \frac{\sqrt{s} + \sqrt{s-4}}{2} \right) \right. \\ &+ \frac{3\sqrt{s-4}}{2\sqrt{s}} \ln \frac{\sqrt{s} + \sqrt{s-4}}{2} - 1 + \frac{s-2}{\sqrt{s^2-4s}} \left[\ln^2 \frac{\sqrt{s} + \sqrt{s-4}}{2} \right. \\ &\left. \left. - \ln \frac{\sqrt{s} + \sqrt{s-4}}{2} \ln(s-4) + \frac{\pi^2}{3} + \text{Li}_2 \left(\frac{s-2-\sqrt{s^2-4s}}{2} \right) \right] \right\}, \end{aligned} \tag{17}$$

and

$$F_2^\tau(s) = \frac{\alpha}{\pi} \frac{1}{\sqrt{s^2-4s}} \left(i\pi - 2 \ln \frac{\sqrt{s} + \sqrt{s-4}}{2} \right), \tag{18}$$

with Li_2 the conventional dilogarithmic function (see also Appendix A). Note that $F_2^\tau(0) = \alpha/2\pi = 1.1614 \cdot 10^{-3}$ is the famous Schwinger correction to the lepton anomalous magnetic moment. At 10 GeV one has a further suppression of $F_2^\tau(s)$ by a factor $-0.23 + i 0.21$.

The last radiative corrections from virtual photon-loops that one has to take into account are the two-photon exchange box diagrams. For the evaluation of the direct and crossed box diagram we follow the decomposition into basic spinorial structures as in section 2 of Ref. [10] and introduce eight complex-valued amplitudes:

$$\begin{aligned} X_1 &= \frac{1}{2} f_0(s, t) - [t \rightarrow u], & Y_1 &= \frac{1}{2} f_0(s, t) + [t \rightarrow u], \\ X_2 &= f_1(s, t) + f_2(s, t) + [t \rightarrow u], \\ Y_2 &= f_1(s, t) + f_2(s, t) - [t \rightarrow u], \\ X_3 &= -f_1(s, t) - 2f_3(s, t) - [t \rightarrow u], \\ Y_3 &= -f_1(s, t) - [t \rightarrow u], & X_4 &= 0, \\ Y_4 &= -f_2(s, t) - [t \rightarrow u], \end{aligned} \tag{19}$$

where the basis functions $f_{0,1,2,3}(s, t)$ are composed of the loop integrals $D_0, \bar{D}_0, C_s, C_t, \bar{C}_t, C_M, B_s - B_M$ and $B_t - B_M$ (see Appendix A for explicit expressions) in the same way as written in Eqs. (7–10) of Ref. [10] (setting $M = 1$).

Using the complex-valued amplitudes $X_{1,2,3}$ and $Y_{1,2,3,4}$, their individual contributions to the spin-density matrix from the interference of the box diagrams with the one-photon exchange read:

$$\mathcal{M}_{\text{box}}[X_1] = \mathcal{M}_{\text{tree}} \frac{\alpha s}{2\pi} \text{Re} X_1, \tag{20}$$

$$\begin{aligned} \mathcal{M}_{\text{box}}[Y_1] &= \frac{\alpha s}{\pi} \sqrt{s-4} \left\{ -\text{Im} Y_1 (\vec{\sigma}_1 - \vec{\sigma}_2^*) \cdot (\hat{p} \times \hat{k}) \right. \\ &+ \text{Re} Y_1 \left[\sqrt{s} \cos \theta \mathbf{1} + (2 - \sqrt{s}) \cos \theta \vec{\sigma}_1 \cdot \hat{p} \vec{\sigma}_2^* \cdot \hat{p} \right. \\ &\left. \left. - \vec{\sigma}_1 \cdot \hat{p} \vec{\sigma}_2^* \cdot \hat{k} - \vec{\sigma}_1 \cdot \hat{k} \vec{\sigma}_2^* \cdot \hat{p} \right] \right\}, \end{aligned} \tag{21}$$

$$\mathcal{M}_{\text{box}}[X_2] = \frac{\alpha s}{4\pi} \sqrt{s^2-4s} \left\{ -\text{Im} X_2 \sqrt{s} (\vec{\sigma}_1 - \vec{\sigma}_2^*) \cdot (\hat{p} \times \hat{k}) \right.$$

$$\begin{aligned} &+ \text{Re} X_2 \left[\left(\frac{s}{2} - 2 \right) \cos \theta \sin^2 \theta (\mathbf{1} - \vec{\sigma}_1 \cdot \vec{\sigma}_2^*) \right. \\ &+ (\sqrt{s} - 2) \left(2 + (\sqrt{s} - 2) \cos^2 \theta \right) \cos \theta \\ &\times \vec{\sigma}_1 \cdot \hat{p} \vec{\sigma}_2^* \cdot \hat{p} + s \cos \theta \vec{\sigma}_1 \cdot \hat{k} \vec{\sigma}_2^* \cdot \hat{k} \\ &\left. - \sqrt{s} \left(1 + (\sqrt{s} - 2) \cos^2 \theta \right) \right. \\ &\left. \times (\vec{\sigma}_1 \cdot \hat{p} \vec{\sigma}_2^* \cdot \hat{k} + \vec{\sigma}_1 \cdot \hat{k} \vec{\sigma}_2^* \cdot \hat{p}) \right] \Big\}, \end{aligned} \tag{22}$$

$$\begin{aligned} \mathcal{M}_{\text{box}}[Y_2] &= \frac{\alpha s^2}{4\pi} \sqrt{s-4} \left\{ -\text{Im} Y_2 (\vec{\sigma}_1 + \vec{\sigma}_2^*) \cdot (\hat{p} \times \hat{k}) \right. \\ &+ \text{Re} Y_2 \left[\vec{\sigma}_1 \cdot \hat{k} \vec{\sigma}_2^* \cdot \hat{p} - \vec{\sigma}_1 \cdot \hat{p} \vec{\sigma}_2^* \cdot \hat{k} \right. \\ &+ \sqrt{s-4} \left(\frac{\sin^2 \theta}{2} (\mathbf{1} + \vec{\sigma}_1 \cdot \vec{\sigma}_2^*) \right. \\ &\left. - \vec{\sigma}_1 \cdot \hat{p} \vec{\sigma}_2^* \cdot \hat{p} - \vec{\sigma}_1 \cdot \hat{k} \vec{\sigma}_2^* \cdot \hat{k} \right. \\ &\left. \left. + \cos \theta (\vec{\sigma}_1 \cdot \hat{p} \vec{\sigma}_2^* \cdot \hat{k} + \vec{\sigma}_1 \cdot \hat{k} \vec{\sigma}_2^* \cdot \hat{p}) \right) \right] \Big\}, \end{aligned} \tag{23}$$

$$\begin{aligned} \mathcal{M}_{\text{box}}[X_3] &= \frac{\alpha s}{4\pi} (s-4) \left\{ -\text{Im} X_3 \sqrt{s} \cos \theta (\vec{\sigma}_1 - \vec{\sigma}_2^*) \cdot (\hat{p} \times \hat{k}) \right. \\ &+ \text{Re} X_3 \left[2 \sin^2 \theta (\vec{\sigma}_1 \cdot \vec{\sigma}_2^* - \mathbf{1}) \right. \\ &\left. - 2 \left(2 + (\sqrt{s} - 2) \cos^2 \theta \right) \vec{\sigma}_1 \cdot \hat{p} \vec{\sigma}_2^* \cdot \hat{p} \right. \\ &\left. \left. + \sqrt{s} \cos \theta (\vec{\sigma}_1 \cdot \hat{p} \vec{\sigma}_2^* \cdot \hat{k} + \vec{\sigma}_1 \cdot \hat{k} \vec{\sigma}_2^* \cdot \hat{p}) \right] \right\}, \end{aligned} \tag{24}$$

$$\begin{aligned} \mathcal{M}_{\text{box}}[Y_3] &= \frac{\alpha s^2}{4\pi} \sqrt{s-4} \left\{ -\text{Im} Y_3 (\vec{\sigma}_1 + \vec{\sigma}_2^*) \cdot (\hat{p} \times \hat{k}) \right. \\ &+ \text{Re} Y_3 (\vec{\sigma}_1 \cdot \hat{k} \vec{\sigma}_2^* \cdot \hat{p} - \vec{\sigma}_1 \cdot \hat{p} \vec{\sigma}_2^* \cdot \hat{k}) \Big\}, \end{aligned} \tag{25}$$

$$\begin{aligned} \mathcal{M}_{\text{box}}[Y_4] &= \frac{\alpha s^2}{4\pi} \left\{ \text{Im} Y_4 \left[\frac{4-s}{\sqrt{s}} \cos \theta (\vec{\sigma}_1 - \vec{\sigma}_2^*) \cdot (\hat{p} \times \hat{k}) \right. \right. \\ &\left. \left. - \sqrt{s-4} (\vec{\sigma}_1 + \vec{\sigma}_2^*) \cdot (\hat{p} \times \hat{k}) \right] + \text{Re} Y_4 \right. \\ &\times \left[4 \vec{\sigma}_1 \cdot \hat{k} \vec{\sigma}_2^* \cdot \hat{k} - 4 + \sqrt{s-4} (\vec{\sigma}_1 \cdot \hat{k} \vec{\sigma}_2^* \cdot \hat{p} \right. \\ &\left. - \vec{\sigma}_1 \cdot \hat{p} \vec{\sigma}_2^* \cdot \hat{k}) + (\sqrt{s} - 2)^2 \frac{\cos \theta}{\sqrt{s}} (\vec{\sigma}_1 \cdot \hat{p} \vec{\sigma}_2^* \cdot \hat{k} \right. \\ &\left. \left. + \vec{\sigma}_1 \cdot \hat{k} \vec{\sigma}_2^* \cdot \hat{p} - 2 \cos \theta \vec{\sigma}_1 \cdot \hat{p} \vec{\sigma}_2^* \cdot \hat{p}) \right] \right\}, \end{aligned} \tag{26}$$

where we have used the relation: $\vec{\sigma}_1 \cdot (\hat{p} \times \hat{k}) \vec{\sigma}_2^* \cdot (\hat{p} \times \hat{k}) = \vec{\sigma}_1 \cdot \vec{\sigma}_2^* \sin^2 \theta - \vec{\sigma}_1 \cdot \hat{p} \vec{\sigma}_2^* \cdot \hat{p} - \vec{\sigma}_1 \cdot \hat{k} \vec{\sigma}_2^* \cdot \hat{k} + (\vec{\sigma}_1 \cdot \hat{p} \vec{\sigma}_2^* \cdot \hat{k} + \vec{\sigma}_1 \cdot \hat{k} \vec{\sigma}_2^* \cdot \hat{p}) \cos \theta$. The total contribution \mathcal{M}_{box} from the box diagrams is given by the sum of the seven individual parts corresponding to $X_1, X_2, X_3, Y_1, Y_2, Y_3$ and Y_4 . According to their construction, the infrared divergence ξ_{IR} is contained in the real parts of the amplitudes X_1, Y_1, X_2, Y_2 and Y_4 . Keeping track of all ξ_{IR} -terms, the total infrared divergence from the box diagrams is given by the piece:

$$\mathcal{M}_{\text{box}}^{\text{IR}} = \mathcal{M}_{\text{tree}} \frac{4\alpha}{\pi} \xi_{\text{IR}} \left[\ln(1-u) - \ln(1-t) \right], \tag{27}$$

where half of it comes from $\text{Re} X_1$ and the rest arranges itself to the same term proportional to $\mathcal{M}_{\text{tree}}$.

Finally, there appears at the end of Eq. (9) the correction factor δ_{soft} from soft photon radiation, which consists of a sum of two pieces: $\delta_{\text{soft}} = \delta_{\text{soft}}^{(\text{uni})} + \delta_{\text{soft}}^{(\text{cm})}$. The universal part:

$$\delta_{\text{soft}}^{(\text{uni})} = \frac{4\alpha}{\pi} \left(\ln \frac{m_\tau}{2\lambda} - \xi_{\text{IR}} \right) \times \left\{ 1 + \frac{2-s}{\sqrt{s^2-4s}} \ln \frac{\sqrt{s} + \sqrt{s-4}}{2} - \frac{1}{2} \ln \frac{s}{r} - \ln(1-t) + \ln(1-u) \right\}, \tag{28}$$

cancels the infrared divergence ξ_{IR} from virtual photon-loops and its remainder depends logarithmically on an infrared cut-off λ for undetected soft photon radiation. For the numerical studies to be presented in Sect. 4 we will choose the value of $\lambda = 100$ MeV. We estimate in Appendix C the effect of λ on the determination of F_2 by scaling it a factor 2 down and up. The other part is specific for assuming in the center-of-mass frame a small momentum sphere $|\vec{\ell}| < \lambda$ for undetected soft photons, and it reads:

$$\delta_{\text{soft}}^{(\text{cm})} = \frac{\alpha}{\pi} \left\{ \frac{2\sqrt{s}}{\sqrt{s-4}} \ln \frac{\sqrt{s} + \sqrt{s-4}}{2} + \ln \frac{s}{r} + \frac{2s-4}{\sqrt{s^2-4s}} \times \left[\ln^2 \frac{\sqrt{s} + \sqrt{s-4}}{2} - \ln \frac{\sqrt{s} + \sqrt{s-4}}{2} \ln(s-4) - \frac{\pi^2}{6} + \text{Li}_2 \left(\frac{s-2-\sqrt{s^2-4s}}{2} \right) \right] - \frac{1}{2} \ln^2 \frac{s}{r} - \frac{\pi^2}{3} + \frac{1}{2} \ln \frac{1-t}{1-u} \ln \frac{s-1+tu}{s^2} + \int_0^1 \frac{dx}{x} \left[\frac{\sqrt{s}(t-1)}{h_t \sqrt{s-4xh_t}} \ln \frac{\sqrt{s} + \sqrt{s-4xh_t}}{\sqrt{s} - \sqrt{s-4xh_t}} + \ln \frac{1-u}{1-t} + \frac{\sqrt{s}(1-u)}{h_u \sqrt{s-4xh_u}} \ln \frac{\sqrt{s} + \sqrt{s-4xh_u}}{\sqrt{s} - \sqrt{s-4xh_u}} \right] \right\}, \tag{29}$$

with the abbreviations $h_t = 1 + t(x-1)$ and $h_u = 1 + u(x-1)$. Note that the singular behavior with respect to the very small parameter $r = (m_e/m_\tau)^2$ has been extracted and terms vanishing in the limit $r \rightarrow 0$ have been dropped. Our calculation of $\delta_{\text{soft}}^{(\text{cm})}$ has been guided by the expression in Eq. (39) of Ref. [11] for muon-pair production, performing the limit $r \rightarrow 0$ and interchanging $t \leftrightarrow u$. The last integral $\int_0^1 dx x^{-1}[\dots]$, together with the (t, u) -dependent logarithmic term before, is solved by the following analytical expression:

$$H(s, t, u) = 2 \ln^2(1-t) + \ln(s+t-2+t^{-1}) \ln \frac{s}{(1-t)^2} - \text{Li}_2 \left(\frac{(1-t)^2}{-st} \right) + 2 \text{Li}_2(t) - 4 \ln \frac{\sqrt{s} + \sqrt{s-4}}{2} \times \ln \frac{\sqrt{-st - (1-t)^2} (2 + \sqrt{-st} + \sqrt{(4-s)t})}{(1-t + \sqrt{-st})(1+t + \sqrt{(4-s)t}) - st - (1-t)^2}$$

$$- \text{Li}_2 \left(\left[\frac{2t - \sqrt{-st} + \sqrt{(4-s)t}}{2 + \sqrt{-st} + \sqrt{(4-s)t}} \right]^2 \right) + \text{Li}_2 \left(\left[\frac{2 - \sqrt{-st} + \sqrt{(4-s)t}}{2t + \sqrt{-st} + \sqrt{(4-s)t}} \right]^2 \right) - [t \rightarrow u]. \tag{30}$$

We note that the master integral $I_y^{(2)}$ in Appendix A of Ref. [12] has been instrumental in deriving this analytical result for $H(s, t, u)$, and we have performed also extensive numerical checks. At this point one should also mention that the QED radiative corrections to $\tau^- \tau^+$ pair-production have also been studied in Ref. [13]. However, the entirely different representation of the spin-density matrix and the absence of analytical expressions for the virtual photon-loop and soft bremsstrahlung corrections hampers a direct and detailed comparison with our results.

In addition to pure QED corrections, we consider also the interference term between one-photon exchange and Z^0 -exchange. Its contribution to the spin-density matrix in the normalization of Eq. (9) reads:

$$\mathcal{M}_{\gamma Z^0} = \frac{s m_\tau^2}{(M_Z \sin 2\vartheta_W)^2} \left\{ \sqrt{s-4} [(\sqrt{s}-2) \cos \theta \vec{\sigma}_1 \cdot \hat{p} \vec{\sigma}_2^* \cdot \hat{p} - \sqrt{s} \cos \theta \mathbf{1} + \vec{\sigma}_1 \cdot \hat{p} \vec{\sigma}_2^* \cdot \hat{k} + \vec{\sigma}_1 \cdot \hat{k} \vec{\sigma}_2^* \cdot \hat{p}] + (1 - 4 \sin^2 \vartheta_W) \left[\left(\sqrt{s-4} \left(\frac{\sqrt{s}}{2} (1 + \cos^2 \theta) - \cos^2 \theta \right) + (s - 2\sqrt{s}) \cos \theta \right) (\vec{\sigma}_1 - \vec{\sigma}_2^*) \cdot \hat{p} + \left(\sqrt{s-4} \cos \theta + 2\sqrt{s} \right) (\vec{\sigma}_1 - \vec{\sigma}_2^*) \cdot \hat{k} \right] - \frac{1}{2} (4 \sin^2 \vartheta_W - 1)^2 \mathcal{M}_{\text{tree}} \right\}, \tag{31}$$

where $M_Z = 91.19$ GeV is the Z^0 boson mass and ϑ_W the Weinberg angle with the value $\sin^2 \vartheta_W = 0.231$ [14]. Note that the ratio between the vector coupling and axial-vector coupling of the Z^0 to leptons is $1 - 4 \sin^2 \vartheta_W$, and thus rather small.

4 Monte-Carlo studies for the measurement of F_2

In order to estimate the effect of the next-to-leading order QED corrections given in Sect. 3, we generate data sets containing 10^6 events each for all possible sixteen combinations of the four decay channels:

$$\pi^- \nu_\tau, \quad \rho^- \nu_\tau, \quad e^- \nu_\tau \bar{\nu}_e, \quad \mu^- \nu_\tau \bar{\nu}_\mu, \tag{32}$$

and their charge-conjugated counterparts. Since B factories currently offer the best way to perform the proposed analysis with real data, we use a center-of-mass energy of $\sqrt{s} = m_{\Upsilon(4S)} = 10.58$ GeV for our numerical studies. We generate the data sets with a simple sampling and rejecting algorithm and analyze them by using the methods laid out

in Sect. 2. Since the kinematics of $\tau^- \tau^+$ pair decays cannot be measured completely due to the escaping neutrinos, we have to account for this missing information. In the case of τ^- and τ^+ decaying both semi-leptonically, i.e. via $\pi \nu$ or $\rho \nu$, the kinematical information can be reconstructed up to a two-fold ambiguity. For the measurement of F_2 , we therefore have to average the intensities over these two ambiguous kinematical solutions, as discussed in Sect. 2.1.

In the case of at least one τ decaying leptonically ($\tau^- \rightarrow \ell^- \nu_\tau \bar{\nu}_\ell$), the invariant mass of the invisible $\nu_\tau \bar{\nu}_\ell$ system, as well as the directions of the individual neutrinos are also unknown. To account for this unknown information, we sample the unknown angles and invariant masses² 100 times for both decays and use the mean value of the intensity given in Sect. 2 over all kinematically valid samples as $\bar{\mathcal{I}}$. Since every sampled value of the invariant mass results in two solutions for the direction of the τ momenta in the purely semileptonic case, we end up with more than 100 kinematical samples for every event on average. This procedure is similar to the one used by the Belle collaboration in the measurement of the electric dipole moment d_τ .

Using these techniques, we list in Table 1 the resolutions obtained for $\text{Re}F_2$ and $\text{Im}F_2$ in the sixteen different combinations of the four decay channels considered in Eq. (32). The obtained values for F_2 are consistent with the input value for every channel. The resolutions using the true kinematical information yields similar values of $(\delta \text{Re}F_2)_{\text{true}} = 7.0 \cdot 10^{-4}$ and $(\delta \text{Im}F_2)_{\text{true}} = 7.2 \cdot 10^{-4}$ for all 16 combinations of decay channels considered and we observe no significant correlation between the two quantities for the case of fully known final-state kinematics.

Using $\mathcal{O}_{\text{Re}}^{\text{opt}}$, the loss of kinematical information in purely semileptonic channels leads to a decrease of the sensitivity of only 10%, while every leptonic decay worsens the sensitivity by more than a factor of two. The loss of kinematical information affects the resolution of $\text{Im}F_2$ for semi-leptonic channels more than for $\text{Re}F_2$, while the resolution of $\text{Im}F_2$ for leptonic channels is less affected by the loss of kinematical information, than in the case of $\text{Re}F_2$. However, semileptonic channels, especially the $\rho \nu$ channels, still offer a better resolution overall. Since we do not observe correlations between $\text{Re}F_2$ and $\text{Im}F_2$ when using the full kinematical information, the strong correlation between $\text{Re}F_2$ and $\text{Im}F_2$ for purely leptonic channels is introduced by the sampling procedure of kinematics. The resolutions in Table 1 are purely statistical and given for a sample of 10^6 events in each case. By varying the sample size N , we have verified that the resolutions scale with $N^{-1/2}$, as it should be.

Table 1 Results obtained from 10^6 simulated events for the resolution of $\text{Re}F_2$ and $\text{Im}F_2$ in the 16 possible combinations of decay channels, and the correlation coefficient $C_{\text{Re,Im}}$ between real and imaginary part of F_2

τ^- mode	τ^+ mode	$10^3 \cdot \delta \text{Re}F_2$	$10^3 \cdot \delta \text{Im}F_2$	$C_{\text{Re,Im}}$
$\pi^- \nu_\tau$	$\pi^+ \bar{\nu}_\tau$	0.76	1.15	0.00886
$\pi^- \nu_\tau$	$\rho^+ \bar{\nu}_\tau$	0.78	0.86	0.00217
$\pi^- \nu_\tau$	$e^+ \bar{\nu}_\tau \nu_e$	1.45	1.26	0.00572
$\pi^- \nu_\tau$	$\mu^+ \bar{\nu}_\tau \nu_\mu$	1.44	1.24	0.00200
$\rho^- \nu_\tau$	$\pi^+ \bar{\nu}_\tau$	0.78	0.86	-0.00524
$\rho^- \nu_\tau$	$\rho^+ \bar{\nu}_\tau$	0.78	0.81	0.00509
$\rho^- \nu_\tau$	$e^+ \bar{\nu}_\tau \nu_e$	1.42	1.29	-0.00396
$\rho^- \nu_\tau$	$\mu^+ \bar{\nu}_\tau \nu_\mu$	1.43	1.29	0.00596
$e^- \nu_\tau \bar{\nu}_e$	$\pi^+ \bar{\nu}_\tau$	1.46	1.30	0.00428
$e^- \nu_\tau \bar{\nu}_e$	$\rho^+ \bar{\nu}_\tau$	1.42	1.26	0.00084
$e^- \nu_\tau \bar{\nu}_e$	$e^+ \bar{\nu}_\tau \nu_e$	2.42	1.64	0.63132
$e^- \nu_\tau \bar{\nu}_e$	$\mu^+ \bar{\nu}_\tau \nu_\mu$	2.61	1.64	-0.69551
$\mu^- \nu_\tau \bar{\nu}_\mu$	$\pi^+ \bar{\nu}_\tau$	1.45	1.29	0.00754
$\mu^- \nu_\tau \bar{\nu}_\mu$	$\rho^+ \bar{\nu}_\tau$	1.42	1.24	0.00310
$\mu^- \nu_\tau \bar{\nu}_\mu$	$e^+ \bar{\nu}_\tau \nu_e$	4.08	1.64	-0.88305
$\mu^- \nu_\tau \bar{\nu}_\mu$	$\mu^+ \bar{\nu}_\tau \nu_\mu$	2.18	1.67	0.51208

4.1 Importance of order α^3 corrections

Since in contrast to the electric dipole moment d_τ , the contributions of the Pauli form factor F_2 to the production spin-density matrix are CP-even, higher order corrections from QED can mimic the effects of F_2 and thus distort a measurement if not properly taken into account. To estimate the effect of higher order QED corrections on the determination of F_2 , we reanalyze the same Monte-Carlo samples as in Sect. 4, but only take the spin-density matrix in tree-level approximation. Thus, the Monte Carlo data set \mathcal{S}_0 defined in Sect. 2 is also generated using only the tree-level spin-density matrix $\mathcal{M}_{\text{tree}}$. Doing so, we find that neglecting higher order effects in the analysis leads to a wrong estimation of $\text{Re}F_2$ with a value of $(-3.46 \pm 0.29) \cdot 10^{-3}$, while the estimation of $\text{Im}F_2$ shows a smaller bias of $(0.94 \pm 0.26) \cdot 10^{-3}$. When using the true kinematical information for every event, this results in $\text{Re}F_2 = (-3.17 \pm 0.14) \cdot 10^{-3}$ and $\text{Im}F_2 = (0.41 \pm 0.19) \cdot 10^{-3}$. Since the prediction for $\text{Im}F_2$ only differs from zero for all channels combined, while for each individual channel it remains consistent with zero for a data-set with 10^6 events, we generate a larger sample of 10^8 events for the $(\pi^- \nu_\tau) \times (\pi^+ \bar{\nu}_\tau)$ mode and perform the analysis, which leads to the result:

$$\begin{aligned} \text{Re}F_2 &= (-3.37 \pm 0.08) \cdot 10^{-3}, \\ \text{Im}F_2 &= (0.20 \pm 0.08) \cdot 10^{-3}. \end{aligned} \tag{33}$$

² With a known lepton energy in the center-of-mass frame, the invisible mass square $m_{\nu\bar{\nu}}^2$ is equally distributed in phase-space.

This bias on the one hand indicates, that corrections of order of α^3 have to be taken into account for a determination of F_2 . However, since these corrections are of the same order of magnitude as the standard model expectation of $F_2^\tau(0) = 1.1772 \cdot 10^{-3}$ [15, 16], corrections of order α^3 seem to suffice to measure a non-zero value of F_2 for the first time. With increasing precision, of course, higher order corrections will become necessary.

5 Summary and conclusion

We have introduced the method of optimal observables and used it to determine the Pauli form factor F_2 of the τ -lepton. In this study we have explicitly included the QED corrections of order α^3 , as well as γZ^0 interference, and determined the resolutions for $\text{Re}F_2$ and $\text{Im}F_2$. We have found, that the method gives a similar resolution for F_2 in all 16 combinations of decay channels, when using the true (complete) kinematical information.

By comparing our results for the resolutions of $\text{Re}F_2$ and $\text{Im}F_2$ with those given in Table 1 of Ref. [2], we find that our obtained resolutions are a somewhat larger. However, since the authors do not give the total event yield, we cannot perform a more detailed comparison of results. In addition, the authors of Ref. [2] state, that their resolutions for the decay channels $(\pi^- \nu_\tau) \times (\rho^+ \bar{\nu}_\tau)$, $(\rho^- \nu_\tau) \times (\pi^+ \bar{\nu}_\tau)$, and $(\rho^- \nu_\tau) \times (\rho^+ \bar{\nu}_\tau)$, are worse by factors of 2 and 4 with respect to the $(\pi^- \nu_\tau) \times (\pi^+ \bar{\nu}_\tau)$ channel and thus only “poorly contribute to increase the precision for the measurement of $\text{Re}F_2$ ”. In contrast to this, we observe a similar resolution for all decay channels, when using the true kinematical information, and moreover the $(\rho^- \nu_\tau) \times (\rho^+ \bar{\nu}_\tau)$ channel yields the best resolution at a given luminosity, due to its high branching fraction [14].

Since the method proposed in Ref. [2] relies on specific integrations of the intensity distributions, the detector acceptance has to be parametrized as a function of the kinematical variables. Otherwise, asymmetric detector effects could bias the measurement. Our method, in contrast, only relies on an event-by-event detector simulation for the data set \mathcal{S}_0 to obtain the optimal observables $\overline{\mathcal{O}}_{\text{Re,Im}}^{\text{opt}}$, as defined in Sect. 2.1. Such an event-by-event simulation is much more readily available than a parametrization of the detector acceptance. Thus, we believe our approach to be more robust with respect to detector effects.

Comparing our results to a similar study performed in Ref. [7] we find, that the quoted improvement due to their reconstruction of the neutrino momenta of a factor of four for d_τ is mostly due to the different formalism they used, rather than due to the resolution of the two-fold kinematical ambiguity. As we show in Appendix B, the ambiguity only increases the

resolution by a factor 1.61 and 1.06 for $\text{Re}(d_\tau)$ and $\text{Im}(d_\tau)$. However, it is difficult to directly compare the resolutions we obtained with the ones in Ref. [7], since we did not perform any detector simulation studies. Additionally, we want to state, that we performed studies separately for the real and imaginary parts of the dipole momenta, while the authors of Ref. [7] solely focused on the real parts. We showed, however, that there are differences in the resolutions of real and imaginary part and both parts are affected differently by the loss of kinematical information, as can be seen in Table 1.

Additionally, we have studied the effect of missing kinematical information due to escaping neutrinos, which we have summarized in Table 1. We find that the resolution of F_2 is worse for leptonic decay channels, since more neutrinos escape in this case.

Finally, we have studied the effect of QED corrections of order α^3 to the spin-density matrix χ^{prod} presented in great detail in Sect. 3. By reanalyzing our generated data sets assuming a tree-level production spin-density matrix $\mathcal{M}_{\text{tree}}$, we find the bias given in Eq. (33) for the measurement of F_2 . The size of this bias indicates that the inclusion of α^3 -corrections allows one to measure the non-zero standard-model value determined in Refs. [15, 16], given a correct treatment of photon radiation.

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Data Availability Statement This manuscript has no associated data or the data will not be deposited. [Authors’ comment: The data have been used to produce the numerical results given in the text and displayed in Tables 1 and 2.]

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Appendix A: Loop functions

After multiplying with the common factor $16\pi^2$ the complex valued scalar loop-integrals encountered in the evaluation of the two-photon exchange box diagrams (with a light electron and/or heavy τ -propagator) [10] read in our notation with dimensionless variables s and t :

$$D_0 = \frac{2}{s(t-1)}(2\xi_{\text{IR}} + \ln s - i\pi) \ln \frac{1-t}{\sqrt{r}}, \tag{34}$$

$$C_s = \frac{1}{s} \left(\frac{1}{2} \ln^2 \frac{s}{r} + \frac{\pi^2}{6} - i\pi \ln \frac{s}{r} \right), \tag{35}$$

$$C_t = \frac{1}{t-1} \left[\text{Li}_2(t) + \ln^2 \frac{1-t}{\sqrt{r}} + (2\xi_{\text{IR}} + \ln r) \ln \frac{1-t}{\sqrt{r}} \right], \tag{36}$$

$$\bar{D}_0 = D_0(t \rightarrow u), \quad \bar{C}_t = C_t(t \rightarrow u), \tag{37}$$

$$C_M = \frac{2}{\sqrt{s^2-4s}} \left[\frac{\pi^2}{12} + \ln^2 \frac{\sqrt{s} + \sqrt{s-4}}{2} + \text{Li}_2 \left(\frac{2-s + \sqrt{s^2-4s}}{2} \right) - i\pi \ln \frac{\sqrt{s} + \sqrt{s-4}}{2} \right], \tag{38}$$

$$B_s - B_M = i\pi - \ln s, \quad B_t - B_M = \frac{1-t}{t} \ln(1-t), \tag{39}$$

where $\text{Li}_2(t) = t \int_1^\infty dx [x(x-t)]^{-1} \ln x$ denotes the dilogarithmic function at argument $t < 1$. Moreover, $\xi_{\text{IR}} = 1/(d-4) + (\gamma_E - \ln 4\pi)/2 + \ln(m_\tau/\mu)$ abbreviates the infrared divergence in dimensional regularization, and r stands for the squared mass ratio $r = (m_e/m_\tau)^2 = 8.271 \cdot 10^{-8}$. We remind that the nomenclature B , C , and D refers to loop-integrals over two, three, and four propagators, respectively.

Appendix B: Results concerning the electric dipole moment

Since the measurement of the electric dipole moment d_τ of the τ -lepton (as performed in Ref. [3]) using the optimal observable method is very similar to the measurement of the anomalous magnetic dipole moment a_τ (in units of $e/2m_\tau$), we give also the results of such a study analogous to Sect. 4. The inclusion of an electric dipole moment through a CP-violating form factor $F_3(s)$ adds the following structure:

$$\frac{F_3(s)}{2m_\tau} \sigma^{\mu\nu} \gamma^5 q_\nu \tag{40}$$

to the Dirac-matrix Γ^μ in Eq. (1). The conversion formula of $F_3(s)$ to the electric dipole moment d_τ , commonly given in units of ecm , reads:

$$d_\tau = \frac{e}{2m_\tau} F_3(0) = 5.44 \cdot 10^{-15} ecm \cdot F_3(0). \tag{41}$$

Clearly, in $\tau^- \tau^+$ pair-production one can only obtain $F_3(s)$ for $s > 4m_\tau^2$.

The resulting resolutions for the extraction of F_3 are given in Table 2. Without loss of kinematical information, the resolutions are similar and take the values of $(\delta \text{Re} F_3)_{\text{true}} = 0.93 \cdot 10^{-3}$ and $(\delta \text{Im} F_3)_{\text{true}} = 0.52 \cdot 10^{-3}$. We observe no significant correlations between $\text{Re} F_3$ and $\text{Im} F_3$. A reanalysis of the generated data neglecting next-to-leading order QED effects, similar to Sect. 4.1, yields no bias for the extraction

Table 2 Results from 10^6 simulated events for the resolutions of $\text{Re} F_3$ and $\text{Im} F_3$ in the 16 possible combinations of decay channels

τ^- mode	τ^+ mode	$10^3 \cdot \delta \text{Re} F_3$	$10^3 \cdot \delta \text{Im} F_3$
$\pi^- \nu_\tau$	$\pi^+ \bar{\nu}_\tau$	1.50	0.55
$\pi^- \nu_\tau$	$\rho^+ \bar{\nu}_\tau$	1.11	0.57
$\pi^- \nu_\tau$	$e^+ \bar{\nu}_\tau \nu_e$	3.57	1.00
$\pi^- \nu_\tau$	$\mu^+ \bar{\nu}_\tau \nu_\mu$	3.47	1.00
$\rho^- \nu_\tau$	$\pi^+ \bar{\nu}_\tau$	1.11	0.57
$\rho^- \nu_\tau$	$\rho^+ \bar{\nu}_\tau$	1.07	0.57
$\rho^- \nu_\tau$	$e^+ \bar{\nu}_\tau \nu_e$	2.96	1.00
$\rho^- \nu_\tau$	$\mu^+ \bar{\nu}_\tau \nu_\mu$	2.95	1.00
$e^- \nu_\tau \bar{\nu}_e$	$\pi^+ \bar{\nu}_\tau$	3.69	1.06
$e^- \nu_\tau \bar{\nu}_e$	$\rho^+ \bar{\nu}_\tau$	3.09	0.95
$e^- \nu_\tau \bar{\nu}_e$	$e^+ \bar{\nu}_\tau \nu_e$	20.3	1.73
$e^- \nu_\tau \bar{\nu}_e$	$\mu^+ \bar{\nu}_\tau \nu_\mu$	10.9	1.71
$\mu^- \nu_\tau \bar{\nu}_\mu$	$\pi^+ \bar{\nu}_\tau$	3.65	1.05
$\mu^- \nu_\tau \bar{\nu}_\mu$	$\rho^+ \bar{\nu}_\tau$	3.02	0.95
$\mu^- \nu_\tau \bar{\nu}_\mu$	$e^+ \bar{\nu}_\tau \nu_e$	11.4	1.72
$\mu^- \nu_\tau \bar{\nu}_\mu$	$\mu^+ \bar{\nu}_\tau \nu_\mu$	13.4	1.70

of F_3 . This is expected, since any contribution from F_3 is CP-odd, while QED contributions are CP-even both at tree-level and one-loop order.

Appendix C: Variation of the soft photon cutoff

In order to study the effect of the soft photon cutoff λ in Eq. (29) we generate 10^8 events for the $(\pi^- \nu_\tau) \times (\pi^+ \bar{\nu}_\tau)$ mode employing the values $\lambda = 50\text{MeV}$ and $\lambda = 200\text{MeV}$ and analyze these data-sets with our method, assuming a value of $\lambda = 100\text{MeV}$, which was used in our studies above. Apart from the different values of λ , the models used for data generation and analysis are equal. Doing so, leads to the following bias in the determination of F_2 :

$$\text{Re} F_2^{\lambda=50\text{MeV}} = (2.53 \pm 0.05) \cdot 10^{-3}, \tag{42}$$

$$\text{Im} F_2^{\lambda=50\text{MeV}} = (-0.16 \pm 0.05) \cdot 10^{-3}, \tag{43}$$

$$\text{Re} F_2^{\lambda=200\text{MeV}} = (-1.75 \pm 0.06) \cdot 10^{-3}, \tag{44}$$

$$\text{Im} F_2^{\lambda=200\text{MeV}} = (0.03 \pm 0.07) \cdot 10^{-3}. \tag{45}$$

Since these values are of similar size as the values found in Eq. (33), a correct treatment of photon radiation turns out to be crucial for a measurement of F_2 . For F_3 we again find no bias in this study, as expected.

Since the value of λ plays a role, it has to be chosen properly to represent the energy resolution of an actual experiment. The treatment of bremsstrahlung in $e^- e^+ \rightarrow \tau^- \tau^+$ beyond the soft photon approximation goes beyond the scope

of the present paper since such a calculation should be tailored to the specific experimental conditions for undetectable photon radiation including detector efficiencies and various other effects.

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