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Energy functionals for Calabi-Yau metrics

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Abstract. We identify a set of “energy” functionals on the space of metrics in a given Kähler class on a Calabi-Yau manifold, which are bounded below and minimized uniquely on the Ricci-flat metric in that class. Using these functionals, we recast the problem of numerically solving the Einstein equation as an optimization problem. We apply this strategy, using the “algebraic” metrics (metrics for which the Kähler potential is given in terms of a polynomial in the projective coordinates), to the Fermat quartic and to a one-parameter family of quintics that includes the Fermat and conifold quintics. We show that this method yields approximations to the Ricci-flat metric that are exponentially accurate in the degree of the polynomial (except at the conifold point, where the convergence is polynomial), and therefore orders of magnitude more accurate than the balanced metrics, previously studied as approximations to the Ricci-flat metric. The method is relatively fast and easy to implement. On the theoretical side, we also show that the functionals can be used to give a heuristic proof of Yau’s theorem.

1. Introduction

Calabi-Yau (CY) manifolds play a pivotal role in complex differential geometry, algebraic geometry, and string theory. Part of the reason for their importance, for both mathematics and physics, lies in the Ricci-flat metrics they admit. The existence of these metrics is guaranteed by Yau’s theorem; this theorem, however, is not constructive, and, outside of flat metrics and orbifolds thereof, no exact solutions to the Einstein equation are known on any CY manifold. This state of affairs naturally leads to the question of whether it is feasible to compute useful numerical approximations to the Ricci-flat metrics, a question that has given rise to a small literature in recent years [1, 2, 3, 4, 5, 6, 7]. A few different methods have been put forward, all of which make crucial use of the advantages offered by Kähler geometry compared to real geometry.¹ The purpose of this paper is to introduce a new method for this problem, that scores well on all three of the major considerations that make such a method useful: ease of implementation, scale of computational resources required, and accuracy of the approximations obtained. The method is based on a mathematical observation that may be of some interest in its own right.

The mathematical and physical motivations for solving the Einstein equation on CYs are discussed at length in the papers cited above, and we will not repeat them here. However, it is important to note that the scope of the potential applications depends strongly on the performance of the methods available. For example, for some problems it is sufficient to compute

¹ Related methods for solving the Einstein equation on toric manifolds were discussed in [8, 9, 10].



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the metric at a single point in the CY moduli space, while for others it is necessary to scan over a part of the moduli space, requiring much faster methods.

Any numerical method for solving a PDE must address two issues. The first is how to represent, or store, the function being solved for—in this case, the Kähler potential—for example using a real-space lattice or a spectral representation. The second is, given that representation, how to solve the equation, or, more precisely, how to find the best approximation within that representation to the exact solution. For the problem at hand, the principal challenges in terms of the representation are the high dimensionality of the manifold (usually four or six real dimensions, so that even for moderate resolutions storage quickly becomes a limiting factor) and its typically very complicated topology. The principal challenge in terms of solving the Einstein equation is its nonlinearity.

The first methods developed for solving the Einstein equation on CYs, by Headrick and Wiseman [1], used a real-space lattice representation of the Kähler potential, and solved the equation by a Gauss-Seidel relaxation method. The method was straightforward, if not particularly elegant. One downside was the messiness of dealing with coordinate patches. The main factor limiting the accuracy of the metrics obtained was storage, due to the large number of lattice points required for high resolution in high dimensions (the method was tested on K3; storage would have severely limited the possible resolutions for threefolds).

Subsequently, Donaldson proposed a radically different set of methods based on a spectral (momentum-space) representation of the metric [2]. Specifically, he advocated the use of so-called algebraic metrics, which had been studied from a theoretical viewpoint by Tian and others. The salient properties for our present purposes are (1) they are based on polynomials, and therefore very easy to work with both theoretically and computationally; (2) they eliminate the messiness of coordinate patches; and (3) within a certain Kähler class they are capable of approximating any given smooth metric—including the Ricci-flat one—exponentially well in k , the degree of the polynomials (k is analogous to the highest mode number of a finite set of Fourier modes used to approximate a function, for example, on S^1). The latter property essentially promises a form of data compression, and, when applied to the Ricci-flat metric, has potential to solve the storage problem mentioned above.

As with any spectral representation, the challenge then comes in solving the PDE. Linear PDEs (especially homogeneous ones) become simpler when written in momentum space, as the different modes decouple from each other. For nonlinear ones such as the Einstein equation, on the other hand, every mode is coupled to every other mode, leading to a complicated system of nonlinear equations. Donaldson proposed two kinds of approximate solutions to the Einstein equation within the framework of the algebraic metrics. The first was the so-called balanced metrics. For each k the balanced metric is the unique solution to a certain integral equation, which can be found by iterating a certain integral map.² Donaldson showed that they approach the Ricci-flat metric as $k \rightarrow \infty$, but only as a power of k . Thus they do not achieve the exponential accuracy promised by the algebraic metrics. The second proposal, called “refined” metrics, were based on a Galerkin method: the error in solving the Einstein equation (more precisely, the equivalent Monge-Ampère equation) was required to be orthogonal to the full set of basis functions (in this case, the degree- k polynomials). The refined metrics are expected to converge exponentially to the Ricci-flat metric. However, efficiently computing the refined metric proved difficult.

Now, it is well known in numerical analysis that it is far easier to minimize a well-behaved function than to solve a non-linear set of equations.³ Here, by “well-behaved”, we mean smooth, bounded below, and having no critical points other than a single global minimum. This is true

² The balanced metrics are of significant mathematical and possibly physical interest quite apart from the problem of solving the Einstein equation [2, 11].

³ For a standard discussion of this point, see e.g. section 9.6.1 of [12].

both from a theoretical viewpoint and in terms of actual available algorithms; while there exist several efficient and robust numerical optimization algorithms, for solving non-linear equations there is essentially only the Newton-Raphson method, which has poor global convergence properties, and a few variants on it. So it is nearly always advantageous, when possible, to convert the latter type of problem into the former. In the case of an elliptic PDE, a well-behaved functional whose variational derivative yields the PDE is usually called an energy functional. Note that most elliptic PDEs do not admit such a variational formulation. (The utility of such a formulation is also of course clear from the mathematical point of view, for example for proving the existence of solutions.) Given an energy functional and a representation scheme (real-space, spectral, or otherwise), one minimizes the energy functional within the corresponding finite-dimensional space of functions.⁴ The prototype for this method is the Rayleigh-Ritz variational method for finding the smallest eigenvalue of a linear operator. Especially for spectral representations, this strategy can vastly simplify the problem of solving the PDE.

In the paper [13], we show how to follow this strategy for case of the Einstein equation on a CY manifold.⁵ We define a family of energy functionals on the space of Kähler metrics on a given CY manifold, that are bounded below and minimized within each Kähler class precisely on the Ricci-flat metric.⁶ The existence of these functionals depends crucially on the magic of Kähler geometry and should by no means be taken for granted—comparable functionals do not, as far as we know, exist in the Riemannian setting (for example, the Einstein-Hilbert action is unbounded below, while the integrated square of the Ricci tensor has many spurious local minima). Note that the functionals we define are not tied to the algebraic metrics, and could be used to solve the Einstein equation given any spectral or other representation scheme. We show, incidentally, that these functionals can also be used to give a heuristic proof of Yau’s theorem. The argument has loopholes, as we discuss, but may nonetheless be useful as a quick way to understand why one should expect Yau’s theorem to hold.⁷ The simplest functional takes the following form:

$$E[J] = \int_X \Omega \wedge \bar{\Omega} (\eta - 1)^2, \quad (1)$$

where Ω is the holomorphic $(n, 0)$ form on the CY X , and

$$\eta = \frac{J^n}{\Omega \wedge \bar{\Omega}}. \quad (2)$$

We then combine the algebraic metrics with one of our energy functionals to define “optimal” metrics. The figures show the results of applying our method to the Fermat quartic and to the following one-real-parameter family of quintics,

$$\sum_{a=1}^5 (z^a)^5 - 5\psi \prod_{a=1}^5 z^a = 0 \quad (3)$$

which includes the Fermat and conifold quintics (at $\psi = 0, 1$ respectively). The integer parameter k is the degree of the polynomial in the projective coordinates used to approximate the Kähler potential; the larger k is, the better the approximation. The method is relatively simple to

⁴ Note that the error in solving the equation will necessarily be orthogonal, in the full function space, to the subspace in which one is working. Hence any method of this type automatically solves some Galerkin-type condition.

⁵ The possibility of following a functional-minimization strategy was briefly mentioned in [4].

⁶ These functionals share some of the properties of the Calabi energy [14], the Mabuchi K-energy [15], and its generalizations [9, 16], although they are somewhat simpler.

⁷ The recent paper [16] uses a variational approach based on the Mabuchi K-energy [15] to prove a weak version of Yau’s theorem.

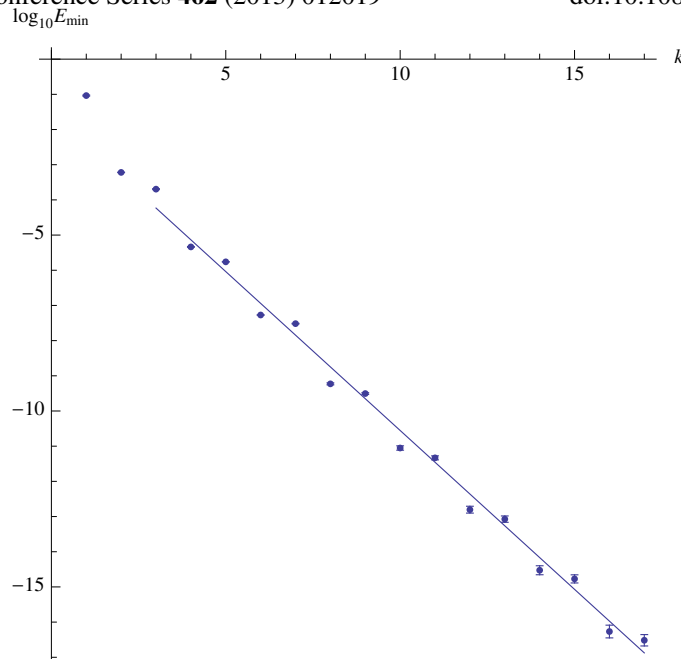


Figure 1. Value of E for the optimal algebraic metric on the Fermat quartic, versus k . The exponential decrease is clear, accompanied by a small even-odd modulation. The line is 0.03×8^{-k} , obtained by fitting to the points with $k = 3$ through 17.

implement (requiring just a few pages of *Mathematica* code), and very efficient (running for a few minutes on a laptop computer, it yields metrics that are likely accurate enough for most applications). In particular, the method achieves the exponential accuracy promised by the algebraic metrics. As a result, the metrics found are far more accurate, for a given value of k , than the balanced metrics. (The high-accuracy metrics we obtain on the compact conifold and deformed conifold, in particular, are perhaps relevant for certain models of string phenomenology; see [17] and references therein).

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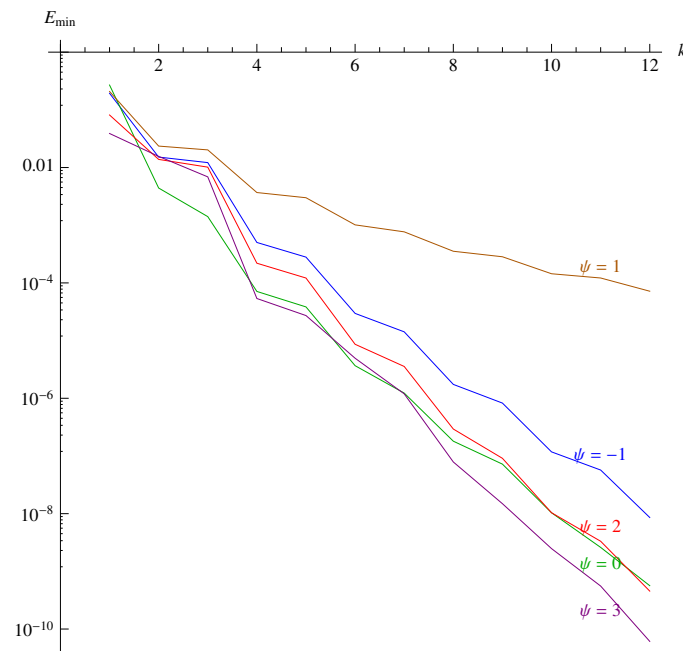


Figure 2. E_{\min} versus k for algebraic metrics on the quintic (3) for the values of ψ indicated on the plot. The exponential decrease is clear for all values of ψ except $\psi = 1$, which is the conifold point in the quintic moduli space. A linear fit of $\log E_{\min}$ gives $E_{\min} \approx 0.07 \times 5^{-k}$ (not plotted) for $\psi = 0$ (the Fermat quintic).

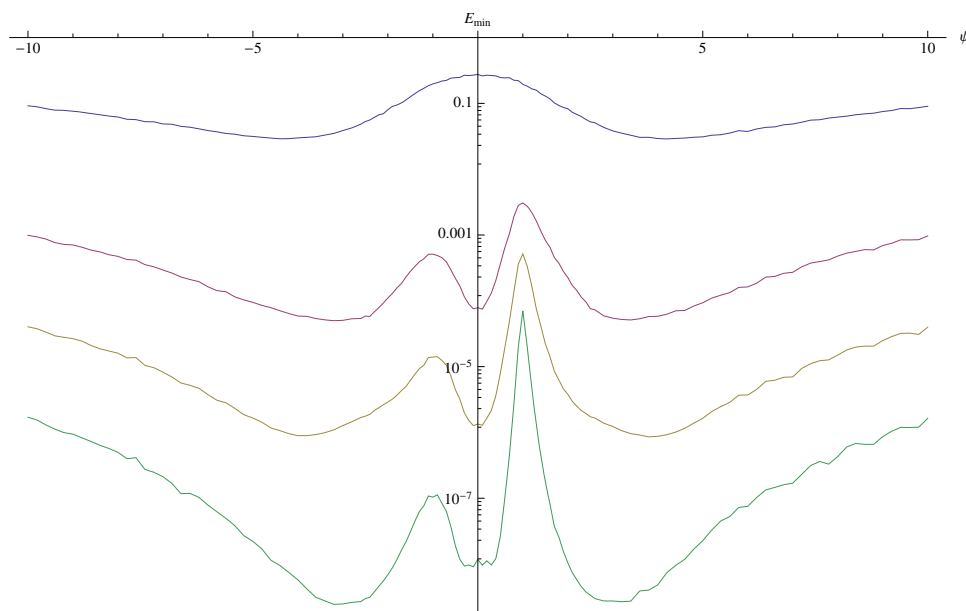


Figure 3. E_{\min} versus ψ for (top to bottom) $k = 1, 4, 7, 10$ (the top curve represents the FS metric). (Error bars are not indicated to avoid cluttering the plot; however, the roughness of the curves, which is due to numerical errors, gives a good indication of their size.)

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