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Strong Differential Subordinations and Superordinations for Riemann–Liouville Fractional Integral of Extended q-Hypergeometric Function

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Abstract: The notions of strong differential subordination and its dual, strong differential superordination, have been introduced as extensions of the classical differential subordination and superordination concepts, respectively. The dual theories have developed nicely, and important results have been obtained involving different types of operators and certain hypergeometric functions. In this paper, quantum calculus and fractional calculus aspects are added to the study. The well-known q-hypergeometric function is given a form extended to fit the study concerning previously introduced classes of functions specific to strong differential subordination and superordination theories. Riemann–Liouville fractional integral of extended q-hypergeometric function is defined here, and it is involved in the investigation of strong differential subordinations and superordinations. The best dominants and the best subordinants are provided in the theorems that are proved for the strong differential subordinations and superordinations, respectively. For particular functions considered due to their remarkable geometric properties as best dominant or best subordinant, interesting corollaries are stated. The study is concluded by connecting the results obtained using the dual theories through sandwich-type theorems and corollaries.

Keywords: Riemann–Liouville fractional integral; extended q-confluent hypergeometric function; strong differential subordination; strong differential superordination; best dominant; best subordinant

MSC: 30C45; 30A20; 34A40

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1. Introduction

Antonino and Romaguera [1] introduced the notion of strong differential subordination while investigating Briot–Bouquet strong differential subordination. The intent motivating this effort was to extend the established notion of differential subordination originated by Miller and Mocanu [2,3]. The newly introduced notion served as the foundation for the theory of strong differential subordination. In a paper published in 2009 [4], the authors elaborated upon the ideas found in the well-known differential subordination theory [5] in order to adapt the concepts to the new notion of strong differential subordination. Furthermore, the dual notion of strong differential superordination was defined in 2009 [6], by applying the pattern set for classical differential superordination theory [7]. Over the following years, both theories developed very well. Methods for determining the best subordinant of a strong differential superordination were given [8], and particular strong differential subordinations and superordinations were taken into consideration for the studies [9]. Various strong differential subordinations and superordinations were investigated by linking different types of operators to the research. The Sălăgean differential operator was employed for introducing a new class of analytic functions and for investigating certain strong differential subordinations in [10]. Multivalent meromorphic functions and the Liu–Srivastava operator were involved in obtaining strong differential subordinations and superordinations in [11]. The Ruscheweyh differential operator is used in [12] for defining a new class of univalent functions and for studying strong differential subordinations. The Sălăgean and Ruscheweyh operators were used together in the study presented in [13], and a multiplier transformation provided new strong differential subordinations in [14]. The Komatu integral operator was applied for obtaining new strong differential subordination results [15,16], and other differential operators proved effective for studying strong differential subordinations and superordinations [17,18]. Citing recent publications such as [19–22] demonstrates that the topic is still of interest today.

Despite being approximately 300 years old, fractional calculus is currently one of the mathematical analysis topics that is expanding the fastest. The great mathematicians G.W. Leibnitz and L. Euler considered the potential benefits of performing non-integer order differentiation. Mathematicians from the XIX to the early XX centuries made significant contributions to the actual establishment and extensive development of fractional calculus. Recent research has greatly benefited from the use of fractional calculus, which has numerous applications in various scientific and technical fields. The review publications [23,24] that address the history of fractional calculus and include references to its numerous applications in science and engineering effectively emphasize the significance of this topic. As part of fractional calculus studies, fractional operators play an important role. Fractional operators are essential tools for studies using fractional calculus. A succinct history detailing fractional calculus operators is provided in [25] and elaborated upon in [26].

In recent years, fractional calculus has advanced significantly and has been shown to be useful in a wide range of scientific fields, including computer graphics, turbulence, physics, engineering, electric networks, biological systems with memory, and computer graphics. For instance, a novel integral transform proposed in the Caputo sense is used in [27] to study the Korteweg–De Vries equation, which was created to reflect a wide range of physical behaviors of the evolution and association of nonlinear waves. Examples of recent research pertaining to biological systems include the fractional calculus analysis of the dengue infection's transmission dynamics, which was observed in [28], and the mathematical modeling of the human liver using the Caputo–Fabrizio fractional derivative, which was proposed in [29]. A useful nonlinear differential equation that is important to both industrial and natural processes is the foam drainage equation. A numerical method for estimating the approximate solution of the nonlinear foam drainage problem with a time-fractional derivative is developed in the study in [30]. New integral inequalities involving fractional integral and convexity properties are investigated in [31,32], and extensions on fractional properties involving the Mittag–Leffler confluent hypergeometric function are given in [33]. Other hypergeometric functions are also effective means in fractional calculus [34,35]. An overview on special functions emphasizing the significance of the advancements made possible by their association with fractional calculus operators is provided in a very recent review [36]. Furthermore, the correlation of fractional calculus and geometric function theory is highlighted in [37].

In early studies [38], fractional calculus was linked to strong differential subordination theory, but this line of investigation was not developed. In a review paper, Srivastava [39] emphasizes how the addition of quantum calculus and elements of fractional calculus in geometric function theory contributed to the theory's advancement. The results presented in this research try to revive the study by including fractional operators and functions familiar to quantum calculus. Inspired by the recent results obtained by embedding hypergeometric functions into the theory of strong differential superordinations seen in [40] and the nice recent findings involving the Riemann–Liouville fractional integral of the q-hypergeometric function in classical differential subordination and superordination theories [41] and in fuzzy differential subordination and superordination theories [42], in this paper, the q-hypergeometric function is extended to certain classes of functions specific to strong differential subordination and superordination theories introduced in [43], and a new operator is defined here by applying the Riemann–Liouville fractional integral to this extended q-hypergeometric function.

The main concepts that were implemented for the investigation are reviewed in Section 2, alongside a list of fundamental lemmas that were employed to demonstrate the main results. The main outcomes of this study are presented in Section 3, in which best subordinants and best dominants are found for strong differential subordinations and for the dual strong differential superordinations involving the Riemann–Liouville fractional integral of the extended q-hypergeometric function, respectively. Interesting corollaries associated with the proven theorems are also presented, when specific functions with particular geometric features are selected as the best dominants and subordinants. As an application, sandwich-type theorems and corresponding corollaries connect the dual new results obtained in this research.

2. Preliminaries

Denote by $\mathcal{H}(U \times \overline{U})$ the class of analytic functions in $U \times \overline{U}$, where $U = \{z \in \mathbb{C} : |z| < 1\}$ and $\overline{U} = \{z \in \mathbb{C} : |z| \leq 1\}$.

In [43], the authors have introduced some special subclasses of $\mathcal{H}(U \times \overline{U})$ that are used only related to the theories of strong differential subordination and its dual, strong differential superordination:

$$\mathcal{A}^*_{n\zeta} = \{f(z,\zeta) = z + a_{n+1}(\zeta)z^{n+1} + \cdots \in \mathcal{H}(U \times \overline{U})\},$$

with $\mathcal{A}^*_{1\zeta} = \mathcal{A}^*_{\zeta}$ and $a_k(\zeta)$ holomorphic functions in $\overline{U}$, $k \geq n+1$, $n \in \mathbb{N}$, and

$$\mathcal{H}^*[a,n,\zeta] = \{f(z,\zeta) = a + a_n(\zeta)z^n + a_{n+1}(\zeta)z^{n+1} + \cdots \in \mathcal{H}(U \times \overline{U})\},$$

with $a_k(\zeta)$ holomorphic functions in $\overline{U}$, $k \geq n$, $a \in \mathbb{C}$, and $n \in \mathbb{N}$.

The next definitions concern the concept of strong differential subordination as it was used in [1] and further developed in [4,43].

Definition 1 ([4])**.** *The analytic function $f(z,\zeta)$ is strongly subordinate to the analytic function $H(z,\zeta)$ if there exists an analytic function w in U, such that $w(0) = 0$, $|w(z)| < 1$ and $f(z,\zeta) = H(w(z),\zeta)$ for all $\zeta \in \overline{U}$. It is denoted $f(z,\zeta) \prec\prec H(z,\zeta)$, $(z,\zeta) \in U \times \overline{U}$.*

Remark 1 ([4])**.** *(i) For analytic function $f(z,\zeta)$ in $U \times \overline{U}$ and univalent in U, for all $\zeta \in \overline{U}$, Definition 1 is equivalently with $f(U \times \overline{U}) \subset H(U \times \overline{U})$ and $f(0,\zeta) = H(0,\zeta)$, for all $\zeta \in \overline{U}$.*

(ii) When $f(z,\zeta) = f(z)$ and $H(z,\zeta) = H(z)$, the strong differential subordination is reduced to the differential subordination.

The following lemma is needed for the investigation related to strong differential subordinations.

Lemma 1 ([44])**.** *Consider the univalent function g in $U \times \overline{U}$ and the analytic functions θ and η in a domain $D \supset g(U \times \overline{U})$ such that $\eta(w) \neq 0$ for $w \in g(U \times \overline{U})$. Assume that function $G(z,\zeta) = zg'_z(z,\zeta)\eta(g(z,\zeta))$ is starlike univalent in $U \times \overline{U}$ and that function $h(z,\zeta) = \theta(g(z,\zeta)) + G(z,\zeta)$ has the property $Re\left(\frac{zh'_z(z,\zeta)}{G(z,\zeta)}\right) > 0$ for $(z,\zeta) \in U \times \overline{U}$.*

If analytic function p with properties $p(0,\zeta) = g(0,\zeta)$ and $p(U \times \overline{U}) \subseteq D$ satisfies the strong differential subordination

$$\theta(p(z,\zeta)) + zp'_z(z,\zeta)\eta(p(z,\zeta)) \prec\prec \theta(g(z,\zeta)) + zg'_z(z,\zeta)\eta(g(z,\zeta)),$$

then

$$p(z,\zeta) \prec\prec g(z,\zeta), \quad (z,\zeta) \in U \times \overline{U},$$

and g is the best dominant.

The next definitions are connected to strong differential superordination theory.

Definition 2 ([6]). *The analytic function $f(z,\zeta)$ is strongly superordinate to the analytic function $H(z,\zeta)$ if there exists an analytic function w in U, such that $w(0)=0$, $|w(z)|<1, z\in U$, and $H(z,\zeta)=f(w(z),\zeta)$, for all $\zeta\in\overline{U}$. It is denoted $H(z,\zeta)\prec\prec f(z,\zeta)$, $(z,\zeta)\in U\times\overline{U}$.*

Remark 2 ([6]). *(i) For analytic function $f(z,\zeta)$ in $U\times\overline{U}$ and univalent in U, for all $\zeta\in\overline{U}$, Definition 2 is equivalently with $H(U\times\overline{U})\subset f(U\times\overline{U})$ and $H(0,\zeta)=f(0,\zeta)$, for all $\zeta\in\overline{U}$.*

(ii) When $f(z,\zeta)=f(z)$ and $H(z,\zeta)=H(z)$, the strong differential superordination is reduced to the differential superordination.

Definition 3 ([45]). *Q^* represents the set of analytic and injective functions on $\overline{U}\times\overline{U}\backslash E(f,\zeta)$, with property $f'_z(y,\zeta)\neq 0$ for $y\in\partial U\times\overline{U}\backslash E(f,\zeta)$, where $E(f,\zeta)=\{y\in\partial U:\lim_{z\to y}f(z,\zeta)=\infty\}$. $Q^*(a)$ represents the subclass of Q^*, with $f(0,\zeta)=a$.*

The following lemma is needed for the investigation related to strong differential superordinations.

Lemma 2 ([44]). *Consider the convex univalent function g in $U\times\overline{U}$ and the analytic functions θ and η in a domain $D\supset g(U\times\overline{U})$ such that $Re\left(\frac{\theta'_z(g(z,\zeta))}{\eta(g(z,\zeta))}\right)>0$ for $(z,\zeta)\in U\times\overline{U}$ and $G(z,\zeta)=zg'_z(z,\zeta)\eta(g(z,\zeta))$ is starlike univalent in $U\times\overline{U}$.*

If function $p(z,\zeta)\in\mathcal{H}^*[g(0,\zeta),1,\zeta]\cap Q^*$, with properties $p(U\times\overline{U})\subseteq D$ and $\theta(p(z,\zeta))+zp'_z(z)\eta(p(z,\zeta))$, is univalent in $U\times\overline{U}$ and satisfies the strong differential superordination

$$\theta(g(z,\zeta))+zg'_z(z,\zeta)\eta(g(z,\zeta))\prec\prec\theta(p(z,\zeta))+zp'_z(z,\zeta)\eta(p(z,\zeta)),$$

then

$$g(z,\zeta)\prec\prec p(z,\zeta),\ \ (z,\zeta)\in U\times\overline{U},$$

and g is the best subordinant.

The q-hypergeometric function used for investigations in [42] has the following extended form when adapted to the special classes defined in [43].

Definition 4 ([42]). *The extended q-hypergeometric function $\phi(m(\zeta),n(\zeta);q,z,\zeta)$ is defined by*

$$\phi(m(\zeta),n(\zeta);q,z,\zeta)=\sum_{j=0}^{\infty}\frac{(m(\zeta),q)_j}{(q,q)_j(n(\zeta),q)_j}z^j,$$

where

$$(m(\zeta),q)_j=\begin{cases}1, & j=0,\\ (1-m(\zeta))(1-qm(\zeta))(1-q^2m(\zeta))...(1-q^{j-1}m(\zeta)), & j\in\mathbb{N},\end{cases}$$

and $m(\zeta),n(\zeta)$ are holomorphic functions depending on the parameter $\zeta\in\overline{U}$, $0<q<1$.

The Riemann–Liouville fractional integral defined in [46,47] and applied to function $f\in\mathcal{A}^*_\zeta$ is described next.

Definition 5 ([46,47]). *For a function f, the fractional integral of order α ($\alpha>0$) is defined by*

$$D_z^{-\alpha}f(z,\zeta)=\frac{1}{\Gamma(\alpha)}\int_0^z\frac{f(t,\zeta)}{(z-t)^{1-\alpha}}dt.$$

Having introduced all the necessary previously known concepts, the next section presents the outcome of the new investigation on strong differential subordinations and strong differential superordinations involving the Riemann–Liouville fractional integral of the extended q-hypergeometric function.

3. Main Results

The investigation begins with defining the new Riemann–Liouville fractional integral of the extended q-hypergeometric function. Definitions 4 and 5 are involved in introducing the new operator.

Definition 6. *The Riemann–Liouville fractional integral of the extended q-confluent hypergeometric function is*

$$D_z^{-\alpha}\phi(m(\zeta),n(\zeta);q,z,\zeta) = \frac{1}{\Gamma(\alpha)}\int_0^z \frac{\phi(m(\zeta),n(\zeta);q,t,\zeta)}{(z-t)^{1-\alpha}}dt = \tag{1}$$

$$\frac{1}{\Gamma(\alpha)}\sum_{j=0}^{\infty}\frac{(m(\zeta),q)_j}{(q,q)_j(n(\zeta),q)_j}\int_0^z \frac{t^j}{(z-t)^{1-\alpha}}dt,$$

where the q-hypergeometric function $\phi(m(\zeta),n(\zeta);q,z,\zeta)$ *is defined by*

$$\phi(m(\zeta),n(\zeta);q,z,\zeta) = \sum_{j=0}^{\infty}\frac{(m(\zeta),q)_j}{(q,q)_j(n(\zeta),q)_j}z^j,$$

with

$$(m(\zeta),q)_j = \begin{cases} 1, & j=0, \\ (1-m(\zeta))(1-m(\zeta)q)\left(1-m(\zeta)q^2\right)\ldots\left(1-m(\zeta)q^{j-1}\right), & j\in\mathbb{N}, \end{cases}$$

$$(d)_k = \frac{\Gamma(d+k)}{\Gamma(d)} = d(d+1)(d+2)\ldots(d+k-1) \text{ and } (d)_0 = 1,$$

and $m(\zeta), n(\zeta)$ *being holomorphic functions depending on the parameter* $\zeta \in \overline{U}$, $\alpha > 0$, $0 < q < 1$.

After some calculations, it can be written using the following form:

$$D_z^{-\alpha}\phi(m(\zeta),n(\zeta);q,z,\zeta) = \sum_{j=0}^{\infty}\frac{(m(\zeta),q)_j}{(q,q)_j(n(\zeta),q)_j(j+1)_\alpha}z^{\alpha+j} \tag{2}$$

and $D_z^{-\alpha}\phi(m(\zeta),n(\zeta);q,z,\zeta) \in \mathcal{H}[0,\alpha,\zeta]$.

The first new result obtained concerns the study of a strong differential subordination obtained by using the Riemann–Liouville fractional integral of the extended q-hypergeometric function for which the best dominant is provided.

Theorem 1. *Let* $g(z,\zeta)$ *be a univalent function in* $U\times\overline{U}$ *such that* $g(z,\zeta)\neq 0$, *for all* $z\in U\backslash\{0\}$, $\zeta\in\overline{U}$ *and* $\frac{z\left(D_z^{-\alpha}\phi(m(\zeta),n(\zeta);q,z,\zeta)\right)'}{D_z^{-\alpha}\phi(m(\zeta),n(\zeta);q,z,\zeta)} \in \mathcal{H}\left(U\times\overline{U}\right)$, *where* $\alpha>0$, $0<q<1$. *Assume that* $\frac{zg_z'(z,\zeta)}{g(z,\zeta)}$ *is a starlike univalent function in* $U\times\overline{U}$ *and*

$$Re\left(\frac{\beta}{\psi}g(z,\zeta) + \frac{2\delta}{\psi}g^2(z,\zeta) + 1 - z\frac{g_z'(z,\zeta)}{g(z,\zeta)} + z\frac{g_{z^2}''(z,\zeta)}{g_z'(z,\zeta)}\right) > 0, \tag{3}$$

for $\varepsilon, \beta, \psi, \delta \in \mathbb{C}$, $\psi \neq 0$, $z \in U\backslash\{0\}$, $\zeta \in \overline{U}$ *and*

$$\Psi_{\alpha}^{q}(\varepsilon,\beta,\delta,\psi;z,\zeta) := \varepsilon + \psi + (\beta-\psi)\frac{z\left(D_z^{-\alpha}\phi(m(\zeta),n(\zeta);q,z,\zeta)\right)_z'}{D_z^{-\alpha}\phi(m(\zeta),n(\zeta);q,z,\zeta)} + \tag{4}$$

$$\delta\left(\frac{z\left(D_z^{-\alpha}\phi(m(\zeta),n(\zeta);q,z,\zeta)\right)_z'}{D_z^{-\alpha}\phi(m(\zeta),n(\zeta);q,z,\zeta)}\right)^2 + \delta\frac{z\left(D_z^{-\alpha}\phi(m(\zeta),n(\zeta);q,z,\zeta)\right)_{z^2}''}{\left(D_z^{-\alpha}\phi(m(\zeta),n(\zeta);q,z,\zeta)\right)_z'}.$$

When g *verifies the strong differential subordination*

$$\Psi_{\alpha}^{q}(\varepsilon,\beta,\delta,\psi;z,\zeta) \prec\prec \varepsilon + \beta g(z,\zeta) + \delta(g(z,\zeta))^2 + \psi\frac{zg_z'(z,\zeta)}{g(z,\zeta)}, \tag{5}$$

for $\varepsilon, \beta, \psi, \delta \in \mathbb{C}$, $\psi \neq 0$, *then*

$$\frac{z\left(D_z^{-\alpha}\phi(m(\zeta),n(\zeta);q,z,\zeta)\right)_z'}{D_z^{-\alpha}\phi(m(\zeta),n(\zeta);q,z,\zeta)} \prec\prec g(z,\zeta),\ (z,\zeta) \in U \times \overline{U}, \tag{6}$$

and the best dominant is the function g.

Proof. Define $p(z,\zeta) := \frac{z\left(D_z^{-\alpha}\phi(m(\zeta),n(\zeta);q,z,\zeta)\right)_z'}{D_z^{-\alpha}\phi(m(\zeta),n(\zeta);q,z,\zeta)}$, $(z,\zeta) \in (U\backslash\{0\}) \times \overline{U}$, and, differentiating it with respect to z, we obtain $p_z'(z,\zeta) = \frac{\left(D_z^{-\alpha}\phi(m(\zeta),n(\zeta);q,z,\zeta)\right)_z'}{D_z^{-\alpha}\phi(m(\zeta),n(\zeta);q,z,\zeta)} - z\left(\frac{\left(D_z^{-\alpha}\phi(m(\zeta),n(\zeta);q,z,\zeta)\right)_z'}{D_z^{-\alpha}\phi(m(\zeta),n(\zeta);q,z,\zeta)}\right)^2 + z\frac{\left(D_z^{-\alpha}\phi(m(\zeta),n(\zeta);q,z,\zeta)\right)_{z^2}''}{D_z^{-\alpha}\phi(m(\zeta),n(\zeta);q,z,\zeta)}$ and

$$\frac{zp_z'(z,\zeta)}{p(z,\zeta)} = 1 - z\frac{\left(D_z^{-\alpha}\phi(m(\zeta),n(\zeta);q,z,\zeta)\right)_z'}{D_z^{-\alpha}\phi(m(\zeta),n(\zeta);q,z,\zeta)} + z\frac{\left(D_z^{-\alpha}\phi(m(\zeta),n(\zeta);q,z,\zeta)\right)_{z^2}''}{\left(D_z^{-\alpha}\phi(m(\zeta),n(\zeta);q,z,\zeta)\right)_z'}. \tag{7}$$

Let $\theta(u) = \delta u^2 + \beta u + \varepsilon$ analytic in $\mathbb{C}$, and $\eta(u) = \frac{\psi}{u}$, analytic in $\mathbb{C}\backslash\{0\}$ with $\eta(u) \neq 0$, $u \in \mathbb{C}\backslash\{0\}$; we consider the starlike univalent functions $G(z,\zeta) = z\eta(g(z,\zeta))g_z'(z,\zeta) = \psi\frac{zg_z'(z,\zeta)}{g(z,\zeta)}$ and $h(z,\zeta) = G(z,\zeta) + \theta(g(z,\zeta)) = \varepsilon + \beta g(z,\zeta) + \delta(g(z,\zeta))^2 + \psi\frac{zg_z'(z,\zeta)}{g(z,\zeta)}$.

Differentiating it with respect to z, we obtain $h_z'(z,\zeta) = \beta g_z'(z,\zeta) + 2\delta g(z,\zeta)g_z'(z,\zeta) + \psi\frac{g_z'(z,\zeta)}{g(z,\zeta)} - \psi z\left(\frac{g_z'(z,\zeta)}{g(z,\zeta)}\right)^2 + \psi z\frac{g_{z^2}''(z,\zeta)}{g(z,\zeta)}$ and $\frac{zh_z'(z,\zeta)}{G(z,\zeta)} = \frac{\beta}{\psi}g(z,\zeta) + \frac{2\delta}{\psi}g^2(z,\zeta) + 1 - z\frac{g_z'(z,\zeta)}{g(z,\zeta)} + z\frac{g_{z^2}''(z,\zeta)}{g(z,\zeta)}$; therefore, we have $Re\left(\frac{zh_z'(z,\zeta)}{G(z,\zeta)}\right) = Re\left(\frac{\beta}{\psi}g(z,\zeta) + \frac{2\delta}{\psi}g^2(z,\zeta) + 1 - z\frac{g_z'(z,\zeta)}{g(z,\zeta)} + z\frac{g_{z^2}''(z,\zeta)}{g(z,\zeta)}\right) > 0$ by relation (3).

Using relation (7), we can write $\varepsilon + \beta p(z,\zeta) + \delta(p(z,\zeta))^2 + \psi\frac{zp_z'(z,\zeta)}{p(z,\zeta)} = \varepsilon + \psi + (\beta-\psi)\frac{z\left(D_z^{-\alpha}\phi(m(\zeta),n(\zeta);q,z,\zeta)\right)_z'}{D_z^{-\alpha}\phi(m(\zeta),n(\zeta);q,z,\zeta)} + \delta\left(\frac{z\left(D_z^{-\alpha}\phi(m(\zeta),n(\zeta);q,z,\zeta)\right)_z'}{D_z^{-\alpha}\phi(m(\zeta),n(\zeta);q,z,\zeta)}\right)^2 + \delta\frac{z\left(D_z^{-\alpha}\phi(m(\zeta),n(\zeta);q,z,\zeta)\right)_{z^2}''}{\left(D_z^{-\alpha}\phi(m(\zeta),n(\zeta);q,z,\zeta)\right)_z'}$.

Taking into account the strong differential subordination (5), we obtain $\varepsilon + \beta p(z,\zeta) + \delta(p(z,\zeta))^2 + \psi\frac{zp_z'(z,\zeta)}{p(z,\zeta)} \prec\prec \varepsilon + \beta g(z,\zeta) + \delta(g(z,\zeta))^2 + \psi\frac{zg_z'(z,\zeta)}{g(z,\zeta)}$ and, applying Lemma 1, we obtain $p(z,\zeta) \prec\prec g(z,\zeta)$, $(z,\zeta) \in U \times \overline{U}$, equivalently with $\frac{z\left(D_z^{-\alpha}\phi(m(\zeta),n(\zeta);q,z,\zeta)\right)_z'}{D_z^{-\alpha}\phi(m(\zeta),n(\zeta);q,z,\zeta)} \prec\prec g(z,\zeta)$, with g as the best dominant. □

Corollary 1. *Assume that relation (3) is true. If*

$$\Psi_{\alpha}^{q}(\varepsilon,\beta,\delta,\psi;z,\zeta) \prec\prec \varepsilon + \beta\frac{Mz+\zeta}{Nz+\zeta} + \delta\left(\frac{Mz+\zeta}{Nz+\zeta}\right)^2 + \frac{\psi(M-N)\zeta z}{(Mz+\zeta)(Nz+\zeta)},$$

where $\alpha > 0$, $0 < q < 1$, $\varepsilon, \beta, \psi, \delta \in \mathbb{C}$, $\psi \neq 0$, $-1 \leq N < M \leq 1$ *and* Ψ_α^q *defined by relation (4), then*

$$\frac{z(D_z^{-\alpha}\phi(m(\zeta), n(\zeta); q, z, \zeta))'_z}{D_z^{-\alpha}\phi(m(\zeta), n(\zeta); q, z, \zeta)} \prec\prec \frac{Mz+\zeta}{Nz+\zeta}, \ (z,\zeta) \in U \times \overline{U},$$

with the best dominant $\frac{Mz+\zeta}{Nz+\zeta}$.

Corollary 2. *Assume the relation (3) is true. If*

$$\Psi_\alpha^q(\varepsilon, \beta, \delta, \psi; z, \zeta) \prec\prec \varepsilon + \beta\left(\frac{z+\zeta}{\zeta - z}\right)^k + \delta\left(\frac{z+\zeta}{\zeta - z}\right)^{2k} + \frac{2k\psi\zeta z}{\zeta^2 - z^2},$$

where $\alpha > 0$, $0 < q < 1$, $\varepsilon, \beta, \psi, \delta \in \mathbb{C}$, $\psi \neq 0$, $0 < k \leq 1$ *and* Ψ_α^q *is defined by relation (4), then*

$$\frac{z(D_z^{-\alpha}\phi(m(\zeta), n(\zeta); q, z, \zeta))'_z}{D_z^{-\alpha}\phi(m(\zeta), n(\zeta); q, z, \zeta)} \prec\prec \left(\frac{z+\zeta}{\zeta - z}\right)^k, \ (z,\zeta) \in U \times \overline{U},$$

with the best dominant $\left(\frac{z+\zeta}{\zeta-z}\right)^k$.

In the following theorem, the best subordinant of a strong differential superordination investigated in correlation to the Riemann–Liouville fractional integral of the extended q-hypergeometric function is obtained.

Theorem 2. *Let* g *be an analytic and univalent function in* $U \times \overline{U}$ *with the properties* $g(z,\zeta) \neq 0$ *and* $\frac{zg'_z(z,\zeta)}{g(z,\zeta)}$ *starlike univalent. Assume that*

$$Re\left(\frac{\beta}{\psi}g(z,\zeta)g'_z(z,\zeta) + \frac{2\delta}{\psi}g^2(z,\zeta)g'_z(z,\zeta)\right) > 0, \text{ for } \beta, \psi, \delta \in \mathbb{C}, \ \psi \neq 0. \tag{8}$$

If $\frac{z\left(D_z^{-\alpha}\phi(m(\zeta),n(\zeta);q,z,\zeta)\right)'_z}{D_z^{-\alpha}\phi(m(\zeta),n(\zeta);q,z,\zeta)} \in \mathcal{H}[g(0,\zeta),1,\zeta] \cap Q^*$ *and* $\Psi_\alpha^q(\varepsilon, \beta, \delta, \psi; z, \zeta)$, *defined by relation (4), is univalent in* $U \times \overline{U}$, *then*

$$\varepsilon + \beta g(z,\zeta) + \delta(g(z,\zeta))^2 + \frac{\psi z g'_z(z,\zeta)}{g(z,\zeta)} \prec\prec \Psi_\alpha^q(\varepsilon, \beta, \delta, \psi; z, \zeta) \tag{9}$$

implies

$$g(z,\zeta) \prec\prec \frac{z(D_z^{-\alpha}\phi(m(\zeta), n(\zeta); q, z, \zeta))'_z}{D_z^{-\alpha}\phi(m(\zeta), n(\zeta); q, z, \zeta)}, \ \ (z,\zeta) \in U \times \overline{U}, \tag{10}$$

and g *is the best subordinant.*

Proof. Define $p(z,\zeta) := \frac{z\left(D_z^{-\alpha}\phi(m(\zeta),n(\zeta);q,z,\zeta)\right)'_z}{D_z^{-\alpha}\phi(m(\zeta),n(\zeta);q,z,\zeta)}$, $(z,\zeta) \in (U \backslash \{0\}) \times \overline{U}$, and consider the analytic functions $\theta(u) = \delta u^2 + \beta u + \varepsilon$ in $\mathbb{C}$ and $\eta(u) = \frac{\psi}{u}$, respectively, in $\mathbb{C}\backslash\{0\}$ with $\eta(u) \neq 0$, $u \in \mathbb{C}\backslash\{0\}$.

Differentiating it with respect to z, we can write $\frac{\theta'_z(g(z,\zeta))}{\eta(g(z,\zeta))} = \frac{[\beta + 2\delta g(z,\zeta)]g(z,\zeta)g'_z(z,\zeta)}{\psi}$ and $Re\left(\frac{\theta'_z(g(z,\zeta))}{\eta(g(z,\zeta))}\right) = Re\left(\frac{\beta}{\psi}g(z,\zeta)g'_z(z,\zeta) + \frac{2\delta}{\psi}g^2(z,\zeta)g'_z(z,\zeta)\right) > 0$, for $\beta, \psi, \delta \in \mathbb{C}$, $\psi \neq 0$, taking into account relation (8).

Strong differential superordination (9) can be written using relation (7) as follows:

$$\varepsilon + \beta g(z,\zeta) + \delta(g(z,\zeta))^2 + \frac{\psi z g'_z(z,\zeta)}{g(z,\zeta)} \prec\prec \varepsilon + \beta p(z,\zeta) + \delta(p(z,\zeta))^2 + \frac{\psi z p'_z(z,\zeta)}{p(z,\zeta)},$$

and applying Lemma 2, we obtain

$$g(z,\zeta) \prec\prec p(z,\zeta) = \frac{z(D_z^{-\alpha}\phi(m(\zeta),n(\zeta);q,z,\zeta))'_z}{D_z^{-\alpha}\phi(m(\zeta),n(\zeta);q,z,\zeta)}, \quad (z,\zeta) \in U \times \overline{U},$$

and g is the best subordinant. □

Corollary 3. *Assume that relation (8) is true. If* $\frac{z\left(D_z^{-\alpha}\phi(m(\zeta),n(\zeta);q,z,\zeta)\right)'_z}{D_z^{-\alpha}\phi(m(\zeta),n(\zeta);q,z,\zeta)} \in \mathcal{H}[g(0,\zeta),1,\zeta] \cap Q^*$ *and*

$$\varepsilon + \beta\frac{Mz+\zeta}{Nz+\zeta} + \delta\left(\frac{Mz+\zeta}{Nz+\zeta}\right)^2 + \frac{\psi(M-N)\zeta z}{(Mz+\zeta)(Nz+\zeta)} \prec\prec \Psi_\alpha^q(\varepsilon,\beta,\delta,\psi;z,\zeta),$$

where $\alpha > 0$, $0 < q < 1$, $\varepsilon,\beta,\psi,\delta \in \mathbb{C}$, $\psi \neq 0$, $-1 \leq N < M \leq 1$, *and* Ψ_α^q *is defined by relation (4), then*

$$\frac{Mz+\zeta}{Nz+\zeta} \prec\prec \frac{z(D_z^{-\alpha}\phi(m(\zeta),n(\zeta);q,z,\zeta))'_z}{D_z^{-\alpha}\phi(m(\zeta),n(\zeta);q,z,\zeta)}, \ (z,\zeta) \in U \times \overline{U},$$

with the best subordinant $\frac{Mz+\zeta}{Nz+\zeta}$.

Corollary 4. *Assume that relation (8) is true. If* $\frac{z\left(D_z^{-\alpha}\phi(m(\zeta),n(\zeta);q,z,\zeta)\right)'_z}{D_z^{-\alpha}\phi(m(\zeta),n(\zeta);q,z,\zeta)} \in \mathcal{H}[g(0,\zeta),1,\zeta] \cap Q^*$ *and*

$$\varepsilon + \beta\left(\frac{z+\zeta}{\zeta-z}\right)^k + \delta\left(\frac{z+\zeta}{\zeta-z}\right)^{2k} + \frac{2k\psi\zeta z}{\zeta^2-z^2} \prec\prec \Psi_\alpha^q(\varepsilon,\beta,\delta,\psi;z,\zeta),$$

where $\alpha > 0$, $0 < q < 1$, $\varepsilon,\beta,\psi,\delta \in \mathbb{C}$, $\psi \neq 0$, $0 < k \leq 1$, *and* Ψ_α^q *is defined by relation (4), then*

$$\left(\frac{z+\zeta}{\zeta-z}\right)^k \prec\prec \frac{z(D_z^{-\alpha}\phi(m(\zeta),n(\zeta);q,z,\zeta))'_z}{D_z^{-\alpha}\phi(m(\zeta),n(\zeta);q,z,\zeta)}, \ (z,\zeta) \in U \times \overline{U},$$

with the best subordinant $\left(\frac{z+\zeta}{\zeta-z}\right)^k$.

Now, as an application of the results obtained so far, a sandwich-type theorem connects the dual results presented in Theorems 1 and 2. The corresponding corollaries follow naturally.

Theorem 3. *Let* g_1, g_2 *be analytic and univalent functions in* $U \times \overline{U}$ *with the properties* $g_1(z,\zeta) \neq 0$, $g_2(z,\zeta) \neq 0$, *for all* $(z,\zeta) \in U \times \overline{U}$, *and* $\frac{z(g_1)'_z(z,\zeta)}{g_1(z,\zeta)}$, $\frac{z(g_2)'_z(z,\zeta)}{g_2(z,\zeta)}$ *are starlike univalent. Assume that* g_1 *verifies (3) and* g_2 *verifies (8). If* $\frac{z\left(D_z^{-\alpha}\phi(m(\zeta),n(\zeta);q,z,\zeta)\right)'_z}{D_z^{-\alpha}\phi(m(\zeta),n(\zeta);q,z,\zeta)} \in \mathcal{H}[g(0,\zeta),1,\zeta] \cap Q^*$ *and* $\Psi_\alpha^q(\varepsilon,\beta,\delta,\psi;z,\zeta)$, *defined by relation (4), is univalent in* $U \times \overline{U}$, $\alpha > 0$, $0 < q < 1$, $\varepsilon,\beta,\psi,\delta \in \mathbb{C}$, $\psi \neq 0$, *then*

$$\varepsilon + \beta g_1(z,\zeta) + \delta(g_1(z,\zeta))^2 + \frac{\psi z(g_1)'_z(z,\zeta)}{g_1(z,\zeta)} \prec\prec \Psi_\alpha^q(\varepsilon,\beta,\delta,\psi;z,\zeta)$$

$$\prec\prec \varepsilon + \beta g_2(z,\zeta) + \delta(g_2(z,\zeta))^2 + \frac{\psi z(g_2)'_z(z,\zeta)}{g_2(z,\zeta)}$$

implies

$$g_1(z,\zeta) \prec\prec \frac{z(D_z^{-\alpha}\phi(m(\zeta),n(\zeta);q,z,\zeta))'_z}{D_z^{-\alpha}\phi(m(\zeta),n(\zeta);q,z,\zeta)} \prec\prec g_2(z,\zeta), \ (z,\zeta) \in U \times \overline{U},$$

and g_1 *and* g_2 *are, respectively, the best subordinant and the best dominant.*

Corollary 5. *Assume that relations (3) and (8) are true. If* $\frac{z\left(D_z^{-\alpha}\phi(m(\zeta),n(\zeta);q,z,\zeta)\right)_z'}{D_z^{-\alpha}\phi(m(\zeta),n(\zeta);q,z,\zeta)} \in \mathcal{H}[g(0,\zeta), 1,\zeta] \cap Q^*$, *and*

$$\varepsilon + \beta\frac{M_1z+\zeta}{N_1z+\zeta} + \delta\left(\frac{M_1z+\zeta}{N_1z+\zeta}\right)^2 + \frac{\psi(M_1-N_1)\zeta z}{(M_1z+\zeta)(N_1z+\zeta)} \prec\prec \Psi_\alpha^q(\varepsilon,\beta,\delta,\psi;z,\zeta)$$

$$\prec\prec \frac{M_2z+\zeta}{N_2z+\zeta} + \delta\left(\frac{M_2z+\zeta}{N_2z+\zeta}\right)^2 + \frac{\psi(M_2-N_2)\zeta z}{(M_2z+\zeta)(N_2z+\zeta)},$$

where $\alpha > 0$, $0 < q < 1$, $\varepsilon,\beta,\psi,\delta \in \mathbb{C}$, $\psi \neq 0$, $-1 \leq N_2 < N_1 < M_1 < M_2 \leq 1$, *and* Ψ_α^q *is defined by relation (4), then*

$$\frac{M_1z+\zeta}{N_1z+\zeta} \prec\prec \frac{z(D_z^{-\alpha}\phi(m(\zeta),n(\zeta);q,z,\zeta))_z'}{D_z^{-\alpha}\phi(m(\zeta),n(\zeta);q,z,\zeta)} \prec\prec \frac{M_2z+\zeta}{N_2z+\zeta}, \quad (z,\zeta) \in U \times \overline{U},$$

with the best dominant $\frac{M_2z+\zeta}{N_2z+\zeta}$ *and the best subordinant* $\frac{M_1z+\zeta}{N_1z+\zeta}$.

Corollary 6. *Assume that relations (3) and (8) are true. If* $\frac{z\left(D_z^{-\alpha}\phi(m(\zeta),n(\zeta);q,z,\zeta)\right)_z'}{D_z^{-\alpha}\phi(m(\zeta),n(\zeta);q,z,\zeta)} \in \mathcal{H}[g(0,\zeta), 1,\zeta] \cap Q^*$, *and*

$$\varepsilon + \beta\left(\frac{z+\zeta}{\zeta-z}\right)^{k_1} + \delta\left(\frac{z+\zeta}{\zeta-z}\right)^{2k_1} + \frac{2k_1\psi\zeta z}{\zeta^2-z^2} \prec\prec \Psi_\alpha^q(\varepsilon,\beta,\delta,\psi;z,\zeta)$$

$$\prec\prec \varepsilon + \beta\left(\frac{z+\zeta}{\zeta-z}\right)^{k_2} + \delta\left(\frac{z+\zeta}{\zeta-z}\right)^{2k_2} + \frac{2k_2\psi\zeta z}{\zeta^2-z^2},$$

where $\alpha > 0$, $0 < q < 1$, $\varepsilon,\beta,\psi,\delta \in \mathbb{C}$, $\psi \neq 0$, $0 < k_1 < k_2 \leq 1$, *and* Ψ_α^q *is defined by relation (4), then*

$$\left(\frac{z+\zeta}{\zeta-z}\right)^{k_1} \prec\prec \frac{z(D_z^{-\alpha}\phi(m(\zeta),n(\zeta);q,z,\zeta))_z'}{D_z^{-\alpha}\phi(m(\zeta),n(\zeta);q,z,\zeta)} \prec\prec \left(\frac{z+\zeta}{\zeta-z}\right)^{k_2}, \quad (z,\zeta) \in U \times \overline{U},$$

with the best dominant $\left(\frac{z+\zeta}{\zeta-z}\right)^{k_2}$ *and the best subordinant* $\left(\frac{z+\zeta}{\zeta-z}\right)^{k_1}$.

Considering the functions $\theta(u) = \varepsilon u$ and $\eta(u) = \psi$, $u \in U$, we obtain other strong subordination and superordination theorems and corollaries.

Theorem 4. *Consider* g *a convex and univalent function in* $U \times \overline{U}$ *with* $g(0,\zeta) = \alpha$, $\zeta \in \overline{U}$ *and* $\frac{z\left(D_z^{-\alpha}\phi(m(\zeta),n(\zeta);q,z,\zeta)\right)_z'}{D_z^{-\alpha}\phi(m(\zeta),n(\zeta);q,z,\zeta)} \in \mathcal{H}(U \times \overline{U})$, *where* $\alpha > 0$, $0 < q < 1$. *Assume that*

$$Re\left(\frac{\varepsilon+\psi}{\psi} + z\frac{g_{z^2}''(z,\zeta)}{g_z'(z,\zeta)}\right) > 0 \tag{11}$$

for $\varepsilon,\psi \in \mathbb{C}$, $\psi \neq 0$ *and*

$$\Psi_\alpha^q(\varepsilon,\psi;z,\zeta) := (\varepsilon+\psi)\frac{z(D_z^{-\alpha}\phi(m(\zeta),n(\zeta);q,z,\zeta))_z'}{D_z^{-\alpha}\phi(m(\zeta),n(\zeta);q,z,\zeta)} - \tag{12}$$

$$\psi\left(\frac{z(D_z^{-\alpha}\phi(m(\zeta),n(\zeta);q,z,\zeta))_z'}{D_z^{-\alpha}\phi(m(\zeta),n(\zeta);q,z,\zeta)}\right)^2 + \psi\frac{z^2(D_z^{-\alpha}\phi(m(\zeta),n(\zeta);q,z,\zeta))_{z^2}''}{D_z^{-\alpha}\phi(m(\zeta),n(\zeta);q,z,\zeta)}.$$

When g verifies the strong differential subordination

$$\Psi_{\alpha}^{q}(\varepsilon, \psi; z, \zeta) \prec\prec \varepsilon g(z, \zeta) + \psi z g'_z(z, \zeta), \tag{13}$$

then

$$\frac{z(D_z^{-\alpha}\phi(m(\zeta), n(\zeta); q, z, \zeta))'_z}{D_z^{-\alpha}\phi(m(\zeta), n(\zeta); q, z, \zeta)} \prec\prec g(z, \zeta), \quad (z, \zeta) \in U \times \overline{U}, \tag{14}$$

and g is the best dominant.

Proof. Define $p(z, \zeta) := \frac{z\left(D_z^{-\alpha}\phi(m(\zeta), n(\zeta); q, z, \zeta)\right)'_z}{D_z^{-\alpha}\phi(m(\zeta), n(\zeta); q, z, \zeta)}$, with $p(0, \zeta) = \alpha$. Differentiating it with respect to z, we obtain

$$p'_z(z, \zeta) = \frac{(D_z^{-\alpha}\phi(m(\zeta), n(\zeta); q, z, \zeta))'_z}{D_z^{-\alpha}\phi(m(\zeta), n(\zeta); q, z, \zeta)} - z\left(\frac{(D_z^{-\alpha}\phi(m(\zeta), n(\zeta); q, z, \zeta))'_z}{D_z^{-\alpha}\phi(m(\zeta), n(\zeta); q, z, \zeta)}\right)^2 \tag{15}$$

$$+z\frac{(D_z^{-\alpha}\phi(m(\zeta), n(\zeta); q, z, \zeta))''_{z^2}}{D_z^{-\alpha}\phi(m(\zeta), n(\zeta); q, z, \zeta)}.$$

Considering the analytic functions $\theta(u) = \varepsilon u$ in $\mathbb{C}$ and $\eta(u) = \psi \neq 0$ in $\mathbb{C}\backslash\{0\}$, we define the starlike univalent function $G(z, \zeta) = z\eta(g(z, \zeta))g'_z(z, \zeta) = \psi z g'_z(z, \zeta)$ in $U \times \overline{U}$ and $h(z, \zeta) = G(z, \zeta) + \theta(g(z, \zeta)) = \varepsilon g(z, \zeta) + \psi z g'_z(z, \zeta)$. Relation (11) can be written $Re\left(\frac{z h'_z(z,\zeta)}{G(z,\zeta)}\right) = Re\left(\frac{\varepsilon+\psi}{\psi} + z\frac{g''_{z^2}(z,\zeta)}{g'_z(z,\zeta)}\right) > 0$ and, using relation (15), we obtain $\varepsilon p(z, \zeta) + \psi z p'_z(z, \zeta) = (\varepsilon + \psi)\frac{z\left(D_z^{-\alpha}\phi(m(\zeta), n(\zeta); q, z, \zeta)\right)'_z}{D_z^{-\alpha}\phi(m(\zeta), n(\zeta); q, z, \zeta)} - \psi\left(\frac{z\left(D_z^{-\alpha}\phi(m(\zeta), n(\zeta); q, z, \zeta)\right)'_z}{D_z^{-\alpha}\phi(m(\zeta), n(\zeta); q, z, \zeta)}\right)^2 + \psi\frac{z^2\left(D_z^{-\alpha}\phi(m(\zeta), n(\zeta); q, z, \zeta)\right)''_{z^2}}{D_z^{-\alpha}\phi(m(\zeta), n(\zeta); q, z, \zeta)}$.

The strong differential subordination (13) can be written $\varepsilon p(z, \zeta) + \psi z p'_z(z, \zeta) \prec\prec \varepsilon g(z, \zeta) + \psi z g'_z(z, \zeta)$ and, applying Lemma 1, we obtain $p(z, \zeta) \prec\prec g(z, \zeta)$, equivalently with $\frac{z\left(D_z^{-\alpha}\phi(m(\zeta), n(\zeta); q, z, \zeta)\right)'_z}{D_z^{-\alpha}\phi(m(\zeta), n(\zeta); q, z, \zeta)} \prec\prec g(z, \zeta)$, $(z, \zeta) \in U \times \overline{U}$, and g is the best dominant. □

Corollary 7. *Assume that relation (11) is true. If*

$$\Psi_{\alpha}^{q}(\varepsilon, \psi; z, \zeta) \prec\prec \varepsilon\frac{Mz + \zeta}{Nz + \zeta} + \frac{\psi(M - N)\zeta z}{(Nz + \zeta)^2},$$

where $\alpha > 0$, $0 < q < 1$, $\varepsilon, \psi \in \mathbb{C}$, $\psi \neq 0$, $-1 \leq N < M \leq 1$ *and* Ψ_{α}^{q} *defined by relation (12), then*

$$\frac{z(D_z^{-\alpha}\phi(m(\zeta), n(\zeta); q, z, \zeta))'_z}{D_z^{-\alpha}\phi(m(\zeta), n(\zeta); q, z, \zeta)} \prec\prec \frac{Mz + \zeta}{Nz + \zeta}, \; (z, \zeta) \in U \times \overline{U},$$

with the best dominant $\frac{Mz+\zeta}{Nz+\zeta}$.

Corollary 8. *Assume that relation (11) is true. If*

$$\Psi_{\alpha}^{q}(\varepsilon, \psi; z, \zeta) \prec\prec \varepsilon\left(\frac{z + \zeta}{\zeta - z}\right)^k + \frac{2k\psi\zeta z}{\zeta^2 - z^2}\left(\frac{z + \zeta}{\zeta - z}\right)^k,$$

where $\alpha > 0$, $0 < q < 1$, $\varepsilon, \psi \in \mathbb{C}$, $\psi \neq 0$, $0 < k \leq 1$, *and* Ψ_{α}^{q} *is defined by relation (12), then*

$$\frac{z(D_z^{-\alpha}\phi(m(\zeta), n(\zeta); q, z, \zeta))'_z}{D_z^{-\alpha}\phi(m(\zeta), n(\zeta); q, z, \zeta)} \prec\prec \left(\frac{z + \zeta}{\zeta - z}\right)^k, \; (z, \zeta) \in U \times \overline{U},$$

with the best dominant $\left(\frac{z+\zeta}{\zeta-z}\right)^k$.

Theorem 5. *Let g be a convex and univalent function in $U \times \overline{U}$ with $g(0,\zeta) = \alpha$, $\zeta \in \overline{U}$, and $\alpha > 0$, $0 < q < 1$. Assume that*

$$Re\left(\frac{\varepsilon}{\psi} g'_z(z,\zeta)\right) > 0, \text{ for } \varepsilon, \psi \in \mathbb{C}, \psi \neq 0. \tag{16}$$

If $\frac{z\left(D_z^{-\alpha}\phi(m(\zeta),n(\zeta);q,z,\zeta)\right)'_z}{D_z^{-\alpha}\phi(m(\zeta),n(\zeta);q,z,\zeta)} \in \mathcal{H}[g(0,\zeta),1,\zeta] \cap Q^*$ *and* $\Psi_\alpha^q(\varepsilon,\psi;z,\zeta)$, *defined by relation (12), is univalent in* $U \times \overline{U}$, *then*

$$\varepsilon g(z,\zeta) + \psi z g'_z(z,\zeta) \prec\prec \Psi_\alpha^q(\varepsilon,\psi;z,\zeta) \tag{17}$$

implies

$$g(z,\zeta) \prec\prec \frac{z(D_z^{-\alpha}\phi(m(\zeta),n(\zeta);q,z,\zeta))'_z}{D_z^{-\alpha}\phi(m(\zeta),n(\zeta);q,z,\zeta)}, \ (z,\zeta) \in U \times \overline{U}, \tag{18}$$

and g is the best subordinant.

Proof. Define $p(z,\zeta) = \frac{z\left(D_z^{-\alpha}\phi(m(\zeta),n(\zeta);q,z,\zeta)\right)'_z}{D_z^{-\alpha}\phi(m(\zeta),n(\zeta);q,z,\zeta)}$, $(z,\zeta) \in U \times \overline{U}$, with $p(0,\zeta) = \alpha$, $\zeta \in \overline{U}$, and consider the analytic functions $\theta(u) = \varepsilon u$ in $\mathbb{C}$ and $\eta(u) = \psi \neq 0$ in $\mathbb{C}\backslash\{0\}$.

Differentiating it with respect to z, we obtain $\frac{\theta'_z(g(z,\zeta))}{\eta(g(z,\zeta))} = \frac{\varepsilon}{\psi} g'_z(z,\zeta)$, and $Re\left(\frac{\theta'_z(g(z,\zeta))}{\eta(g(z,\zeta))}\right) = Re\left(\frac{\varepsilon}{\psi} g'_z(z,\zeta)\right) > 0$, for $\varepsilon, \psi \in \mathbb{C}, \psi \neq 0$, taking into account relation (16).

Applying Lemma 2 for the strong differential superordination (17) written in the following form

$$\varepsilon g(z,\zeta) + \psi z g'_z(z,\zeta) \prec\prec \varepsilon p(z,\zeta) + \psi z p'_z(z,\zeta),$$

we obtain

$$g(z,\zeta) \prec\prec p(z,\zeta) = \frac{z(D_z^{-\alpha}\phi(m(\zeta),n(\zeta);q,z,\zeta))'_z}{D_z^{-\alpha}\phi(m(\zeta),n(\zeta);q,z,\zeta)}, \ (z,\zeta) \in U \times \overline{U},$$

and g is the best subordinant. □

Corollary 9. *Assume that relation (16) is true. If* $\frac{z\left(D_z^{-\alpha}\phi(m(\zeta),n(\zeta);q,z,\zeta)\right)'_z}{D_z^{-\alpha}\phi(m(\zeta),n(\zeta);q,z,\zeta)} \in \mathcal{H}[g(0,\zeta),1,\zeta] \cap Q^*$ *and*

$$\varepsilon \frac{Mz+\zeta}{Nz+\zeta} + \frac{\psi(M-N)\zeta z}{(Nz+\zeta)^2} \prec\prec \Psi_\alpha^q(\varepsilon,\psi;z,\zeta),$$

where $\alpha > 0$, $0 < q < 1$, $\varepsilon, \psi \in \mathbb{C}$, $\psi \neq 0$, $-1 \leq N < M \leq 1$, and Ψ_α^q is defined by relation (12), then

$$\frac{Mz+\zeta}{Nz+\zeta} \prec\prec \frac{z(D_z^{-\alpha}\phi(m(\zeta),n(\zeta);q,z,\zeta))'_z}{D_z^{-\alpha}\phi(m(\zeta),n(\zeta);q,z,\zeta)}, \ (z,\zeta) \in U \times \overline{U},$$

with the best subordinant $\frac{Mz+\zeta}{Nz+\zeta}$.

Corollary 10. *Assume that relation (16) is true. If* $\frac{z\left(D_z^{-\alpha}\phi(m(\zeta),n(\zeta);q,z,\zeta)\right)'_z}{D_z^{-\alpha}\phi(m(\zeta),n(\zeta);q,z,\zeta)} \in \mathcal{H}[g(0,\zeta),1,\zeta] \cap Q^*$ *and*

$$\varepsilon\left(\frac{z+\zeta}{\zeta-z}\right)^k + \frac{2k\psi\zeta z}{\zeta^2-z^2}\left(\frac{z+\zeta}{\zeta-z}\right)^k \prec\prec \Psi_\alpha^q(\varepsilon,\psi;z,\zeta),$$

where $\alpha > 0$, $0 < q < 1$, $\varepsilon, \psi \in \mathbb{C}$, $\psi \neq 0$, $0 < k \leq 1$, *and* Ψ_α^q *is defined by relation (12), then*

$$\left(\frac{z+\zeta}{\zeta - z}\right)^k \prec\prec \frac{z(D_z^{-\alpha}\phi(m(\zeta), n(\zeta); q, z, \zeta))'_z}{D_z^{-\alpha}\phi(m(\zeta), n(\zeta); q, z, \zeta)}, \; (z,\zeta) \in U \times \overline{U},$$

with the best subordinant $\left(\frac{z+\zeta}{\zeta-z}\right)^k$.

Together, Theorems 4 and 5 imply the following sandwich theorem.

Theorem 6. *Let* g_1, g_2 *be convex and univalent functions in* $U \times \overline{U}$ *such that* $g_1(z,\zeta) \neq 0$, $g_2(z,\zeta) \neq 0$, *for all* $(z,\zeta) \in U \times \overline{U}$, *and* $\frac{z(g_1)'_z(z,\zeta)}{g_1(z,\zeta)}$, $\frac{z(g_2)'_z(z,\zeta)}{g_2(z,\zeta)}$ *are starlike univalent. Assuming that* g_1 *satisfies (11) and* g_2 *satisfies (16), if* $\frac{z\left(D_z^{-\alpha}\phi(m(\zeta),n(\zeta);q,z,\zeta)\right)'_z}{D_z^{-\alpha}\phi(m(\zeta),n(\zeta);q,z,\zeta)} \in \mathcal{H}[g(0,\zeta), 1, \zeta] \cap Q^*$, *and* $\Psi_\alpha^q(\varepsilon, \psi; z, \zeta)$, *defined by relation (12), is univalent in* $U \times \overline{U}$, $\alpha > 0$, $0 < q < 1$, *then*

$$\varepsilon g_1(z,\zeta) + \psi z(g_1)'_z(z,\zeta) \prec\prec \Psi_\alpha^q(\varepsilon, \psi; z, \zeta) \prec\prec \varepsilon g_2(z,\zeta) + \psi z(g_2)'_z(z,\zeta),$$

for $\varepsilon, \psi \in \mathbb{C}$, $\psi \neq 0$, *implies*

$$g_1(z,\zeta) \prec\prec \frac{z(D_z^{-\alpha}\phi(m(\zeta), n(\zeta); q, z, \zeta))'_z}{D_z^{-\alpha}\phi(m(\zeta), n(\zeta); q, z, \zeta)} \prec\prec g_2(z,\zeta), \quad (z,\zeta) \in U \times \overline{U},$$

and g_1 *and* g_2 *are, respectively, the best subordinant and the best dominant.*

Corollary 11. *Assume that relations (11) and (16) are true. If* $\frac{z\left(D_z^{-\alpha}\phi(m(\zeta),n(\zeta);q,z,\zeta)\right)'_z}{D_z^{-\alpha}\phi(m(\zeta),n(\zeta);q,z,\zeta)} \in \mathcal{H}[g(0,\zeta), 1, \zeta] \cap Q^*$ *and*

$$\varepsilon\frac{M_1 z+\zeta}{N_1 z+\zeta} + \frac{\psi(M_1 - N_1)\zeta z}{(N_1 z+\zeta)^2} \prec\prec \Psi_\alpha^q(\varepsilon, \psi; z, \zeta) \prec\prec \varepsilon\frac{M_2 z+\zeta}{N_2 z+\zeta} + \frac{\psi(M_2 - N_2)\zeta z}{(N_2 z+\zeta)^2},$$

where $\alpha > 0$, $0 < q < 1$, $\varepsilon, \psi \in \mathbb{C}$, $\psi \neq 0$, $-1 \leq N_2 < N_1 < M_1 < M_2 \leq 1$, *and* Ψ_α^q *is defined by relation (12), then*

$$\frac{M_1 z+\zeta}{N_1 z+\zeta} \prec\prec \frac{z(D_z^{-\alpha}\phi(m(\zeta), n(\zeta); q, z, \zeta))'_z}{D_z^{-\alpha}\phi(m(\zeta), n(\zeta); q, z, \zeta)} \prec\prec \frac{M_2 z+\zeta}{N_2 z+\zeta}, \; (z,\zeta) \in U \times \overline{U},$$

with the best dominant $\frac{M_2 z+\zeta}{N_2 z+\zeta}$ *and the best subordinant* $\frac{M_1 z+\zeta}{N_1 z+\zeta}$.

Corollary 12. *Assume that relations (11) and (16) are true. If* $\frac{z\left(D_z^{-\alpha}\phi(m(\zeta),n(\zeta);q,z,\zeta)\right)'_z}{D_z^{-\alpha}\phi(m(\zeta),n(\zeta);q,z,\zeta)} \in \mathcal{H}[g(0,\zeta), 1, \zeta] \cap Q^*$ *and*

$$\varepsilon\left(\frac{z+\zeta}{\zeta-z}\right)^{k_1} + \frac{2k_1\psi\zeta z}{\zeta^2 - z^2}\left(\frac{z+\zeta}{\zeta-z}\right)^{k_1} \prec\prec \Psi_\alpha^q(\varepsilon, \psi; z, \zeta)$$
$$\prec\prec \varepsilon\left(\frac{z+\zeta}{\zeta-z}\right)^{k_2} + \frac{2k_2\psi\zeta z}{\zeta^2 - z^2}\left(\frac{z+\zeta}{\zeta-z}\right)^{k_2},$$

where $\alpha > 0$, $0 < q < 1$, $\varepsilon, \psi \in \mathbb{C}$, $\psi \neq 0$, $0 < k_1 < k_2 \leq 1$, *and* Ψ_α^q *is defined by relation (12), then*

$$\left(\frac{z+\zeta}{\zeta-z}\right)^{k_1} \prec\prec \frac{z(D_z^{-\alpha}\phi(m(\zeta), n(\zeta); q, z, \zeta))'_z}{D_z^{-\alpha}\phi(m(\zeta), n(\zeta); q, z, \zeta)} \prec\prec \left(\frac{z+\zeta}{\zeta-z}\right)^{k_2}, \; (z,\zeta) \in U \times \overline{U},$$

with the best dominant $\left(\frac{z+\zeta}{\zeta-z}\right)^{k_2}$ *and the best subordinant* $\left(\frac{z+\zeta}{\zeta-z}\right)^{k_1}$.

4. Discussion and Concluding Remarks

Motivated by the inspiring outcomes of integrating aspects of quantum and fractional calculus into geometric function theory studies, the theories of strong differential superordination and its dual, strong differential subordination, incorporate such aspects in an attempt to resurrect a study started in [38] but not yet followed. As a result of this research, new results regarding strong differential subordination and dual new strong differential superordinations are obtained in this paper. Specifically, the definition of the Riemann–Liouville fractional integral of the extended q-hypergeometric function is introduced in Definition 6, given by relations (1) and (2). In each theorem established, the best dominants and best subordinants are provided. Significant corollaries follow when notable functions with respect to their geometric features are employed as the best dominant or best subordinant in the theorems. The new results derived from the research conducted in this study, which examined the two dual theories of strong differential subordination and strong differential superordination, are connected via sandwich-type theorems and corollaries. The purpose of the paper is to offer a new direction for the study of strong differential superordination and its dual, strong differential subordination, by integrating quantum calculus associated with fractional calculus. By applying the ideas addressed in this paper to different hypergeometric functions and operators developed using them, further intriguing operators could be obtained.

Considering the geometrical properties derived from the results presented in the corollaries, future studies could result in the introduction of new subclasses of functions using the Riemann–Liouville fractional integral of the extended q-hypergeometric function as seen in [48].

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