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Strings and D-branes on orbifolds:

from boundary states to geometry

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Contents

The following guidelines may prove helpful during the reading of the manuscript.

- ☞ Except in the introductory Section 2.1, a basic acquaintance with superstring theory in general, and CFT in particular has been assumed. The short glossary in Appendix D provides supplementary material to Section 2.1 and may be useful when you are browsing through the document.
- ☞ Throughout, established theorems (conjectures) are output inside a THEOREM (CONJECTURE) environment. In contrast, a FACT environment is reserved for results that are true to most string theorists, even though a rigorous mathematical proof is still lacking.
- ☞ Items marked with '$^\flat$' are further explained in the **Glossary**.

1

Planner

In recent times, string theory has opened new perspectives on geometry. Of fundamental importance, is the understanding that space-time becomes a secondary concept, from the viewpoint of string theory, rather than a primary given. Armed with this insight, it was hoped, first, that a deeper understanding of the nature of space-time would be found within string theory. Unfortunately, a complete understanding, if within reach at all, is still lacking, but some progress has been made. Various parts in the present thesis must be viewed in the light of this on-going search.

That string theory can generalise classical geometrical notions, finds its origin in the way perturbative string theory is defined: it is studied as (a set of) twodimensional field theories, that, moreover, must be conformal (CFT) for consistency. One says that CFTs define string vacua. In this framework, space-times that obey Einstein equations give rise to particular examples of CFTs; on the other hand, there exists a huge collection of CFT beyond those. Hence, the CFT string vacua enlarge the class of classical space-times. That conformal field theory happens to be a well-studied subject in mathematical physics, is a convenience that makes them attractive for applications in string theory.

Open strings are associated to extended objects, different from strings, which go under the name of 'D-branes'. The latter will be the subject of Chapter 2. Here, we motivate the use of the boundary state formalism to study D-branes. This algebraic formalism enables one to extend the notion of D-branes in classical geometries, to D-branes in arbitrary CFTs, where, a priori, the meaning of D-branes need not be clear. That they are the appropriate tools to define D-branes of the latter type nevertheless, is the main motivation for the study of boundary states.

1.1 Motivation

1.1.1 Why string theory ?

Originally designed in an attempted description of the strong interaction, string theory has gradually evolved into a research discipline strongly interacting with various other branches of physics and mathematics. As of today, it covers a vast area ranging from 'quantum gravity' to 'quantum geometry'. Quotes used here must remind the reader that these notions are still in a somewhat premature stage despite the huge efforts spent during the past decades. Among various motivations for string theory, one is found extremely attractive: string theory may contain the ingredients to answer the question

"What is the nature of space-time?"

In Einstein's theory of general relativity, gravity is intimately tied to geometry: general relativity, that is, classical gravity, is most concisely formulated in differential geometric terms. In that framework, one is looking for manifolds $\mathcal{M}$ with metrics g that solve Einsteins equations. It is precisely such pairs $(\mathcal{M}, g)$ that legally acquire the epithet 'space-time'. In comparison with Newton's formulation of gravity, of which general relativity is a refinement, two features are new: firstly, space-time now becomes a dynamical entity, the evolution of which is governed by the Einstein equations. For example, physical particles do not move in a fixed space-time background arena; rather, they modify this background as dictated by the cited equations. Secondly, general relativity reveals the strong entanglement between gravity and geometry, thus connecting physical and mathematical data. One could say that in Einstein gravity, Nature shows its geometrical face.

Since theoretical physicists appreciate a flavour of elegance in such physics-geometry links, Einstein's theory stimulated a geometrisation trend in (theoretical) physics: classical geometry was probed to 'explain' features of theoretical models. In more recent times, developments in superstring theory have pushed this process to its limits. This has culminated in a number of surprising results, often defying one's intuition. Out of a plethora of such stringy geometric phenomena, let us lift some remarkable examples:

(a) *Space-time is a derived concept.*

The quantum theory of strings in its present formulation, is such that space-time is rather secondary: measurable quantities are derived from auxiliary two-dimensional quantum field theories of a peculiar type, the so-called conformal field theories (CFTs). Let us elaborate these statements some further.

In a given space-time $(\mathcal{M}, g)$, a string sweeps out a two-dimensional surface $\mathcal{W}$ as it evolves in time; $\mathcal{W}$ describes the history of the string, so to speak. To separate world-sheet and space-time has turned out to be a powerful convenience: given a two-dimensional surface Σ, the world-sheet as it is called, one studies how it is embedded in the space-time manifold $(\mathcal{M}, g)$, the target space for short. Let us denote the embedding map(s) $\Phi : \Sigma \to \mathcal{M}$.

In the quantum theory, contributions of maps Φ with the appropriate boundary conditions (initial and final string states) are weighted by the exponentiated action S, where

$$S = \frac{1}{\alpha'} \int_{\Sigma} ||\partial \phi||^2 \ . \tag{1.1.1}$$

The norm is taken w.r.t. the metric g on $\mathcal{M}$, and ∂ symbolically denotes derivatives w.r.t. world-sheet coordinates, obviously. In Polyakov's formulation, the string path integral schematically takes the form

$$\int \mathcal{D}h \int \mathcal{D}\Phi \exp(iS[\Phi, h]) \ . \tag{1.1.2}$$

In words: the prescription is that one should sum over all possible metrics on Σ (the 'h' were left implicit in Eq. (1.1.1) for notational convenience), and all embeddings Φ. Further, note that S is manifestly invariant under reparametrisations and Weyl rescalings of Σ.

Potentially, there is a long story here, which can be summarised, fortunately, as follows: after working one's way through the gauge-fixing for the path-integral to make sense, one would discover a residual conformal symmetry. It is the latter that turns out to be the guiding principle in string theory. An elaborate analysis further reveals that the initial, classical conformal symmetry becomes anomalous in the quantum theory, unless some conditions are obeyed. Among these, the ones that are most relevant for our point are that g should obey the Einstein equation, and further that the space-time dimension be 26 (10) in the bosonic string (superstring) case. This means that string theory is only well-defined quantum mechanically on *physical* space-times of the right dimension.

In conclusion, it is only through a backdoor that space-time effectively pops up in the picture: the constraints on the target space are derived entirely from consistency requirements on the two-dimensional, auxiliary world-sheet theory.

(b) *String theory and non-geometric phases.*

Intuition, both physical and geometrical, can easily be appealed to in the non-linear sigma model[♭] (NLSM) approach, that is, the picture of strings drawn thus far. As said, strings can be understood there as pieces of rope evolving in some background space-time. Recall that consistency requirements singled out physical space-times, i.e., those obeying Einstein equations, to be the ones of relevance. Crudely stated, the string only cares about its associated two-dimensional field theory being a conformal field theory (CFT). This observation paves the way for abstraction: could one replace the non-linear sigma model by some abstract CFT that need not have a geometrical meaning? The answer is in the affirmative. Deprived of imagination, the strings are then said to propagate in a CFT background, as opposed to the geometrical one of the NLSM.

So-called minimal models are examples of CFTs where in many cases, a lagrangian realisation is not known to exist. This state of affairs blurs the immediate geometric significance of such models, since there is no action, let alone embedding maps Φ. Rather, the CFTs are specified by the energy-momentum tensor, T, the operator spectrum and the OPEs that encode the short-distance behaviour of colliding operators. These data suffice, in principle, to compute correlators and thus to solve the theory.

Awkward as they appear, a large class of CFT backgrounds have been argued to be continuously connected to non-linear sigma models (geometric backgrounds), however, without the theory going awry. By this, we mean to say that there exists a one-parameter family of CFTs, $C_t, t \in [0,1]$ say, with respective energy-momentum tensors T_t, such that $t = 0$ corresponds to the abstract CFT, whereas $t = 1$ yields the geometric NLSM. As such, this circumstance puts both types of backgrounds on an equal footing. Put differently, CFTs enlarge the space of string vacua formed by non-linear sigma-models. Moreover, the continuous connection to the classical geometry of the NLSM phase allows us to give 'stringy geometry' a meaningful content in the abstract CFT phase.

What is the true nature of space-time, then? Probably the best option here, is to give up insisting on a single well-defined notion. Rather, the way space-time reveals itself depends on the scale (energy) and object (point particle, string, D-brane) it is being probed with. Compare the situation to a hydrogen nucleus being probed with electrons, say. At large distances, i.e.,using low-energy probes, the nucleus is discovered as a localised positive charge, whereas high-energy probes reveal a non-trivial internal structure.

1.1.2 Why CFT ?

Consistent geometric backgrounds are singled out by obeying Einstein's equation, which may be derived from world-sheet (non-linear sigma model) perturbation theory; from Eq. (1.1.1), this is a perturbation theory in α'/R^2, where R, a typical curvature radius of the target space, makes this parameter dimensionless. If one looks more carefully, Einsteins's equation acquires subleading corrections in α'/R^2, due to two-dimensional quantum effects. One says that such α'-corrections are the string theory modification to general relativity: sending α'/R^2 to zero with fixed R corresponds to taking the point particle limit, indeed (see 'worldsheet' in the Glossary for more details). Anyhow, the main point here, is that quantum conformal invariance of the non-linear sigma model is only achieved order-by-order in α'/R^2. That is, inherently to the approach, the consistency equations, among which Einstein's equation, will receive corrections to all orders in α', generically. This has the effect that the classical string backgrounds, whence all quantities in the theory, have to be adjusted order-by-order in α' accordingly, for the string theory to be well-defined.

This situation sharply contrasts with the case of genuine CFT backgrounds: the latter being conformal to all orders of α' from the start, there is no need for/notion of (perturbative) corrections of the stated type. This feature lends a certain degree of superiority to exactly conformal field theories over approximately conformal geometric backgrounds (NLSMs).

1.1.3 Why boundary states ?

Since boundary states[♭] will constitute a whole chapter in this thesis, let us motivate them. Roughly speaking, boundary states are an approach to D-branes in a closed-string framework. As will be clarified in Chapter 2, the geometric picture of D-branes is a locus in space-time where open strings can end. Features of these open strings, namely, the spectrum of physical open-string states, are precisely what boundary states encode, thereby keeping any closed-string symmetry manifest. If the massless open superstrings in geometric D-branes are all one is after, a systematic study of boundary states would seem like a waste of time and effort at first sight: given the massless states in the flat space case, combined with supersymmetry, one simply extrapolates to curved space. On second thoughts, however, this procedure is not beyond question:

(a) *Does it remain valid in regions where curvatures are large?*

Orbifold spaces♭, or slight perturbations thereof, are examples of spaces where Riemann curvatures are sharply peaked. D-branes on such spaces will be studied in Chapter 4.

(b) *What about non-geometric CFT phases?*

In such phases, space-time is (partially) replaced by some abstract CFT, hence all geometric intuition disappears. Obviously, the space-time picture of D-branes as space-time submanifolds then breaks down, since there is no such thing as space-time. However, as it turns out, boundary states are still sensible constructs. Following general principles, Cardy was able to write down a consistent class of such states, given a generic closed-string CFT. Assembling this information, boundary states will be the key objects in extending the notion of D-branes from the geometric to the non-geometric CFT phase. Boundary states do well deserve to be studied, since they contain information on non-geometric D-branes that appears to be hard to get one's hands on, with the presently available techniques in string theory.

(c) *What if no supersymmetry is present?*

The subclass of non-BPS D-brane states constitutes yet another piece of motivation for the study of boundary states, for presently, other string theory techniques usually rely on BPS♭(space-time supersymmetry) properties explicitly, and hence fail to apply to non-BPS situations. Boundary states can be constructed without appealing to any space-time supersymmetry at all, which is why they persist in the absence of the latter.

Finally, let us stress that D-brane boundary states have the same flavour of exactness w.r.t. D-brane SUGRA solutions as a closed-string CFT has w.r.t. the effective supergravity description: for example, where the first capture the complete string theory spectrum, the second only deal with the massless states, effectively.

Also, boundary states enable us to understand features of orbifold string theories, that otherwise follow from heuristic arguments only. In particular, from the orbifold boundary states, it will become clear in Section 4.4 why string theory actually manages to remain well-behaved even though the target space geometry may be singular.

1.2 A guided tour

Let us give an outline of the structure of this thesis, meanwhile pointing out the results that were obtained.

In the introductory Section 2.1, D-branes are discussed from three complementary perspectives. Additionally, some links between them are briefly touched upon. Next, Section 2.2 sets the CFT framework that will be used throughout this volume.

Chapter 3 collects various facts in the literature concerning geometry in general, and orbifolds in particular. This chapter is conceived as if point particles, strings and D-branes were used as probes, which is reflected by its structure. In the end, we hope to provide the reader with a sufficient background to tackle the remaining chapters. Section 3.1 deals with aspects of classical geometry, thereby paying particular attention to reduced holonomy issues and desingularisation of orbifolds. Since so-called ADE orbifolds provide the simplest examples, one such example is worked out in detail; this should stimulate the digestion of the presented material. Next, Section 3.2 contains a review of orbifold CFTs, in the framework set in Section 2.2. During the preparation of this text, explicit formulas for traces were obtained as a nice side-result. These do not seem to have appeared previously. Also here, ADE cases are the main source of examples. However, Section 3.2.3 sheds a light on Calabi-Yau threefold singularities. Toric geometry is argued to support an intermediate version of McKay correspondence, to be explored in Chapter 5; this argument appears to be new. Finally, Section 3.3 recalls basic facts concerning (fractional) D-branes on orbifolds, for convenience.

In Chapter 4, we develop the systematics of orbifold boundary states, extracting material from our paper [1] mostly. Through the example of the $N = 4$ superconformal algebra (SCA), Section 4.1 demonstrates the relevance of symmetry-preserving boundary conditions. In turn, this sets the stage for Section 4.2 that reviews Cardy's construction of consistent boundary states in rational conformal field theory (RCFT). Preliminary to orbifold boundary states, are the simpler flat space boundary states. Hence, these are treated first, in Section 4.3. The fermion sectors and absolute normalisation of the states, two features that appear not to have attracted special interest before, receive special attention. Finally, in Section 4.4 we construct consistent states describing fractional orbifold D-branes. Cardy's initial prescription only needs slightly modifying for the latter to fit within the framework.

In Chapter 5, McKay correspondence is explored. A short digression in Section 5.1 on the mathematics involved, must in fact prepare for the subsequent Section 5.2. In the latter, the issue is tackled from the physics viewpoint: D-branes and massless open strings are argued to realise McKay correspondence, whereby some emphasis is put on the key rôle of the spin bundle. It is hoped that the presentation adds to a clarification of various points in the existing literature.

Chapter 6 deals with discrete torsion. After a review of the closed-string side in Section 6.1, open-string issues are the subject of Section 6.2. The $\mathbb{Z}_6 \times \mathbb{Z}_6$ example which is worked out there, and which allows non-minimal torsion, presents the possibility of multiple discrete charges. This conjectural state of affairs would require further investigation. Finally, the geometry-CFT correspondence is reviewed in Section 6.3, where toric geometry is also briefly touched upon.

To tie up a few loose ends, two appendices are added. Appendix A contains a primer on sheaves, whereas Appendix B gives an explicit account of chiral traces that show up in orbifold partition functions.

Conclusions and outlook

Rounding up, let us recall the main result that was discussed in the thesis: far and foremost, we have been able to generalize Cardy's consistent boundary state prescription to orbifold CFTs. Among other things, this means that we have explicitly constructed closed-string descriptions of D-branes in orbifold string theories, thereby not leaving the well-trodden path of CFT. The power of the proper generalisation of Cardy's prescription resides in the fact that it allows us to treat all geometric orbifold boundary states in a unified manner.

The following results were obtained as side-products of the manipulations involved.

(a) Toric geometry has been demonstrated to support an intermediate version of McKay correspondence, realised in closed-string CFT.

(b) As a by-product of the manipulations involved, the decomposition of $\mathrm{SO}(2)_1$-modules into G-modules was explicitly found at the chiral-trace level.

More interestingly, the reader must be aware that the subject is not completely closed. The most prominent open questions that remain, are listed below:

(a) discrete torsion D-branes require further investigation: a McKay like correspondence and K-theoretic understanding of discrete charges would seem desirable goals to endeavour;

(b) thus far, McKay correspondence has not been investigated in connection with orientifold backgrounds, or equivalently, unoriented type I string theories, either. The observation that the latter can accommodate $N = 1, d = 4$ gauge theories, similarly to the oriented type II strings in CY threefold backgrounds, seems to point towards a modified correspondence, whereby a key question involves the orientifold projection.

It is hoped that the presented results contribute to a deeper insight in the geometry–string theory correspondence. Beyond the CFT techniques in the thesis, other approaches look equally attractive. Topological strings, in particular, seem to be promising tools for major progress along these lines.

On the other hand, if one thing, recent years' developments have taught string theorists that immediate progress sometimes hides in unsuspected corners...

2

Pointers

The present chapter serves two purposes. In Section 2.1, we provide the reader with an elementary introduction to D-branes. Rather than to go technical, we have opted to sketch a global picture of present-day views on D-branes. In the same run, the presentation enables us to comment on boundary states and their range of applicability. Next, in Section 2.2 a number of conformal field theory basics are reviewed, with the single goal to set the framework for subsequent chapters. The topics covered include the Hilbert space structure and fusion rules, CFT moduli spaces and supersymmetry, each of which will be recurrent throughout this thesis.

2.1 Pictures of D-branes

In the first half of the last decade, string theory was observed to contain extended objects, other than perturbative strings. Since the memorable Strings '95 Conference, it has become clear that the way towards progress in the understanding of nonperturbative string theory may well be paved with so-called D-branes. So far, the D-brane paradigm has essentially rested upon three major cornerstones, to know: supergravity (SUGRA) solutions, gauge theories and boundary conditions. Although different in character, the three approaches are grossly complementary, and as such make D-branes a rich set of tools to study a lot of string and field theory aspects previously believed beyond reach. We shall briefly review each of the approaches here, and provide some obvious links.

Before a discussion of D-branes, it is instructive to look at an example in SU(2) QCD with adjoint scalar fields. At large-distance scales, the quantum theory is effectively described by a classical gauge theory. It was realised that there exist classical static solutions that, besides spontaneously breaking SU(2) → U(1), also carry magnetic charge under the unbroken gauge group: the 't Hooft–Polyakov monopoles. We recall some basic features that will become relevant below:

(a) This class of solutions has finite energy that is moreover localised; as such, they can be thought of as particles, with a mass proportional to $1/e$ (e = the electric coupling) as it turns out[1]. Moreover, their charge is quantised in units of $1/e$ for the quantum theory to make sense.

(b) The solutions come with integration 'constants', collectively denoted by $\{c_i\}$. Sometimes, the latter are referred to as collective coordinates, moduli or zero-modes. Typically, they are Goldstone modes of broken global symmetries: e.g., the center-of-mass reflects broken translation invariance. One proceeds by making the $\{c_i\}$ vary with time t; the $\{c_i(t)\}$ are then interpreted as encoding the dynamics of the monopole(s), where t is a coordinate along the monopole world-line. In a semiclassical treatment, the $\{c_i(t)\}$ are canonically quantised.

(c) Inherently to the approach, the elementary excitations are treated differently from the monopole particles. Where QFT of the former continues to apply in a classical monopole background, the latter can at best be treated semiclassically as in (b).

2.1.1 D-branes as SUGRA solutions

At low energy scales, perturbative type II string♭ theories have effective supergravity descriptions. They are the string theory analogues of the classical gauge theory in the QCD example, with Ramond-Ramond♭ fields the higher-p-form counterparts of the unbroken-U(1) gauge fields there. This observation ignited the search for static solutions that are charged wrt. those gauge fields, D-brane solutions, say. The success of this program [2] signalled the existence of corresponding nonperturbative objects (D-branes) in the full string theory in particular. In the simplest class of solutions, there is only one nontrivial Ramond-Ramond charge and space-time is asymptotically Minkowski. A common feature of those simple solutions is that they partially break global translation invariance, and half of the global space-time supersymmetry. Equivalently, the remaining half of the supersymmetries are

[1] In fact, for the notion of particle to make sense, one should verify that the solutions possess a sufficient degree of coherence as well.

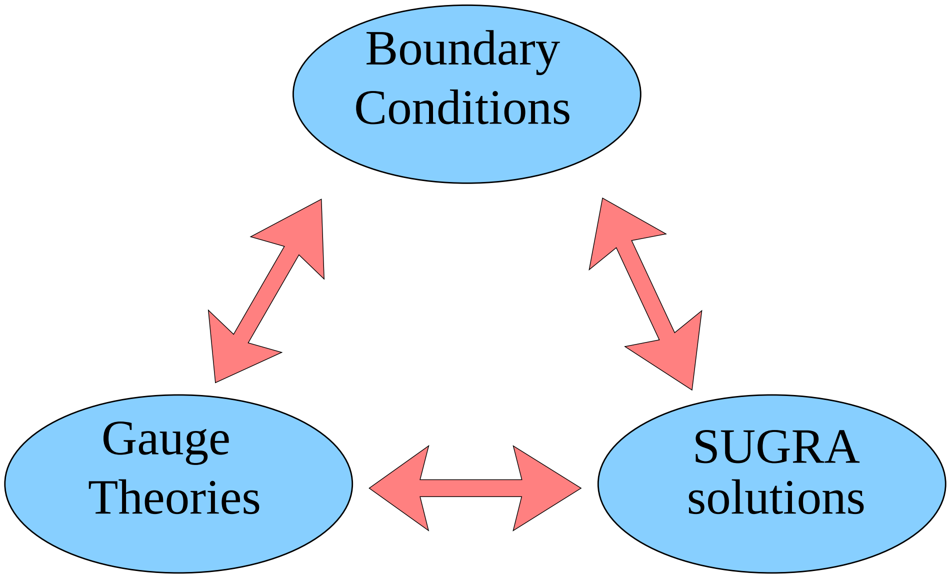

Figure 2.1: Three possible approaches to D-branes.

preserved: they are BPS$^\flat$ states. The simplicity of such basic solutions makes them useful building blocks to construct more complicated ones. Of the latter, solutions with multiple charges are particularly easily obtained by superposition. Even though the gravity field equations are nonlinear, the BPS key property[2] ensures that (linear) superposition is a viable procedure to generate new solutions from old ones. Typically, configurations with multiple charges would further reduce the amount of preserved SUSY.

From the fact that the field equations encode the consistency requirements on the closed string background, any solution to those equations defines a consistent string vacuum. Particularly so, D-brane solutions may be viewed as new nonperturbative arenas where the strings can live. Like in the SU(2) example, the quantum theory of closed strings in such vacua is defined through perturbation theory around such classical backgrounds.

[2]A more thorough analysis reveals that the range of parameters where the SUGRA solutions can be trusted requires that $g_s N$ be large; at weak string coupling g_s, this condition implies that the number of branes, N, be large. As such, the no-force BPS condition is a crucial ingredient for the SUGRA solutions to make sense.

2.1.2 D-branes as boundary conditions

In an approach orthogonal to the above, D-branes made their appearance in string theory via considerations of boundary conditions. Boundary conditions come naturally with open strings, and vice versa. For one thing, the latter sweep out surfaces with boundaries. In the simplest setup, the strings are imagined to be moving in a Minkowski background. The requirement that the endpoints of the string be confined to a given hypersurface $\mathcal{D}$ imposes Dirichlet boundary conditions on the (bosonic) two-dimensional fields that parametrise the space transverse to $\mathcal{D}$, with a natural generalisation to worldsheet fermionic fields, if any. Accordingly, $\mathcal{D}$ becomes a spacetime locus 'where open strings can end'. From the particular boundary condition type, these hypersurfaces borrowed their name 'D(irichlet)-branes'. In a somewhat loose language, D-branes get identified with (sets of) boundary conditions. Among all possible boundary conditions, those preserving conformal invariance at the worldsheet boundaries acquire a special status in string theory, as will be seen in Chapter 4.

Actually, nothing prevents the logic from being turned around. Ultimately, D-branes will be defined by a set of conformal symmetry preserving boundary conditions, indeed (see Chapter 4). In contrast with nonlinear sigma models, which are generically conformal order by order in α' only, genuine CFTs are *exactly* conformal by construction. If judiciously exploited, the latter symmetry is in fact the key property that makes closed string correlation functions accessible to computation after all. In a similar vein, the treatment of D-branes as conformal symmetry preserving boundary conditions lends a power to them that is superior to the supergravity solutions discussed in the previous section. In many respects, D-branes would be far less useful constructs in string theory if the approximate (in α') backgrounds above remained without this 'microscopic' supplement.

As an illustration, let us find out about immediate implications of allowing strings to have their ends moving on the given hypersurface $\mathcal{D}$. Quantisation of such open strings produces an infinite tower of states that are confined to $\mathcal{D}$; that is, unlike the closed-string modes, which live in the bulk spacetime, open-string modes can only live where the D-branes are. In turn, the effective dynamics of these states is governed by open string theory, and can be encoded in a quantum field theory on $\mathcal{D}$. As a result, open strings turn the hypersurfaces they are ending on into a dynamical object: not only do they endow it with degrees of freedom, they also supply the dynamics.

2.1.3 D-branes and gauge theories

As to the massless degrees of freedom associated to D-branes, the massless worldvolume fields for short, they interact according to a gauge theory. To see this, note that the D-brane worldvolume bosonic fields involve massless vectors among other fields. A consistent theory of such fields is believed to be necessarily a gauge theory. Under favourable circumstances, namely, when the branes are BPS$^\flat$ objects embedded in superstring theories, the fields furnish multiplets$^\flat$ of a supersymmetric gauge theory. The fact that the leading order effective theory is of super-Yang–Mills (SYM) type in effect, has been confirmed in various ways (see Ref. [3] and references therein). As such, localised gauge theories are accomodated inside type II (and type I/I') superstring theories through D-branes.

Gradually, it has become clear that the SYM action is only the leading order (in a derivative expansion) approximation to the so-called Born–Infeld (DBI) action. Moreover, consistency considerations were used to argue that DBI had to be supplemented further with a topological Wess-Zumino term. Presently, it is beyond reasonable doubt that the D-brane long-wavelength dynamics is indeed provided by appropriate versions of SYM (DBI) with a topological term.

D-branes and string theory have played a prominent rôle in major parts of the progress in the area of ordinary quantum field theory. The following examples were lifted out of an extensive list:

(a) Relations between seemingly unrelated field theories find a natural realisation in terms of branes. A prominent example involves Hanany-Witten [4] dualities. Without the notion of branes, it seems unlikely that those would have been put forward.

(b) 'Old' features of field theories allow for a modern brany reinterpretation. As an example, certain objects in gauge theories, such as magnetic monopoles, vortices, and the like, can be understood as particular brane set-ups.

Let us end with some comment here. In type II$^\flat$ theories two kinds of nonabelian gauge symmetries [3] can be traced back to D-branes, even though

[3] Other known mechanisms to generate nonabelian gauge symmetries are:

(a) Torus compactifications lead to nonabelian enhanced symmetries whenever the radii are tuned appropriately. Here, a subset of the winding and momentum states of closed strings provide the nonabelian massless vectors. This is the old well-known route.

(b) More recently, multiple coincident NS-fivebranes in type IIA/B have been argued to display nonabelian gauge symmetries equally well [5, 6]. However, since NS-fivebranes are not as well-understood in perturbative string worldsheet terms as D-branes, all arguments that have appeared so far suffer from a degree of indirectness. This is to be their nature is very different.

(a) First, there are the aforementioned gauge theories localised on coincident D-brane worldvolumes. As indicated, these result from open strings that end there. In this scenario, the branes are rather passive: they do nothing more than to provide an arena for the open string modes to live.

(b) Next, D-branes figure in space-time gauge symmetry enhancement in a completely disparate manner. In compactified type II theories the Ramond-Ramond[♭] closed string sector typically yields a set of abelian U(1) vectors. The spacetime W-bosons charged under these R-R U(1)'s are necessarily D-branes, since the latter are the only objects in the theory that carry R-R charges. In appropriate circumstances these W-bosons go massless, thus triggering the nonabelian gauge symmetry enhancement. In contrast with (a), the D-branes now play an active rôle in the enhancement. Also, since the R-R fields involve closed, rather than open strings, the gauge fields live in the bulk noncompact space-time. In particular, it is not confined to any brane world-volume whatsoever.

2.1.4 Links

So far, the focus has been on the three corners in Fig. 2.1. we now move on to discuss the arrows in the picture. To keep the digression within reasonable bounds, we shall provide one nontrivial link between any pair of corners only. It is hoped that this will get the reader convinced that the picture drawn so far is an extremely rich source of new ideas.

SUGRA ↔ Gauge theories

One of the fruitful links between these two corners is the celebrated AdS-CFT correspondence [7, 8, 9]. In this conjectured correspondence, gravitation (closed string theory) in the near-horizon geometry of supergravity solutions is believed to be dually described by a gauge theory (open strings, SYM) with a large number of colors. This picture stirred quite some excitement, for if true, strongly coupled (quantum) gravity might be a gauge theory, after all! For a review, see Ref. [10].

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contrasted with D-branes, where an exact CFT description is available, as discussed previously.

SUGRA ↔ Open strings/Boundary conditions

Undoubtedly, the string theory microscopic derivation of the classical Bekenstein–Hawking black hole entropy must be put on this link. This entropy, identified with the area of the black hole horizon, has long remained a formal quantity without a convincing underlying microscopic picture. However, the identification of classes of D-brane solutions as particular (supersymmetric) black holes on the one hand, and the picture of theories of open strings on the other, led Strominger and Vafa [11] to the bold conjecture that the latter provide the microscopic states accounting for the entropy of the former. This conjecture was subsequently generalised in various ways, but real-world nonsupersymmetric black holes have stayed out of reach, so far.

Gauge theory ↔ Open strings

String theory and brane configurations provide a very effective setting for space-time interpretations of classical solutions to the Yang–Mills equations. Quite surprisingly, a class of magnetic monopoles in 4d YM theory allows for an interpretation as D1-branes ending on a D3-brane. This may be seen as to add to the high degree of consistency of the picture.

Also, besides ordinary gauge theories, the noncommutative ones also find a natural home in open string theories. The noncommutativity parameter in the gauge theory is in fact argued to be related to the closed string Kalb-Ramond field. This remarkable observation has revived the interest of string theorists in noncommutative gauge theories [12].

2.1.5 Boundary states

Now how do boundary states fit in? Shortly stated, boundary states have two faces: they are sources for closed strings that simultaneously encode information about open string spectra. If the open string *dynamics* is the issue, one has to invoke techniques beyond these states. As a result, they will apply in two corners of Fig. 2.1, and will be of no obvious use in the third one ('Gauge theories').

(a) SUGRA solutions
As demonstrated in Ref. [13] one possible view on boundary states is to see them as classical sources for the bulk fields. The physical picture is that of a corresponding D-brane emitting massless closed strings. In turn, the latter generate the associated closed string background, i.e., the solution to the field equations. This viewpoint makes it clear that

only those fields that couple to the D-brane stand a chance of being nontrivially excited in the solution.

(b) Boundary conditions
'New' types of D-branes show up in nongeometric CFT phases of string theory. With an exact closed string CFT at hand, there exists a standard procedure to construct consistent boundary states. This recipe will be reviewed in Section 4.2. In view of the lack of a spacetime interpretation, boundary conditions, and accordingly, boundary states, constitute the easiest defining principle of D-branes in such phases. Since they are designed so as to encode the open strings associated to such D-branes, the boundary states are in fact the only known available tool to find out about the gauge theory spectrum [14].

2.2 Elements of CFT : (not) a primer

Quantum theory is the physicist's way to deal with the mathematical concept of Hilbert spaces (of states), particularly so for quantum field theories (QFTs). Moreover, the introduction of correlation functions assigns a certain reality to the Hilbert space, at least in the physicisist's mind. To 'solve a QFT' is commonly understood as to find its spectrum and the exact correlation functions. For generic QFTs, including those that are considered to be realistic, this program is presently beyond reach: we have to do with perturbation theory, supplemented with nonperturbative insights in fortunate cases.

An operator-state correspondence, i.e., a set of maps

$$\begin{array}{ccc} \text{Hilbert space } \mathcal{H} & \leftrightarrow & \text{Algebra } \mathcal{B} \\ \text{state } |\psi\rangle & & \text{(vertex) operator } \mathcal{O}_\psi \end{array}$$

allows one to trade operators for states and vice-versa, depending on needs. This means in particular that properties of states ought to have a counterpart in operator terms and the other way around. For example: gauge-invariant states in BRST-quantised theories are created by operators that (anti-)commute with the BRST-operator. In QFTs, subsets of single operators get organised into quantum fields, resulting in an increased degree of transparancy. Loosely speaking[4], quantum fields are $\mathcal{B}$-valued functions on the relevant space(-time).

Symmetry generators form an interesting subclass of the algebra of operators. They close into a subalgebra, henceforth referred to as the symmetry

[4] A more precise statement is that quantum fields are sections of a sheaf of operator algebras. If anything, this can hardly be argued to shed any new light on the present discussion.

algebra, $\mathcal{A}$. Further, it is a fact that the Hilbert space should furnish a representation of $\mathcal{A}$.

In the remainder of this section, we shall review some consequences of these ideas in the context of two-dimensional conformal field theories. Since this is not meant to be a primer, the interested reader is referred to the excellent textbook, Ref. [15].

2.2.1 Algebraic structures

Hilbert space and chiral algebra

Consider chiral CFT on the complex plane $\mathbb{C}$ first. This comes with an algebra $\mathcal{B}$ of operators that have no dependence on $\bar{z}$, the anti-holomorphic coordinate in a patch around the origin of $\mathbb{C}$. Of particular importance is the holomorphic stress-tensor, $T(z)$. It is any operator that has an OPE with itself of the form:

$$T(z)T(w) \sim \frac{c/2}{(z-w)^4} + \frac{2T(w)}{(z-w)^2} + \frac{\partial T(w)}{(z-w)}\,, \tag{2.2.1}$$

where c is a c-number, the central charge. In terms of the modes L_n of T, Eq. (2.2.1) says that they obey a Virasoro algebra with central charge c.

The algebra $\mathcal{B}$ of operators in the CFT decomposes into (Virasoro) quasi-primary, primary and descendant fields (operators). These notions attributed to fields refer to properties of the latter under OPE product with T. Primary operators are characterised as those that obey

$$T(z)\phi(w) \sim \frac{h\phi(w)}{(z-w)} + \frac{\partial\phi(w)}{(z-w)} \tag{2.2.2}$$

where the c-number h is called the conformal dimension of ϕ. In the operator-state correspondence, such operators create highest-weight states of possibly reducible Virasoro modules. Among primary operators, the unit operator $\phi_0 = \mathbb{1}$, and its corresponding module[5] $[\phi_0]$, the so-called vacuum-module, are singled out: the *chiral (symmetry) algebra* $\mathcal{A}$ is defined through operator-state mapping as that particular set of operators in $\mathcal{B}$ corresponding to the states in $[\phi_0]$. Besides containing T, $[\phi_0]$ enjoys the following properties:

(a) (locality) All operators in $\mathcal{A}$ have integral conformal dimensions;

(b) $\mathcal{A}[\phi] \subset [\phi]$ for any $\mathcal{A}$-module $[\phi]$ by definition, implying tacitly that $\mathcal{A}$ should be local wrt. operators in $[\phi]$.

[5] Modules will conventionally be labelled by their primaries here and below.

Since $T(z)$ is part of the symmetry algebra of a chiral CFT, the chiral Hilbert space decomposes into Virasoro modules, i.e., formally one may write

$$\mathcal{H}^{chiral} = \bigoplus_{\alpha} n_\alpha [\phi_\alpha] \ , \tag{2.2.3}$$

where n_α denote incidental multiplicities and the sum runs over primaries.

Of all possible correlators in a (chiral) CFT, a special rôle is played by the three-point chiral blocks (3-point functions). They encode the structure of the algebra, that is, the OPEs that turn $\mathcal{A}$ into an associative operator algebra.

A less refined but useful concept arises from taking operators $\phi_{(\alpha)}$ inside modules $[\phi_\alpha]$, and considering 3-point blocks

$$\langle \phi_{(\alpha)} \phi_{(\beta)} \phi_{(\gamma)} \rangle \ . \tag{2.2.4}$$

For fixed α, β, γ, let $N_{\alpha\beta\gamma}$ count the number of independent sets of such operators yielding a nonvanishing result. From this data an algebra of modules is defined by

$$[\phi_\alpha] \times [\phi_\beta] = N_{\alpha\beta}{}^{\gamma} [\phi_\gamma] \ . \tag{2.2.5}$$

Eq. (2.2.5) defines the so-called *fusion algebra*. From its very definition this algebra comes with the following properties:

(a) commutativity : $N_\alpha N_\beta = N_\beta N_\alpha$, i.e., the $\{N_\alpha\}$ are normal and may be diagonalised simultaneously.

(b) associativity : $N_\alpha N_\beta = N_{\alpha\beta}{}^{\gamma} N_\gamma$, i.e., the $\{N_\alpha\}$ furnish a representation of the fusion algebra Eq. (2.2.4).

The matrices used here are defined as $(N_\alpha)_\beta{}^\gamma := N_{\alpha\beta}{}^\gamma$.

In the present text, the mild assumption will be made that the relevant CFTs have a sufficiently nice split into chiral and anti-chiral parts:

(a) The full CFT Hilbert space has a decomposition

$$\mathcal{H} = \bigoplus_{\alpha,\beta} n_{\alpha\beta} [\phi_\alpha] \otimes [\tilde{\phi}_\beta] \ , \tag{2.2.6}$$

with $n_{\alpha\beta} \in \mathbb{N}_0$ denoting incidental multiplicities.

(b) Correlation functions (n-point conformal blocks) are *finite* sums of chiral-anti-chiral conformal blocks.

Modular invariance and partition functions

Hitherto, the CFT has been assumed to live on the complex plane. One is then lead to ask the question

"What can one learn from the corresponding CFT on the torus ? "

As it turns out, the nontrivial input here resides in one-loop modular invariance. Recall that the modular group $PSL(2,\mathbb{Z})$ consists of reparametrisations that are globally defined on the torus. In that respect, it is the counterpart of $SL(2,\mathbb{C})$ on the plane. Since the physics can only be ignorant about any choice of coordinates, no outcome of any meaningful computation must depend on this choice. Particularly so, the torus vacuum amplitude had better be modular invariant.

In the operator formalism, the torus amplitude is expressed as a trace over the Hilbert space. As such it is equally referred to as the closed string partition function. The idea here is that the chiral (anti-chiral) states are propagated over a Schwinger time τ $(\bar{\tau})$ by resp. Hamiltonians L_0 $(\bar{L}_0)$ before being glued so as to produce the trace. According to the closed-string Hilbert space decomposition Eq. (2.2.6) one finds a partition function

$$\mathcal{Z}_{1-loop} = \sum_{\alpha\beta} \chi^{\alpha}(q)\ n_{\alpha\beta}\ \tilde{\chi}^{\beta}(\bar{q})\ . \qquad (2.2.7)$$

The chiral Virasoro characters entering this expression are defined as

$$\chi_{\alpha} \equiv \mathrm{T}r_{[\phi_{\alpha}]}(q^{L_0-\frac{c}{24}})\ , \qquad (2.2.8)$$

where $q = \mathrm{e}^{\mathrm{i}\pi\tau}$ and τ is the torus modular parameter.

The modular $PSL(2,\mathbb{Z})$ group has a concise presentation in terms of generators and relations : $\langle S, T | S^2 = \mathbb{1}, (ST)^3 = \mathbb{1}\rangle$. The geometric action of S, T on the torus induces a representation on the Virasoro characters :

$$\chi_{\alpha}(-\frac{1}{\tau}) = \sum_{\beta} S_{\alpha}{}^{\beta}\chi_{\beta}(\tau)\ ; \qquad (2.2.9)$$

$$\chi_{\alpha}(-\tau+1) = \sum_{\beta} \mathcal{T}_{\alpha}{}^{\beta}\chi_{\beta}(\tau)\ ; \qquad (2.2.10)$$

Actually, the relevant relations here are $\langle S, \mathcal{T} | S^2 = C, (S\mathcal{T})^3 = \mathbb{1}\rangle$ where C is charge conjugation, such that the space of characters carries only a *projective* representation of $PSL(2,\mathbb{Z})$, generically.

From this modular action on the chiral and anti-chiral building blocks of $\mathcal{Z}_{1-loop}$, modular invariance is directly observed to impose a strong constraint on the operator content of the theory. More precisely, it restricts the allowed chiral-anti-chiral gluings in Eq. (2.2.6). This furnishes a precise answer to the question raised at the beginning of the present section.

As remarked earlier, the normal property of the fusion matrices N_α makes them simultaneously diagonalisable. The highly nontrivial fact is that the modular matrix S is the object actually doing the job:

$$N_{\alpha\beta}{}^{\gamma} = \sum_{\delta} S_{\alpha}{}^{\delta} \left(\frac{S_{\beta}{}^{\delta}}{S_{0}{}^{\delta}} \right) (S^{-1})_{\delta}{}^{\gamma} \ . \tag{2.2.11}$$

Eq. (2.2.11) is known as the *Verlinde formula*, and has been demonstrated to hold for rational theories [16]. In Section 3.2.1 a similar although modified expression will be found to exist for orbifold theories.

2.2.2 Moduli spaces

Conformal field theories with central charges $c \geq 1$ tend to come in continuous families : rather than being isolated, they allow for deformations to 'nearby' CFTs. In such cases one says that there is a moduli space of conformal field theories. Let me review an easy example here, that of the free compact boson ($c = 1$) [17]. However simple, the corresponding moduli space will already illustrate generic features of CFT moduli spaces.

In the CFT of a free boson compactified on a circle, the circle radius R is an obvious free parameter, as can be seen from the lagrangian

$$\mathcal{L} = -\frac{R^2}{2\alpha'}(\partial\phi)^2 \ . \tag{2.2.12}$$

The moduli space $\mathcal{M}_{c=1}$ of inequivalent CFTs thus appears to be the real line. However, this conclusion is too quick in two ways. Firstly, there is an involution $\phi \to -\phi$ which is actually a symmetry; it leaves the Virasoro algebra, in particular its central charge $c = 1$ untouched. There exist therefore two lines of CFTs: the original circle theories and the same theories orbifolded by the $\mathbb{Z}_2$ symmetry. They are parametrised by real moduli R_{circ}, (R_{orb}), respectively. Secondly, T-duality $R \leftrightarrow \alpha'/R$ relates isomorphic theories. As a result there are only two half-lines of inequivalent theories; what's more, a careful analysis reveals that these branches actually touch as depicted in Fig. 2.2. A set of three isolated CFTs then conjecturally completes the full $c = 1$ moduli space [17].

From the two-dimensional point of view, perturbations correspond to marginal operators, which have conformal weights $(h, \tilde{h}) = (1, 1)$ by definition. In the superstring theory, those will be interpreted as massless scalars.

Which features should catch your eye here ?

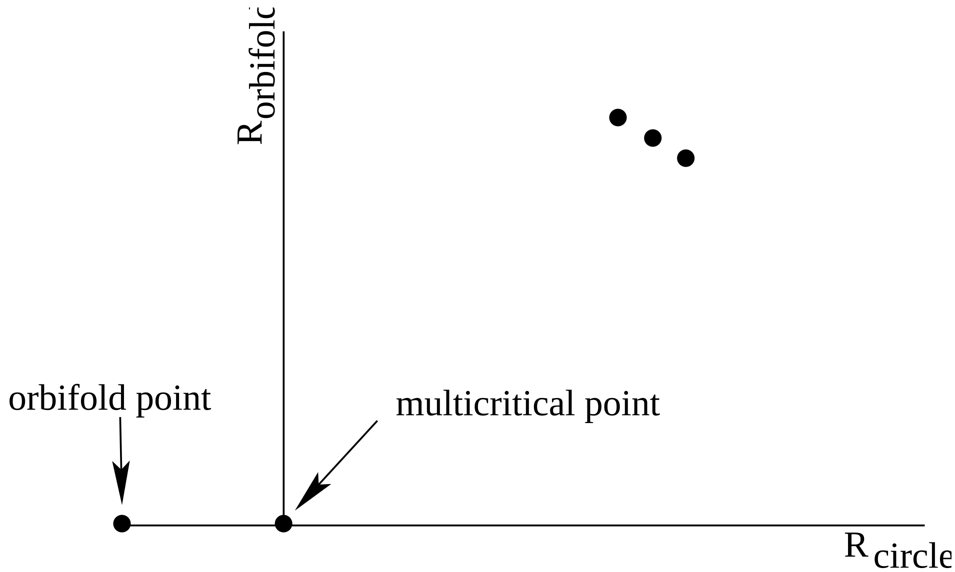


Figure 2.2: The moduli space of $c = 1$ conformal field theories consists of two branches touching at a multicritical point.

(a) The moduli space looks like a *manifold* at generic points, meaning that the number of exactly marginal operators is locally constant.

(b) There are *multicritical points* where new branches develop, due to the appearance of new exactly marginal operators in the CFT. In the free boson case, this is the point where the orbifold and circle branches join.

(c) There are *orbifold singularities*. This happens whenever there is an "enhanced" symmetry that relates otherwise inequivalent deformations. As an example of this phenomenon, you can think of the end-point corresponding to the critical radius in the $c = 1$ series where a δR deformation is equivalent to a $-\delta R$ one.

Evidently, free bosons only comprise a tiny subclass of all possible conformal field theories. The seemingly interesting question

"What is the moduli space of all CFTs ?"

is presently too ambitious to allow for an answer, unfortunately. Slightly lowering our ambitions, we should rather hope to find a satisfactory description of subsets at most of the total CFT moduli space. Therefore, a more modest issue concerns finding appropriate choices of discrete data shared by a given family of CFTs, such that the family becomes tractable while retaining interest beyond triviality. Given that a (S)CFT is specified by 1) its central charge, 2) the spectrum of primaries $\{(h, \tilde{h})\}$, possibly with multiplicities, 3) the OPE's and 4) the amount of 2d supersymmetry, it is reasonable to stratify the complete moduli space $\mathcal{M}$ by the first two pieces of data, while the remaining two are allowed to vary over the stratum. As an example, one may wish to study the moduli space of $c = 6, N = 4$ SCFTs (see e.g., Ref. [18]).

Some caution is appropriate here. Of all marginal operators, it is only the *exactly marginal* ones that effectively give rise to nearby conformal field theories. In other words, marginal but not exactly marginal deformations spoil conformal invariance. A necessary condition for exact marginality of $\mathcal{O}_i$ is that

$$\langle \mathcal{O}_i \mathcal{O}_i \mathcal{O}_j \rangle = 0, \tag{2.2.13}$$

for all marginal $\mathcal{O}_j$. Infinitesimal deformations by such $\mathcal{O}_i$ yield the tangent space to the moduli space in the point corresponding to the undeformed theory. In this light, the condition Eq. (2.2.13) must be viewed as an integrability condition. If it is satisfied, there is no obstruction for these first-order deformations to be extended to second order. It is not clear, however, if the condition guarantees the absence of any higher order obstructions. In the remainder of this text, though, I will adopt the point of view of a 'working physicist', and not bother about this subtle point. In what follows, I will be concerned mostly with the dimension of the (stratum of the) moduli space only, which is locally given by the number of unobstructed first-order deformations.

2.2.3 Worldsheet and spacetime SUSY

In this section we wish to expose some tight connections between worldsheet and target space phenomena, namely issues involving supersymmetry. There are three points to make:

(a) Which space-time geometries allow worldsheet supersymmetric versions of bosonic nonlinear sigma-models?

(b) Does worldsheet SUSY yield space-time supersymmetry ?

(c) In a string compactification on a manifold $\mathcal{M}$, say, does the requirement of (extended) space-time supersymmetry constrain the possible classes of worldsheet theories ?

In some respect, the questions (b) and (c) can be thought of as each other's converses. I will provide you with an outline of an answer to the first two questions, postponing a discussion of the (c) to Section 3.1.

A. Supersymmetric σ-models

Before answering the first question, recall the concept of nonlinear bosonic σ-models. They involve maps $\phi : \Sigma \to X$, describing embeddings of the two-dimensional worldsheet Σ into a target manifold X. At the classical level, the maps should be such that they extremise the action

$$S = \int_{\Sigma} ||\partial\phi||^2 + \phi^* B \ , \tag{2.2.14}$$

which is the familiar Polyakov action for bosonic strings; the second term is the coupling of the string to $B \in \Omega^2(X)$, the Kalb-Ramond two-form on X. The natural question that comes up here: can this model be made supersymmetric, and if not generically so, which are the conditions on X in order that the programme should work? This issue was dealt with a long time ago [19]. Therefore, we only give a summary of a well-known answer. First note that the supersymmetry will always be assumed to be rigid, that is, its parameters are constant on Σ. The amount of supersymmetry, denoted here as N, is shorthand for the nonchiral (N, N), where N counts the number of conserved real supercharges. Also, the B-field is set to zero. Since our interest will primarily be in nonchiral supersymmetric models, I shall further restrict to those only.

A beautiful analysis then reveals that minimal worldsheet supersymmetry, $N = 1$, imposes no constraints on the Riemannian manifold $\mathcal{M}$. Minimally extended supersymmetry, $N = 2$, on the other hand, requires that the manifold be complex Kähler, while to realise $N = 4$, $\mathcal{M}$ must be hyperkähler. The latter two cases trivially impose a condition on the dimension of the target manifold, which has to be a multiple of two ($N = 2$) or four ($N = 4$). Prototypes of such spaces are $\mathbb{P}^N$ (Kähler) and the Eguchi-Hanson gravitational instanton (hyperkähler). The main lesson to be drawn from this: every extension requires the existence of an independent complex structure on X (see Section 3.1.1 for an elementary review of complex structures.) Since closure of the algebra of two complex structures automatically generates a third one: $N = 3 \to N = 4$.

So far for the supersymmetry. What about conformal invariance? In the absence of torsion (B-field), absence of the conformal anomaly in the two-dimensional quantum theory requires that the target space metric be Ricci-flat, to leading order in α'. All this is summarised as follows, for compact manifolds:

SCA	↔	$\mathcal{M}$
N=1		Riemannian
N=2		Calabi-Yau
N=4		Hyperkähler

In fact, there is an intimate connection between geometrical objects, complex structures, on the one hand, and enhanced (super)symmetry currents in the superconformal algebra (SCA) on the other. This correspondence will be further explored when discussing Riemannian holonomy groups in Section 3.1.1 (see also Section 4.1 for an explicit example).

B. From the worldsheet to space-time

Let us next turn to space-time supersymmetry. Careful analysis will reveal that in order that there be space-time supersymmetry, one has to start with a SCA that is an extension of the $N = 1$ SCA. The main ingredients are GSO-projection and the existence of spectral flows. The absence of the latter in purely $N = 1$ settings is responsible for the absence of space-time supersymmetry there.

For the reader's convenience, we recall some facts about $N = 2 \subset N = 4$ SCAs and their representations. As is well-known, the $N = 2$ SCA contains a $U(1)$ current J and supercurrents $G^{\pm}$ charged positively (negatively) under J. Upon introduction of an auxiliary boson ϕ, such that $J = i\sqrt{c/3}\partial\phi$ one constructs a spectral flow operator $\mathcal{U}_1^{int} = e^{q_{\max}\sqrt{3/c}\phi}$, in a CFT with central charge c. When realised on the Hilbert space by some unitary operator, the flow takes NS → NS, R → R sectorwise. This is therefore called unit spectral flow. A half-spectral flow operator $\mathcal{U}_{1/2}^{int}$ is defined as a square-root of $\mathcal{U}_1^{int}$, i.e., $q_{\max} \to q_{\max}/2$. The newly obtained operator obviously takes NS ↔ R. In fact, the more precise statement here is that in abstract CFTs the free theory notions of 'NS' and 'R' (as resulting from boundary conditions) are replaced by the respective periodicities of $G^{\pm}$. Notice further that the conformal weight of $\mathcal{U}_{1/2}^{int}$ equals $h = \frac{d}{8}$.

A consistent combined space-time and internal CFT has respective central charges $c_{st} = 15/2 - 3d, c_{int} = 3d$ where d is the abstract CFT equivalent of the complex dimension of an internal geometric manifold. With these conventions, $q_{\max} = d$, above. The total half-spectral flow is the operator

$$\mathcal{U}_{1/2} = e^{-\frac{\varphi}{2}} S \ \mathcal{U}_{1/2}^{int} \ , \tag{2.2.15}$$

constructed from the bosonised superghost, free-fermion space-time spin-field, and internal parts. In particular, the NS massless ('ground') states, corresponding to the operators

$$\mathcal{O}_0 = e^{-\varphi} \psi \ \mathbb{1}^{int} \tag{2.2.16}$$

yield upon OPE with $\mathcal{U}_{1/2}$ Ramond states. The latter are easily verified to be massless from the conformal weights. This should not come as a surprise : after all, half spectral flows correspond to space-time supersymmetry generators. In the left-right combined (closed string) sector, massless NS-NS states are mapped to massless R-NS states, i.e., gravitini states, by left half-spectral flow. Upon further left flow these are taken back to the NS-NS sector.

In generic cases, there will be more than one independent spectral flow (with corresponding half flow) in the internal CFT. That is, not all spectral flows arise from the specific $N = 2$ ϕ boson chosen. On the other hand, the SCAs of Spin(7) and G_2 compactifications have been demonstrated to be particular extensions of the $N = 1$ SCA by an Ising, and tricritical Ising model, respectively. Both extensions were further shown to possess a corresponding Ising (tricritical Ising) spectral flow operator. In all, we have thus found the following:

FACT 2.1

> *Type II symmetric compactifications with an internal SCA that is at least $N = 1$ extended by n spectral flows, yields $\mathcal{N} = n$ space-time supersymmetry.*

As such it is a plain observation that (half) spectral flows are the abstract CFT counterparts of parallel spinors in geometric compactifications, (see Section 3.1.1).

2.2.4 Example 2.1 : Free fermions and $SO(n)_1$ current algebra

A nice example to illustrate part of the ideas is provided by a system of $2n$ free Majorana-Weyl fermions ψ^i $(i = 1, \dots 2n)$. The energy-momentum tensor is $T = \sum_i \psi^i \partial \psi^i$ and as such the central charge equals $c = n$. One may now build fermion bilinears $J^{ij} = \psi^i \psi^j$; they are verified to realise an affine $SO(2n)$ algebra at level $k = 1$:

$$J^{ij}(z)J^{kl}(w) \quad \sim \quad \frac{1}{(z-w)^2} + f^{ij,kl}{}_{mn}\frac{J^{mn}}{z-w} \,, \qquad (2.2.17)$$

where $f^{ij,kl}{}_{mn}$ are the $SO(8)$ Lie algebra structure constants. In short, the algebra of operators $\mathcal{B}$ is generated by $\mathbb{1}$ and $\{\partial^k \psi^i\}_{k\geq 0}$. The *chiral algebra* is the subalgebra $\mathcal{A} = \langle \mathbb{1}, \partial^k J^{ij} \rangle_{k\geq 0}$.

The irreducible $\mathcal{A}$-modules are obtained with equal ease: they are four in number, to know $[\mathbb{1}], [\psi^i], [S], [C]$ where the primary (ground state) labels the module. In more standard notation, they are also denoted as (o, v, s, c)

or $(NS-, NS+, R-, R+)$, respectively. Note that they are in one-to-one correspondence with the SO(8) Lie-algebra conjugacy classes, as a consequence of their ground states building the corresponding highest-weight representation of the J_0^{ij}.

Finding out about the modular representations on the affine characters χ_a $(a = o, v, s, c)$ precedes finding out about modular invariants. From standard manipulations, it is found that,

$$\begin{aligned}\mathcal{T}_{(2n)} &= \mathrm{diag}\,(-\mathrm{e}^{-\frac{n\pi\mathrm{i}}{12}}, \mathrm{e}^{-\frac{n\pi\mathrm{i}}{12}}, \mathrm{e}^{\frac{n\pi\mathrm{i}}{6}}, \mathrm{e}^{\frac{n\pi\mathrm{i}}{6}})\ , \\ S_{(2n)} &= \frac{1}{2}\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & (-\mathrm{i})^n & -(-\mathrm{i})^n \\ 1 & -1 & -(-\mathrm{i})^n & (-\mathrm{i})^n \end{pmatrix}\ .\end{aligned} \tag{2.2.18}$$

For simplicity, let us restrict attention to the case $n = 4$, that is, $\mathrm{SO}(8)_1$. The significance of the latter in superstring theory stems from strings moving in a flat Minkowski background, where the ψ^i are actually the physical light-cone worldsheet fermion degrees of freedom. In fact, it is only a small step to the covariant SO(9,1) formalism. In the latter, taking the superghosts β, γ from the worldsheet superdiffeomorphisms into account, the combined $\mathrm{SO}(9,1)_1 \times C^{\beta\gamma}$ characters coincide with their $\mathrm{SO}(8)_1$ counterparts. This should not come as a surprise, since the superghosts thus effectively cancel the light-cone fermion (unphysical) contributions. However, there is a subtlety here : the rôles played by the $\mathrm{SO}(8)_1$ character o (v) are taken over by the covariant v (o), respectively. Also, the spinorial s, c of SO(8) acquire a minus sign in the SO(9,1) case.

It is a well-established fact that the $\mathrm{SO}(9,1)_1 \times C^{\beta\gamma}$[6] ("superstring") theory allows four inequivalent torus modular invariants, to know IIB/IIA,0B,0A theories, with respective one-loop partition functions:

$$\mathcal{Z}_{IIB} = |\chi_v - \chi_s|^2\ , \tag{2.2.19}$$
$$\mathcal{Z}_{IIA} = (\chi_v - \chi_s)(\bar{\chi}_v - \bar{\chi}_c)\ , \tag{2.2.20}$$
$$\mathcal{Z}_{0B} = |\chi_0|^2 + |\chi_v|^2 + |\chi_s|^2 + |\chi_c|^2\ , \tag{2.2.21}$$
$$\mathcal{Z}_{0A} = |\chi_0|^2 + |\chi_v|^2 + \chi_s\,\bar{\chi}_c + \chi_c\,\bar{\chi}_s\ . \tag{2.2.22}$$

From Verlinde's formula Eq. (2.2.11) and the explicit S modular matrix, it only takes a few lines to find [16]:

$$N_{ab}{}^c = \sum_m \frac{(S_{(8)})_a^m\,(S_{(8)})_b^m\,((S_{(8)})_c^m)^*}{(S_{(8)})_v^m} \tag{2.2.23}$$

[6] Only the corresponding worldsheet fermions are being considered, here. As such, heterotic string considerations do not enter the discussion at any point.

$$= \quad 2 \sum_m (S_{(8)})_a^m \, (S_{(8)})_b^m \, (S_{(8)})_c^m \quad .$$

With this expression, the fusion rules can be verified to be given by

$$\begin{array}{lll} [v] \times [a] = [a] \; , & [a] \times [\alpha] = [v] \; , & [o] \times [s] = [c] \; , \\ [o] \times [c] = [s] \; , & [s] \times [c] = [o] \; , & \end{array}$$

for $a = o, v, s, c$ as before. That is, we have $n^i_{v\,a} = \delta^i_a$, $N_{aa}{}^i = \delta^i_v$, $N_{oc}{}^j = \delta^i_s$, and so on. These fusion rules coincide with the algebra of the conjugacy classes o, v, s, c of SO(8) representations where o and v have been exchanged according to the discussion of the previous paragraph (see e.g., Refs. [20, 21, 22]).

3

Landscapes

Geometrical aspects of strings and D-branes constitute the central theme in the present chapter. Even though most of the material has been known for quite some time, facts and fiction seem to be spread throughout the existing literature. Here, we aim at providing a useful overview/collection of the state of affairs. Besides this, a number of examples are worked out explicitly, which should help the reader to digest the presented material.

String theory possesses three potential probes of geometries: point particles, closed strings and D-branes. In turn, these are naturally linked with classical geometry, conformal field theories and gauge theories, respectively. This chapter is organised accordingly, and treats each of the mentioned topics more or less separately.

In Section 3.1, aspects of classical geometry are reviewed. After a general discussion of structures and moduli spaces, issues regarding orbifold spaces receive particular emphasis, e.g. desingularisation. In addition, a digression on special Riemannian holonomy and how this is reflected in superconformal algebras (SCAs) then sets the stage for Section 3.2.

In that section, a particular desingularisation technique, namely, blow-up, is reviewed from the CFT viewpoint. Orbifold CFT will be argued to organise the classical blow-up. Also given is a preliminary explanation why the CFT remains well-behaved in the orbifold limit, where point particle theory does not. As a final topic, abelian Calabi-Yau threefold singularities are approached via toric geometry. Armed with this tool, we can visualise a remarkable correspondence between CFT and geometry in a number of explicit examples. At the same time, the latter must compensate the brevity of the general exposition on toric methods.

Finally, Section 3.3 gives an elementary discussion of D-branes as probes of orbifold geometry. Certain classes of gauge theories will be observed to reproduce the target space geometry via their associated moduli spaces (of vacua), a remarkable feature at first sight. Further, precisely these gauge theories will also figure in Chapter 5, where they will be seen to provide a physical realisation of a mathematical construct, known as McKay correspondence.

3.1 From afar – The classical geometry

3.1.1 Elements of differential and algebraic geometry

A. Hierarchy of structures

Below, we give a quick account of some 'structures' on manifolds. The choice is motivated by further developments in the present text, rather than personal preference. A manifold $\mathcal{M}$ will be understood to be in the C^∞-category, i.e., there is an underlying topological (Haussdorff) space admitting a covering with C^∞ transition functions. $\mathcal{M}$ is a *Riemannian* manifold if endowed with a metric tensor g; we reserve the obvious notation $(\mathcal{M}, g)$ for such manifolds. With or without a metric specified, a manifold can be further decorated with structures, most conveniently encoded in associated tensorial quantities defined on $\mathcal{M}$.

Complex structures

Of the various equivalent notions of complex structure, the following one is found most convenient. Let I be a section of $T^*\mathcal{M} \otimes T\mathcal{M}$, for a given manifold $\mathcal{M}$; then I is a complex structure on $\mathcal{M}$, provided that

$$I^2 = -\mathbf{1} \,, \tag{3.1.1}$$

$$N_I(-,-) = \mathbf{0} \,, \tag{3.1.2}$$

where the Nijenhuis tensor N_I is defined in terms of the Lie-bracket $[-,-]$ of vector-fields:

$$N_I(v,w) \equiv [v,w] + I([Iv,w] + [v,Iw]) - [Iv,Iw] \,. \tag{3.1.3}$$

Plainly, at each tangent space $T_p\mathcal{M}$, I restricts to an endomorphism I_p, that squares to minus the identity, and satisfies some integrability condition. As such, I_p introduces the notion of 'i' locally on $T_p\mathcal{M}$, endowing it with a complex structure. A manifold with a complex structure is called a complex manifold, and is denoted as $(\mathcal{M}, I)$. As a real manifold, clearly $\dim \mathcal{M}$ must be even. The prototype of a complex manifold is $\mathbb{P}^N$, the complex projective N-plane.

Complex-symplectic structures

A complex-symplectic structure $(I, \omega_{\mathbb{C}})$ consists of a complex structure and a closed $(2,0)$-form $\omega_{\mathbb{C}}$, such that $\omega_{\mathbb{C}}^m$ is a non-vanishing $(2m,0)$-form. A complex $2m$-dimensional manifold $\mathcal{M}$ endowed with this structure is a complex symplectic manifold.

Observe that none of the structures introduced so far refers to any metric. The existence of a complex structure I is sufficient to fix the notion of (p,q)-forms: I defines a split of the complexified cotangent bundle into holomorphic and anti-holomorphic subbundles:

$$T^*_{\mathbb{C}}\mathcal{M} := T^*\mathcal{M} \otimes_{\mathbb{R}} \mathbb{C} = T^*_h\mathcal{M} \oplus \bar{T}^*_h\mathcal{M}. \tag{3.1.4}$$

A (p,q)-form is an element of $\Gamma(X, \Lambda^p T^*_h\mathcal{M} \otimes \Lambda^q \bar{T}^*_h\mathcal{M})$. The notation employed here, is standard: $\Gamma(X,E)$ is the space of sections of a vector bundle $E \overset{\pi}{\to} X$. No reference to any metrics being made here, complex and complex-symplectic structures are suitable concepts in algebraic geometry.

Next introduce metrics g. Here and below, ∇ will be the torsion-free connection derived from g.

Kähler structures

A metric g on a complex manifold $(\mathcal{M}, I)$ is hermitian provided $g(I-, I-) = g(-,-)$; the latter property expresses the compatibility between the complex and metric structures. For an hermitian metric to be Kähler, any of the following equivalent conditions must hold:

$$\nabla I = 0\,; \tag{3.1.5}$$

$$\nabla K = 0\,; \tag{3.1.6}$$

$$\mathrm{d}K = 0\,, \tag{3.1.7}$$

where in the latter two equations $K \in \Gamma(X, \Lambda^2 T^*_{\mathcal{M}})$ is the Kähler form associated to I : $K(-,-) := g(-, I-)$. A nice property of Kähler manifolds resides in their harmonic analysis. From the observation that $\Delta_{\mathrm{d}} = 2\Delta_{\bar{\partial}}$[1], it follows

[1] As well-known, $\Delta_{\mathrm{d}} \equiv \mathrm{d}^\dagger \mathrm{d} + \mathrm{d}\,\mathrm{d}^\dagger$, and likewise for $\Delta_{\bar{\partial}}$. The adjoint is defined w.r.t. the metric g.

that d-harmonic forms are $\bar{\partial}$-harmonic and vice-versa. As a consequence, the de Rham (d-) cohomology has a Hodge decomposition (into Dolbeault ($\bar{\partial}$-) cohomology):

$$H^r_{\mathrm{d}}(\mathcal{M}, \mathbb{C}) = \oplus_{p+q=r} H^{p,q}_{\bar{\partial}}(\mathcal{M}) \ . \tag{3.1.8}$$

The betti numbers can be refined likewise : $b_r = \sum_p h^{p,r-p}$, and the hodge numbers can be arranged into a diamond (e.g. for a 3-dimensional manifold):

$$\begin{array}{ccccccc} & & & h^{0,0} & & & \\ & & h^{1,0} & & h^{0,1} & & \\ & h^{2,0} & & h^{1,1} & & h^{0,2} & \\ h^{3,0} & & h^{2,1} & & h^{1,2} & & h^{0,3} \\ & h^{3,1} & & h^{2,2} & & h^{1,3} & \\ & & h^{3,2} & & h^{2,3} & & \\ & & & h^{3,3} & & & \end{array} \tag{3.1.9}$$

Hyperkähler structures

A hyperkähler structure (I_a, g) consists of a triplet of complex structures I_a that close into a quaternion algebra,

$$I_a\ I_b = -\delta_{ab} + \epsilon_{abc} I_c \ ; \tag{3.1.10}$$

furthermore, g must be Kähler wrt. each of the I_a. Notice that each I_a gives rise to a corresponding Kähler form K_a, by lowering an index with the metric. Examples here are the K3 surfaces.

B. Manifold moduli spaces

A superior insight in manifolds with structures is often gained from consideration of continuous families $\mathcal{M}_t$ of those in one go, rather than the study of isolated examples. Meanwhile, this approach partially solves manifold classification problems.

For example, Riemann surface theory is probably singled out as the case where this program has proven extremely successful. Topologically speaking, compact Riemann surfaces without boundaries are completely specified by an integer, their genus g. Endowing them with complex structures, however, makes a finer distinction possible. It is a well-known fact that the space of compact genus $g \geq 2$ inequivalent complex manifolds is parametrised by $3g-3$ complex numbers [2]. The latter are viewed as local coordinates (moduli) of the associated moduli space $\mathcal{M}_g$, which thus has complex dimension $3g-3$.

[2] For $g = 1$, it is a single complex number.

Moreover, beyond this local exploration, a global description of $\mathcal{M}_g$ is known as an orbifold by the discrete mapping class group (see e.g., Ref. [23]).

For higher-dimensional manifolds, the situation is generically less favourable. Given some manifold $\mathcal{M}$ with structure, mathematicians would ideally want to find the moduli space of such structures on $\mathcal{M}$. Contrary to naive expectations, perhaps, this program tends to fail if the structure is too rich. For example, the moduli spaces of hyperkähler structures are not under complete control (yet). If necessary, therefore, part of the structure is typically disregarded. In the hyperkähler example, the hypercomplex structure is suppressed, and the manifolds can be viewed as complex manifolds only. Even though the question may now find an answer, the latter potentially fails to match the objective: e.g., with the Kähler class fixed over the family, some deformations of the complex structure potentially destroy the Kähler property initially present. From this example, it must become clear that tuning the moduli problem, i.e., formulating a problem that possesses a right balance between tractability and relevance (genericness), is prior to any satisfactory solution.

It is hoped that these have given the reader a flavour of the vast and difficult subject. Below, some general aspects concerning two particular instances, the moduli spaces of Kähler and complex structure deformations are briefly reviewed; so far, in string theory they have played the dominant rôles.

Kähler cone

What does it mean to consider all possible Kähler structures on a manifold with a given complex structure I? Stated otherwise, how can one parametrise the set of viable metrics that are Kähler wrt. I ? The answer is most naturally phrased in terms of the Kähler forms. Combine the following two ingredients:

(a) THEOREM 3.1 (WIRTINGER) *On an m-dimensional Kähler manifold $\mathcal{M}$ with Kähler form K, the volume $Vol(\mathcal{M}) = \frac{1}{m!}\int_{\mathcal{M}} K^m$ is measured by the appropriate wedge power of K.*

(b) Any n-dimensional complex submanifold of a Kähler manifold is Kähler in its own right. From the closedness of $\wedge^n K$, it follows that volumes are topological invariants of the submanifolds: they depend only on the homology class inside $H_{2n}(\mathcal{M})$. Of all real manifolds in a given such class, the holomorphic submanifolds are the volume-minimising ones.

Since volumes are real, positive quantities, we must have that $\int_C K^n > 0$ for $n = 1, \dots, m$, and all C for which the expression makes sense. This constraint makes the geometrically relevant moduli space appear as a real convex polyhedral cone of dimension $h^{1,1}(\mathcal{M})$. This real-cone structure arising from classical geometry is naturally complexified in string theory, where any Kähler form K acquires an imaginary part from the Kalb-Ramond two-form field, $J = K + \mathrm{i}B$. As such, the string theory complexified-Kähler moduli space has complex dimension $h^{1,1}(\mathcal{M})$.

An alternative approach, based on infinitesimal metric deformations, is outlined in Ref. [24].

Complex-structure moduli space

As to deformations of complex structures, the story goes under the name of Kodaira-Spencer theory, a nice account of which can be found in, e.g., Ref. [25]. In short it entails the following facts: take $\bar\partial$ as a measure of the complex structure. A new operator is then formed by $\bar\partial' := \bar\partial + A\cdot\partial$, where A is a $(0,1)$-form valued in $(1,0)$ vectorfields, i.e., $A = A^i_{\bar j}\partial_i \otimes d\bar z^{\bar j}$ in a holomorphic frame. However, there is an integrability condition here, namely $(\bar\partial')^2 = 0$; it is equally well expressed as

$$\bar\partial A + [A, A] = 0\ , \tag{3.1.11}$$

where $[-,-]$ involves both wedge product and bracket of vector-fields. Second, some would-be deformations can be undone by reparametrisations, and should not count as true deformations accordingly. In a finite deformation,

$$A = \sum_{n\geq 1} \lambda^n\, A_{(n)}\ , \tag{3.1.12}$$

the $A_{(1)}$ part is identified as the infinitesimal (first-order) deformation. To this order, the integrability condition reads

$$\bar\partial A_{(1)} = 0\ , \tag{3.1.13}$$

whereas $A_{(1)} = \bar\partial\omega$ should be discarded since they are generated by reparametrisation. The upshot is that first-order complex structure deformations of a manifold M are given by T_M-valued $\bar\partial$ cohomology.

The hard issue, here, is to check whether first-order deformations are unobstructed, i.e., that given some first order $A_{(1)}$, a corresponding finite A can actually be found to all orders. Second point is to see if the A thus generated is 'unique', that is, modulo the reparametrisation ambiguity. For $c_1 = 0$ complex manifolds these facts were demonstrated by Tian and others [26], so the Calabi-Yau complex-structure moduli space is unobstructed.

Given that M is Calabi-Yau, the $(n,0)$-form can be used to set up isomorphisms between tangent-bundle-valued cohomology and Hodge cohomology

$$\phi^{(q,p)}\ :\ \Omega^{0,p}(\wedge^q T_M) \to \Omega^{n-q,p}(M)\ , \tag{3.1.14}$$

whereby the $(n, 0)$-form is evaluated on the wedge-product of vector fields. It follows that complex-structure deformations are in one-to-one correspondence with $(n-1, 1)$ forms.

In the process of deformation, the original holomorphic $(n, 0)$ form Ω will generically pick up components of mixed Hodge type. With a deformation A it may be shown that e.g. for $n = 3$ [25]:

$$\Omega' = \Omega + \phi^{(1,1)}(A) + \phi^{(2,2)}(A \wedge A) + \phi^{(3,3)}(A \wedge A \wedge A) \,. \tag{3.1.15}$$

Therefore, complex-structure deformation is intimately tied to the *middle cohomology*.

C. Holonomy, SUSY and SCAs

For a long time, it was believed that compact manifolds $\mathcal{M}$ had to have special holonomy groups $\mathrm{Hol}^0(\nabla)$ in order to make interesting candidates for superstring compactification. Nowadays, mechanisms such as brane-world scenarios have been proposed to side-step the issue of compactification. Even so, the issue of special holonomy is not completely out of the question, as briefly explained in the introduction.

Geometries with reduced holonomy

Recall first what (reduced) Riemannian holonomy groups $Hol^0(g)$ are:

$$Hol^0(g) = \{P_\gamma | \gamma \text{ a contractible loop based at } p\} \ , \tag{3.1.16}$$

where P_γ is the parallel transport map, $P_\gamma \in \mathrm{End}(T_p\mathcal{M})$. Note that $Hol^0(g)$ is not really an abstract Lie group, but rather a concrete matrix subgroup of $GL(n, \mathbb{R})$, i.e., the group is specified by the natural representation in the tangent space. Also, the matrix representatives get conjugated inside $GL(n, \mathbb{R})$ into an isomorphic group if some other base point is chosen.

Which are the subgroups of $GL(n, \mathbb{R})$ that can occur as restricted holonomy groups of a Riemannian connection on an n-dimensional manifold?

First observe that since $\nabla g = 0$, the holonomy group must be a subgroup of $\mathrm{O}(n, \mathbb{R})$ only, or even $\mathrm{SO}(n, \mathbb{R})$ if the manifold is oriented. Excluding direct-product metrics and maximally symmetric spaces, the complete answer is provided by Berger's list in Table 3.1 [27].

A few useful observations here:

(a) In Table 3.1, the Kähler metrics are those with holonomy groups $\mathrm{U}(m)$, $\mathrm{SU}(m)$, $\mathrm{Sp}(m)$, while the Ricci-flat ones have holonomy inside $\mathrm{SU}(m)$, $\mathrm{Sp}(m)$, G_2, $\mathrm{Spin}(7)$.

Holonomy group	Dimension	Features
SO(m)	m	oriented
U(m)	2m	Kähler
SU(m)	2m	Calabi-Yau
Sp(m)	4m	Hyperkähler
Sp(1).Sp(m)	4m	quaternionic-Kähler
G_2	7	exceptional, Joyce-7
Spin(7)	8	exceptional, Joyce-8

Table 3.1: Reduced holonomy groups and the (real) dimensions of the manifolds realising them.

(b) Considering the division algebras $\mathbb{R}^m, \mathbb{C}^m, \mathbb{H}^m, \mathbb{O}$ of real and complex numbers, quaternions and octonions, respectively, the groups O(m), U(m), Sp(1).Sp(m), Spin(7) are precisely the respective automorphism groups. Moreover, an appropriate determinant-one like constraint reduces them to SO(m), SU(m), Sp(m), G_2.

Granted the established fact that manifolds with holonomy groups that allow parallel spinors can actually be shown to be spin, how do covariantly constant tensors and spinors relate to special holonomy? The following theorem deals with that question:

THEOREM 3.2 *Holonomy singlets are in one-to-one correspondence with covariantly constant tensors [28] (spinors).*

This being given, all that is further needed, is how the holonomy is embedded, i.e., how $Hol(g) \hookrightarrow \mathrm{SO}(d)$ for a real d-dimensional manifold. Table 3.2 contains a schematic overview (for $d = 7, 8$).

Holonomy	$\mathbf{8}_v$	$\mathbf{8}_s$	$\mathbf{8}_c$	(N_+, N_-)
Spin(7)	$\mathbf{8}$	$\mathbf{1} \oplus \mathbf{7}$	$\mathbf{8}$	(1,0)
SU(4)	$\mathbf{4} \oplus \bar{\mathbf{4}}$	$\mathbf{1} \oplus \mathbf{1} \oplus \mathbf{6}$	$\mathbf{4} \oplus \bar{\mathbf{4}}$	(2,0)
G_2	$\mathbf{1} \oplus \mathbf{7}$	$\mathbf{1} \oplus \mathbf{7}$	$\mathbf{1} \oplus \mathbf{7}$	(1,1)
Sp(2)	$\mathbf{4} \oplus \mathbf{4}$	$\mathbf{1} \oplus \mathbf{1} \oplus \mathbf{1} \oplus \mathbf{5}$	$\mathbf{4} \oplus \mathbf{4}$	(3,0)
SU(2)× SU(2)				(4,0)
SU(3)	$\mathbf{1} \oplus \mathbf{1} \oplus \mathbf{3} \oplus \bar{\mathbf{3}}$			(2,2)
SU(2)				(4,4)

Table 3.2: Embeddings of the holonomy representations. (N_+, N_-) counts the (positive,negative)-chirality MW-spinors in $d = 2$.

An easy way to find out about the spinor embeddings in the complex cases (excluding G_2, Spin(7)) employs the correspondence between spinors and $(p,0)$-forms on complex n-manifolds with holonomy inside SU(n). In a holomorphic basis, let unbarred greek indices label holomorphic coordinates and vice-versa, while taking roman labels for the underlying real coordinates. With this convention, define $\gamma = \prod_{m=1}^{n} \gamma^m$, and let a covariantly constant spinor ζ (which is known to exist, see the theorem on p. 40) be chosen such that

$$\begin{aligned} \nabla\zeta &= 0\,; \\ \gamma\zeta &= \zeta\,. \end{aligned}$$

Then, an arbitrary spinor χ has the following decomposition [29]:

$$\chi = \phi^{(0)}\zeta + \phi^{(1)}_{\mu}\gamma^{\mu}\zeta + \ldots + \phi^{(n)}_{\mu_1\mu_2\ldots\mu_n}\gamma^{\mu_1\mu_2\ldots\mu_n}\zeta\,, \tag{3.1.17}$$

with coefficients $\phi^{(r)} \in \Omega^{0,r}(\mathcal{M})$. It follows that if Q is the holonomy representation of the holomorphic cotangent bundle ($Q + \bar{Q} \hookrightarrow \mathbf{2n}_v$), then the spinor transforms as

$$\bigoplus_{p=0}^{n} \Lambda^p Q\,. \tag{3.1.18}$$

Moreover, there is a split into chiral and anti-chiral spinors (for convenience, expressed in terms of the relevant bundles):

$$\Gamma(\mathcal{M}, \mathrm{Spin}^+) \approx \bigoplus_{q\ \mathrm{even}} \Omega^{0,q}(\mathcal{M})\,; \tag{3.1.19}$$

$$\Gamma(\mathcal{M}, \mathrm{Spin}^-) \approx \bigoplus_{q\ \mathrm{odd}} \Omega^{0,q}(\mathcal{M})\,. \tag{3.1.20}$$

From this data, the spinor holonomy representations in Table 3.2 follow at once. Notice, e.g., that one of the singlets in the Sp(2) case is the trace part of $Q \wedge Q$.

The final column uses the decomposition of spinors according to $SO(1,9) \to SO(1,1)\times SO(8)$, where e.g.

$$\mathbf{16}_s \to (+, \mathbf{8}_s) \oplus (-, \mathbf{8}_c)\,. \tag{3.1.21}$$

Given $\mathbf{16}_s$, N_+ (N_-) counts the holonomy singlets in $\mathbf{8}_s$ ($\mathbf{8}_c$).

Beyond geometry : SCAs and space-time SUSY

In summary, geometries with a reduced holonomy are thus observed to yield space-time supersymmetric supergravity theories. Trying to generalise to non-geometric CFT backgrounds, one is led to the question:

"Which properties are the abstract CFT counterparts of reduced holonomy in the geometric phase?"

Put differently, we wish to complete the following diagram:

GEOMETRIC PHASE	NON-GEOMETRIC PHASE
Reduced holonomy	????
Parallel spinors	????

(EXTENDED) SUPERGRAVITY
$\mathcal{R}$-symmetry

As it turns out, $\mathcal{R}$-symmetry in the space-time SUSY algebra contains the clue. The argument is short and parallels the reasoning in [30]. Consider the NS-R sector in a type II compactification, and neglect the R-NS sector for a while. The gravitino states are created by the vertex operator

$$V^{\text{gravitino}} = V_L \otimes V_R = \mathrm{e}^{-\phi}\psi_\mu\, W_L \otimes e^{-\phi/2}\Sigma_\alpha\, W_R. \qquad (3.1.22)$$

where $\mu\,(\alpha)$ labels components of the vector (spinor) of the non-compact space-time part, and $W_{L,R}$ are internal CFT completions. Further, ψ_μ and Σ are the worldsheet fermion and spinfield, respectively, and ϕ is de bosonised superghost. At zero-momentum, the ground states created by the anti-chiral Ramond vertex in Eq. (3.1.22) are in one-to-one correspondence with the space-time supercharges Q_α; they close into a super-Poincaré algebra, encoded in their OPE:

$$\{Q_{\alpha,R}, Q_{\beta,R}\} = \oint_{\bar{w}} d\bar{z}\; e^{-\phi/2}\Sigma_\alpha\, W_R(\bar{z})\; e^{-\phi/2}\Sigma_\beta\, W_R'(\bar{w})\;. \qquad (3.1.23)$$

When applied to a state created by a vertex Φ_{p_L,p_R} where $p_{L,R}$ label momentum and winding, the rhs. evaluates (up to a constant tensor factor, see below) to $\not{p}_R$; as already announced, this is the super-Poincaré algebra.

In geometric compactifications on a d-dimensional manifold, say, the ('chiral') $\mathcal{R}$-symmetry for the supercharges in the non-compact space-time SUSY algebra coincides with the commutant[3] of the holonomy group inside $\mathrm{SO}(d)$: recall that supersymmetries correspond to covariantly constant spinors and therefore, holonomy singlets. The reduction of the d-dimensional spinor according to $\mathrm{SO}(d) \to Hol^0 \times U$ will contain $(\mathbf{1},\mathbf{r})$, where $\mathbf{r}$ is a U irrep. As such,

[3] I should warn the reader that this is an improper use of the terminology. Here, the "commutant of H inside G" (for a given embedding) is the maximal subgroup K of G, such that $G \supseteq H \times K$.

U is identified as the $\mathcal{R}$-symmetry. Clearly, U is intimately tied to the internal geometry.

Similarly, in Eq. (3.1.23), the internal CFT vertex operators W_R, W'_R should represent the $\mathcal{R}$-symmetry. This implies in particular that its generators correspond to worldsheet currents $j(z)^A$, closing under OPE into an affine algebra $\mathcal{A}$. A worldsheet symmetry algebra containing an $N = 1$ SCA together with this $\mathcal{A}$ is necessarily an N-extended SCA. The proper identification is quickly made going through the classification of the latter: if $\mathcal{A}$ =U(1), SU(2) then the N-extended algebra has to be $N = 2, 4$ respectively. This completes the argument for the chiral $\mathcal{R}$-symmetry groups. For low-dimensional irreducible compactifications, we give a list in Table 3.3, which may be compared to Table 3.2; the cases with reducible holonomy follow from this upon dimensional reduction.

The story for the R-NS gravitino that was hitherto left out, completely parallels the one above: the associated supercharges close into an algebra likewise, and they come with an anti-chiral $\mathcal{R}$-symmetry. Moreover, on *perturbative* closed string states, the OPE dictates that $\{Q_{\alpha,L}, Q_{\beta,R}\} = 0$, since left-movers and right-movers decouple. The upshot is that only $\mathcal{R} = U_L \times U_R$ is manifest in the OPEs. However, when worldsheet boundaries, or equivalently, D-branes, are present, the left-right decoupling no longer holds. First of all, this has the effect that central charges show up in the $\{Q_{\alpha,L}, Q_{\beta,R}\}$ anti-commutator. More importantly for the point we wish to make, the manifest $\mathcal{R}$-symmetry gets enhanced. The resulting algebra must match one of the space-time super-Poincaré algebras in the well-known list. For example, $d = 4$, $(N = 1, N = 1)$ with $\mathcal{R}$-symmetries (U(1),U(1)) yields $N = 2$ with U(1)× SU(2).

d	Space-time SUSY	$\mathcal{R}$	Holonomy	Worldsheet SUSY
$d = 6$	$N = 1$	SU(2)	SU(2)	$N = 4$
$d = 4$	$N = 1$	U(1)	SU(3)	$N = 2$
$d = 3$	$N = 1$	$-^*$	G_2	$N = 1$
$d = 2$	$N = (1, 0)$	$-^*$	Spin(7)	$N = 1$
	$N = (2, 0)$	U(1)	SU(4)	$N = 2$
	$N = (3, 0)$	SU(2)	Sp(2)	$N = 4$

Table 3.3: Low-dimensional space-time vs. worldsheet SUSY.

In Table 3.3, the entries with $-^*$ actually fall outside this argument, since the commutant of the holonomy fails to be a group. As argued in Ref. [31], a modified notion of commutant leads at best to a *coset* rather than a group. A related reasoning shows that particular extensions of the $N = 1$ SCA lying in between $N = 1$ and $N = 2$ actually correspond to those cases, thus completing the list.

In summary, in the present paragraph we have established a link between the specific extension of the $N = 1$ SCA and the holonomy of the geometric compactification. In combination with the previous paragraph, the results are summarised in the following scheme:

GEOMETRY
Irreducible holonomy with commutant algebra $\mathcal{A}$
and n covariantly constant spinors
$\updownarrow$
CFT
Extended SCA containing an affine algebra $\mathcal{A}$
with n spectral flows

From CFT back to geometry (I)

Thus far, the CFT-geometry correspondence has been considered in a fairly abstract setting: no reference was made to concrete models. That is to say, the correspondence took place at the level of SCAs and spectral flows.

If the CFT is more concretely taken to be a non-linear sigma-model, however, a more refined picture of the correspondence emerges. It includes the following ingredients:

FACT 3.1

> *R-R ground states in the CFT are in one-to-one correspondence with cohomology classes of the manifold.*

FACT 3.2

> *A subset of the marginal deformations of the CFT is mapped to deformations of structures on the manifold.*

Observe that Facts 3.1 and 3.2 are related by spectral flow [32]. Since spectral flow operators are space-time supersymmetry generators, morally speaking, this implies that states in 3.1 and 3.2 sit in the same multiplet of the unbroken SUSY. By itself, this is a first manifestation of the close connection between worldsheet and space-time features.

Further, the geometric deformations in Fact 3.2 involve metric perturbations; together with the perturbations of the B-field background, they fill out the NS-NS truly marginals in the CFT, while 3.1 corresponds to R-R background deformations. Therefore, space-time SUSY relates metric moduli and cohomology.

A detailed picture of point (a) is obtained if the $N = 2$ structure[4] on the worldsheet is judiciously exploited: Fact 3.1 can be established via topological twisting of the worldsheet theory. However elegant, we shall refrain from a detailed analysis here. The interested reader is referred to, e.g., Ref. [33].

With less sophistication, the same conclusion is reached via simple Kaluza-Klein reduction of the supergravity and the non-linear sigma model, both on a smooth target $\mathcal{M}$, say. The basic observation is that R-R p-form fields yield a massless $(p-q-r)$-form for each $\bar{\partial}$-harmonic (q,r)-form ω on $\mathcal{M}$. Dolbeault cohomology classes are in one-to-one correspondence with such forms[5].

The present section's discussion presumed a smooth target $\mathcal{M}$ throughout. However, the correspondence will shortly be argued to persist in singular orbifold limits (see p. 67).

D. Orbifolds and desingularisation

Since a large part of the present volume is going to deal with so-called orbifold spaces, orbifolds for brevity, some generic features of the classical geometry are collected here. The selected topics should give the reader a sufficiently detailed overview to see the links with CFT and D-branes (to be explored in Section 3.2 and Section 3.3).

An orbifold $(\mathcal{M}, g)$ is a singular manifold M/G of real dimension n whose singularities are locally isomorphic to quotient singularities $\mathbb{R}^n/G$, where G is some finite subgroup of $GL(n,\mathbb{R})$; each non-identity element of G fixes a subspace of at least real codimension 2 in $\mathbb{R}^n$. In a similar vein, a complex orbifold is a singular complex manifold of dimension m whose singularities are locally isomorphic to quotient singularities $\mathbb{C}^m/G$, for finite subgroups $G \subset GL(m,\mathbb{C})$.

The condition on the codimension of the fixed-point set is technical (it makes orbifolds behave very similarly to manifolds) and will not bother us anyway.

The orbifold M/G is obtained by identifying G-orbits of points on M, the singular set is characterised to consist of points $xG \in M/G$ that have a non-

[4] The target is assumed to be Kähler at least. The the G_2 and Spin(7) cases require only minor modifications, see [31].

[5] See also p. 36.

trivial stabiliser group.

Let a discrete subgroup $G \subset GL(n, \mathbb{R})$ be such that it is embedded in one of the special holonomy representations of Table 3.2. It is clear that the holonomy is discrete, accordingly. However, in the desingularisation process, this initial discrete holonomy will gradually become "enhanced": in the completely desingularised situation, some continuous subgroup of $GL(n, \mathbb{R})$ will be filled out. Of special interest are resolutions that preserve the original structure, e.g. desingularisations taking initial SU(3) discrete holonomy into SU(3) continuous holonomy. In fortunate cases (including all $n \leq 6$ cases) such structure-preserving desingularisations are known to exist. Unfortunately, this does not seem to be the general pattern. For example, some Calabi-Yau fourfold orbifolds have been demonstrated not to have a crepant blow-up, that is, a resolution that does not affect the Calabi-Yau property. Complete control over higher-dimensional singularities and their structure-preserving resolutions remains an open problem.

Basically, known ways to get rid of orbifold singularities fall into two classes: blow-up and deformation (resolution). We collect general features of both procedures below, leaving detailed examples to Section 3.1.2 (SU(2) holonomy) and Section 3.2.3 (SU(3) case).

Blow-up

Blowing up along subvarieties of a given variety X is a common technique in complex algebraic geometry. A priori, it is unrelated to singularities (one may well blow up non-singular points) but rather trades 'new' varieties for 'old' ones. More to the point, the newly obtained varieties are closely related to the original one, it is so-called birational to the latter. Loosely speaking, a birational map is an isomorphism almost everywhere.

In essence, to blow up a non-singular point $p : (x = 0)$ in $X := \mathbb{C}^N$ is the replacement of p by a $\mathbb{P}^{N-1}$, as follows from the definition of $\widetilde{X}$:

$$\widetilde{X} = \{((x); [y]) \in \mathbb{C}^N \times \mathbb{P}^{N-1} |\ x \wedge y = 0\} \ . \tag{3.1.24}$$

Coordinates (x) are affine, whereas $[y]$ are projective; further, $x \wedge y := x^{[i}y^{j]}$. It is elementary to see that there is a bijective correspondence $\pi : \widetilde{X} \to X$ away from the origin of X, while π^{-1} maps the origin to the entire $\mathbb{P}^{N-1}$. The latter is referred to as the exceptional divisor $\mathcal{E}_X$ (codimension one subvariety) of the blow-up.

Next, consider a subvariety Y embedded as an algebraic subset in X, i.e., Y is locally determined by a set of polynomial constraints. The blow-up $\widetilde{X} \to X$ induces $\widetilde{Y} \to Y$ with an exceptional set $\mathcal{E}_Y := \widetilde{Y} \cap \mathcal{E}_X$. Judiciously chosen blow-ups of X will partially smooth out initially singular points of Y. Then, if $\mathcal{E}_Y$ still contains singularities, successive blow-ups are required to remove all of them. In fact, it is an established fact that by blow-up, all orbifold singularities can actually be removed (Hironaka's theorem).

A number of comments are appropriate here:

(a) In blowing up, one is not guaranteed to preserve all properties enjoyed by the original singular variety. Typically, one is interested in blow-ups that are crepant, i.e. such that $K_{\widetilde{Y}} = \pi^* K_Y$, where K's are the associated canonical bundles. In particular, if Y is Calabi-Yau (i.e. K_Y is trivial) it means that the defining property is not destroyed in the blow-up to $\widetilde{Y}$. For one thing, Hironaka's theorem does not guarantee that there exists a *crepant* blow-up undoing the singularity. Similar reasonings apply to the preservation of additional structures (see, e.g., Section 3.1.2 for a discussion of the hyperkähler case).

(b) The blown-up completely non-singular $\widetilde{Y}$ is not unique. Firstly, one can go on indefinitely blowing up non-singular points. In fact, the blow-up of the origin in $\mathbb{C}^N$ above demonstrates this explicitly. However, since such a procedure necessarily destroys crepancy in the end, one usually restricts to 'minimal' resolutions, containing no spurious exceptional (-1)-divisors[6]. Blow-ups of non-singular points give rise to such creeps. The surface singularities in Section 3.1.2 will have crepant blow-ups that contain only (-2)-curves. Secondly, it is well-known that in dimension three and higher, crepancy does not even suffice to single out unique resolutions: typically, there are a number of them related by flop-transitions (see, e.g., Ref. [34]).

(c) Being defined by holomorphic algebraic equations, the exceptional divisors are actually complex submanifolds of $\widetilde{Y}$. As such their volumes are measured by the appropriate wedge power of the Kähler form, as follows from Wirtinger's theorem on p. 37. In more precise terms, one should think of the new Kähler class to be related to the original one as $\widetilde{\omega}_Y = \omega_Y + \omega_{\mathcal{E}}$. The second summand is a two-form that governs the size of the exceptional set; in particular, $\int_C \omega_{\mathcal{E}}$ is the size of a two-cycle C that is dual to the exceptional divisor $\mathcal{E}_Y \in H_{2d-1}(\widetilde{Y})$. As such, the physical picture of blowing up and down had better be thought of as curves expanding or shrinking.

(d) Since complex submanifolds are added in the process, a blow-up is said to add to $\oplus_p H^{p,p}$. In fancy terminology, blow-up takes place in the holomorphic category.

[6] More generally, $(-n)$-divisors arise as sections of a line-bundle $\mathcal{L}$ with $c_1(\mathcal{L}) = -n$.

Deformation

Deformation or resolution is an alternative way of removing singularities. With quasi-projective varieties, i.e., those embedded by polynomial constraints in projective spaces or toric varieties there are two potential sources of singularities.

(a) The ambient space may already display singular points. A subvariety crossing those will typically inherit some of those. As a typical example, weighted projective spaces (and general toric varieties) are prone to having orbifold singularities (see e.g., Ref. [35]).

(b) The subvariety can acquire additional singularities through the embedding polynomial constraints F. Singular points p of this type are characterised by

$$\begin{cases} F(p) = 0 \\ dF(p) = 0 \,. \end{cases}$$

Deformation removes the singularities by adding monomials (of appropriate homogeneity type, in the projective case) to the defining F so as to end up with an empty zero-set to the system of equations Eq. (3.1.25).

A variety defined as the zero-set of holomorphic polynomial constraints naturally inherits the complex structure from the ambient space. Intuitively, it seems reasonable to assume that deformed polynomials give rise to deformed induced complex structures. From this one would be tempted to set up a map

$$\begin{pmatrix}\text{polynomial}\\ \text{deformations}\end{pmatrix} \leftrightarrow \begin{pmatrix}\text{complex structure}\\ \text{deformations}\end{pmatrix} \tag{3.1.25}$$

However, some care is required, since polynomial deformations can be *ineffective* on the one hand, and *deficient* on the other hand (see e.g. Ref. [36]). For example, reparametrisations of the ambient space are observed to kill some of the potential complex-structure deformations.

Up to these trivial removals, the map in Eq. (3.1.25) can be shown to be 1 : 1 in the hypersurface case. In fact, the Griffiths residue map may then be used to establish a correspondence between monomials and added middle cohomology classes [37].

3.1.2 ADE orbifolds I : a case study

Throughout, ADE orbifolds will be the main source of illustrations, to come to grips with the presented material. Here and below, orbifolds with the epithet

'ADE' are shorthand for $\mathbb{C}^2/G$, where $G \subset \mathrm{SU}(2)$; such discrete G are known to fall into an $A-D-E$-classification. Apart from simplicity considerations, which allow an explicit demonstration of most operations involved, there is a second motivation for the study of such orbifolds: the associated type of singularities also figure in higher-dimensional orbifolds, where they are no longer isolated. Simplicity has its price, however: ADE orbifolds may fail to be generic enough, and display an atypical behaviour in certain respects. For the present section, this will not become relevant.

The singular A_2 case

The A_2 singularity is described by the following polynomial equation in $\mathbb{C}^3$:

$$P(x_1, x_2, x_3) \equiv x_1^2 + x_2^2 - x_3^3 = 0 \; . \tag{3.1.26}$$

This describes a hypersurface with a singularity in the origin : the singular character is signalled by $P = dP = 0$ at that point. A change of coordinates, such that $x_1^2 + x_2^2 = uv$, makes G manifest: to this end, consider the parametrisation:

$$(t_1, t_2) \to (u, v, x_3) = (t_1^3, t_2^3, t_1 t_2) \; . \tag{3.1.27}$$

The full set $(t_1, t_2) \in \mathbb{C}^2$ manifestly obeys Eq. 3.1.26, and moreover, enjoys a $\mathbb{Z}_3$ invariance, $(t_1, t_2) \to (\mathrm{e}^{2\pi \mathrm{i}/3}\, t_1, \mathrm{e}^{2\pi \mathrm{i}/3}\, t_2)$. Thus, $(t_1, t_2) = (0, 0)$ is the only point that is fixed by this action. All in all, the singularity described by Eq. (3.1.26) is recognised to be locally modelled by $\mathbb{C}^2/\mathbb{Z}_3$.

Blowing up

This is probably the next-to-easiest example of a surface singularity to blow up. Consider the space

$$\{((x_1, x_2, x_3); [y_1, y_2, y_3]) \in \mathbb{C}^3 \times \mathbb{P}^2 | \forall i, j, x_{[i} y_{j]} = 0\} \; , \tag{3.1.28}$$

which is the affine three-plane $\mathbb{C}^3$ with the origin blown-up. This total space may be viewed in two ways:

(a) Either as $\mathbb{C}^3$ with the origin replaced by $\mathbb{P}^2$,

(b) or as the total space of the line-bundle $\mathcal{O}_{\mathbb{P}^2}(-1)$. In fact, this is the line-bundle over $\mathbb{P}^2$ dual to the hyperplane bundle: sections of the latter are the homogeneous functions of degree 1 on $\mathbb{P}^2$. Alternatively, the projectivisation $x \wedge y = 0$ of associates an affine line L_p, determined

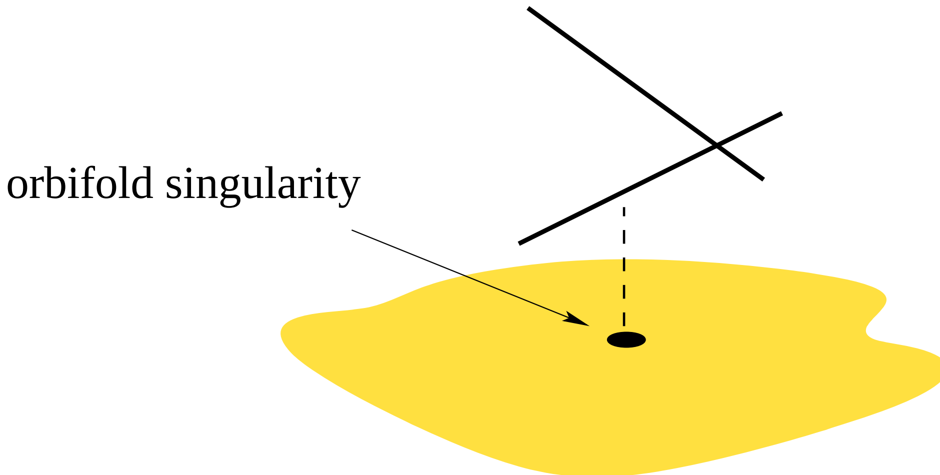


Figure 3.1: Blow-up of an A_2-singular point. The irreducible exceptional components are two projective lines $\mathbb{P}^1$ intersecting in a point.

by the constraints, to each point p outside the origin of $\mathbb{C}^3$. As such, $\mathcal{O}_{\mathbb{P}^2}(-1)$ constitutes the space of affine lines in $\mathbb{C}^3$ (see also Appendix A).

Let us next see how this blow-up in the ambient space affects the hypersurface singularity.

The blow-up $(\mathbb{A}^3)_i \xrightarrow{\sigma} \mathbb{C}^3$ involves three patches, e.g., for $(\mathbb{A}^3)_1$,

$$(x_1, y_1, z_1) \overset{\sigma}{\hookrightarrow} (x_1, x_1y_1, x_1z_1) , \tag{3.1.29}$$

and likewise for the other two. Clearly, in $(\mathbb{A}^3)_1$, the exceptional set in the ambient space blow-up, $\sigma^{-1}(0)$, is the affine two-plane $(\mathbb{A}^2)_1 : (x_1 = 0)$. In this patch, the blow-up $X_{(1)}$ of $(P(x, y, z) = 0)$ is therefore given by

$$P(x_1, x_1y_1, x_1z_1) = (x_1)^2(1 + y_1^2 + x_1z_1^3) = 0 . \tag{3.1.30}$$

The first factor is an artefact, whereas the second is the so-called *proper* transform, or blow-up, of X. Observe that the surface has become non-singular, and the exceptional set is a set of two disjoint lines $(y_1 = \pm \mathrm{i})$ in $(\mathbb{A}^2)_1$. Similarly, in the second patch, the blow-up is given by local equations

$$(1 + x_2^2 + y_2z_2^3) = 0 , \tag{3.1.31}$$

with an exceptional set consisting of two lines, $(x_2 = \pm \mathrm{i})$, in $(\mathbb{A}^2)_2$. Finally, in the third patch, one has

$$(x_3^2 + y_3^2 + z_3) = 0 , \tag{3.1.32}$$

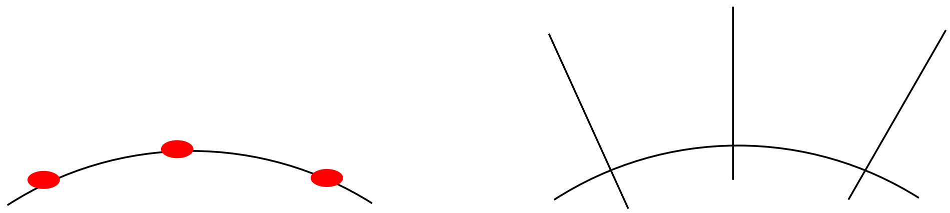


Figure 3.2: Blow-up of a $\mathbb{D}_2$-singular point. An exceptional $\mathbb{P}^1$ with three A_1-singular points results after the first step; further blow-ups yields three more lines.

with two exceptional lines $(x_3 \pm iy_3 = 0)$, that intersect in the origin of $(\mathbb{A}^2)_3$.

The exceptional set produced in the blow-up of $\mathbb{C}^3$ was $\cup_i(\mathbb{A}^2)_i = \mathbb{P}^2$. Since $\mathbb{P}^2$ is compact, the affine non-compact exceptional lines above are in fact glued together to form two $\mathbb{P}^1$s (projective lines), intersecting in one point.

Since the local equations used in the process have been holomorphic throughout, the exceptional $\mathbb{P}^1$ divisors are complex submanifolds of $\tilde{X}$.

Even though the above steps display the exceptional set and its properties explicitly, it must have crossed your mind that this explicit procedure becomes hopelessly tedious in more complicated cases. In fact, only very recently[7] was the procedure carried out for three-dimensional generalisations of the ADE-singularities [38]. Luckily, more powerful tools are often available, such as toric geometry for abelian orbifold singularities (see Section 3.2.3).

Deformation

Polynomial deformations of Eq. (3.1.26) provide a second method of desingularisation. The vanishing cycles are easily visualised from the steps below: the initial polynomial constraint defining the singular surface, is deformed into

$$uv = Q_3(x_3) \; , \tag{3.1.33}$$

meaning that the r.h.s. gets replaced by a generic $Q_3(x_3)$ of degree 3. Non-singularity holds whenever all roots e_i of Q_3 are distinct, as is most easily verified.

The deformed equation Eq. (3.1.33) eliminates v, and the surface is hence parametrised by (u, x_3); u parametrises an algebraic torus $\mathbb{C}^\times$ over the x_3-plane. At the roots e_i of Q_3, this is seen to degenerate.

[7] That is, to my best knowledge.

Two-cycles come in two types, now. First, there are cycles parametrised by (t, θ)

$$\begin{aligned} u &= e^{i\theta} f(t) \ , \theta \in [0, 2\pi] \ ; \\ x_3 &= e_i t + (1-t) e_j \ , t \in [0, 1] \ , \end{aligned} \tag{3.1.34}$$

for an arbitrary real function f vanishing at $t = 0, 1$ and nowhere in $(0, 1)$. Denote the corresponding cycle as C^f_{ij}. These are obviously cycles, i.e., $\partial C^f_{ij} = 0$; moreover, their homology class is non-trivial since u takes values in $\mathbb{C}^\times$: therefore, there doesn't exist any finite set of three-balls $\{B_a\}_a$ such that

$$C^f_{ij} = \partial \left(\sum_a B_a \right) \ . \tag{3.1.35}$$

Clearly, the homology class $[C^f_{ij}] \in H_2(X)$ has a spherical representative, i.e., cycles of this type are two-spheres, topologically speaking.

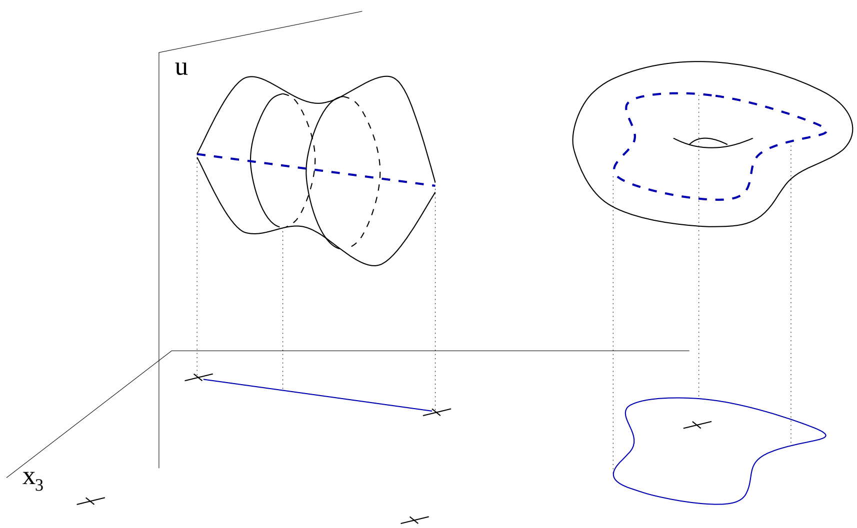


Figure 3.3: Cycles arising from deformation. On the left the spherical cycles of Eq. (3.1.34), on the right, toroidal ones. Marks in the x_3-plane are zeroes of Q_N.

Besides these, there exist toroidal cycles

$$u = e^{i\theta} f(t) \ , \theta \in [0, 2\pi] \ ; \tag{3.1.36}$$

$$x_3 = g(t), t \in [0, 1] \ , \tag{3.1.37}$$

for some complex function g such that $g(0) = g(1)$ not taking values in the root set of Q; further, for the cycle not to be trivial, g must enclose at least one root of Q_3. Cycles of this type can in fact be demonstrated to be homologous to linear combinations of the spherical ones above.

The picture drawn makes the following points intuitively clear:

(a) Homology classes $[C_{ij}], [C_{jk}]$ where $k \neq i$ have topological intersection number ± 1, depending on relative orientations.

(b) Upon deformation back to the original orbifold set-up, i.e. when all roots of Q become coincident whereby the $\mathbb{Z}_3$ symmetry is restored, the spherical cycles manifestly vanish.

(c) These vanishing cycles are moreover linearly dependent: their homology classes are generated by two independent ones.

More generally, (deformations of) A_{N-1}-singularities are treated similarly: there, the initial x_3^N is replaced by a degree-N polynomial $\mathcal{Q}_N(x_3)$. Since there are then $N-1$ monomials x_3^i of order strictly less than $N-1$, the complex structure moduli space will be $N-1$ complex dimensional [8]: a natural local coordinate system is given by the complex coefficients in the deformed $Q(x_3)$. Furthermore, the counting is seen to be in 1-1 correspondence with the number of vanishing cycles. This is in accord with the general hypersurface result, p. 48.

ALE metrics

Let us quickly collect some properties of Asymptotically Locally Euclidean (ALE) metrics on four-dimensional manifolds. Asymptotically, such spaces look like $\mathbb{C}^2/G$, with G a discrete subgroup of $SU(2)$. More is true, in fact: these smooth spaces are diffeomorphic to resolutions of the corresponding $\mathbb{C}^2/G$.

In Ref. [39] ALE manifolds M_G have been shown to result from a hyperkähler quotient construction. Being hyperkähler spaces by construction, the M_G come equipped with 3 covariantly constant symplectic forms ω_a. Moreover, their holonomy sits inside $SU(2)$.

Further, the M_G develop singularities whenever $\int_C \omega_\alpha = 0, \forall \alpha$. That is, whenever the volume of $C \in H_2(M_G)$ vanishes wrt. any of the Kähler forms. In this picture, $\mathbb{C}^2/G$ results in the limit where all of $H_2(M_G)$ obeys the stated condition.

[8] The order $N-1$ monomial does not represent a true deformation: adding it to $\mathcal{Q}_N$ corresponds to picking an origin in the x_3-plane.

Compactification

Only non-compact orbifolds have been the issue in the above: the local models $\mathbb{C}^N/G$ represented the full space under study. Here are three ways how such local models show up when dealing with spaces that are compact:

(a) Weighted projective spaces and their generalisations, toric varieties, are quite apt to developing orbifold singularities. These spaces may serve as ambient spaces for defining Calabi-Yau manifolds as complete intersections. As such, the latter can inherit local orbifold singularities from the ambient space, which makes a thorough understanding of the local models worthwile. Further, it is a well-known fact that the embedding itself is a potential source of additional orbifold singularities [35].

(b) In the case $N = 2$, to add points at infinity yields an orbifold of the compact $\mathbb{P}^2$ space. There are now two singularities, each of which is sitting at a pole. In fact, the resolution of the space can be shown to yield a K3 surface (see e.g., Refs. [40, 41]).

(c) A compact higher-dimensional complex torus can be modded out by discrete rotations. Alternatively, this space results from dividing $\mathbb{C}^N$ by the combined action of discrete translations (encoded in a discrete lattice Λ) and discrete rotations (G). For the procedure to make sense, Λ must be invariant under the action of G. Since the main purpose is illustrative, I will be brief and only comment on the $N = 2$ case below (see Refs. [42] for a list of $N = 3$ torus orbifolds, however).

K3 surface from the torus

The only rank four lattices left invariant by discrete subgroups of $\mathrm{SU}(2)$ are collected in Table 3.4.

Let me comment on the entries: $[g]$ stands for the weight lattice of the Lie-algebra g and $\oplus$ is the orthogonal sum. With multiplicities in front, the summands need not be orthogonal wrt. each other. Moreover, the multiplicity equals the number of independent scale factors multiplying the lattice, yielding an equal amount of independent size parametres in the quotient space. The second column contains the global rotation orbifold group G, and the third entry contains the fixed point types that occur, together with their multiplicities. The data in the third column will be reproduced from string theory in Section 3.2.2. Suffice it here to add only the observation that the binary dihedral groups have not shown up in string theory until quite recently [43]. Independently, these were under study by the author and M. Billó. Actually,

Λ	G	Fixed points
$4[su(2)]$	$\mathbb{Z}_2$	$16\ A_1$
$2[su(2)\oplus su(2)]$	$\mathbb{Z}_4$	$6\ A_1$, $4\ A_3$
$[su(2)\oplus su(2)\oplus su(2)\oplus su(2)]$	$\mathbb{D}_2$	$2\ D_4$, $3\ A_3$, $2\ A_1$
$[so(8)]$	$\mathbb{D}_2$	$4\ D_4$, $3\ A_1$
$2[su(3)]$	$\mathbb{Z}_3$	$9\ A_2$
	$\mathbb{Z}_6$	$1\ A_5$, $4\ A_2$, $5\ A_1$
$[su(3)\oplus su(3)]$	$\mathbb{D}_3$	$1\ D_5$, $3\ A_3$, $2\ A_2$, A_1

Table 3.4: Four-tori, discrete rotation groups and the associated fixed-point structure.

the entry with $so(8)$ was discovered precisely from string theory considerations by M. Billó, and appears to be missing from the table in Ref. [43].

EXAMPLE 3.1 *:*
It is instructive to see how things work in an explicit example, the $\mathbb{D}_2$ case, say. To that end, take Λ the hypercube lattice, as indicated in Table 3.4. In local holomorphic coordinates $(z^{1,2})$, the action of the $\mathbb{D}_2$ generators a, b is left multiplication by $\mathrm{i}\sigma_3, \mathrm{i}\sigma_1$, respectively. Moreover, the point group has one subgroup of order two ($\langle a^2\rangle$), and three of order four ($\langle a\rangle, \langle b\rangle, \langle ab\rangle$). On the square four-torus $\langle a^2\rangle$ fixes 16 points $(\star,\star)$, with $\star = 0, \frac{1}{2}, \frac{1}{2}\mathrm{i}, \frac{1}{2}(1+\mathrm{i})$. Of these, the index-two subgroups fix 4 points each, respectively

$$\begin{aligned} \langle a\rangle &: (0,0), (m,m), (0,m), (m,0)\ ; \\ \langle b\rangle &: (0,0), (m,m), (\frac{1}{2}, \frac{1}{2}\mathrm{i}), (\frac{1}{2}\mathrm{i}, \frac{1}{2})\ ; \\ \langle ab\rangle &: (0,0), (m,m), (\frac{1}{2}, \frac{1}{2}), (\frac{1}{2}\mathrm{i}, \frac{1}{2}\mathrm{i})\ , \end{aligned} \tag{3.1.38}$$

where $m = \frac{1}{2}(1+\mathrm{i})$. Finally, both $(0,0)$ and (m,m) are invariant under the full dihedral group. As to e.g. $(\frac{1}{2}, \frac{1}{2}), (\frac{1}{2}\mathrm{i}, \frac{1}{2}\mathrm{i})$, they get identified under the action of the full $\mathbb{D}_2$; in other words, they arrange themselves into orbits of the large group. A similar story applies to the remaining fixed points: those fixed under an index p subgroup combine into length-p orbits. In all, one thus ends up with 2 length-1, 3 length-2 and 2 length-4 orbits. Locally, the regions in the vicinity of fixed points look like $\mathbb{C}^2/\mathbb{D}_2$, $\mathbb{C}^2/\mathbb{Z}_4$, $\mathbb{C}^2/\mathbb{Z}_2$, respectively. Gluing in smooth ALE spaces with the right asymptotics, it is thus found that 2 D_4, 3 A_3 and 2A_1 spaces will do. This is the result listed in Table 3.4. Performing a count of the (-2)-exceptional curves, it is readily seen that 19 of those are produced in the resolution. Their Poincaré-duals together with the three invariant hyperkähler forms yield a total of 22, which is precisely suitable for a K3! □

3.2 A closer look – The closed-string picture

In the present section, orbifolds will be reviewed in a CFT approach. As such, it will become clear how closed strings effectively manage to deal with the classically singular geometry: contrary to expectations, perhaps, the orbifold CFT remains well-behaved. This section concludes with an explanation for this seemingly striking feature.

In the various possible approaches to orbifold CFTs [44, 42, 45, 46, 47], the algebraic aspects that will be needed further on are best illustrated in the treatment of [48]. Hence, this path will be followed closely in Section 3.2.1.

3.2.1 Aspects of the CFT

Consider a well-defined CFT C with symmetry algebra $\mathcal{A}_L \times \mathcal{A}_R$. Let further a finite group G act chirally on the CFT C, i.e., G does not mix left- and rightmovers. Under those circumstances G is a symmetry group of the CFT, provided it commutes with the Virasoro algebras. There exists a canonical procedure, called orbifolding, to build a (possibly) new consistent CFT $C' = C/G$ out of the original one. To spend a few lines on the main points in the construction will prove to be a useful preliminary for the construction of D-branes in the new theories.

A. Hilbert space structure

First, let $\mathcal{A}^G$ be the G-invariant part of the chiral $\mathcal{A}$. Since $\mathcal{A}^G$ commutes with G and $\mathcal{A}^G$ is a submodule of $\mathcal{A}$, every $\mathcal{A}$-module $[\phi]$ has a decomposition into $\mathcal{A}^G \times G$-modules:

$$[\phi] = \bigoplus_{\alpha,I} m^{(e)}_{\alpha,I}\, [\phi^e_{\alpha,I}] \otimes R^{(e)}_I \ , \tag{3.2.1}$$

where e is the unit element in G. This extra label, whose meaning will become clear shortly, denotes the so-called untwisted sector. Further, $R^{(e)}_I$ are irreducible G representations: it captures the *complete* group-theoretic behaviour. Besides the modules of Eq. (3.2.1), there exist g-twisted $\mathcal{A}^G$-modules, characterised by the fact that fields obey boundary conditions twisted by g. Modules of the latter kind will comprise the g-twisted sectors with corresponding Hilbert spaces $\mathcal{H}_g$. Further, the action of elements $h \in G$ takes $\mathcal{H}_g \to \mathcal{H}_{hgh^{-1}}$. Let C_g be the conjugacy class that contains g. It follows then that $\oplus_{h\in C_g}\mathcal{H}_h$ is taken into itself by the orbifold action, hence carries some representation of G. In contrast, individual sectors $\mathcal{H}_g$ only furnish representations of the normaliser $N_g \subset G$, where

$$N_g = \{h \in G | [h, g] = e\} \ . \tag{3.2.2}$$

Thus, with individual elements labelling boundary conditions, one has a decomposition

$$\mathcal{H}_g = \bigoplus_{\alpha, I} m^{(g)}_{\alpha, I} \ [\phi^g_{\alpha, I}] \otimes R^{(g)}_I \ , \tag{3.2.3}$$

where I runs over the irrepses of N_g. Alternatively, one can take conjugacy classes C_g as labels for boundary conditions, where representations of the full G then organise the twisted sector

$$\mathcal{H}_{[g]} = \bigoplus_{\alpha, I} m^{(g)}_{\alpha, I} \ [\phi^g_{\alpha, I}] \otimes R_I \ . \tag{3.2.4}$$

The number of twisted sectors is thus seen to equal the number of conjugacy classes in the group. If you adhere to the picture where elements label the sectors, recall that different such sectors are grouped into orbits so as to make G invariants, thus making both counts agree. In summary, the structure of the chiral Hilbert space takes one of the equivalent forms:

$$\mathcal{H} \quad = \quad \bigoplus_{g \in G} \mathcal{H}_g \ , = \bigoplus_{[g]} \mathcal{H}_{[g]} \tag{3.2.5}$$

where each component can be further decomposed into $\mathcal{A}^G \times N_g$, $\mathcal{A}^G \times G$-modules, respectively. In fact, the Hilbert space structure in Eq. (3.2.5) will prove to be the crux for modular invariance shortly.

EXAMPLE 3.2 *: $SO(2d)_1$ current algebra*
The following is a simple illustration. Let $\mathcal{A}$ be the algebra generated by free fermions

$$\mathcal{A} := \langle \mathbb{1}, \partial^k \psi^\mu \rangle \tag{3.2.6}$$

where $k \in \mathbb{N}$ and $\mu = 1, \dots, 2d$. Clearly, there is a $G = \mathbb{Z}_2$ symmetry, generated by $(-)^F$, with F the worldsheet fermion number. Now

$$\mathcal{H}_{[e]} = [\mathbb{1}]_+ \oplus [\psi^\mu]_- \ , \tag{3.2.7}$$

where the subscript in the rhs. denotes the $G = \mathbb{Z}_2$ irrep. The modules are $\mathcal{A}^G = SO(2d)$ level-1 modules, each containing integrally spaced conformal weight operators only. Observe that the elementary ψ_μ obey anti-periodic (NS-) boundary conditions in both modules of Eq. (3.2.7).

A twist by $g := (-)^F$ effectively turns NS into periodic (Ramond) boundary conditions. Additionally, the corresponding twisted module has a decomposition

$$\mathcal{H}_{[g]} = [S]_+ \oplus [C = \Gamma^\mu S]_- \ , \tag{3.2.8}$$

containing the spinor and anti-spinor as the highest weight states.

In all, the full chiral Hilbert space decomposes as in Eq. (3.2.5):

$$\mathcal{H} = [\mathbb{1}]_+ \oplus [\psi^\mu]_- \oplus [S]_+ \oplus [C]_- \ . \tag{3.2.9}$$

To keep only G-invariant states is observed to effect a chiral GSO-projection. □

B. Chiral traces and partition function

Next, we turn to traces in the chiral theory. For notational convenience, here and below two shorthand notations for chiral blocks (=traces over a full (un-)twisted sector) will be employed interchangeably:

$$Z^c(h,g) \equiv h \underset{g}{\Box} \equiv \mathrm{Tr}_{\mathcal{H}_g}(h\, q^{L_0 - \frac{c}{24}}) \ , \tag{3.2.10}$$

where the superscript '^{c}' must remind you that the traces are taken over the chiral space only. Given the structure of the Fock space, Eq. (3.2.3) and Eq. (3.2.4), the chiral traces in the $[\phi^g_\alpha]$ modules are easy to extract:

$$\chi^g_I = \frac{1}{|G|} \sum_{h\in G,[g,h]=e} \rho_I(h^{-1})\, h \underset{g}{\Box} \ , \tag{3.2.11}$$

where ρ_I is the character of the associated representation. Observe that the operator

$$P_I \equiv \frac{1}{|G|} \sum_{h\in G,[g,h]=e} \rho_I(h^{-1})\ h \tag{3.2.12}$$

precisely projects onto states in R_I in the C_g-twisted sector [48].

How about the full, rather than chiral, CFT then? The procedure to construct a one-loop modular invariant proceeds in three canonical steps.

(a) Project the original C theory onto its G-invariant subspace, i.e. keep only G invariants;

(b) Add twisted sectors;

(c) Project the twisted sectors onto their G-invariant parts.

Note that the full Hilbert space, rather than the chiral space, is now the relevant object. Point (a) is accomplished by

$$\frac{1}{|G|}\sum_{g\in G} Z(e,g)\ ; \tag{3.2.13}$$

where $Z(-,-)$ denotes the trace over the combined chiral $\times$ anti-chiral space. This procedure generically destroys modular invariance (of the initial CFT C). Steps (b) and (c) are performed next so as to restore it. The complete orbifold partition function at one-loop therefore thus reads schematically,

$$\sum_{g}\frac{1}{|N_g|}\sum_{h\in N_g} Z(g,h)\ . \tag{3.2.14}$$

In Ref. [48], the chiral blocks were shown to carry a representation of the modular group where [9]

$$\mathcal{S} : g \underset{h}{\square} \ \to\ \sigma(g,h)\ h^{-1} \underset{g}{\square}\ ; \tag{3.2.15}$$

$$\mathcal{T} : g \underset{h}{\square} \ \to\ \tau(g,h)\, gh \underset{h}{\square}\ , \tag{3.2.16}$$

with phase factors $\sigma(g,h), \tau(g,h)$. From the structure of the partition function, modular invariance is formally restored by inspection.

C. Of fusion rules and algebras

The action Eq. (3.2.15) of $\mathcal{S}$ on the space of orbifold chiral blocks implies that

$$S_{g,h'}^{h,g'} = \sigma(g,h)\ \delta_g^{g'}\ \delta_{h'}^{h^{-1}}\ . \tag{3.2.17}$$

The base-change Eq. (3.2.11) trading orbifold chiral blocks $Z^c(g,h)$ for one-loop characters χ_I^g turns $\mathcal{S}$ into

$$S_{IJ}^{\alpha\beta} = \sum_{h\in N_g} \rho_I(h)\rho_J(h^{-1})\sigma(\alpha,h)\delta^{\alpha,\beta}\ . \tag{3.2.18}$$

It is this matrix, rather than Eq. (3.2.17), that will produce the orbifold fusion rules from Verlinde's formula, Eq. (2.2.11).

[9] Strictly speaking, less is true. The literal statement in Ref. [48] was that the modular generators acted on the *chiral* $Z^c(g,h)$ as given with additional g,h dependent phases. Assuming a left-right symmetric Hilbert space and action, the phases cancel from $Z(g,h)$, which is the content of Eq. (3.2.14).

Let us first think about three-index objects that could be meaningful in the present context. Therefore, consider the following subset of $C^\alpha \times C^\beta \times C^\gamma$:

$$S^{\alpha\beta\gamma} = \{(g_1, g_2, g_1 g_2)\} = \bigsqcup_{\mu=1}^{n_{\alpha\beta\gamma}} O^\mu \ , \tag{3.2.19}$$

where the second equality is the stratification into $n_{\alpha\beta\gamma}$ G-orbits O^μ. From these numbers, the linear space of conjugacy classes may be turned into a commutative, associative algebra: the *class algebra*, where multiplication is defined as:

$$C^\alpha \star C^\beta \equiv n_{\alpha\beta\gamma}\, C^\gamma \ . \tag{3.2.20}$$

Since fusion rules for ϕ_I^α are what we are after, the representation theory of G must still be taken care of properly. For a given triplet $(g_1, g_2, g_1 g_2) \in O^\mu$, denote the normaliser $N^\mu \equiv N_{g_1} \cap N_{g_2}$. Let furthermore $n_{IJK}^{(\mu)}$ be the multiplicity of the trivial representation in the tensor product $R_I \otimes R_J \otimes R_K$ of representations of N^μ. Then, from the numbers

$$N_{IJK}^{\alpha\beta\gamma} \equiv \sum_{\mu=1}^{n_{\alpha\beta\gamma}} n_{IJK}^{(\mu)} \tag{3.2.21}$$

the *orbifold fusion algebra* is defined as in Eq. (2.2.5).

For rational CFTs, the numbers in Eq. (3.2.21) have been demonstrated in Ref. [48] to follow from Eq. (3.2.18) upon application of Verlinde's formula Eq. (2.2.11), that is,

$$N_{IJK}^{\alpha\beta\gamma} = \sum_\alpha S_I^\alpha \frac{S_J^\alpha}{S_0^\alpha} (S^{-1})_J^\alpha \tag{3.2.22}$$

It is an illustration of the intruiging CFT property that "S diagonalises the fusion rules".

3.2.2 ADE orbifolds II

Let us pause here, and illustrate how the general orbifold formalism of the previous section applies in the concrete case of geometric flat space orbifolds. Rather than to pursue generality, it is believed that to demonstrate pro's and con's in particular examples is more instructive. Therefore, whenever appropriate, we shall feel free below to restrict the discussion to particular subsets of the zoo of all possible orbifolds.

The general setting will always be as follows: take four free $N = 1$ complex superfields $\mathcal{Z}^i = Z^i + \theta\psi$ that are the superstring extensions of holomorphic coordinates on $\mathbb{C}^4$. These fields make up the $c = 12$ superstring matter CFT in the light-cone. In complex orbifolds, the $\mathcal{Z}^i$ ($\bar{\mathcal{Z}}^{\bar{\imath}}$) carry a representation Q (Q^*) of the orbifold group G that is embedded in the **4** ($\bar{\mathbf{4}}$) of SU(4). In fact, if G is abelian, Q is specified by four irrepses of G: $Q = \oplus_i R_{I_i} \hookrightarrow \mathbf{4}$.

Example 1

Consider the embeddings of abelian subgroups $\mathbb{Z}_N \subset \mathrm{SU}(2) \subset \mathrm{SU}(4)$. In that case, Q can be chosen as $Q = 2R_0 \oplus R_1 \oplus R_1^*$, where R_0 and R_1 are the trivial and defining representations, respectively. As such, the original chiral $\mathrm{SO}(8)_1$ affine algebra splits into $\mathrm{SO}(4)_1\times \mathrm{SO}(4)'_1$, and accordingly, one has

$$O_8 \to O_4 O_4' + V_4 V_4' \; ; \tag{3.2.23}$$

$$V_8 \to V_4 O_4' + O_4 V_4' \; ; \tag{3.2.24}$$

$$S_8 \to S_4 S_4' + C_4 C_4' \; ; \tag{3.2.25}$$

$$C_8 \to S_4 C_4' + V_4 S_4' \; . \tag{3.2.26}$$

The $\mathrm{SO}(4)'_1$ currents can be conveniently rewritten so as to make the underlying $\mathrm{SU}(2)_1\times\mathrm{SU}(2)_1$ structure manifest:

$$\langle J^{3\bar{4}}, J^{\bar{3}4}, J^{3\bar{3}} - J^{4\bar{4}} \rangle \times \langle J^{34}, J^{\bar{3}\bar{4}}, J^{3\bar{3}} + J^{4\bar{4}} \rangle \; , \tag{3.2.27}$$

where, e.g., $J^{34} := \mathrm{i}\psi^3\psi^4$, etc. Orbifold projection removes two currents, and the invariant algebra we are left with thus becomes

$$\mathcal{A}^G = \mathrm{SO}(4)_1 \times \mathrm{U}(1) \times \mathrm{SU}(2)_1 \; . \tag{3.2.28}$$

Under this reduction, it is observed that

$$V_4' = [\psi^3, \psi^4, \bar{\psi}^{\bar{3}}, \bar{\psi}^{\bar{4}}] \to [\psi^3, \bar{\psi}^{\bar{4}}] \oplus [\bar{\psi}^{\bar{3}}, \psi^4] \oplus \dots \; . \tag{3.2.29}$$

At the ground state level, this is simply the group-theoretical $\mathbf{4} \to \mathbf{2} + \bar{\mathbf{2}}$. Further, for $N \neq 2$, the ellipsis is non-empty and stands for modules with higher conformal weight primary states. The argument is short and simple: assigning a $\mathbb{Z}_N$ charge $+1$ (-1) to ψ^3 (ψ^4), the non-invariant currents have charges ± 2. As such, when the positively-charged current acts on ψ^3 it generates a descendant state in the initial V_4', that ceases to be so in the reduction: the charge of the new state is $+3$ and therefore it is the lowest state in a distinct $\mathrm{U}(1)\times\mathrm{SU}(2)$-module. Observe that its conformal weight is one unit higher than that of ψ^3.

Clearly, a similar analysis applies to O_4', with the result

$$O_4' = [\mathbb{1}] \to [\mathbb{1}] \oplus [\psi^3 \bar{\psi}^{\bar{4}}] \oplus [\bar{\psi}^{\bar{3}} \psi^4] \oplus \dots \; . \tag{3.2.30}$$

In all, the reduction of V_8 into $\mathcal{A}^G \times G$-modules therefore reads

$$\begin{array}{rcl} V_8 & \to & O_4 \times \left\{ [\psi^3, \bar{\psi}^{\bar{4}}] \times R_1 \oplus [\bar{\psi}^{\bar{3}}, \psi^4] \times R_{-1} \oplus \ldots \right\} \\ & & \quad \bigoplus \\ & & V_4 \times \left\{ [\mathbb{1}] \times R_0 \oplus [\psi^3 \bar{\psi}^{\bar{4}}] \times R_2 \oplus [\bar{\psi}^{\bar{3}} \psi^4] \times R_{-2} \oplus \ldots \right\} \end{array}$$

This expression makes painfully clear that the explicit $\mathcal{A}^G$-module structure is quite intricate. Moreover, the results derived here are seen only to hold for A_{N-1} orbifolds. These drawbacks are hardly compensated by the manifest exposure of the $\mathcal{A}^G \times G$ structure. Therefore, we shall not pursue an analysis along similar lines of the R-sector and the twisted modules.

Rather, a formalism will be presented below where only the $\mathrm{SO}(2)_1^4$ initial structure is presumed. At the price of hiding part of the $\mathcal{A}^G$-module structure, the results derived there will be valid for all complex orbifolds.

EXAMPLE 2: (PART I : $\mathrm{SO}(2)_1$ FERMIONS)

Consider a complex fermion ψ that, when combined with its complex conjugate, generates an affine $\mathrm{SO}(2)_1$ algebra. Pick a cyclic subgroup $G' \equiv \langle g | g^{K_g} = e \rangle$ inside G, and let ψ carry a one-dimensional representation R_I of G'; that is, $g \cdot \psi = \exp(2\pi \mathrm{i} n_I / k_I) \psi$, with $\gcd(n_I, k_I) = 1$. Setting $\nu_I = n_I / k_I$, one checks easily that

$$\mathrm{Tr}_{v}(g\, q^{L_0 - \frac{1}{24}}) = \frac{1}{2\eta} \left[\vartheta_3(\nu_I|\tau) - \vartheta_4(\nu_I|\tau) \right] ; \qquad (3.2.31)$$

$$\mathrm{Tr}_{o}(g\, q^{L_0 - \frac{1}{24}}) = \frac{1}{2\eta} \left[\vartheta_3(\nu_I|\tau) + \vartheta_4(\nu_I|\tau) \right] ; \qquad (3.2.32)$$

$$\mathrm{Tr}_{s}(g\, q^{L_0 - \frac{1}{24}}) = \frac{1}{2\eta} \left[\vartheta_2(\nu_I|\tau) - \mathrm{i}\vartheta_1(\nu_I|\tau) \right] ; \qquad (3.2.33)$$

$$\mathrm{Tr}_{c}(g\, q^{L_0 - \frac{1}{24}}) = \frac{1}{2\eta} \left[\vartheta_2(\nu_I|\tau) + \mathrm{i}\vartheta_1(\nu_I|\tau) \right] ; \qquad (3.2.34)$$

in the untwisted sector of the chiral theory. Their anti-chiral counterparts are obtained by replacing $\tau \to \bar{\tau}$.

As to the g^r-twisted sector, it is explained in Appendix B that the corresponding chiral blocks follow from replacing $\nu_I \to \nu_I + \tau n_I / r k_I$ in Eq. (3.2.31)-Eq. (3.2.34).

What does this teach us? First, the $\mathrm{SO}(2)_1 \times G'$ decomposition of the initial modules is

$$\mathcal{H} = \oplus_j \; [j] \times R_I^j \; ; \qquad (3.2.35)$$

and e.g. in the untwisted NS+ (v) sector the trace decomposes accordingly (for $k_I \in 2\mathbb{Z}$) $\mathrm{Tr}_v\,(q^{L_0-\frac{1}{24}}) = \sum_j \chi_v^j(q)$, where

$$\chi_v^j(q) = \frac{q^{2j^2}}{\eta(q)} \sum_{m=0}^{\infty} q^{2m^2 k_I{}^2} (q^{4jmk_I} + q^{-4jmk_I}) \; ; \tag{3.2.36}$$

$$= \Theta_{2k_I(2j),4k_I^2}(q,0,0) \; . \tag{3.2.37}$$

This follows at once from the series representations of the ϑ-functions. In fact, as shown in Appendix B, a closed form is known for each o, v, s, c and g-twisted sector: in all cases, there is a decomposition into level k_I^2 theta-functions.

EXAMPLE 3: (PART II : $\mathrm{SO}(2)_1$ BOSONS)

Now move on to the bosonic partners of $\psi, \bar{\psi}$. Zero-modes are the only possible complication beyond a discussion that completely parallels that of the fermions. First note that closed string zero-modes only occur in the untwisted sector.

In a noncompact orbifold $\mathbb{C}/G$ the insertion of an order k_I element in the partition function yields

$$g^m \underset{e}{\square} = \int d^2k \langle k| g^m |q|^{\alpha' p^2} |k\rangle \times g^m \underset{e}{\square}' \; ;$$

$$= (\det(1-g^m))^{-1} \int d^2k \delta^2(k)\; |q|^{\alpha' k^2} \times g^m \underset{e}{\square}' \; ;$$

$$= \left| \frac{\eta(q)}{\vartheta_1(\nu_{k_I})} \right|^2 , \tag{3.2.38}$$

where the $'$ denotes the trace over non-zero modes.

In a compact orbifold T^2/G, however, the situation changes in two respects: not only can there be non-zero windings, but the momenta k now take on discrete values only. Therefore, one rather has

$$g^m \underset{e}{\square} = \sum_{k,w} \langle k,w| g^m q^{\frac{\alpha'}{2}(p+\frac{w}{\alpha'})} \bar{q}^{\frac{\alpha'}{2}(p-\frac{w}{\alpha'})} |k,w\rangle \times g^m \underset{e}{\square}' \; ;$$

$$= 4\sin^2 \pi\nu_{k_I} \left| \frac{\eta(q)}{\vartheta_1(\nu_{k_I})} \right|^2 \tag{3.2.39}$$

where the extra factor, as compared to Eq. (3.2.38), is seen to result from the absence of continuous momentum.

The present discussion, when combined with the modular properties of the non-zero-mode piece (see Appendix B), implies a modular S-matrix:

$$g \underset{e}{\Box} \quad \overset{S}{\rightarrow} \quad e \underset{g}{\Box} \,, \ (\text{non} - \text{compact}) \ ; \tag{3.2.40}$$

$$g \underset{e}{\Box} \quad \overset{S}{\rightarrow} \quad (2 \sin \pi \nu_{k_I})\, e \underset{g}{\Box} \,, \ (\text{compact}) \ ; \tag{3.2.41}$$

where the numerical prefactor in Eq. (3.2.39) has been equally distributed over chiral and anti-chiral parts.

A. From CFT to geometry II

As explained on p. 44, the massless states are the central characters in the correspondence CFT-geometry. It is instructive to see how this general fact is realised in the present orbifold setting. Basically, the closed-string partition function contains the required data; an explicit count entails the following recipe:

(a) Find out about the multiplicities of fixed sets of a given type. These will shortly be shown to be governed by modular invariance. Moreover, this step is empty in the case of non-compact orbifolds (i.e., local orbifolds).

(b) For each fixed point type, determine the number of massless states separately. Rather than to expand the partition function, it is generically faster to reason in terms of shifted modings and ground state energies.

(c) As a check, in the superstring case the found states should arrange themselves naturally into massless multiplets of the unbroken supersymmetry, at least if some portion is left unbroken.

EXAMPLE 3.3 *:*
Let me illustrate Step (a) by the K3 $T^4/\mathbb{D}_2$ example. A simpler example was worked out in [1]. Recall that the binary dihedral group is defined by: $\mathbb{D}_2 = \langle a, b \mid a^4, b^4, bab^3a \rangle$. It has three order four $\mathbb{Z}_4^g$ subgroups, generated by $g = a, b, ba$, respectively. With a defining $\mathcal{Q} \hookrightarrow \mathbf{2}$ such that $\mathcal{Q}(a) = \mathrm{diag}(\mathrm{i}, -\mathrm{i})$ and $\mathcal{Q}(b) = \mathrm{i}\sigma_1$, the four-dimensional hypercube lattice defining T^4 is seen to be left invariant. Modular invariance under S, Eq. (3.2.41), requires that sectors twisted by g be added with multiplicities m_g, where:

$$g^2 = e \quad : \quad m_g = (4 \sin^2 \frac{\pi}{2})^2 = 16 \ ; \tag{3.2.42}$$

$$g^4 = e \quad : \quad m_g = (4 \sin^2 \frac{\pi}{4})^2 = 4 \ . \tag{3.2.43}$$

The modular invariant can be rearranged as follows, therefore:

$$\mathcal{Z} = \frac{1}{|\mathbb{D}_2|} \sum_{g;h\in N_g} h \underset{g}{\square} ; \qquad (3.2.44)$$

$$= \mathcal{Z}_{untwisted} + 2\mathcal{Z}_{\mathbb{D}_2} + 3\mathcal{Z}_{\mathbb{Z}_4} + 2\mathcal{Z}_{\mathbb{Z}_2} . \qquad (3.2.45)$$

The meaning of the last line is as follows: the $\mathcal{Z}_H$ are the twisted sector partition functions of noncompact (local) orbifolds with group H, while the integers in front count the number of $\mathbb{D}_2$-orbits *of H-type fixed points.* □

In fact, the content of the above CFT procedure is a well-known mathematical property, a.k.a. the Lefschetz fixed point theorem. A version of the latter asserts that for $g \in SO(2d)$, the number of points fixed under g is given by

$$\det(\mathbb{1} - g) = \prod_{i=1}^{n} (4 \sin \pi \nu_g)^2 , \qquad (3.2.46)$$

where the r.h.s. applies to the situation at hand. On the CFT side, these were integers counting multiplicities of twisted sectors in order to establish modular invariance. As a matter of fact, the requirement that these be integers actually puts strong restrictions on the possible point groups G that can be used. As an example, in the four-dimensional case, only $\mathbb{Z}_{2,3,4,6}$, $\mathbb{D}_{2,3}$ are valid, as already asserted in Section 3.1.2 (see Table 3.4). It is rather remarkable how one-loop consistency manages to reproduce Lefschetz' result.

As for Step (b), the analysis has a local nature. Let me go back to the general orbifold case and pick coordinates on $\mathbb{C}^4$ that are adapted to the analysis in the, say, g-twisted sector. With $\mathcal{Q}$ such that a twist by $\mathcal{Q}(g)$ results in modings of holomorphic $(SO(2)_1)$ coordinates shifted by ν_g, the NS twisted sector ground state energy is found from Appendix B: $E_{0,g} = -\frac{1}{2} + \sum_{i=1}^{4} \nu_g$, whereas it vanishes in the corresponding R sector. Accordingly, massless NS-states are created by fermions with mode numbers $-E_{0,g}$. In the ALE $G \subset SU(2)$ case, the number of G-invariant closed string NS-NS states can thus be shown to be four (see e.g., Ref. [49]) in *each* twisted sector. This is to be contrasted with the CY $G \subset SU(3)$ case (see Section 3.2.3).

Once it has been found out *which* twisted sectors can effectively contribute massless states, and what the nature of such states is, Step (c) is fairly easy. For type IIA/IIB compactification on an ADE singularity, I list the $d = 6$ twisted sector (bosonic) spectrum in Table 3.5. Anticipating a discussion of D-brane actions in the next section, we also list the dimensionally reduced field content.

Applying the ideas on p. 44 to our concrete orbifold examples now only remains as a short exercise. First, the orbifold projection in the CFT untwisted

Type	NS-NS	R-R	SUSY multiplet
IIA ($d = 6$)	4 scalars	1 vector	$N = (1, 1)$ VM
IIB ($d = 6$)	4 scalars	1 scalar 1 SD two-form	$N = (2, 0)$ VM
IIA ($d = 5$)	3 scalars	1 scalar	$N = 2$ LHM
	3 scalar	1 vector	$N = 2$ VM
IIB ($d = 4$)	3 scalars	1 scalar	$N = 2$ LHM
	2 scalars	1 vector	$N = 2$ VM

Table 3.5: The massless bosonic $d = 6$ closed-string states, and their organisation into vectormultiplets (VM) and linear hypermultiplets (LHM).

sector parallels projection onto invariant cohomology on $\mathbb{C}^2$, by inspection. The non-trivial part of the connection resides in the CFT twisted sectors. From Table 3.5 you read off that in 6 dimensions, each RR twisted sector contributes either one massless vector (IIA) or one scalar and a self-dual two-form (IIB). This is exactly the field content that is obtained upon dimensional reduction of the 10d threeform (IIA) or the two-form and self-dual four-form (IIB) on a harmonic two-form. Therefore, the pattern is suggestive of additional harmonic forms equal in number to the conjugacy classes in $G \subset SU(2)$. On the other hand, blow-up is the geometrical procedure that produces cycles which are precisely Poincaré-dual such harmonic (1,1)-forms. In fact, for the SU(2) discrete subgroups it is a well-established property that the number of exceptional cycles in the blow-up matches the number of conjugacy classes of G. This remarkable coincidence may be seen as a weak version of a deep mathematical conjecture/statement known as McKay correspondence (see Chapter 5). We bring the following fact to the reader's attention, though: string theory does *not* require the orbifold to be resolved to be well-defined, in spite of the correspondence.

Besides the correspondence cohomology - RR ground states, the relation geometry - CFT is further tightened by a count of moduli. Call n the number of non-trivial conjugacy classes in G. The geometrical significance of the twisted sector NS-NS SO$(1, 5)$ singlets is straightforward: given the triplet of Kähler forms ω^I on the resolved smooth ALE space, it is observed that the NS-NS scalars can be arranged into $(\int_a \omega^I, \int_a B)$, for every cycle ($\approx$ twisted sector) C_a. This makes the count just right. First, notice that there are additional fourth components that involve the stringy B field besides the pure metric data. Second, the latter can be verified to transform as a triplet of SU$(2)_{\mathcal{R}}$ in the dimensionally reduced cases, e.g., from the explicit vertex operators (alternatively, see e.g. Ref. [49]).

So here is the preliminary conclusion:

FACT 3.3

(NS-NS) The G orbifold CFTs have a number of truly marginal NS-NS operators that equals the number of geometric moduli supplemented with the B-fields.

FACT 3.4

(R-R) The R-R orbifold ground states match the geometric number of exceptional cycles in a blow-up.

Observe that this exactly reproduces the scheme on p. 44, as anticipated there.

More is true, actually. Taking the untwisted sectors as well as the twisted sector contributions properly into account in each of the examples in Table 3.4, it is found that type IIB comes with 21 tensor multiplets by inspection, irrespective of the example. Luckily, this is the right matter content that ensures absence of a gravitational anomaly in 6-d, a non-trivial conspiracy of facts, indeed. Likewise, IIA string theory on the very same orbifolds is seen to yield 80 scalars. They account for the 58 metric deformations, supplemented with the 22 $\int_{C_a} B$, and the dilaton.

B. Stringy effects – Why is the CFT well-defined?

From the geometrical interpretation of the massless scalars in the foregoing, it is a simple conclusion that closed type II strings contain the necessary ingredients to resolve the *geometric* orbifold singularities. The 'physical' mechanism consists in giving nonzero vev's to a subset of the twisted sector triplet scalars. In the geometric moduli space, this moves one away from the point(s) corresponding to the singular orbifold, provided no $SU(2)_{\mathcal{R}}$ triplet vanishes identically. However, the CFT remains well-behaved even *before* turning on any vev's! This feature was explained in Ref. [50] to follow from some nonvanishing B-flux on each of the blow-up cycles. Rather than the vague statement that 'strings can cope with the singular geometry in view of their spatial extension', we now dispose of a *physical* explanation as to why things work so well. For example, in type IIA massless $U(1)$ vectors show up in the R-R twisted sectors. On the other hand, D-branes are the objects that are charged under the corresponding gauge symmetries. Further, the orbifold limit is such that all cycles have vanishing volume as measured by any of the (triplet of) Kähler forms. As such, a D-brane wrapped around, say, a two-cycle C_a would couple to the Poincaré dually reduced $U(1)$ vector A_a. The mass of the wrapped brane goes to zero with the size of the cycle, thus causing symmetry enhancement $U(1) \to SU(2)$. Since these non-perturbative massless states were not present in the original description, this would explain the *description* to go singular. However, a non-vanishing B-flux on the two-cycle is seen to prevent the outlined scenario, since the wrapped-D-brane mass is bounded from below by a non-zero number proportional to that flux. In a nutshell, this is the physical picture explaining why the CFT behaves nicely, in spite of the geometry.

In principle, though, one could imagine wandering in the CFT moduli space by perturbing the B-field to a vanishing value. Accordingly, symmetry enhancement would occur, causing the CFT description to become singular or useless. As a matter of fact, the moduli subspace hinted at does exist, and the symmetry enhancement is an important ingredient to set up heterotic-type IIA duality. I shall refrain from any further comments here, since that subject is somewhat outside the scope of the presentation (See e.g. Ref. [51], however). The main point here, is

FACT 3.5

> *The CFT orbifold point is disjoint from the enhanced symmetry locus in moduli space.*

Apart from the above considerations, there can be further stringy and quantum effects, such as worldsheet instanton corrections. Let me suffice here with the fact that effects of the named type are argued to be absent in the ADE case, due to the high amount of preserved spacetime supersymmetry. In CY compactifications, on the contrary, those effects in combination with Mirror Symmetry have played a central rôle in the correct prediction of the number of rational curves on certain CYs.

3.2.3 Calabi-Yau orbifolds beyond ADE

Not surprisingly, higher-codimension singularities are subtler than the codimension-2 ADE cases dealt with thus far. The present section illustrates some features that will shed some light on discrete torsion, to be discussed in greater detail in Chapter 6. So far, the technical machinery seems to be well-developed for abelian singularities, while the story is far less clear for nonabelian cases. Since abelian orbifolds will largely suffice to make my point, we restrict to those.

A. Shifted fermion number and GSO-projection

In the CFT approach to orbifolds, the fermion number F of twisted ground states requires some care. Subtle points concerning F are manifestly displayed in the bosonised formulation of the fermion sectors: working in the light-cone formalism, the SO(8) $\mathbf{8}_v$ fermions are grouped into four complex pairs, with an associated boson each:

$$\psi^i \bar{\psi}^{\bar{\imath}} \approx \mathrm{i}\partial\phi^i \ , i = 1, \ldots, 4 \ . \tag{3.2.47}$$

With the assumption that these fermions actually figure in an $N = 2$ SCA, the fermion number F is recognised as the charge q under the U(1) current $J = \mathrm{i} \sum_i \partial\phi^i$ of that algebra.

The state with the lowest conformal weight in a twisted sector is commonly referred to as a twisted ground state. Denote this ground state in a sector where mode numbers of ψ^i get shifted by ν_i as $|\{\nu_i\}\rangle$. In the operator formalism, the shift is due to an insertion of a twist field $\prod_i \mathrm{e}^{\mathrm{i}\nu_i\phi^i}(0)$ in the origin of the complex plane. This insertion has two effects: the corresponding ground state gets charged w.r.t. J and the fermion number of this twisted vacuum acquires a shift: $F \to F + q$. Evidently, the shift may affect the GSO-projection, since the latter involves $\mathrm{e}^{\mathrm{i}\pi F}$: the projection flips in sectors with q odd, while it stays unaltered otherwise. Besides this, the conformal weight of the ground state is modified to $\frac{1}{2}\sum_i \nu_i^2$. For $|\{\nu_i\}\rangle$ to have the lowest weight in the given sector, the ν_i must take values in $\nu_i \in [-\frac{1}{2}, \frac{1}{2})$.

Examples

Here are some examples. In appropriate coordinates, the action on $\mathbb{C}^4$ amounts to multiplication by powers of n-th roots of unity $\mathrm{e}^{2\pi\mathrm{i}k_i/n} := (\frac{k_i}{n})$. For abelian threefold orbifolds we adopt the shorthand self-evident notation $\frac{1}{n}(1, k_2, k_3)$ for both the group and its generator g. After appropriate integer shifts, each of the entries is in fact one of the ν_i in the g-twisted sector.

EXAMPLE 3.4 : $\frac{\mathbf{1}}{\mathbf{6}}(\mathbf{1}, \mathbf{1}, \mathbf{4})$
From Table 3.6, the GSO-projection is unaltered in the sectors twisted by g, g^{-1} only. The corresponding lowest-mass NS–NS states surviving GSO- and orbifold projections are listed in the third column. A computation of the ground-state energy (see App. B) reveals that each state listed is in fact massless: the negative ground-state energy of twisted oscillators coincides precisely with the mode-number of the indicated oscillator. □

g^k	$q \bmod 2$	Age	Lowest NS–NS state
1	0	1	$\psi^3_{-\frac{1}{6}}\tilde{\bar{\psi}}^3_{-\frac{1}{6}}\lvert(1,1,-2)\rangle$
2	1	1	$\lvert(2,2,2)\rangle$
3	1	1	$\lvert(-3,-3,0)\rangle$
4	1	2	$\lvert(-2,-2,-2)\rangle$
5	0	2	$\bar{\psi}^3_{-\frac{1}{6}}\tilde{\psi}^3_{-\frac{1}{6}}\lvert(-1,-1,2)\rangle$

Table 3.6: Fermion number shifts q and lowest states for the $\frac{\mathbf{1}}{\mathbf{6}}(\mathbf{1},\mathbf{1},\mathbf{4})$ orbifold (the meaning of 'age' will become clear from p. 75).

g^k	$q \bmod 2$	Age	Lowest NS–NS state
1	0	1	$\psi^3_0\tilde{\bar{\psi}}^3_0\lvert(1,2,-3)\rangle$
2	0	1	$\psi^2_{-\frac{1}{6}}\tilde{\bar{\psi}}^2_{-\frac{1}{6}}\lvert(2,-2,0)\rangle$
3	1	1	$\bar{\psi}^3_0\tilde{\psi}^3_0\lvert(-3,0,-3)\rangle$
4	0	1	$\psi^1_{-\frac{1}{6}}\tilde{\bar{\psi}}^1_{-\frac{1}{6}}\lvert(-2,2,0)\rangle$
5	1	2	$\lvert(-1,-2,-3)\rangle$

Table 3.7: Fermion number shifts q and lowest states for the $\frac{\mathbf{1}}{\mathbf{6}}(\mathbf{1},\mathbf{2},\mathbf{3})$ orbifold.

EXAMPLE 3.5 *:* $\frac{\mathbf{1}}{\mathbf{6}}(\mathbf{1},\mathbf{2},\mathbf{3})$
Here, the GSO-projection changes in g- and g^4-twisted sectors, as indicated in Table 3.7. □

EXAMPLE 3.6 *:* $\frac{\mathbf{1}}{\mathbf{7}}(\mathbf{1},\mathbf{2},\mathbf{4})$
In this third example, all twisted sectors keep the original GSO-projection. The ground-state energies can be computed to be $E_{0,g} = \frac{1}{2} - \frac{3}{7}$, $\forall g$. This is precisely right to have one massless closed-string state in each sector, see Table 3.8. □

A pattern reveals itself here: the NS-NS sectors twisted by g and g^{-1} each yield one massless scalar (modulus). On the other hand, we knew what to expect: every element in $H^2(X)$ of the smooth space should yield one complexified Kähler form $J + \mathrm{i}B$, where the geometric quantity J controls the size of the divisor, and B is the stringy Kalb-Ramond field. Below, evidence from toric geometry will be argued to support this observation, even though the map : (Conjugacy classes) $\longrightarrow$ (Moduli) is noncanonical.

g^k	$q \bmod 2$	Age	Lowest NS–NS state
1	0	1	$\psi^3_{-\frac{1}{14}}\tilde{\bar{\psi}}^3_{-\frac{1}{14}}\vert(1,2,-3)\rangle$
2	0	1	$\psi^2_{-\frac{1}{14}}\tilde{\bar{\psi}}^2_{-\frac{1}{14}}\vert(2,-3,1)\rangle$
3	0	1	$\bar{\psi}^1_{-\frac{1}{14}}\tilde{\psi}^1_{-\frac{1}{14}}\vert(3,-1,-2)\rangle$
4	0	2	$\psi^1_{-\frac{1}{14}}\tilde{\bar{\psi}}^1_{-\frac{1}{14}}\vert(-3,1,2)\rangle$
5	0	2	$\bar{\psi}^2_{-\frac{1}{14}}\tilde{\psi}^2_{-\frac{1}{14}}\vert(-2,3,-1)\rangle$
6	0	2	$\bar{\psi}^3_{-\frac{1}{14}}\tilde{\psi}^3_{-\frac{1}{14}}\vert(-1,-2,3)\rangle$

Table 3.8: Fermion number shifts q and lowest states for the $\frac{1}{7}(\mathbf{1},\mathbf{2},\mathbf{4})$ orbifold.

B. Toric resolution of abelian singularities

As has been mentioned at various occasions, the explicit blow-up procedure of Section 3.1.1 becomes hopelessly tedious for higher-codimension singularities. In case the orbifold group is abelian, toric geometry comes to the rescue, though. This subject would easily occupy a book by itself [52, 53], although shorter accounts are available in, e.g, Refs [54, 55, 24]. Let us suffice here with a number of how-to-rules, borrowing mainly from Ref. [54]; this should largely do to make our point, namely, to provide a graphical picture of the CFT-geometry correspondence for abelian Calabi–Yau singularities.

Toric fans and how to read them

Toric geometry describes a class of spaces known as toric varieties. Of the various equivalent formulations, we select the one based on fans. Here is an intuitive sketch of how an n-dimensional complex toric variety is defined. The starting point is a Euclidean lattice $N \approx \mathbb{Z}^n$; define further $N_{\mathbb{R}} \equiv N \otimes_{\mathbb{Z}} \mathbb{R}$. Convex r-cones σ with apex at the origin of $N_{\mathbb{R}}$ are specified by r linearly independent lattice points in N. A fan Σ then consists of a collection of cones, where faces of cones are also in Σ. The toric variety X_Σ associated to Σ arises as follows: each large (i.e., $n-$) cone σ corresponds to a coordinate patch U_σ on X_Σ. Further, patches $U_{i,j}$ are glued together whenever $\sigma_{i,j}$ share a common face. The precise prescription is encoded in the way $\sigma_{i,j}$ are adjoined, see Ref. [24].

For low values of n the fan is concisely visualised as, e.g., in Fig. 3.4.

To proceed, how do properties of Σ translate into geometric data of X_Σ? The vertex set $\{\mathbf{n}_i^\sigma\}$ 'associated' to a cone σ is a minimal (finite) set of lattice

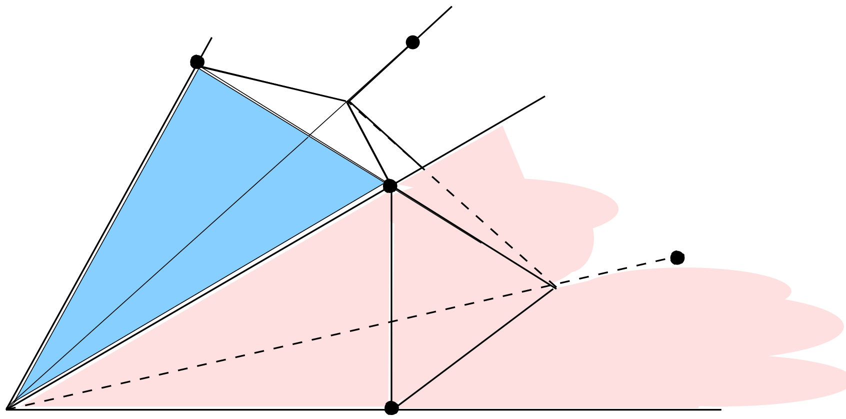


Figure 3.4: A toric fan Σ describing a 3-dimensional variety. Shaded are a 2-cone (dark) and a 3-cone (light).

points generating all lattice points inside σ over the positive integers, i.e.

$$N \cap \sigma = \sum_{i=1}^{q} \mathbb{Z}^+ \mathbf{n}_i^\sigma \ .$$

With this understanding, the following properties can be established:

(a) Compactness
X_Σ is compact iff the fan Σ spans the whole of N.

(b) Non-singularity
X_Σ is smooth iff every r-subcone σ with associated vertex set $\{\mathbf{n}_i^\sigma\}$ is such that

$$\sigma = \mathbb{R}^+ \mathbf{n}_1^\sigma + \ldots + \mathbb{R}^+ \mathbf{n}_r^\sigma ,$$

in particular, $q = r$.

(c) Calabi-Yau property
If X_Σ is smooth, then the canonical class K_{X_Σ} vanishes iff $\mathbf{n}_i^\sigma$ lie in a hyperplane of $\mathbb{R}^n$.

(d) Divisors
Real codimension-p subcones correspond to toric complex p-dimensional subvarieties of X_Σ. In particular, divisors are represented by 1-cones (rays).

Rule (b) deserves some comment: a given set of cones, $\{\sigma\}$, will generically fail to meet the smoothness criterion. Therefore, new cones are added by so-called star-subdivision: adding (integral) vertices that subdivide larger cones σ. However, as will shortly be exemplified, this procedure need not be unique, causing the existence of different resolutions. In the threefold case, these are birationally related through flop-transitions, though.

We end this summary with an observation that was put forward in Ref. [56]: if crepant resolutions exist, the so-called G-Hilbert scheme appears to have a preferred status[10] from the viewpoint of algebraic geometry. Ref. [56] explains how to construct the relevant fan. It is not obvious whether and if so, why G-Hilb is to be singled out by string theory.

Examples

In Ref. [54] it was shown that the toric fan describing a $\frac{1}{n}(1, a, b)$ orbifold is specified by the three-simplex with vertices

$$(n, -a, -b), \ (0, 1, 0), \ (0, 0, 1) \tag{3.2.48}$$

and $(0, 0, 0)$. The three points in Eq. (3.2.48) specify a hyperplane, thus ensuring the CY-property (c). To facilitate the drawing, in Fig. 3.5 only the face lying inside this plane is displayed.

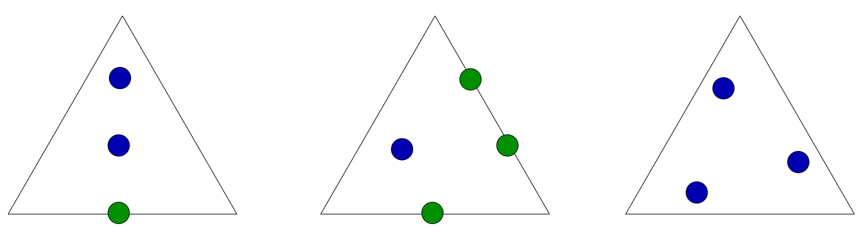


Figure 3.5: Toric fans for the singularities in the examples. Vertices are the intersections of the hyperplane with N. According to Rule (d) above, they represent exceptional divisors produced in the blow-up.

Blow-ups of these singularities are produced in a straightforward manner, by so-called star-subdivision. This two-step program proceeds as follows:

[10] Imprecisely stated, Hilbert schemes of p points on a variety X are generalisations of $S^p X$, the symmetric p-product of X. The G-Hilbert scheme of $|G|$ points parametrises G-invariant configurations of $|G|$ points, thereby keeping track not only of positions, but also of 'directions' (see e.g. Ref. [57] for a more detailed account); the latter feature is responsible for the smoothness of the Hilbert scheme.

first, integral vertices besides those defining the original fan are added. According to rule (d), they represent (exceptional) divisors of the blow-up. Next, one-by-one each of the new vertices is connected by lines to all of its neighbours that can be reached; that is, no lines must cross during this stage. Two possible outcomes in the middle example of Fig. 3.5 are visualised in Fig. 3.6: the original big triangle is subdivided into smaller ones. At the level of cones, the figures specify particular subdivisions of the original n-cone into smaller n-cones. Therefore, the second step specifies the nature of the divisors: the added lines dictate how the new patches (subcones) are to be glued together in the resolved space. Note that the final result will generically depend on the order in which the second step is performed. For example, in the $\frac{1}{6}(1,1,4)$ and $\frac{1}{7}(1,2,4)$ examples the resolution is unique, while different resolutions exist for $\frac{1}{6}(1,2,3)$. In Fig. 3.6 below, two birationally related blow-ups are displayed. It can further be checked that with the displayed vertices and lines added, the singularities are smoothed out completely. Since all lie in a common hyperplane, Rule (c) tells us that the blow-up takes place within the CY-category.

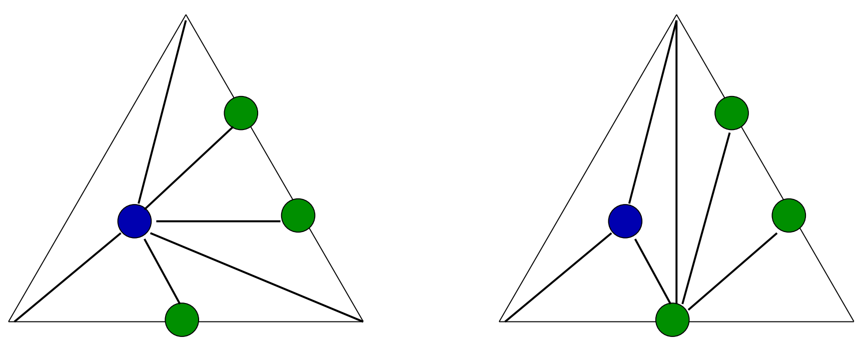


Figure 3.6: Toric fans describing two different blow-ups of the $\frac{1}{6}(\mathbf{1},\mathbf{2},\mathbf{3})$ singularity. To the left: stardivision starts with the central vertex; to the right: stardivision begins with the lower-central vertex.

Below, the following elementary observation will become relevant: exceptional divisors lying on a face of the original fans in Fig. 3.5 yield *non-compact* exceptional divisors $\mathbb{C} \times \mathbb{P}^1$. In fact, the rules of the toric game equally apply to the subcone formed by that face. Since the face is a two-cone only, it represents a (non-isolated) codimension-2 singularity locally modelled after $\mathbb{C} \times [\mathbb{C}^2/\mathbb{Z}_q]$. Recall from Section 3.1.2 that the blow-up of the singularity in $\mathbb{C}^2$ produces rational curves $\mathbb{P}^1$, from where the result follows. In contrast, blow-ups associated to internal vertices are *compact*.

C. Tying it up: from CFT to geometry III

Armed with insights from toric geometry, however rudimentary, it is shown next, how these are related to CFT properties.

Age-grading and cohomology

The CFT-geometry correspondence in the RR-sector has an alternative explanation besides the one presented on p. 44. To cut a fairly long story [58] short, the procedure boils down to the following steps:

(a) Associate a number $s_g \in \mathbb{N}$ to each $g \in G$ as follows: if modings in g-twisted sectors receive a shift by $\{\nu_i\}$ $\nu_i \in [-\frac{1}{2}, \frac{1}{2})$, then

$$s_g \equiv \sum \tilde{\nu}_i \ ,$$

where $\tilde{\nu}_i = \nu_i \bmod 1$ and $\tilde{\nu}_i \in [0, 1)$.

(b) Project the cohomology of the fixed-point set $H^*(\mathcal{F}_g)$ of g onto its G-invariant part $H^*(\mathcal{F}_g)^G$.

(c) Then, $H^{p,q}(\mathcal{F}_g)^G$ contributes to $H^{p+s,q+s}(\mathcal{F}_g)$ of the resolved space.

In a separate development, the authors of Ref. [59] assigned numbers, termed *ages* there, to individual group elements as well. With this concept of age, a conjecture was put forward:

Conjecture 3.1 (Ito–Reid)
Age-1 group elements are in one-to-one correspondence with crepant exceptional divisors (i.e., elements of $\mathrm{H}_4(X;\mathbb{Z})$*); in particular, their Poincaré-duals base* $H^2_c(X)$ *of the blown-up space.*

Comparison teaches that age(g)=s_g, even though that link was not made. In the special case of threefolds, age-2 elements were argued to yield a basis for $H^4(X)$, and as such, they are related to 2-homology.

Let us quickly verify this in the examples studied so far. The ages in Table 3.6-Table 3.8 are suggestive of the correspondence depicted in Fig. 3.7: the added vertices in the figure are in one-to-one correspondence with age-1 elements in the Tables, although not canonically so. Observe, however, that vertices lying on faces contribute a non-compact $\mathbb{C} \times \mathbb{P}^1$ to $\mathrm{H}_4(X)$. It is soon realised that the non-compactness is entirely due to the $\mathbb{C}$-factor being invariant under the subgroup G', where the 2-cone represents the blown-up $\mathbb{C}^2/G'$.

As such, the blow-up produces a *compact* $\mathbb{P}^1$; the conjecture only requires an obvious minor modification to account correctly for the situation. As to the remaining age-2 elements, string theory provides an alternative interpretation: they complexify the Kähler forms in $H^2_c(X)$ with the B-field.

The issue seems worth further investigation, especially a generalisation to the four-fold and non-abelian orbifold cases.

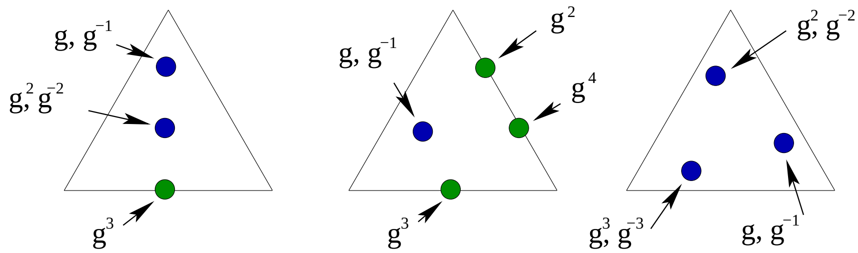


Figure 3.7: Toric fans for the singularities in the examples. Dots are the intersections of the hyperplane with N. According to Rule (d) on p. 72, they represent exceptional divisors produced in the blow-up.

3.3 Zooming in yet closer – How do open strings and D-branes discover space ?

This third section contains a review of the basic ingredients of orbifold D-branes and their worldvolume theories. No essentially new material appears here, since it was largely distilled from Ref. [60]

3.3.1 D-branes on orbifolds : the gauge theory picture

Definition

Given that orbifold CFTs make sensible type II vacua, and the fact that type II theories contain D-brane objects, one may well wonder how to combine both pieces of data. That is, we seek to answer the question:

"What makes a D-brane on an orbifold space ?"

The answer appears most naturally from the covering space point of view. With $\widetilde{\mathcal{M}}$ a simply connected cover of the orbifold space, the latter is recovered by $\mathcal{M} = \widetilde{\mathcal{M}}/G$, where G is a discrete group. For the cases of interest to us $\widetilde{\mathcal{M}} \approx \mathbb{C}^N$, so I restrict to that case. To any point $x \in \mathcal{M}$ corresponds an orbit $G \cdot \tilde{x} \in \widetilde{\mathcal{M}}$. As such a particle (D-brane) in the orbifold space corresponds to $|G|$ particles on the cover. From this picture, the $|G|$ particles are permuted by the orbifold action, and as such they naturally furnish the regular representation $\mathbb{C}G$. This program was first explored in the context of D-branes by Ref. [61] for discrete translation groups and by Ref. [60] for discrete rotations.

Spectrum + interactions

The issue of the open string spectrum clearly factorises into two parts. First, the oscillator Fock space $\mathcal{H}_{ij}$ of open strings between branes i and j, is observed to split into simple G-modules,

$$\mathcal{H}_{ij} = \bigoplus_K \mathcal{H}_{ij}^{(K)} \otimes R_K \qquad (3.3.1)$$

where R_K are G-irrepses, which may well be only projective. (In this notation, the $\mathcal{H}_{ij}^{(K)}$ are inert under G). The action on the oscillators descends from the action on the coordinates, after identifying the latter with the 2d string fields. In fact, this side of the story completely parallels the discussion of Section 3.2.1.

Secondly, the branes i, j get permuted. So far i, j have been an arbitrary abstract set of labels. The geometrical covering space point of view suggests a natural set of elementary branes. Burnside's theorem asserts that the regular representation decomposes into irreducibles

$$\mathbb{C}G \approx \bigoplus_I \left(\oplus^{d_I} R_I\right) \ , \tag{3.3.2}$$

with the dimensions d_I of the irrepses providing the multiplicities. As such, the indecomposable constituents ('elementary branes') get identified with irrepses R_I. Beware that no true evidence besides the above handwaving-style arguments has appeared here. Ultimately, it requires careful verification that labelling boundary conditions with irrepses is a consistent procedure after all. The steps involved in this program are postponed until the next Chapter, the outcome of which will support the viewpoint adopted, indeed.

Henceforth, constituent branes will as such be labelled by I. In a more traditional language, the I label the Chan–Paton factors arising from the brane degeneracy on the covering space. By definition G acts by conjugation $\gamma \cdot \mathbb{C}G \cdot \gamma^{-1}$ on these Chan–Paton factors.

In all, the orbifold open string Hilbert space is given by

$$\mathcal{H}^G = \bigoplus_{I,J} \left[\left(\oplus_{K \in I_{IJ}} \mathcal{H}_{IJ}^{(K)} \otimes R_K \right) \otimes \left(R_I^\dagger \times R_J \right) \right]^G \ , \tag{3.3.3}$$

i.e., only the G-invariant states are being kept. This orbifold projection in fact guarantees that they descend to well-defined states on the quotient space. As an aside, the reader's attention is drawn on the fact that the complete open string space is not a tensor product of a single oscillator space and some Chan–Paton vector space. Rather, it is a sum (direct product) of such tensor products.

A full specification of the D-brane theory requires the spectrum to be supplemented with interactions. The recipe is straightforward enough and parallels the reasoning above, at least initially: start from the covering space picture where the interactions are supposedly known. In the supersymmetric case this is maximally supersymmetric $U(|G|)$ Yang–Mills theory with adjoint matter. Next, this theory is orbifold-projected, as given in Eq. (3.3.3). However, interaction terms that were originally forbidden in the parent theory may cease to be so in the orbifold theory, due to the reduced (super)symmetry. Such terms do not descend from the original theory, and they have to be checked/argued separately. In favourable cases, remaining supersymmetry can in fact be set to work in order to infer their presence. The latter feature will be illustrated in Section 3.3.2.

Besides the Yang–Mills terms, which is the lowest-order-in-derivatives approximation to the Born–Infeld action, the D-brane action is well-known to contain a Wess–Zumino term. The presence of the latter has two important reasons. First of all, it is required in order that the worldvolume action be κ-symmetric, i.e. it is necessary for a supersymmetric worldvolume spectrum to exist. Secondly, it induces anomaly inflow [62], thus preventing inconsistencies in the quantum worldvolume theory. For a detailed exposition, see e.g., Ref. [63]. It is precisely the interplay of supersymmetry with the Wess–Zumino term that provides a short-cut to guessing the right action. In principle, interaction terms can always be verified from brute-force string amplitude evaluation [64, 65, 66]. With less sophistication, supersymmetry will shortly be demonstrated to do the same job, basically.

Given the (classical) action, one would typically like to actually solve the orbifold gauge theory. Vacua are found by solving the equations of motion. A detailed example will be worked out in the next section. The main result to note will be that the orbifold geometry is recovered as (part of) the D-brane moduli space. In other words, the following is true, if D-branes are being used to probe the geometry:

FACT 3.6

Space-time is recovered as the moduli space of D-branes.

This can hardly come as a surprise: it will be clear from the analysis below that this moduli space in fact parametrises the G-equivariant D-brane configurations.

3.3.2 ADE orbifolds III

In this section, it is shown explicitly how things work in the by now familiar ADE orbifold case. To this end, put a regular representation of G on the origin of the covering space $\mathbb{C}^2$. As before, supersymmetry preservation requires that the orbifold group be holomorphically embedded, i.e. $(z_1, z_2) \to (z_1, z_2)\mathcal{Q}$, where $\mathcal{Q}$ is the defining representation of $G : \mathcal{Q} \hookrightarrow \mathbf{2}$ of $\mathrm{SU}(2)$. This action is extended linearly on the corresponding complex $N = 1$ string superfield. For definiteness, pick $G = \mathbb{Z}_N$, and coordinates z_i diagonalising the action.

The D0 brane picture

Following the outlined strategy, the gauge theory massless spectrum is recovered from the following considerations:

(a) Fock space

The ground state energy in the NS-sector is $-\frac{1}{2}$, since all open string oscillators are half-integrally moded. Massless NS-states thus correspond to the lowest excitations of the fermionic fields. On the other hand, massless Ramond ground states represent the space-time and transverse Clifford algebras as usual. Table 3.9 concisely summarises the low-energy spectrum.

Sector	Fields	$SO(1,5) \times SU(2) \times SU(2)$
NS	ψ^μ	$(6,1,1)$
	$\psi^1, \bar{\psi}^{\bar{2}}$	$(1,1,2)$
	$\psi^2, \bar{\psi}^{\bar{1}}$	$(1,1,\bar{2})$
R	$\mathbf{16}_s$	$(4,2,1) \oplus (4,1,2)$

Table 3.9: The massless open string spectrum in ADE orbifolds with G embedded via $\mathcal{Q}$ in the second SU(2) factor.

(b) Chan-Paton factor

With a regular brane, there is an N-dimensional Chan-Paton vector space, decomposing into N one-dimensional G simple modules $R_I : g \leftarrow \omega^I$ (with ω a primitive root of unity and g a generator of G). As such, the $N \times N$ Chan-Paton factor falls apart into 1×1 blocks and under the regular action the (I,J)-entry gets multiplied by $R_I^\dagger(g) \times R_J(g)$.

The massless spectrum is conveniently encoded in the $N \times N$ matrix below (in $d = 6$ type IIB):

$$\begin{pmatrix} V & H & 0 & & \dots & H \\ H & V & H & 0 & \dots & 0 \\ 0 & H & V & H & & \\ & & \dots & & & \\ & & & H & V & H \\ H & 0 & \dots & & H & V \end{pmatrix} \tag{3.3.4}$$

where V (H) are $d = 6, N = 1$ vector multiplets ((half-)hypermultiplets) In summary, the spectrum is that of half-maximally supersymmetric $U(1)^N$ SYM with bifundamental matter. Note that the D-brane(s) break another half of the one-half supersymmetry left unbroken by the background.

As to type IIA, the D4-branes come with the $d = 5, N = 2$ massless spectrum found upon dimensional reduction of Eq. (3.3.4), yielding a real massless scalar a. In five dimensions, the off-shell completion of the $N = 2$ vectormultiplet consists of an $SU(2)_{\mathcal{R}}$ triplet of auxiliary scalars, yielding a bosonic content (A_μ, a, D_i). As such, it is natural to guess that the WZ-term linear in A_μ, has an $N = 2$ supersymmetric completion:

$$\sum_k \int C_{3,k} \wedge \mathrm{T}r\left[\gamma(g^k) dA + A \wedge A\right] + \int \phi_k \cdot \mathrm{Tr}\left[\gamma(g^k) D\right] \ , \qquad (3.3.5)$$

where the second term contains the 'completion'. This follows from the observation that $C_{3,k}$ is the Ramond three-form potential Poincaré-dual to a scalar, and the fact that the latter scalar together with the triplet metric moduli ϕ_i fill out the bosons in an $N = 2$ linear hypermultiplet. The bulk-to-gauge-theory couplings involving ϕ_k are Fayet–Iliopoulos terms and are in one-to-one correspondence with the central traceless generators of the gauge group [67].

Similarly, in the dimensionally reduced IIB theory, the D3-brane $d = 4, N = 2$ completion of the WZ-term leads to:

$$\sum_k \int C_{2,k} \wedge \mathrm{Tr}\left[\gamma(g^k)(dA + A \wedge A)\right] + \int \phi_k \cdot \mathrm{Tr}\left[\gamma(g^k) D\right] \ , \qquad (3.3.6)$$

where likewise, $C_{2,k}$ is dualised into a scalar; together with the metric moduli the latter comprises the bosonic content of a linear hypermultiplet.

Besides the above couplings, the effective low-energy dynamics now follows from the combined dimensional reduction + orbifold projection of the ten-dimensional $U(N)$ gauge theory, supplemented with terms involving D_Λ (SU(2) indices are suppressed, henceforth) . More explicitly, the terms relevant for the HM scalars are

$$\mathcal{L} = \mathcal{L}_{kin} + \sum_\Lambda \int D_\Lambda \cdot (D_\Lambda + \mu_\Lambda + \phi_\Lambda) \qquad (3.3.7)$$

where Λ runs over the gauge group and $\phi_\Lambda \equiv 0$ for non-central generators. Further, the hypermultiplet scalars $(z_{1,\alpha}, z_2{}^\alpha)$, transforming in $(R, R^\dagger)$ of the gauge group are conveniently organised into "quaternions":

$$X_\alpha = \begin{pmatrix} z_{1,\alpha} & \bar{z}_2^\alpha \\ -z_2{}^\alpha & z_{1,\alpha} \end{pmatrix} \ ; \qquad (3.3.8)$$

from X_α, one defines the $\mathrm{SU}(2)_{\mathcal{R}}$-triplet of sesquilinears $\mu_\Lambda = \mathrm{tr}(\sigma X^\dagger T_\Lambda X)$, where T_Λ is the representation matrix of the gauge algebra. The second term in Eq. (3.3.7) yields the potential energy after integrating out the auxiliary D_Λ.

Let us finally come to the moduli space of vacua, and let me for simplicity reduce the IIB theory further to 1 + 0 dimensions. That is, we shall be using (effective) D0 branes as probes of the geometry. The class of classical vacua consists of constant field configurations, such that extremising the action corresponds to finding minima of the potential. As such, the field equations become effectively,

$$\mu_\Lambda = \langle \phi_\Lambda \rangle \ . \tag{3.3.9}$$

However, the solutions present an overcomplete description of the actual moduli space, in view of the gauge-equivalence present: the actual moduli space is a further reduction to the space of gauge-equivalent *orbits* of solutions.

The simplest examples, or ADE revisited

To gain some confidence, let us deal with the $\mathbb{C}^2/\mathbb{Z}_N$ orbifolds very explicitly. From Eq. (3.3.4) the gauge group is $U(1)^N$, with bifundamental hypermultiplets:

$$\mathcal{Z}_1 = \begin{pmatrix} 0 & z_{1,1} & & & \\ & 0 & z_{1,2} & & \\ & & \ddots & \ddots & \\ & & & 0 & z_{1,N-1} \\ z_{1,N} & & & & 0 \end{pmatrix} ; \tag{3.3.10}$$

and likewise $\mathcal{Z}_2^T = \mathcal{Z}_1[z_1 \leftrightarrow z_2]$. Next,

(a) constant *finite* gauge transformations can be used to set all phases equal: $\arg(z_{1,i}) = \theta_1$;

(b) With Fayet–Iliopoulos terms set to zero, the equations of motion Eq. (3.3.9) read

$$[Z_1, Z_2] = 0 \ ; \ [Z_1, Z_1^\dagger] + [Z_2, Z_2^\dagger] = 0 \ , \tag{3.3.11}$$

and impose further that $\arg({z_2}^i) = \theta_2, |z_{1,i}| = r_1, |{z_2}^i| = r_2$. As such, setting $u_i := r_i e^{\mathrm{i}\theta_i}$, the moduli space appears to be $\mathbb{C}^2$. However, the gauge-fixing leaves a *residual* $\mathbb{Z}_N$ symmetry that identifies $(u_1, u_2) \approx (e^{2\pi\mathrm{i}/N}u_1, e^{-2\pi\mathrm{i}/N}u_2)$. Therefore, the moduli space is rather $\mathbb{C}^2/\mathbb{Z}_N$.

In this limit, the D-brane moduli space indeed recovers the singular target space as the moduli space of the gauge-theory. Once more, space-time shows up as a secondary issue. Intuitively, it should be rather clear that turning on non-zero FI-terms will smooth out the singularity.

Also, for $\mathbb{Z}_{2,3,4,6}$ this type of explicit analysis also allows for discrete translation groups besides the discrete rotations (see Ref. [67] for an implicit description). Similarly, the D-brane moduli space is then found to be $T^4/\mathbb{Z}_N$, as expected.

As a final comment, it must be noted that a non-trivial solution only exists, provided (a number of copies of) the regular representation D-brane is put on the orbifold space. Otherwise, the Higgs-branch does not open up. In physical terms, this means that the branes are stuck at the orbifold point, hence cannot move away. The latter feature will be in agreement with charge quantisation, as discussed in Section 5.2.3.

3.3.3 Beyond the simplest orbifolds: tools

From the rather explicit exposition of the moduli spaces, it should have become clear that the procedure outlined is tedious and lengthy, if not impossible, for more complicated orbifolds. Therefore, additional tools and technology are more than welcome. Lack of space forces us to pick two prominent examples, thereby leaving out symplectic quotients (toric geometry) and further techniques from algebraic geometry (coherent sheaves) that have appeared to be valuable in the recent literature [68].

A. Hyperkähler quotients

As to orbifolds of $\mathbb{C}^2$ already, the explicit treatment above becomes undoable as soon as the point group gets non-abelian. In fact, it remains an open problem to do even the simplest $\mathbb{D}_2$ along those lines[11]. Schematically, the moduli space was found as follows:

(a) Begin with *free* massless hypermultiplets, and couple those to Yang–Mills theory through Eq. (3.3.7). This way, a potential is generated upon integrating out the auxiliaries.

(b) Assume the gauge group $\mathcal{G}$ to sit inside $\mathrm{SU}(n)$, for some n.

(c) The moduli space is then

$$\mathcal{M}_{\langle\phi\rangle} = \frac{\{\text{solutions of } \mu = \langle\phi\rangle\}}{\mathcal{G}\text{-equivalence}} \,. \tag{3.3.12}$$

In fact, this is a physical realisation of a mathematical construct, known as a *hyperkähler quotient*. This procedure yields an implicit description of $\mathcal{M}_{\langle\phi\rangle}$, irrespective of the orbifold group being abelian or not. From the construction, non-singularity of the quotient space can be argued, provided $\langle\phi\rangle \neq 0$ [39].

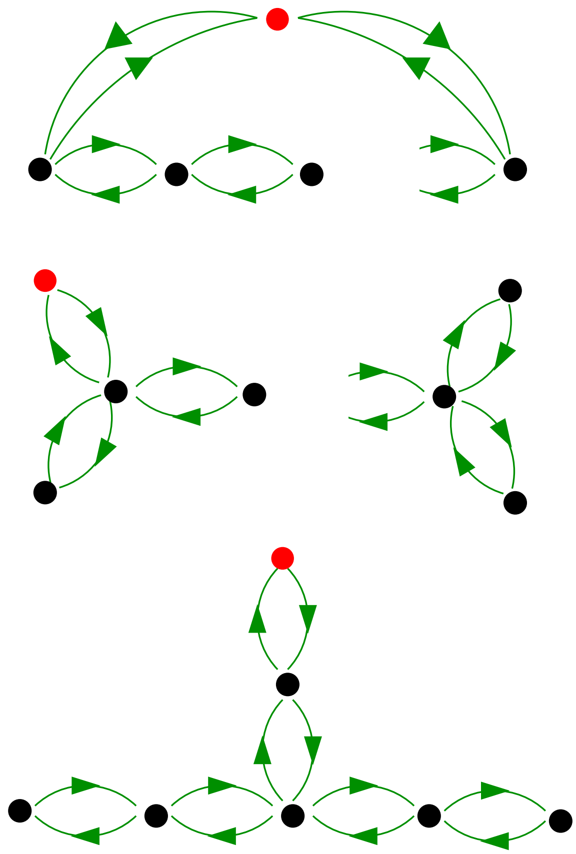


Figure 3.8: Three ADE quivers: A_n (top), $\mathbb{D}_n$ (middle) and $\mathbb{E}_6$ (bottom).

B. Quivers and what they can do for you

The data describing a quiver consist of the following:

(a) A collection of dots (vertices) $\{V_a\}$, with correspondingly a vector of positive integers (n_a).

(b) A collection of arrows (oriented edges) I_{ab} connecting a dot a with a dot b, and a vector of positive integers (m_{ab});

All of this has a concise graphical representation, where dots a, b are connected with $m_{ab}(m_{ba})$ arrows $a \to b (a \leftarrow b)$.

[11]Quite likely, this is closely related to the fact that the A_n-series ALE metrics are explicitly known, whereas the $D_n, E_{6,7,8}$ are not.

Quiver representations, then, are any set of objects (corresponding to the vertices) and relations or homomorphisms (arrows) corresponding to the quiver. For now, the quiver representations that will become most relevant are the so-called

Quiver gauge theories
Quiver gauge theories are such that the gauge group is $\mathcal{G} = \prod_a U(n_a)$, with matter transforming in the bifundamental $(\bar{n}_a, n_b)$, $\forall (a,b) \leftrightarrow I_{ab}$. Pictorially, this is the following assignment:

$$\bigcirc \Longleftrightarrow \begin{matrix}\text{(GAUGE GROUPS)}\\ \text{vector multiplets}\end{matrix} \qquad \longrightarrow \Longleftrightarrow \begin{matrix}\text{(MATTER)}\\ \text{chiral multiplets}\end{matrix}$$

Loosely speaking, one may interpret the quiver diagram quite physically, by taking the $\bigcirc$ for branes and the $\longrightarrow$ for (massless) strings stretching between them.

Of all possible quiver diagrams, a subset presents a concise graphical representation of the matter content of orbifold gauge theories. The $\mathbb{C}^2/G$ quivers are listed in Fig. 3.8. Inspection teaches that they closely resemble affine Dynkin diagrams. To be more precise, it is the corresponding unoriented quiver, obtained by replacing pairs of oppositely oriented edges by lines, that coincides with the affine diagram. In fact, this feature is related to McKay correspondence (see Chapter 5).

4

Edges

In this chapter, we review the systematic approach to boundary states. The recurrent theme throughout will be that boundary conditions are strongly organised by the symmetries they preserve. In the particular case of conformal-symmetry-preserving boundary conditions, the associated boundary states will even turn out to be purely encoded in CFT data.

Basically, the boundary state issue involves an answer to the twofold question:

(a) Classify all (sets of) boundary conditions.

(b) Construct the corresponding boundary states and obtain thus all solutions implementing the conditions.

Section 4.1 illustrates question (a) through the example of the $N = 4$ superconformal algebra, while the remainder of the chapter is devoted to (b). The complete classification of boundary conditions for general CFTs is an unsolved problem, as far as we are aware. Therefore, Section 4.1 serves illustrative purposes solely by means of flat-space D-branes. The main point to be made is that the $N = 4$ worldsheet SCA is the organising principle in the classification of boundary conditions in this specific setting.

Boundary states in rational CFTs (RCFTs) are the subject of Section 4.2. Cardy's original derivation [69] of the open-closed consistency constraint on boundary states, Eq. (4.2.16), and his construction of a generating set of solutions, Eq. (4.2.19), are reviewed.

Next, as a warm-up, a detailed account of Cardy's construction of flat-space D-brane boundary states in type II theories is given in Section 4.3. Particular attention is paid to the construction of the fermion components and the global normalisation of the boundary states: in contrast to the well-studied bosonic piece, both issues do not seem to have been studied from the Cardy point of view before Ref. [1] appeared.

Finally, Section 4.4 contains the results of Ref. [1], which constitute the main novelty of the presented material. In short, Cardy's construction is appropriately generalised so as to yield orbifold D-brane boundary states, the so-called BPS fractional D-brane states.

The setting for the sequel will be as follows : in the 2d worldsheet theory one has a symmetry algebra $\mathcal{A}_L \times \mathcal{A}_R$ of left- and right-moving symmetries, consisting of two copies of $\mathcal{A}$. The global automorphism group of $\mathcal{A}$ is denoted by $Aut(\mathcal{A})$.

The unravelling of the systematics of boundary conditions comprises at least two components:

(a) Which are the boundary conditions preserving a given subalgebra $\mathcal{B}$ of the diagonal $\mathcal{A}_\Delta$?

(b) Which are mutually consistent sets of boundary conditions ?

By 'diagonal', we shall mean an algebra that may be identified with the first factor, while it is identified with the second factor only up to some automorphism $\Omega \in Aut(\mathcal{A})$.

4.1 Step up: boundary conditions in SCFTs

Take $\mathcal{A}$ to be the $N = 4$ SCA. The preserved diagonal subalgebra $\mathcal{B}$ of $\mathcal{A}_L \times \Omega(\mathcal{A}_R)$ will generically contain some fraction of preserved worldsheet supersymmetry out of the total $N_L + N_R = 4 + 4$ present. Besides $\mathcal{B}$, additional operators ('extensions') of the SCA often play a distinguished rôle too. In particular, since spectral flow operators are in one-to-one correspondence with space-time supersymmetries, any question addressing the latter necessarily involves the former [70, 71]. In the case of some non-zero fraction of unbroken space-time SUSY, the resulting states are called BPS.

First thing to note here, is that $Aut(\mathcal{A}) = \mathrm{SO}(4) \approx \mathrm{SU}(2)_{int} \times \mathrm{SU}(2)_{out}$, consisting of an internal (local) and an outer SU(2) group. The four supercurrents $G^a_\pm$ transform as $(\mathbf{2}, \mathbf{2})$, and the $N = 1$ supercurrent is identified as $G^0_+ + G^0_-$. Since the corresponding $N = 1$ SCA is gauged its generators should be form-invariant:

$$T_L = T_R \tag{4.1.1}$$

$$G^0_{+,L} + G^0_{-,L} \quad = \quad \pm(G^0_{+,R} + G^0_{-,R}) \tag{4.1.2}$$

while the remaining generators are allowed to get mixed. As a result, only the diagonal $SU(2) \subset SO(4)$, i.e., the $SU(2)$ subgroup under which the $N = 1$ supercurrent is a singlet, may be used to glue left- and right-moving symmetry currents. The remaining currents form a **3** representation of this $SU(2)$. Choosing a basis G, G^I adapted to this decomposition, we are effectively left with an $SO(3)$ choice of automorphisms encoded in the matrix $R^I{}_J$:

$$G_L = \pm G_R \; ; \qquad G^I_L = \pm R^I{}_J G^J_R \; , \tag{4.1.3}$$

if we wish to preserve some diagonal $N = 4$ SCA. Likewise, the affine $SU(2)$ currents must then be related by

$$J^I{}_L = R^I{}_K J^K{}_R \; ; \tag{4.1.4}$$

Select a particular current J^0, say. In bosonised form, $J_0 = \mathrm{i}\partial\phi$. The operation $\phi \to \phi + \theta$ leaves J_0 manifestly untouched, yielding an eigenvalue 1 in Eq. (4.1.4). For $R \in SO(3)$ the remaining two eigenvalues have to be such that their product equals 1 if some diagonal $N = 4$ SCA is to be preserved. That this is the case, follows straightaway, since $J_\pm = \mathrm{e}^{\pm \mathrm{i}a\phi}$. In fact, shifting ϕ by a constant is an automorphism of the full $N = 4$ SCA.

As to the extensions (spectral flows) of the $N = 4$ SCA, they require a separate analysis [70]. Also, it is possible, in principle, to extend the foregoing analysis to $\mathcal{B} \subsetneq \mathcal{A}$. In string theory, boundary conditions of this type are expected to be realised by superpositions of D-branes at best, rather than single D-brane configurations. Since single constituent branes are the objects of primary interest, we refrain from a further analysis here (but see e.g., Ref. [70]).

Flat space and $N = 4$

Instead, we now turn to an explicit model by picking up the example of flat space free bosons and fermions. To make life easy, set $\mathbb{R}^{1,9} \approx \mathbb{R}^{1,5} \times \mathbb{C}^2$ and neglect the first factor for a while. From the worldsheet $N = 1$ superfields $\hat{X}^\mu = X^\mu + \theta\psi^\mu$ ($\mu = 1, \ldots, 4$), define new worldsheet fields that take values in the target space tangent bundle $T\mathbb{C}^2$:

$$\mathcal{X} \quad \equiv \quad X^\mu \otimes \partial_\mu \; ; \tag{4.1.5}$$

$$\Psi \quad \equiv \quad \psi^\mu \otimes \partial_\mu \; . \tag{4.1.6}$$

The latter make a compact notation for the various symmetry generators in the $N = 4$ SCA possible [72]:

$$T \quad = \quad G(\partial\mathcal{X}, \partial\mathcal{X}) + G(\Psi, \partial\Psi) \; ; \tag{4.1.7}$$

$$G^i = \begin{cases} G(\Psi,\partial\mathcal{X}) + \mathrm{i}K^3(\Psi,\partial\mathcal{X}) \\ K^1(\Psi,\partial\mathcal{X}) + \mathrm{i}K^2(\Psi,\partial\mathcal{X}) \end{cases} \tag{4.1.8}$$

$$\bar{G}^{\bar{\imath}} = \begin{cases} G(\Psi,\partial\mathcal{X}) - \mathrm{i}K^3(\Psi,\partial\mathcal{X}) \\ K^1(\Psi,\partial\mathcal{X}) - \mathrm{i}K^2(\Psi,\partial\mathcal{X}) \end{cases} \tag{4.1.9}$$

$$J^a = K^a(\Psi,\Psi) \tag{4.1.10}$$

where G, K^a are the relevant metric and complex structures on the target. Furthermore, the spectral flows,

$$\psi^1\psi^2 \approx \mathcal{U}^{\pm}_{-1} = e^{\mathrm{i}\sqrt{2}\phi} \; ; \tag{4.1.11}$$

$$\psi^1\bar{\psi}^2 \approx \mathcal{V}^{\pm}_{-1} = e^{\mathrm{i}\sqrt{2}\chi} \; , \tag{4.1.12}$$

and their conjugates are 4 when counted chirally, yielding a total of 8. The canonically normalised ϕ in Eq. (4.1.11) is such that $J^0 = \mathrm{i}\sqrt{2}\partial\phi$.

Instead of the real basis of Eq. (4.1.5), let us choose a complex coordinate system: $\hat{Z}^i = \hat{X}^{2i-1} + \mathrm{i}\hat{X}^{2i}$. Through Eq. (4.1.7)-Eq. (4.1.12), boundary conditions on the symmetry generators and spectral flows are induced from those imposed on the elementary superfields $\hat{Z}^i$. The choice

$$\hat{Z}^1_L = \mathrm{e}^{\mathrm{i}\alpha}\hat{Z}^1_R \; , \hat{Z}^2_L = \mathrm{e}^{\mathrm{i}\beta}\hat{Z}^2_R \; , \tag{4.1.13}$$

leads to $\phi_L = \phi_R + (\alpha+\beta)/\sqrt{2}$. As such, the matrix R in Eq. (4.1.4) is diagonal

$$J_L = \begin{pmatrix} \mathrm{e}^{\mathrm{i}(\alpha+\beta)} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \mathrm{e}^{-\mathrm{i}(\alpha+\beta)} \end{pmatrix} J_R \; , \tag{4.1.14}$$

and likewise for the supercurrent triplet. In all, each pair (α,β) identifies a particular preserved diagonal $N = 4$ SCA.

How about the preserved space-time supersymmetries ? Since the left- and right-moving spectral flows are related by four linear conditions, only half of them are preserved at the boundary. As such, the corresponding D-brane is said to be $\frac{1}{2}$-BPS.

It is an obvious fact that the analysis of boundary conditions is useful beyond conformal field theory applications. As an example, one may wish to inquire properties of boundary states in gauged linear sigma models [73], where two-dimensional worldsheet supersymmetry, rather than superconformal symmetry, is the crucial preserved symmetry. Since it lies somewhat outside the scope, we shall not pursue this here.

4.2 And beyond : Cardy's approach to boundary states

In this section, Cardy's construction of consistent boundary states is reviewed. Basically, such states are constructed from knowledge of the modular S-matrix, following a canonical recipe. The whole procedure can be traced back to implementing open-closed consistency in the CFT on the cylinder. The original derivation in [69], applicable to rational CFTs, is closely followed. In subsequent sections, the procedure will be demonstrated to be generalisable to specific instances of non-rational CFTs as well.

4.2.1 The safe ground : rational CFT

Rational CFT is a realisation of holomorphic and anti-holomorphic symmetry algebras $\mathcal{A}_L \times \mathcal{A}_R$, both containing the Virasoro algebra, with only a *finite* number of primaries. In the case of diagonal modular invariants, the resulting closed string Hilbert space decomposes as

$$\mathcal{H} = \bigoplus_i [\phi_i] \otimes [\tilde{\phi}_i] \tag{4.2.1}$$

The Virasoro characters in the corresponding irreducible modules are defined by

$$\chi_i(q) = \mathrm{tr}_i q^{L_0 - \frac{c}{24}} \ , \tag{4.2.2}$$

where $q = e^{2\pi i\tau}$, with τ the modular parameter. I shall only consider purely imaginary τ.

As already outlined in Section 4.1 a diagonal subalgebra $\mathcal{B}$ of $\mathcal{A}_L \times \Omega(\mathcal{A}_R)$ is picked. The boundary states to be constructed should preserve this $\mathcal{B}$. As such the choice of $\mathcal{A}$ will largely determine what kind of D-branes (open strings, boundary states) one wishes to keep in the theory.

At world-sheet boundaries the generators of the holomorphic and the anti-holomorphic embeddings of $\mathcal{A}$ are related through gluing conditions. This is rephrased alternatively : if $\mathcal{B}$ is identified as a subalgebra of $\mathcal{A}_L$, it need only be so in $\mathcal{A}_R$ up to automorphisms of the latter.

Not to make things messier than strictly needed, the preserved diagonal symmetry algebra $\mathcal{A}$ will be assumed to be isomorphic to the algebras $\mathcal{A}_L$ and $\mathcal{A}_R$ Furthermore, a diagonal modular invariant will be assumed, i.e.,

$$Z(q,\bar{q}) = \sum_j \chi_j(q) \left(\chi_j(q)\right)^* \ . \tag{4.2.3}$$

Consider the cylinder amplitude with boundary conditions labeled by α and β. In the loop channel, one has

$$Z_{\alpha\beta} = \sum_i n^i_{\alpha\beta}\chi_i(q) \ , \tag{4.2.4}$$

where the integers $n^i_{\alpha\beta}$ denote the multiplicities of the representations i running in the loop. Alternatively, this same amplitude can be viewed as a closed string tree level amplitude. It then reads

$$Z_{\alpha\beta} = \langle\alpha|\tilde{q}^{\frac{1}{2}(L_0+\tilde{L}_0-\frac{c}{12})}|\beta\rangle \ , \tag{4.2.5}$$

where $\tilde{q} = e^{-2\pi \mathrm{i}/\tau}$. Eq. (4.2.5) is the closed-string exchange between D-branes α, β, denoted symbolically by the corresponding boundary states. These boundary states $\langle\alpha|$ and $|\beta\rangle$ are assumed to impose the boundary condition

$$\left(W^{(r)}_n - (-)^s\Omega(\tilde{W}^{(r)}_{-n})\right) = 0 \tag{4.2.6}$$

on the Fourier modes of the symmetry generators; that is,

$$\left(W^{(r)}_n - (-)^s\Omega(\tilde{W}^{(r)}_{-n})\right)|\beta\rangle = 0 \ , \tag{4.2.7}$$

where s is the spin of $W^{(r)}$ and $\Omega \in Aut(\mathcal{A}_R)$[1] of the preserved algebra $\mathcal{A}$. As an example, take $W^{(r)}$ to be J^+ of Section 4.1. Since the spin $s = 1$, the boundary condition in Eq. (4.2.7) becomes in terms of Fourier-modes

$$J^+_n + e^{\mathrm{i}(\alpha+\beta)}\tilde{J}^+_{-n} = 0 \ , \ n \in \mathbb{Z} \ , \tag{4.2.8}$$

where Ω acts as multiplication by $\exp(\mathrm{i}(\alpha+\beta))$ as in the previous section. Invoking simplicity of the exposition once more, Ω will be assumed to be the trivial automorphism.

The way to proceed from here is to choose a convenient basis of solutions to Eq. (4.2.7), the so-called *Ishibashi states* [74]. The *consistent boundary states* are then built as particular linear combinations of such states. Consider a highest weight module of $[\phi_j]$ of $\mathcal{A}_L$ and the corresponding (isomorphic) module $[\tilde{\phi}_j]$ of $\mathcal{A}_R$. The states of $[\phi_j]$ are linear combinations of states of the form $\prod_I W^{(r_i)}_{-n_I}|j;0\rangle$, where the $W^{(r_i)}_{-n_I}$ are lowering operators and $|j;0\rangle$ is the highest weight state. Denote the elements of an orthonormal basis of the module $[\phi_j]$ by $|j;N\rangle$ and the corresponding basis of $[\tilde{\phi}_j]$ by $\widetilde{|j;N\rangle}$. In terms of the anti-unitary operator U defined by

$$U\widetilde{|j;0\rangle} = \widetilde{|j;0\rangle}^* \ ; \ U\tilde{W}^{(r_I)}_{-n_I}U^{-1} = (-)^{s_{r_I}}\tilde{W}^{(r_I)}_{-n_I} \ , \tag{4.2.9}$$

[1] For instance, in the case of a free boson, choosing Ω the trivial automorphism corresponds to Neumann boundary conditions, while a non-trivial one gives Dirichlet conditions (see Section 4.3 below).

the states

$$|j\rangle\!\rangle \equiv \sum_N |j;N\rangle \otimes U\widetilde{|j;N\rangle} \qquad (4.2.10)$$

clearly solve Eq. (4.2.7) (for $\Omega = 1$). Now these are the Ishibashi states.

4.2.2 Cardy's condition

With the Ansatz that consistent boundary states are in the linear span of the space of Ishibashi states of Eq. (4.2.10):

$$|\alpha\rangle = \sum_j B_\alpha^j |j\rangle\!\rangle \;, \qquad (4.2.11)$$

Eq. (4.2.5) takes the form

$$Z_{\alpha\beta} = \sum_j (B_\alpha^j)^* B_\beta^j \chi_j(\tilde{q}) \;. \qquad (4.2.12)$$

Here,

$$\chi_j(\tilde{q}) = \langle\!\langle j|\tilde{q}^{\frac{1}{2}(L_0+\tilde{L}_0-\frac{c}{12})}|j\rangle\!\rangle = \mathrm{tr}_j \tilde{q}^{L_0-\frac{c}{24}} \;. \qquad (4.2.13)$$

is consistent with Eq. (4.2.2). In deriving Eq. (4.2.12), we have used that the Ishibashi states are orthogonal, in the sense that

$$\langle\!\langle j'|\tilde{q}^{\frac{1}{2}(L_0+\tilde{L}_0-\frac{c}{12})}|j\rangle\!\rangle = 0 \text{ if } j \neq j' \;. \qquad (4.2.14)$$

Eq. (4.2.4) is transformed to the tree channel by the modular S transformation:

$$Z_{\alpha\beta} = \sum_{i,j} n_{\alpha\beta}^i S_i^j \chi_j(\tilde{q}) \;. \qquad (4.2.15)$$

Recall that the $n_{\alpha\beta}^i$ count multiplicities of the characters running in the loop, hence $n_{\alpha\beta}^i \in \mathbb{N}$. To impose open-closed consistency of the cylinder amplitude amounts to demanding equality of Eq. (4.2.12) and Eq. (4.2.15). Eventually, this yields the key equation

$$\sum_i S_i^j n_{\alpha\beta}^i = (B_\alpha^j)^* B_\beta^j \;, \qquad (4.2.16)$$

at least, if no two representations of the holomorphic algebra have the same Virasoro character. Eq. (4.2.16) is commonly referred to as *Cardy's equation*. The requirement that the multiplicities $n_{\alpha\beta}^i$ be nonnegative integer numbers is a strong condition on the coefficients B_α^j. Furthermore, this constraint is nonlinear: multiplying a consistent boundary state by a noninteger number will generically not yield a consistent boundary state. The task of finding mutually consistent sets $\{B_\alpha^j\}$ defining consistent boundary states (D-branes) is taken up in the next subsection.

4.2.3 Cardy's solution

In [69], Cardy gave a set of solutions to his equation Eq. (4.2.16). In a first step, one consistent boundary state, $|\mathbf{0}\rangle$, is singled out by the requirement that $n^i_{00} = \delta^i_0$ in Eq. (4.2.15); that is, the only representation running in the loop channel is the identity representation. From Eq. (4.2.16), such a state necessarily satisfies

$$|B^j_0|^2 = S^j_0 \ . \tag{4.2.17}$$

The entries S^j_0 of the modular transformation matrix are positive [69], so Eq. (4.2.17) is consistent. It implies

$$|\mathbf{0}\rangle = \sum_j \sqrt{S^j_0}\,|j\rangle\!\rangle \tag{4.2.18}$$

(up to the relative phases of the coefficients, which are not fixed by these considerations).

Additional boundary states $|\mathbf{l}\rangle$ are built next, with the distinguishing property that $n^i_{0l} = \delta^i_l$, for every primary ϕ_l. Using Eq. (4.2.16), Eq. (4.2.17) and the fact that $S^j_0 > 0$, these states are found to be

$$|\mathbf{l}\rangle = \sum_j \frac{S^j_l}{\sqrt{S^j_0}}\,|j\rangle\!\rangle \ . \tag{4.2.19}$$

In Cardy's solution, the multiplicities $n^i_{\alpha\beta}$ coincide with the fusion rule coefficients of the algebra $\mathcal{A}$. That the boundary states Eq. (4.2.19) solve Eq. (4.2.16) is then verified as a consequence of Verlinde's formula Eq. (2.2.11) [16, 69].

At this point, let us look back and summarise what has been found thus far:

FACT 4.1

A set of consistent boundary states is in the linear span of the Ishibashi states, which in turn provide a basis of solutions to the gluing conditions.

FACT 4.2

The particular coefficients B^i_α come for free with rational CFTs; more particularly, they are encoded in the modular S-matrix and the mutual consistency is an immediate consequence of the Verlinde formula.

These will remain the cornerstones in the generalisation of Cardy's construction to a class of CFTs that are not strictly rational.

It remains an important open question whether the consistent (Cardy) states comprise an integral basis of the space of *all* consistent boundary states.

4.3 Boundary states for flat space D-branes

Cardy's construction is clearly valid for arbitrary rational CFTs. Nevertheless, it is instructive to see how it applies to free bosons and fermions. Strictly speaking, the free bosons do not generically constitute rational CFTs, except in specific cases such as bosons on rational tori. As it will come out, the validity of the prescription persists; in hindsight, this should not come as a surprise since free bosons are in a sense the simplest CFTs beyond rational models. The discussion borrows largely from Ref. [14, 75] for the bosons, while the fermions were treated in Ref. [1].

4.3.1 Boson boundary state

Generic features

Consider strings moving in a Minkowski background. The corresponding CFT is a tensor-product of single boson CFTs and Cardy's construction factorises accordingly. As such, we can do with the discussion of a single boson X first. Also, to take the safe road, X is assumed to be compactified on a circle of radius R : $X(\sigma + 2\pi) = X(\sigma) + 2\pi R n$ for some $n \in \mathbb{Z}$. The chiral symmetry algebra $\mathcal{A}_L$ contains the unit operator, and for generic values of R it is generated by the chiral energy-momentum tensor T and the $U(1)$ symmetry current $J_L = \partial X$ (and likewise for the anti-chiral counterpart). Further, $Aut(\mathcal{A}_L) = \mathbb{Z}_2$, where the non-trivial element acts as $J \to -J$.

In the holomorphic and anti-holomorphic sectors, the Fourier modes α_n and $\tilde{\alpha}_n$ are defined by

$$J_L(z) = -\mathrm{i}\sqrt{\frac{\alpha'}{2}}\sum_{m=-\infty}^{\infty} \alpha_m z^{-m-1}\ ;\ \ J_R(\bar{z}) = -\mathrm{i}\sqrt{\frac{\alpha'}{2}}\sum_{m=-\infty}^{\infty} \tilde{\alpha}_m \bar{z}^{-m-1} \qquad (4.3.1)$$

and they obey the algebra $[\alpha_m, \alpha_n] = [\tilde{\alpha}_m, \tilde{\alpha}_n] = m\,\delta_{m+n}$

On a generic circle, this CFT has an infinite number of highest weight states $|(k, w)\rangle$ $(k, w \in \mathbb{Z})$. These have no oscillators excited and are defined by

$$\hat{p}|(k,w)\rangle = \frac{k}{R}|(k,w)\rangle ; \quad (4.3.2)$$
$$\hat{w}|(k,w)\rangle = w|(k,w)\rangle , \quad (4.3.3)$$

with $\hat{p} = (\alpha_0 + \tilde{\alpha}_0)/\sqrt{2\alpha'}$, and $\hat{w} = \sqrt{\frac{\alpha'}{2}}(\alpha_0 - \tilde{\alpha}_0)/R$.

Automorphisms Ω of $\mathcal{A}_R$, impose the following gluing conditions at world-sheet boundaries (see Eq. (4.2.7)):

$$(\alpha_n + \Omega(\tilde{\alpha}_{-n}))\,|i\rangle\!\rangle_\Omega = 0 \,. \quad (4.3.4)$$

Further, on $|i\rangle\!\rangle_\Omega$ the left-moving and right-moving closed string Virasoro operators get identified: $(L_0^c - \tilde{L}_0^c)|i\rangle\!\rangle_\Omega = 0$, irrespective of Ω.

Neumann boundary states

With $\Omega = \mathrm{Id}_{\mathcal{A}}$, the boundary conditions Eq. (4.3.4) are recognised as Neumann boundary conditions, whereas Dirichlet conditions are realised by $\Omega(J_R) = -J_R$. Generalised Ishibashi states for Dirichlet or Neumann gluing conditions will be denoted by $|i\rangle\!\rangle_D, |i\rangle\!\rangle_N$, respectively.

With Neumann boundary conditions, the Ishibashi states are

$$|(0,w)\rangle\!\rangle_N = \exp\left(-\sum_{n=1}^{\infty}\frac{1}{n}\alpha_{-n}\tilde{\alpha}_{-n}\right)\,|(0,w)\rangle \quad (4.3.5)$$

with corresponding Ishibashi characters as in Eq. (4.2.13)

$$\chi_{N,w}(\tilde{q}) = \frac{\tilde{q}^{\frac{R^2w^2}{4\alpha'}}}{\eta(\tilde{q})} \,. \quad (4.3.6)$$

The Ishibashi states of Eq. (4.3.5) satisfy the orthogonality condition Eq. (4.2.14). Notice that the highest weight representations $[\phi_j]$ of the chiral preserved chiral algebra are labeled by the winding number $j = (0,w)$ here. The orthonormal set of basis vectors $|j;N\rangle$ becomes $|(0,w);\{m_n\}\rangle$ in this concrete setting, where m_n denote the α_{-n} oscillator numbers.

With Neumann boundary conditions along the compact X direction, open strings are allowed to carry non-zero momentum along that direction, such that the corresponding open string character $\chi(q)_N = \mathrm{Tr}_N(q^{L_0^o})$ involves a sum over discrete momenta. Moreover, one can turn on a Wilson line $A_X = \frac{\theta}{2\pi R}$ with $\theta \in [0,2\pi)$; this is equivalent to shifting the open string momenta: $\frac{n}{R} \to \frac{n}{R} - \frac{\theta}{2\pi R}$. Thus, the generic open string character $\chi(q)_{N,\theta}$ reads

$$\begin{aligned}\chi(q)_{N,\theta} &= \sum_{n\in\mathbb{Z}} \frac{q^{(\frac{n}{R}-\frac{\theta}{2\pi R})^2}}{\eta(q)} \\ &= \frac{R}{\sqrt{2\alpha'}} \sum_{w\in\mathbb{Z}} \mathrm{e}^{\mathrm{i}\theta w} \frac{\tilde{q}^{\frac{R^2 w^2}{4\alpha'}}}{\eta(\tilde{q})} = \frac{R}{\sqrt{2\alpha'}} \sum_{w\in\mathbb{Z}} \mathrm{e}^{\mathrm{i}\theta w}\, \chi_{N,w}(\tilde{q}) \;,\end{aligned} \tag{4.3.7}$$

where in the second line a Poisson resummation was performed. Correspondingly, the consistent boundary states are

$$|\theta\rangle_N = \left(\frac{R}{\sqrt{2\alpha'}}\right)^{1/2} \sum_{w\in\mathbb{Z}} e^{\mathrm{i}\theta w} |(0,w)\rangle\!\rangle_N \;. \tag{4.3.8}$$

These states take the form of Eq. (4.2.19), indeed, with $S_\theta^w = \frac{R}{\sqrt{2\alpha'}} e^{\mathrm{i}\theta w}$. The rôle of the distinguished Cardy's state $|\mathbf{0}\rangle$ is clearly played by the state describing a D-brane with no Wilson line, $|\mathbf{0}\rangle_N$.

Dirichlet boundary states

Let me turn to Dirichlet boundary conditions next. As said, this amounts to taking $\Omega = -\mathbb{1}$ on J_R. This is recognised as an implementation of T-duality (which is a one-sided parity transformation by definition). Instead of Eq. (4.3.5), the generalised Ishibashi states now read

$$|(k,0)\rangle\!\rangle_D = \exp\left(\sum_{n=1}^{\infty} \frac{1}{n} \alpha_{-n}\tilde{\alpha}_{-n}\right) \; |(k,0)\rangle \;, \tag{4.3.9}$$

and the corresponding Ishibashi characters Eq. (4.2.13) become

$$\chi_{D,k} = \frac{\tilde{q}^{\frac{\alpha' k^2}{4R^2}}}{\eta(\tilde{q})} \;. \tag{4.3.10}$$

Dirichlet boundary conditions at both ends set the open string momenta along X to zero but allow non-zero windings, which must accordingly be summed over in the character. Open strings stretching between two D-branes that are separated by a distance Δx, result in a character $\mathrm{Tr}_D(q^{L_0^o})$

$$\begin{aligned}\chi(q)_{D,\Delta x} &= \frac{q^{\frac{(2\pi R w+\Delta x)^2}{4\pi^2\alpha'}}}{\eta(q)} \\ &= \frac{\sqrt{\alpha'}}{\sqrt{2}R} \sum_{k\in\mathbb{Z}} \mathrm{e}^{\mathrm{i}\Delta x \frac{k}{R}} \frac{\tilde{q}^{\frac{\alpha' k^2}{4R^2}}}{\eta(\tilde{q})} = \frac{\sqrt{\alpha'}}{\sqrt{2}R} \sum_{k\in\mathbb{Z}} \mathrm{e}^{\mathrm{i}\Delta x \frac{k}{R}} \chi_{D,k}(\tilde{q}) \;,\end{aligned} \tag{4.3.11}$$

where in the second line we went to the tree channel once more by Poisson resummation.

Following Cardy, the consistent Dirichlet boundary states are then

$$|x\rangle_D = \left(\frac{\sqrt{\alpha'}}{\sqrt{2}R}\right)^{1/2} \sum_{k\in\mathbb{Z}} \mathrm{e}^{\mathrm{i}x\frac{k}{R}} |(k,0)\rangle\!\rangle_D \ , \tag{4.3.12}$$

and the distinguished state $|\mathbf{0}\rangle$ is nothing but $|x=0\rangle_D$.

The close analogy between Eq. (4.3.8) and Eq. (4.3.12) reflects the fact that turning on Wilson lines and shifting positions of branes are T-dual operations.

Let us conclude here with a number of remarks:

(a) The above discussion was restricted to the CFT of one free boson to keep matters simple. Already here, it is nice to find that the ratio of the coefficients in front of $|\theta=0\rangle_N$ and $|x=0\rangle_D$ is $(2\pi R)/(2\pi\sqrt{\alpha'})$, since it confirms [76] that the relative tensions differ by $\frac{1}{2\pi\sqrt{\alpha'}}$ [77].

(b) Regarding the states in Eq. (4.3.8) and Eq. (4.3.12) as constituents, the generalisation to the multi-boson CFT is obvious and consists of taking tensor products of the appropriate constituents, where an independent choice of Neumann or Dirichlet condition is left for each boson separately. Moreover, with n_D (n_N) Dirichlet (Neumann) boundary conditions chosen, it is clear that the $\prod_i \mathbb{Z}_2$ automorphism group of $\mathcal{A}_R$ gets enhanced to the space-time rotation group $\mathrm{SO}(1,n_N-1)\times\mathrm{SO}(n_D)$. Since this extension is trivial, we won't pursue it further.

(c) In the decompactification limit $R\to\infty$, the states $|(k,0)\rangle\rangle$ appearing in Eq. (4.3.9) are normalised to have unit-norm. As such, they differ from the states $e^{\mathrm{i}k\hat{x}}|0\rangle$ that are usually considered in string theory by a factor of $1/\sqrt{2\pi R}$. When the manifest $1/\sqrt{R}$ factor in Eq. (4.3.12) is properly taken into account, the R-dependence will be just right to turn the discrete momentum sum in Eq. (4.3.12) into a momentum integral. This integral is the delta function in position space: it localises the brane in the transverse directions (see, for instance, Refs [13, 78]).

4.3.2 Fermion boundary state

Let us next focus on the world-sheet fermions. Like in Section 2.2.4 fermions will be dealt with in a covariant formalism, thus combining them with the superghost-system. It was further explained there that the covariant chiral characters coincide with their light-cone $\mathrm{SO}(8)_1$ counterparts, apart from the subtlety that in fusion rules etc. the rôles of o and v are exchanged, and the

s, c characters pick up a minus sign. In short, the Ishibashi and Cardy *states* to be constructed below will be so in the covariant framework, even though the characters obtained from them appear only SO(8) (light-cone)-like.

Let us first consider type 0B theory where the bulk partition function is diagonal:

$$\mathcal{Z}_{0B}(q) \propto \left(|\chi_o(q)|^2 + |\chi_v(q)|^2 + |\chi_s(q)|^2 + |\chi_c(q)|^2\right) , \tag{4.3.13}$$

The contribution of the bosonic string fields completes this expression to the full type 0B torus partition function, but this is suppressed temporarily.

Given this bulk CFT, boundaries may be introduced in string world-sheets: attention will be focused on D9-branes, preserving the full Lorentz, whence SO(8) invariance in light-cone gauge. The Dp-branes with $p < 9$ are obtained by T-duality.

As to boundary conditions, in both the NS-NS and R-R sectors, two gluing conditions are possible:

$$\psi_r = \mathrm{i}\eta\tilde{\psi}_{-r} , \tag{4.3.14}$$

where $\eta = \pm 1$. The Ishibashi states solving these conditions as in Eq. (4.2.10) are

$$|\sigma;\eta\rangle\!\rangle = \prod_{\mu} \exp\left[\mathrm{i}\eta \sum_{r>0} \psi^{\mu}_{-r}\tilde{\psi}^{\mu}_{-r}\right] |\sigma,\eta;0\rangle\!\rangle , \tag{4.3.15}$$

with σ = NS, R indicating the NS-NS or R-R sector and μ running over the directions transverse to the light-cone. In the R-R (NS-NS) sector, the mode numbers r are (half-)integer. Also, there is a non-trivial zero-mode part in the R-R sector:

$$|\mathrm{R},\eta;0\rangle\!\rangle = \mathcal{M}^{(\eta)}_{AB}\,|A\rangle|\tilde{B}\rangle \ , \tag{4.3.16}$$

where

$$\mathcal{M}^{(\eta)} = C\Gamma^0\Gamma^{l_1}\ldots\Gamma^{l_p}\left(\frac{1+\mathrm{i}\eta\Gamma_{11}}{1+\mathrm{i}\eta}\right) \ ; \tag{4.3.17}$$

C is the charge conjugation matrix and l_i label the space directions along the D-brane world-volume. The vacuum states $|A\rangle|\tilde{B}\rangle$ for the fermionic zero-modes ψ^{μ}_0 and $\tilde{\psi}^{\mu}_0$ transform in the 32-dimensional Majorana representation.

The states in Eq. (4.3.15) fail to meet the condition in Eq. (4.2.13), a situation which is remedied by taking linear combinations:

$$|v\rangle\!\rangle = \frac{1}{2}\left(|\mathrm{NS},+\rangle\!\rangle - |\mathrm{NS},-\rangle\!\rangle\right) \ ; \tag{4.3.18}$$

$$|o\rangle\!\rangle = \frac{1}{2}\left(|\mathrm{NS},+\rangle\!\rangle + |\mathrm{NS},-\rangle\!\rangle\right) \ ; \tag{4.3.19}$$

$$|s\rangle\!\rangle = \frac{1}{2}\left(|\mathrm{R},+\rangle\!\rangle + |\mathrm{R},-\rangle\!\rangle\right) \ ; \tag{4.3.20}$$

$$|c\rangle\!\rangle = \frac{1}{2}\left(|\mathrm{R},+\rangle\!\rangle - |\mathrm{R},-\rangle\!\rangle\right) \; . \tag{4.3.21}$$

Not only are they mutually orthogonal, they survive the type 0B GSO projection $\frac{1}{2}(1+(-)^{F+\tilde{F}})$. The above labeling follows the general convention of Eq. (4.2.13); the corresponding chiral blocks may indeed be verified to be:

$$\langle\!\langle m|\tilde{q}^{\frac{1}{2}(L_0+\tilde{L}_0-\frac{c}{12})}|n\rangle\!\rangle = \delta_{mn}\,\chi_m(\tilde{q}) \; , \tag{4.3.22}$$

with $m,n=o,v,s,c$. Thus we are done with the Ishibashi states. From these Ishibashi states consistent boundary states $|a\rangle$, with $a=v,o,s,c$ are derived using Cardy's prescription. After all, the fermion CFT is rational, so there is no reason to doubt the validity of the procedure.

Taking the covariant versus light-cone subtleties into account, it follows from the S matrix of Section 2.2.4 that consistent boundary states are

$$\begin{aligned}
|v\rangle &= \frac{1}{\sqrt{2}}\sum_m |m\rangle\!\rangle = \frac{1}{\sqrt{2}}(|v\rangle\!\rangle + |o\rangle\!\rangle + |s\rangle\!\rangle + |c\rangle\!\rangle) \\
&= \frac{1}{\sqrt{2}}(|NS,+\rangle\!\rangle + |R,+\rangle\!\rangle) \; , \\
|o\rangle &= \sqrt{2}\sum_m (S_{(8)})_v^m\, |m\rangle\!\rangle = \frac{1}{\sqrt{2}}(|v\rangle\!\rangle + |o\rangle\!\rangle - |s\rangle\!\rangle - |c\rangle\!\rangle) \\
&= \frac{1}{\sqrt{2}}(|NS,+\rangle\!\rangle - |R,+\rangle\!\rangle) \; , \\
|s\rangle &= \sqrt{2}\sum_m (S_{(8)})_s^m |m\rangle\!\rangle = \frac{1}{\sqrt{2}}(|v\rangle\!\rangle - |o\rangle\!\rangle + |s\rangle\!\rangle - |c\rangle\!\rangle) \\
&= \frac{1}{\sqrt{2}}(-|NS,-\rangle\!\rangle + |R,-\rangle\!\rangle) \; , \\
|c\rangle &= \sqrt{2}\sum_m (S_{(8)})_s^m |m\rangle\!\rangle = \frac{1}{\sqrt{2}}(|v\rangle\!\rangle - |o\rangle\!\rangle - |s\rangle\!\rangle + |c\rangle\!\rangle) \\
&= \frac{1}{\sqrt{2}}(-|NS,-\rangle\!\rangle - |R,-\rangle\!\rangle) \; .
\end{aligned} \tag{4.3.23}$$

These states are the type 0B boundary states that may be found in the literature [79, 80, 81]. The states $|v\rangle$ and $|o\rangle$ are commonly referred to as electric D9-brane and anti-D9-brane, respectively, while $|s\rangle$ and $|c\rangle$ are called magnetic D9-brane and anti-D9-brane.

Next, move on to supersymmetric type IIB theory, with a one-loop partition function

$$\mathcal{Z}_{\mathrm{IIB}}(q) \propto |\chi_v(q) + \chi_s(q)|^2 \; . \tag{4.3.24}$$

This is an example where Cardy's prescription cannot be implemented straightaway. Let me first proceed with a modified construction and postpone comments temporarily.

The open string characters running in the loop channel can be organised into the column vector $\hat{\chi}_A$ ($A = 0, 1, 2, 3$) where

$$\hat{\chi}_A(q) \equiv (\chi_v + \chi_s, \chi_o + \chi_c, \chi_v - \chi_s, \chi_o - \chi_c) \ , \tag{4.3.25}$$

while the tree-channel $\hat{\chi}_M(\tilde{q}) = (\chi_v, \chi_s, \chi_o, \chi_c)$. The net result of the base-change operation is that the modular S matrix, defined by $\hat{\chi}_A(q) = \hat{S}_A^M \hat{\chi}_M(\tilde{q})$, takes the form

$$\hat{S} = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & -1 \end{pmatrix} . \tag{4.3.26}$$

Following Cardy's formula for the B_α^i coefficients, the consistent states are built from the newly defined $\hat{S}$:

$$|\hat{A}\rangle = \frac{\hat{S}_A^M}{\sqrt{\hat{S}_0^M}} |\hat{M}\rangle\!\rangle \ . \tag{4.3.27}$$

Only the Ishibashi components $|v\rangle\!\rangle, |s\rangle\!\rangle$ in Eq. (4.3.18) and Eq. (4.3.20) survive the closed-string GSO-projection. Equivalently, only the upper-left block in $\hat{S}$ yields properly projected closed-string states. With this restriction, the first row in $\hat{S}$ identifies $\chi_v + \chi_s$ as the character playing the rôle of the identity in Cardy's setup.

$$\begin{aligned} |v - s\rangle &= \sum_{\hat{M}} |\hat{M}\rangle\!\rangle = |v\rangle\!\rangle + |s\rangle\!\rangle \\ &= \frac{1}{2}(|NS,+\rangle\!\rangle - |NS,-\rangle\!\rangle + |R,+\rangle\!\rangle + |R,-\rangle\!\rangle) \ , \\ |o - c\rangle &= \sum_{\hat{M}} \hat{S}_1^{\hat{M}} |\hat{M}\rangle\!\rangle = |v\rangle\!\rangle - |s\rangle\!\rangle \\ &= \frac{1}{2}(|NS,+\rangle\!\rangle - |NS,-\rangle\!\rangle - |R,+\rangle\!\rangle - |R,-\rangle\!\rangle) \ , \end{aligned} \tag{4.3.28}$$

where the labeling is as in Section 4.2.3 In particular, these yield

$$Z_{v-s,v-s} = Z_{o-c,o-c} = \chi_v - \chi_s \ ; \tag{4.3.29}$$

$$Z_{v-s,o-c} = Z_{o-c,v-s} = \chi_o - \chi_c \ , \tag{4.3.30}$$

which are the $D9$-$D9$ and $D9$-$\bar{D}9$ amplitudes, indeed. Consistently, $|v - s\rangle$ and $|o - c\rangle$ are observed to coincide with the common expressions for IIB D9 and anti-D9 boundary states that had been derived on other grounds already in [82].

This Cardy-like derivation of consistent type II theory boundary states does not seem to display the same degree of rigour as that in the original derivation. Observe, though, that this situation could have been expected on the following grounds. If string theory is believed to contain a certain physical truth, it must not come as a complete surprise that Cardy's prescription, intrinsic to CFT, has to be supplemented by *additional*, extrinsic physical considerations at some point. In other words, the initial recipe produces D-brane states that are perfectly valid from the CFT point of view, but are discarded on physical grounds. For example:

(a) States with the *opposite* GSO-projection could not possibly couple to the graviton. Therefore, such objects would not have a proper notion of energy-density, whence mass; this seems to be viable enough a reason to discard those.

(b) Had such states been kept in the theory nevertheless, there would be open string channels where space-time fermion characters contribute with the same sign as the bosons, thus violating space-time spin-statistics.

In the foregoing analysis, the mild modification to project onto proper closed-string-GSO states is observed to kill the potentially unphysical states.

Thus far, only parallel D-branes of like dimensions have been the issue. Whenever more general configurations involving D-branes of different dimensions are considered, mutual consistency must be checked independently. Generically, this will impose additional constraints. For example, one learns that in type IIA theory with BPS Dp branes for even p, only non-BPS Dq-branes with odd q can be added consistently (see, for instance, [83]).

4.3.3 String theory : bosonic zero-modes and global normalisation

As they stand, the Cardy boundary states are consistent as far as the CFT is concerned. However, for them to become meaningful in a string theory context, two more pieces of input are required:

(a) Modular integration
Since string theory involves *families* of CFTs, rather than the single CFTs considered so far, one must revise open-closed consistency. The proper string theory loop channel quantity becomes

$$\mathcal{Z}_{1,2} = V_{p+1} \int_0^{i\infty} \frac{d\tau}{\tau} Z_{1,2}^{(d)}(q) \ , \tag{4.3.31}$$

where the modulus τ parametrises inequivalent worldsheet cylinders. In the tree channel, the modular integral produces a closed-string propagator:

$$\langle B_1 | \frac{\alpha'}{4\pi} \int_{|z|<1} \frac{d^2 z}{|z|^2} z^{L_0 - a} \bar{z}^{\tilde{L}_0 - \tilde{a}} |B_2\rangle \ . \tag{4.3.32}$$

Open-closed consistency now requires that the quantities in Eq. (4.3.31) and Eq. (4.3.32) match. Note that the canonical propagator in Eq. (4.3.32) involves a normalisation that Cardy's prescription could not possibly tell, whence the need for this extra input.

(b) Global normalisation
Although not manifest, the amplitudes in Eq. (4.3.31) and Eq. (4.3.32) involve integrals over momenta (bosonic zero-modes). These will shortly be shown to result in additional normalisation factors.

Let us deal with point (b) first. In the open-string channel non-trivial momenta are present for Neumann–Neumann boundary conditions, thus yielding a total contribution

$$\left[\int_{-\infty}^{\infty} \frac{dk}{2\pi} e^{\frac{1}{2}\alpha' \pi \tau k^2} \right]^{p+1} = \left(\frac{\sqrt{2}}{\pi \sqrt{\alpha'}} \right)^{p+1} \times \ldots \tag{4.3.33}$$

for pp-strings. The ellipsis denotes the appropriate inverse power of τ, which is most easily verified to play no rôle in this story. On the closed-string side, however, transverse (Dirichlet) momentum integrals are implicit in the construction of the boundary state (see Eq. (4.3.11)). In d space-time dimensions, we therefore find

$$\begin{aligned} & N_p^2 \times \frac{\alpha'}{4\pi} \times \left[\int_{-\infty}^{\infty} \frac{dk}{2\pi} e^{2\alpha' \pi \tilde{\tau} k^2} \right]^{d-(p+1)} \\ = \ & N_p^2 \times \frac{(2\pi\sqrt{\alpha'})^2}{8\pi} \times \left(\frac{1}{2\pi\sqrt{2\alpha'}} \right)^{d-(p+1)} \times \ldots \end{aligned} \tag{4.3.34}$$

where N_p is the Dp-brane boundary state normalisation to be determined, and the second factor results from the propagator. Further, in the last line there was an insertion of the Jacobian factor $2\pi^2$ obtained upon change of variables from $z = e^{-\pi\tilde{\tau}} \to \tilde{\tau}$ in Eq. (4.3.32).

Collecting factors and equating both sides, it is found that

$$N_p = \frac{\sqrt{\pi}}{2} 2^{\frac{10-d}{4}} (2\pi\sqrt{\alpha'})^{\frac{d-4}{2} - p} \ . \tag{4.3.35}$$

This *intrinsic* normalisation fixes the physical tension T_p/κ of the Dp-brane. Here, κ is the gravitational coupling constant. From the NS-NS graviton-dilaton exchange in the long-wavelength limit of the closed-string amplitude

[77], the tension becomes $T_p = 2N_p$. Similarly, measuring the BPS R-R charge of the type-II branes w.r.t. a canonically normalised $(p+1)$-form A_{p+1}, leads one to

$$\mu_p = 2\sqrt{2}N_p = \sqrt{2\pi}(2\pi\sqrt{\alpha'})^{3-p} \tag{4.3.36}$$

4.4 Cardy states in geometric orbifolds

Cardy's construction can be generalised further so as to produce consistent boundary states for branes on orbifolds. In the present section, the focus will be on geometric orbifolds that are moreover complex. The route followed will not be different from the one that ought to be familiar by now: from the Ishibashi[74] basis to consistent Cardy states via the modular S-matrix.

In a first approach, the discussion is restricted to branes that are pointlike in the orbifold, while extended along the transverse directions. Branes of this type were the subject of Section 3.3 in the open-string viewpoint. Stated otherwise, Neumann boundary conditions will be taken in the transverse spacetime, Dirichlet conditions along the orbifold. Besides the untwisted closed string sector there are now twisted sectors. Accordingly, there exist Ishibashi states solving these gluing conditions in each of the sectors. For future reference, the corresponding Ishibashi states in a sector twisted by some element g will be denoted symbolically as $|g\rangle\rangle$.

Loosely speaking, the story is similar to the one told before and it falls apart in three pieces: a purely group-theoretical one, a bosonic and a fermionic part.

4.4.1 Ishibashi states

Consider the worldsheet bosons first. The Ishibashi components in the g-twisted sector are fairly similar to the one in the untwisted sector:

$$|g\rangle\rangle = \prod_{l=1}^{n} \exp\left[\sum_{\kappa_l} \frac{\bar{\alpha}^l_{-\kappa_l}\tilde{\alpha}^l_{-\kappa_l}}{\kappa_l} + \sum_{\bar{\kappa}_l} \frac{\alpha^l_{-\bar{\kappa}_l}\tilde{\bar{\alpha}}^l_{-\bar{\kappa}_l}}{\bar{\kappa}_l}\right] |0,g\rangle \ , \tag{4.4.1}$$

where modings are now shifted: $\kappa_l \in \mathbb{Z} + \nu_l$, while $\bar{\kappa}_l \in \mathbb{Z} - \nu_l$; $\exp(2\pi i\nu_l)$ is the eigenvalue of $g \in C_g \equiv [g]$ on the complex field X^l. Further, $|0,g\rangle$ is the twisted sector ground state. Note here that the untwisted Ishibashi state is the only component that can emit closed strings carrying momentum in any of the orbifold directions,

In case the orbifold group G is nonabelian, the Ishibashi states $|g\rangle\rangle$ are not G-invariant. Rather, they mix with Ishibashi states in sectors twisted by conjugate group elements. The invariant Ishibashi state associated to a conjugacy class C^α is in general

$$|\alpha\rangle\!\rangle = \frac{1}{\sqrt{n_\alpha}} \sum_{g^{(\alpha)} \in C^\alpha} |g^{(\alpha)}\rangle\!\rangle \ . \tag{4.4.2}$$

Such states $|\alpha\rangle\!\rangle$ are orthogonal, in the sense that they satisfy:

$$\langle\!\langle \alpha | \tilde{q}^{\frac{1}{2}(L_0 + \bar{L}_0 - \frac{c}{12})} | \beta \rangle\!\rangle = \delta_{\alpha\beta}\, \chi_\alpha(\tilde{q}) \ . \tag{4.4.3}$$

As to the fermions, the story is similar. For a single twisted complex fermion the Ishibashi state counterpart of expression Eq. (4.4.1) is found to be

$$|g;\sigma;\eta\rangle\!\rangle = \prod_{l=1}^{n} \exp\left[\mathrm{i}\eta \sum_{r>0} \psi^l_{-r} \tilde{\bar{\psi}}^l_{-r} + \bar{\psi}^l_{-r'} \tilde{\psi}^l_{-r'} \right] |g;\sigma,\eta;0\rangle\!\rangle \ , \tag{4.4.4}$$

with additional $\mathrm{SO}(2n)_1$ labels $\sigma = \mathrm{NS}, \mathrm{R}$ and $\eta = \pm$ as in the flat space case of Section 4.3.1. They solve the respective overlap conditions that read in oscillators:

$$\psi^l_{r_l} = \mathrm{i}\eta\, \tilde{\psi}^l_{-r_l} \ , \tag{4.4.5}$$

where $l = 1, \ldots n$. Again, modings r_l, r'_l are appropriately shifted and depend on the order of $g^{(\alpha)}$ and the R or NS sector ($r_l \in \mathbb{Z} + \nu_{\alpha,l}, r'_l \in \mathbb{Z} - \nu_{\alpha,l}$ in the R sector, and an extra shift of 1/2 in the NS sector).

A proper set of Ishibashi states involves first forming the counterparts $|\alpha, m\rangle\!\rangle$ ($m = v, o, s, c$) of the flat space case in each twisted sector. Next, a G-invariant orthogonal basis of Ishibashi states is obtained by summing twisted Ishibashi states within each conjugacy class, as in Eq. (4.4.2).

4.4.2 S-matrix and group theory factors

The next ingredient involves the modular S-matrix. Recall some facts from Section 3.3 that will prove relevant for the present discussion:

(a) The open string Fock space decomposes under $Vir \times G$ as:

$$\mathcal{H}^{osc} = \bigoplus_K \left(\sum_{\alpha_K} [\phi_{\alpha_K}] \otimes R_K \right) \ . \tag{4.4.6}$$

Moreover, the open string oscillators do not acquire shifted modings, unlike the closed strings.

(b) Through the G-action on Chan-Paton factors, boundary conditions get naturally labeled by irreducible G-representations R_I; open strings with boundary conditions I, J at the respective endpoints thus form

$$\mathcal{H}_{IJ} = \mathcal{P}_G \left(\mathcal{H}^{osc}_{(IJ)} \otimes R_I^{\vee} \otimes R_J \right) \ , \tag{4.4.7}$$

where $\mathcal{P}_G$ projects onto G-invariants. The full open-string Hilbert space is a direct sum of modules $\mathcal{H}_{IJ}$.

In terms of the orbifold chiral blocks $Z(g,h)$ of Section 3.2 (see also Appendix B) item (a) implies that only the $Z(g,e)$ occur in open string traces. When combined with (b), the net outcome is that

$$\mathcal{Z}_{IJ} = \mathrm{Tr}_{\mathcal{H}_{IJ}} (\mathcal{P}_G \, q^{L_0 - \frac{c}{24}}) \tag{4.4.8}$$

$$= \frac{1}{|G|} \sum_g \rho^I(g^{-1}) \rho^J(g) \; Z(g,e) \ , \tag{4.4.9}$$

with ρ_I the character of R_I.

In the chiral orbifold CFT, the natural modular matrix that appears is $S^{h,g'}_{g,h'}$, where

$$\chi^h_g(q) = \sum_{g',h'} S^{h,g'}_{g,h'} \chi^{h'}_{g'}(\tilde{q}) \ . \tag{4.4.10}$$

For ease of notation, set $S^{e,h}_{g,e} = \sigma(g,h)$ as in Section 3.2.

Similarly to the discussion of type-II boundary states there, a change of basis in the space of open string traces seems to be forced upon us by Eq. (4.4.8). The modular transform from the tree to the loop channel is accordingly modified:

$$\chi_I(q) = \sum_\alpha S^\alpha_I \, \chi_\alpha(\tilde{q}) \ , \tag{4.4.11}$$

where a shorthand notation $\chi_\alpha(\tilde{q}) = \chi^e_{g^{(\alpha)}}(\tilde{q})$, $g^{(\alpha)} \in C^\alpha$ is introduced. The change of basis is reflected in

$$S^\alpha_I = \frac{n_\alpha \, \rho^\alpha_I}{|G|} \, \sigma(e, g^{(\alpha)}) \ . \tag{4.4.12}$$

Observe that Eq. (4.4.12) is nothing but a discrete Fourier transform.

In summary, we have thus obtained that:

(a) the modified matrix S^α_I has acquired the announced group-theoretical factor, besides the chiral CFT contribution $\sigma(e,g)$ already present.

(b) In contrast with $S^{h,g'}_{g,h'}$, the newly defined matrix S^α_I contains a row with strictly positive only. This is essential in Cardy's procedure since it is precisely (the square-roots of) these that appear in the denominator of $\mathcal{B}^\alpha_I$ expressions.

As in Section 4.2.3, Cardy's construction of consistent boundary states can then be followed literally, provided that this new S is being used. This nicely results in

$$|I\rangle = \sum_\alpha \sqrt{\frac{n_\alpha}{|G|}\,\sigma(e, g^{(\alpha)})}\,\rho_I^\alpha\,|\alpha\rangle\!\rangle\ . \tag{4.4.13}$$

In terms of the boundary states $|I\rangle, |J\rangle$, the tree-channel expression of the open-string one-loop amplitude, Eq. (4.4.8), reads

$$Z_{IJ}(\tilde{q}) = \langle I|\tilde{q}^{\frac{1}{2}(L_0+\bar{L}_0-\frac{c}{12})}|J\rangle\ . \tag{4.4.14}$$

Summarising, up to global normalisations to be dealt with shortly, we have derived the following

---

FACT 4.3

> *The boundary states in Eq. (4.4.13) give rise to the appropriate open string traces Eq. (4.4.14). Moreover, the closed string twisted sector contributions implement open-string orbifold projection after modular transformation. As such, their presence insures G-invariant open string states.*

4.4.3 Normalisations and string theory

Like in flat space, as discussed in Section 4.3.3, bosonic zero-modes largely determine the overall normalisation of the states given in Eq. (4.4.13). Following a similar reasoning as there, the closed-open string duality becomes subtle at worst in a proper account of the orbifold directions. To start with, it is readily seen that Eq. (4.3.35) generalises to the present case

$$N_p^{(\alpha)} = \frac{\sqrt{\pi}}{2}\,2^{\frac{10-d^{(\alpha)}}{4}}\,(2\pi\sqrt{\alpha'})^{\frac{d^{(\alpha)}-4}{2}-p}\ , \tag{4.4.15}$$

where $d^{(0)} = d$ in the untwisted sector, while in twisted sectors[2] $d^{(\alpha\neq 0)} = d - 2n$ for $\mathbb{C}^n$ orbifolds. As an example, the twisted sector charges that are the analogues of Eq. (4.3.36) are

$$\mu_p^{(\alpha)} = 2\sqrt{2}\,2^{-\frac{n}{2}}\,N_p^{(\alpha)} = \sqrt{2\pi}\,(2\pi\sqrt{\alpha'})^{3-n-p}\ . \tag{4.4.16}$$

[2] For simplicity, we are assuming that in the twisted sector zero-modes are absent in any of the orbifold directions, see the discussion after Eq. (4.4.1).

How about the (physical) tensions and charges? The *untwisted* fields emitted by the Dp-brane at the orbifold fixed point propagate in the full bulk, particularly so along the orbifold. To have canonically normalised bulk kinetic terms for those fields in $d-2n$ dimensions, they are defined as $\sqrt{|G|}$ times their canonically normalised counterparts on the covering space $\mathbb{R}^{1,d-1}$. Consequently, the tension and charge normalisations $T_p^{(G)}$ and $\mu_p^{(G)}$ as probed by canonically normalised fields on the orbifold space are related to the flat space expressions, Eq. (4.3.35) and Eq. (4.3.36), through

$$T_p^{(G)} = \frac{T_p}{\sqrt{|G|}} \,, \qquad \mu_p^{(G)} = \frac{\mu_p}{\sqrt{|G|}} \,. \tag{4.4.17}$$

Taking the factor $1/\sqrt{|G|}$ of Eq. (4.4.13) into account, the untwisted tension and charge are thus seen to be *fractional*, i.e. rational non-integer. In contrast, no redefinition of this kind is required for twisted sector fields, since those do not propagate in the orbifold. Further, the twisted charges as they stand superficially fail to be real, let alone integer. An appropriate basis change in field space cures this apparent short-coming, as will be discussed in Section 5.2.3.

5

Bridges

In the present chapter, we shall introduce the celebrated McKay correspondence. This beautiful mathematical truth that relates the geometry of singularity resolutions to representation theory of discrete groups, has been formulated in three versions, partial proofs of which were obtained only relatively recently [84, 85]. As of today, the general correspondence remains conjectural and begs for a better understanding.

In Section 5.1.2 we present some of the mathematics lifted from Ref. [86], hoping to supply a sufficient background to make an outline of the K-theoretic picture. As has been known for a while by now [87], K-theory is the higher-level[1] organising principle of D-branes and their charges. By no means do we wish to build the full mathematical toolshed there, but a minimal amount of technical notions seems unavoidable in order to appreciate the correspondence in its strongest form.

Moreover, the mathematical side of the exposition is motivated by recent developments in the physics field [88], where D-branes build a parallel story. The physical picture of McKay correspondence will thus constitute the second part of the present chapter.

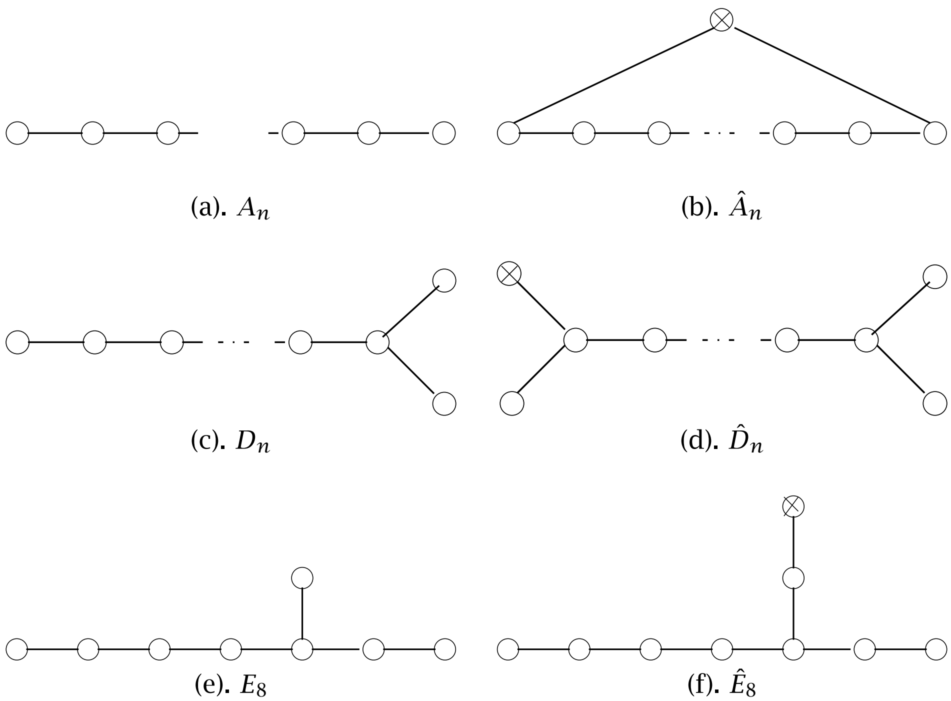


Figure 5.1: Non-extended and affine simply-laced Dynkin diagrams, corresponding to discrete subgroups $\mathbb{Z}_n$ $((a),(b))$, $\mathbb{D}_n$ $((c),(d))$, $\mathbb{O}$ $((e),(f))$ of $\mathrm{SU}(2)$.

5.1 McKay correspondence

5.1.1 The ingredients

In its original form, the McKay correspondence was observed as some curious relation between the geometry of the exceptional set in blow-ups of $\mathbb{C}^2/G$ orbifolds on the one hand, and the representation theory of G on the other hand [89].

Recall from Section 3.1.1 that blow-ups of $\mathbb{C}^2/G$, $G \subset \mathrm{SU}(2)$, produce one of the exceptional sets in the left column of Fig. 5.1. Dynkin diagrams of A_n, D_n, E_n Lie-algebras thus encode the geometrical data. On the other hand, with irreducible representations R_I of G forming a closed set under tensor

[1] Underlying K-theory, derived categories (of coherent sheaves and/or Fukaya categories) have recently been argued to provide a more refined picture, see e.g. [68] and references therein.

product, $Q \otimes [-]$ is an endomorphism of the representation ring $\mathrm{Rep}(G)$:

$$Q \otimes R_I = \bigoplus_J a_{IJ} R_J \ , \tag{5.1.1}$$

in the basis of irreducible representations R_I. A little exercise yields the result that a_{IJ} coincides with the incidence matrix of one of the diagrams in the right column of Fig. 5.1. These are also the affine Dynkin diagrams of the $\hat{A}_n, \hat{D}_n, \hat{E}_n$ affine Lie-algebras. Dropping the ⊗-node that corresponds to the trivial representation, the left-hand side is recovered.

Presently, the most satisfactory understanding of this miraculous correspondence appeals to K-theory. For the SU(2) and SU(3) (abelian) subgroups the mathematical construction was first put forward in Ref. [90] resp. Ref. [91]. An outline will be given shortly.

In the mathematics literature, McKay correspondence shows up in various guises, the common denominator being that each establishes a link between representation-theoretical and geometric data [92, 93, 94, 59]. For our purposes, the following three versions seem to be most relevant:

CONJECTURE 5.1 (WEAK CORRESPONDENCE) *The Euler number $\chi(X)$ of the resolved space equals the number of conjugacy classes (irreducible representations) of G.*

CONJECTURE 5.2 (INTERMEDIATE CORRESPONDENCE) *The exceptional divisors in a blow-up are in one-to-one correspondence with conjugacy classes of age 1.*

CONJECTURE 5.3 (STRONG CORRESPONDENCE) *Equivariant K-theory $K(Y)^G$ on the covering space is dual to compactly supported K-theory on the blown-up quotient space $K^c(X)$.*

Let us place a comment here: some CFT arguments in favour of the former two versions have been the subject of Section 3.2.1 and Section 3.2.3, respectively. Briefly, this means that *closed-string* CFT is teaching us about de Rham cohomology ($\mathrm{H}^*_D R(X)$), or the more refined Dolbeault cohomology ($\mathrm{H}^*_{\bar\partial}(X)$) at best. For one thing, recall that the Euler characteristic computes the index of the de Rham complex. As such, the geometric data involved, the cotangent bundle in essence, is intrinsically given with the manifold X itself. In contrast, D-branes are intrinsically defined with vectorbundles E supported on submanifolds $\mathcal{M} \subset X$. Thus, additional structure besides the cotangent bundle goes into the specification of D-branes. Since K-theory classes are vectorbundles, morally speaking, it is only to be suspected that the Strong Correspondence finds a natural realisation in the D-brane and associated *open-string* picture; the relevant cohomology will turn out to be of twisted Dolbeault type:

(CLOSED-STRING) CFT	D-BRANES
d-cohomology	$\bar{\partial}_E$-twisted cohomology

5.1.2 Outline of the K-theoretic picture

In the present discussion, various rings and groups will enter the stage. They are,

(a) the representation ring $\mathrm{Rep}(G)$ of G, with obvious direct sum and tensor product. Irreducible representations R_I provide a convenient generating set.

(b) the Grothendieck ring $K(X)$ of algebraic vector bundles on X. Elements here are equivalence classes $[E]$ of vectorbundles on the smooth space X, where $E \sim E + F - F' - F''$ whenever these fit into a short exact sequence
$$0 \to F' \to F \to F'' \to 0$$
(see e.g. Ref. [95]).

(c) the Grothendieck group $K^c(X)$ of *complexes* of vector bundles; the complexes are exact outside $\pi^{-1}(0)$, from where the superscript ' ^{c}' ($X \stackrel{\pi}{\to} Y$ is a blow-up).

(d) the group of coherent sheaves $\mathrm{Coh}(\pi^{-1}(0))$ with support on the exceptional set.

Further, two isomorphisms play a key rôle. The first is between $\mathrm{Rep}(G) \approx K(X)$, associating irreducible representations R_I to bundles $\mathcal{R}_I$ (see [86]), termed 'tautological bundles' in Ref. [96]. The fibres of $\mathcal{R}_I$ have the same dimensionality as the space that carries the corresponding representation. Next, there exists an isomorphism between the last two groups in the list, defined as follows:
$$\begin{array}{ccc} K^c(X) & \longrightarrow & \mathrm{Coh}(\pi^{-1}(0)) \\ [E_\bullet] & \mapsto & \sum_{i=0}^{n} (-1)^i [H_i(E_\bullet)] \, , \end{array}$$
where $[-]$ means: 'take the K-theory class'. The inverse mapping takes a sheaf on $\pi^{-1}(0)$ to (the K-class of) its locally free resolution. The isomorphism enables an invariable switch between coherent sheaves and complexes

of vectorbundles. Let me remind you that the result of applying $H_i(-)$ at the i-th term of the resolution complex yields a coherent *sheaf*, and further, that exactness outside $\pi^{-1}(0)$ means that the sheaves are supported on $\pi^{-1}(0)$. In the whole setup, the main technical difficulty resides in proving that sequences like the one in Eq. (5.1.5) below are truly exact outside the exceptional set (see Ref. [91]).

The ring (a) (and through the isomorphism, also (b)) in the list implements representation theory manifestly. On the other hand, rings (c) and (d) have a geometrical nature. So both sides of the correspondence seem to be present already. There is a natural pairing that establishes the desired link:

$$\begin{aligned} \langle -,-\rangle : K(X)\times K^c(X) &\rightarrow \mathbb{Z} \\ ([E],[S]) &\mapsto P_*([E\otimes S])\ ; \end{aligned} \tag{5.1.2}$$

where P is the projection of $\pi^{-1}(0)$ to a point.

In $\mathrm{Rep}(G)$, taking tensor products with Q as in Eq. (5.1.1) induces associated bundles

$$\mathcal{R}_I \rightarrow \mathcal{Q}\otimes\mathcal{R}_I = \bigoplus_J a_{IJ}^{(1)}\mathcal{R}_J . \tag{5.1.3}$$

The notation is such that straight symbols are representations, while calligraphic ones are reserved for the corresponding vector bundles. Repeated application and anti-symmetrisation yield likewise

$$\wedge^m\mathcal{Q}\otimes\mathcal{R}_I = \bigoplus_J a_{IJ}^{(m)}\mathcal{R}_J\ . \tag{5.1.4}$$

Given the $\{a_{IJ}^{(m)}\}$, construct the following sequences:

$$S^I : \mathcal{R}_I^\vee \longrightarrow \bigoplus_J a_{IJ}^{(1)}\mathcal{R}_J^\vee \longrightarrow \bigoplus_J a_{IJ}^{(2)}\mathcal{R}_J^\vee \longrightarrow \ldots \longrightarrow \mathcal{R}_J^\vee\ . \tag{5.1.5}$$

Appropriate differentials B can be demonstrated to exist [91], turning these sequences into *complexes*. After some work, the complexes are also demonstrated to be exact outside $\pi^{-1}(0)$. Correspondingly, they get associated to coherent sheaves with support on the exceptional set.

The lengthy exposition of objects in the above allows a more precise formulation of the correspondence:

Conjecture 5.4 (Strong McKay correspondence)
(a). **Part I**
The sets $\{\mathcal{R}_I\}$, and $\{S^I\}$ base $K(X)$ and $K^c(X)$, respectively. Moreover, they are dual *w.r.t. the pairing Eq. (5.1.2).*

(b). **Part II**
With $\theta : K^c(X) \to K(X)$ the natural homomorphism, an inner product (intersection product) on $K^c(X)$ is defined:

$$\begin{array}{rcl}(-,-): K^c(X)\times K^c(X) & \to & \mathbb{Z} \\ (S^I, S^J) & \mapsto & \langle \theta S^I, S^J\rangle\end{array}$$

Moreover,

$$(S^{I\vee}, S^J) = \sum_{m=0}^{n} (-)^m a_{IJ}^{(m)} \ , \qquad (5.1.6)$$

where $n = \dim_{\mathbb{C}}(X)$.

This, then, is the proposed link between K-theory (sheaves, intersection product) and representation theory (irrepses, $a^{(m)}$). Proofs were obtained for the cases $n = 2$ ([90], see also Ref. [86]), and $n = 3$ abelian [91] and non-abelian orbifold groups [84]; in all cases, the so-called G-Hilbert schemes were singled out as resolution spaces X.

If you wish, (a) is Poincaré-duality in K-theory on non-compact spaces, whence the nomenclature 'intersection pairing'. In particular, the pairing Eq. (5.1.2) is non-degenerate. As to (b), when applied to $SU(2)$ one finds that

$$(S^{I\vee}, S^J) = 2\delta_{IJ} - a_{IJ}^{(1)} \qquad (5.1.7)$$

while for $SU(3)$,

$$(S^{I\vee}, S^J) = a_{IJ}^{(2)} - a_{IJ}^{(1)} \qquad (5.1.8)$$

is obtained.

5.2 Dévissage of the correspondence

5.2.1 D-brane realisation of McKay

After this mathematical detour, let us next try and approach the above correspondence in a more physically-inspired fashion. To this end, we recall that the Chern character is a map from K-theory to rational cohomology:

$$Ch : K(X) \to \mathrm{H}^*(X, \mathbb{Q}) \ . \qquad (5.2.1)$$

In terms of Ch, the Hirzebruch-Riemann-Roch theorem asserts that

$$(E, F) = \int_X Ch(E^\vee \otimes F)\ Td(X). \qquad (5.2.2)$$

This number is the index of the $\bar{\partial}_{E-F}$-operator acting in the twisted complex $\Omega^{0,q}(X) \otimes E^\vee \otimes F$. In other words, it counts the number of holomorphic maps. The Todd class $Td(X)$ coincides with the $\hat{A}$-roof genus for $c_1 = 0$ manifolds [88]. Then, Eq. (5.2.2) becomes the index of the twisted Dirac operator $\not{D}_{E-F}$ acting in the twisted spin-complex. In plain language, it counts the net number of chiral fermions coupled to gauge bundles $E^\vee \otimes F$. Note that the above holds for smooth bundles (K-theory) on smooth manifolds X. When applied to $\mathcal{R}_I \in K(X)$ the latter classes (bundles) are naturally associated to D$2n$-branes, that is, branes wrapping the whole of X.

The $a_{IJ}^{(m)}$ emerge quite naturally in the smooth (large-volume) Calabi-Yau phase. From the decomposition of the spin-bundle, $\mathrm{Spin}^+ \approx \bigoplus_q \Omega^{(0,q)}(X)$, one finds that

$$\mathrm{Spin}^+ \approx \bigoplus_q \Lambda^q \mathcal{Q} \ . \tag{5.2.3}$$

From this, the twisted Dirac index readily follows:

$$\mathrm{Tr}_{IJ}(\Gamma) = \sum_{q=0}^{n} (-)^q a_{IJ}^{(q)} \ , \tag{5.2.4}$$

where the trace is taken over the massless fermion ground states, and Γ is the analog of the four-dimensional γ_5 for the internal X-space; it causes the crucial $(-)^q$ factor on the rhs of Eq. (5.2.4). In all, in the smooth phase, the twisted Dirac-index is observed to realise Part II in the correspondence.

Further, notice that $\mathcal{I}_{IJ} \equiv (\mathcal{R}_I, \mathcal{R}_J)$, as given by Eq. (5.2.2), is (anti-)symmetric for even (odd) dimensional X when applied to CY-spaces X. This observation led the authors of Ref. [97] to the following conjecture:

CONJECTURE 5.5 (DOUGLAS–FIOL) *On CY-threefolds, $\mathcal{I}_{IJ}$ describes intersections of three-cycles on the mirror Calabi-Yau space.*

If true, this conjecture would establish a direct correspondence between purely geometrical data and group theory.

5.2.2 Massless open strings and the correspondence

Let us next move on to the orbifold limit. There, one sees the natural emergence of fractional D-branes S^I, say, and the associated gauge theories of Section 3.3. Besides those, D$2n$-branes R_I extended along the orbifold will also enter the stage. Of particular relevance are the massless fermions resulting from open $2n - 0$ and $0 - 0$ strings. In the former case, a computation of the Dirac index establishes that $\langle R_I, S^J \rangle = \delta_I^J$, i.e. $\{R_I\}$ and $\{S^I\}$ are 'dual'. On the other hand, the $a_{IJ}^{(m)}$ show up in the Dirac index for $0-0$ strings. Similarly to the discussion above, the key rôle is played by the spin-bundle: spinors on the covering space decompose into G-modules as

$$\Sigma = \bigoplus_q \Lambda^q Q \ . \tag{5.2.5}$$

As such, the net number of chiral fermionic open strings with Chan-Paton spaces S^I and S^J[2], is computed straightaway, with the result

$$\mathrm{Ind}_{\not{D}_{I-J}} = \mathrm{Tr}_{R,IJ}((-)^F) \ ; \tag{5.2.6}$$

$$= \sum_{q=0}^{n} (-)^q a_{IJ}^{(q)} \ . \tag{5.2.7}$$

The operator $(-)^F$ in the first line is the worldsheet counterpart of Γ in Eq. (5.2.4), and the second line is the orbifold D-brane realisation of Eq. (5.1.6).

The next step would then be to continue the results to the smooth phase, whereby R_I get identified with appropriate bundles $\mathcal{R}_I$. Notice that all computed quantities involve chiral objects that cannot disappear from the spectrum. Particularly so, the relation between R_I and S^I is expected to go over unchanged if the blow-up is performed. This makes an identification of the S^I possible in terms of coherent sheaves in large volume.

5.2.3 A final visit to ADE orbifolds: fractional branes as wrapped branes

In the present section the boundary states constructed following Cardy's prescription will be demonstrated to nicely contend with the geometric interpretation of D-branes wrapped around exceptional cycles. It is good to keep in mind that all of this section will take place in the orbifold limit, i.e., all cycles have zero-volume. Recall what we have obtained in the foregoing chapters: on the one hand, there are twisted sector RR-potentials, naturally labelled by non-trivial conjugacy classes; on the other hand, there are boundary states, that are in one-to-one correspondence with irreducible representations. Moreover, evidence from open string considerations suggest that the associated Dp-branes are honest D$(p+2)$-branes wrapped around exceptional cycles of the ALE. The objective here is to present an alternative argument involving closed strings only. In passing, this may be viewed as a consistency check of the boundary state coefficients.

[2] For simplicity of notation, the representation is identified with its carrier vector space.

Let us start with the geometry. There, the intersection form is defined as

$$\mathcal{I}_{ij} :=^{\#} (C_i \cdot C_j) \ , \ [C_i], [C_j] \in H^2(X, \mathbb{Z}) \ . \tag{5.2.8}$$

Alternatively, it is expressed as $\mathcal{I}_{ij} = \int_X \omega_i \wedge \omega_j$, where $[\omega_i]$ is Poincaré-dual to $[C_i] : \int_{C_i} \eta = \int_X \eta \wedge \omega_i$, for all closed two-forms η. Apart from ω_i, it will be convenient to likewise define $\omega^i := (\mathcal{I}^{-1})^{ij}\omega_j$, such that $\int_{C_i} \omega^j = \delta_i^j$. In short, Einstein summation convention will be assumed below, and $\mathcal{I}_{ij}$ will serve as a metric.

Now to the low-energy effective field theory. In a geometric Kaluza–Klein-reduction[3], the RR-$(p+3)$-form has an expansion into harmonic two-forms on X:

$$\hat{A}_{(p+3)} = \mathcal{A}^i_{(p+1)} \wedge \omega_i \ . \tag{5.2.9}$$

Thirdly, the boundary states $|i\rangle$ come into play. Mind the notation: i runs over non-trivial irreducible representations only, as opposed to I. The objective is now to associate $|i\rangle$ to a D-brane $\mathcal{B}_i$, such that:

(a) the associated brane $\mathcal{B}_i$ wraps the cycle C_i ;

(b) the brane has a WZ-coupling to the (canonically normalised) twisted sector $A^{(\alpha)}_{(p+1)}$ as dictated from the boundary state.

Condition (a) is the *assumption* that

$$\int_{D_{(p+2);i}} \hat{A} = \int_{M \times C_i} \hat{A} \ . \tag{5.2.10}$$

In other words, this coupling reduces to $\int_M \mathcal{A}_{(p+1),i}$, where M is transverse to the orbifold.

Condition (b) requires a WZ-coupling $\sum_\alpha \mu_{i\alpha} \int_X A^\alpha_{(p+1)}$. As such, the change of basis relating geometric to twisted-sector fields, assumes the form

$$\mathcal{A}_{(p+1),i} = \sum_\alpha \mu_{i\alpha} A^\alpha_{(p+1)} \ ; \ \mu_{i\alpha} = \sqrt{\frac{n^\alpha}{|G|} \sigma(e, g^{(\alpha)})} \, \rho_i(g^{(\alpha)}) \ , \tag{5.2.11}$$

as follows from Eq. (4.4.13).

[3] More rigorously, we should write the decomposition as

$$\hat{A}_{(p+3)} = p^*_{\mathbb{R}^{5,1}} \mathcal{A}^i_{(p+1)} \wedge p^*_X \omega_i \ ,$$

using the pullbacks induced by projections of $\mathbb{R}^{5,1} \times X$ onto the first and second factors. However, this notation is obviously clumsy and will therefore be suppressed.

It is an elementary exercise to derive that

$$\begin{aligned}\int d\hat{A}\wedge^* d\hat{A} &= \int_M d\bar{\mathcal{A}}_{(p+1),i}\wedge^* d\mathcal{A}^i_{(p+1)}\ ,\\ &= \bar{\mu}_{\alpha i}(\mathcal{I}^{-1})^{ij}\mu_{j\beta}\int_M d\bar{A}^{\alpha}_{(p+1)}\wedge^* dA^{\beta}_{(p+1)}\ .\end{aligned}$$

With equally straightforward manipulations[4], one finds that the coupling matrix in the second line reduces to $\delta_{\alpha\beta}$, provided μ assumes the form given in Eq. (5.2.11). More precisely, starting from a canonical kinetic term for $\hat{A}$ in $d = 10$, the boundary state coefficients conspire to yield a canonical kinetic term likewise for the twisted sector (KK-reduced) RR-fields. That the kinetic coupling be diagonal in such fields is a minimal requirement, since twisted sectors are mutually orthogonal! In summary, the Cardy states survive a first consistency check, given the assumption that they are associated to branes wrapped on exceptional cycles. The non-triviality here resides in the consistent combination of two separate pieces of information: boundary state coefficients, obtained from Cardy, on the one hand, and geometrical data, the intersection numbers of the exceptional set, on the other.

Besides the above, the $|i\rangle$-associated D-brane also couples to the untwisted sector $A_{(p+1)}$ RR-field, as encoded in the WZ-coupling

$$\mu_{p+2}\int_{M\times C_i} A_{(p+1)}\wedge\left[2\pi\alpha'\mathcal{F}+B\right], \tag{5.2.12}$$

where the assumption about the wrapping was plugged in once more. From the fact that the first Chern class, $\int_{C_i}\frac{\mathcal{F}}{2\pi} = k_i \in \mathbb{Z}$, is quantised, Eq. (5.2.12) is rewritten as

$$\mu_{p+2}\int_M A_{(p+1)}\left[4\pi^2\alpha' k_i + B_i\right]\ , \tag{5.2.13}$$

with a B-field $B = \sum_i B_i\omega^i$, i.e. $\int_{C_i}\omega_j = B_j\delta^i_j$. Consistency with the boundary state untwisted sector coefficient is established, provided

$$B_i = \frac{\mu_p}{\mu_{p+2}}\frac{d_i}{|G|}\ ;\qquad n_i = 0\ . \tag{5.2.14}$$

Finally, let us discuss the brane associated to the trivial irrep. Repeating the reasoning above, one is led to conclude that it corresponds to a brane

[4] In fact, this entails showing that

$$\sum_i \bar{\rho}^i(g^{(\alpha)})\rho_i(g^{(\beta)}) = \frac{|G|}{n_\alpha\sigma(e,g^{(\alpha)})}\,\delta_{\alpha\beta}\ .$$

A proof of this property relies on the fact that ρ_I are eigenvectors of the *extended* Cartan matrix, see also Ref. [1].

wrapping $C_0 \equiv -\sum_i d_i C_i$. From here, the untwisted fractional charge is correctly accounted for by setting $n_0 = 1$, i.e., the associated brane carries a non-trivial bundle supported on C_0. Only then, one has

$$B_0 = -\sum_i B_i = \frac{\mu_p}{\mu_{p+2}} \frac{1 - |G|}{|G|} \,, \tag{5.2.15}$$

adding up correctly with the first Chern class contribution to the untwisted charge.

The results obtained here are summarised as follows:

(a) The coefficients obtained from Cardy are not in disagreement with the assumption that the associated states are fractional branes wrapping the exceptional cycles.

(b) With the same assumption, the Cardy states constitute a piece of evidence for quantised B-flux on the vanishing cycles.

(c) The charges of a regular representation D-brane cannot be distinguished from those of a bulk brane: the fractional untwisted charges add up to a unit charge, whereas the twisted sector charges vanish: such a brane can consistently be pulled off the singularity.

6

Discrete torsion

In this chapter, aspects of discrete torsion are discussed. Systems of phases may yield different modular invariant closed-string partition functions. Consistency demands that such sets of phase factors be organised by the group cohomology group $H^2(G; U(1))$. This is the basic content of Section 6.1.

Next, in Section 6.2 this same group is shown to be related to the modified *open* string orbifold projection: the orbifold group is now realised on the Chan–Paton factors via projective representations, as opposed to the vector representations in the no-torsion case. For self-containedness, a short digression on projective representations is added. The BPS D-branes that result from this prescription have been termed 'projective fractional branes' [98].

Section 6.2 then discusses boundary states for these projective fractional branes: they are demonstrated to follow from Cardy's recipe. As a result, the states actually *prove* that the associated branes induce projective Chan–Paton representations. This purely algebraic proof is considerably simpler than the original motivation in Ref. [99].

Finally, the CFT-geometry correspondence needs reviewing in this new light. The example $\mathbb{C}^3/\mathbb{Z}_6 \times \mathbb{Z}_6$ is studied in detail, and serves as an illustration that a modified correspondence persists.

6.1 Closed strings

6.1.1 Modular invariance with phases

Modular invariance alone does not fix the closed string partition function uniquely. The left-over ambiguity involves an assignment $\varepsilon(g, h) \in U(1)$ of phase factors to every pair (g, h) of commuting elements. With those factors, the torus vacuum amplitude becomes a weighted sum:

$$\mathcal{Z} = \sum_{g,h} \varepsilon(g,h) \left| g \underset{h}{\square} \right|^2 , \tag{6.1.1}$$

where a left-right symmetric action of G was assumed. Not any assignment ε will result in a consistent expression, however. This consistency issue was addressed in Ref. [100], yielding the following conditions: $\varepsilon : G \times G \to U(1)$ should be such that

(a) they are well-defined on modular orbits, i.e., orbits of $\mathrm{PSL}(2, \mathbb{Z})$, in the sense that

$$\varepsilon(g,h) = \varepsilon(g^a h^b, g^c h^d) \ , \ \forall \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{PSL}(2,\mathbb{Z}) \ ;$$

this follows from modular invariance on the torus.

(b) They furnish a representation of $N_k \subset G$ below; further, they are antisymmetric, and normalised. In formulas, these requirements boil down to

$$\varepsilon(gh,k) = \varepsilon(g,k)\varepsilon(h,k) \ ; \tag{6.1.2}$$
$$\varepsilon(g,h) = \varepsilon(h,g)^{-1} \ ; \tag{6.1.3}$$
$$\varepsilon(g,g) = 1 \ , \tag{6.1.4}$$

respectively. These conditions are in fact a rewriting of the ones that ensure higher-loop modular invariance and factorisation (see Ref. [100]).

What about solutions of conditions Eq. (6.1.2) - Eq. (6.1.4)? As it turns out, the group cohomology $H^2(G; U(1))$ provides a set of them [100]. The required group cohomology is defined in terms of 2-cochains $\alpha : G \times G \to U(1)$ and 1-cochains $\beta : G \to U(1)$, with coboundary operators[1] (in multiplicative notation)

$$\delta_3 \alpha(g,h,k) := \frac{\alpha(g,hk)\alpha(h,k)}{\alpha(gh,k)\alpha(g,h)} \ ; \tag{6.1.5}$$
$$\delta_2 \beta(g,h) := \frac{\beta(g)\beta(h)}{\beta(gh)} \ . \tag{6.1.6}$$

[1] A note on notation: $\delta_n : C_{n-1}(G; U(1)) \to C_n(G; U(1))$; e.g., $\delta_3\alpha(g,h,k) := (\delta_3\alpha)(g,h,k)$, i.e., the coboundary yields a 3-cochain which is afterwards evaluated in (g,h,k).

It is easily checked that $\delta_3\delta_2 = 1$, such that the δ-cohomology is meaningful:

$$H^2(G; U(1)) \equiv \frac{\ker \delta_3}{\mathrm{im}\, \delta_2} \ . \tag{6.1.7}$$

In terms of a representative α of some class in $H^2(G; U(1))$ the discrete torsion phases are defined by $\varepsilon(g, h) = \alpha(g, h)\alpha(h, g)^{-1}$. Each of the consistency requirements is readily verified to be met.

The foregoing analysis thus reveals the following

FACT 6.1

G-orbifold theories with discrete torsion are classified by $H^2(G; U(1))$.

A modified spectrum

The physical content of the torus partition function Eq. (6.1.1) in the presence of discrete torsion phases is an altered closed string spectrum. In particular, the ground states in twisted sectors are subject to a modified projection, compared to the no-torsion orbifolds, i.e.,

$$g \cdot |h\rangle = \varepsilon(g, h)|h\rangle \ . \tag{6.1.8}$$

Since the phases $\varepsilon(g, h)$ form a one-dimensional representation, say R_ε of N_g, the g-twisted sector full CFT Hilbert space now decomposes as

$$\mathcal{H}_g = \oplus[\phi^g_\alpha] \times [\tilde{\phi}^g_\beta] \ (R_\alpha \otimes R_\beta \otimes R_\varepsilon) \tag{6.1.9}$$

where the group theory factor results from the chiral (α) and anti-chiral (β) primary transformation properties, and the modified G-realisation on the ground-states (ϵ). Orbifold projection now amounts to keeping only invariants in $R_\alpha \otimes R_\beta \otimes R_\varepsilon$, resulting in a modified closed string spectrum. Note that the case $R_{\varepsilon=0}$ reduces to the orbifold without torsion.

6.1.2 Example 6.1

As was already exemplified on p. 75, a short-cut allows for a quick count of massless twisted-sector RR-states. Recall that given an action of G on the target space, encoded in a representation $\mathcal{Q}$, the relevant data consist of (a) the age s, or equivalently, the fermion number shift associated to g, and (b) the fixed set $\mathcal{M}_g$ of g. In the desingularisation, the g-twisted sector then adds to the cohomology $\left[H^{*+s,*+s}(\mathcal{M}_g)\right]^G$.

An instructive example is $G = \mathbb{Z}_6 \times \mathbb{Z}_6$, where $H^2(G; U(1)) = \mathbb{Z}_6$. Further, the defining representation $\mathcal{Q}$ on $\mathbb{C}^3$ sends the generators $g_{1,2}$ of G into diagonal matrices:

$$g_1 \overset{\mathcal{Q}}{\to} \mathrm{Diag}(\omega, \omega^{-1}, 1)\ ; \qquad g_2 \overset{\mathcal{Q}}{\to} \mathrm{Diag}(1, \omega, \omega^{-1})\ ; \tag{6.1.10}$$

Now G has three distinct $\mathbb{Z}_6$ subgroups, generated by $g_1, g_2, g_1 g_2$, respectively, that each leave a complex line fixed. The remaining twenty elements leave a codimension three set fixed. As to the ages, ten out of these twenty have $s = 2$, while any other element in $G \setminus \{e\}$ has $s = 1$. As a final ingredient, the cohomology of the fixed line yields a Hodge diamond

$$\begin{array}{ccc} & h^{0,0} & \\ h^{1,0} & & h^{0,1} \\ & h^{1,1} & \end{array} = \begin{array}{ccc} & 1 & \\ 1 & & 1 \\ & 1 & \end{array} \tag{6.1.11}$$

It is a simple exercise to verify that the codimension-2 (resp. -3) twisted sectors add to the cohomology as summarised in Table 6.1. A note on the terminology: the 'order' of the torsion refers to its order inside the cyclic $H^2(G; U(1)) = \mathbb{Z}_6$; 'minimal' is attributed to the generator of the latter.

$\begin{array}{ccc} & h^{1,1} & \\ h^{2,1} & & h^{1,2} \\ & h^{2,2} & \end{array}$	Codimension 2	Codimension 3
(no torsion)	$\begin{array}{ccc} & 15^* & \\ 0 & & 0 \\ & 15 & \end{array}$	$\begin{array}{ccc} & 10 & \\ 0 & & 0 \\ & 10 & \end{array}$
(minimal torsion)	$\begin{array}{ccc} & 0 & \\ 15 & & 15 \\ & 0 & \end{array}$	$\begin{array}{ccc} & 0 & \\ 0 & & 0 \\ & 0 & \end{array}$
(torsion of order 3)	$\begin{array}{ccc} & 3^* & \\ 12 & & 12 \\ & 3 & \end{array}$	$\begin{array}{ccc} & 0 & \\ 0 & & 0 \\ & 0 & \end{array}$
(torsion of order 2)	$\begin{array}{ccc} & 6^* & \\ 9 & & 9 \\ & 6 & \end{array}$	$\begin{array}{ccc} & 1 & \\ 0 & & 0 \\ & 1 & \end{array}$

Table 6.1: Contributions to the cohomology of the (partially) desingularised space. Entries with * do not have compact support.

6.2 Open strings and D-branes

Before reconsidering the issue of D-branes in the presence of non-trivial discrete torsion, it will prove worthwile to insert an interlude on projective representations (of finite groups, basically).

Projective representations

An n-dimensional projective representation of a group G is a map $R : G \to GL(n, \mathbb{C})$ such that $R(gh) = \alpha(g,h)R(g)R(h), \forall g, h \in G$. The phases involved form a so-called factor system, following from an assignment $\alpha : G \times G \to U(1)$. Evidently, not any assignment will do. Associativity requires

$$\alpha(gh,k)\alpha(g,h) = \alpha(g,hk)\alpha(h,k) \ . \tag{6.2.1}$$

When viewed as a condition on a 2-cochain, associativity exactly translates into $\delta_3\alpha = 1$, hence α must in fact be a cocycle. Further, the rescaling $R(g) \to \beta(g)R(g)$ induces $\alpha \to \alpha.\delta_2\beta$, i.e. it adds a coboundary. The newly obtained factor system is said to be equivalent to the given one. Inequivalent factor systems are thus in one-to-one correspondence with cohomology classes in $H^2(G; U(1))$.

Given a representative of such a class, i.e. a factor system α, one should wonder about inequivalent projective representations realising that α. The notion of equivalence is taken over unmodified from the context of linear (vector, true) representations; specifically, equivalence is induced by change of basis in the carrier vector space. Let me state a few facts without proof here (see, e.g., Ref. [101] and references there);

(a) The number of inequivalent projective representations equals the number of α-regular elements in G. The latter are g such that $\alpha(g,h) = \alpha(h,g), \forall h \in N_g$.

(b) As to the projective characters, they are no longer class functions, i.e. $\check\rho_I(g)$ need not equal $\check\rho_I(hgh^{-1})$. Rather,

$$\check\rho_I(hgh^{-1}) = \frac{\alpha(h^{-1},h)\alpha(g,e)}{\alpha(h,g)\alpha(hg,h^{-1})}\ \check\rho_I(g) \ . \tag{6.2.2}$$

On the other hand, they remain orthogonal, in the sense that

$$\frac{1}{|G|}\sum_g \check\rho_I^*(g)\check\rho_J(g) = \delta_{IJ} \ . \tag{6.2.3}$$

From this equation, the fact that $\sum_I d_I^2 = |G|$ is immediate, as with linear representations.

(c) There always exists some group $\tilde{G}$, a covering group (= representation group), defined by the sequence

$$1 \to K \xrightarrow{\iota} \tilde{G} \xrightarrow{\pi} G \to 1 \ , \tag{6.2.4}$$

which is split-exact, i.e., there exists a section $\tilde{G} \xleftarrow{\sigma} G$ acting as a right-inverse to π. Moreover, this extension of G by K is central, meaning that the $\iota(K)$ sits in the centre of $\tilde{G}$. The linear representations of $\tilde{G}$ then descend to projective or linear representations of G. Equivalently, any representation of G, be it projective or not, lifts to a linear representation of the representation group of G. This property has been worked out in Ref. [102], and proves useful in practical calculations. In the discussion of boundary states, this point of view will become of some relevance as well. Perhaps the simplest example here is

$$1 \to \mathbb{Z}_2 \to \mathbb{D}_4 \to D_4 \to 1 \ , \tag{6.2.5}$$

expressing the binary dihedral as a double covering of the ordinary dihedral group.

(d) The ordinary representation ring $\text{Rep}_0(G)$ of G is enlarged to $\text{Rep}_*(G)$. The latter now contains linear as well as projective representations. The underlying $H^2(G;U(1))$ structure is apparent from the tensor product

$$[-] \otimes [-] : \text{Rep}_\alpha(G) \times \text{Rep}_\beta(G) \quad \longrightarrow \quad \text{Rep}_{\alpha\cdot\beta}(G)$$

D-branes I – Aspects of the gauge theory

Let us now turn back to D-branes. Since $H^2(G;U(1))$ governs both the discrete torsion phases and projective representations, it seems temptive to conclude the following

---

FACT 6.2

D-branes in orbifolds with discrete torsion associated with $[\alpha] \in H^2(G;U(1))$ give rise to Chan-Paton factors carrying representations with a factor system α.

This proposal was put forward in Ref. [99] and it was verified to be consistent for abelian orbifolds, meaning that open and closed string interactions figure in a single coherent picture. Moreover, an important implicit assumption there was that the D-branes were *pointlike*[2] along the orbifold directions. An alternative proof, involving boundary state considerations, will be presented shortly; it has the advantage of encompassing the technical obstructions to non-abelian generalisation of the evidence given in Ref. [99].

Given this piece of information, D-branes in discrete torsion orbifolds can be defined along the lines of Section 3.3. Let the linear defining representation on the string coordinates be $Q \hookrightarrow \mathbf{3}$.[3] Further, for a choice of discrete torsion associated to $[\alpha] \in H^2(G;U(1))$, define the generalised regular representation on the Chan-Paton factor as

$$\mathcal{R}^{(\alpha)}_{\text{regular}} = \oplus_I d_I \; \check{R}^{(\alpha)}_I \,, \tag{6.2.6}$$

where $d_I = \dim R_I$, and $\check{R}^{(\alpha)}_I$ realise the cocycle α. The open string spectrum is then found upon projection, i.e.

$$\left(\mathcal{Q} \otimes \mathcal{R}^{(\alpha)}_{\text{regular}} \otimes \mathcal{R}^{*,(\alpha)}_{\text{regular}}\right)^G \,. \tag{6.2.7}$$

Next, the interactions, in particular, the superpotential, are modified when discrete torsion is non-trivial:

$$\mathcal{W}^{N=1} = \text{Tr}(Z^i[Z^j,Z^k])_\alpha \epsilon_{ijk} \,, \tag{6.2.8}$$

where the subscript denotes α-twisted products. Z^i are $N = 1, d = 4$ chiral superfields (or their appropriate dimensional reductions) that parametrise the space transverse to the D-brane, that is, the orbifold covering space.

Then, the classical equations of motion follow from the superpotential Eq. (6.2.8) as usual, and determine the tree-level moduli space. For the cases $\mathbb{Z}_2 \times \mathbb{Z}_2, \mathbb{Z}_3 \times \mathbb{Z}_3$, this program was carried out in detail in Ref. [97].

It is instructive to take up Example 6.1 again. The defining (linear) representation of Eq. (6.1.10) is more succinctly written as:

$$\mathcal{Q} = R_1 \times R_0 \oplus R_5 \times R_1 \oplus R_0 \times R_5 \,. \tag{6.2.9}$$

As pointed out previously, $H^2(G;U(1)) = \mathbb{Z}_6$. Denote the cocycle-representative by α, and its order by n_α. Given n_α, there are then $|G|/n_\alpha^2$ inequivalent projective irrepses $\check{R}^{(\alpha)}_I$ of dimension n_α. More explicitly, with minimal torsion, i.e., $n_\alpha = 6$, the single projective representation $\check{R}^{(1)}$ is given by

$$g_1 \to \begin{pmatrix} 0 & 1 & 0 & \\ & 0 & 1 & \vdots \\ & & \ddots & \\ 1 & 0 & \cdots & 0 \end{pmatrix} \quad ; \quad g_2 \to \begin{pmatrix} 1 & & & \\ & \omega & & \\ & & \ddots & \\ & & & \omega^5 \end{pmatrix} \quad ; \tag{6.2.10}$$

[2] Or (even,even,even)-dimensional in the notation of Ref. [98].
[3] In the present section only CY threefolds will be considered.

In the non-minimal case, e.g., $n_\alpha = 3$, we rather have four $\check{R}^{(2)}_{j,k}$

$$g_1 \to \omega^j \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} ; \qquad g_2 \to \omega^k \begin{pmatrix} 1 & & \\ & \omega^2 & \\ & & \omega^4 \end{pmatrix} ; \qquad (6.2.11)$$

where $j, k = 0, 1$, or with $n_\alpha = 2$, the nine irrepses $\check{R}^{(3)}_{j,k}$:

$$g_1 \to \omega^j \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix} ; \qquad g_2 \to \omega^k \begin{pmatrix} 1 & \\ & \omega^3 \end{pmatrix} ; \qquad (6.2.12)$$

If now a regular projective representation, $\mathbb{C}^\alpha G = \oplus n_\alpha \check{R}^{(\alpha)}_I$ is put transverse to the origin, the gauge theory spectrum is encoded in the quivers of Fig. 6.1.

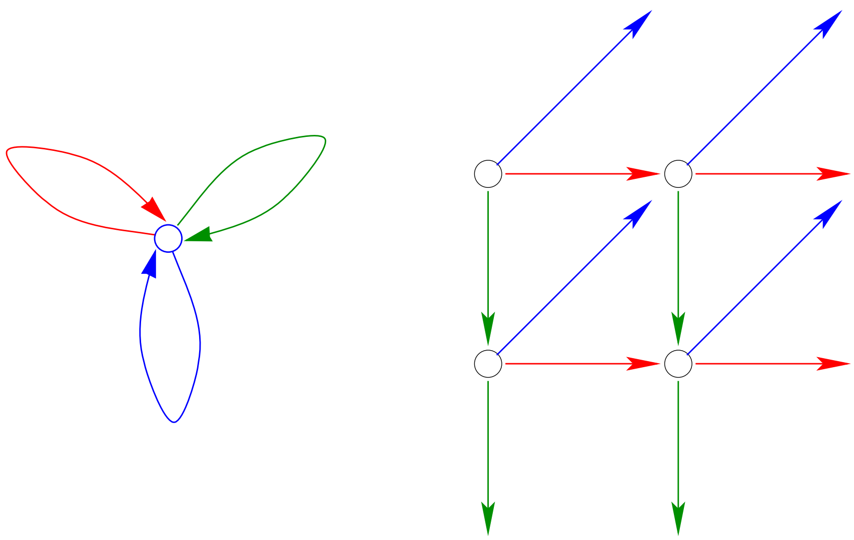


Figure 6.1: Quivers encoding the massless open-string spectrum in $\mathbb{Z}_6 \times \mathbb{Z}_6$ orbifolds with minimal torsion (l.) and torsion of order 2 (r.). See the main text for further details regarding the Figure on the right.

As in Section 3.3, p. 85, vertices are gauge groups, and oriented edges stand for chiral $d = 4$, $N = 1$ multiplets. For the non-minimal order two torsion case, a unit cell of a periodic array is displayed. Furthermore, the order three torsion comes with a similar quiver consisting of a unit cell with 9 vertices.

Moreover, an analysis of the dynamics, similar to the one of Ref. [97] (see also Ref. [103]) reveals a moduli space $\mathcal{M}$ displaying quite some structure. A regular representation probe brane is free to move anywhere on the 3d orbifold; this yields the conventional 3–dimensional branch of $\mathcal{M}$, describing the position of the probe brane when it moves as a coherent unit along the orbifold. However, lower-dimensional branches open up equally well, since subrepresentation branes are allowed to explore e.g., the g_1- and g_2- fixed lines independently. For a complete account of the branches, the reader is referred to Ref. [103].

If only a subrepresentation, not necessarily irreducible, is put, the freedom to move away from the singularity is reduced. In the extreme case of a single projective irrep brane, the Higgs branch collapses to a point. It takes n_α *identical* branes for a non-trivial branch to open up. As such, projective branes have a (possibly restricted) freedom to move in bunches of n_α.

Physically, this circumstance is interpreted as the single branes being stuck at the fixed point. That they can move away unless a sufficient number of copies are placed is reminiscent of them carrying a twist charge, similar to the no-torsion case. However, the distinctive feature here is that all copies must be identical, in contrast to the familiar situation: the twist charge has become discrete rather than additive, i.e., it is only defined modulo n_α. Accordingly, there is no corresponding RR gauge field[4].

In all, the open string analysis reveals the existence of a discrete conserved quantum number, that is defined modulo n_α. This was the main motivation for the present discussion.

D-branes II – Boundary states

Finally, let me come to the boundary states corresponding to the aforementioned D-branes. As usual by now, one starts the construction with the Ishibashi states. In the present situation, these are no different from those in the orbifold without torsion. As to the coefficients B_I^g that perform the transform to the Cardy basis, they are obtained from Eq. (4.4.13) upon replacing $\rho_I \to \breve{\rho}_I^{(\alpha)}$. A minor subtlety hides in that projective characters are no longer class functions on G. Therefore, twisted sector Ishibashi components come labelled by individual elements, rather than conjugacy classes. Carrying out these steps, you will discover that the projective fractional brane states are

[4]As a matter of fact, it may well turn out to be the case that there exist *multiple* discrete charges. A first possibility, investigated in Ref. [102], is that $H^2(G;U(1))$ has more than one generator; a second option, though, seems to reside in turning on non-minimal discrete torsion, but would require further analysis.

given by

$$|I\rangle = \sum_g \sqrt{\frac{\sigma(e,g)}{|G|}}\, \check{\rho}_I^{(\alpha)}(g)\, |g\rangle\!\rangle \ . \tag{6.2.13}$$

Discrete torsion removes some of the twisted sector Ishibashi states from the physical orbifold projected spectrum. How then is this to be reconciled with Eq. (6.2.13)? The answer is simple and short: from Eq. (6.2.2) and associativity, Eq. (6.1.2), it follows that

$$\check{\rho}(hgh^{-1}) \quad = \quad \varepsilon(g,h)\ \check{\rho}(g)\ , \tag{6.2.14}$$

for commuting g, h. Whence the conclusion

$$\exists h \in N_g : \varepsilon(h,g) \neq 1 \Rightarrow \check{\rho}(g) = 0 \tag{6.2.15}$$

follows. As such, the only Ishibashi components that contribute to the projective Cardy states in Eq. (6.2.13) are precisely those labelled by α-regular elements. Stated differently, unphysical components are projected out by construction. This provides an easy and direct piece of evidence for the connection between projective representation Chan-Paton factors and discrete torsion.

From Eq. (6.2.13) observe that the left-hand side consists of objects labelled by projective irrepses, while on the right-hand side only α-regular conjugacy classes contribute. As will be argued in a moment, these are in one-to-one correspondence; hence we must conclude that all the *additive* twisted sector charges are carried by one of the branes in Eq. (6.2.13).

How about the discrete charges, then? It must not come as a surprise that those cannot possibly be detected by boundary state methods. A heuristic explanation is that they generate the open string spectrum, nothing more, nothing less. In particular, they have no means to tell you about the open string *interactions*; hitherto, the latter have been the single indication of the existence of non-trivial discrete charges.

6.3 Geometrical picture

Due to the modified projection in the closed string sector, the correspondence geometry-CFT seems to have gone lost. It is the purpose of the present section to shed some light on a modified correspondence.

6.3.1 Geometry-CFT correspondence

First, the massless RR-states that are being kept are in one-to-one correspondence with so-called α-regular conjugacy classes. These are the ordinary conjugacy classes containing group elements g such that $\varepsilon(g,h) = 1, \forall h \in N_g$. From the property that ε furnish N_g representations, this notion is seen to be well-defined on conjugacy classes. Therefore, the RR-twisted sector CFT marginal deformations are counted by (#α-regular conjugacy classes -1), where the -1 subtraction accounts for the untwisted sector.

On the other hand, the number of inequivalent projective representations realising a cocycle α also equals the number of α-regular conjugacy classes. Therefore, there is a numerical match:

$$\text{\# twisted sector RR-forms} \quad \Leftrightarrow \quad \text{(\# irrepses) - 1}$$

Since D-branes are labelled by (projective) irreducible representations, the twisted sector additive charges, i.e. those that couple to a gauge-potential, can distinguish between them, in complete analogy to the case without torsion.

It is beyond doubt that a geometric interpretation of the projective branes in terms of wrapped higher-dimensional objects would make the correspondence more direct. Before working out some details in a specific instance, recall the following facts:

(a) Theorem 6.1 (Schlessinger [104]) *$\mathbb{C}^m/G$-singularities of codimension at least three are rigid. This means that they admit no non-trivial deformations.*

(b) Blow-up adds to even cohomology, whereas deformation adds to middle cohomology.

6.3.2 The example revisited

Take up the example $\mathbb{C}^3/(\mathbb{Z}_6 \times \mathbb{Z}_6)$ again. From the middle column in Table 6.1, it is seen that sectors twisted by elements leaving a codimension-2 singular fixed set can contribute to both middle and even cohomology, depending on the type of discrete torsion turned on. This is easily understood in terms of the local geometry. To be more precise, the fixed-line orbifold geometry is locally modelled by $\mathbb{C} \times \mathbb{C}^2/\mathbb{Z}_6$, i.e., a complex plane times an A_5-singularity (see Section 3.1.2). From the analysis there, it is known that such singularities can be blown-up with no torsion present, with a net yield of 5 exceptional curves. The Poincaré-duals of each of these adds to $H^{2,2}$, whereas the Poincaré-duals of the type ($\mathbb{C}$)×(curve) do so[5] to $H^{1,1}$. At the other extreme, A-type singularities can be deformed, as discussed likewise in Section 3.1.2, yielding an equal amount of homology cycles. Taking the action of g_2 into account, one is led to the conclusion that the duals of such curves add to $H^{2,1} \oplus H^{1,2}$, as appropriate for deformations. From Table 6.1, each of these scenarioes is realised, either without torsion or with minimal torsion. Besides these, the Table suggests two intermediate cases, with non-minimal torsion turned on. Under the mild assumption that each of these desingularisations smoothes out the singularity completely, this example illustrates once more the fact already argued in Ref. [105], namely,

---

FACT 6.3

> *Codimension-2 (non-isolated) singularities can be removed completely by deformation and/or resolution.*

Now on to the codimension-3 singular point (the origin). Schlessinger's theorem shows that it cannot be undone by deformations, so blow-up is what one is left over with. However, even though geometrically speaking blow-up is a valid procedure, discrete torsion removes the required marginal deformations from the CFT. Stated otherwise, the necessary blowing-up modes are not (all) present in the string theory. Hence,

---

FACT 6.4

> *Codimension-3 (isolated) singularities can at most be partially undone by resolution in string theory.*

Alternatively, this says that the discrete torsion orbifold is totally disconnected from its smooth (large-volume) phase.

In fact, the above features can be illustrated by toric diagrams. Following the rules of Section 3.2.3, the fans in Fig. 6.2 are found in a straightforward manner. Let us suffice with the observation that discrete torsion indeed removes available blow-up cycles (i.e. nodes corresponding to divisors). If the count is performed, agreement will be found between the toric fans and the results in Table 6.1.

[5] If only compactly-supported cohomology is to be taken into account, there are no contributions to $H^{1,1}$

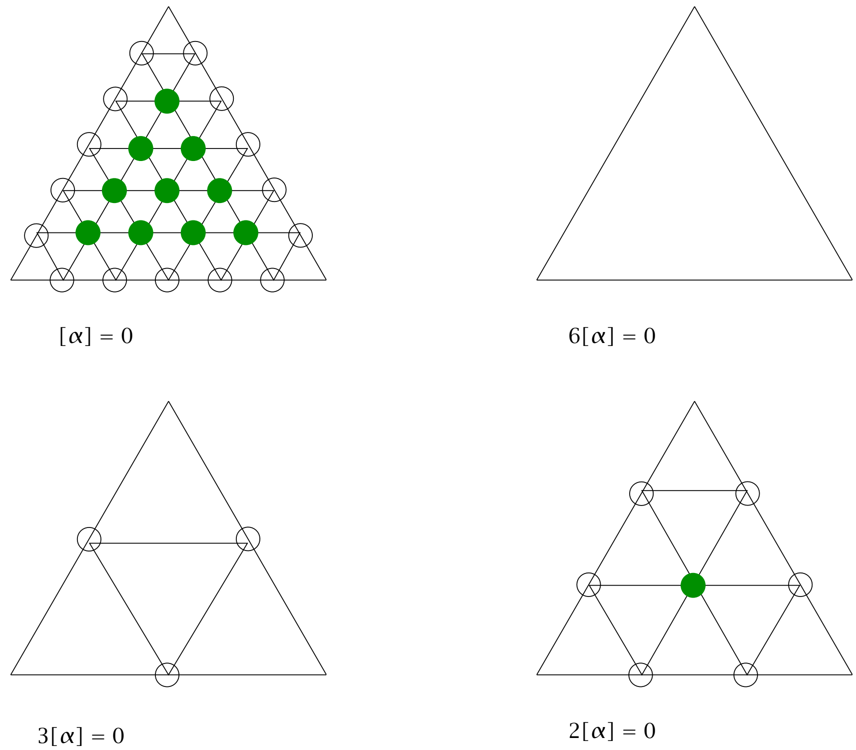


Figure 6.2: The 4 toric $\mathbb{Z}_6 \times \mathbb{Z}_6$ fans with the available blow-up cycles. The number of such cycles increases with decreasing order of the torsion.

As to the D-branes, the picture that emerges is that they do indeed correspond to branes wrapping compact cycles that result from the blow-ups[6]. Equivalently, the following is true:

Fact 6.5

> *Projective fractional branes couple to RR-potentials coming from KK-reduction on compactly-supported exceptional cohomology.*

---

[6] D-branes that are related to non-compact cycles have been explored in Refs [106, 98]. Two distinctive features are: 1. they are not BPS, and 2. they are not point-like but extended in some of the orbifold directions.

6.3.3 Torsion in homology ?

Besides possible additive charges, projective branes also carry discrete charges, as suggested by the open string picture. Can these be accounted for as well in a geometric framework? This issue was explored only quite recently [103] and we shall have very little to say here. Suffice it to point out the main ideas and a (hitherto) unanswered question.

(a) FACT 1. From Ref. [87] it has been known that D-brane charges are encoded in the K-theory of the space. This idea was pursued by the authors of [103], and $H^2(G; U(1))$ was argued to be a torsion subgroup of the K-homology. In this sense, the discrete torsion can be accomodated in a geometric context.

(b) FACT 2. When this result is mapped to integral homology, the fact whether this torsion subgroup is to be identified as a two- or four-cycle is obscure. This makes the situation still rather unsatisfactory at present. On the other hand, it is known that the torsion in K-theory may be different from that in homology. One could therefore question the quest for a purely singular-homological interpretation.

(c) QUESTION. Rather than in K-theory (i.e., smooth vector bundles), is there room for an interpretation in terms of sheaves (singular bundles) and/or complexes of bundles, or derived categories, as in Chapter 5? In other words, does a modified version of McKay correspondence survive if discrete torsion is turned on?

A

Elements of sheaf theory

Contrary to vector bundles, sheaves seem to have resisted wide acceptance by string theorists so far. Most often, one can do without them, and to introduce them would merely seem like a fancy rephrasing. However, in Chapter 5 coherent sheaves played a key rôle. The main objective of this section is to provide the reader with some intuition of sheaves, in a way to convince him/her of their power that takes them beyond fanciness. We learned about sheaves mainly from Refs. [107, 108, 109], whereby Ref. [57] is recommended for background material on ring and module theory. Lack of space forces us to be brief, but nonetheless we hope to give a flavour of the subject.

In essence, sheaves on a manifold (variety, scheme) X consist of a covering $\cup_\alpha U_\alpha$ of X, equipped with an assignment $\mathcal{F}$ of sets $\mathcal{F}(U)$ to each open set. Typically, the sets $\mathcal{F}(U)$ are groups, rings or modules. Further, for any pair of opens U, V, such that $U \subset V$, there are maps ρ_U^V taking $\mathcal{F}(U)$ into $\mathcal{F}(V)$ with a number of conditions making $\mathcal{F}$ into a sensible object. Typically, these maps will be homomorphisms of groups, rings, and modules, whence structure-preserving morphisms. [1]

A.1 Structure sheaves

A first important class of sheaves are so-called structure sheaves $\mathcal{O}_X$. They provide an underlying space X with *rings* encoding its structure. Depending

[1] Actually, the $\mathcal{F}$ introduced are only pre-sheaves. To make a sheaf, $\mathcal{F}$ has to obey patching conditions, essentially stating that $\mathcal{F}(X)$-valued objects are completely determined by local data $\mathcal{F}(U_\alpha)$. See Ref. [109]

on one's needs, $\mathcal{O}(U)$ are rings of continuous functions (topology), C^∞ functions (differential geometry), rational functions on X that are regular on U (algebraic geometry).

As an example, consider the sheaf $\mathcal{O}_{\mathbb{P}^N}(n)$ over (complex) projective N-space. It is the sheaf of homogeneous degree n rational functions:

$$\mathcal{O}_{\mathbb{P}^N}(n)(U) = \left\{ \frac{f}{g} | \deg(f) = d + n; \deg(g) = d; g(x) \neq 0 \forall x \in U \right\} .$$

Equivalently, it is the n-th tensor power of $\mathcal{O}_{\mathbb{P}^N}(1)$. Also, for an algebraic subvariety X of $\mathbb{P}^N$, i.e. X is a so-called projective variety, the structure sheaf $\mathcal{O}_X$ is the sheaf of regular (rational) functions as already explained. Consider next

$$\mathcal{O}_X(1) := \mathcal{O}_X \otimes_{\mathcal{O}_{\mathbb{P}^N}} \mathcal{O}_{\mathbb{P}^N}(1) . \tag{A.1.1}$$

from which new sheaves are defined through $\mathcal{O}_X(n) := \otimes^n \mathcal{O}_X(1)$.

A.2 Sheaf homomorphisms, (co)kernels and stalks

Preliminary to the exploration of the differences and resemblances between vector bundles and sheaves of modules (see below), is a short digression on maps between sheaves. A sheaf map consists of a system of homomorphisms $f_U : \mathcal{E}(U) \to \mathcal{F}(U)$, that is moreover compatible with the restriction maps. The kernel, Ker f, is the *sheaf* built from the local sections Ker f_U; the cokernel sheaf, Cok f, is somewhat subtler, since generically the naive process only yields a *pre*sheaf (see e.g. Ref. [107]).

Next, let me discuss the stalk $\mathcal{F}_p$ at a point p. Given two opens U, U' containing p, two sections $\sigma_U, \sigma_{U'}$ are identified whenever they have some open V in common where they coincide. The set of equivalence classes (i.e., the set of the germs of local sections at p) is $\mathcal{F}_p$. Now here is why stalks are any good: a sheaf map f is called injective, resp. surjective, if the property holds at the level of stalks. In other words, given

$$\mathcal{E}(U) \stackrel{f_U}{\to} \mathcal{F}(U) , \tag{A.2.1}$$

the homomorphism f may be injective, resp. surjective, even though f_U fails to meet the requirement for some U. This weaker notion turns out to be precisely right, i.e., not too rigid nor completely empty, in further developments of the theory. Exactness of sequences of sheaves is defined in a similar spirit, that is, through the notion of stalks.

Let us illustrate these points with an example: consider a hypersurface $(Y : F = 0)$ defined as the zero locus of some polynomial F inside an (affine) variety X. The structure sheaf $\mathcal{O}_Y$ may then be defined in terms of $\mathcal{O}_X$, by demanding that the sequence

$$0 \to \mathcal{O}_X \xrightarrow{F} \mathcal{O}_X \to \mathcal{O}_Y \to 0 \tag{A.2.2}$$

be exact; the indicated map is multiplication by F. The stalks $\mathcal{O}_{X,p}$ can be characterised as indicated above, but in this concrete case, they are equivalently defined as

$$\mathcal{O}_{X,p} := \left\{ \frac{f}{g} \mid g(p) \neq 0 \right\} , \tag{A.2.3}$$

the so-called local ring at p. Now, two possible situations are discerned:

(a) $p \notin Y$. Then, the first map is surjective on stalks, since any $f/g \in \mathcal{O}_{X,p}$ is the image of $f/(Fg)$. Accordingly, $\mathcal{O}_{Y,p} = 0$.

(b) $p \in Y$. Here, multiplication by F is no longer surjective, leaving a nontrivial stalk $\mathcal{O}_{Y,p}$.

The following point deserves special attention: even though F vanishes along Y, multiplication by F remains a nontrivial operation between the stalks $\mathcal{O}_{X,p\in Y}$: in the local ring, only 0 is mapped to 0. Stated alternatively, F does not vanish slightly off Y, and as such the multiplication is non-trivial on germs (recall that the latter are defined on open sets that necessarily contain points off Y). What is to be learnt from this example?

(a) Stalks may jump at points. This situation is to be contrasted with the case of fibre bundles, where all fibres are isomorphic.

(b) Even though F formally vanishes at $p \in Y$, injectivity still holds, since both observations are disparate.

A.3 Sheaves of modules

Sheaves of modules $\mathcal{F}$ assign *modules* to opens U, rather than rings. In particular, $\mathcal{F}(U)$ will be $\mathcal{O}_X(U)$-modules. Of course, any ring $\mathcal{O}_X(U)$ can be viewed as a module in its own right, and as such, sheaves of modules generalise structure sheaves.

In what respect do sheaves of modules generalise vector bundles? The answer comes in two pieces. Firstly, starting from any vectorbundle E, one builds a corresponding sheaf $\mathcal{E}$ as the space $\Gamma(E)$ of sections of the former, i.e. for every open U, $\mathcal{E}(U)$ is the module of local sections of E, with natural restriction maps ρ_U^V. The class of sheaves thus obtained are the locally-free sheaves. This correspondence can be shown to be one-to-one and onto.

Secondly, just beyond locally-free sheaves are the so-called *coherent sheaves.* Recall that the fiber of a vector bundle is a vectorspace. The latter can always be freely presented: there is a basis of generators with no relations among them. Consider instead a module M that is presented by a set of generators F and a set of relations R_1. The relations generate a new module that will be denoted by the same symbol, for convenience. In turn, the module R_1 need not be free, i.e., there can be additional relations R_2 between the relations R_1, and so forth. This information is concisely encoded in an exact sequence:

$$\ldots \to R_2 \to R_1 \to F \to M \to 0 \,. \tag{A.3.1}$$

For sheaves $\mathcal{M}$ of $\mathcal{O}_X$-modules, the corresponding sequence is

$$\ldots \to \mathcal{O}_X^{r_2} \to \mathcal{O}_X^{r_1} \to \mathcal{O}_X^{r} \to \mathcal{M} \to 0 \,. \tag{A.3.2}$$

This is called a *projective resolution*; the resolving sheaves $\mathcal{O}_X^r$ are locally-free (or, $\mathcal{O}_X^r(U)$ are projective modules). Coherent sheaves are singled out by their associated exact sequences being finite. Locally-free sheaves are in fact trivial examples of coherent sheaves: the sequence Eq. (A.3.2) consists of two terms only, and such sheaves are thus freely and finitely generated as $\mathcal{O}_X$-modules. In summary, coherent sheaves are finitely but not necessarily freely generated and come with a finite sequence of relations.

We conclude this digression by listing some of the generic features of coherent sheaves [108] that should develop one's intuition sufficiently to follow the discussion in Chapter 5.

(a) It can be shown that for any coherent sheaf $\mathcal{F}(X)$, there exists a dense open subset W such that $\mathcal{F}|_W$ is free[2]. More interestingly, the torsion[3] is supported inside the complement of W. That is, coherent sheaves are 'free modulo sheaves with support distinct from X'.

(b) Conversely, a coherent sheaf is a torsion sheaf iff its support is distinct from X.

(c) For any subvariety D defined by r polynomial constraints in an affine variety X, the structure sheaf $\mathcal{O}_D$ may be viewed as a sheaf on X by

[2] That is, locally free. In other words,

$$0 \to \mathcal{O}_W^r \to \mathcal{F}|_W \to 0 \,,$$

is the associated sequence, for some r.

[3] An R-module M is torsion iff its annihilator $\mathrm{Ann}_R(M)$ is non-trivial, i.e., if $\exists r \in R, \forall m \in M : rm = 0$; likewise, a sheaf $\mathcal{M}$ is a torsion sheaf, if $\forall U$, $\mathcal{M}(U)$ is torsion as an $\mathcal{O}_X(U)$-module.

'extending by zero'. This is an immediate generalisation of the example given previously:

$$\mathcal{O}_X^r \xrightarrow{F} \mathcal{O}_X \to \mathcal{O}_D \to 0\ , \tag{A.3.3}$$

where $\mathcal{O}_D$ is defined as the cokernel of F. When viewed as a sheaf on X, $\mathcal{O}_D$ is torsion as the support is not the whole of X. In vector bundle terminology, this would be signalled by transition matrices with ranks jumping on X.

B

Orbifold chiral blocks

This appendix collects explicit expressions for orbifold chiral traces in twisted and untwisted sectors, based on Ref. [1]. The purpose is to provide an easy reference. For conventions on ϑ-series and the Dedekind η, the reader is referred to Ref. [76], Chapter 6, or Ref. [1]. It must be clear that no originality is claimed here.

B.1 Orbifold blocks

B.1.1 Twists, shifts and zero-point energies

The setting will be as follows: let there be given a complex boson X and a complex fermion ψ together with an element h of the orbifold group G, such that h and all $g \in N_h$[1] are simultaneously diagonal. Further, let the orbifold be specified by

$$h \cdot X = e^{2\pi i \nu'} X \; ; \tag{B.1.1}$$
$$g \cdot X = e^{2\pi i \nu} X \; , \tag{B.1.2}$$

and likewise for the fermions.

Then, the h-twisted sector comes with boundary conditions along the space-direction:

$$X(\sigma + 2\pi) = e^{2\pi i \nu'} X \; ; \tag{B.1.3}$$
$$\psi(\sigma + 2\pi) = e^{2\pi i (\nu' + \zeta)} \psi \; , \tag{B.1.4}$$

[1] N_h is the normaliser subgroup of G, see Section 3.2.1

where $\zeta = 0$ (resp. $\frac{1}{2}$) is the Ramond- (resp. NS-) sector. The modified conditions, Eq. (B.1.4), induce two effects, to know:

(a) the modings are shifted away from their reference values in the untwisted sector; the mode numbers for X, ψ take values in $\mathbb{Z} + \nu' + \zeta'$[2]. Similarly, those for $\bar{X}, \bar{\psi}$ acquire shifts with $\nu' \to -\nu'$.

(b) A subtler point is the shifted ground-state energy a:

$$\begin{aligned} a^{(X)} &= \frac{1}{24} - \frac{1}{8}(2\nu' - 1)^2 \ ; \\ a^{(\psi)}_{\zeta'} &= -\frac{1}{24} + \frac{1}{8}(2(\tilde{\nu}' + \zeta') - 1)^2 \ . \end{aligned} \tag{B.1.5}$$

These formulas hold for $\nu', \tilde{\nu}' + \zeta' \in [0, 2\pi)$; the value of $\tilde{\nu}'$ is then the appropriate integer shift of ν'.

The combined ground-state energy vanishes in the R-sector.

B.1.2 The tale of the complex boson ...

The bosonic chiral blocks are obtained by inspection. If $h \neq e$, one finds

$$\begin{aligned} \chi^g_h(q) &= \mathrm{Tr}_{\mathcal{H}_h}(g\, q^{L_0 - \frac{c}{24}}) \ ; \\ &= q^{a^{(X)}} \left[\prod_{n=0}^{\infty} (1 - e^{2\pi i \nu} q^{n+\nu'})(1 - e^{-2\pi i \nu} q^{n+(1-\nu')}) \right]^{-1} \ ; \\ &= i\, e^{-\pi i \nu} q^{-\frac{\nu'^2}{2}} \frac{\eta(\tau)}{\vartheta_1(\nu + \tau\nu' | \tau)} \ . \end{aligned} \tag{B.1.6}$$

The second line decomposes into an X-oscillator and an $\bar{X}$-oscillator factor, besides the $a^{(X)}$ contribution to L_0.

The untwisted sector sees the appearance of bosonic zero-modes, i.e., momenta (+ windings) for noncompact (compact) bosons. These are shared between chiral and anti-chiral sectors, and there is no natural telling to which they belong. A discussion of the zero-mode contribution was made on p. 63. As to the nonzero modes, with a nontrivial g-insertion they yield

$$\begin{aligned} \hat{\chi}^g_e(q) &= q^{-\frac{1}{12}} \prod_{n=1}^{\infty} (1 - e^{2\pi i \nu} q^n)^{-1} (1 - e^{-2\pi i \nu} q^n)^{-1} \\ &= 2 \sin \pi\nu \, \frac{\eta(\tau)}{\vartheta_1(\nu|\tau)} \ . \end{aligned} \tag{B.1.7}$$

[2] Set $\zeta' = 0$ in the bosonic sector.

B.1.3 ... and that of the fermion

The story for the fermion is only slightly more complicated than that of the boson. Similar to the two possible boundary conditions (R/NS), the fermion number operator $(-)^F$ may or may not be inserted inside the trace. A trace with (without) a $(-)^F$ inserted will be labelled by $\zeta = 0\,(\frac{1}{2})$. With the zero-point energies in Eq. (B.1.5), the ψ-$\bar{\psi}$-pair yields

$$\chi^{\zeta,g}_{\zeta',h}(q) = q^{\frac{\tilde{\nu}'^2}{2}}\, \mathrm{e}^{-2\pi \mathrm{i}(\zeta'-\frac{1}{2})(\nu+\zeta-\frac{1}{2})}\, \frac{\vartheta\left[\begin{smallmatrix}1-2\zeta'\\ 1-2\zeta\end{smallmatrix}\right](\tau+\tilde{\nu}'\tau|\tau)}{\eta(\tau)}\,. \tag{B.1.8}$$

Often, it is convenient to trade the (ζ,ζ')-basis for $\mathrm{SO}(2)_1$-characters. The latter come with labels o, v, s, c, the conjugacy classes of $\mathrm{SO}(2N)$, indeed. This basis-change reads explicitly:

$$\begin{aligned}
(\chi_o)^g_h &= q^{\frac{(\tilde{\nu}')^2}{2}}\, \frac{\vartheta_3(\nu+\tau\tilde{\nu}'|\tau)+\vartheta_4(\nu+\tau\tilde{\nu}'|\tau)}{2\eta(\tau)}\,,\\
(\chi_v)^g_h &= q^{\frac{(\tilde{\nu}')^2}{2}}\, \frac{\vartheta_3(\nu+\tau\tilde{\nu}'|\tau)-\vartheta_4(\nu+\tau\tilde{\nu}'|\tau)}{2\eta(\tau)}\,,\\
(\chi_s)^g_h &= q^{\frac{(\nu')^2}{2}}\, \mathrm{e}^{\mathrm{i}\pi\nu}\, \frac{\vartheta_2(\nu+\tau\nu'|\tau)-\mathrm{i}\vartheta_1(\nu+\tau\nu'|\tau)}{2\eta(\tau)}\,,\\
(\chi_c)^g_h &= q^{\frac{(\nu')^2}{2}}\, \mathrm{e}^{\mathrm{i}\pi\nu}\, \frac{\vartheta_2(\nu+\tau\nu'|\tau)+\mathrm{i}\vartheta_1(\nu+\tau\nu'|\tau)}{2\eta(\tau)}\,.
\end{aligned} \tag{B.1.9}$$

Like in the bosonic situation, the presence of zero-modes requires some care. Fermion zero-modes exist either in the untwisted R and $\mathrm{R}(-)^F$ sectors, or in the NS and $\mathrm{NS}(-)^F$ sectors twisted by an element of order two (*i.e.*, such that $\nu' = 1/2$). The zero-modes $e^+ = \psi^i_0/\sqrt{2}$ and $e^- = \psi^{\bar{i}}_0/\sqrt{2}$, satisfy a Clifford algebra: $\{e^+, e^-\} = 1$. This produces a two-state spectrum $|\downarrow\rangle, |\uparrow\rangle$, defined by

$$e^-|\downarrow\rangle = 0 \quad ; \qquad |\uparrow\rangle = e^+|\downarrow\rangle\,; \tag{B.1.10}$$

In this basis, the zero-mode part of the relevant Lorentz rotation generator reads $J = J_{12} = \frac{1}{2}[e^+, e^-] = \frac{1}{2}\sigma^3$. As such, g as given in Eq. (B.1.2) is effected on the degenerate ground states by $\exp(2\pi \mathrm{i}\nu J)$, while $(-)^F = -\sigma_3$ similarly. As a result, a g-insertion with (without) $(-)^F$ produces a zero-mode factor $-2\mathrm{i}\sin\pi\nu$ (resp. $2\cos\pi\nu$) in the trace.

B.2 Modular transformations

The modular properties of orbifold chiral blocks were dealt with in Ref. [48] in the context of rational conformal field theories, in which cases the generators

S, T are realised on the chiral blocks χ^g_h as given by:

$$g \underset{h}{\square} \xrightarrow{S} \sigma(h|g) \times h^{-1} \underset{g}{\square} \,; \tag{B.2.1}$$

$$g \underset{h}{\square} \xrightarrow{T} \mathrm{e}^{-\pi \mathrm{i} \frac{c}{12} \tau_h} \times hg \underset{h}{\square} \,. \tag{B.2.2}$$

Since free bosons do not generically build a rational CFT, it is not guaranteed, a priori, that the modular properties described above carry over unchanged. The explicit analysis below will reveal that they do, though.

In view of S playing the key rôle in this thesis, we restrict the discussion to that. For an account of T and, relatedly, twist field conformal weights, see Ref. [1].

Consider the bosons first. The characters given in Eq. (B.1.6) behave under S as

$$\chi^{(X)}{}^{g}_{h}(q') = \mathrm{i}\mathrm{e}^{2\pi \mathrm{i}\left(\nu\nu' - \frac{\nu+\nu'}{2}\right)} \chi^{(X)}{}^{h^{-1}}_{g}(q) \,, \tag{B.2.3}$$

where $\tau' = -1/\tau$ and $q' = \exp(2\pi \mathrm{i}\tau')$; this follows from standard Poisson resummation. As such, the phase factor in Eq. (B.2.1) is

$$\sigma(h|g) = \mathrm{e}^{2\pi \mathrm{i}(\nu' - \frac{1}{2})(\nu - \frac{1}{2})} \,. \tag{B.2.4}$$

Untwisted characters require a separate treatment, basically due to the possible presence of zero-modes. For a compactified boson, with momenta taking only discrete values, the untwisted characters are the $\hat{\chi}^g_e(q)$ of Eq. (B.1.7), and accordingly S, Eq. (3.2.41) in the text, produces a zero-mode factor

$$\hat{\sigma}(e|g) = 2 \sin \pi\nu \tag{B.2.5}$$

in addition to the phase in Eq. (B.2.3).

The fermion twisted sector characters ($h \neq e$, and not of order two) yield an S-transformation

$$\chi^{\zeta,g}_{\zeta',h}(q') = \mathrm{e}^{-2\pi \mathrm{i}(\tilde{\nu}' + \zeta' - \frac{1}{2})(\tilde{\nu} + \zeta - \frac{1}{2})} \chi^{\zeta',h^{-1}}_{\zeta,g}(q) \,, \tag{B.2.6}$$

with $\tilde{\nu}$ being an integer shift of ν such that $\tilde{\nu} + 1/2 < 1$, in analogy with $\tilde{\nu}'$. This transformations exhibits phases

$$\sigma(\zeta', h|\zeta, g) = \mathrm{e}^{-2\pi \mathrm{i}(\tilde{\nu}' + \zeta' - \frac{1}{2})(\tilde{\nu} + \zeta - \frac{1}{2})} \,. \tag{B.2.7}$$

In the untwisted case, as well as in the in the case in which h is of order two, the expressions become simple:

$$\sigma(\zeta', e|\zeta, g) = \mathrm{e}^{-\pi \mathrm{i}(\zeta' - \frac{1}{2})(\zeta - \frac{1}{2})} \,, \tag{B.2.8}$$

that is, only the $\mathrm{R}(-)^F$ sector acquires a $-\mathrm{i}$ factor. Chiral traces in NS-sectors twisted by a nontrivial h of order two, behave similarly:

$$\sigma(\zeta', h|\zeta, g) = \sigma(\zeta' + \frac{1}{2} \bmod 1, e|\zeta, g) \qquad (h^2 = e) \ . \tag{B.2.9}$$

Throughout the main text, the o, v, s, c-character basis was used most frequently. Organising phases into 4×4 matrices $S(h|g)$ acting on a vector $(\chi_a)_h^g$, with $a = v, o, s, c$, one has

$$(\chi_a)_h^g \xrightarrow{S} [S(h|g)]_a^{\ b} (\chi_b)_g^{h^{-1}} \ , \tag{B.2.10}$$

or more explicitly, for generic h,

$$S(h|g) = \frac{1}{2} \begin{pmatrix} \sigma(\frac{1}{2},h|\frac{1}{2},g) & \sigma(\frac{1}{2},h|\frac{1}{2},g) & \sigma(\frac{1}{2},h|0,g) & \sigma(\frac{1}{2},h|0,g) \\ \sigma(\frac{1}{2},h|\frac{1}{2},g) & \sigma(\frac{1}{2},h|\frac{1}{2},g) & -\sigma(\frac{1}{2},h|0,g) & -\sigma(\frac{1}{2},h|0,g) \\ -\sigma(0,h|\frac{1}{2},g) & \sigma(0,h|\frac{1}{2},g) & \sigma(0,h|0,g) & -\sigma(0,h|0,g) \\ -\sigma(0,h|\frac{1}{2},g) & \sigma(0,h|\frac{1}{2},g) & -\sigma(0,h|0,g) & \sigma(0,h|0,g) \end{pmatrix} . \tag{B.2.11}$$

Again, this expression is considerably simplified in the untwisted sector, where it coincides with the S matrix for the $\mathrm{SO}(2)_1$ chiral blocks (see Eq. (2.2.18)): $S(e|g) = S_{(2)}$. An analogous expression is obtained when the twist h is of order 2, with $(\chi_o)_h^g \leftrightarrow (\chi_s)_e^g$ and $(\chi_v)_h^g \leftrightarrow (\chi_c)_e^g$.

B.3 Characters of submodules

Given the $\mathrm{SO}(2)_1$ o, v, s, c characters in Eq. (B.1.9), can these be of any use in telling how the o, v, s, c modules decompose into $\mathcal{A}^G \times G$-modules? A short analysis given below yields an affirmative answer.

For a given h-twisted sector, fix a cyclic subgroup $\langle g \mid g^{M'} \rangle \subset N_h$. Let ν in Eq. (B.1.9) result from an insertion of g in the chiral trace, i.e.

$$g \cdot \psi = \mathrm{e}^{2\pi \mathrm{i} \nu} \psi \equiv z\psi \ . \tag{B.3.1}$$

Assume first that the order of g is even; then, one has that

$$\begin{aligned} &\frac{1}{2}\left(\vartheta_3(\nu + \tilde{\nu}'\tau) + \vartheta_4(\nu + \tilde{\nu}'\tau)\right) \\ &\quad = \ \frac{1}{2} \sum_{n \in \mathbb{Z}} q^{\frac{n^2}{2} + \tilde{\nu}' n} z^n (1 + (-1)^n) \ ; \\ &\quad = \ \sum_{j \in \mathbb{Z}_M} z^{2j} q^{-\frac{(\tilde{\nu}')^2}{2}} \sum_{l \in \mathbb{Z}} q^{2(j + lM + \frac{\tilde{\nu}'}{2})^2} \ ; \end{aligned}$$

$$= \sum_{j\in\mathbb{Z}_M} z^{2j} q^{-\frac{(\tilde{v}')^2}{2}} \sum_{m\in\mathbb{Z}+\frac{2M(2j+\tilde{v}')}{4M^2}} q^{2M^2m^2} \; ;$$

$$= \sum_{j\in\mathbb{Z}_M} z^{2j} q^{-\frac{(\tilde{v}')^2}{2}} \Theta_{2M(2j+\tilde{v}'),2M^2}(\tau,0,0) \; .$$

In this derivation, use was made of the series representation of ϑ-functions. The last equation follows from the definition of the level-N SU(2) Θ-functions (see e.g. [110]):

$$\Theta_{m,N}(\tau,\nu,w) := \mathrm{e}^{-2\pi \mathrm{i} N w} \sum_{n\in\mathbb{Z}+\frac{m}{2N}} \mathrm{e}^{2\pi\mathrm{i}\tau n^2 - 4\pi\mathrm{i}\nu N n} \; . \tag{B.3.2}$$

Altogether, the above result yields concisely that

$$(\chi_o)_h^g = \frac{1}{\eta(\tau)} \sum_{j\in\mathbb{Z}_M} z^{2j}\, q^{-\frac{(\tilde{v}')^2}{2}}\, \Theta_{2M(2j+\tilde{v}'),2M^2}(\tau,0,0) \; . \tag{B.3.3}$$

Eq. (B.3.3) encodes the desired decomposition into $\langle g\rangle$-modules:

$$[\mathbb{1}]_h = \bigoplus_{j=1}^{M} \mathcal{H}^{(j)}_{\mathbb{1},h} \times R_{2j} \; , \tag{B.3.4}$$

and the chiral trace falls apart accordingly:

$$(\chi_o)_h = \frac{1}{\eta(\tau)} \sum_{j\in\mathbb{Z}_M} q^{-\frac{(\tilde{v}')^2}{2}}\, \Theta_{2M(2j+\tilde{v}'),2M^2}(\tau,0,0) \; . \tag{B.3.5}$$

Analogous expressions for v, s, c and for g of odd order are derived in a similar vein. They are summarised in Table B.1.

$$\boxed{\mathbf{z^{2M} = 1}}$$

$$(\chi_o)^g = \frac{q^{-\frac{(\tilde{v}')^2}{2}}}{\eta(\tau)} \sum_{j\in\mathbb{Z}_M} \Theta_{2M(2j+\tilde{v}'),2M^2}(\tau,0,0)$$

$$(\chi_v)^g = \frac{q^{-\frac{(\tilde{v}')^2}{2}}}{\eta(\tau)} \sum_{j\in\mathbb{Z}_M} \Theta_{2M(2j+1+\tilde{v}'),2M^2}(\tau,0,0)$$

$$(\chi_c)^g = \frac{q^{-\frac{(\tilde{v}')^2}{2}}}{\eta(\tau)} \sum_{j\in\mathbb{Z}_M} \Theta_{2M(2j-\frac{1}{2}+\tilde{v}'),4M^2}(\tau,0,0)$$

$$(\chi_c)^g = \frac{q^{-\frac{(\tilde{v}')^2}{2}}}{\eta(\tau)} \sum_{j\in\mathbb{Z}_M} \Theta_{2M(2j+\frac{1}{2}+\tilde{v}'),4M^2}(\tau,0,0)$$

$$\boxed{\mathbf{z^{2M+1} = 1}}$$

$$(\chi_o)^g = \frac{q^{-\frac{(\tilde{v}')^2}{2}}}{\eta(\tau)} \sum_{j\in\mathbb{Z}_{2M+1}} \Theta_{2(2M+1)(2j+\tilde{v}'),2(2M+1)^2}(\tau,0,0)$$

$$(\chi_v)^g = \frac{q^{-\frac{(\tilde{v}')^2}{2}}}{\eta(\tau)} \sum_{j\in\mathbb{Z}_{2M+1}} \Theta_{2(2M+1)(2j+1+\tilde{v}'),2(2M+1)^2}(\tau,0,0)$$

$$(\chi_c)^g = \frac{q^{-\frac{(\tilde{v}')^2}{2}}}{\eta(\tau)} \sum_{j\in\mathbb{Z}_{2M+1}} \Theta_{2(2M+1)(2j-\frac{1}{2}+\tilde{v}'),4(2M+1)^2}(\tau,0,0)$$

$$(\chi_c)^g = \frac{q^{-\frac{(\tilde{v}')^2}{2}}}{\eta(\tau)} \sum_{j\in\mathbb{Z}_{2M+1}} \Theta_{2(2M+1)(2j+\frac{1}{2}+\tilde{v}'),4(2M+1)^2}(\tau,0,0)$$

Table B.1: Decomposition of (un)twisted $SO(2)_1$ modules.

C

Samengevat

C.1 Inleiding

Bij een eerste kennismaking lijkt snaartheorie doorgaans gehuld in een waas van mysterie, niet in het minst te wijten aan een geëigend jargon. Eens de mist opgetrokken evenwel, ontwaar je een fascinerend spel met een interne consistentie die grenst aan het ondenkbeeldige. In de ban van de fascinatie kun je al gauw uit het oog verliezen waar het allemaal om begonnen was: een consistente theorie die de waargenomen natuurkrachten geünificeerd beschrijft. Uiteraard blijft snaartheorie de ultieme kandidaat, al blijft experimentele bevestiging voorlopig uit. Anders gezegd, tot op vandaag is theoretische consistentie het enige houvast gebleken. In afwachting van tastbare experimentele evidentie, een mogelijkheid waarvoor sinds kort stemmen opgaan, is de theorie verder geëxploreerd. Niet alleen zijn er heel wat onvermoede fysische aspecten van de theorie naar boven gekomen, gaande van (snaar)dualiteiten tot de Maldacena conjectuur, maar waren er ook nevenproducten van een meer zuiver wiskundige aard, zoals kwantummeetkunde. Telkens is snaartheorie een ideale 'context-of-discovery' gebleken, een gedroomd theoretisch laboratorium, zeg maar. Of snaartheorie al dan niet gerealiseerd wordt in de natuur, wordt, gezien de glansrol die ze speelt in het vernoemde scenario, in zeker opzicht van ondergeschikt belang. De toekomst moet uitwijzen of dit aan snaartheorie voldoende bestaansrecht geeft, indien nodig.

Voor we overgaan tot een overzicht van deze thesis in Sectie C.2, ondernemen we een poging om een beeld te schetsen van enkele centrale concepten.

C.1.1 De hoofdrolspelers (in deze thesis)

Snaren

Conceptueel zijn snaren vrij eenvoudig, en veralgemenen ze op een natuurlijke wijze de notie van puntdeeltjes: waar laatstgenoemde een eendimensionaal traject volgen in ruimte-tijd, de zgn. wereldlijn, volgen snaren een tweedimensionaal wereldoppervlak, Σ. Wiskundig wordt dit traject vastgelegd door een afbeelding $\Phi : \Sigma \to (\mathcal{M}, g)$. De variëteit $\mathcal{M}$ met metriek g wordt de snaarachtergrond genoemd. Bij gegeven begin- en eindconfiguraties van de snaar, wordt de snaardynamica bepaald door een actie, voor gesloten bosonische snaren bv.

$$S = \frac{1}{\alpha'} \int_\Sigma ||\partial \Phi||_g^2, \tag{C.1.1}$$

waarbij de norm afgeleid is van de metriek g.

In de kwantumsnaartheorie van Polyakov worden amplitudes gegeven door een padintegraal, voor de vacuümamplitude bv.

$$\int \mathcal{D}h \int \mathcal{D}\Phi \exp(\mathrm{i}S[h, \Phi]) \tag{C.1.2}$$

waarbij geïntegreerd wordt over metrieken h op Σ, en afbeeldingen Φ die voldoen aan gepaste randvoorwaarden. Merk dat in de uitdrukking voor S hierboven, de expliciete h-afhankelijkheid impliciet gelaten werd, om de notatie niet onnodig gecompliceerd te maken. Verder is duidelijk dat S invariant blijft onder coördinaatverandering en herschaling van Σ. Na ijkfixatie van deze symmetrieën blijft residueel conforme symmetrie over. Men zegt dat Φ een conforme-veldentheorie vastlegt. Met andere woorden, perturbatieve snaartheorie is intrinsiek tweedimensionale conforme-veldentheorie, die zich in eerste instantie weinig inlaat met de ruimte-tijd variëteit $(\mathcal{M}, g)$.

In de wetenschap dat conforme symmetrie hét organiserende principe van snaartheorie is, groeit het besef dat algemenere, consistente snaarachtergronden mogelijk zijn: de concrete $(\Phi, \mathcal{M}, g)$ data kunnen probleemloos vervangen worden door abstracte conforme-veldentheorieën (CVTs) C. Het voornaamste datum bij een dergelijke CVT, is de energie-momentum tensor, T, die de conforme symmetrie genereert. Verder wordt de theorie gespecifieerd door een (operator)spectrum $\mathcal{O}_\alpha$ en hun OPE-producten onderling en met T. Conforme symmetrie laat dan in principe toe willekeurige correlatoren te berekenen. Als dusdanig behoren modellen van dit type tot de exact oplosbare categorie.

Enkele kanttekeningen hierbij: bij abstracte CVTs zijn formuleringen in termen van een Lagrangiaan, of de geïntegreerde vorm, een actie, niet bekend.

Bijgevolg gaat het intuitieve beeld van een minuscule snaar die beweegt in een gegeven ruimte-tijd verloren. Daar staat evenwel onmiddellijk tegenover dat conforme symmetrie exact en manifest is, wat niet het geval is met de initiële, meer meetkundige achtergronden (zie bv. 'NLSM', p. 163 in de **Glossary** voor een uitgebreidere discussie).

Snaartheorie nu, plaatst abstracte conforme-veldentheorieën op gelijke hoogte als de intuïtief meer voor de hand liggende meetkundige achtergronden. In de afgelopen jaren zijn heel wat prominente voorbeelden bestudeerd waarin geargumenteerd wordt dat een abstracte CVT, met energie-momentumtensor (e.m.-tensor) T_{abs}, zeg maar, continu vervormd kan worden naar een geometrisch model met e.m.-tensor T_{geom}. Dit wil zeggen, er is (minstens) een eenparameterfamilie $T(s), s \in [0,1]$ van e.m.-tensoren en bijhorende spectra, zodat $T(0) = T_{abs}$ en $T(1) = T_{geom}$. Elke tussenliggende $T(s)$ definieert een CVT op zich. We spreken in dat geval van een moduli-ruimte $\mathcal{T}$ van conforme theorieën: $\mathcal{T}$ is een lokaal samenhangende verzameling punten die elk corresponderen met een welbepaalde conforme-veldentheorie, een parameterruimte van theorieën dus.

D-branen

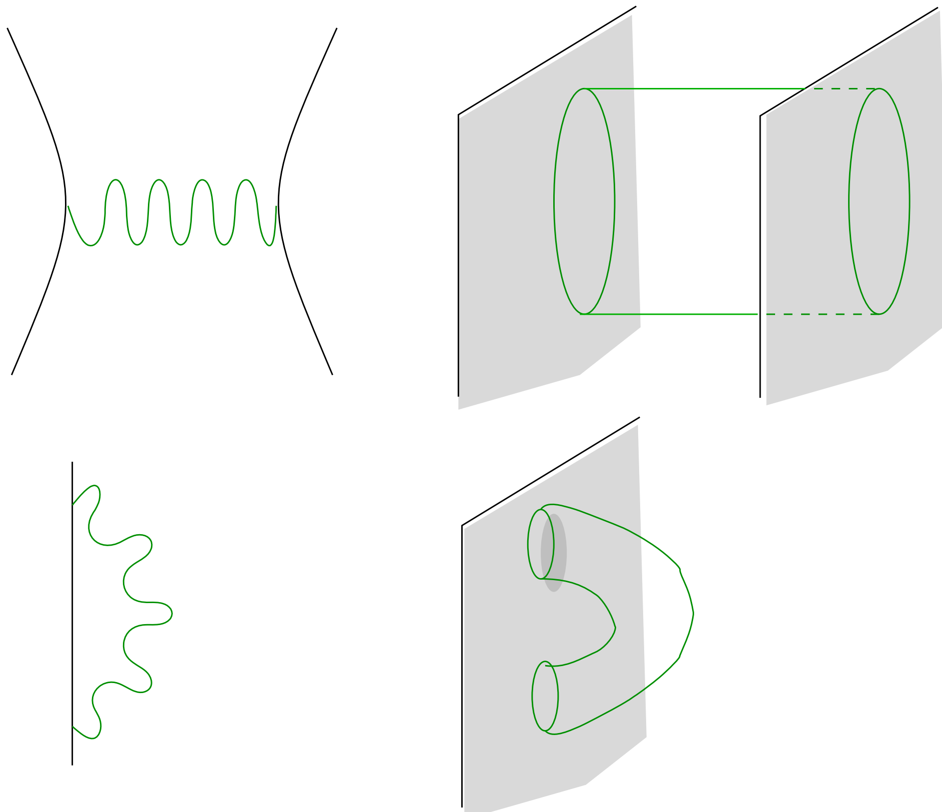
Sedert enige tijd is snaartheorie niet enkel een theorie van snaren: ze bevat ook andere ruimtelijk uitgebreide objecten, D-branen genoemd. Feitelijk zijn het deze laatste die een ware revolutie ontketend hebben in het midden van het afgelopen decennium. Laten we kort nagaan waar het om gaat.

In een meetkundige achtergrond zijn D-branen hogerdimensionale oppervlakken waar eindpunten van open snaren op bewegen. Dit initieel eenvoudige concept brengt evenwel het volgende teweeg:

(a) in tegenstelling tot dat van gesloten snaren, heeft het wereldoppervlak van open snaren randen: de wereldlijnen $L_{1,2}$ van de eindpunten. Als dusdanig introduceren D-branen randvoorwaarden die verzekeren dat $L_{1,2}$ op het D-braan liggen. In meetkundige achtergronden leidt dit typisch tot D(irichlet) randvoorwaarden, vanwaar de terminologie.

(b) De kwantisatie van de open snaren die met een D-braan geassocieerd worden, maken die laatste dynamisch. Intuitief kun je je inbeelden dat het hogerdimensionale oppervlak gaat bewegen en vervormd wordt. In het lage-energie regime legt de Born–Infeld theorie, een veralgemening van de conventionele Yang–Mills ijktheorie, de effectieve dynamica vast.

De termen open snaar, D-braan en randvoorwaarde zijn op die manier onlosmakelijk verbonden.



Figuur C.1: Fotonen mediëren de interactie tussen elektronen onderling en de elektronzelfinteraktie (links); D-branen interageren via gesloten-snaar uitwisseling (rechts).

Randtoestanden

Randtoestanden beschrijven aspecten van D-branen vanuit het perspectief van gesloten snaren. Na de uitleg in de vorige paragraaf kan dit vreemd lijken, maar Fig. C.1 moet verduidelijking brengen: waar elektronen met elkaar of met zichzelf interageren door uitwisseling van fotonen, doen D-branen dit met behulp van gesloten snaren. D-branen kunnen dus gesloten snaren uitzenden of absorberen. In beide gevallen in Fig. C.1 is het wereldoppervlak van de gesloten snaren een cylinder, met randen op de branen. In formules wordt de bijhorende amplitude

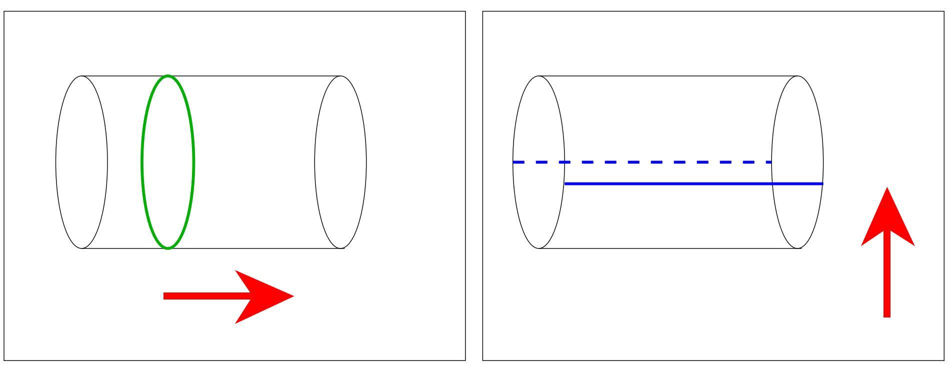
$$\mathcal{A}^{\text{gesloten}}(\tilde{\tau}) := \langle D_1 | \mathrm{e}^{\pi \mathrm{i} \tilde{\tau} H^{cl}} | D_2 \rangle \ , \qquad \text{(C.1.3)}$$

waar H^{cl} de gesloten snaar propageert over een gesloten-snaartijd $\tilde{\tau}$; d.w.z. D-braan 1 zendt een gesloten snaar, $|D_1\rangle$ uit, die geabsorbeerd wordt door D-braan 2, $\langle D_2|$. In feite zijn zowel $\langle D_2|$ als $|D_1\rangle$ heel speciale gesloten-snaartoestanden: meer bepaald gaat het hier om coherente superposities van perturbatieve éénsnaartoestanden. Dit valt als volgt te interpreteren: zoals elektronen zowel gravitonen als fotonen uitzenden, zijn D-branen niet alleen een bron van snaargravitonen, maar een hele resem gesloten snaren, zoals massievere gesloten-snaar modes (tot dusver hebben die laatste geen eigen naam gekregen).

Laten we nu proberen een verband te leggen tussen de beschrijvingen in de secties 'D-branen' en 'Randtoestanden'. Uit Fig. C.2 blijkt dat het cylinderdiagram waarvan hierboven sprake (links in de figuur), even goed alternatief te interpreteren valt : met een andere keuze voor de tijdrichting (rechts in de figuur), wordt het een één-lus-vacuumdiagram voor open snaren, met randvoorwaarden afhankelijk van de D-branen geassocieerd aan $|D_{1,2}\rangle$. In dit open-snaarbeeld wordt de tegenhanger van vgl. (C.1.3):

$$\mathcal{A}^{\text{open}}(\tau) := \text{tr}_{1,2}(\mathrm{e}^{\pi\mathrm{i}\tau H^{op}}) \,, \tag{C.1.4}$$

waar een open-snaar over de eigentijd $\tau = -1/\tilde{\tau}$ wordt gepropageerd in de lus. Vanwege de tr in vgl. (C.1.4) lees je dit best als een partitiefunctie voor de open-snaren met de gestelde randvoorwaarden $1, 2$; deze uitdrukking beschrijft dan ook het spectrum van de open snaren tussen D-branen 1 en 2.



Figuur C.2: Tweemaal hetzelfde wereldvolume: links beschouwd met de gesloten-snaartijd (Born-benadering), rechts in het open-snaarbeeld (één-lus niveau).

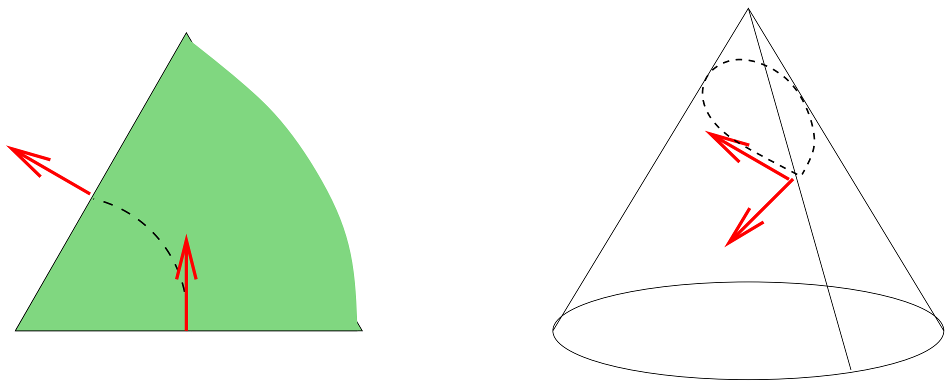
De overgang van het gesloten- naar het open-snaarbeeld komt technisch neer op een modulaire S-transformatie: ze bewerkstelligt $\tilde{\tau} \to \tau$. Stel nu dat we erin slagen toestanden $|D_{1,2}\rangle$ op te schrijven, zodat de S-getransformeerde $\mathcal{A}^{\text{gesloten}}$ te interpreteren valt als een open-snaar amplitude. Zonder in de details te treden, is dit een hoogst niet-triviale voorwaarde: ze legt o.m. op dat $|D_{1,2}\rangle$ welbepaalde coherente superposities zijn van perturbatieve éénsnaartoestanden, zoals eerder al aangegeven, maar pas nu toegelicht. Gesteld dat dit het geval is, kunnen we zeggen dat de randtoestanden D-branen definiëren: de D-branen worden m.a.w. *vastgelegd* door de randtoestanden, eerder dan ze een a priori gegeven zijn. Voor de klasse van zgn. rationale conforme-veldentheorieën heeft Cardy een canoniek voorschrift uitgewerkt dat dergelijke randtoestanden oplevert. Een kerngegeven hierbij, is dat de procedure intrinsiek is aan de conforme-veldentheorie.

Vanwaar de nadruk hierop? Ik herinner de lezer eraan dat het beeld van D-branen als hoger-dimensionale oppervlakken in de ruimte-tijd (zie sectie 'D-branen') heel sterk refereert naar een ruimtelijke interpretatie. Anderzijds, laat snaartheorie evengoed abstracte CVTs (sectie 'Snaren') toe als consistente snaarachtergronden. De totale afwezigheid van enige ruimtelijke interpretatie bij die laatste klasse achtergronden vormt de aanleiding tot de vraag hoe dat dan zit met D-branen. Wel dan, Cardy's strategie breidt de D-braan notie uit van meetkundige naar algemene conforme-veldentheorie snaarachtergronden, precies doordat ze in intrinsieke CVT-data geformuleerd is, zonder enige verwijzing naar ruimtetijd.

Orbifold ruimten

Een laatste voorname ingrediënt in deze thesis zijn de zgn. orbifold ruimten. Dit zijn ruimten M die er lokaal uitzien als een quotient $\mathbb{R}^n/G$ waarbij G een eindige puntgroep van n-dimensionale rotaties is. Zo je wil, correspondeert een punt van M met een G-orbiet van punten in $\mathbb{R}^n$. In twee dimensies kun je je alternatief het volgende voorstellen: neem een segment met een openingshoek $\alpha = 2\pi/n$ ($n \in \mathbb{N}_0$). De kegel die ontstaat door de randen van het segment te identificeren, is een model voor $\mathbb{R}^2/\mathbb{Z}_n$. In het bijzonder is de orbifold ruimte lokaal vlak, en vertoont ze een konische singulariteit in de top van de kegel (zie Fig. C.3).

Hoe zou een tweedimensionale waarnemer de singulariteit kunnen vaststellen? Eenvoudigweg, door een stok voor zich uit te houden, en bv. rechtdoor te wandelen langs een cirkelsegment, zoals aangegeven in Fig. C.3. Terug op zijn beginpositie, zou hij vaststellen dat de richting van de stok gedraaid is over een hoek $2\pi/n$! Dit fenomeen zou zich in feite voordoen voor elk gesloten traject rond de singulariteit: n keren rond het traject lopen brengt de stok terug in de initiële richting. Dit illustreert dat de ruimte niet globaal vlak is. Technisch gesproken, ondergaan raakvectoren bij parallel transport rond de singulariteit een holonomie-transformatie, toe te schrijven aan de opgehoopte Riemannse kromming in de singulariteit.



Figuur C.3: Een tweedimensionale vlakke orbifold ruimte. Bij identificatie van de randen van het segment (l.) ontstaat de kegel (r.). Parallel-transport van de vectoren langs het traject aangegeven in ... detecteert een konische singulariteit.

Ondanks de singulariteiten, is snaartheorie perfect zinvol op orbifold achtergronden, allicht tegen de verwachtingen in. Historisch is dit op twee manieren uitgelegd. Ten eerste bevat gesloten-snaartheorie extra toestanden tov. snaren in vlakke ruimte: de zgn. getwiste sectoren, waar gesloten snaren feitelijk enkel gesloten zijn modulo G; dwz. dat de snaarvelden in dergelijke sectoren aan gewijzigde periodiciteitsvoorwaarden voldoen, typisch

$$\phi(\sigma + 2\pi, \tau) = g_* \phi(\sigma, \tau), \ (g \in G), \tag{C.1.5}$$

waar σ, τ het wereldoppervlak parametriseren. Deze sectoren zijn niet echt een optie: consistentie, met name, modulaire invariantie, *vereist* dat deze deel uit maken van de theorie. Verder is het zo dat die getwiste sectoren de nodige ingrediënten bevatten om de singulariteit ongedaan te maken (zie Hoofdstuk 3), en dit gegeven werd lange tijd beschouwd als een verklaring van het verbazend goede gedrag van de snaartheorie.

Is er meer fysisch begrip mogelijk? Laten we eerst aangeven dat een alternatief, equivalent beeld van orbifold ruimten mogelijk is: er kan aangetoond worden dat het ontstaan van de konische singulariteit gepaard gaat met cykels, deelruimten, zeg maar, die minuscuul klein worden. Stel nu dat een D-braan rond zo een cykel gewonden is. Een eindig-volume cykel geeft zo aanleiding tot een massieve D-braantoestand, via de eindige spanning van deze laatste. In de orbifold limiet, waar het volume van de cykel naar nul gaat, vinden we zodoende massaloze toestanden. Het zojuist geschetste beeld is echter niet volledig. Naast de volumefactor, genereert ook de aanwezigheid van een eindige B-flux een D-braanmassa. De ruimten die overeenkomen met de niet-singuliere conforme-veldentheorie zijn precies van het type waarbij cykels een nulvolume hebben, maar waar een eindige B-flux opgehoopt zit in de singulariteit. De bijhorende D-braantoestanden zijn dan ook massief. Hoe past dit nu in het plaatje? Wel, stel zowel volumes als B-flux nul waren, dan zouden er massaloze D-braantoestanden bestaan, geassocieerd met branen die rond nulvolume-cykels gewonden zitten. In de effectieve supergravitatie-beschrijving van snaartheorie zijn de bijhorende velden echter niet aanwezig: er zijn aldus *extra* massaloze toestanden, buiten diegene, beschreven door de effectieve-veldentheorie. Typisch wordt een dergelijke situatie gesignaleerd door divergenties in diverse amplitudes; m.a.w., de theorie gedraagt zich slecht. Niets van dit alles echter, in de orbifold conforme-veldentheorie. Dit wordt dan ook uitgelegd door de aanwezigheid van B, die voorkomt dat D-braantoestanden massaloos worden.

Dit alles, gecombineerd met de wetenschap dat orbifold ruimten *exacte* conforme-veldentheorieën genereren, maakt die laatste tot het ideale studiedomein voor diverse aspecten van snaartheorie.

C.2 Overzicht en samenvatting

In deze thesis staan D-branen en orbifold meetkunde centraal. In een reeks publicaties [1, 81, 111] werden randtoestanden voor orbifold D-branen systematisch ontwikkeld. We vonden het zinvol de resultaten te situeren binnen het ruimere kader van snaarmeetkunde, die in feite uiteenvalt in twee delen: bevat Hoofdstuk 3 het 'klassieke' gezichtspunt, dan vormt McKay correspondentie de bulk van Hoofdstuk 5. Inzicht in zowel snaartheoretische als puur wiskundige aspecten van die correspondentie is relatief jong, en nog volop in ontwikkeling. Bij elkaar genomen, neemt een overzicht van dit referentiekader zowat de rest van deze thesis in beslag. Dit impliceert dan meteen dat een aantal onderwerpen waarover we gepubliceerd hebben, geen plaats vinden binnen het bestek: we besteden geen aandacht aan speciale Kählermeetkunde [112, 113, 114], anomale koppelingen van D-branen en oriëntifold vlakken [64, 65, 66, 81], of NS-branen in type-0 theorieën [115]. Voor een behandeling van de laatstegenoemde twee onderwerpen kun je terecht bij [63].

In wat volgt, geven we een overzicht van de verschillende Hoofdstukken.

C.2.1 Hoofdstuk 2 : Aspecten van conforme-veldentheorie

Hoofdstuk 2 vormt in zekere zin een aanloop tot deze thesis. In Sectie 2.1 worden drie gezichtspunten op D-branen ontwikkeld. Ten eerste bestaan er zogenaamde *D-braanoplossingen* in supergravitatie, verder kun je D-branen zien als *randvoorwaarden*, en tenslotte zijn D-branen onlosmakelijk verbonden met *ijktheorieën*.

Sectie 2.2 belicht een aantal voornamelijk algebraische aspecten van conforme-veldentheorieën. Aan bod komen de structuur van de Hilbertruimte van toestanden, fusie-regels en de Verlinde-formule, en modulaire transformaties. Uiteraard is dit standaardmateriaal, en de sectie wil voornamelijk een referentiekader voor het vervolg van de thesis scheppen.

Tot slot is er in Sectie 2.2.3 ook aandacht voor verbanden tussen supersymmetrieën in wereldvolume- en ruimte-tijdtheorieën. In feite is dit een eerste voorbeeld van het centrale thema in de thesis, nl. een antwoord op de vraag hoe ruimte-tijd- en wereldvolumetheorieën elkaar weerspiegelen.

C.2.2 Hoofdstuk 3 : Meetkunde: deeltjes, snaren en D-branen

Een uitgebreid Hoofdstuk 3 brengt een collage van (snaar-)meetkundige aspecten in de bestaande literatuur samen, met een tweevoudig objectief: enerzijds wil ze de coherentie tussen klassieke meetkunde, conforme-veldentheorie en D-braantechnieken onder het voetlicht plaatsen, terwijl ze anderzijds explicietheid nastreeft. Verder zijn de volgende nevenresultaten nieuw:

(a) In een voorbereidende fase die de thesis voorafging, vonden we expliciete uitdrukkingen voor de chirale sporen over de orbifold submodules als Z_N-representaties. Deze zijn samengevat in Tabel B.1.

(b) We leggen een verband tussen het wereld-volume fermiongetal en het concept 'age-grading', wat ons toestaat een intermediaire versie van McKay correspondentie in een snaarfysische context te identificeren. De vaststelling wordt ondersteund door inzichten uit de torusmeetkunde.

Sectie 3.1 brengt een aantal klassiek-meetkundige noties in herinnering. Speciale aandacht gaat eerst uit naar aspecten van speciale holonomie, en hoe deze weerspiegeld wordt in termen van superconforme symmetrie-algebra's enerzijds, en ruimte-tijd supersymmetrie anderzijds. De resultaten zijn samengevat in Tabellen 3.2 en 3.3. Een tweede onderwerp dat uitgebreid behandeld wordt, betreft desingularisatie van orbifold ruimten. Kort gesteld, zijn twee methodes voorhanden: opblazen en vervormen. Beide worden gedetailleerd geïllustreerd aan de hand van ADE-singulariteiten.

In Sectie 3.2 bekijken we de formele structuur van de conforme-veldentheorie geassocieerd met orbifold ruimten. Het stramien in Hoofdstuk 2 volgend, spitsen we de aandacht vooral toe op de toestands-Hilbertruimte en de fusieregels. Ook hier dient een uitgebreide analyse van ADE-orbifolds als concreet referentiekader voor de lezer. Meer in het bijzonder, wordt de aandacht gevestigd op een correspondentie tussen conforme-veldentheorie en de meetkunde van opgeblazen orbifold ruimten, zoals eerder al uitgewerkt. Omdat ADE-singulariteiten in menig opzicht te eenvoudig zijn, werken we tenslotte een correspondentie uit voor abelse zgn. Calabi-Yau orbifold groepen. Technieken uit torusmeetkunde maken de overgang van veldentheorie naar meetkunde expliciet zichtbaar.

Sectie 3.3 tenslotte, behandelt branen op orbifold ruimten, de zgn. fractionele D-branen. Dat representatietheorie een sleutelrol speelt bij de identificatie van elementaire fractionele branen, komt niet totaal uit de lucht gevallen. Met dit gegeven exploreren we verder hoe desingularisatie in het algemeen, en voor ADE-gevallen in het bijzonder, *fysisch* gerealiseerd wordt in de ijktheorie op de branen.

C.2.3 Hoofdstuk 4 : Randtoestanden

Dit Hoofdstuk ontwikkelt een systematische aanpak van randtoestanden. Daartoe bestuderen we in Sectie 4.1 eerst randvoorwaarden: de klasse van randvoorwaarden die een centrale rol gaan spelen zijn die, die een deelalgebra van de volledige symmetrie-algebra bewaren. Dit principe wordt toegepast op een $N = 4$ superconforme symmetrie-algebra in twee dimensies.

Sectie 4.2 herneemt Cardy's constructie van consistente randvoorwaarden en -toestanden in algemene rationale modellen. Zoals uitgelegd op p. 153, betekent "consistent"hier dat een open-snaar interpretatie zinvol is. Cardy's voornaamste resultaat is bevat in een uitdrukking voor de randtoestanden in termen van de modulaire S-matrix. Anders gezegd, geeft formule (4.2.19) consistente randvoorwaarden (D-branen) in termen van intrinsieke conforme-veldentheoriedata.

Vooraleer orbifold-randtoestanden aan te pakken, worden randtoestanden voor supersnaar D-branen in vlakke ruimte in detail uitgewerkt. Hierbij is de analyse van de fermionische component in Sectie 4.3.2 een nieuw gegeven. Verder wordt uitgelegd hoe de absolute normering van de randtoestanden, en bijgevolg ook de fysische spanning en lading van de D-branen, *afgeleid* wordt uit beschouwingen van bosonische nulmodes.

In de afsluitende Sectie 4.4 laten we zien hoe Cardy's voorschrift veralgemeend wordt, zodat het de randtoestanden voor fractionele orbifold D-branen produceert. Oorspronkelijk waren de randtoestanden op een primitievere manier geconstrueerd in onze artikels [81, 111], door handmatig de consistentie te gaan opleggen. Omdat het Cardy-kader deze resultaten impliceert, hernemen we de resultaten in die publicaties niet.

C.2.4 Hoofdstuk 5 : McKay correspondentie

Hier komen verschillende formuleringen van de McKay correspondentie aan bod. Eenvoudig gesteld, legt ze een niet-triviaal verband tussen de meetkunde van gedesingulariseerde orbifold ruimtes enerzijds, en representatietheorie van de orbifold groep G anderzijds. Dit gebeurt op verschillende niveaus:

(a) Het topologische Euler getal χ van de orbifold resolutie is gelijk aan het aantal conjugatieklassen van G.

(b) Noem $K(X)$ de K-theorie ring van (equivalentieklassen van) holomorfe vectorbundels op de resolutie, en $K^c(X)$ de ring van bundels met compacte drager, dan bestaat er een reguliere paring op de produktgroep. Verder bestaan er duale bases $\{\mathcal{R}_I\}, \{S^J\}$ tov. dit inprodukt, zodanig dat (S^I, S^J) uitgedrukt kan worden in termen van de representatietheorie van G.

Naast deze zwakke en sterke vorm, vind je nog een intermediaire formulering terug in Sectie 5.1.

In Sectie 5.2 gaan we na hoe D-branen de Sterke McKay Correspondentie realiseren in een fysische context. Een sleutelrol is daarbij weggelegd voor de spinbundel over X. Een dergelijke expliciete uitwerking was o.i. tot dusver een hiaat in de bestaande literatuur. We hopen dan ook dat deze sectie enige verduidelijking brengt.

Tenslotte dan, vormen fractionele D-branen op ADE-singulariteiten het onderwerp van Sectie 5.2.3. Een grondige analyse met de randtoestanden als uitgangspunt, laat zien hoe de fractionele D-branen McKay correspondentie realiseren via topologische intersecties. Eens te meer is dit een illustratie van een niet-triviaal verband tussen a priori geometrische en veldentheoretische eigenschappen. In minder uitgebreide vorm is deze analyse terug te vinden in ons artikel [1].

C.2.5 Hoofdstuk 6 : Discrete torsie

Modulaire invariantie, een sterke consistentievoorwaarde op gesloten-snaarmodellen, blijkt dikwijls niet voldoende om orbifold groepen eenduidig te identificeren met de genoemde snaarmodellen: deze ambiguïteit wordt discrete torsie genoemd; ze komt erop neer dat de toruspartitiefunktie in orbifoldmodellen een keuzevrijheid toelaat wat relatieve gewichten betreft:

$$\mathcal{Z} = \sum_{g,h} \varepsilon(g,h) \left| g \, \underset{h}{\square} \right|^2 , \tag{C.2.1}$$

met een afbeelding $\varepsilon : G \times G \to U(1)$. Omdat de partitiefunktie het gesloten-snaarspectrum beschrijft, stellen we vast dat eenzelfde orbifoldgroep geassocieerd kan worden met uiteenlopende spectra. In Sectie 6.1 hernemen we Vafa's [100] oorspronkelijke analyse van het fenomeen, met als resultaat dat de groepcohomologie $H^2(G;U(1))$ de keuzevrijheid qua fasefactoren onder controle brengt. Anders gezegd, zijn de inequivalente modellen bij een gegeven groep G eenduidig te identificeren met cohomologieklassen. Een expliciet voorbeeld, $G = \mathbb{Z}_6 \times \mathbb{Z}_6$ moet de lezer enige houvast bieden.

Sectie 6.2 onderzoekt de implicaties van discrete torsie op D-branen en open snaren. Waar gesloten snaren in verband gebracht werden met groepcohomologie, nemen projectieve representaties de organiserende rol over bij open snaren. Dit resultaat werd al vooropgesteld door Douglas in [99]. In termen van de randtoestanden, geconstrueerd in [1], is een alternatief bewijs algebraïsch en eenvoudig; we geven dit nieuwe argument in Sectie 6.2. Verder maakt Douglas' analyse van de ijktheorieën duidelijk dat D-branen nu discrete ladingen dragen. Dit wil zeggen dat de ladingen torsie-elementen zijn in het ladingsrooster: identieke niet-nul discrete ladingen, voldoende in aantal, annihileren elkaar. Dit bizarre fenomeen is toe te schrijven aan de gewijzigde orbifoldprojectie, die de geassocieerde niet-invariante R-R potentiaal uit het spectrum verwijdert.

Afsluitend, wordt de correspondentie tussen conforme-veldentheorie en meetkunde herzien voor modellen met torsie. Het meest in het oog springend, is de vaststelling dat niet langer alle snaarmodes aanwezig zijn om de orbifold singulariteit teniet te doen, zoals oorspronkelijk opgemerkt door Vafa en Witten [105]. Tot dusver fungeerden voornamelijk $G = \mathbb{Z}_n \times \mathbb{Z}_n$-modellen voor $n = 2, 3$ als voorbeelden in de literatuur. Het $n = 6$ voorbeeld, dat een rijker spectrum van modellen toestaat vanwege het acyclische karakter van $H^2(G;U(1)) = \mathbb{Z}_2 \times \mathbb{Z}_3$, moet de eerste stappen in een meer gedetailleerde ontwikkeling van de correspondentie illustreren. We beperken ons tot een discussie in termen van torusmeetkunde, en laten een nadere analyse voor verder onderzoek.

C.2.6 Ter conclusie

Om af te ronden lijkt het zinvol na te gaan wat precies bereikt werd in deze thesis. Ontegensprekelijk staat de uitbreiding van Cardy's constructie van randtoestanden naar orbifold CVTs voorop. De aldus geconstrueerde toestanden zijn de gesloten-snaar beschrijving van orbifold D-branen. Cardy's formalisme heeft ons in staat gesteld D-branen in willekeurige meetkundige orbifold achtergronden te behandelen in één enkel geünificeerd kader.

Hiernaast werden ook nog enkele nevenresultaten bereikt, te weten:

(a) Torusmeetkunde weerspiegelt een versie van McKay correspondentie tussen de sterke en zwakke vorm in; de fysische realisatie gebeurt door gesloten snaren (zie Hoofdstuk 3).

(b) Bij een G-orbifold zijn de $SO(2)_1$-modules manifest te ontbinden in G-representaties. De bijhorende expliciete uitdrukkingen voor de karakters vind je terug in Appendix B.

Van een intrinsiek interessantere aard is de wetenschap dat het onderwerp niet 'af' is: de voornaamste open problemen zijn o.i.

(a) D-branen in orbifold achtergronden met discrete torsie vallen bezwaarlijk onder de noemer 'volkomen begrepen'. Het blijft een uitdaging uit te leggen hoe McKay correspondentie, discrete ladingen en K-theorie onder één dak te brengen zijn.

(b) Tot dusver is weinig concreet onderzoek verricht naar de relatie tussen McKay correspondentie en orientifold achtergronden. Aangezien laatstgenoemde in staat zijn $N = 1, d = 4$ ijktheorieën te produceren, lijkt een gewijzigde correspondentie niet compleet uit de lucht gegrepen. De sleutel ligt wellicht in een beter begrip van de orientifold projectie.

D

Glossary

NLSM In the non-linear sigma-model approach to string theory, which is the two-dimensional QFT governed by the action

$$S = \frac{1}{\alpha'} \int_\Sigma G_{\mu\nu}(X) \partial X^\mu \bar{\partial} X^\nu \ + \ \ldots \ ,$$

the fields can be expanded:

$$X = x_0 \mathbb{1} + X_{qu} \ ;$$

that is, $\langle X \rangle = x_0$, or the classical and quantum pieces have been separated. The metric on the target space is likewise expanded around x_0, yielding effectively

$$S = \frac{1}{\alpha'} \int_\Sigma G_{\mu\nu}(x_0) \partial X^\mu_{qu} \bar{\partial} X^\nu_{qu} + \ldots \ .$$

where '...' involves subleading (higher order) terms. Classical string backgrounds are as such specified by a choice of a target space metric $G_{\mu\nu}$ (and other fields). The corresponding NLSM describes strings propagating in that background.

worldsheet In the worldsheet point of view, string theory consists of a collection of auxiliary two-dimensional quantum field theories, living on Riemann surfaces. In these two-dimensional theories, the rôle of $\hbar$ is played by α', the inverse tension of the string: α' governs the loop expansion of the two-dimensional correlation functions. Sending α' to zero, i.e. taking a limit where all dimensionful quantities p are such that $\alpha'^r p \to 0$, corresponds physically to treating strings as point particles.

boundary state Boundary states are closed-string states that preserve a diagonal subalgebra $\mathcal{B}$ of the chiral $\times$ anti-chiral symmetry algebra, i.e. they typically solve conditions like $(\mathcal{O}_L + \mathcal{O}_R)|B\rangle = 0$ for $\mathcal{O}_L + \mathcal{O}_R \in \mathcal{B}$. As such, they are observed to impose boundary (gluing) conditions that relate left- and right-moving closed-string worldsheet fields.

BPS States are called BPS if they preserve supersymmetry. With central charges Z in the supersymmetry algebra setting a lower bound for the masses M in that charge sector, $M \geq |Z|$, states saturating the bound are guaranteed to be BPS. Since they are annihilated by the preserved SUSY generators, such states come in smaller irrepses (shorter multiplets) of the original supersymmetry algebra. A collection of states is called mutually BPS if all preserve the *same* supersymmetry subalgebra.
Examples:
1. $\frac{1}{2}$-BPS D-brane states in flat Minkowski space, having a mass density equal to their RR-charge density.
2. Chiral primary states of the $N = 2, d = 2$ SCA. These states saturate $h = q$, i.e. their conformal weight equals their $U(1)$-charge.

space-time singularity A space-time singular point is characterised by some curvature scalar, e.g. $R_{\mu\nu}R^{\mu\nu}$ diverging there. In general relativity, the solutions to Einstein's equations are believed to be such that their singularities are hidden behind a so-called event horizon; i.e. no naked singularities occur. The prototypical example is the Schwarzschild black hole.

orbifold singularity (Flat space) orbifolds considered in the present thesis locally look like $\mathbb{R}^d/G$, where G is some finite subgroup of the group of rotations centered at the origin. Except at the origin therefore, they are locally isomorphic to $\mathbb{R}^d$; at the origin, however, there is a singularity of conical type.

type II strings Closed-string theories with a gauged $(1,1)$ worldsheet supersymmetry, and $(c, \tilde{c}) = (15, 15)$ matter CFTs exist in four types: IIA/IIB and 0A/0B, of which only the first two realise ($N = 2$) space-time supersymmetry. The A and B types are distinguished by their left-moving Ramond ground state having the same (B) or opposite (A) space-time chirality as the right-moving one.

Ramond-Ramond states In the simplest type II theories, boundary conditions can be imposed indepently on the left- and rightmoving worldsheet fermions:

if periodic (anti-periodic), the corresponding sector is called Ramond, R, (Neveu-Schwarz, NS). Accordingly, type II theories have four sectors: NS-NS, R-NS, NS-R and R-R, labeled by left- and right-moving sectors. The R ground state is a space-time spinor, and therefore, the R-R ground state is a bispinor. Upon use of Fierz identities, the latter is decomposed into differential p-forms, with p even/odd in IIA/IIB, due to the relative chirality of the left- and right-moving spinors.

SUGRA multiplet A supersymmetry multiplet is a collection of fields ϕ_i, such that the supersymmetry algebra is realised on ϕ_i and their derivatives. The multiplet is called off-shell provided the algebra closes without invoking the equation of motion for the ϕ_i, and on-shell otherwise. Besides the SUGRA multiplet, that contains the graviton, there exist so-called matter multiplets, e.g. the hypermultiplet, vector multiplet and tensor multiplet ($d \leq 6$).

divisor Given a complex manifold $\mathcal{M}$ of dimension n, a formal linear combination $\sum_i n_i V_i$ ($n_i \in \mathbb{Z}$) of analytic codimension 1 subvarieties is called a divisor. Equivalently, a divisor is associated to a meromorphic section s of a holomorphic line bundle L over M: V_i are the sets of points where s vanishes or has poles, and n_i are the orders of the zeroes ($n_i > 0$) or the poles ($n_i < 0$).

In a blow-up $\tilde{X} \xrightarrow{\pi} X$ of a point $p \in X$, a divisor $\mathcal{E} \subseteq \tilde{X}$ is called exceptional, if $\mathcal{E} \subseteq \pi^{-1}(p)$.

birational map A birational map $Y \xrightarrow{f} X$ is an isomorphism on dense subsets of Y, X. In particular, f or f^{-1} need not be well-defined on the complements of these subsets. Even though a weaker equivalence than homeomorphism, birational equivalence is a useful notion in algebraic geometry. Blow-ups $Y \xrightarrow{\pi} X$ are canonical examples of birational maps.

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