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J=0 STATES OF THREE BOSONS BOUND BY NON-RELATIVISTIC LOCAL POTENTIALS*†

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ABSTRACT

Binding energies and approximate wave functions of the J=0 bound states of three identical bosons have been computed under the assumption that only two-body forces in the ℓ=0 state derived from local potentials of the Yukawa and exponential form are important. In both cases, only three bound states occur for interaction strengths below or comparable to that required to give a second bound state in two-body subsystems, the first excited state appearing when the two-body subsystems bind, and the second close to the interaction strength where the two-body subsystems have zero scattering length. Also close to this point, the Yukawa interaction produces a collapse of the ground state to a tightly bound system of dimensions small compared to the range of the interaction, while the exponential interaction does not; separable interactions of the Bander or Yamaguchi form fitted to the same two-body binding energy and scattering length give infinite binding energy to the three-body system before this point is reached, at an interaction strength only 40% greater than that required to fit the n-p triplet scattering length and deuteron binding energy using central forces.

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The quantum mechanical three-body problem and in particular the problem of determining the binding energy of three strongly interacting nucleons has received the attention of physicists for a number of years. A variety of techniques have been devised to treat the problem. Variational methods applied to the three-nucleon problem still yield upper and lower bounds for the binding energy which differ by an order of magnitude[1]. The special dynamical technique used by Baker, Gammel, et al.[2] to obtain a three-body ground state wave function is valid only for the ground state and has so far not been followed up. The most promising development has been that of Faddeev[3] who presented a well defined set of integral equations which treat on an equal footing all possible non-relativistic processes: bound states, elastic scattering, exchange interactions and breakup. Typically[4] Faddeev's equations have only been exploited by studying problems in which the two-body interaction is separable, i.e. $I(p,q) = f(p)f(q)$, and consequently non-local. Employing separable potentials in Faddeev's equations is mathematically advantageous in that it reduces the problem to that of solving a one variable integral equation. While this is a compelling reason for using separable potentials, it is not equivalent to justifying their use as an accurate representation of a physical interaction. In fact it is the local potential description of interactions which is known to be applicable to high accuracy in atomic problems, to the high angular momentum states of the nucleon-nucleon system[5], and even to the long-range part of the s-wave nucleon-nucleon interaction[6]. Thus it would be extremely useful to have a reliable and numerically accurate method for solving the three-body problem via Faddeev's equations for local interactions. Having such a method we can then use the non-relativistic three-

body problem as a theoretical laboratory in which to test the consequences of various two-body models. We can examine whether or not the three-body problem is sensitive to the local or non-local character of the interactions for models which give comparable agreement with the two-body subsystems. Finally, we should be able to obtain information from physical three-body systems about features of the interaction not probed by two-body scattering. We present below a calculation of this type.

The reduction of the Faddeev equations to integral equations in two variables with $3 \times (L + 1) \times \min(2J + 1, 2L + 1)$ coupled components has already been given by Osborn and Noyes[7] and by Ahmezadeh and Tjon[8], where J is the total angular momentum and L the maximum orbital angular momentum included in the two-body subsystems. We will use the variables $E = \omega_1 + \omega_2 + \omega_3$ and ω_i as defined in Reference 7. The problem we describe here in detail is the simplest of all three-body problems: that of three identical spinless particles in the J=0 bound states. For the purely attractive forces considered here the most deeply bound states will occur for J=0. It is reasonable to assume as a first approximation that centrifugal shielding in the two-body subsystems will prevent any state except the ℓ=0 from making an important contribution to the binding energy. Thus we are left with three integral equations in two variables, which because of bose symmetry reduce to just one integral equation.

Part of our purpose is to study the consequences of separability in three body systems. So we want to reformulate (without changing its physical content, of course) our integral equation in such a way that the separable approximation manifests itself in an obvious way. This is easily done. One of the most general separable formulations of the two-body partial wave transition amplitude, $t_\ell(p,q;s + i\epsilon)$, is the Kowalski-Noyes[9]

representation which reads

$$t_\ell(p,q;s + i\epsilon) = f_\ell(q,s)t_\ell(k,k;s + i\epsilon)f_\ell(p,s) + \frac{\pi}{2}\frac{q^2 - s}{s} R_\ell(p,q;s) \quad (1)$$

Here q and p describe the magnitude of the incoming and outgoing momenta in the relative two-body C.M. system; s is the energy and $k = \sqrt{-2\mu s}$, for $s < 0$. The $f_\ell(p,s)$ appearing in this relation is the half-off-shell extension function defined by $f_\ell(p,s) = t_\ell(p,k;s + i\epsilon)/t_\ell(k,k;s + i\epsilon)$. For any separable potential the R_ℓ term is identically zero. Using this two-body formulation of separability our three-body integral equation may be rewritten so that the dependence of the resulting eigenfunction on E is just a constant when separability is valid. Writing $E = k^2/m$ and $\omega_i = q^2/2m$ then our final integral equation is

$$f(k'^2 - \tfrac{3}{4}q'^2,\ z - \tfrac{3q'^2}{4m})\, M(k'^2,q') = \text{inhomogeneous term}$$

$$+ \frac{4}{\pi}\int_0^\infty q''^2 dq''\ \frac{1}{2q'q''}\int_{q'^2 + q''^2 - q'q''}^{q'^2 + q''^2 + q'q''} dk''^2 \times \quad (2)$$

$$\times \frac{t(k'^2 - \frac{3}{4}q'^2,\ k''^2 - \frac{3}{4}q'^2;\ z - \frac{2q'^2}{4m})}{mz - k''^2} f(k''^2 - \tfrac{3}{4}q''^2,\ z - \tfrac{3q''^2}{4m}) \times M(k''^2,q'')$$

For convenience we have dropped the $\ell=0$ subscript on all functions. The whole left hand side of Eq. (2) is the wave function, and M is constant in k'^2 if t is separable. The details of the inhomogeneous term are not important here since we are looking just at eigenvalue solutions. When M is constant in k'^2 it is trivial to reduce (2) to an integral equation involving the single variable q'.

Our technique for solving these integral equations is to transform

them to matrix form by approximating the integrals by a finite sum. In the separable case the one variable integral equations for M may be easily handled by making a change of variable to give finite limits to the integral. The integral is efficiently converted to a sum by using Gaussian quadrature. Seven points in the q dimension is sufficient to give an accuracy of 10^{-3} in the binding energy. The general two variable equation for M is more difficult. For a fixed point mesh in the k dimension few (or even zero!) points may fall between the variable upper and lower limits. We get around this difficulty in two ways. First, we employ a quadrature rule that uses neighboring points outside the interval of integration. Second, we adjust the weights in our quadrature rule for k to evaluate exactly all the rapidly varying terms in the integral. The computer used (a Burrough's B5500) has about 10K of fast memory available. This limited the k mesh to 15 points and the q mesh to 7. One check of the accuracy of this procedure was to solve the two variable integral equation for a separable problem. Then independently we solve the resulting one variable separable equation. The bound state energies (accurately known) from the one variable equation always agree to better than $\frac{1}{2}\%$ with the energy obtained from the two variable version of the problem. Another check is to do just the k" integration in the two variable problem. This leaves us with the kernel of the resulting separable problem. The matrix elements of this kernel were all determined to better than 2% and all but 3 of the 49 elements are given to better than 1%. Our experience indicates that our three-body solutions for local potentials are accurate to better than $\frac{1}{2}\%$. Solutions for a Yukawa potential in the limit of small coupling constants can be compared to the published results of Wong and Zambotti[10] and the results do not agree. However, a new calculation by Wong and Ball[11]

using an expansion in separable Stürmian functions[12] of comparable numerical complexity to the calculation presented here, agrees to better than $\frac{1}{2}\%$ in the binding energy. These authors have also calculated the correction to the binding arising from the $\ell=2$ states and find that it is less than $\frac{1}{2}\%$.

We have summarized our results in Fig. 1 and 2. These figures contain our solutions for two frequently studied potentials - the Yukawa and exponential. The units for these curves are fixed by choosing the nucleon mass for the three bosons. In momentum space the Yukawa potential has the form $v(p,q) = (\lambda/2pq)Q_0([p^2+ q^2+ \mu^2]/2pq)$. The value for parameter $\mu = 0.633\ F^{-1}$ is chosen from a calculation which fits this potential to the low energy triplet N-P scattering. Consequently at $\lambda = -1.58$ we approximate the deuteron. In fact, as is seen from Fig. 1, our two-body bound state energy is (as it must be) about $\alpha = -.23\ F^{-1}$. The exponential potential with $\mu = 1.49\ F^{-1}$ is similarly chosen and approximates the deuteron for $\lambda = -6.0$. The solid curves in both figures represent energy eigenvalue solutions plotted as a function of the coupling constant λ. Both potentials have three different three-body bound states. In both cases the second excited state emerges from the continuum midway between the appearance of the first and second two-body bound states. Around $\lambda = -2.6$ the Yukawa potential problem exhibits a striking behavior. The bound states trajectories which are constant up until this value of λ undergo a rapid change with emergence of the second excited state. For $\lambda < -2.6$ the rate of binding of the ground state is seven-fold greater than for the first section of this trajectory. Simultaneously the first excited state jumps down and then assumes a path which is just a linear continuation of the first portion of the ground state trajectory. Finally, the newly emerged second excited state continues along a linear extension

of the original first excited state trajectory. None of this behavior is observed in the exponential problem.

In order to facilitate the interpretation of this phenomena we have calculated the eigenfunctions M. These eigenfunctions characteristically have only a very mild dependence on k^2 so we will describe them as though they depended only on q. As is expected the ground state eigenfunction is nodeless, the first excited state has one node and the second two. For $\lambda > -2.6$ the ground state M is highly concentrated near q=0 rapidly falling off as q becomes large. At $\lambda \approx -2.6$ it expands and is nearly constant, falling off to zero only for $q \gg \mu$. Thus it resembles a delta function in coordinate space. Our interpretation then of the change in the ground state curve is that the wave function has fallen down the 1/r portion of the potential well. Examining the second excited state wave function for $\lambda \approx -2.6$ we see that it has come to resemble the weakly bound ground state wave function by having its zero move out to very large values of q. Where the wave function is large it has a similar shape and value to the ground state wave function for $\lambda > -2.6$. A similar correspondence exists between the first excited state before the break and the second excited state after the break. In this case the second zero of the second excited state has moved out to very large momenta and the first zero has moved to the position of the zero in the weakly bound first excited state.

The separable approximations studied were the Yamaguchi[13] interaction $v(p,q) = -\lambda/(p^2+\mu^2)(q^2+\mu^2)$, the interaction used by Bander[14] $v(p,q) = -\lambda/(p^2+\mu^2)^{\frac{1}{2}}(q^2+\mu^2)^{\frac{1}{2}}$, and the Kowalski-Noyes approximation obtained by setting R_ℓ equal to zero in Eq. (2). In the first two cases the parameters were fitted by requiring the separable interaction to give the same scattering length and binding energy as the local two-body interaction.

Comparing the three-body binding energies gives a simple test of the ability of the separable potentials to reproduce the physics of our local potentials. The results given in Fig. 1 and 2 show that even at interaction strengths comparable to the N-P triplet even state, the separable approximations deviate from the local potential predictions by about 10%, and in the first two cases lead to infinite binding for the three-body system at an interaction strengh only 40% greater than that required to bind the deuteron. Since the deuteron is very weakly bound, in the sense that 90% of the time the neutron and proton are outside the range of the force, this shows immediately that the single term separable approximations can only be used with confidence for very weak binding. The Kowalski-Noyes approximation does not produce this catastrophy, but on the other hand it fails to reproduce the sudden collapse given by the local Yukawa potential, even though the t-matrix used is identical to the exact one when taken half-on-shell. We conclude that at least two (and probably more) separable terms will be needed to reproduce the three-body features of local potentials even for simple monotonic forms like those considered here, in which case it would seem better to do a little more work and adopt the Wong approach[11], or that pursued here.

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FOOTNOTES AND REFERENCES

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FIGURE CAPTIONS

Figure 1 - Binding energies for two and three particle states of identical spinless particles, interacting via S-wave Yukawa potentials, as a function of the interaction strength. Results from Bander, Yamaguchi, and Kowalski-Noyes separable approximations are given for comparison.

Figure 2 - Binding energies for two and three particle states of identical spinless particles, interacting via S-wave exponential potentials, as a function of the interaction strength. Results from Bander and Yamaguchi separable approximations are given for comparison.

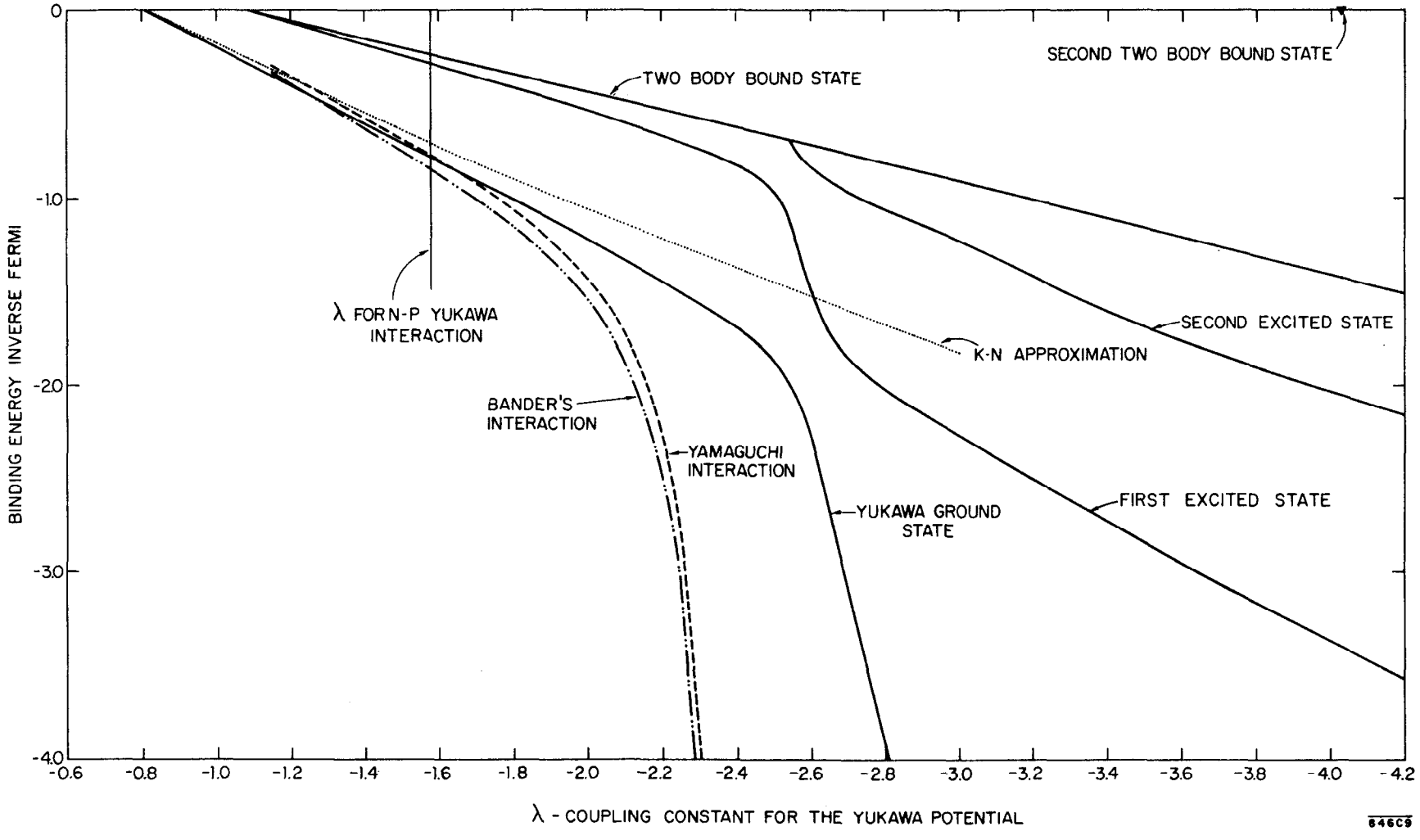
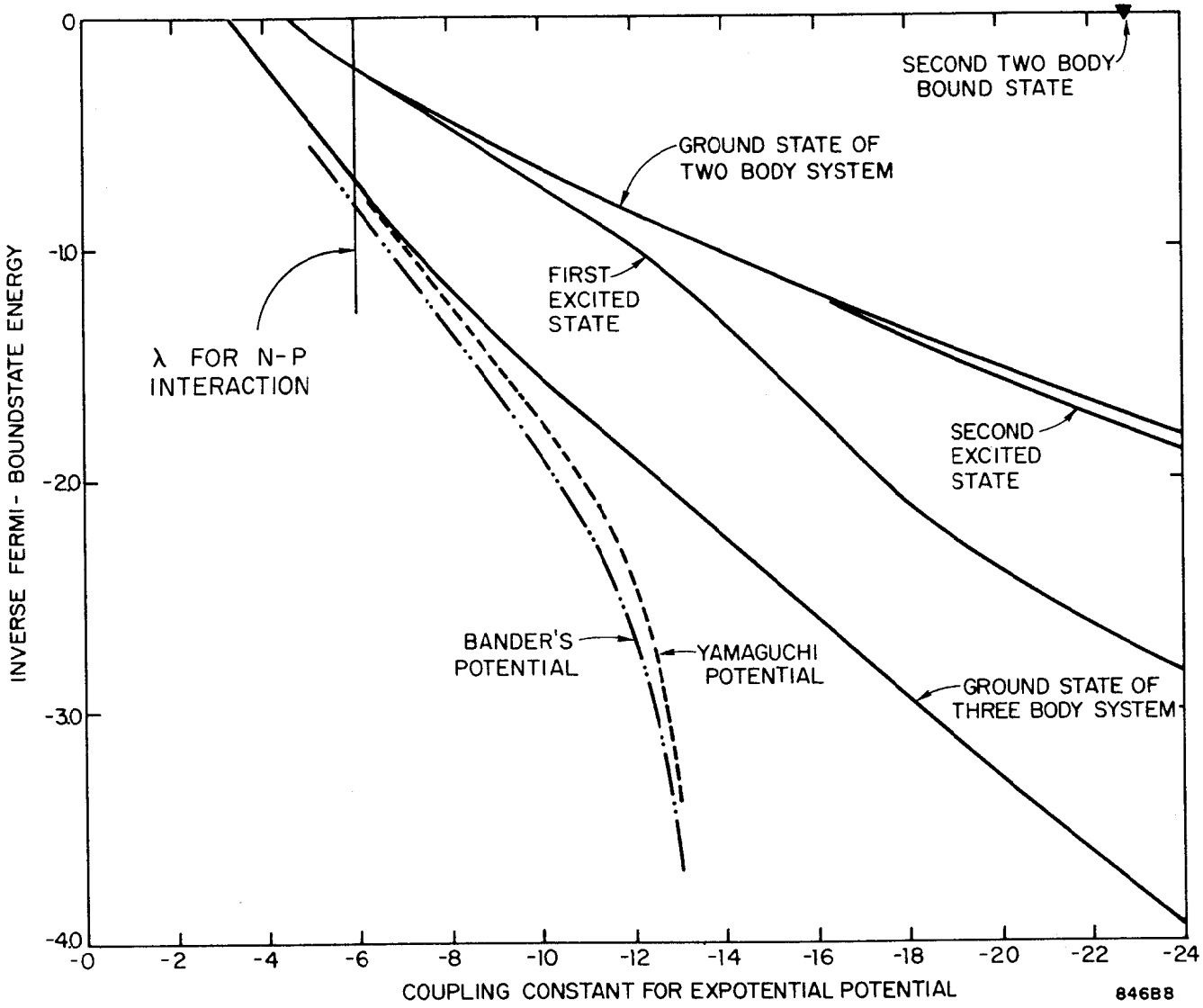

0
-1.0
-2.0
-3.0
-4.0
BINDING ENERGY INVERSE FERMI
-0.6
-0.8
-1.0
-1.2
-1.4
-1.6
-1.8
-2.0
-2.2
-2.4
-2.6
-2.8
-3.0
-3.2
-3.4
-3.6
-3.8
-4.0
-4.2
λ - COUPLING CONSTANT FOR THE YUKAWA POTENTIAL
SECOND TWO BODY BOUND STATE
TWO BODY BOUND STATE
SECOND EXCITED STATE
K-N APPROXIMATION
λ FOR N-P YUKAWA INTERACTION
BANDER'S INTERACTION
YAMAGUCHI INTERACTION
YUKAWA GROUND STATE
FIRST EXCITED STATE
646C9

FIG. 1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-1.0
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-8
-10
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-14
-16
-18
-20
-22
-24
INVERSE FERMI - BOUNDSTATE ENERGY
COUPLING CONSTANT FOR EXPOTENTIAL POTENTIAL
SECOND TWO BODY BOUND STATE
GROUND STATE OF TWO BODY SYSTEM
FIRST EXCITED STATE
SECOND EXCITED STATE
λ FOR N-P INTERACTION
BANDER'S POTENTIAL
YAMAGUCHI POTENTIAL
GROUND STATE OF THREE BODY SYSTEM
84688

FIG. 2