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## Particle motion and electromagnetic fields of rotating compact gravitating objects with gravitomagnetic charge

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**Abstract** The exact solution for the electromagnetic field occurring when the Kerr–Taub–NUT compact object is immersed (i) in an originally uniform magnetic field aligned along the axis of axial symmetry (ii) in dipolar magnetic field generated by current loop has been investigated. Effective potential of motion of charged test particle around Kerr–Taub–NUT gravitational source immersed in magnetic field with different values of external magnetic field and NUT parameter has been also investigated. In both cases presence of NUT parameter and magnetic field shifts stable circular orbits in the direction of the central gravitating object. Finally we find analytical solutions of Maxwell equations in the external background spacetime of a slowly rotating magnetized NUT star. The star is considered isolated and in vacuum, with monopolar configuration model for the stellar magnetic field.

**Keywords** General relativity, Kerr–Taub–NUT spacetime, Electromagnetic fields, Particle motion

### 1 Introduction

The existence of strong electromagnetic fields is one of the most important features of rotating neutron stars observed as pulsars. Starting the pioneering paper of Deutsch [1] it was proved that due to the rotation of highly magnetized star the electric field is induced. The general relativistic effect of dragging of inertial frames is very important in pulsar magnetosphere [2; 3] and is considered to be a source of additional electric field of general relativistic origin (see, e.g., [4; 5; 6; 7]).

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It was first shown by Ginzburg and Ozernoy [8] that an electrically neutral black hole can not have an intrinsic magnetic field. However, assuming that black hole can be inserted in the external uniform magnetic field created by nearby source as neutron star or magnetar, Wald [9] found for the first time the exact solution of the vacuum Maxwell equations for an asymptotically uniform magnetic field. Then the properties of black holes immersed in external magnetic field were extensively studied by different authors (see, e.g., [10; 11; 12; 13; 14] and for more references [15]).

Despite the absence of observational evidence for the existence of gravitomagnetic monopole, that is of exotic space–time, called NUT space (Newman, Unti and

Tamburino [16]) at present, it is interesting to study the electromagnetic fields and particle motion in NUT space with the aim to get new tool for studying new important general relativistic effects which are associated with nondiagonal components of the metric tensor and have no Newtonian analogues.

Here we study the electromagnetic fields in the Kerr–Taub–NUT spacetime and in the surrounding space–time of slowly rotating magnetized NUT star. Our approach is based on the reasonable assumption that the metric of spacetime is known, i.e. neglecting the influence of the electromagnetic field on the gravitational one and finding analytical solutions of Maxwell equations on a given, fixed background (estimations of contribution of electromagnetic field energy into total energy momentum could be found e.g., in paper [17]).

We also study motion of test particle around Kerr–Taub–NUT source, which is immersed either (i) in an originally uniform magnetic field aligned along the axis of axial symmetry or (ii) in dipolar magnetic field generated by current loop. We use Hamilton–Jacobi equation to find influence of both NUT parameter and magnetic field on the effective potential of motion of test charged particles. Moreover here we completely ignore the pathologies (existence of singularities along the axis or periodicity of the time coordinate to avoid them and spacetime regions containing closed timelike curves) of the spacetime metric due to fact that NUT parameter is considered as small one.

The outline of the paper is as follows. In Sect. 2 we calculate the electric and magnetic fields generated in the vicinity of Kerr–Taub–NUT spacetime following the method of construction of the vacuum solution to Maxwell equations in axially-symmetric stationary spacetime suggested by Wald [9].

In Sects. 3 and 4 we consider the separation of variables in the Hamilton–Jacobi equation and derive the effective potential for the motion of charged particles around Kerr–Taub–NUT source in a uniform and dipolar magnetic field generated by current loop. These results are used to obtain the basic equations governing the region of marginal stability of the circular orbits and their associated energies and angular momenta.

In Sect. 5 we look for stationary solutions of Maxwell equations when the stellar magnetic field has monopolar configuration which allows to find exact analytical solution. Section 6 is devoted to analysis of obtained solutions for electric and magnetic fields exterior to the rotating compact objects with NUT parameter.

Throughout, we use a space-like signature  $(-, +, +, +)$  and a system of units in which  $G = 1 = c$  (However, for those expressions with an astrophysical application we have written the speed of light explicitly). Greek indices are taken to

run from 0 to 3 and Latin indices from 1 to 3; covariant derivatives are denoted with a semi-colon and partial derivatives with a comma.

## 2 Kerr–Taub–NUT Source in a uniform magnetic field

We consider electromagnetic fields of compact astrophysical objects in Kerr–NUT spacetime which in a spherical coordinate system  $(ct, r, \theta, \phi)$  is described by the metric (see, e.g., [18; 19])

$$ds^2 = -\frac{1}{\Sigma}(\Delta - a^2 \sin^2 \theta) dt^2 + \frac{2}{\Sigma} [\Delta \chi - a(\Sigma + a\chi) \sin^2 \theta] dt d\phi \\ + \frac{1}{\Sigma} [(\Sigma + a\chi)^2 \sin^2 \theta - \chi^2 \Delta] d\phi^2 + \frac{\Sigma}{\Delta} dr^2 + \Sigma d\theta^2, \quad (1)$$

where parameters  $\Sigma, \Delta$  and  $\chi$  are defined by

$$\Sigma = r^2 + (l + a \cos \theta)^2, \quad \Delta = r^2 - 2Mr - l^2 + a^2, \quad \chi = a \sin^2 \theta - 2l \cos \theta, \quad (2)$$

$l$  is the gravitomagnetic monopole momentum,  $a = J/M$  is the specific angular momentum of metric source with total mass  $M$ .

Here we will exploit the existence in this spacetime of a timelike Killing vector  $\xi_{(t)}^\alpha$  and spacelike one  $\xi_{(\phi)}^\alpha$  being responsible for stationarity and axial symmetry of geometry, such that they satisfy the Killing equations

$$\xi_{\alpha;\beta} + \xi_{\beta;\alpha} = 0, \quad (3)$$

and consequently the wave-like equations (in vacuum spacetime)

$$\square \xi^\alpha = 0, \quad (4)$$

which gives a right to write the solution of vacuum Maxwell equations  $\square A^\mu$  for the vector potential  $A_\mu$  of the electromagnetic field in the Lorentz gauge in the simple form

$$A^\alpha = C_1 \xi_{(t)}^\alpha + C_2 \xi_{(\phi)}^\alpha. \quad (5)$$

The constant  $C_2 = B/2$ , where gravitational source is immersed in the uniform magnetic field  $\mathbf{B}$  being paralel to its axis of rotation. The value of the remaining constant  $C_1$  can be easily calculated from the asymptotic properties of spacetime (1) at the infinity.

Indeed in order to find the remaining constant one can use the electrical neutrality of the source

$$4\pi Q = 0 = \frac{1}{2} \oint F^{\alpha\beta} {}_*dS_{\alpha\beta} = C_1 \oint \Gamma_{\beta\gamma}^\alpha \mu_\alpha m^\beta \xi_{(t)}^\gamma(uk) dS \\ + \frac{B}{2} \oint \Gamma_{\beta\gamma}^\alpha \mu_\alpha m^\beta \xi_{(\phi)}^\gamma(uk) dS \quad (6)$$

evaluating the value of the integral through the spherical surface at the asymptotic infinity. Here the equality  $\xi_{\beta;\alpha} = -\xi_{\alpha;\beta} = -\Gamma_{\alpha\beta}^\gamma \xi_\gamma$  following from the Killing

equation was used, and element of an arbitrary 2-surface  $dS^{\alpha\beta}$  is represented in the form [20]

$$dS^{\alpha\beta} = -u^\alpha \wedge m^\beta (uk) dS + \eta^{\alpha\beta\mu\nu} u_\mu n_\nu \sqrt{1 + (uk)^2} dS, \quad (7)$$

and the following couples

$$m_\alpha = \frac{\eta_{\lambda\alpha\mu\nu} u^\lambda n^\mu k^\nu}{\sqrt{1 + (uk)^2}}, \quad n_\alpha = \frac{\eta_{\lambda\alpha\mu\nu} u^\lambda k^\mu m^\nu}{\sqrt{1 + (uk)^2}}, \quad (8)$$

$$k^\alpha = -(uk)u^\alpha + \sqrt{1 + (uk)^2} \eta^{\mu\alpha\rho\nu} u_\mu m_\rho n_\nu$$

are established between the triple  $\{\mathbf{k}, \mathbf{m}, \mathbf{n}\}$  of vectors,  $n^\alpha$  is normal to 2-surface, space-like vector  $m^\alpha$  belongs to the given 2-surface and is orthogonal to the four-velocity of observer  $u^\alpha$ , a unit spacelike four-vector  $k^\alpha$  belongs to the surface and is orthogonal to  $m^\alpha$ ,  $dS$  is invariant element of surface,  $\wedge$  denotes the wedge product,  $*$  is for the dual element,  $\eta_{\alpha\beta\gamma\delta}$  is the pseudo-tensorial expression for the Levi-Civita symbol  $\varepsilon_{\alpha\beta\gamma\delta}$ .

Then one can insert  $u_0 = -(1 - M/r)$ ,  $m^1 = (1 - M/r)$ , and asymptotic values for the Christoffel symbols  $\Gamma_{10}^0 = M/r^2$  and  $\Gamma_{13}^0 = -3J \sin^2 \theta / r^2 - l(1 - 2M/r) \cos \theta / r$  in the flux expression (6) and get the value of constant  $C_1 = aB$ . Parameter  $l$  does not appear in constant  $C_1$  because the integral  $\int_0^\pi \cos \theta \sin \theta d\theta = 0$  vanishes.

Finally the 4-vector potential  $A_\alpha$  of the electromagnetic field will take a form

$$A_0 = -\frac{B}{\Sigma} \left\{ \Delta \left( a - \frac{\chi}{2} \right) + a \left[ \frac{1}{2} (\Sigma + a\chi) - a^2 \right] \sin^2 \theta \right\} = -\frac{BK}{\Sigma}, \quad (9)$$

$$A_3 = \frac{B}{\Sigma} \left\{ \Delta \chi \left( a - \frac{\chi}{2} \right) + (\Sigma + a\chi) \left[ \frac{1}{2} (\Sigma + a\chi) - a^2 \right] \sin^2 \theta \right\} = \frac{BL}{\Sigma}. \quad (10)$$

Using solution for vector potential (5) in spacetime (1) one could easily find expression for change of the electrostatic energy of the charged particle, which is lower to the horizon of gravitational source:

$$\varepsilon = eA^\mu (\xi_{\mu(t)} + \Omega|_{hor} \xi_{\mu(\varphi)})|_{hor} - eA^\mu \xi_{\mu(t)}|_{inf} = elB - eaB. \quad (11)$$

At the horizon new timelike Killing vector [21; 14]

$$\begin{aligned} \Psi_\mu|_{hor} &= \xi_\mu(t) + \Omega|_{hor} \xi_{\mu(\varphi)}, \\ \Omega|_{hor} &= \frac{a}{2Mr_+}, \quad r_+ = M + \sqrt{M^2 - a^2 + l^2} \end{aligned} \quad (12)$$

is introduced since the timelike Killing vector  $\xi$  becomes spacelike inside ergoregion defined by  $g_{00} = 0$ .

Upper limit for electric charge

$$Q = 2aMB - 2lMB \quad (13)$$

accreted by gravitational source will include in addition to the contribution from Faraday induction effect arising from angular momentum  $a$  (see e.g. [9], [22]) the

new term being proportional to additional rotation coming from NUT parameter  $l$ . It is interesting to note that each parameter will accrete the charges of opposite sign.

The orthonormal components of the electromagnetic fields measured by zero angular momentum observers (ZAMO) with four velocity components

$$(u^\alpha)_{\text{ZAMO}} \equiv \left( \sqrt{\frac{(\Sigma + a\chi)^2 \sin^2 \theta - \chi^2 \Delta}{\Delta \Sigma \sin^2 \theta}}, 0, 0, -\frac{\Delta \chi - a(\Sigma + a\chi) \sin^2 \theta}{\sqrt{\Delta \Sigma [(\Sigma + a\chi)^2 \sin^2 \theta - \chi^2 \Delta]} \sin \theta} \right);$$

$$(u_\alpha)_{\text{ZAMO}} \equiv \left( -\sqrt{\frac{\Delta \Sigma \sin^2 \theta}{(\Sigma + a\chi)^2 \sin^2 \theta - \chi^2 \Delta}}, 0, 0, 0 \right) \quad (14)$$

are given by expressions

$$E^{\hat{r}} = -\frac{2rB \sin \theta}{\Sigma^2 \sqrt{(\Sigma + a\chi)^2 \sin^2 \theta - \chi^2 \Delta}} \left\{ \left[ \Delta - \left( 1 - \frac{M}{r} \right) \Sigma - a^2 \sin^2 \theta \right] \right.$$

$$\left. \times (\Sigma + a\chi) \left( a - \frac{\chi}{2} \right) - \frac{\Sigma}{2} [\chi \Delta - a(\Sigma + a\chi) \sin^2 \theta] \right\}, \quad (15)$$

$$E^{\hat{\theta}} = \frac{B \sin^2 \theta}{\Sigma^2 \sqrt{\Delta((\Sigma + a\chi)^2 \sin^2 \theta - \chi^2 \Delta)}}$$

$$\times \left[ \left\{ \Delta(l + a \cos \theta) + 2a(\Sigma + a\chi - 2a^2) \cos \theta \right\} \Sigma(\Sigma + a\chi) \right.$$

$$\left. - \left\{ \Sigma(\Sigma + a\chi - 2a^2) - 2K(l + a \cos \theta) \right\} \frac{\chi \Delta}{\sin^2 \theta} \right], \quad (16)$$

$$B^{\hat{r}} = \frac{B \sin \theta}{\Sigma \sqrt{(\Sigma + a\chi)^2 \sin^2 \theta - \chi^2 \Delta}} \left[ \chi \Delta(l + a \cos \theta) \right.$$

$$\left. - (\Sigma - a\chi)(\Sigma + a\chi - 2a^2) \cos \theta - \frac{2K}{\Sigma}(\Sigma + a\chi)(l + a \cos \theta) \right], \quad (17)$$

$$B^{\hat{\theta}} = -\frac{2rB\Delta}{\Sigma^2 \sqrt{\Delta((\Sigma + a\chi)^2 \sin^2 \theta - \chi^2 \Delta)}} \times \left\{ \left[ \Delta - \left( 1 - \frac{M}{r} \right) \Sigma \right. \right.$$

$$\left. \left. - a^2 \sin^2 \theta \right] \left( a - \frac{\chi}{2} \right) \chi - \frac{1}{2} \Sigma^2 \sin^2 \theta \right\}, \quad (18)$$

which depend on angular momentum and NUT parameter in complex way. Astrophysically it is interesting to know the limiting cases of expressions (15)–(18), for example in either linear or quadratic approximation  $\mathcal{O}(a^2, l^2/r^2)$  in order to give

physical interpretation of possible physical processes near the slowly rotating relativistic compact stars, where they take the following form:

$$E^{\hat{r}} = \frac{B}{r} \left\{ (2l + a \cos \theta) \cos \theta - \frac{M(12l \cos \theta + a(1 + 3 \cos 2\theta))}{2r} \right\}, \quad (19)$$

$$E^{\hat{\theta}} = \frac{B \sin \theta}{r} \left\{ l \left( 1 + \frac{2 \cos \theta}{\sin^2 \theta} \right) + a(3 \cos \theta - 1) \right\}, \quad (20)$$

$$B^{\hat{r}} = -B \cos \theta \left\{ 1 - \frac{1}{2r^2} \left( -4(l^2 - a^2 + al \cos \theta) - 4l^2 \cot^2 \theta + a(2l + 3a \cos \theta) \sin \theta \tan \theta \right) \right\}, \quad (21)$$

$$B^{\hat{\theta}} = B \sin \theta \left\{ 1 - \frac{M}{r} + \frac{1}{16r^2 \sin^2 \theta} (-a^2 + 4l^2 - 4M^2 - 8al \cos \theta + 4(7l^2 + M^2) \cos 2\theta + 8al \cos 3\theta + a^2 \cos 4\theta) \right\}. \quad (22)$$

In the limit of flat spacetime, i.e. for  $M/r \rightarrow 0$ ,  $Ma/r^2 \rightarrow 0$  and  $l^2/r^2 \rightarrow 0$ , expressions (15)–(18) give

$$\lim_{M/r, Ma/r^2, l^2/r^2 \rightarrow 0} B^{\hat{r}} = -B \cos \theta, \quad \lim_{M/r, Ma/r^2, l^2/r^2 \rightarrow 0} B^{\hat{\theta}} = B \sin \theta, \quad (23)$$

$$\lim_{M/r, Ma/r^2, l^2/r^2 \rightarrow 0} E^{\hat{r}} = \lim_{M/r, Ma/r^2, l^2/r^2 \rightarrow 0} E^{\hat{\theta}} = 0. \quad (24)$$

As expected, expressions (23)–(24) coincide with the solutions for the homogeneous magnetic field in Newtonian spacetime.

Finally we would like to show that two form of electromagnetic field tensor will take more simplified form:

$$\begin{aligned} F = & \frac{B}{\Sigma^2} [2Kr - \Sigma(r - M)(2a - \chi)] \omega^1 \wedge \omega^0 + \frac{B(l + a \cos \theta) \sin \theta}{\Sigma^2 \Delta^{1/2}} \\ & [\Delta(\Sigma + 2a^2 - a\chi) + a^2(\Sigma + a\chi - 2a^2) \sin^2 \theta - 2aK] \omega^2 \wedge \omega^0 \\ & - \frac{Br\Delta^{1/2} \sin \theta}{\Sigma} \omega^1 \wedge \omega^3 + \frac{B \cos \theta}{\Sigma^2} [-a\chi(\Sigma + a\chi - 2a^2) \cos \theta + \\ & (l + a \cos \theta) [a(\Sigma + a\chi - 2a^2) \sin^2 \theta + 2\Delta a - \Delta\chi]] \omega^2 \wedge \omega^3 \end{aligned} \quad (25)$$

in the orthonormal Carter-type frame [22]:

$$\begin{aligned} \omega^0 &= \left( \frac{\Delta}{\Sigma} \right)^{1/2} (dt - \chi d\varphi), \quad \omega^1 = \left( \frac{\Sigma}{\Delta} \right) dr, \\ \omega^2 &= \Sigma^{1/2} d\theta, \quad \omega^3 = \frac{\sin \theta}{\Sigma^{1/2}} [adt - (r^2 + a^2 + l^2)d\varphi]. \end{aligned} \quad (26)$$

In the limiting case when NUT parameter  $l \rightarrow 0$  the components of field tensor (1) almost coincide with the expressions (3.9) of paper [9]. In the limit of flat spacetime, expressions for components of electromagnetic field derived using tensor (25) take Newtonian limits (23)–(24).

### 3 Motion of charged particles

It is very important to investigate in detail the motion of charged particles around a rotating compact source with NUT parameter in an external magnetic field given by 4-vector potential (9) and (10) with the aim to find astrophysical evidence for the existence of gravitomagnetic charge.

The Hamilton–Jacobi equation

$$g^{\mu\nu} \left( \frac{\partial S}{\partial x^\mu} + eA_\mu \right) \left( \frac{\partial S}{\partial x^\nu} + eA_\nu \right) = m^2, \quad (27)$$

for motion of the charged test particles with mass  $m$  and charge  $e$  is applicable as a useful computational tool only when separation of variables can be effected.

Since Kerr–Taub–NUT spacetime admits such separation of variables (see e.g. [18]) we shall study the motion around source described with metric (1) using the Hamilton–Jacobi equation when the action  $S$  can be decomposed in the form

$$S = -\mathcal{E}t + \mathcal{L}\varphi + S_{r\theta}(r, \theta), \quad (28)$$

since the energy  $\mathcal{E}$  and the angular momentum  $\mathcal{L}$  of a test particle are constants of motion in spacetime (1). This is an extension of approach developed in the paper [10] to the case with nonvanishing NUT parameter.

Therefore the Hamilton–Jacobi equation (27) with action (28) implies the equation for inseparable part of the action:

$$\begin{aligned} & \Delta \left( \frac{\partial S_{r\theta}}{\partial r} \right)^2 + \left( \frac{\partial S_{r\theta}}{\partial \theta} \right)^2 - \frac{(\Sigma + a\chi)^2 \sin^2 \theta - \chi^2 \Delta}{\Delta \sin^2 \theta} \left( \mathcal{E} + \frac{eBK}{\Sigma} \right)^2 \\ & - \frac{2(\Delta\chi - a(\Sigma + a\chi) \sin^2 \theta)}{\Delta \sin^2 \theta} \left( \mathcal{E} + \frac{eBK}{\Sigma} \right) \left( \mathcal{L} + \frac{eBL}{\Sigma} \right) \\ & + \frac{\Delta - a^2 \sin^2 \theta}{\Delta \sin^2 \theta} \left( \mathcal{L} + \frac{eBL}{\Sigma} \right)^2 - m^2 \Sigma = 0. \end{aligned} \quad (29)$$

It is not possible to separate variables in this equation in general case but it can be done for the motion in the equatorial plane  $\theta = \pi/2$  when the equation for radial motion takes form

$$\left( \frac{dr}{d\sigma} \right)^2 = \mathcal{E}^2 - 1 - 2V(\mathcal{E}, \mathcal{L}, r, b, a, l), \quad (30)$$

Here  $\sigma$  is the proper time along the trajectory of a particle,  $\mathcal{E}$  and  $\mathcal{L}$  are energy and angular momentum per unit mass  $m$  and

$$\begin{aligned} V(\mathcal{E}, \mathcal{L}, r, b, a, l) = & - \left( \mathcal{E} + \frac{bK}{M\Sigma} \right) \frac{bK}{M\Sigma} + \frac{b^2 K^2}{2M^2 \Sigma^2} - \frac{a^2}{\Sigma} \left( 1 + \frac{a^2}{2\Sigma} \right) \left( \mathcal{E} + \frac{bK}{M\Sigma} \right)^2 \\ & - \frac{2l^2 + 2Mr - a^2}{2\Sigma} + \frac{\Delta - \Sigma - a^2}{\Sigma^2} a \left( \mathcal{E} + \frac{bK}{M\Sigma} \right) \left( \mathcal{L} + \frac{bL}{M\Sigma} \right) \\ & - \frac{\Delta - a^2}{2\Sigma^2} \left( \mathcal{L} + \frac{bL}{M\Sigma} \right)^2 \end{aligned} \quad (31)$$

**Fig. 1** Radial dependence of the effective potential of radial motion of charged particle near the Kerr–Taub–NUT gravitational source immersed in external uniform magnetic field for different values of NUT parameter  $\tilde{l} = l/M$  for two cases of magnetic parameter: **a**  $b = 0.1$  and **b**  $b = 0.15$

**Fig. 2** Radial dependence of the effective potential of radial motion of charged particle near the Kerr–Taub–NUT gravitational source immersed in external uniform magnetic field for the different values of magnetic field in the case when NUT parameter  $\tilde{l} = 0.5$

can be thought of as an effective potential of the radial motion which depends on additional dimensionless parameter

$$b = \frac{eBM}{m} \quad (32)$$

being responsible for the relative influence of a uniform magnetic field on the motion of the charged particles which may be valuable even for small values of the magnetic field strength [10].

Figure 1 shows radial dependence of effective potential (31) for different values of NUT parameter  $\tilde{l} = l/M$ . From this dependence one can obtain modification of radial motion of charged particle in the equatorial plane in the presence of NUT parameter. As it is seen from the figure the gravitomagnetic monopole momentum changes the shape of effective potentials when external magnetic field is not strong (Fig. 1a). In the case of strong external magnetic field (Fig. 1b), influence of gravitomagnetic monopole momentum is negligible.

Figure 2 shows radial dependence of effective potential (31) for different values of  $b$  for fixed value of NUT parameter  $\tilde{l} = 0.5$ . Motion of charged particle in the presence of this kind of effective potential can be explained as follows: in the presence of external magnetic field in addition to stable circular orbits unstable circular orbits could appear due to the appearance of maximum in the graphs of effective potential. From the potential we can infer the qualitative structure of the particles orbits. As it is seen from the figure the potential carries the repulsive character. It means that the particle coming from infinity and passing by the source will not be captured: it will be reflected and will go to infinity again. For small values of electromagnetic field particles can follow both bound and unbound orbits depending on their energy. As external electromagnetic field increases interesting feature arises: the orbits start to be only parabolic or hyperbolic and no more circular or elliptical orbits exist.

### 3.1 Marginally stable circular orbits

Special interest for the accretion theory of test particles around a rotating compact source with NUT parameter in a magnetic field is related to study of circular orbits which are possible in the equatorial plane  $\theta = \pi/2$  when  $dr/d\sigma$  is zero. It is well known that in the presence of NUT parameter trajectory of particle lies in the cone with  $\cos \theta = \pm |2l\mathcal{E}/\mathcal{L}|$  (see e.g. [19]). In the case of energy and momentum of the particle are equal to  $\mathcal{E} = 0.9$ ,  $\mathcal{L} = 4.3$  respectively, and NUT parameter  $l \approx 0.5$ , then we get  $90^\circ - 10^\circ \leq \theta \leq 90^\circ + 10^\circ$ , which allows us to neglect deviation

and consider the motion in the almost equatorial plane. Consequently the right hand side of equation (31) vanishes:

$$\mathcal{E}^2 - 1 - 2V(\mathcal{E}, \mathcal{L}, r, b, a, l) = 0 \quad (33)$$

along with its first derivative with respect to  $r$

$$\frac{\partial V(\mathcal{E}, \mathcal{L}, r, b, a, l)}{\partial r} = 0. \quad (34)$$

In papers [23] and [24] the authors have appealed to a numerical analysis of analogues of Eqs. (33) and (34), when NUT parameter is equal to zero, for different values of the constants of motion, orbital radii, the rotation parameter  $a$ , as well as the strength parameter of the magnetic field. The problem of existence of stable orbits in NUT space-time [25] and other properties of particles motion are also discussed in our preceding paper [26].

The radius of marginal stability, the associated energy and angular momentum of the circular orbits can be derived from the simultaneous solution of the condition

$$\frac{\partial^2 V(\mathcal{E}, \mathcal{L}, r, b, a, l)}{\partial r^2} \leq 0, \quad (35)$$

and equation (33).

Combining equations (33) and (34) one could find the energy

$$\mathcal{E} = -\frac{ab}{M} + \gamma \pm \sqrt{\lambda}, \quad (36)$$

and the angular momentum

$$\mathcal{L} = -\frac{b}{2M}(l^2 + a^2) + \eta \pm (4ab - 12r\gamma)\sqrt{\lambda}, \quad (37)$$

of a test particle, where the following notations

$$\gamma = \frac{2ab(r - \beta)}{3r^2 - l^2 - 6\beta r}, \quad \beta = r \frac{6r - 4M}{12r - 4M}, \quad (38)$$

$$\lambda = [3r^2 - l^2 - 6\beta r]^{-1} \left[ \frac{5b^2 r^3}{4M^2}(r - 4\beta) - \frac{2b^2 r^2}{M}(r - 3\beta) + 3r \left( 1 - \frac{3b^2 l^2}{4M^2} \right) (r - 2\beta) + 2 \left( \frac{l^2 - a^2}{M} b^2 - 2M \right) (r - \beta) + a^2 - 2l^2 \right] \quad (39)$$

and

$$\eta = 4ab\gamma - 6r\lambda - 6r\gamma^2 - \frac{5b^2 r^3}{M^2} + \frac{6b^2 r^2}{M} - 6r \left( 1 - \frac{3b^2 l^2}{4M^2} \right) - 2 \left( \frac{l^2 - a^2}{M} b^2 - 2M \right) \quad (40)$$

**Table 1** Marginally circular orbits around non-rotating NUT gravitational source immersed in uniform magnetic field

$\bar{l}$	0	0.01	0.1	0.3	0.5
$b = 0.1$	3.66650	3.66460	3.66154	3.62072	3.53294
$b = 0.2$	2.83292	2.83288	2.82876	2.79448	2.72020
$b = 0.3$	2.55504	2.55500	2.55162	2.52360	2.46338
$b = 0.4$	2.41614	2.41610	2.41326	2.38984	2.33994
$b = 0.5$	2.33282	2.33280	2.33036	2.31032	2.26794

were introduced. Inserting now (36) and (37) into equation (35) one could obtain the basic equation

$$\begin{aligned}
& (r^3 - l^2 r + 2a^2 M) (\gamma \pm \sqrt{\lambda})^2 + 2ab(2l^2 - r^2) (\gamma \pm \sqrt{\lambda}) \\
& + 4aM (\gamma \pm \sqrt{\lambda}) (\eta \pm (4ab - 12r\gamma) \sqrt{\lambda}) \\
& + 2br^2 \left( \frac{r}{M} - 1 \right) (\eta \pm (4ab - 12r\gamma) \sqrt{\lambda}) \\
& + \frac{b^2 r^5}{4M^2} - \frac{b^2 r^4}{2M} + \left( 1 - \frac{3b^2 l^2}{4M^2} \right) r^3 + \left( \frac{l^2 - a^2}{M} b^2 - 2M \right) r^2 \\
& + (a^2 - 2l^2) r + 2Ml^2 - \frac{4a^2 l^2 b^2}{M} = 0.
\end{aligned} \tag{41}$$

The numerical solutions of this equation will determine the radii of the marginally stable circular orbits for non-rotating NUT source immersed in uniform magnetic field as functions of the NUT parameter  $l$ , the angular momentum  $a$ , as well as of the influence parameter of the magnetic field  $b$ . In the Table 1 it is shown list of numerical solutions for radii of stable circular orbits of particles for different values of NUT parameter and external magnetic field. With the increase of the gravitomagnetic monopole momentum, radii of stable circular orbits shifts to gravitational object, while external field also displace orbits to gravitational source.

#### 4 Motion of charged particle in the field of current carrying loop located near the Kerr–NUT Black Hole

Consider a motion of test particle in the electromagnetic field created by toroidal currents of ionized matter rotating in accretion discs [27] around Kerr–Taub–NUT spacetime. We shall employ internal solution of [28]. In particular, of the full multipole solution for magnetic field provided in [28], we shall here focus our attention on the dominant (dipolar) term, which is given by

$$A_\alpha = -\frac{3}{8} \delta_\alpha^\varphi \frac{\mu r^2 \sin^2 \theta}{M^3} \left[ \ln \left( 1 - \frac{2M}{R} \right) + \frac{2M}{R} \left( 1 + \frac{2M}{R} \right) \right], \tag{42}$$

where

$$\mu = \pi R^2 \left( 1 - \frac{2M}{R} \right)^{1/2} I$$

**Fig. 3** Radial dependence of the effective potential of radial motion of charged particle near the Kerr–Taub–NUT gravitational source in the presence of current carrying loop around it for the different values of NUT parameter  $\tilde{l} = l/M$  for two cases of modulus of the dipolar magnetic field **a**  $\varepsilon = 0.3$  and **b**  $\varepsilon = 0.7$

is the modulus of the dipole moment due to the current  $I$  in the loop and  $R$  is the radius of the loop, which is considered to be approximately equal to  $6M$  (because of small values of  $l$  and  $a$  it will be not essentially modified) (see [28]).

Using Hamilton–Jacobi Eq. (27) and potential (42) one can find expression for effective potential for radial motion of charged particle in the equatorial plane ( $\theta = \pi/2$ ). It takes the following form

$$V_{\text{eff}} = -\frac{a^2}{\zeta} \left( \frac{1}{2} + \frac{\eta}{\zeta} \right) \mathcal{E}^2 - \frac{2\eta - a^2}{2\zeta} + \frac{2a\mathcal{E}\eta}{\zeta^2} \left( \mathcal{L} - \frac{\varepsilon}{M} r^2 \right) + \frac{\zeta - 2\eta}{2\zeta^2} \left( \mathcal{L} - \frac{\varepsilon}{M} r^2 \right)^2, \quad (43)$$

where we use the notations

$$\zeta = r^2 + l^2, \quad \eta = Mr + l^2,$$

and

$$\varepsilon = -\frac{3}{8} \frac{e\mu}{mM^2} \left[ \ln \left( 1 - \frac{2M}{R} \right) + \frac{2M}{R} \left( 1 + \frac{2M}{R} \right) \right].$$

In the case, when rotational parameter  $a$  is very small effective potential takes following form:

$$V_{\text{eff}} = -\frac{\eta}{\zeta} + \frac{\zeta - 2\eta}{2\zeta^2} \left( \mathcal{L} - \frac{\varepsilon}{M} r^2 \right)^2, \quad (44)$$

which coincides with effective potential of radial motion in the equatorial plane in the Schwarzschild metric (see e.g., [29]) if one puts  $\zeta = r^2$ ,  $\eta = Mr$ , and  $\varepsilon = 0$  into Eq. (44).

Plots in Fig. 3 show the radial dependence of effective potential, which is governed by Eq. (43), for the different values of  $\tilde{l}$ . Like in the case of uniform external magnetic field, described in the previous section, in this case gravitomagnetic monopole momentum has an influence onto the effective potential, if magnetic field is weak (Fig. 3a). If magnetic field of loop is strong influence of gravitomagnetic monopole momentum is negligible (Fig. 3b).

Consider now stable circular orbits of charged particles as it was done in Sect. 3. We shall repeat calculations, which are done in Sect. 3.1, in this case. Using expression (44) as effective potential we find numerical solutions for radii of stable circular orbits of charged particles. From results shown in the Tables 2 and 3 (for anti-Larmor and Larmor orbits, respectively) one can obtain that with the increase of the gravitomagnetic monopole momentum radii of anti-Larmor and Larmor orbits shift to loop while with the increase of the electric current (creating dipolar magnetic field) of loop the stable orbits shift to the gravitational object. These results may be useful to determine NUT parameter from astrophysical observations.

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## 5 External electromagnetic field of slowly rotating NUT star for a special monopolar magnetic field configuration

In this section we will look for stationary solutions of the Maxwell equation, i.e. for solutions in which we assume that the magnetic moment of the star does not vary in time as a result of the infinite conductivity of the stellar interior. Below we suggest that external electric field is generated by the magnetic field, taking as a special monopolar configuration. For this case we can obtain and investigate an analytical solution with detail consideration of the contributions from the dragging effects and nonvanishing NUT charge in the magnitude of the external electric field of the slowly rotating magnetized NUT star.

Our main approximation is in the specific form of the background metric which we choose to be that of a stationary, axially symmetric system truncated at the first order in the angular momentum  $a$  and in gravitomagnetic monopole moment  $l$ . The “slow rotation metric” for exterior space–time of a rotating relativistic star with nonvanishing gravitomagnetic charge is

**Table 2** Stable orbits of test particles near the non-rotating gravitational source with gravito-magnetic monopole momentum immersed in the external electromagnetic field of current carrying loop (anti-Larmor orbits)

$\bar{l}$	0	0.01	0.1	0.3	0.5
$\varepsilon = 0.1$	3.65296	3.65305	3.66106	3.72475	3.84692
$\varepsilon = 0.2$	3.09353	3.09360	3.10045	3.15506	3.26056
$\varepsilon = 0.3$	2.83420	2.83427	2.84064	2.89154	2.99030
$\varepsilon = 0.5$	2.57592	2.57598	2.58190	2.62931	2.72175
$\varepsilon = 0.7$	2.44347	2.44353	2.44922	2.49488	2.58422

**Table 3** Stable orbits of test particles near the non-rotating gravitational source with gravito-magnetic monopole momentum immersed in the external electromagnetic field of current carrying loop (Larmor orbits)

$\bar{l}$	0	0.01	0.1	0.3	0.5
$\varepsilon = 0.1$	4.50043	4.50058	4.51456	4.62526	4.83569
$\varepsilon = 0.2$	4.36234	4.36248	4.37677	4.48985	4.70439
$\varepsilon = 0.3$	4.33040	4.33054	4.34495	4.45890	4.67491
$\varepsilon = 0.5$	4.31295	4.31310	4.32758	4.44206	4.65896
$\varepsilon = 0.7$	4.30800	4.30815	4.32265	4.43729	4.65445

$$ds^2 = -N^2 dt^2 + N^{-2} dr^2 + r^2 d\theta^2 + r^2 \sin^2 \theta d\varphi^2 - 2 [\omega(r) r^2 \sin^2 \theta + 2lN^2 \cos \theta] dt d\varphi, \quad (45)$$

that is, the Schwarzschild metric plus the Lense-Thirring and Taub-NUT terms. Parameter  $N \equiv (1 - 2M/r)^{1/2}$  is the lapse function,  $\omega(r) \equiv 2J/r^3$  can be interpreted as the Lense-Thirring angular velocity of a free falling (inertial) frame.

As a toy model we could consider the following magnetic field configuration [30]

$$B^{\hat{r}} = B^{\hat{r}}(r) \neq 0, \quad B^{\hat{\theta}} = 0. \quad (46)$$

Although this form of magnetic field can not be considered as a realistic, we will show that this toy model can be used to obtain first estimates of the influence of gravitational field of the NUT charge on the external electromagnetic field of the star. For this case, the relevant Maxwell equation reduces to

$$(r^2 B^{\hat{r}})_{,r} = 0. \quad (47)$$

The solution admitted by this equation is

$$B^{\hat{r}} = \frac{\mu}{r^2}, \quad (48)$$

where  $\mu$  is integration constant being responsible for the source of monopolar magnetic field.

Radial magnetic field is continuous at the stellar surface and it is reasonable to assume that only  $\theta$  component of electric field will survive since it will be produced by cross product of velocity and magnetic field inside neutron star due to the assumed infinite conductivity of stellar matter. According to [17] the interior electric field is

$$E_{\text{in}}^{\hat{\theta}} = -e^{-\Phi} v^{\hat{\phi}} B_{\text{in}}^{\hat{r}}, \quad (49)$$

where  $v^{\hat{\theta}}$  is the velocity of the stellar matter which is equal to  $\Omega r \sin \theta$  for the Newtonian uniformly rotating star with angular velocity  $\Omega$ ,  $g_{00} = -e^{2\Phi}$ .

Then the electric field created by monopolar magnetic field is defined by the following Maxwell equation

$$\left( rNE^{\hat{\theta}} \right)_{,r} - \mu \sin \theta (\omega)_{,r} - \frac{2\mu l \cos \theta}{\sin \theta} \left( \frac{N^2}{r^2} \right)_{,r} = 0. \quad (50)$$

The analytical solution

$$E^{\hat{\theta}} = \frac{\omega - \Omega}{cN} \frac{\mu}{r} \sin \theta + \frac{2lN \cos \theta}{r \sin \theta} \frac{\mu}{r^2} \quad (51)$$

of Eq. (50) is responsible for the electric field of NUT star with the monopolar magnetic field (48). The integration constant  $-(\Omega \mu / c) \sin \theta$  has been found from the matching of the exterior solution ( $C_3 / rN$ ) with the interior one (49) in the Newtonian case taking into account that the tangential components of electric field and the radial component of magnetic field are continuous across the surface of the star. Vector potential being responsible for the fields (48) and (51) is defined as

$$A_{\alpha} \equiv \left( 0, 0, 0, -\mu \cos \theta \right). \quad (52)$$

In the Fig. 4 we plot the radial dependence of the ratio of the electric field to that when parameter  $l = 0$  for the different values of the NUT parameter. In this analysis we take typical parameters for neutron star as radius of star  $R = 10^6$  cm,  $M = 2 \times 10^5$  cm,  $\Omega = 2\pi / (0.1s)$ ,  $\omega = 4MR^2\Omega / (5r^3)$ , magnetic field at the stellar surface is  $10^{12}$  G. Due to the fact that in the right hand side of the expression (51) the first and second terms have different signs the ratio is less than one. The results show the strong dependence of the electric field on NUT parameter.

## 6 Conclusion

Derived exact expressions (15)–(18) for electromagnetic field in the vicinity of the Kerr–Taub–NUT spacetime indicate that electromagnetic field will be affected by the gravitomagnetic charge. However, the induced electric field (15), (16) depends on NUT parameter  $l$  linearly while the magnetic field (17), (18) depends on  $l$  quadratically.

Analytic general relativistic expressions for the electromagnetic fields external to a slowly rotating magnetized neutron star with nonvanishing gravitomagnetic charge  $l$  are presented. The star is considered isolated and in vacuum, and for simplicity with the monopolar magnetic field directed along the radial coordinate.

We have shown that the general relativistic corrections due to the dragging of reference frames and gravitomagnetic charge are not present in the form of the magnetic fields similar to dipolar case [8; 31] but emerge only in the form of the

**Fig. 4** Radial dependence of the ratio of the electric field to that when parameter  $l = 0$  for the different values of the NUT parameter. The value of NUT parameter  $l$  is given in cm

electric fields. In particular, we have shown that the frame-dragging and gravito-magnetic charge provide an additional induced electric field which is analogous to the one introduced by the rotation of the star in the flat spacetime limit [6].

Motion of charged particles around Kerr–Taub–NUT source immersed in either (i) external uniform or (ii) dipolar magnetic field have been investigated using Hamilton–Jacobi equation. We have shown that in the presence of NUT parameter and magnetic field the shape of effective potential will be changed. However, modifications caused by external electromagnetic field are dominating. Investigation of the stability of motion of charged particles shows, that external magnetic field shifts orbits of test particles to gravitational source in both cases, while NUT parameter shifts to black hole in the case of uniform magnetic field and towards loop (in opposite direction) in the case of the presence of current loop around the gravitational source.

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