LANDAU-GINZBURG HAMILTONIANS, STRING THEORY AND CRITICAL PHENOMENA

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Abstract

The identification of fixed points of N=2 supersymmetric Landau-Ginzburg models with N=2 superconformal minimal models is reviewed. We also discuss renormalization group flows, the role of singularity theory and the Gepner conjecture.

The subject of this talk has a bearing on a number of different areas. It is relevant to our understanding of string theory, the theory of critical phenomena and, more surprisingly, to the mathematical theory of manifolds. We begin by outlining the connection to the first two subjects.

When string theory first emerged in the 70's an obstacle to their being considered as realistic models of nature was that they were only consistent in 26 dimensions for the bosonic string and 10 dimensions for the superstring. In more recent years it has become apparent that consistent strings exist in other dimensions including four dimensions. We now believe that given any two-dimensional conformal system we can form a superstring theory by coupling it to supergravity. Such a string is thought to be consistent if there are no worldsheet anomalies, such as the conformal anomaly, and, in particular, if it is modular invariant. The former condition is to be expected as we have a local conformal symmetry while modular invariance ensures that certain space-time anomalies such as gauge and gravitational anomalies are absent.* Unfortunately, there are very, very many such string theories, corresponding to the vast number of different two-dimensional conformal systems. There is no known method of picking out a preferred string theory. However, as we shall see, Landau-Ginzburg models provide a framework in which a number of the interesting conformal models occur in one system depending upon which fixed point is selected. As such, they may provide a framework for addressing the above problem of the many theories or "vacua".

There may be some reasons to believe that the resulting space-time theory should possess space time supersymmetry [1]. This implies that the string should possess N=2 world-sheet supersymmetry. In this case, we are interested in N=2 supersymmetric Landau-Ginzburg Lagrangians and these form the subject of this review.

We begin with an elementary account of critical phenomena. We include it as an understanding of this subject is basic to the understanding of supersymmetric Landau-Ginzburg theory, and because it is perhaps not familiar to a number of theoretical physicists working in this field. Indeed, there are a number of incorrect statements in the literature on $N = 2$ Landau-Ginzburg mod els and we shall comment on some of these later.

A familiar example of a critical point is the point $T_c = 647^o K, p_c = 218 atm$ for H_2O where water and steam coexist. The transition that takes place at this point is a second- order phase transition, i.e. one involving no latent heat. In general more than two phases can coexist at a critical point; such critical points are known as multi-critical points.

Critical theory is the study of a system in the neighbourhood of its critical point. For H_20 one finds that if ρ is the density of water and ρ_c is the density at the critical point then $\rho - c\rho \sim (T_c - T)^\beta$ when measured along the water steam transition line and $\rho_c - \rho \sim (p - p_c)^{\frac{1}{\delta}}$ at T_c. The study of critical systems has revealed that they possess two remarkable features, non- analyticity and universality. The power law behaviour of certain quantities in the neighbourhood of the critical point is given by the critical indices $\beta, \delta, \ldots$, which turn out to be non- integral thus implying

* More recently, the consistency of non- critical strings has been the subject of much study.

non-analyticity. It was also found that the critical indices $\beta, \delta ...$ are the same for systems which have nothing in common except their dimensionality and symmetry. This property is referred to as universality. For example, the critical indices for the appropriate quantities for the CO_2 liquid-gas transition at $304^o K$ and for the binary alloy Co - Zn at $759^o K$ are the same.

The explanation for these features was found in a series of papers culminating in the work of Wilson [2] who made essential use of the renormalization group. He supposed that the system lies on the critical surface of an infra-red stable fixed point. This means that when one looks at the system at larger and larger scales, the Hamiltonian approaches a scale invariant one. In other words, the system at the fixed point is scale invariant and is dominated by long range correlations. Thus, at the critical point, short distance effects are unimportant, which is the reason for universal behaviour. In addition it is possible to give a good description of the theory in terms of an effective continuum field theory. Such field theories are called Landau-Ginzburg theories.

A good illustration of this is provided by the Ising model in three dimensions. The Landau-Ginzburg Hamiltonian given by

$$\mathcal{H}^{L-G} = \int d^3x \{ \frac{1}{2}(\partial_i \phi)^2 + \frac{1}{2} m^2(T)\phi^2 + \frac{\lambda}{4!}\phi^4 \}$$

where $m^2(T) \propto (T - T_c) + O((T - T_c)^2)$. The partition function Z is defined by

$$Z = \int \mathcal{D}\phi e^{-\beta \mathcal{H}^{L-G}}$$

where $\beta = (kT)^{-1}$. Although we are dealing with a classical system we can use the techniques of quantum field theory to evaluate the path integral. In the tree approximation, we are left with Landau theory [3], which although not correct in detail, provides at least some of the correct general features.

To follow the strategy of Wilson we must believe that the above theory possesses a β-function with an infra-red stable fixed point even though the theory is not even classically conformally invariant. It is easily seen that such a fixed point is outside the range of usual perturbation theory, for such an expansion must be in a dimensionless parameter $u = \frac{g}{m}$, however at the fixed point we have $m \to 0$ or $u \to \infty$.

Within the field theory context, there are two known techniques for finding the fixed point and calculating critical indices in its vicinity: the Wilson Fisher ε-expansion and the Parisi technique. We refer the reader to the reviews of references [4] and [5]. In the Wilson Fisher ε-expansion, the theory is considered in d dimensions and we perform a perturbation expansion in the coupling and $\varepsilon = d_c - d$ where d_c is the critical dimension of the theory, i.e. the dimension in which the coupling constant is dimensionless and the theory classically scale invariant. At the end of the calculation we set d to be the actual dimension of the system. For the three dimensional Ising model $d_c = 4$ and so $\varepsilon = +1$. The Parisi technique relies on the introduction of an appropriate coupling constant which is well-defined at the fixed point. The perturbation expansion and the renormalisation group calculations can then be done using the Callan-Symanzik equation. For the Ising model the anomalous dimension of the field ϕ was found by these techniques to be 1.250 $\pm$ 0.005 leading to a critical index which is correct to a few parts in 10^4. A similar, but much less precise result was found for the two dimensional Ising model [6]. The anomalous dimensions of various, in general, composite, operators in the field theory are related in a straightforward way to the critical indices of the model. As the former are non-integral, so are the latter and hence the behaviour is non-analytic.

We now turn to the discussion of two dimensional supersymmetric critical phenomena. Our N=2 supersymmetric Landau-Ginzburg Hamiltonian is given by

$$\mathcal{H}^{L-G} = \int d^2x d^4\theta \phi \phi + g\{ \int d^2x d^2\theta \frac{\phi^n}{n!} + h.c.\}.$$

where the superfield ϕ consists of the component fields (a, χ_A, f) which are a complex spin 0, a complex spin $\frac{1}{2}$ and an auxiliary field f. The super Ising model corresponds to taking n = 3; but for larger n we have n-1 phases coexisting corresponding to a multi-critical point.

We were able to show [7], [8], [9], using the ϵ -expansion and also the Parisi technique that this N=2 supersymmetric model has an infra-red stable fixed point at which the superconformal theory is described by the nth member of the N=2 supersymmetric minimal series.

This result had been previously conjectured by Zamalodchikov [10] for N=0 and N=1 multicritical models and by Kastor Martinec and Shenker [11] for the N=2 case. An alternative, but subsequent, derivation using different methods was given in reference [12].

We do not have the space in this review to give a full account of these proofs and we refer the reader to references [7], [8] and [9] for more details. The essential steps are as follows. We first adapt the method of interest, i.e. the ϵ-expansion or Parisi technique, to the supersymmetric case. We then prove the analogue of the four dimensional non-renomalization theorem. For the Wilson-Fisher ϵ-expansion technique the proof of the non-renormalization theorem is along similar lines to the standard four-dimensional case, making use of superspace Feynman diagram techniques. In the Parisi method the non-renormalization theorem is established by using the Callan-Symanzik equations. The non-renormalization theorem is then used to derive a relation between the β function and the anomalous dimension γ of the field ϕ allowing us to compute γ at the fixed point.

To give the reader some insight we outline this last step for the Wilson-Fisher ϵ -expansion technique. Let us perform the wave function and coupling constant renormalizations

$$\phi_0 = Z^{\frac{1}{2}}\phi, g_0 = \mu^{(\frac{n-2}{2})} g Z_g$$

where μ is the renormalization scale, ϕ_0 and g_0 the bare fields and coupling and Z and Z_g the wavefunction and coupling renormalizations. The perturbation theory is an expansion in g and $\epsilon = \frac{2}{n-2}$ where the critical dimension of the theory is $2+\epsilon$. The non-renormalization theorem implies the relation $Z^{n/2} Z_g = 1$.

Using the definitions $\beta = \mu\frac{\partial}{\partial_\mu} g \mid_{g_0,\Lambda}, \gamma = \frac{1}{2}\mu\frac{\partial}{\partial_\mu} lnZ \mid_{g_0,\Lambda}$, we readily find that

$$0 = (\frac{n-2}{2})\epsilon g + \beta - ng\gamma.$$

At the fixed point $\beta = 0$ and so, to all orders of perturbation theory,

$$\gamma_* = (\frac{n-2}{2})\epsilon = \frac{1}{n}.$$

Hence, in this case, unlike in previous examples of the ϵ-expansion, we find an exact result for the anomalous dimension.

The last step in the derivation is to identify the field theory at a given fixed point with a member of the N=2 supersymmetric minimal series which have $c = 3(1-\frac{2}{n}), n = 3,4,...$ The primary fields, $\phi_{j,m}$, of these models, are labelled by two integers j and m which are related to the eigenvalues of L_0 and the U(1) symmetry generator T_0 by [13]

$$H_{j,m} = \frac{j(j+1)-m^2}{4n} \qquad , T_{j,m} = \frac{m}{n}$$

respectively. The "chiral" primary fields are those with $m = \pm j$. In any conformal field theory, the two point functions are known and in these theories those for the chiral primaries are given by $< \phi_{H,T}\bar{\phi}_{H,-T} > \sim \frac{1}{|z\bar{z}|^{2H}}$. On the other hand, in the Landau- Ginzburg model, the renormalisation group summed propagator goes as $\sim \frac{1}{|z\bar{z}|^{\gamma_*}}$. Consequently we identify $H = \frac{\gamma_*}{2}$. Substituting our value for γ_* we find $H = \frac{1}{2n}$ which agrees with the dimension of the lowest primary field $\phi_{1,1}$ in the n^{th} minimal model.

It was also shown that at the fixed point the central charge is $c = 3(1-\frac{2}{n})$, the anomalous dimension of the compposite operator ϕ^m is m/n, the operator product expansion $\phi^n\phi^m \sim \phi^{n+m}$ [7] and finally that the chiral correlators $< \phi(z_1)...\phi(z_n) >$ calculated in the Landau-Ginzburg model and the conformal field theory corresponding to the n^{th} member of the N=2 minimal series agree [8].

It had been first observed in references [14] and [15] at an empirical level that, if the relation between $N=2$ superconformal models and Landau- Ginzburg models is assumed to hold, then the central charge can be written as $c = 3(1-2q_L)$ where $q_L = \frac{1}{n}$ is the $U_L(1)$ charge of ϕ. In references [14] and [16] no calculations starting from a Landau-Ginzburg Hamiltonian were performed. However, it was assumed that the perturbation series relevant to criticality satifies a strong version of the non-renormalisation theorem, namely that there are no quantum corrections to the effective action of the form of subsuperspace integrals of local functionals. This assumption, repeated in other papers, is in fact not true as was shown by an explicit calculation in [8]. A more detailed account of this will be given elsewhere.

We can generalise the discussion of Landau-Ginzburg Hamiltonians given in this review to the case of many superfields $\phi^i, i = 1,2,...,n$. We consider

$$\mathcal{H} = \int d^2x d^4\theta \sum_i \phi_i \bar{\phi}_i + \{\int d^2x d^2\theta \frac{m_{ij}}{2} \phi^i \phi^j$$
$$+ + W(\phi) + h.c.\}.$$

where

$$W(\phi) = \sum_{p=3} \sum_{\{i_1,\ldots,i_p\}} \lambda_{i_1,\ldots,i_p} \phi^{i_1 \ldots i_p}$$

To examine the fixed points and corresponding critical behaviour of such a Hamiltonian one must use the Parisi method since the ϵ-expansion relies on the theory possessing a dimension for which all the coupling are dimensionless. Carrying out the analogue of the previous calculations for the many field case [9] we find that the β -functions are given by

$$\beta_{i_1 \ldots i_n} = -\lambda_{i_1 \ldots i_n} + \sum_n \sum_p \gamma^k_{ip} \lambda_{i_1 \ldots k \ldots i_n}$$

where γ^k_{ip} is the matrix of anomalous dimensions. The reader may recover the results for the one field case given earlier. At a fixed point $\beta_{i_1 \ldots i_n} = 0$ and one obtains a relation between the anomalous dimensions γ^j_i and the couplings. In fact, γ^j_i is a positive definite symmetric matrix which may be diagonalized by a field redefinition. This relation then has a simple interpretation in terms of W, namely at the fixed point, W satisfies

$$W(\lambda^{\gamma_i} \phi_i) = W(\phi_i).$$

This means W is a quasi-homogeneous function of degree 1, a result conjectured in reference [14].

There exists a considerable literature on the relation between critical phenomena and singularity theory [17]. It was realised that at the critical point the effective potential of the Landau-Ginzburg model is massless ($i.e. \frac{\partial^2 V^{eff}}{\partial \phi^i \partial \phi^j} |_{\phi=0} = 0$) and the equation of motion at zero momentum implies $\frac{\partial v^{eff}}{\partial \phi^i} |_{\phi=0} = 0$. In singularity theory [18] a potential is called degenerate and critical if the above two criteria are met. All such potentials (of low modality) have been classified up to field redefinitions. This relation between singularity theory and critical phenomena was also reiterated more recently [14,15] within the context of Landau-Ginzburg models.

However, it is not clear that these two such disparate subjects are related in the sense that every potential which is in the singularity theory classification list necessarily corresponds to a fixed point and so to critical behaviour in the physical sense. Indeed, one finds that there exist examples of potentials in the singularity listing that are not quasi-homogeneous and so according to the previous discussion cannot correspond to fixed points.

For an N=2 supersymmetric with one superfield, however, one can show [9] that all fixed points are of the form ϕ^n and this does coincide with the list of such potentials given in singularity theory. In this case one can also carry out explicitly the renormalization group flow from one fixed point to another [9].

We end this review with two unsolved problems. Zamalodchikov has conjectured [10] a relation between the N=0 and N=1 minimal series and fixed points of the corresponding Landau-Ginzburg models. The techniques described here which established this conjecture for the N=2 case were also used for the N=0 case, but the low order results of the ϵ-expansion were far away from the minimal series results[19]. It is not possible to conclude one way or the other from these low order results, but these authors know of no concrete argument that supports the Zamolodchikov conjecture for N=0 and 1.

Finally, we will discuss to what extent the interesting conjecture of Gepner [20] concerning the relation between strings propagating on Calabi-Yau manifolds and strings constructed from tensor products of N=2 superconformal field theories is true.

We now know [7,8,9] that N=2 superconformal theories are related to fixed points in Landau-Ginzburg Hamiltonians. For example, 5 tensored copies of $c = \frac{9}{5}$ N=2 minimal model corresponds to the potential $\sum_i \phi_i^5$. It was observed empirically in reference [11] that the potential that occurs in the Landau-Ginzburg model is the same as the embedding polynomial of the corresponding Calabi-Yau manifold in CP^n. For the example above, the Calabi Yau manifold is defined by setting the embedding polynomial $\sum_i z_i^5 = 0$ in complex 5-space. Thus one could prove the Gepner conjecture if one could show that the Landau-Ginzburg Hamiltonian at the fixed point was such that it constrained the potential to vanish and the kinetic term had the appropriate metric, although it is not entirely clear what metric this should be. In this context we note that the majority of the considerations of the proof of references [7] [8] [9] apply to the chiral operators and we did not identify the non-chiral primary fields with Landau-Ginzburg operators. Although we took a canonical kinetic term our considerations were rather insensitive to this assumption. This would not be the case for the non-chiral operators whose identification would presumably fix the kinetic term, and hence establish in what sense the Gepner conjecture is true. A clear demonstration of this point would provide an interesting connection between quantum field theory (critical phenomena) and the geometry of Calabi Yau manifolds. In [21] Green, Vafa and Warner assumed the connection between Landau-Ginzburg models and minimal N=2 super conformal models and purported to show the relation of Landau-Ginzburg models to Calabi-Yau manifolds. An essential step in this "proof" was setting the kinetic term to zero at the fixed point. In this case, however, we have a system for which interactions at different spatial points are not correlated. In the language of the lattice, the spins at different lattice sites are not correlated. The system is effect frozen and there is no renormalization group flow from this Hamiltonian. The most important characteristic of a system at its critical point is that it is a system which is dominated by long range correlations. Thus the Hamiltonian with zero kinetic term has nothing to do with the fixed points corresponding to the N=2 minimal series nor is it in the same universality class.

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