July, 2016

# Renormalizable theories with symmetry breaking [1]

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**Abstract**

The description of symmetry breaking proposed by K. Symanzik within the framework of renormalizable theories is generalized from the geometrical point of view. For an arbitrary compact Lie group, a soft breaking of arbitrary covariance, and an arbitrary field multiplet, the expected integrated Ward identities are shown to hold to all orders of renormalized perturbation theory provided the Lagrangian is suitably chosen. The corresponding local Ward identity which provides the Lagrangian version of current algebra through the coupling to an external, classical, Yang-Mills field, is then proved to hold up to the classical Adler-Bardeen anomaly whose general form is written down. The BPHZ renormalization scheme is used throughout in such a way that the algebraic structure analyzed in the present context may serve as an introduction to the study of fully quantized gauge theories.

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[1] The present manuscript contains a partial revision and update of the results of a work done in collaboration with A. Rouet and R. Stora in 1974-5 and published after six years in [1]. The author of the present manuscript has tried to keep unchanged the largest possible part of the original paper and takes full responsibility for any mistakes.

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# 1 Introduction

Besides the well-known relevance of broken symmetries to elementary particle physics, further investigations of renormalizable models exhibiting broken symmetries are amply justified by the present understanding of gauge and even super-gauge theories. The present status of the subject is well represented by K. Symanzik's 1970 Cargése lectures where the fundamental phenomena are discussed [2]. Since much of the structure analyzed there can also be found in the study of gauge theories, it is of interest to complete Symanzik's analysis from the algebraic point of view, both in considering an arbitrary compact Lie group as describing the symmetry to be broken, and in analyzing the perturbative analog of current algebra, namely the coupling with an external Yang-Mills field, which, as is well-known, leads to the definition of the celebrated Schwinger-Bell-Jackiw-Adler-Bardeen anomaly [3]. In fact, although this program had been announced by K. Symanzik, [2], it has not been carried out until now following the stimulation [4] provided by the advent of gauge theories. Also, it seems that recent progress in renormalization theory has allowed a more tractable general treatment than the techniques known in 1970 would have permitted. Most of the present analysis relies on general properties of the perturbative series which stem from locality and power counting, as summarized by the renormalized action principle of Lowenstein and Lam [5] whose detailed form is one of the highlights of the Bogoliubov-Parasiuk-Hepp-Zimmermann [6] renormalization scheme. Within this framework, explicit bases of local operators of given dimensions are constructed [7], together with the linear relations connecting operators with different dimensions (the so-called Zimmermann identities). To avoid inessential technical complications, we shall only treat cases in which no massless field is involved. Our analysis can, however, be extended without essential modifications to a wide class of models involving massless fields exploiting Lowenstein's and Zimmermann's extension of BPHZ renormalization scheme [8]. Whereas these elementary tools, which are best exploited by means of a repeated application of the implicit function theorem for formal power series [1], suffice to solve most of the algebraic problems at hand, the elimination of some possible anomalies is occasionally performed by looking more deeply into the behavior of the theory under scaling transformations, which provides some new non-renormalization type statements similar to that which leads to the non-renormalization of the Adler-Bardeen anomaly coefficient [9][10]. This article is divided into two main parts: Section 2 is devoted to the proof of the integrated Ward identity which expresses symmetry breaking for an arbitrary compact Lie group, with an arbitrary dimension ($< 4$) and covariance. Section 3 is devoted to a discussion of 'current algebra' which, in the present framework amounts to the proof of a local Ward identity, in the presence of an external Yang-Mills field, and leads to the definition of the Adler-Bardeen anomaly.

[1] For a brief summary see [4] (b), Appendix II.

A number of appendices are devoted to the treatment of some technical questions, among which the elimination from the integrated Ward identities of algebraically allowed anomalies consistent with power counting, and details about the cohomology of the gauge Lie algebra associated with the symmetry group (i.e. the Wess-Zumino [11] consistency conditions).

# 2 Broken global symmetries

This chapter is devoted to the proof of the perturbative renormalizability of a generic model built on a set of quantized field variables and characterized by a softly broken invariance under field transformations belonging to a compact Lie group. We shall systematically use a functional formulation in which, e.g., Green's functions are obtained as functional derivatives of their functional generator, and the classical Lagrangian is a local field functional.[2] The need of describing local operators such as e.g. the terms breaking the invariance of the classical Lagrangian, requires the introduction, together with the quantized fields, of further functional variables, that we call external fields, coupled to the relevant operators.

It might be useful to shortly remind the general properties of a perturbatively renormalizable theory, in particular, in the chosen, regularization independent, framework based on the BPHZ scheme.

First of all, the perturbative construction is based on the Feynman diagram expansion.[3] The kernel of Feynman's construction is the calculation of 1-particle irreducible (1-P.I.) diagrams amputated of their external legs. Their functional generator is called the *effective action* and denoted by $\Gamma(\varphi)$. A $n$ loop 1-P.I. diagram corresponds to an amplitude proportional to $\hbar^n$, thus $\Gamma(\varphi)$ is a formal power series in $\hbar$. In the classical limit Feynman diagrams correspond to tree diagrams, those without loops, and $\Gamma(\varphi)$ corresponds to the classical action. In our scheme in the fully quantized limit, not only Green's functions, but also many important quantities, as e.g. Lagrangian parameters, are formal power series in $\hbar$.

Renormalizability is based on power counting. A canonical (power counting) dimension is associated with any field, in particular, in the case of quantized fields, this dimension is determined by the maximum derivative degree of the free, bilinear part of the Lagrangian, or else, of the higher derivative part of the wave operator which is assumed non-degenerate. As is well known the short distance behavior of the causal Green's function, the propagator, is determined by the dimension of the

[2] Thus quantized fields can also be interpreted as functional variables.

[3] In Feynman amplitudes quantized fields propagate while external ones do not. In the functional formalism to every quantized field one associates a further functional variable called the field source which plays the role of Legendre conjugate variable to the quantum field. The Legendre transform of the classical action is the functional generator of the tree-approximation Feynman diagrams.

corresponding fields. The general necessary condition for renormalizability is that the canonical dimension of the Lagrangian, also including the contribution of derivatives, should not exceed four.[4]

In many important cases, once the classical Lagrangian is given, one builds Feynman diagrams, and hence the effective action, computing suitably regularized 1-P.I. diagrams, so avoiding divergent results. A clever choice of regularization my help in preserving symmetry properties of Green's functions. In reality this works well in some cases, while it is not a universal method. Choosing BPHZ subtraction method we have a systematic construction of Green's functions, but symmetry might be broken by loop corrections. The aim of the present paper is to show how symmetry can be restored even in the BPHZ framework and hence independently of regularization. Zimmermann's subtraction method associates with every vertex in a Feynman diagram a quantized field dependent monomial $M(\varphi)$ equipped with the prescription that the non trivial (sub)-diagrams containing the vertex should be subtracted at zero momenta of the external legs together with their Taylor expansion up to total dimension $\delta \geq \dim M$. The operator corresponding to the 'subtracted' monomial is denoted by $N_\delta[M]$ and $\delta$ is called Zimmermann's index.[5] For the terms of the effective Lagrangian $\mathcal{L}_{\text{eff}}$, which contains all the prescriptions for the Green function construction, an $N_4$ subtraction is understood. A second basic point is Lowenstein-Lam's quantum action principle [5] according to which the variation of $\Gamma(\varphi)$ under infinitesimal parameter and field transformations corresponds to the *insertion* into $\Gamma(\varphi)$ of a (possibly integrated) local vertex whose Zimmermann's index is the maximum canonical dimension of the variation of $\mathcal{L}_{\text{eff}}$, four in our case. The insertion of the vertex $V(\varphi)$ into $\Gamma(\varphi)$ corresponds to the introduction into every 1-P.I. diagram contributing to the expansion of $\Gamma(\varphi)$ of a further local vertex which is specified by the form of $V$. In general $V$ may depend on both external and quantum fields. The insertion of the vertex $V(\varphi)$ into $\Gamma(\varphi)$ is denoted by $V\Gamma(\varphi)$ which is a new $\hbar$ formal power series valued functional satisfying the equation

$$V\Gamma(\varphi) = V(\varphi) + O(\hbar V)$$

where $V(\varphi)$ is interpreted as a local functional and $O(\hbar V)$ lumps the contributions of the non trivial loop diagrams together. There are exceptional situations which correspond to operators, either independent, or linear in the quantized fields. In these

[4]Note that e.g. in the case of a massive vector field where gauge invariance is broken by a mass term the higher derivative part of the wave operator is degenerate due to gauge invariance and the model is not renormalizable.

[5]The great advantage of Zimmermann's subtraction method is its precise definition and the identification of complete bases of local operators with well defined power counting behavior. In spite of the very careful and detailed form of the original formulation, it is possible to show that essentially the same properties are obtained by other methods, e.g. renormalization group evolution equations, in which extra subtractions correspond to stronger initial conditions.[12]

cases the corresponding vertices cannot be inserted into 1-P.I. diagrams and hence the insertion of an exceptional operators into $\Gamma(\varphi)$ is purely additive $V\Gamma(\varphi) = V(\varphi)$.

A general quantization condition for any system is the stability of its dynamics under infinitesimal changes of parameters and consistent deformations of symmetry conditions. Dealing with $\hbar$ formal power series, implicit function theorem says that the mentioned stability properties are guaranteed if they hold true at the zeroth order, that is, in the classical theory. Thus if, e.g., we want to construct a perturbation theory for which a particle interpretation exists, we must assume that there exists an invertible change of variables between the parameters of the classical Lagrangian and the physical ones. In general we shall precede the analysis of any quantum property by a discussion of the classical case and of its stability under change of parameters and symmetry conditions. We shall denote by an upper ring the classical quantities with the exception of the the classical Lagrangian density functional/operator $\mathcal{L}$. Thus we have the functional equation

$$\mathring{\Gamma} = \int dx\ \mathcal{L},$$

and the operator equation

$$\mathring{\mathcal{L}}_{\text{eff}} = N_4[\mathcal{L}].$$

## 2.1 The Classical Theory

The general situation is as follows: G is a compact Lie group, $\mathcal{G}$ its Lie Algebra: $\mathcal{G} = \mathcal{S} + \mathcal{A}$, $\mathcal{S}$ semi-simple, $\mathcal{A}$ Abelian. $\boldsymbol{\varphi}$ is a field multiplet belonging to a fully reduced finite-dimensional unitary representation $D$ of $G$, $d_\varphi > 0$ is the canonical dimension of $\varphi$. Given $X \in \mathcal{G}$ the corresponding infinitesimal transformation of $\boldsymbol{\varphi}$ is:

$$\delta_X \boldsymbol{\varphi} = -\mathring{t}(X)\boldsymbol{\varphi} \tag{2.1}$$

where $X \to \mathring{t}(X)$ is the representation of $\mathcal{G}$ induced by $D$.

Let $\boldsymbol{\beta}$ be a classical field to which is assigned dimension $d_\beta < 4$, belonging to a multiplet characterized by another representation $\mathcal{D}$ of $G$ (finite dimensional, fully reduced, unitary, with no identity component) and

$$X \to \mathring{\theta}(X), \quad X \in \mathcal{G}, \tag{2.2}$$

be the corresponding representation of $\mathcal{G}$. The symmetry $G$ will be said to be broken with dimension $4 - d_\beta$, covariance $\mathring{\boldsymbol{b}}$, belonging to multiplet $\mathcal{D}$, if there exists a Lagrangian $\mathcal{L}(\boldsymbol{\varphi}, \boldsymbol{\beta})$ of maximum dimension four invariant under the simultaneous transformation

$$\boldsymbol{\varphi} \to {}^g\boldsymbol{\varphi} = D(g^{-1})\boldsymbol{\varphi}, \tag{2.3}$$

$$\boldsymbol{\beta} + \boldsymbol{b} \to {}^g(\boldsymbol{\beta} + \boldsymbol{b}) = \mathcal{D}(g^{-1})(\boldsymbol{\beta} + \mathring{\boldsymbol{b}}). \tag{2.4}$$

The classical field $\boldsymbol{\beta}$ is introduced as an auxiliary item, in order to characterize the breaking described by the space-time independent $\mathring{\boldsymbol{b}}$ according to its dimension, a concept which is meaningful in the renormalizable framework we have in mind. The theory will be truly renormalizable, i.e. $\mathcal{L}(\boldsymbol{\varphi}, \boldsymbol{\beta})$ will be a polynomial, if

$$d_{\boldsymbol{\beta}} > 0, \tag{2.5}$$

This criterion, introduced by Symanzik [2], leaves the broken theory with an asymptotic memory of the initial symmetry group. However, the limiting case

$$d_{\boldsymbol{\beta}} = 0,$$

can also be considered since $\mathcal{L}(\boldsymbol{\varphi}, \mathring{\boldsymbol{b}})$ is invariant under simultaneous transformation of $\boldsymbol{\varphi}$ and $\mathring{\boldsymbol{b}}$ and is not, in general, the most general Lagrangian which is invariant under the residual symmetry group, namely the stability group $H_{\mathring{\boldsymbol{b}}}$ of $\mathring{\boldsymbol{b}}$. Clearly, the notion we have introduced only depends on the equivalence classes of $D$, $\mathcal{D}$ and the orbit of $\mathring{\boldsymbol{b}}$. We shall assume that in the tree approximations of the corresponding Green functions and for some values of the parameters characterizing $\mathcal{L}$, a particle interpretation is possible and that there is an invertible change of parameters between the coefficients of $\mathcal{L}$ and those occurring in normalization conditions through which masses, coupling constants, etc., are defined. We shall furthermore assume that no vanishing mass parameter appears in the theory. When the Lagrangian has a term linear in the quantized field, the particle interpretation requires a field translation

$$\boldsymbol{\varphi} \to \boldsymbol{\varphi} + \mathring{\boldsymbol{F}}$$

through which the linear term is eliminated. $\mathring{\boldsymbol{F}}$ is then defined by:

$$\partial_{\varphi} \mathcal{L}(\boldsymbol{\varphi}, \mathring{\boldsymbol{b}})|_{\varphi = \mathring{F}} = 0$$

which certainly has a solution continuous in the parameters of $\mathcal{L}$ if the mass matrix

$$\mathcal{M} = \partial^2_{\varphi\varphi} \mathcal{L}(\boldsymbol{\varphi}, \mathring{\boldsymbol{b}})|_{\varphi = \mathring{F}}$$

is non-degenerate. As shown in Appendix A, $\mathring{\boldsymbol{F}}$ is then a covariant function of $\mathring{\boldsymbol{b}}$, and, consequently, the coefficients of the Lagrangian expressed in terms of the translated fields are also covariant. From now on, we shall still denote by $\boldsymbol{\varphi}$ and $\boldsymbol{\beta}$ the translated field and by

$$\tilde{\mathcal{L}}(\boldsymbol{\varphi}, \boldsymbol{\beta}) \equiv \mathcal{L}(\boldsymbol{\varphi} + \mathring{\boldsymbol{F}}, \boldsymbol{\beta} + \mathring{\boldsymbol{b}}).$$

At this point, the action

$$\mathring{\Gamma}(\boldsymbol{\varphi}, \boldsymbol{\beta}) = \int dx \; \tilde{\mathcal{L}}(\boldsymbol{\varphi}, \boldsymbol{\beta}) \tag{2.6}$$

fulfills the *integrated* Ward identity

$$W(X)\mathring{\Gamma}(\boldsymbol{\varphi},\boldsymbol{\beta}) \equiv \quad -\int dx\Big\{\frac{\delta\mathring{\Gamma}}{\delta\boldsymbol{\varphi}}\mathring{t}(X)(\boldsymbol{\varphi}+\mathring{\boldsymbol{F}}) + \frac{\delta\mathring{\Gamma}}{\delta\boldsymbol{\beta}}\mathring{\theta}(X)(\boldsymbol{\beta}+\mathring{\boldsymbol{b}})\Big\} = 0, \qquad X\in\mathcal{G} \tag{2.7}$$

which expresses its invariance under the infinitesimal transformation

$$\begin{aligned} \delta_X\boldsymbol{\varphi} &= -\mathring{t}(X)(\boldsymbol{\varphi}+\mathring{\boldsymbol{F}}), \\ \delta_X\boldsymbol{\beta} &= -\mathring{\theta}(X)(\boldsymbol{\beta}+\mathring{\boldsymbol{b}}). \end{aligned} \tag{2.8}$$

Now we note that if we wants to construct a perturbation theory for which a particle interpretation exists, it is necessary to assume the following: Let the Lagrangian be written in the form

$$\mathcal{L} = \mathring{\boldsymbol{C}}_\sharp\boldsymbol{\mathcal{L}}_\sharp = \sum_i \mathring{C}^i_\sharp\mathcal{L}^i_\sharp \tag{2.9}$$

where the $\mathring{C}^i_\sharp$'s are numerical coefficients and $\mathcal{L}^i_\sharp$ are all possible local monomials invariant under Equation (2.8), consistent with the renormalizability requirement. Then there must exist an invertible change of variables between, on the one hand, the $\mathring{C}^i_\sharp$'s and $\mathring{\boldsymbol{b}}$ and, on the other hand, a set of physical parameters (masses, wave function normalizations, coupling constants) occurring in normalization conditions imposed on $\mathring{\Gamma}$. These normalization conditions must be consistent with the symmetry expressed by the Ward identity, but not constrained by power counting. This implies in particular that power counting does not restrict Equation (2.9) compared to the most general solution of Equation (2.7), as far as these normalization conditions are concerned (e.g. power counting does not enforce mass rules). Of course, the fulfillment of normalization conditions is only necessary if a particle interpretation is required, the Ward identity being sufficient if only a theory of Green's functions is aimed at.

Secondly one might object that the prescription of the Ward identity Equation (2.7) does not seem to define the theory in a natural way from the point of view of power counting: in a more general scheme one would have a Ward identity with the following structure:

$$W(X)\mathring{\Gamma}(\boldsymbol{\varphi},\boldsymbol{\beta}) \equiv \quad -\int dx\Big\{\frac{\delta\mathring{\Gamma}}{\delta\boldsymbol{\varphi}}(\mathring{T}(X)\boldsymbol{\varphi}+\mathring{\boldsymbol{F}}(X)) + \frac{\delta\mathring{\Gamma}}{\delta\boldsymbol{\beta}}(\mathring{\Theta}(X)\boldsymbol{\beta}+\mathring{\boldsymbol{b}}(X))\Big\} = 0, \qquad X\in\mathcal{G} \tag{2.10}$$

subject to the algebraic constraint

$$[W(X),W(Y)] = W([X,Y]), \qquad X,Y\in\mathcal{G} \tag{2.11}$$

thus expressing the invariance of $\mathring{\Gamma}$ under the transformation

$$\begin{aligned}
\delta_X \boldsymbol{\varphi} \quad &= -\left[\mathring{T}(X)\boldsymbol{\varphi} + \mathring{\boldsymbol{F}}(X)\right], \\
\delta_X \boldsymbol{\beta} \quad &= -\left[\mathring{\Theta}(X)\boldsymbol{\beta} + \mathring{\boldsymbol{b}}(X)\right].
\end{aligned} \tag{2.12}$$

with coefficients constrained by:

$$\begin{aligned}
&a) \quad \left[\mathring{T}(X), \mathring{T}(Y)\right] = \mathring{T}([X,Y]), \\
&b) \quad \left[\mathring{\Theta}(X), \mathring{\Theta}(Y)\right] = \mathring{\Theta}([X,Y]), \\
&c) \quad \mathring{T}(X)\mathring{\boldsymbol{F}}(Y) - \mathring{T}(Y)\mathring{\boldsymbol{F}}(X) - \mathring{\boldsymbol{F}}([X,Y]) = 0, \\
&d) \quad \mathring{\Theta}(X)\mathring{\boldsymbol{b}}(Y) - \mathring{\Theta}(Y)\mathring{\boldsymbol{b}}(X) - \mathring{\boldsymbol{b}}([X,Y]) = 0,
\end{aligned} \tag{2.13}$$

according to which

$$\begin{aligned}
X &\rightarrow \mathring{T}(X), \\
X &\rightarrow \mathring{\Theta}(X),
\end{aligned}$$

are representations of $\mathcal{G}$ and $\mathring{\boldsymbol{b}}(X)$, $\mathring{\boldsymbol{F}}(X)$ are $\mathcal{G}$ Lie algebra cocycles [6] with values in the representation spaces $E$ and $\mathcal{E}$ of $\mathring{T}$ and $\mathring{\Theta}$ respectively. It is shown in Appendix B that the requirements which allow a particle interpretation and take into account our definition of symmetry breaking put quite severe restrictions on $\mathring{T}(X)$, $\mathring{\Theta}(X)$, namely they can be lifted to the group $G$, and thus, in particular, they are fully reducible. They can thus be obtained from representatives of their equivalence classes which are related to $\mathring{t}$, $\mathring{\theta}$, through suitable field renormalizations:

$$\begin{aligned}
\mathring{T}(X) = &\quad Z^{-1}\mathring{t}(X)Z, \\
\mathring{\Theta}(X) = &\quad Z^{-1}\mathring{\theta}(X)Z,
\end{aligned} \tag{2.14}$$

It is then shown in Appendix B that $\mathring{\boldsymbol{F}}(X)$, is a Lie algebra coboundary

$$\mathring{\boldsymbol{F}}(X) = \mathring{t}(X)\mathring{\boldsymbol{F}} \tag{2.15}$$

for some fixed $\mathring{\boldsymbol{F}}$, up to invariant components

$$\mathring{\boldsymbol{F}}_\sharp(X)$$

which vanish for $X \in \mathcal{S}$, the semi-simple part of $\mathcal{G}$. It is finally shown in Appendix B that $\mathring{\boldsymbol{F}}_\sharp(X) \neq 0$ contradicts the assumption that the mass matrix is non-degenerate. Thus the Ward identity (Equation (2.7)) is actually the most general in the present context, since the Lie algebra coboundary structure of $\mathring{\boldsymbol{b}}(X)$:

$$\mathring{\boldsymbol{b}}(X) = \mathring{\theta}(X)\mathring{\boldsymbol{b}},$$

is implied by our picture of symmetry breaking.

[6] For a brief summary of Lie algebra cohomology, in particular the meaning of coboundary and cocycle, see [4] (c), Appendix A

## 2.2 Radiative Corrections

The description of radiative corrections proceeds via the construction of an effective dimension four Lagrangian

$$\mathcal{L}_{\text{eff}} = N_4[\mathcal{L} + \hbar\Delta\mathcal{L}] \tag{2.16}$$

without a term linear in the quantized fields, with coefficients formal power series in $\hbar$ such that the renormalized Ward identity holds:

$$\begin{aligned} W(X)\Gamma \equiv \quad & -\int dx \Big\{ \frac{\delta\Gamma}{\delta\boldsymbol{\varphi}}(t(X)\boldsymbol{\varphi} + \boldsymbol{F}(X)) \\ & + \frac{\delta\Gamma}{\delta\boldsymbol{\beta}}(\theta(X)\boldsymbol{\beta} + \boldsymbol{b}(X)) \Big\} = 0, \qquad X \in \mathcal{G} = 0 \end{aligned} \tag{2.17}$$

subject to the algebraic constraints strictly analogous to those given in Equation (2.11). Equation (2.17) expresses the invariance of $\mathcal{L}_{\text{eff}}$ in the sense of the renormalized action principle under the renormalized transformation

$$\begin{aligned} \delta_X\boldsymbol{\varphi} \quad & = -(t(X)\boldsymbol{\varphi} + \boldsymbol{F}(X)), \\ \delta_X\boldsymbol{\beta} \quad & = -(\theta(X)\boldsymbol{\beta} + \boldsymbol{b}(X)). \end{aligned} \tag{2.18}$$

which coincides with Equation (2.8) in the lowest order in $\hbar$. The analysis performed in this section will actually lead to the conclusion that there exists a quantum extension in which the almost naive Ward identity holds:

$$\begin{aligned} W(X)\Gamma(\boldsymbol{\varphi},\boldsymbol{\beta}) \equiv \quad & -\int dx \Big\{ \frac{\delta\Gamma}{\delta\boldsymbol{\varphi}}\mathring{t}(X)(\boldsymbol{\varphi} + \boldsymbol{F}) \\ & + \frac{\delta\Gamma}{\delta\boldsymbol{\beta}}\mathring{\theta}(X)(\boldsymbol{\beta} + \boldsymbol{b}) \Big\} = 0, \qquad X \in \mathcal{G} \end{aligned} \tag{2.19}$$

where $\boldsymbol{F}$ is determined by the requirement that $\mathcal{L}_{\text{eff}}$ has no term linear in $\boldsymbol{\varphi}$, and $\boldsymbol{b}$, which picks up radiative corrections due to normalization conditions, has the same stability group $H_{\mathring{\boldsymbol{b}}}$ of $\mathring{\boldsymbol{b}}$. This is to say that one can fulfill quantum invariance under the renormalized transformation

$$\begin{aligned} \delta_X\boldsymbol{\varphi} \quad & = -\mathring{t}(X)(\boldsymbol{\varphi} + \boldsymbol{F}), \\ \delta_X\boldsymbol{\beta} \quad & = -\mathring{\theta}(X)(\boldsymbol{\beta} + \boldsymbol{b}), \end{aligned} \tag{2.20}$$

as a consequence of the algebraic constraints, Equation (2.11). It is shown in Appendix B that the consistency conditions (Equations (2.13)) on $t(X)$ and $\theta(X)$ can be used to replace them by $\mathring{t}(X)$ and $\mathring{\theta}(X)$ up to terms which can be interpreted as invariant anomalies to the Abelian Ward identities, i.e. Equation (2.17) with $X$ restricted to $\mathcal{A}$, the Abelian part of $\mathcal{G}$. Similarly the consistency condition on $\boldsymbol{b}(X)$ leads to

$$\boldsymbol{b}(X) = \mathring{\theta}(X)\boldsymbol{b}$$

where $\boldsymbol{b}$ is arbitrary, and can trivially be chosen to keep the same stability group of its classical limit (the residual symmetry group). Finally $\boldsymbol{F}(X)$ is found to be of the form $\mathring{t}(X)\boldsymbol{F}$ for some fixed $\boldsymbol{F}$, up to terms which again can be interpreted as invariant anomalies to the Abelian Ward identities.

It will be shown, see in particular Appendix C, that the Ward identity, Equation (2.19), cannot be broken exclusively by invariant Abelian anomalies, one concludes that, modulo a field renormalization, all solutions of Equation (2.17) are solutions of Equation (2.19).

We thus proceed to analyze the validity of Equation (2.19). Applying the action principle to the case of the field variations given in Equation (2.20), we find

$$W(X)\Gamma = -\Delta(X)\Gamma \tag{2.21}$$

where $\Delta(X)$ denotes the dimension four vertex insertion [7]

$$-\Delta(X) = \int dx N_4[\delta_X \mathcal{L}_{\text{eff}} + \hbar Q(X)](x) \tag{2.22}$$

where $\hbar Q(X)$ lumps the radiative corrections together. Note that the first one-particle irreducible diagrams appearing in the expansion of $-\Delta(X)\Gamma$ are the tree diagrams with a single vertex whose functional generator is

$$-\Delta(X) = \int dx \{\delta_X \mathcal{L}_{\text{eff}} + \hbar Q(X)\}(x) = W(X) \int dx \mathcal{L}_{\text{eff}}(x) + O(X, \hbar\mathcal{L}_{\text{eff}}), \tag{2.23}$$

the second term being linear in $X$. Furthermore, because adding a loop to a diagram introduces a factor $\hbar$, we have the functional equation

$$-\Delta(X)\Gamma = -\Delta(X) + O(\hbar\Delta(X)). \tag{2.24}$$

The first step of our analysis will consist in deriving consistency conditions on $\Delta(X)$ which stem from the algebraic properties of $W(X)$ (Equation (2.20)). Iterating Equation (2.21) we get

$$\begin{aligned} [W(X), W(Y)]\,\Gamma \quad &= W([X,Y])\Gamma \\ &= -\Delta([X,Y])\Gamma \\ &= -\left[W(X)\Delta(Y)\Gamma - W(Y)\Delta(X)\Gamma\right], \end{aligned} \tag{2.25}$$

therefrom, using Equation (2.24), we get

$$W(X)\Delta(Y) - W(Y)\Delta(X) = \Delta([X,Y]) + \hbar O(\Delta(X), \Delta(Y), \Delta([X,Y])). \tag{2.26}$$

[7] Here $\delta_X \mathcal{L}_{\text{eff}}$ is considered a functional.

Since $\mathcal{L}_{\rm eff}$ and $Q$ belong to finite-dimensional representation spaces of $G$, $\Delta$ can be reduced into irreducible components.

Equation (2.26) is a perturbed Lie algebra cocycle condition.[8] Having split $\mathcal{G}$ into its Abelian part and its semi-simple part: $\mathcal{G} = \mathcal{A} + \mathcal{S}$ and $\Delta(X)$ into its invariant and non-invariant parts:

$$\Delta(X) = \Delta^\sharp(X) + \Delta^\flat(X),$$

let

$$X = X^\alpha e_\alpha, \quad W(e_\alpha) \equiv \mathcal{T}_\alpha, \quad \Delta(e_\alpha) \equiv B_\alpha \tag{2.27}$$

$e_\alpha$ being a basis in $\mathcal{G}$. Due to its linearity in $X$ and $Y$ Equation (2.26) can be rewritten

$$\mathcal{T}_\alpha B_\beta - \mathcal{T}_\beta B_\alpha - f^\gamma_{\alpha\beta} B_\gamma = \hbar \mathcal{M}_{\alpha\beta}(B), \tag{2.28}$$

Let $\{X, X\}$ be a symmetric, positive definite, invariant form on $\mathcal{G}$ (e.g. $\mathrm{Tr}(W(X)W(X))$) which can be used to raise and lower indices. Let $\{\mathcal{T}, \mathcal{T}\} = \mathcal{T}_\alpha \mathcal{T}^\alpha$, we get from Equation (2.28)

$$\{\mathcal{T}, \mathcal{T}\} B_\beta - \mathcal{T}_\beta \mathcal{T}^\alpha B_\alpha = \hbar \mathcal{T}^\alpha \mathcal{M}_{\alpha\beta}(B) \tag{2.29}$$

where commutation relations have been used together with the antisymmetry of $f_{\alpha\beta\gamma}$ which is due to the invariance of $\{X, X\}$. Positive definiteness of $\{X, X\}$ insures that $\{\mathcal{T}, \mathcal{T}\}$ is strictly positive on the non-invariant ($\flat$) part, so that using again invariance, which insures that

$$[\mathcal{T}_\beta, \{\mathcal{T}, \mathcal{T}\}] = 0,$$

we get

$$B^\flat_\beta = \mathcal{T}_\beta \frac{\mathcal{T}^\alpha}{\{\mathcal{T}, \mathcal{T}\}^\flat} B^\flat_\alpha + \hbar \frac{\mathcal{T}^\alpha}{\{\mathcal{T}, \mathcal{T}\}^\flat} \mathcal{M}^\flat_{\alpha\beta}(B) \tag{2.30}$$

i.e.

$$\Delta^\flat(X) = W(X)\hat{\Delta} + O(\hbar\Delta) \tag{2.31}$$

where $\hat{\Delta}$ is linear in $\Delta^\flat$. Furthermore, for $e_\alpha, e_\beta \in \mathcal{S}$

$$f^\gamma_{\alpha\beta} B^\sharp_\gamma = \hbar \mathcal{M}^\sharp_{\alpha\beta}(B). \tag{2.32}$$

Thus, using the non-degeneracy and invariance of the Killing form for $\mathcal{S}$, we have

$$B^\sharp_\alpha = O(\hbar B), \qquad e_\alpha \in \mathcal{S}. \tag{2.33}$$

Assuming temporarily that also

$$B^\sharp_\alpha = O(\hbar B), \qquad e_\alpha \in \mathcal{A}, \tag{2.34}$$

[8] The following analysis consists in a perturbed version of the construction of the first class Lie algebra cohomology which is discussed in Appendix B.

as we shall demonstrate in a moment, we have

$$\Delta(X) = W(X)\hat{\Delta} + O(\hbar\Delta). \tag{2.35}$$

Now we show that it is possible to choose $\mathcal{L}_{\rm eff}$ and $F$ in such a way that

$$\hat{\Delta} \equiv \mathcal{T}^\alpha B^\flat_\alpha = 0. \tag{2.36}$$

Indeed Equation (2.22) reads

$$B_\alpha = \mathcal{T}_\alpha \int dx\ \mathcal{L}_{\rm eff} + \hbar Q_\alpha$$

and separating in $\mathcal{L}_{\rm eff}$ and $Q(X)$ the invariant and non-invariant parts, Equation (2.36) reads

$$\int dx\ \mathcal{L}^\flat_{\rm eff} + \hbar \frac{\mathcal{T}_\alpha}{\{\mathcal{T},\mathcal{T}\}^\flat} Q^\flat_\alpha = 0, \text{[9]} \tag{2.37}$$

which is soluble for $\int dx\ \mathcal{L}^\flat_{\rm eff}$ in terms of $\int dx\ \mathcal{L}^\sharp_{\rm eff}$, $\boldsymbol{F}$ and $\boldsymbol{b}$.[10]

Once $\mathcal{L}_{\rm eff}$ and $\boldsymbol{F}$ are so adjusted, Equation (2.26) is of the form

$$\Delta = O(\hbar\Delta)$$

whose solution is

$$\Delta = 0.$$

The breaking parameter $\boldsymbol{b}$, which has been so far left arbitrary, will eventually be determined together with $\mathcal{L}^\sharp_{\rm eff}$ in terms of the physical parameters.

Thus, there remains to prove that

$$\Delta^\sharp = O(\hbar\Delta), \tag{2.38}$$

which requires a more detailed analysis than that provided by power counting used up to now.

The idea is to order $\Delta^\sharp$ according to terms of decreasing dimensions and analyze the various terms successively [9][10]. For this purpose, let us consider the linear space spanned by the integrated monomials in the components of $\boldsymbol{\varphi}$, $\boldsymbol{\beta}$ and their derivatives. Denoting altogether these functional variables by $\boldsymbol{\Phi}$, we define

$$\boldsymbol{M}_{I,J,\mu(I\cup J)} \equiv \int dx \prod_{i\in I\cup J} \prod_{\sigma=0}^{3} \partial_\sigma^{\mu_\sigma(i)} \Phi_i(x) \tag{2.39}$$

[9] Which one may check to be independent of the choice of $\{X, X\}$.

[10] Note that renormalizabilty implies that $\mathcal{L}_{\rm eff}$ depends on a finite number of parameters which are formal power series in $\hbar$ and $Q^\flat$ can be written as a formal power series in $\mathcal{L}_{\rm eff}$ and $\hbar$.

where $I$ and $J$ denote sets of, possibly repeated, components of $\boldsymbol{\varphi}$ and $\boldsymbol{\beta}$ respectively and $\mu_\sigma(i)$ is a 4-vector valued function on the union of these sets whose components are integers identifying the degree of the $x^\sigma$-derivative on the $i$-th element.

It is clear that, on the one hand, different functions $\mu$ and $\nu$ must be identified if they coincide after permutations of elements of $I$ and $J$ corresponding to the same component of the fields and, on the other hand, that linear combinations of $\boldsymbol{M}$'s are trivial if the corresponding linear combinations of the monomials appearing in Equation (2.39) are equal to a total derivative. For this reason we fix a unique basis of the space spanned by the monomials by ordering in a given sequence the components of $\boldsymbol{\Phi}$ and we identify one element in the equivalence class up to a total derivative choosing the monomial in which $\Phi_{i_M}$, the last component of $\Phi$ belonging to $I \cup J$, appears at least once without derivatives.

The set of integrated monomials $\boldsymbol{M}$ with canonical dimension bounded by $d$ (we shall consider in particular the case $d = 4$) span a finite dimensional linear space in much the same way as polynomials of bounded degree are elements of a finite dimensional linear space. The *dual space* of the space of polynomials is spanned by multiple derivatives at the origin. In our case we introduce the dual functional differential operators $\boldsymbol{X}$ defined by

$$\boldsymbol{X}_{I,J,\mu(I\cup J)}(\boldsymbol{q}) \equiv \frac{\delta}{\delta\Phi_{i_M}(0)} \prod_{i\in I\cup J,\ i\neq i_M} \prod_{\sigma=0}^{3} \partial_\sigma^{\mu_\sigma(i)} \frac{\delta}{\delta\tilde{\Phi}_i(q_i)}, \tag{2.40}$$

where $\tilde{\Phi}$ denotes the Fourier transformed field. It is easy to see that one has the following orthogonality property

$$\boldsymbol{X}_{I,J,\mu(I\cup J)}(\boldsymbol{0})\boldsymbol{M}_{I',J',\mu'(I'\cup J')}|_{\boldsymbol{\Phi}=0} = N_{I,J,\mu(I\cup J)}\delta_{I,I'}\delta_{J,J'}\delta_{\mu,\mu'}, \tag{2.41}$$

indeed, in particular, the right-hand side of Equation (2.41) vanishes unless $I = I', J = J'$ and hence $i_M = i'_M$. $N$ is a non vanishing normalization factor. Furthermore,

$$\boldsymbol{X}_{I,J,\mu(I\cup J)}(\boldsymbol{q})\boldsymbol{M}_{I',J',\mu'(I'\cup J')}|_{\boldsymbol{\Phi}=0} = \boldsymbol{X}_{I,J,\mu(I\cup J)}(\boldsymbol{0})\boldsymbol{M}_{I',J',\mu'(I'\cup J')}|_{\boldsymbol{\Phi}=0} \tag{2.42}$$

provided that the canonical dimensions

$$\dim\boldsymbol{M}_{I,J,\mu(I\cup J)} \geq \dim\boldsymbol{M}_{I',J',\mu'(I'\cup J')}. \tag{2.43}$$

Let $\mathcal{G}$ act on $\boldsymbol{X}$ according to:

$$\boldsymbol{X}_{I,J,\mu(I\cup J)}(\boldsymbol{q})\mathcal{T}_\alpha^H\bar{\Gamma}|_{\boldsymbol{\Phi}=0} \equiv \left(\mathcal{T}_\alpha^H\boldsymbol{X}_{I,J,\mu(I\cup J)}(\boldsymbol{q})\right)\bar{\Gamma}|_{\boldsymbol{\Phi}=0} \tag{2.44}$$

for any integrated local functional $\bar{\Gamma}$. We have set

$$\mathcal{T}_\alpha^H \equiv \int dx \Big\{\boldsymbol{\varphi}\mathring{t}_\alpha^T\frac{\delta}{\delta\boldsymbol{\varphi}} + \boldsymbol{\beta}\mathring{\theta}_\alpha^T\frac{\delta}{\delta\boldsymbol{\beta}}\Big\}, \tag{2.45}$$

and $\mathcal{T}_\alpha^H$ is the homogeneous part of $\mathcal{T}_\alpha$ obtained by putting $\boldsymbol{\beta} = \boldsymbol{F} = 0$ in Equation (2.20). Let

$$\boldsymbol{X}^\sharp(\boldsymbol{q}) = \sum_{I,J,\mu(I\cup J)} C^\sharp_{I,J,\mu(I\cup J)} \boldsymbol{X}_{I,J,\mu(I\cup J)} \tag{2.46}$$

be an element of a basis of $\mathcal{T}_\alpha^H$-invariant test operators corresponding to dimension four local polynomial functionals, one has

$$\begin{aligned}
\boldsymbol{X}^\sharp(\boldsymbol{q})\mathcal{T}_\alpha\Gamma|_{\boldsymbol{\Phi}=0} &= -\boldsymbol{X}^\sharp(\boldsymbol{q})B_\alpha\Gamma|_{\boldsymbol{\Phi}=0} \\
&= -\boldsymbol{X}^\sharp(\boldsymbol{q})\int dx\Big\{\boldsymbol{F}\mathring{t}_\alpha^T\frac{\delta}{\delta\boldsymbol{\varphi}} + \boldsymbol{b}\mathring{\theta}_\alpha^T\frac{\delta}{\delta\boldsymbol{\beta}}\Big\}\Gamma|_{\boldsymbol{\Phi}=0} \\
&= -\boldsymbol{X}^\sharp(\boldsymbol{q})B_\alpha|_{\boldsymbol{\Phi}=0} + O(\hbar B) \\
&= -\boldsymbol{X}^\sharp(0)B_\alpha|_{\boldsymbol{\Phi}=0} + O(\hbar B).
\end{aligned} \tag{2.47}$$

The first line is a consequence of the anomalous Ward identity, the second one makes use of the $\mathcal{T}_\alpha^H$-invariance of $\boldsymbol{X}^\sharp(\boldsymbol{q})$, the last one follows from Equations (2.24, 2.42, 2.43). Now for $\boldsymbol{q}$ large in the Euclidean region power counting insures that the expression

$$\boldsymbol{X}^\sharp(\boldsymbol{q})\int dx\Big\{\boldsymbol{F}\mathring{t}_\alpha^T\frac{\delta}{\delta\boldsymbol{\varphi}} + \boldsymbol{b}\mathring{\theta}_\alpha^T\frac{\delta}{\delta\boldsymbol{\beta}}\Big\}\Gamma|_{\boldsymbol{\Phi}=0}$$

is asymptotically negligible, because $\dim\boldsymbol{\varphi} > 0$ and $\dim\boldsymbol{\beta} > 0$, and hence, this expression is a linear combination of multiple derivatives of one-particle irreducible Feynman amplitudes with global dimension, including the field and momentum derivatives, smaller than minus four. It must vanish for large, linearly independent, $\boldsymbol{q}$'s in the Euclidean region. Thus

$$\boldsymbol{X}^\sharp_{I,J,\mu(I\cup J)}(0)B_\alpha = O(\hbar B), \quad \forall(I,J,\mu(I\cup J)), \tag{2.48}$$

and hence the dimension four part of $B_\alpha^\sharp$ is $O(\hbar B)$. The analysis of the lower dimension terms of $B_\alpha^\sharp$ is slightly more sophisticated and is given in Appendix C.

This analysis completes the proof of Equation (2.38)

At this point, we have completed the construction of an effective action fulfilling the Ward identity (2.19). The free power series parameters $C_\sharp$, and $\boldsymbol{b}$ can then be used to fulfill the normalization conditions which allow a particle interpretation of the theory under the assumptions stated in section (2.1), namely the existence in the tree approximation of an invertible transformation from $\mathring{C}_\sharp$ and $\mathring{\boldsymbol{b}}$ to the physical parameters involved in the normalization conditions.

# 3 The local Ward identity (*current algebra*)

## 3.1 The Classical Theory [14]

Given a Lagrangian $\tilde{\mathcal{L}}(\boldsymbol{\varphi}, \boldsymbol{\beta})$ invariant under the global transformation Equation (2.20), it is easy to introduce an external gauge field $\boldsymbol{a}_\mu$ of dimension 1, and construct a Lagrangian $\mathcal{L}(\boldsymbol{\varphi}, \boldsymbol{\beta}, \boldsymbol{a}_\mu)$ invariant under the local gauge transformation:

$$\begin{aligned}
\delta_{\boldsymbol{\omega}}\boldsymbol{\varphi}(x) &= -\mathring{t}(\boldsymbol{\omega}(x))(\boldsymbol{\varphi}(x) + \boldsymbol{F}), \\
\delta_{\boldsymbol{\omega}}\boldsymbol{\beta}(x) &= -\mathring{\theta}(\boldsymbol{\omega}(x))(\boldsymbol{\beta}(x) + \boldsymbol{b}), \\
\delta_{\boldsymbol{\omega}}\boldsymbol{a}_\mu(x) &= \partial_\mu\boldsymbol{\omega}(x) - [\boldsymbol{\omega}(x), \boldsymbol{a}_\mu(x)],
\end{aligned} \tag{3.49}$$

where we have considered $\boldsymbol{a}_\mu(x)$ as well as $\boldsymbol{\omega}(x)$ as elements of $\mathcal{G}$: it is enough to replace the derivatives occurring in $\mathcal{L}(\boldsymbol{\varphi}, \boldsymbol{\beta})$ by covariant derivatives:

$$\begin{aligned}
\partial_\mu\boldsymbol{\varphi} \to D_\mu\boldsymbol{\varphi} &= \partial_\mu\boldsymbol{\varphi} + \mathring{t}(\boldsymbol{a}_\mu)(\boldsymbol{\varphi} + \mathring{\boldsymbol{F}}), \\
\partial_\mu\boldsymbol{\beta} \to \Delta_\mu\boldsymbol{\beta} &= \partial_\mu\boldsymbol{\beta} + \mathring{\theta}(\boldsymbol{a}_\mu)(\boldsymbol{\beta} + \mathring{\boldsymbol{b}}),
\end{aligned} \tag{3.50}$$

and to include gauge invariant terms constructed with $\boldsymbol{a}_\mu$, through the antisymmetric covariant tensor

$$\boldsymbol{G}_{\mu\nu} = \partial_\mu\boldsymbol{a}_\nu - \partial_\nu\boldsymbol{a}_\mu - [\boldsymbol{a}_\mu, \boldsymbol{a}_\nu].$$

The local Ward identity which expresses the invariance of $\mathcal{L}(\boldsymbol{\varphi}, \boldsymbol{\beta}, \boldsymbol{a}_\mu)$ under the local gauge transformation Equation (3.49) is:

$$\begin{aligned}
\mathcal{W}(\boldsymbol{\omega})\mathring{\Gamma}(\boldsymbol{\varphi}, \boldsymbol{\beta}, \boldsymbol{a}_\mu) &= \int dx \left[ \frac{\delta\mathring{\Gamma}}{\delta\boldsymbol{a}_\mu}(\partial_\mu\boldsymbol{\omega}(x) - [\boldsymbol{\omega}(x), \boldsymbol{a}_\mu(x)]) \right. \\
&\quad - \frac{\delta\mathring{\Gamma}}{\delta\boldsymbol{\varphi}}\mathring{t}(\boldsymbol{\omega})(\boldsymbol{\varphi} + \mathring{\boldsymbol{F}}) \\
&\quad \left. - \frac{\delta\mathring{\Gamma}}{\delta\boldsymbol{\beta}}\mathring{\theta}(\boldsymbol{\omega})(\boldsymbol{\beta} + \mathring{\boldsymbol{b}}) \right] = 0, \qquad \boldsymbol{\omega}(x) \in \mathcal{G}.
\end{aligned} \tag{3.51}$$

The relationship between the integrated Ward identity for $\mathcal{G}$ and the local Ward identity for the associated gauge group is:

$$W(\boldsymbol{\omega}) = \mathcal{W}(\boldsymbol{\omega}) \quad \text{for } \boldsymbol{\omega} \text{ space}-\text{time independent.}$$

Note that the introduction of the external gauge field $\boldsymbol{a}_\mu$, which globally transforms under the adjoint representation of $\mathcal{G}$ whose generators are denoted by $\{f_\alpha\}$, does not spoil the conclusions of the previous section because nowhere Lorentz covariance of the fields was used.

## 3.2 Radiative Corrections

Defining

$$\mathcal{W}_\alpha(x) \equiv \frac{\delta \mathcal{W}}{\delta \omega^\alpha(x)},$$

we are going to give a general proof of an anomalous local Ward identity:

$$\mathcal{W}_\alpha(x)\Gamma = G_\alpha(x)$$

where $G_\alpha$ is a dimension four polynomial in the classical gauge field $\boldsymbol{a}_\mu$ and its derivatives. Taking into account the remark at the end of section 2 we have already proved the integrated Ward identity in the presence of the gauge field:

$$\int dx \Big\{ \boldsymbol{a}_\mu f^T_\alpha \frac{\delta}{\delta \boldsymbol{a}_\mu} + (\boldsymbol{\varphi} + \boldsymbol{F}) \mathring{t}^T_\alpha \frac{\delta}{\delta \boldsymbol{\varphi}} + (\boldsymbol{\beta} + \boldsymbol{b}) \mathring{\theta}^T_\alpha \frac{\delta}{\delta \boldsymbol{\beta}} \Big\}(x)\Gamma = 0.$$

On the contrary, performing a local gauge transformation yields

$$\mathcal{W}_\alpha(x)\Gamma = \mathcal{K}_\alpha(x)\Gamma$$

where $\mathcal{K}_\alpha(x)$ is a dimension four local insertion. It follows from the validity of the integrated Ward identity that

$$\int dx\ \mathcal{K}_\alpha(x) = 0,$$

hence

$$\mathcal{K}_\alpha(x) = \partial_\mu \mathcal{K}^\mu_\alpha(x)$$

where $\mathcal{K}^\mu_\alpha$ a dimension three local operator. Now the quantum action principle implies that $\mathcal{K}_\alpha$ fulfills the perturbed compatibility condition [11]

$$\frac{\delta \mathcal{K}_\beta(y)}{\delta \omega^\alpha(x)} - \frac{\delta \mathcal{K}_\alpha(x)}{\delta \omega^\beta(y)} - f^\gamma_{\alpha\beta}\delta(x-y)\mathcal{K}_\gamma(x) = O(\hbar \mathcal{K}) \tag{3.52}$$

where $O(\hbar\mathcal{K})$ lumps the radiative corrections together.

In Appendix D it is shown that the solution to the unperturbed compatibility condition

$$\frac{\delta \hat{\mathcal{K}}_\beta(y)}{\delta \omega^\alpha(x)} - \frac{\delta \hat{\mathcal{K}}_\alpha(x)}{\delta \omega^\beta(y)} - f^\gamma_{\alpha\beta}\delta(x-y)\hat{\mathcal{K}}_\gamma(x) = 0 \tag{3.53}$$

is of the form

$$\hat{\mathcal{K}}_\alpha = \frac{\delta}{\delta \omega^\alpha(x)} \int dy\ K(y) + G_\alpha(x) \tag{3.54}$$

where $G_\alpha(x)$ does not depend on the quantized fields and is not the gauge variation of any local functional of dimension less than or equal to four. $K(x)$ is a local dimension four functional. Furthermore the insertion of the vertex $G_\alpha(x)$ into the effective action is additive and does not contribute any radiative correction[11]

$$G_\alpha(x)\Gamma = G_\alpha(x). \tag{3.55}$$

Therefore, the solution of Equation (3.52) is provided by

$$\mathcal{K}_\alpha(x) = \hat{\mathcal{K}}_\alpha(x) + O(\hbar(\mathcal{K} - G_\alpha)). \tag{3.56}$$

Furthermore, since $\mathcal{K}_\alpha$ is a divergence, so is $\hat{\mathcal{K}}_\alpha$. Now we recall that, according to the quantum action principle,

$$\mathcal{K}_\alpha(x) = \frac{\delta}{\delta\omega_\alpha(x)} \int dy \mathcal{L}_{\text{eff}}(y) + \mathcal{Q}_\alpha(x) \tag{3.57}$$

with $\mathcal{Q}_\alpha = O(\hbar\mathcal{L}_{\text{eff}})$. From this equation, considering Equations (3.54), (3.56) and (3.57), we have

$$\begin{aligned} \mathcal{K}_\alpha(x) \quad &= \frac{\delta}{\delta\omega_\alpha(x)} \int dy\ \mathcal{L}_{\text{eff}}(y) + \mathcal{Q}_\alpha(x) \\ &= \hat{\mathcal{K}}_\alpha(x) + O(\hbar(\mathcal{K} - G)) \\ &= \frac{\delta}{\delta\omega_\alpha(x)} \int dy\ K(y) + G_\alpha(x) + O(\hbar(\mathcal{K} - G_\alpha)). \end{aligned} \tag{3.58}$$

From which we have

$$\begin{aligned} \mathcal{Q}_\alpha(x) - G_\alpha(x) \quad &= \frac{\delta}{\delta\omega_\alpha(x)} \int dy\, [K(y) - \mathcal{L}_{\text{eff}}(y)] + O(\hbar(\mathcal{K} - G_\alpha)) \\ &\equiv \frac{\delta}{\delta\omega_\alpha(x)} \int dy\ N(y) + O(\hbar(\mathcal{K} - G)), \end{aligned} \tag{3.59}$$

where $N(y)$ is a term generated by radiative corrections and hence is $O(\hbar\mathcal{L}_{\text{eff}}(y))$. It follows that the equation

$$\frac{\delta}{\delta\omega_\alpha(x)} \int dy\, [N(y) + \mathcal{L}_{\text{eff}}(y)] = \frac{\delta}{\delta\omega_\alpha(x)} \int dy K(y) = 0$$

can be solved in terms of the parameters in $\mathcal{L}_{\text{eff}}$ and hence the system (3.58) reduces to

$$\mathcal{K}_\alpha(x) - G_\alpha(x) = O(\hbar(\mathcal{K} - G))$$

[11] That is: to the right-hand side of Equation (3.52). Concerning the radiative corrections to $G$ see also [13][14] .

whose unique solution is

$$\mathcal{K}_\alpha(x)\Gamma = G_\alpha(x)\Gamma.$$

At this point, taking into account Equation (3.55), we have

$$\mathcal{K}_\alpha(x)\Gamma = G_\alpha(x)\Gamma = G_\alpha(x),$$

namely we have proved the anomalous Ward identity

$$\mathcal{W}_\alpha(x)\Gamma = G_\alpha(x).$$

As shown in Appendix D, $G_\alpha(x)$ can always be chosen in the form:

$$G_\alpha(x) = \partial_\mu K^\mu_\alpha(x) \tag{3.60}$$

with

$$K^\mu_\alpha(x) = \epsilon^{\mu\nu\rho\sigma}\left[D_{\alpha\beta\gamma}(\partial_\nu a^\beta_\rho)a^\sigma_\gamma + F_{\alpha\beta\gamma\delta}a^\beta_\nu a^\gamma_\rho a^\delta_\sigma\right] \tag{3.61}$$

and

$$F_{\alpha\beta\gamma\delta} = \frac{1}{12}\left[D_{\alpha\beta\eta}f^\eta_{\gamma\delta} + D_{\alpha\delta\eta}f^\eta_{\beta\gamma} + D_{\alpha\gamma\eta}f^\eta_{\delta\beta}\right]. \tag{3.62}$$

$D_{\alpha\beta\gamma}$ is a symmetric invariant rank three tensor on $\mathcal{G}$, it parametrizes the general form of the Adler-Bardeen anomaly.

# 4 Conclusion

We have completed a number of points of Symanzik's program on the renormalization of theories with symmetry breaking.

For models without massless particles, we have been able to deal with an arbitrary compact internal symmetry Lie group, and prove the integrated Ward identities characteristic of a super-renormalizable breaking with given covariance. The corresponding anomalous local Ward identity - the functional expression of current algebra - is then proved in full generality and a compact formula exhibited for the corresponding Adler-Bardeen anomaly. Our perturbative treatment fails if power counting mixes with geometry to produce e.g. mass rules, since in this case a particle interpretation of the theory is no longer possible. The breakdown of our treatment generated by this phenomenon is quite more dramatic in models involving massless particles. This happens in particular if, due to tree approximation mass rules there are more massless scalar fields than Goldstone bosons (pseudo-Goldstone bosons [15]). In this case, even the construction of a Green function theory needs a deep modification of the perturbative scheme [16]. Another limiting case which is worth mentioning occurs when the breaking has dimension four and, given the direction $\boldsymbol{b}$ which characterizes the breaking, the most general invariant Lagrangian formed with the quantized field is not the most general Lagrangian invariant under the residual symmetry group $H_{\boldsymbol{b}}$.

## 5 Acknowledgements

The present paper contains a revision of a work published 35 years ago[1] whose subject was inspired by K. Symanzik, in particular, through its exchange of correspondence with R. Stora. For this reason this paper is dedicated to the memory of both R. Stora and K. Symanzik. The author is indebted to his friends A. Blasi, C. Imbimbo, S. Lazzarini and N. Magnoli for careful readings of different versions of the manuscript.


## A $\mathring{\boldsymbol{F}}$ is a covariant function of $\mathring{\boldsymbol{b}}$

A classical action is viewed as an integrate local functional whose argument is indefinitely differentiable with fast decrease. In the present case

$$\mathring{\Gamma}(\boldsymbol{\varphi},\boldsymbol{\beta}) = \int dx\mathcal{L}(\boldsymbol{\varphi}+\mathring{\boldsymbol{F}},\boldsymbol{\beta}+\mathring{\boldsymbol{b}})(x)$$

where $\mathcal{L}$ is a classical Lagrangian density without a constant term, i. e. $\mathcal{L}(\mathring{\boldsymbol{F}},\mathring{\boldsymbol{b}}) = 0$, defined up to a divergence. We shall limit ourselves to renormalizable Lagrangians, according to the conventional power counting theory through which fields $\boldsymbol{\varphi}$ are assigned dimensions connected with the structure of the quadratic part of $\mathcal{L}$, the dimension of $\boldsymbol{\beta}$ being a priori given, namely Lagrangians of positive dimension smaller than or equal to four. If $\dim\boldsymbol{\varphi} > 0$, renormalizable Lagrangians are polynomials. Assuming that $\mathcal{L}$ has no term linear in $\boldsymbol{\varphi}$, we see that the integrated Ward identity, Equation (2.7), is only meaningful if

$$\frac{\delta\mathring{\Gamma}}{\delta\boldsymbol{\beta}}|_{\boldsymbol{\varphi}=\boldsymbol{\beta}=\mathbf{0}}\mathring{\theta}(X)\mathring{\boldsymbol{b}} = 0 \tag{A.1}$$

which we shall assume. The field translation parameter appropriate to get rid of the term linear in $\boldsymbol{\varphi}$ from a Lagrangian which is an invariant formed with $\boldsymbol{\varphi}+\mathring{\boldsymbol{F}}$ and $\boldsymbol{\beta}+\mathring{\boldsymbol{b}}$, of course, depends on $\mathring{\boldsymbol{b}}$ . For constant $\boldsymbol{\varphi}$ and vanishing $\boldsymbol{\beta}$, $\mathcal{L}(\boldsymbol{\varphi}+\mathring{\boldsymbol{F}},\mathring{\boldsymbol{b}})$ is an invariant polynomial which we denote by $\mathcal{F}$. Hence $\mathring{\boldsymbol{F}}$ is implicitly defined by

$$\frac{\partial\mathcal{F}}{\partial\boldsymbol{\varphi}}(\mathring{\boldsymbol{F}},\mathring{\boldsymbol{b}}) = 0 \tag{A.2}$$

and the Ward identity implies

$$\frac{\partial\mathcal{F}}{\partial\boldsymbol{\varphi}}\mathring{t}(X)(\boldsymbol{\varphi}+\mathring{\boldsymbol{F}}) + \frac{\partial\mathcal{F}}{\partial\mathring{\boldsymbol{b}}}\mathring{\theta}(X)\mathring{\boldsymbol{b}} = 0. \tag{A.3}$$

Differentiating Equation (A.2) with respect to $\mathring{\boldsymbol{b}}$ and Equation (A.3) with respect to $\boldsymbol{\varphi}$ at $\boldsymbol{\varphi} = \mathring{\boldsymbol{F}}$ yields

$$\frac{\partial^2\mathcal{F}}{(\partial\boldsymbol{\varphi})^2}|_{\boldsymbol{\varphi}=\mathring{\boldsymbol{F}}}\left[\mathring{t}(X)\mathring{\boldsymbol{F}} - \frac{\partial\mathring{\boldsymbol{F}}}{\partial\mathring{\boldsymbol{b}}}\mathring{\theta}(X)\mathring{\boldsymbol{b}}\right] = 0$$

which, under the assumption that the mass matrix $\frac{\partial^2 \mathcal{F}}{(\partial \boldsymbol{\varphi})^2}|_{\boldsymbol{\varphi}=\mathring{\boldsymbol{F}}}$ be non-degenerate, implies that $\mathring{\boldsymbol{F}}$ is a covariant function of $\mathring{\boldsymbol{b}}$ :

$$\mathring{t}(X)\boldsymbol{F} = \frac{\partial \boldsymbol{F}}{\partial \boldsymbol{b}} \mathring{\theta}(X)\boldsymbol{b}.$$

Similarly the other coefficients of $\mathcal{L}$ are covariant functions of $\mathring{\boldsymbol{b}}$.

# B Canonical form of the Ward Identity

This appendix is devoted to the reduction of the Ward identity to canonical form.

## B.1 The tree approximation

We have assumed that $\mathcal{G} = \mathcal{S} + \mathcal{A}$ be a compact Lie algebra and hence that $\{X, X\}$ be a symmetric, positive definite, invariant form on $\mathcal{G}$ which can be used to raise and lower indices. We first show that the representations

$$X \to \mathring{t}(X), \quad X \to \mathring{\theta}(X), \quad X \in \mathcal{G}, \tag{B.1}$$

are fully reducible. This is automatic for $X \in \mathcal{S}$, the semi-simple part of G . For $X \in \mathcal{A}$, the Abelian part of G , this is a consequence of the assumption that the kinetic part of $\mathcal{L}$ be Hermitian non-degenerate, which insures that $X \to \mathring{t}(X)$ is fully reducible. Then, the Lie algebra cocycle condition Equation (2.13 c) can be solved as follows. Reducing Equation (2.13 c) to components (see Equation (2.27)):

$$\mathring{t}_\alpha \mathring{\boldsymbol{F}}_\beta - \mathring{t}_\beta \mathring{\boldsymbol{F}}_\alpha - f^\gamma_{\alpha\beta} \mathring{\boldsymbol{F}}_\gamma = 0$$

yields

$$\{\mathring{t}, \mathring{t}\}\mathring{\boldsymbol{F}}_\beta \equiv (\mathring{t}_\alpha \mathring{t}^\alpha)\mathring{\boldsymbol{F}}_\beta = \mathring{t}_\beta(\mathring{t}^\alpha \mathring{\boldsymbol{F}}_\alpha).$$

Thus restricting $\mathring{\boldsymbol{F}}_\alpha$ to its non-invariant part $\mathring{\boldsymbol{F}}^\flat_\alpha$we have:

$$\mathring{\boldsymbol{F}}^\flat_\alpha = \frac{1}{\{\mathring{t}, \mathring{t}\}} \mathring{t}_\alpha(\mathring{t}^\beta \mathring{\boldsymbol{F}}_\beta) \equiv \mathring{t}_\alpha \mathring{\boldsymbol{F}}^\flat.$$

Similarly, using the non-degeneracy of the Killing form of $\mathcal{S}$, we also get

$$\mathring{\boldsymbol{F}}^\sharp_\alpha = 0, \qquad e_\alpha \in \mathcal{S}$$

so that only $\mathring{\boldsymbol{F}}^\sharp_\alpha$ for $e_\alpha \in \mathcal{A}$ is left undetermined.

Thus one has to find a polynomial Lagrangian $\mathcal{L}$ invariant under

$$\delta_X \boldsymbol{\varphi}^\flat = -\mathring{t}(X)(\boldsymbol{\varphi}^\flat + \mathring{\boldsymbol{F}}^\flat), \quad \delta_X \boldsymbol{\varphi}^\sharp = \mathring{\boldsymbol{F}}^\sharp(X), \quad \delta_X \boldsymbol{\beta}^\flat = -\mathring{\theta}(X)(\boldsymbol{\beta}^\flat + \mathring{\boldsymbol{b}}^\flat).$$

According to the mathematical meaning of $\boldsymbol{\beta}$, which characterizes the symmetry breaking, any $\boldsymbol{\beta}^\sharp$ component is excluded. Thus the last of the above equations is proved in the same way as the first one.

It is easy to see that, due to the polynomial character of $\mathcal{L}$, those components of $\boldsymbol{\varphi}^\sharp$ for which $\mathring{\boldsymbol{F}}^\sharp(X) \neq 0$ *do not couple.*

## B.2 Radiative Corrections

We shall first show that the representation property Equation (2.13a) [12] with

$$t(X) = \mathring{t}(X) + O(\hbar)$$

where $t(X)$ is a formal power series in $\hbar$, implies that

$$t(X) = Z^{-1}\mathring{t}(X)Z$$

for some formal power series $Z$:

$$Z = \mathbb{1} + O(\hbar).$$

Let first $X \in \mathcal{S}$ the semi-simple part of $\mathcal{G}$, let

$$t(X) = \sum_{n=0}^{\infty} t_n(X)$$

$$Z = \sum_{n=0}^{\infty} Z_n$$

be the formal power series for $t(X)$ and $Z$, respectively. We have chosen

$$t_0(X) \equiv \mathring{t}(X), \quad Z_0 = \mathbb{1}$$

thanks to a symmetric wave function renormalization. The possibly non-trivial first order term in the expansion of Equation (2.13a) reads

$$[\mathring{t}(X), t_1(Y)] - [\mathring{t}(Y), t_1(X)] - t_1([X,Y]) = 0$$

[12] We replace into Equation (2.13a) $\mathring{T}(X)$ by $t(X)$ because we are now considering radiative corrections and hence formal power series in $\hbar$.

which is a Lie algebra cocycle condition strictly analogous to Equation (2.13c)[13] and can be solved in the same way; hence, due to the semi-simplicity of $\mathcal{G}$

$$t_1(X) = [\mathring{t}(X), Z_1]$$

for some $Z_1$.

Let us now assume that

$$t(X) = (Z^{(n-1)})^{-1}\mathring{t}(X)Z^{(n-1)} + \tau_n(X)$$

with $\tau_n(X) = O(\hbar^n)$ and $Z^{(n-1)} = \sum_0^{n-1} Z_k$ which is true for $n = 2$ with

$$Z^{(1)} = \mathbb{1} + Z_1.$$

The term in Equation (2.13a) at the lowest non-vanishing order reads:

$$\begin{aligned}&\left[\mathring{t}(X), \{Z^{(n-1)}\tau_n(Y)(Z^{(n-1)})^{-1}\}_n\right]\\ &- \left[\mathring{t}(Y), \{Z^{(n-1)}\tau_n(X)(Z^{(n-1)})^{-1}\}_n\right]\\ &-\{Z^{(n-1)}\tau_n([X,Y])(Z^{(n-1)})^{-1}\}_n = 0,\end{aligned} \tag{B.2}$$

where, given a $\hbar$ formal power series $X$, $\{X\}_n$ denotes the term of order $n$. This is a further cocycle condition whose solution is

$$\{Z^{(n-1)}\tau_n(X)(Z^{(n-1)})^{-1}\}_n = [\mathring{t}(X), Z_n]$$

for some $Z_n$, so that

$$\tau_n(X) = (Z^{(n-1)})^{-1}[\mathring{t}(X), Z_n]Z^{(n-1)} + O(\hbar^{n+1}) = [\mathring{t}(X), Z_n] + O'(\hbar^{n+1})$$

thus

$$t_n(X) = \{(Z^{(n)})^{-1}\mathring{t}(X)Z^{(n)}\}_n$$

with

$$Z^{(n)} = Z^{(n-1)} + Z_n.$$

As a conclusion, we may choose $t(X) = \mathring{t}(X)$ for $X \in \mathcal{S}$ up to a field renormalization identified by $Z$.

Now, for $X \in \mathcal{A}$, the Abelian part of $\mathcal{G}$, any anomaly in the Ward identity can be considered as a breaking of the canonical Ward identity through a term which, up to $O(\Delta^2)$, is Abelian invariant, and thus, cannot occur as a consequence of the argument at the end of section 2 for the anomaly in $t(X)$, and of the argument in next Appendix C for the anomaly in $\boldsymbol{F}(X)$.

[13]It just refers to a different representation, the adjoint, of $\mathcal{G}$.

# C Elimination of Soft Invariant Anomalies from the Integrated Ward Identity

Once the dimension four anomalies have been eliminated as indicated in the text, one might remain with a Ward identity of the form:

$$W(X)\Gamma = \sum_{\delta=1,2,3} \Delta^{\sharp}_{\delta}(X)\Gamma \quad \text{for} \quad X \in \mathcal{A}$$

where the breaking insertions $\Delta^{\sharp}_{\delta}$ are invariant and have power counting dimension (Zimmermann's index) $\delta$. Let now $\lambda$ be any parameter of the theory (every parameter identifies an independent term of $\mathcal{L}$), and let

$$D_\lambda = \partial_\lambda - \int dx\; \partial_\lambda \mathcal{B} \frac{\delta}{\delta\Phi(x)} \tag{C.1}$$

where

$$\mathcal{B} = (\boldsymbol{F}, \boldsymbol{b}), \quad \Phi = (\boldsymbol{\varphi}, \boldsymbol{\beta}).$$

Then

$$[D_\lambda, W(X)] = 0.$$

In particular, let $m\partial_m$ be the operator which scales all the parameters of the theory according to their mass dimensions (the first term in the Callan-Symanzik equation) and $D_m$ the associated invariant operator [10] (as in Equation (C.1)). In the tree approximation $D_m\mathcal{L}$ is invariant and soft.

A differential scaling equation is written introducing into $\mathcal{L}$ an invariant external fields $\eta$, with dimension $d = 1$, coupled to soft invariant terms constrained by the condition for the classical action $\mathring{\Gamma}$

$$D_m \mathring{\Gamma}(\eta) = \int dx\; m \frac{\delta \mathring{\Gamma}(\eta)}{\delta \eta(x)},$$

where $m$ defines a reference mass scale. If this equation is satisfied $\mathcal{L}$ is a linear combination of dimension four independent invariant local polynomials in $\Phi$ and in $\eta - m$. The coefficients of this linear combination, that we label by $\xi$, are dimensionless and are constrained by a sum rule which follows from the already stated condition that $\mathcal{L}$ must vanish when all the quantized and external fields vanish.[14]

[14]It is important to note here that this condition, holding true in the tree approximation, remains fulfilled also by the loop corrections since the subtraction prescription does not contribute any constant term.

After the introduction of $\eta$, repeating the analysis shown in the text, we see that the Ward identity becomes

$$W(X)\Gamma(\eta) = \sum_{\delta=1,2,3} \Delta^{\sharp}_{\delta}(X,\eta)\Gamma(\eta) \tag{C.2}$$

where $\Delta^{\sharp}_{\delta}(X,\eta)$ are the new, soft, $\eta$-dependent, invariant breaking insertions and $\Gamma(\eta)$ the new $\eta$-dependent effective action functional.

Considering the scaling equation beyond the tree approximation, we deduce from the quantum action principle[15]

$$\left[D_m - \int dx\ m\frac{\delta}{\delta\eta(x)}\right]\Gamma(\eta) = \hbar\left[M_4^{m\sharp}(\eta)\Gamma(\eta) + M_4^{m\flat}(\eta)\Gamma(\eta)\right] \tag{C.3}$$

where $M_4^{m\sharp/\flat}\Gamma(\eta) = \int dx N_4[M^{m\sharp/\flat}(x)]\Gamma(\eta)$ correspond to the insertion into $\Gamma$ of a linear combination of, invariant/non-invariant, integrated local vertices among which there are some which are $\eta$ dependent.

Furthermore we have

$$D_\xi\Gamma(\eta) = M_4^{\xi\sharp}(\eta)\Gamma(\eta) + \hbar M_4^{\xi\flat}(\eta)\Gamma(\eta), \tag{C.4}$$

where the non-invariant ($\flat$) operators appear because the Ward identity is broken.

The mentioned operator set (i.e. that spanned by the linear combinations of the $M_4^{\xi\sharp}(\eta)$'s) being complete, there must be a linear relation among $M_4^{m\sharp}$ and the $M_4^{\xi\sharp}$'s. Thus Equation (C.3) reads

$$\left[D_m - \int dx\ m\frac{\delta}{\delta\eta(x)} + \hbar\sum c_\xi D_\xi\right]\Gamma(\eta) = \hbar\tilde{M}_4^{m\flat}(\eta)\Gamma(\eta), \tag{C.5}$$

where $\tilde{M}_4^{m\flat}$ lumps the non invariant insertions appearing in the right-hand side of Equations (C.3) and (C.4) together.

It is obvious that, if the Ward identity were unbroken, the right-hand side of Equation (C.5) would vanish because it is not invariant, while the left-hand side is. In that case Equation (C.5) coincides with the Callan-Symanzik equation of the theory ([10]). If, on the contrary, we have the broken Equation (C.2), combining this

[15] Which in the present case corresponds to the Zimmermann identities giving the expansion of local operators with a weaker subtraction prescription in terms of local operators with stronger subtraction prescriptions, such as those coupled to $\eta - m$. This difference vanishes in the tree approximation because there is no diagram to subtract.

equation with Equation (C.5) we get

$$\begin{aligned}
&W(X)\left[D_m - \int dx\ m\frac{\delta}{\delta\eta(x)} + \hbar\sum c_\xi D_\xi\right]\Gamma(\eta) \\
&= \left[D_m - \int dx\ m\frac{\delta}{\delta\eta(x)}\right]\sum_{\delta=1,2,3}\Delta^\sharp_\delta(X,\eta) + O(\hbar\Delta^\sharp) \\
&= -\hbar T(X)\tilde{M}_4^{m\flat}(\eta) + O(\hbar^2\tilde{M}_4^{m\flat}).
\end{aligned} \tag{C.6}$$

Indeed the last term in the second line is due to the action of $\hbar\sum c_\xi D_\xi$ on the breaking $\Delta^\sharp(\eta)$ and to the loop diagrams with the insertion of $\Delta^\sharp$. The last line in Equation (C.6) accounts for the action of $W(X)$ on the right-hand side of Equation (C.5) whose first order approximation is given by the action of $T(X)$ on $\hbar\tilde{M}_4^{m\flat}(\eta)$.

Equation (C.6) is equivalent to a system of equations involving terms with different dimensions and covariances. For the non-invariant part we have

$$T(X)\tilde{M}_4^{m\flat}(\eta) \sim \tilde{M}_4^{m\flat}(\eta) = O(\Delta^\sharp, \hbar M_4^{m\flat}(\eta))$$

which implies

$$\tilde{M}_4^{m\flat}(\eta) = O(\Delta^\sharp).$$

Thus we have

$$\left[D_m - \int dx\ m\frac{\delta}{\delta\eta(x)}\right]\sum_{\delta=1,2,3}\Delta^\sharp_\delta(X,\eta) = O(\hbar\Delta^\sharp).$$

Now, considering in the order the terms with decreasing powers of $\eta$ and decreasing dimension $d$, none of which is annihilated by the differential operator $D_m$[16], we finally get

$$\Delta^\sharp_\delta(X,\eta) \equiv 0$$

and the unbroken integrated Ward identity is proved.

# D Cohomology of the Gauge Lie Algebra

We shall analyze the structure of

$$\hat{\mathcal{K}}_\alpha(x) = \partial_\mu K^\mu_\alpha \tag{D.7}$$

[16] Indeed the breaking $\Delta_\delta$ has physical (mass) dimension 4 and power counting dimension $\delta \leq 3$, this implies the presence of coefficient with mass dimension larger than one.

solution of the gauge algebra[17] cocycle condition (Cf. Equation (80))

$$\frac{\delta\hat{\mathcal{K}}_\beta(y)}{\delta\omega_\alpha(x)} - \frac{\delta\hat{\mathcal{K}}_\alpha(x)}{\delta\omega_\beta(y)} - f^\gamma_{\alpha\beta}\delta(x-y)\hat{\mathcal{K}}_\gamma(x) = 0 \tag{D.8}$$

(the Wess-Zumino consistency condition [11]). Integrating first Equation (D.8) over $x$ shows that $\partial_\mu K^\mu_\alpha$ transforms like the adjoint (regular) representation, under global transformations. Indeed, upon $x$-integration, Equation (D.2) reduces to

$$T_\alpha\hat{\mathcal{K}}_\beta(y) - f^\gamma_{\alpha\beta}\hat{\mathcal{K}}_\gamma(y) \tag{D.9}$$

where $T_\alpha$ is the infinitesimal generator of the global transformations. Due to its definition, Equation (D.7), $K^\mu_\alpha$ is a local polynomial in the fields and their derivatives identified up to terms which belong to the kernel of $\partial_\mu$. The mentioned polynomials carry completely reducible representations of the compact Lie algebra $\mathcal{G}$ which commutes with $\partial_\mu$. Thus, writing $K^\mu_\alpha$ as a combination of terms, each belonging to a different irreducible representation $\mathcal{G}$, we should find two different combinations corresponding to the kernel of $\partial_\mu$ and to the rest of $K^\mu_\alpha$ which must belong to the adjoint representation. In the following we shall only consider this rest which we shall persist denoting by $K^\mu_\alpha$ and which belongs to the adjoint representation of $\mathcal{G}$ . We shall now expand $K^\mu_\alpha$ in increasing powers of $\boldsymbol{a}_\mu$, obviously every term of this expansion belongs to the adjoint representation of $\mathcal{G}$. Let $K^\mu_{0\alpha}$ be the term independent of $\boldsymbol{a}_\mu$. We may write

$$\partial_\mu K^\mu_{0\alpha} = \frac{\delta}{\delta\omega^\alpha(x)}\int dy\ K^\nu_{0\beta}(y)a^\beta_\nu(y) + \partial_\mu L^\mu_\alpha \tag{D.10}$$

where $L^\mu_\alpha$ subtracts the homogeneous part of the gauge transformation of the first term and hence is linear in $\boldsymbol{a}_\mu$. The same decomposition can be repeated for the terms of higher degree.

Let $K^\mu_1$ be the term of $K^\mu + L^\mu$ linear in $\boldsymbol{a}_\mu$. We can similarly write

$$\partial_\mu K^\mu_{1\alpha}(x) = \frac{1}{2}\frac{\delta}{\delta\omega^\alpha(x)}\int dy\ K^\nu_{1\beta}(y)a^\beta_\nu(y) + \partial_\mu Q^\mu_\alpha$$

provided that

$$K^{\mu\nu}_{1\alpha\beta}(x,y) \equiv \frac{\delta K^\nu_{1\beta}(y)}{\delta a^\alpha_\mu(x)} - \frac{\delta K^\mu_{1\alpha}(x)}{\delta a^\beta_\nu(y)} = 0. \tag{D.11}$$

$Q^\mu_\alpha$ is now quadratic in $\boldsymbol{a}_\mu$. From the $\boldsymbol{a}_\mu$ independent part of Equation (D.8) we get

$$\partial^x_\mu\partial^y_\nu K^{\mu\nu}_{1\alpha\beta}(x,y) = 0. \tag{D.12}$$

[17]The gauge Lie algebra discussed in the present paper is an infinite dimensional generalization of a Lie algebra. The analysis shown in this section has been extended to a more general situation in [17].

The only possible $K^{\mu\nu}_{1\alpha\beta}(x,y)$ consistent with power counting, symmetry, and condition (D.12), is:

$$K^{\mu\nu}_{1\alpha\beta}(x,y) = (\Box g^{\mu\nu} - \partial^\mu\partial^\nu)\delta(x-y)A_{\underline{\alpha\beta}}$$

for some $A_{\underline{\alpha\beta}}$ anti-symmetric invariant tensor on the Lie algebra. Then

$$\tilde{K}^\mu_{1\alpha}(x) = K^\mu_{1\alpha}(x) + (\Box a^\mu_\beta(x) - \partial^\mu\partial_\nu a^\nu_\beta(x))A_{\underline{\alpha\beta}}$$

does fulfill Equation (D.11) and

$$\partial_\mu \tilde{K}^\mu_{1\alpha}(x) = \partial_\mu K^\mu_{1\alpha}(x).$$

Thus

$$\partial_\mu K^\mu_{1\alpha}(x) = \frac{1}{2}\frac{\delta}{\delta\omega^\alpha(x)}\int dy\ \tilde{K}^\nu_{1\beta}(y)a^\beta_\nu(y) + \partial_\mu R^\mu_\alpha(x) \tag{D.13}$$

for some $R^\mu_\alpha(x)$ quadratic in $\boldsymbol{a}_\mu$.

Similarly we proceed considering the terms $K^\mu_{2\alpha}(x) = K^\mu_\alpha(x) + R^\mu_\alpha(x)$ quadratic in $\boldsymbol{a}_\mu$.

It is convenient to continue our analysis after Fourier transformation of fields and local functionals. To simplify our formulae and calculations for a generic quantity $f(x)$ (or $y$) we denote its Fourier transform by $f(p)$ (or $q$ or else $k$), only changing the variables. The most general form of $K^\mu_{2\alpha}(-p)$ which is not orthogonal to $p$ is

$$\begin{aligned}K^\mu_{2\alpha}(-p) = i\int dk\, \big[&k^\mu a^\nu_\beta(k)a_{\nu\gamma}(-p-k))Z^{\alpha\beta\gamma} + k^\nu a^\mu_\beta(k)a_{\nu\gamma}(-p-k))X^{\alpha\beta\gamma}\\ &+k^\nu a_{\nu\beta}(k)a^\mu_\gamma(-p-k))Y^{\alpha\beta\gamma} + \epsilon^{\mu\nu\rho\sigma}D^{\alpha\beta\gamma}k_\nu a_{\rho\beta}(k)a_{\sigma\gamma}(-p-k)\big]\,,\end{aligned} \tag{D.14}$$

where $D^{\alpha\beta\gamma}$ must be symmetric in $\beta$ and $\gamma$ [18]. Furthermore all the coefficient are invariant tensors on $\mathcal{G}$.

Now

$$\begin{aligned}ip_\mu K^\mu_{2\alpha}(-p) = -\int dk\, \big[&p\cdot k a^\nu_\beta(k)a_{\nu\gamma}(-p-k))Z^{\alpha\beta\gamma} + p_\mu k^\nu a^\mu_\beta(k)a_{\nu\gamma}(-p-k))X^{\alpha\beta\gamma}\\ &+p_\mu k^\nu a_{\nu\beta}(k)a^\mu_\gamma(-p-k))Y^{\alpha\beta\gamma} + \epsilon^{\mu\nu\rho\sigma}D^{\alpha\beta\gamma}k_\mu a_{\nu\beta}(k)p_\rho a_{\sigma\gamma}(-p-k)\big]\,,\end{aligned} \tag{D.15}$$

The part of the cocycle (consistency) condition on $p_\mu K^\mu_{2\alpha}(-p)$ which is linear in $\boldsymbol{a}_\mu$ requires the symmetry under simultaneous interchange of $p$ and $q$ and $\alpha$ and $\beta$ of

$$\begin{aligned}q_\nu\frac{\delta\ p_\mu K^{\mu\alpha}_2(-p)}{\delta\ a_{\nu\beta}(q)} &= q\cdot a_\gamma(-p-q)[p\cdot q(Z^{\alpha\beta\gamma} - Z^{\alpha\gamma\beta} - Y^{\alpha\gamma\beta} + X^{\alpha\beta\gamma}) - p^2 Z^{\alpha\gamma\beta}]\\ &+p\cdot a_\gamma(-p-q)[p\cdot q(-Y^{\alpha\gamma\beta} - X^{\alpha\gamma\beta}) - q^2(X^{\alpha\gamma\beta} - Y^{\alpha\beta\gamma})]\end{aligned} \tag{D.16}$$

[18] In order to verify these properties it is useful to have occasionally recourse to the change of the integration variable $k \to -p-k$.

where we have performed twice the partial change of variables mentioned in the footnote. From Equation (D.16) we get

$$\begin{aligned}
&Z^{\alpha\gamma\beta} = X^{\beta\gamma\alpha} - Y^{\beta\alpha\gamma} \\
&Z^{\alpha\beta\gamma} - Z^{\alpha\gamma\beta} - Y^{\alpha\gamma\beta} + X^{\alpha\beta\gamma} + Y^{\beta\gamma\alpha} + X^{\beta\gamma\alpha} = 0.
\end{aligned} \tag{D.17}$$

We now consider $K_3$, the most general integrated local functional of dimension four and cubic in $\boldsymbol{a}_\mu$, it is

$$\begin{aligned}
K_3 \quad &= \int dx \ \big[\partial^\mu a_{\mu\beta}(x) a^\nu_\gamma(x) a_{\nu\delta}(x) A^{\beta\gamma\delta} + a_{\mu\beta}(x)\partial^\mu a^\nu_\gamma(x) a_{\nu\delta}(x) B^{\beta\gamma\delta} \\
&+\epsilon^{\mu\nu\rho\sigma} E^{\beta\gamma\delta} \partial_\mu a_{\nu\beta}(x) a_{\rho\gamma}(x) a_{\sigma\delta}(x)\big]
\end{aligned} \tag{D.18}$$

where $A^{\beta\gamma\delta}$ is symmetric in $\gamma$ and $\delta$ and $E^{\beta\gamma\delta}$ is anti-symmetric in the same indices. Computing $p_\mu \frac{\delta \ K_3}{\delta \ a_{\mu\alpha}(p)}$ we get the same expression as that in the right-hand side of Equation (D.14) where however

$$Z^{\alpha\beta\gamma} = B^{\alpha\beta\gamma} - 2A^{\alpha\beta\gamma} = -Y^{\beta\alpha\gamma}, \quad \text{and} \quad X^{\alpha\beta\gamma} = Z^{\gamma\beta\alpha} - Z^{\gamma\alpha\beta}. \tag{D.19}$$

These are consistent with Equation (D.17). $B$ being arbitrary, although invariant, we can choose $B$ and $A$ satisfying Equation (D.19). With this choice and using Equation (D.15) we have, for some $Q_3$ cubic in $\boldsymbol{a}_\mu$

$$\begin{aligned}
&ip_\mu K_2^{\mu\alpha}(-p) = \frac{\delta}{\delta\omega_\alpha(p)} K_3 + Q_3^\alpha(p) \\
&-\int dk \ p_\mu k^\nu \ \big[a^\mu_\beta(k) a_{\nu\gamma}(-p-k))(X^{\alpha\beta\gamma} - Z^{\gamma\beta\alpha} + Z^{\gamma\alpha\beta}) \\
&+a_{\nu\beta}(k) a^\mu_\gamma(-p-k))(Y^{\alpha\beta\gamma} + Z^{\beta\alpha\gamma})\big] \\
&-\int dk \ \epsilon^{\mu\nu\rho\sigma}(D^{\alpha\beta\gamma} - 2(E^{\alpha\beta\gamma} + E^{\beta\alpha\gamma}))k_\mu a_{\nu\beta}(k) p_\rho a_{\sigma\gamma}(-p-k)
\end{aligned} \tag{D.20}$$

Using the first Equation (D.16) we get $X^{\alpha\beta\gamma} - Z^{\gamma\beta\alpha} + Z^{\gamma\alpha\beta} = Y^{\alpha\gamma\beta} + Z^{\gamma\alpha\beta} \equiv W^{\alpha\gamma\beta}$ which is antisymmetric in the last two indices due to Equation (D.16). Now it is not difficult to verify, using again the above mentioned change of variables, that the third term in the right-hand side of Equation (D.20) vanishes. We are still free to choose $E$; we set $E^{\alpha\beta\gamma} = \frac{1}{3}[D^{\beta\gamma\alpha} - D^{\gamma\alpha\beta}]$ ($D$ is symmetric in the last two indices). Then the coefficient in the third term in the right-hand side of Equation (D.20) reads

$$\frac{1}{3}(D^{\alpha\beta\gamma} + 4D^{\gamma\alpha\beta}) - \frac{2}{3}D^{\beta\gamma\alpha}.$$

However we must remind that the non-vanishing contribution to Equation (D.20) of its fourth term corresponds to the part of this tensor which is $\beta-\gamma$-symmetric, that is

$$\tilde{D}^{\alpha\beta\gamma} \equiv \frac{1}{3}(D^{\alpha\beta\gamma} + D^{\gamma\alpha\beta} + D^{\beta\gamma\alpha}), \tag{D.21}$$

this is an invariant fully symmetric tensor on the Lie algebra $\mathcal{G}$. Thus we get

$$\partial_\mu K_2^{\mu\alpha}(x) = \frac{\delta}{\delta\omega_\alpha(x)} K_3 + Q_3^\alpha(x) + \epsilon^{\mu\nu\rho\sigma}\tilde{D}^{\alpha\beta\gamma}\partial_\mu a_{\nu\beta}(x)\partial_\rho a_{\sigma\gamma}(x). \tag{D.22}$$

In order to perform the last step we put together Equations (D.10), (D.13) and (D.22) and, omitting the ˜ sign above $D$, we obtain

$$\partial_\mu K^{\mu\alpha}(x) = \frac{\delta}{\delta\omega_\alpha(x)}\left[\int dy\ (K_{0\beta}^\nu(y)a_\nu^\beta(y) + \frac{1}{2}\ \tilde{K}_{1\beta}^\nu(y)a_\nu^\beta(y)) + K_3\right]$$
$$+\partial_\mu S^{\mu\alpha}(x) + \epsilon^{\mu\nu\rho\sigma}D^{\alpha\beta\gamma}\partial_\mu a_{\nu\beta}(x)\partial_\rho a_{\sigma\gamma}(x) = \frac{\delta}{\delta\omega_\alpha(x)}M + \partial_\mu J^{\mu\alpha}(x), \tag{D.23}$$

for some $S^{\mu\alpha}$ and hence $J^{\mu\alpha}$ of dimension four and cubic in $\boldsymbol{a}_\mu$. Thus

$$\partial_\mu J_\alpha^\mu(x) = \partial_\mu\Big[\epsilon^{\mu\nu\rho\sigma}(\frac{1}{2}D^{\alpha\beta\gamma}a_{\nu\beta}(x)\overleftrightarrow{\partial}_\rho a_{\sigma\gamma}(x) + F^{\alpha\beta\gamma\delta}a_{\nu\beta}(x)a_{\rho\gamma}(x)a_{\sigma\delta}(x))$$
$$+G^{\alpha\beta\gamma\delta}a_{\mu\beta}(x)a_{\nu\gamma}(x)a_\delta^\nu(x)\Big], \tag{D.24}$$

for some $F^{\alpha\beta\gamma\delta}$ and $G^{\alpha\beta\gamma\delta}$ invariant tensors on the Lie algebra $\mathcal{G}$. $F^{\alpha\beta\gamma\delta}$ is antisymmetric in its last three indices while $G^{\alpha\beta\gamma\delta}$ is symmetric in its last two indices. Furthermore $\partial_\mu J^{\mu\alpha}(x)$ must satisfy the consistency condition (D.8).

This condition generates a system of algebraic equations for the coefficients $F^{\alpha\beta\gamma\delta}$ and $G^{\alpha\beta\gamma\delta}$. In particular, the parts containing the antisymmetric four dimensional Ricci symbol give three independent equations that we now write in terms of spacetime functionals

$$\epsilon^{\mu\nu\rho\sigma}\partial_\mu\delta(x-y)\partial_\nu(a_{\rho\gamma}a_{\sigma\delta})(y)\left[3(F^{\alpha\beta\gamma\delta} + F^{\beta\alpha\gamma\delta}) + D^{\alpha\omega\gamma}f_\omega^{\beta\delta} + D^{\beta\omega\gamma}f_\omega^{\alpha\delta}\right] = 0, \tag{D.25}$$

$$\epsilon^{\mu\nu\rho\sigma}\partial_\mu\delta(x-y)(a_{\rho\gamma}\overleftrightarrow{\partial}_\nu a_{\sigma\delta})(y)\left[D^{\alpha\omega\gamma}f_\omega^{\beta\delta} - \frac{1}{2}D^{\delta\omega\gamma}f_\omega^{\alpha\beta}\right] = 0, \tag{D.26}$$

$$\epsilon^{\mu\nu\rho\sigma}\partial_\mu\delta(x-y)(a_{\nu\gamma}a_{\rho\delta}a_{\sigma\eta})(y)\left[3f_\omega^{\beta\delta}F^{\alpha\omega\gamma\eta} - f_\omega^{\alpha\beta}F^{\omega\delta\gamma\eta}\right] = 0. \tag{D.27}$$

The equations for the coefficients in Equations (D.26) and (D.27) are

$$\frac{1}{2}\left[D^{\alpha\omega\gamma}f_\omega^{\beta\delta} + D^{\alpha\omega\delta}f^{\beta\gamma} + D^{\delta\omega\gamma}f_\omega^{\beta\alpha}\right] = 0$$

and

$$f_\omega^{\beta\delta}F^{\alpha\omega\gamma\eta} + f_\omega^{\beta\gamma}F^{\alpha\delta\omega\eta} + f_\omega^{\beta\eta}F^{\alpha\delta\gamma\omega} + f_\omega^{\beta\alpha}F^{\omega\delta\gamma\eta} = 0$$

which are trivially satisfied due to the invariance of $D$ and $F$.

The equation for the coefficient in Equation (D.25) is

$$\begin{aligned} 6(F^{\alpha\beta\gamma\delta}+F^{\beta\alpha\gamma\delta}) &= D^{\alpha\omega\gamma}f_{\omega}^{\delta\beta}+D^{\beta\omega\gamma}f_{\omega}^{\delta\alpha}+D^{\alpha\omega\delta}f_{\omega}^{\beta\gamma}+D^{\beta\omega\delta}f_{\omega}^{\alpha\gamma} \\ &= \frac{1}{2}\left[D^{\alpha\omega\gamma}f_{\omega}^{\delta\beta}+D^{\alpha\omega\delta}f_{\omega}^{\beta\gamma}+D^{\beta\omega\delta}f_{\omega}^{\alpha\gamma}+D^{\beta\omega\gamma}f_{\omega}^{\delta\alpha}+2D^{\beta\omega\alpha}f_{\omega}^{\gamma\delta}\right], \end{aligned} \tag{D.28}$$

which is apparently solved by Equation (3.62).

It is clear that this is a particular solution of Equation (D.28) whose general solution is obtained by adding to the right-hand side of Equation (3.62) a solution of the corresponding homogeneous equation ($D=0$). This must be antisymmetric in $\alpha-\beta$. But $F^{\alpha\beta\gamma\delta}$ is also completely antisymmetric in its last three indices, thus the solution of the homogeneous equation must be completely antisymmetric in all its indices. However the contribution to $\partial\cdot K^{\alpha}(x)$ corresponding to a generic invariant totally antisymmetric $F_A^{\alpha\beta\gamma\delta}$ is just equal to

$$\frac{1}{4}\frac{\delta}{\delta\omega_{\alpha}(x)}\int dy\epsilon^{\mu\nu\rho\sigma}(a_{\mu\alpha}a_{\nu\beta}a_{\rho\gamma}a_{\sigma\delta})(x)F_A^{\alpha\beta\gamma\delta}\equiv\frac{\delta}{\delta\omega_{\alpha}(x)}N.$$

$N$ can be added to $M$ in Equation (D.23).

Still we have to discuss the last term in Equation (D.24), that is the consistency condition (D.8) for $K_G^{\alpha}(x)=G^{\alpha\beta\gamma\delta}\partial^{\mu}(a_{\mu\beta}(x)a_{\nu\gamma}(x)a_{\delta}^{\nu}(x))$. We find, once again a system of algebraic equations for the coefficient $G^{\alpha\beta\gamma\delta}$. Selecting the independent parts we have

$$\partial_{\mu}^{(x)}\partial^{(y)\mu}(\delta(x-y)a_{\nu\gamma}(x)a_{\delta}^{\nu}(x))\left[G^{\alpha\beta\gamma\delta}-G^{\beta\alpha\gamma\delta}\right]=0, \tag{D.29}$$

$$\partial_{\mu}^{(x)}\partial_{\nu}^{(y)}(\delta(x-y)a_{\gamma}^{\mu}(x)a_{\delta}^{\nu}(x))\left[G^{\alpha\gamma\beta\delta}-G^{\beta\delta\alpha\gamma}\right]=0, \tag{D.30}$$

$$\partial^{\mu}\delta(x-y)(a_{\mu\delta}a_{\nu\gamma}a_{\eta}^{\nu})(y)\left[f_{\omega}^{\beta\delta}G^{\alpha\omega\gamma\eta}+2f_{\omega}^{\beta\gamma}G^{\alpha\delta\omega\eta}-f_{\omega}^{\alpha\beta}G^{\omega\delta\gamma\eta}\right]=0, \tag{D.31}$$

from which we see that $G^{\alpha\beta\gamma\delta}$ must be an invariant tensor on the Lie algebra $\mathcal{G}$, that must be symmetric in its first (and last) two indices and it must be left invariant by the exchange of the first pair of indices with the second one. Therefore we have

$$K_G^{\alpha}(x)=\frac{\delta}{\delta\omega_{\alpha}(x)}\frac{1}{4}\int dy\ G^{\beta\gamma\delta\epsilon}(a_{\mu\beta}a_{\gamma}^{\mu}a_{\nu\delta}a_{\epsilon}^{\nu})\equiv\frac{\delta}{\delta\omega_{\alpha}(x)}P.$$

Also $P$ can be added to $M$ in Equation (D.23).

In this way we have proved Equation (3.54) with $\int dx\ K(x)=M+N+P$ and $G_{\alpha}(x)$ satisfying Equations (3.60), (3.61) and (3.62).

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