Action-angle variables for spherical mechanics related to near horizon extremal Myers–Perry black hole

Anton Galajinsky[1], **Armen Nersessian**[2] **and Armen Saghatelian**[2]

[1]Tomsk Polytechnic University, Lenin Ave. 30, 634050 Tomsk, Russia
[2] Yerevan State University, 1 Alex Manoogian St., Yerevan, 0025, Armenia

E-mail: galajinsky@tpu.ru, arnerses@ysu.am, armen.saghatelian@ysu.am

Abstract. The action-angle formulation for the spherical part of the conformal mechanics describing a massive relativistic particle moving near the horizon of an extremal rotating black hole in arbitrary dimension is presented for the special case that all rotation parameters are equal.

An extremal rotating black hole in arbitrary dimension exhibits the conformal symmetry $SO(2,1)$ in the near horizon limit (see, e.g., [1] and references therein). Because to each Killing vector of a background geometry there corresponds the first integral of the geodesic equations, a model of a massive relativistic particle propagating near the horizon of an extremal rotating black hole is automatically conformal invariant. Conformal mechanics originating from the near horizon extremal black hole geometry has been extensively investigated in the past [2]–[16] with a particular emphasis on its dynamics, a relation to nonrelativistic conformal mechanics, and the construction of supersymmetric extensions. Quite recently, based on an earlier work [17], such a conformal mechanics has been used to build new superintegrable models [18, 19]. In particular, in Ref. [19] a maximally superintegrable spherical mechanics associated with the black hole in $d = 2n + 1$ dimensions has been constructed. It was also demonstrated that its counterpart related to the black hole in $d = 2(n + 1)$ dimensions lacks for only one integral of motion to be maximally superintegrable.

The goal of this note is to extend the analysis in [19] and to construct the action-angle variables. There are several reasons for this study. The action–angle formulation of an integrable system gives a comprehensive geometric description of its dynamics and provides a useful means for developing the perturbation theory [20]. The use of the action-angle variables suggests a criterion of the (non)equivalence of two integrable systems. The variables provide a systematic way to reveal hidden symmetries and to construct new non–trivial examples of maximally superintegrable models (see, e.g., [21]). Worth mentioning also is that the formulation in terms of the action-angle variables is a base for the (semiclassical) quantization, which implies imposing on the action variables the Bohr–Sommerfeld quantization conditions.

Because only integrable systems with a compact phase space admit the action–angle formulation, the full conformal mechanics associated with the near horizon geometry of an extremal rotating black hole is not amenable to such a description. However, one can separate the radial (noncompact) motion from the angular variables dynamics, which is compact and can be considered on its own. Note that it is in this way that the action–angle variables were constructed recently for the Calogero model [22] and for some variants of the conformal mechanics related to the near horizon extremal black holes [14, 15, 16]. The present work extends the list of such examples and also illuminates the origin of the superintegrability of the models constructed recently in [19]. Below we first discuss the model originating from a black hole geometry in $d = 2n + 1$ dimensions and then turn to the case of $d = 2(n + 1)$.

The Hamiltonian of a spherical mechanics related to the extremal rotating black hole in $d = 2n + 1$ is a kind of matryoshka doll [19]

$$\mathcal{I}_0 = F_{n-1}, \tag{1}$$

where F_{n-1} is derived from the recurrence relation

$$F_i = p^2_{\theta_i} + \frac{g^2_{i+1}}{\cos^2\theta_i} + \frac{F_{i-1}}{\sin^2\theta_i}. \tag{2}$$

Here (θ_i, p_{θ_i}), with $i = 1, \dots, n-1$, constitute the canonical pairs, g_i are coupling constants, and $F_0 = g^2_1$. As is obvious from (1) and (2), the functionally independent integrals of motion in involution F_i ensure the integrability of (1). Let us demonstrate the maximal superintegrability (and in fact exact solvability) of this system by constructing the action–angle variables. Following the standard procedure [20], one has to write down the generating function

$$S^{odd}_0(F_i, |g_i|, \theta_i) = \sum_{i=1}^{n-1} \int p_{\theta_i}(F_1, \dots, F_{n-1}, \theta_i)\mathrm{d}\theta_i, \tag{3}$$

where $p_{\theta_i}(F_1, \dots, F_{n-1}, \theta_i)$ are to be expressed from (2). For the action variables one gets (for the details of evaluation of the integrals see, e.g., Ref. [21])

$$I_i = \frac{1}{2\pi}\oint \mathrm{d}\theta_i \left[\sqrt{F_i - \frac{F_{i-1}}{\sin^2\theta_i} - \frac{g^2_{i+1}}{\cos^2\theta_i}}\right] = \frac{1}{2}(\sqrt{F_i} - \sqrt{F_{i-1}} - |g_{i+1}|), \tag{4}$$

with $i = 1, \dots, n-1$. Inverting these expression, one finds

$$F_i = \left(2\sum_{k=1}^{i} I_k + \sum_{k=1}^{i+1}|g_k|\right)^2, \tag{5}$$

while the angle variables read

$$\Phi^{odd}_i = \frac{\partial S}{\partial I_i} = \sum_{k=i}^{n-1}\arcsin X_k + 4\sum_{k=i+1}^{n-1}\arctan Y_k, \tag{6}$$

where we denoted

$$X_k = \frac{\left(F_k + F_{k-1} - g^2_{k+1}\right) - 2F_k\sin^2\theta_k}{\sqrt{\left(-F_k + F_{k-1} - g^2_{k+1}\right)^2 - 4F_k g^2_{k+1}}}, \tag{7}$$

$$Y_k = \frac{\left(F_k + F_{k-1} - g^2_{k+1}\right)\sqrt{F_k\sin^2\theta_k\cos^2\theta_k - F_{k-1}\cos^2\theta_k - g^2_{k+1}\sin^2\theta_k}}{\sqrt{F_{k-1}}\left((F_k + F_{k-1} - g^2_{k+1}) - 2F_k\sin^2\theta_k\right)} -$$

$$- \frac{\sin^2\theta_k\sqrt{F_k\left(F_k + F_{k-1} - g^2_{k+1}\right)^2 - F^2_k F_{k-1}}}{\sqrt{F_{k-1}}\left((F_k + F_{k-1} - g^2_{k+1}) - 2F_k\sin^2\theta_k\right)}. \tag{8}$$

Thus, being rewritten in the action–angle variables, the Hamiltonian has the following form:

$$\mathcal{I}_0 = \left(2\sum_{k=1}^{n-1} I_k + \sum_{k=1}^{n} |g_k|\right)^2 . \tag{9}$$

Up to the shift of the action variables, it coincides with the Hamiltonian of a free particle moving on the $(n-1)$–dimensional sphere [21]. So the two systems differ only by the range of $\sum_i I_i$ ($[0,\infty)$ and $[\sum_{k=1}^{n}|g_k|,\infty)$, respectively). The former model thus possesses $SO(n)$ symmetry and is obviously maximally superintegrable.

The action–angle formulation allows one to immediately construct functionally independent hidden constants of the motion. Because the evolution of the angle variables is governed by the equation (cf. [21, 23])

$$\frac{d\Phi_i}{dt} = 2\left(2\sum_{k=1}^{n-1} I_k + \sum_{k=1}^{n} |g_k|\right), \tag{10}$$

the expressions $\cos(\Phi_i - \Phi_j + \text{const})$ determine (globally defined) constants of the motion $n-2$ of which are functionally independent

$$G_i = \cos\left(\Phi_i - \Phi_{i+1}\right) = \frac{\sqrt{1-X_i^2}(1-Y_{i+1}^2) - 2X_iY_{i+1}}{1+Y_{i+1}^2}, \tag{11}$$

with $i = 1,\ldots,n-2$. Hence, the system (1) with $(n-1)$ configuration space degrees of freedom possesses $2n-3$ functionally independent constants of the motion and, as thus, is maximally superintegrable. That it is exactly solvable follows from the fact that the Hamiltonian is expressed via the action variables in terms of elementary functions only.

Now let us turn to the case of $d = 2(n+1)$. The Hamiltonian of a spherical mechanics related to the extremal rotating black hole in $d = 2(n+1)$ dimensions reads [19]

$$\mathcal{I}_0 = 2np_{\theta_n}^2 + \nu\sin^2\theta_n + \left(\frac{2n}{\sin^2\theta_n} - 2n + 1\right)F_{n-1}, \tag{12}$$

where F_{n-1} is given in (2). As we have seen above, F_{n-1} is the Hamiltonian of a particle moving on $\mathcal{S}^{n-1}$. This sector provides $2(n-1)-1$ functionally independent integrals of motion. Because (12) involves one more canonical pair (θ_n, p_{θ_n}) and only one extra integral of motion (the Hamiltonian (12) itself), the full theory lacks for only one integral of motion to be maximally superintegrable.

Let us construct the action-angle variables for this system. One starts with the generating function

$$S_0^{even} = \sum_{i=1}^{n}\int p_{\theta_i}(\mathcal{I}_0, F_1,\ldots,F_{n-1},\theta_i)\mathrm{d}\theta_i = \int p_{\theta_n}(\mathcal{I}_0, F_{n-1},\theta_n)d\theta_n + S_0^{odd}, \tag{13}$$

where S_0^{odd} is defined in (3) and p_{θ_n} is to be derived from (12). The action variables $I_1,\ldots I_{n-1}$ coincide with those in (4), while I_n reads

$$I_n = \sqrt{\frac{-a^-\nu}{8n}}a^+\mathcal{F}_1\left(\frac{1}{2},1,-\frac{1}{2},2,a^+,\frac{a^+}{a^-}\right), \tag{14}$$

where $\mathcal{F}_1$ is the Appell's first hypergeometric series[1] and

$$a^\pm = 1 - \frac{\mathcal{I}_0}{2\nu} - \frac{2n-1}{2\nu}F_{n-1} \pm \sqrt{\left(1 - \frac{\mathcal{I}_0}{2\nu} - \frac{2n-1}{2\nu}F_{n-1}\right)^2 + \frac{\mathcal{I}_0}{\nu} - \frac{F_{n-1}}{\nu} - 1}. \tag{15}$$

[1] Here and in what follows we use the notations in [24, 25].

Inverting these expressions, one gets the Hamiltonian in terms of the action variables

$$\mathcal{I}_0 = \mathcal{I}_0\left(I_n, F_{n-1}\right), \quad F_{n-1} = \left(2\sum_{k=1}^{n-1} I_k + \sum_{k=1}^{n} |g_k|\right)^2. \tag{16}$$

Obviously, this cannot be done in terms of elementary functions. We thus conclude that the system under consideration is integrable but not exactly solvable.

The angle variable conjugated to I_n reads

$$\Phi_n^{even} = \frac{\partial \mathcal{I}_0}{\partial I_n}\frac{1}{\sqrt{8\nu n a^+}}\mathcal{F}\left(\arcsin\sqrt{\frac{a^+}{a^+ - \cos^2\theta_n}}, 1 - \frac{a^-}{a^+}\right), \tag{17}$$

while the remaining angle variables are defined by the following expressions ($i = 1, \ldots, n-1$):

$$\begin{aligned}\Phi_i^{even} = \Phi_i^{odd} \quad &- \quad A\,\Pi\left(1 - \frac{1}{a^+}, \arcsin\sqrt{\frac{a^+}{a^+ - \cos^2\theta_n}}, 1 - \frac{a^-}{a^+}\right) + \\ &+ \quad B\,\mathcal{F}\left(\arcsin\sqrt{\frac{a^+}{a^+ - \cos^2\theta_n}}, 1 - \frac{a^-}{a^+}\right),\end{aligned} \tag{18}$$

where Φ_i^{odd} are given in (6), $\mathcal{F}(\phi|m)$ is the elliptic integral of the first kind, $\Pi(n;\phi|m)$ is the elliptic integral of the third kind, and

$$A = \sqrt{\frac{8nF_{n-1}}{\nu}}\frac{1}{\sqrt{a^+}(a^+ - 1)}, \quad B = A + \frac{\sqrt{2F_{n-1}}}{\sqrt{n\nu a^+}}\left(\frac{\partial \mathcal{I}_0}{\partial F_{n-1}} + 2n - 1\right). \tag{19}$$

Since the Hamiltonian of our system is of the form (16), the ratio of the effective frequencies $\omega_1 = \partial\mathcal{I}/\partial I_n$ and $\omega_2 = \partial\mathcal{I}/\partial F_{n-1}$ is not a rational number but a function of the action variables (for a similar situation see [21, 23]). Hence, although $(\omega_2\Phi_n - \omega_1\Phi_i)$ commutes with the Hamiltonian $\mathcal{I}_0$, it is not a periodic function. All hidden symmetries of our model are thus exhausted by (11). The system with n configuration space degrees of freedom thus has $n + (n-2) = 2n - 2$ constants of the motion and lacks for only one constant of the motion to be maximally superintegrable.

Acknowledgments

This work was supported by the RFBR grant 13-02-90602-Arm, MSE Program "Nauka" under the grant 1.604.2011 (A.G.), the Armenian State Committee of Science under the grant 13RF-018 (A.N.,A.S.), by the ANSEF Grant hepth-3267 (A.S.) and Volkswagenstiftung under the contracts 86 260 (A.S.) and I/84 496 (A.N.).

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