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Article

A New Route to Symmetries through the Extended Dirac Equation

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Abstract: A new route to the Dirac equation and its symmetries is outlined on the basis of the four-vector representation of the Lorentz group (LG). This way permits one to linearize the first Casimir operator of the LG in terms of the energy–momentum four-vector and enables one to derive an extended Dirac equation that naturally reveals the $SU(2)$ symmetry in connection with an isospin associated with the LG. The procedure gives a spin-one-half fermion doublet, which we interpret as the electron and neutrino or the up-and-down quark doublet. Similarly, the second Casimir operator can be linearized by invoking an abstract isospin that is not connected with the LG, but with the two basic empirical fermion types. Application of the spinor helicity formalism yields two independent singlet and triplet fermion states—which we interpret as being related to $U(1)$ and the lepton, respectively—to the $SU(3)$ symmetry group of the three colors of the quarks. The way in which we obtain these results indicates the genuine yet very different physical natures of these basic symmetries. This new notion does not need the idea of grand unification. However, by still combining them in the product group $SU(4) = SU(3) \otimes U(1)$ and then further combining all groups into $SU(2) \otimes SU(4)$, one may get a symmetry scheme that perhaps supports the notion of unification by the group $SU(8)$. We also argue that the simpler $SO(4)$ group—instead of $SU(4)$—seems more appropriate for achieving unification.

Keywords: Lorentz group; extended Dirac equation; isospin; $SU(2)$, $SU(3)$, and $SO(4)$ symmetries; fermion unification

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1. Introduction

The mathematical $SU(N)$ symmetries related to Lie groups are key elements of the standard model of elementary particle physics (SM) [1,2], which describes the electroweak ($N = 2$) and strong ($N = 3$) interactions among lepton and quark fermions. The way of including the associated gauge bosons in the quantum mechanical framework of modern field theory was first described in the seminal paper by Yang and Mills [3]. However, the physical origin of these important symmetries has remained somewhat obscure or of a mainly empirical nature until today. They were chosen in the case of $SU(2)$ to describe the effects of parity violation by assuming that weak interactions only involve the left-handed fields [4] of massless fermions or were just introduced ad hoc in the case of $SU(3)$ by Gell-Mann [5] in order to bring some schematics into the complex variety of hadrons found empirically in accelerator experiments.

The aim of this paper is to cast somewhat more light on the genuine nature of the $SU(2)$ symmetry (weak interactions) and $SU(3)$ symmetry (strong interactions) in particle physics and mathematical physics with the help of (1) the mathematical and physical principles of Lorentz invariance, (2) the two associated Casimir operators of the Lorentz group (hereafter, LG), and (3) their relations to the notion of isospin. This paper is meant to extend and deepen the recent work by Marsch and Narita [6,7] on these topics. A similar approach to understanding space-time hidden symmetry was used by Hestenes in his development of space-time algebra, e.g., interpreting electron spin direction and spin magnitude as geometric properties [8].

We first prepare some necessary ingredients of the theory in a section on the generators of the Lorentz group for the transformation of four-vectors in Minkowski space-time. Then, new versions of an extended Dirac equation and the associated Clifford algebra are presented. The related physical spin and rapidity of the resulting fermion spinor field are discussed. A version of the extended Dirac equation on the Weyl basis is also derived in the Appendix A. The notion of isospin is introduced and derived in the context of the spinorial representation of the Lorentz group. The consequences for the origin of $SU(2)$ and the related fermion lepton doublet, such as an electron and a neutrino, are elucidated.

Furthermore, on the basis of the Pauli–Lubański operator, we develop a concept for the possible origin of the $SU(3)$ color symmetry. It may stem from a new intrinsic isospin of $1/2$, which is assumed to describe the lepton–quark fermion doublet. Thereby, use is made of the powerful concept of spinor helicity by employing the Pauli matrices, which permit one to decompose the square of any three-vector into a Pauli matrix product. The related symmetry turns out to be $SO(4)$, which contains $SO(3)$ as a subgroup.

Finally, we discuss some possible schemes for the unification of these symmetries. A short discussion section concludes the paper.

2. The Generators of the Lorentz Group

To begin with, we discuss the generators of the Lorentz group in the vectorial representation. The Lie algebra for the Lorentz group [9–11] is decomposed into two commuting independent sub-algebras as $so(3,1) = su(2) \otimes su(2)$. Here, so denotes the algebra associated with the special orthogonal group SO, and su denotes the algebra associated with the special unitary group SU. That is, they define the generators of the irreducible $SU(2) \otimes SU(2)$ representation of the LG. We introduce the standard hermitian rotation operator $\mathbf{J} = (J_x, J_y, J_z)$ and the anti-hermitian boost operator $\mathbf{K} = (K_x, K_y, K_z)$. These three-vector generators of the LG are component-wise 4×4 matrices in Minkowski space-time. According to their definitions, the rotation and boost operators obey the well-known linked three-vector equations of the Lorentz algebra:

$$\mathbf{J} \times \mathbf{J} = \mathrm{i}\mathbf{J}, \quad \mathbf{K} \times \mathbf{K} = -\mathrm{i}\mathbf{J}, \quad \mathbf{J} \times \mathbf{K} = \mathbf{K} \times \mathbf{J} = \mathrm{i}\mathbf{K} \tag{1}$$

where the cross-product sign stands as an abbreviation for the commutator $[,]$. The four-vector LG generators can be written as tensors in Minkowski space-time [12], a well-known subject. In the Appendix A, the components of the matrix vectors $\mathbf{J}$ and $\mathbf{K}$ are quoted for completeness and reference. From them, we straightforwardly obtain the absolute value of the rotation operator as

$$\mathbf{J}^2 = J_x^2 + J_y^2 + J_z^2 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 2 \end{pmatrix}. \tag{2}$$

and that of the boost matrix as

$$\mathbf{K}^2 = K_x^2 + K_y^2 + K_z^2 = \begin{pmatrix} -3 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}. \tag{3}$$

Consequently, we find $\mathbf{J}^2 - \mathbf{K}^2 = 3 1_4$. Then, for any three-vector $\mathbf{V} = (x, y, z)$, one obtains

$$\mathbf{J} \cdot \mathbf{V} = \mathrm{i} \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -z & y \\ 0 & z & 0 & -x \\ 0 & -y & x & 0 \end{pmatrix}, \quad (\mathbf{J} \cdot \mathbf{V})^2 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & y^2 + z^2 & -xy & -xz \\ 0 & -xy & x^2 + z^2 & -yz \\ 0 & -xz & -yz & x^2 + y^2 \end{pmatrix}, \tag{4}$$

and similarly, one obtains

$$\mathbf{K} \cdot \mathbf{V} = \mathrm{i} \begin{pmatrix} 0 & x & y & z \\ x & 0 & 0 & 0 \\ y & 0 & 0 & 0 \\ z & 0 & 0 & 0 \end{pmatrix}, \quad (\mathbf{K} \cdot \mathbf{V})^2 = - \begin{pmatrix} x^2 + y^2 + z^2 & 0 & 0 & 0 \\ 0 & x^2 & xy & xz \\ 0 & xy & y^2 & yz \\ 0 & xz & yz & z^2 \end{pmatrix}. \tag{5}$$

Therefore, one finds that $(\mathbf{K} \cdot \mathbf{V})(\mathbf{J} \cdot \mathbf{V}) = (\mathbf{J} \cdot \mathbf{V})(\mathbf{K} \cdot \mathbf{V}) = 0$. Moreover, we obtain the relation

$$(\mathbf{J} \cdot \mathbf{V})^2 - (\mathbf{K} \cdot \mathbf{V})^2 = (x^2 + y^2 + z^2) 1_4 = \mathbf{V}^2 1_4. \tag{6}$$

This important result will play a key role and be exploited in the subsequent section.

3. New Versions of the Dirac Equation and Clifford Algebra

Recently, Marsch and Narita [13] derived an extended Dirac equation on the basis of the vector representation of the Lorentz group. In this section, we present a new route to obtain such an extension of the standard Dirac equation [14]. Historically, the key question was that of how to derive a linear relativistic wave equation. This task requires linearizing the kinetic energy for a massive particle, which goes with the momentum squared in the basic relativistic dispersion relation and is given by

$$E^2 - \mathbf{p}^2 = m^2 = P^\mu P_\mu. \tag{7}$$

This is the so-called mass–shell condition for a free particle of mass m, energy E, and momentum $\mathbf{p}$, and it is just the first Casimir operator of the Lorentz group. Here, we use the covariant four-momentum $P_\mu = (E, -\mathbf{p})$ as a variable in Fourier space. According to relativistic quantum mechanics [1,2], the covariant four-momentum operator is associated with a temporal or spatial derivative:

$$P_\mu = (E, -\mathbf{p}) = \mathrm{i}\partial_\mu = \mathrm{i}(\frac{\partial}{\partial t}, \frac{\partial}{\partial \mathbf{x}}). \tag{8}$$

We abbreviate the contravariant space-time coordinates $x^\mu = (t, \mathbf{x})$ with x and conveniently use the units for which $c = 1$ and $\hbar = 1$. The differential operator P_μ will be used later when we discuss the desired relativistic wave equation.

If we now multiply the above relativistic dispersion relation (7) by the four-dimensional unit matrix, by means of (6), we obtain the algebraic matrix result:

$$E^2 1_4 + (\mathbf{K} \cdot \mathbf{p})^2 - (\mathbf{J} \cdot \mathbf{p})^2 = m^2 1_4. \tag{9}$$

This equation should then be linearized in the energy and momentum variables E and $\mathbf{p}$. For that purpose, we introduce three 2×2 Pauli-type matrices:

$$\lambda_0 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \lambda_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \lambda_2 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \tag{10}$$

which are real, obey $\lambda_{0,1}^2 = 1_2$ and $\lambda_2^2 = -1_2$, and mutually anticommute with each other. Thus, we can write the linearized dispersion equation as follows:

$$\lambda_0 1_4 E + \lambda_1 \mathbf{K} \cdot \mathbf{p} + \lambda_2 \mathbf{J} \cdot \mathbf{p} = m 1_8. \tag{11}$$

When squaring this equation and exploiting Equations (6) and (9), we retain the original dispersion relation (7) times the 8×8 unit matrix, which corresponds to the eight degrees of freedom obtained by the linearization procedure. Two of them belong to the lambda matrices. Their two dimensions relate to the two possible signs of the energy and, thus, correspond to the particle and antiparticle, as in the standard Dirac equation. The other four degrees of freedom stem from the space-time coordinates of Minkowski space and the Lorentz transformation. Their physical meaning will become clear below.

It is convenient and appropriate to rewrite Equation (11) in covariant form by introducing the subsequent Lambda matrices corresponding to Dirac's gamma matrices. We obtain

$$\Lambda_0 = \begin{pmatrix} 1_4 & 0 \\ 0 & -1_4 \end{pmatrix}, \quad \mathbf{\Lambda} = \begin{pmatrix} 0 & \mathbf{K} - \mathbf{J} \\ \mathbf{K} + \mathbf{J} & 0 \end{pmatrix}. \tag{12}$$

They obey $\Lambda_0^2 = 1_8$ and $\mathbf{\Lambda}^2 = -3\,1_8$, where use was made of Equations (4) and (5). The three space components and the single time component of the contravariant Lambda four-vector $\Lambda^\mu = (\Lambda_0, \mathbf{\Lambda})$ anticommute, which is obvious for Λ_0 and $\mathbf{\Lambda}$, and this follows for the x, y, z components after some lengthy calculations with the help of the Lorentz algebra of the three-vectors $\boldsymbol{J}$ and $\boldsymbol{K}$ after Equation (1). Therefore, as shown in the subsequent section, we then obtain the Clifford algebra for the Lambdas:

$$\Lambda^\mu \Lambda^\nu + \Lambda^\nu \Lambda^\mu = 2 g^{\mu\nu} 1_8, \tag{13}$$

from which it follows that one can take, so to speak, the root of (7) and obtain

$$\Lambda^\mu P_\mu = m 1_8. \tag{14}$$

Finally, by inserting the differential operator of Equation (8) here, we obtain the Dirac equation in a new non-standard form as

$$\Lambda^\mu \mathrm{i} \partial_\mu \Psi(x) = m \Psi(x). \tag{15}$$

The spinor wave function $\Psi(x)$ has eight components, of which two correspond to particles and antiparticles. The other four are related to their associated spin and isospin doublets, as shown in the next section.

We would like to emphasize that other equivalent forms of the Lambda four-vector $\Lambda^\mu = (\Lambda_0, \mathbf{\Lambda})$ are possible. They have been extensively discussed by Marsch and Narita [6]. In the present framework, there are essentially three options, which correspond to the possible ways in which the $\lambda_{0,1,2}$ matrices can occur in Equation (11). They are obtained through the cyclic permutation of the lambda positions in that equation. It was argued in [6,15] that these three possibilities reflect the physical fact that there exist exactly three families of fermions in the SM. The number three just corresponds to the three dimensions of the real physical space, which are revealed by the three Pauli matrices (or lambdas in our case) as the generators of the rotation group in its fundamental bi-spinor representation.

4. Spin and Rapidity

The aim of this section is to first calculate the spinorial analogs of the Lorentz group generators $\mathbf{J}$ and $\mathbf{K}$ in Minkowski space. All one needs to do is use the properties of the Clifford algebra (13). The cartesian spin x component is given by

$$S_{\mathrm{x}} = \frac{\mathrm{i}}{2} \Lambda_{\mathrm{y}} \Lambda_{\mathrm{z}}, \tag{16}$$

and the y and z components are obtained through cyclic index permutation. Multiplication of the three spin components gives

$$S_\mathrm{x} S_\mathrm{y} S_\mathrm{z} = (\frac{\mathrm{i}}{2})^3 \Lambda_\mathrm{y}\Lambda_\mathrm{z}\Lambda_\mathrm{z}\Lambda_\mathrm{x}\Lambda_\mathrm{x}\Lambda_\mathrm{y} = \frac{\mathrm{i}}{8} 1_8. \tag{17}$$

Similarly, we obtain

$$S_\mathrm{x} S_\mathrm{y} - S_\mathrm{y} S_\mathrm{x} = (\frac{\mathrm{i}}{2})^2 (\Lambda_\mathrm{y}\Lambda_\mathrm{z}\Lambda_\mathrm{z}\Lambda_\mathrm{x} - \Lambda_\mathrm{z}\Lambda_\mathrm{x}\Lambda_\mathrm{y}\Lambda_\mathrm{z}) = \mathrm{i} S_\mathrm{z}. \tag{18}$$

Thereby, we only used $\Lambda_j^2 = -1_8$ for $j = \mathrm{x, y, z}$ and the fact that the Lambdas anticommute. Moreover, one gets $S_\mathrm{x}^2 = \frac{1}{4} 1_8$ and the same result for the y and z components. In conclusion, here, we are dealing with a spin one-half fermion.

The rapidity or boost operator is adequately defined as

$$R_\mathrm{x} = \frac{\mathrm{i}}{2} \Lambda_0 \Lambda_\mathrm{x}, \tag{19}$$

from which it follows that

$$R_\mathrm{x} R_\mathrm{y} - R_\mathrm{y} R_\mathrm{x} = (\frac{\mathrm{i}}{2})^2 (\Lambda_0\Lambda_\mathrm{x}\Lambda_0\Lambda_\mathrm{y} - \Lambda_0\Lambda_\mathrm{y}\Lambda_0\Lambda_\mathrm{x}) = -\mathrm{i} S_\mathrm{z}. \tag{20}$$

Similarly, we also find that

$$R_\mathrm{x} S_\mathrm{y} - S_\mathrm{y} R_\mathrm{x} = (\frac{\mathrm{i}}{2})^2 (\Lambda_0\Lambda_\mathrm{x}\Lambda_\mathrm{z}\Lambda_\mathrm{x} - \Lambda_\mathrm{z}\Lambda_\mathrm{x}\Lambda_0\Lambda_\mathrm{x}) = \mathrm{i} R_\mathrm{z}. \tag{21}$$

As a result of all of these calculations, we obtain, in full analogy to Equation (1), the spinorial Lorentz algebra

$$\mathbf{S} \times \mathbf{S} = \mathrm{i}\mathbf{S}, \;\; \mathbf{R} \times \mathbf{R} = -\mathrm{i}\mathbf{S}, \;\; \mathbf{S} \times \mathbf{R} = \mathbf{R} \times \mathbf{S} = \mathrm{i}\mathbf{R}. \tag{22}$$

Let us then evaluate more explicitly the components of spin and rapidity matrix vectors. For the spin x component (the y and z components are obtained through cyclic index permutations), we get the matrix

$$S_\mathrm{x} = \frac{\mathrm{i}}{2} \begin{pmatrix} (K_\mathrm{y} - J_\mathrm{y})(K_\mathrm{z} + J_\mathrm{z}) & 0 \\ 0 & (K_\mathrm{y} + J_\mathrm{y})(K_\mathrm{z} - J_\mathrm{z}) \end{pmatrix}, \tag{23}$$

and for the rapidity, we obtain the matrix three-vector

$$\mathbf{R} = \frac{\mathrm{i}}{2} \begin{pmatrix} 0 & (\mathbf{K} - \mathbf{J}) \\ -(\mathbf{K} + \mathbf{J}) & 0 \end{pmatrix}. \tag{24}$$

At this point, we should explicitly calculate the involved matrices $\mathbf{K} \pm \mathbf{J}$. By using their expressions in Equations (A1) and (A2) of the Appendix A, with the definition $\mathbf{K} \pm \mathbf{J} = \mathrm{i}\boldsymbol{\lambda}^{\pm}$, we obtain the three real Lambda 4×4 matrices, which are also quoted in the Appendix A. In terms of these matrices, we can rewrite the spatial Lambda matrix vector in the form

$$\boldsymbol{\Lambda} = \mathrm{i} \begin{pmatrix} 0 & \boldsymbol{\lambda}^- \\ \boldsymbol{\lambda}^+ & 0 \end{pmatrix}. \tag{25}$$

The components (with $i, j = \mathrm{x, y, z}$) of the Lambda vector obey the important metric condition:

$$\lambda_i^- \lambda_j^+ + \lambda_j^- \lambda_i^+ = 2\delta_{ij} 1_4, \tag{26}$$

which guarantees that the spatial Lambda components obey the Clifford algebra. At this point, we can also define the chiral Lambda matrix:

$$\Lambda_5 = \mathrm{i}\Lambda_0\Lambda_x\Lambda_y\Lambda_z = \begin{pmatrix} 0 & \lambda_x^-\lambda_y^+\lambda_z^- \\ -\lambda_x^+\lambda_y^-\lambda_z^+ & 0 \end{pmatrix}. \quad (27)$$

Multiplying the above product of the lambdas out, one obtains

$$\Lambda_5 = \begin{pmatrix} 0 & \mathrm{i}\Delta \\ -\mathrm{i}\Delta & 0 \end{pmatrix}, \quad (28)$$

where the new diagonal matrix Delta corresponds to the metric in Minkowski space and reads as follows: $\Delta = \mathrm{diag}[1, -1, -1, -1]$, which obeys $\Delta^2 = 1_4$. This permits one to switch the sign of the superscripts attached to the lambdas because one finds that $\Delta\boldsymbol{\lambda}^{\pm}\Delta = -\boldsymbol{\lambda}^{\mp}$. In terms of the matrix vector lambda, the spin and rapidity can be written as

$$S_x = -\frac{\mathrm{i}}{2}\begin{pmatrix} \lambda_y^-\lambda_z^+ & 0 \\ 0 & \lambda_y^+\lambda_z^- \end{pmatrix}, \ \mathbf{R} = \frac{1}{2}\begin{pmatrix} 0 & -\boldsymbol{\lambda}^- \\ \boldsymbol{\lambda}^+ & 0 \end{pmatrix}. \quad (29)$$

Upon the insertion of the expressions of (25), we obtain the spin and rapidity vectors in the form

$$\mathbf{S} = \frac{1}{2}\begin{pmatrix} \boldsymbol{\Sigma}^- & 0 \\ 0 & \boldsymbol{\Sigma}^+ \end{pmatrix}, \ \mathbf{R} = \frac{\mathrm{i}}{2}\begin{pmatrix} 0 & \boldsymbol{\Sigma}^-\Delta \\ \boldsymbol{\Sigma}^+\Delta & 0 \end{pmatrix}. \quad (30)$$

The connection of the Sigmas to the previously used vector lambdas is as follows: $\boldsymbol{\lambda}^{\pm} = \pm\mathrm{i}\boldsymbol{\Sigma}^{\pm}\Delta$. These matrices are defined in terms of $\lambda_{0,1,2}$ and are quoted in the Appendix A. Both of the Pauli-type (but 4×4) matrices that appear here obey $\boldsymbol{\Sigma}^{\pm} \times \boldsymbol{\Sigma}^{\pm} = 2\mathrm{i}\boldsymbol{\Sigma}^{\pm}$, yet for opposite superscripts, they commute component-wise with each other: $[\boldsymbol{\Sigma}^{\pm}, \boldsymbol{\Sigma}^{\mp}] = 0$. Moreover, we get $\Sigma_x^{\pm}\Sigma_y^{\pm}\Sigma_z^{\pm} = \mathrm{i}1_4$. One finds that $\Delta\boldsymbol{\Sigma}^{\pm}\Delta = \boldsymbol{\Sigma}^{\mp}$, which is a relation that is very useful for validating the commutation relations of the spin and isospin addressed below. Finally, note that $(\boldsymbol{\Sigma}^{\pm}\Delta)^{\dagger} = \boldsymbol{\Sigma}^{\mp}\Delta$. This is required to show that $-\mathrm{i}\mathbf{R}$ is hermitian.

Using the Sigma matrices is appropriate to rewrite the previous Lambda matrices of Equation (12) in a new form. We obtain

$$\Lambda_0 = \begin{pmatrix} 1_4 & 0 \\ 0 & -1_4 \end{pmatrix}, \ \boldsymbol{\Lambda} = \begin{pmatrix} 0 & \boldsymbol{\Sigma}^-\Delta \\ -\boldsymbol{\Sigma}^+\Delta & 0 \end{pmatrix}, \ \Lambda_5 = \begin{pmatrix} 0 & \Delta \\ \Delta & 0 \end{pmatrix}. \quad (31)$$

This version of the extended Dirac equation is formally the same as that derived recently by Marsch and Narita [6], yet it differs slightly in the definition of the Sigma matrices. However, their way of obtaining this result was also somewhat different. In the Appendix A of their paper, they provided six equivalent versions of Equation (31), which are all connected by similarity transformations.

5. Isospin and $SU(2)$

Using the results on the Sigma matrices of the previous section, we are now in the position to define a new entity, the isospin, as follows:

$$\tilde{\mathbf{S}} = \frac{1}{2}\begin{pmatrix} \boldsymbol{\Sigma}^+ & 0 \\ 0 & \boldsymbol{\Sigma}^- \end{pmatrix}. \quad (32)$$

This obviously commutes with the spin because Sigma matrices with opposite superscripts commute. However, it is less trivial to show that $[\tilde{\mathbf{S}}, \boldsymbol{\Lambda}] = 0$, yet when using the algebraic properties of $\lambda_{0,1,2}$, with some lengthy matrix multiplications, one can show that this is the case. Moreover, we find more easily that $[\mathbf{S}, \Lambda_0] = [\tilde{\mathbf{S}}, \Lambda_0] = 0$, and therefore, S_x, $\tilde{S}_x$, and Λ_0 can have common eigenfunctions in the rest frame of the fermion. Consequently, the multiplet state Ψ can, in that frame, be fully determined by the quantum numbers of those matrix operators. Since their squares are proportional to the unit matrix 1_8, according to Equations (12), (A7), and (A8), we obtain the eigenvalues $E = \pm m$ for the energies of the Dirac equation (14) in the fermion rest frame corresponding to particles and antiparticles. The values $\pm\frac{1}{2}$ for the spin (or isospin) are obtained as evaluated by means of their block-diagonal x components, which correspond to the physical spin up and down states and to the up and down components of the isospin doublet. Marsch and Narita [6] extensively discussed a possible gauge theory based on the isospin (32) to which we refer and which, therefore, shall not be elucidated any further here.

6. Intrinsic Spin and $SU(3)$

In their seminal work, Wigner [9] and Bargman and Wigner [10] emphasized the important role played by the two Casimir operators of the Poincaré and Lorentz group, namely, the four-momentum squared (7) and the square of the Pauli–Lubański operator $W^\mu = (\mathbf{I}\cdot\mathbf{p}, E\mathbf{I} + \mathrm{i}\mathbf{I}\times\mathbf{p})$, which involves the intrinsic spin $\mathbf{I}$ of the particle, where spin just means a physical quantity that obeys the usual angular momentum algebra

$$\mathbf{I}\times\mathbf{I} = \mathrm{i}\mathbf{I} \tag{33}$$

The square of the Pauli–Lubański operator is the product of the squared mass with $\mathbf{I}^2$, and thus, for a massive particle, it reads:

$$-W^\mu W_\mu = m^2\mathbf{I}^2 = (E^2 - \mathbf{p}^2)\mathbf{I}^2 \tag{34}$$

Here, the squares of the space-time Fourier variables E and $\mathbf{p}$ and the intrinsic isospin three-vector $\mathbf{I}$ are connected in a simple multiplicative way, which is consistent with the constraint placed by the Coleman–Mandula theorem [16]. The Casimir operators turn out to be relevant in the derivation of covariant and first-order (with respect to the derivatives) relativistic wave equations for massive charged particles of arbitrary intrinsic spin $\mathbf{I}$.

It was Dirac [14] who noticed that the first Casimir operator (7) can be written as the square of an expression that is linear P_μ with the help of the famous contravariant gamma matrices that he introduced. Here, we have extended his approach to include the $SU(2)$-isospin related to the Lorentz group in the Lambda matrices of (12), which leads to (14). Furthermore, Dirac [17] later also noticed how one can write the square of the intrinsic spin $\mathbf{I}$ with the quantum number l as the following product:

$$1_2\mathbf{I}^2 = (\boldsymbol{\sigma}\cdot\mathbf{I})(\boldsymbol{\sigma}\cdot\mathbf{I} + 1_{2(2l+1)}) = 1_{2(2l+1)}l(l+1), \tag{35}$$

which uses the concept of what we now call [2] spinor helicity $\boldsymbol{\sigma}\cdot\mathbf{I}$. To define it, we use the standard Pauli matrix three-vector $\boldsymbol{\sigma}$. We can thus express the three-vector $\mathbf{I}$ as a bi-spinor in 2×2-matrix form. Then, we obtain the following result for the spinor helicity:

$$\boldsymbol{\sigma}\cdot\mathbf{I} = \begin{pmatrix} I_z & I_x - \mathrm{i}I_y \\ I_x + \mathrm{i}I_y & -I_z \end{pmatrix}. \tag{36}$$

It is important to note that for any intrinsic spin operator $\mathbf{I}$ in its standard representation, the x and z components are real, and the y component is purely imaginary. Consequently, the above spinor helicity is a real $(2(2l+1)\times 2(2l+1))$-matrix operator and has real eigenvalues. We abbreviate the two operators involved in (35) as

$$H_0(l) = (\boldsymbol{\sigma}\cdot\mathbf{I})/l, \quad H_1(l) = (\boldsymbol{\sigma}\cdot\mathbf{I} + 1_{2(2l+1)})/(l+1). \tag{37}$$

It is obvious that $[H_0(l), H_1(l)] = 0$. Consequently, we can rewrite, with the help of (14) and (35), the Pauli–Lubański operator (34) in an algebraically new form as

$$(\Lambda^\mu P_\mu)^2 H_0(l) H_1(l) = m^2 1_8 1_{2(2l+1)}. \tag{38}$$

We recall that all operators appearing in this equation commute with each other. In addition to the eight kinetic degrees of freedom associated with the particle–antiparticle doublet, the spin up and down doublet, and the isospin up and down doublet, we now have the spinor helicity multiplet of $\mathbf{I}$ with $2(2l+1)$ degrees of freedom. This is reflected in the product of the unit matrices at the mass term.

However, in what follows, we shall restrict the discussion to $l = 1/2$, i.e., we shall consider only two intrinsic configurational or species degrees of freedom. Empirically, the spin-one-half fermions come in only two species, namely, as leptons and quarks, which we can accommodate in the doublet given by $\mathbf{I} = \frac{1}{2}\boldsymbol{\sigma}$. It was shown by Marsch and Narita [7] that the operators $H_{0,1}$ can, for $l = 1/2$, be brought into diagonal forms based on their four common orthogonal eigenfunctions, and thus, they take the simple form

$$H_0 = \mathrm{diag}[1, 1, 1, -3], \quad H_1 = \left[1, 1, 1, -\frac{1}{3}\right]. \tag{39}$$

It is interesting that H_0 is apart from the normalization identical to the fifteenth element of the $SU(4)$ Lie group. H_0 can be considered as the hypercharge matrix operator of a unified lepton–quark gauge theory, which was discussed by Marsch and Narita [18] in their study of the connections existing in the Dirac equation between the Clifford algebra of Lorentz invariance and the Lie algebra of $SU(N)$ gauge symmetry. Apparently, the state space of the spinor helicity operator H_0 decomposes into two orthogonal Hilbert spaces with dimensions of three and one. The corresponding wave equation applies to the four-component super-spinor $\Phi^\dagger = (\Psi_1^\dagger, \Psi_2^\dagger, \Psi_3^\dagger, \Psi_4^\dagger)$, where the spinor fields Ψ_j are solutions of the extended Dirac wave Equation (15). By using Equation (37), we can thus write a new second-order (in P_μ) wave equation for Φ as follows:

$$(\Lambda^\mu P_\mu)^2 H_0(l) H_1(l) \Phi = m^2 \Phi, \tag{40}$$

which can, again, be decomposed into two linear wave equations:

$$\begin{aligned} H_0 \Lambda^\mu P_\mu \Phi_0 &= m \Phi_1 \\ H_1 \Lambda^\mu P_\mu \Phi_1 &= m \Phi_0. \end{aligned} \tag{41}$$

We recall that all operators commute with each other and relate to the 32 independent inner and kinetic degrees of freedom. Close inspection of the above two equations shows that they are, in fact, identical if we choose $\Psi_{0,4} = -\frac{1}{3}\Psi_{1,4}$ and $\Psi_{0,j} = \Psi_{1,j}$ for $j = 1, 2, 3$. $H_{0,1}$ then just act as unit matrices in their respective subspaces, and in this way, we obtain two entirely decoupled Dirac equations. One has three intrinsic degrees of freedom, which we associate with the three colors of the quarks, and the other has one intrinsic degree of freedom, which we associate with the “single color" of the lepton. So, we write the corresponding Dirac equations as

$$\begin{aligned} \Lambda^\mu P_\mu \tilde{\Phi}_\mathrm{q} &= m \tilde{\Phi}_\mathrm{q} \\ \Lambda^\mu P_\mu \tilde{\Phi}_\mathrm{l} &= m \tilde{\Phi}_\mathrm{l}. \end{aligned} \tag{42}$$

Here, the spinors are $\tilde{\Phi}_\mathrm{l} = \Psi_\mathrm{l}$ for the lepton and $\tilde{\Phi}_\mathrm{q}^\dagger = (\Psi_\mathrm{r}^\dagger, \Psi_\mathrm{g}^\dagger, \Psi_\mathrm{b}^\dagger)$ for the quarks, with the usual color indices of red, green, and blue being used as in the SM. Of course, the related symmetry gauge groups are $U(1)$ for the leptons and $SU(3)$ for the quarks. In conclusion, each of the four fields associated with the four degrees of freedom of the spinor helicity obeys the extended Dirac Equation (15) that yields the $SU(2)$ symmetry for the isospin doublet. Thus, we have covered all three known fundamental fermion symmetries of the SM. The color symmetry is a consequence of the Pauli–Lubański operator involving the simplest possible isospin with quantum number $l = 1/2$, corresponding to a fermion doublet, which splits into four states by means of the spinor helicity mechanism being applied to $\mathbf{I}^2$ according to (35).

7. Symmetry Unification Schemes

In the previous section, we showed that the simplest possible internal isospin $\mathbf{I}$ for a fermion doublet with a quantum number of $l = \frac{1}{2}$ yields, via the spinor helicity formalism, a quadruplet assembling three colored quarks and a single lepton into a four-component super-spinor, $\tilde{\Psi}^\dagger = (\Psi_r^\dagger, \Psi_b^\dagger, \Psi_g^\dagger, \Psi_l^\dagger)$, where each of the four spinor fields obeys the extended Dirac wave Equation (15). So, we can concisely rewrite the unified extended Dirac equation as

$$\Lambda^\mu P_\mu \tilde{\Psi} = m\tilde{\Psi}. \tag{43}$$

This equation has the symmetry $SU(2)$ stemming from Lorentz invariance, which determines the four-vector Λ^μ, and the symmetry $SU(4)$ stemming from the spinor helicity $\boldsymbol{\sigma} \cdot \mathbf{I}$ of (36), which is associated with the internal isospin. It formally describes a quark–lepton couple that splits into a quadruplet involving a single lepton (with symmetry $U(1)$) and three colored quarks (with symmetry $SU(3)$). As a result, we obtain the unified fermion symmetry $SU(8) = SU(2) \otimes SU(4)$. Such a unification model was developed before by Marsch and Narita [19,20], but on the basis of combinatorial symmetries of the standard Dirac equation, which comes in two main versions, as evaluated on the basis of Dirac and Weyl.

The $SU(8)$ symmetry suggested here is equivalent to the symmetry of the orthogonal $SO(10)$ group as described by Fritzsch [21,22]. The related 16-component spinor representation of $SO(10)$ includes all spin one-half fermions of the first generation in the SM. The symmetry $SU(4)$ was proposed long ago by Pati and Salam [23,24], who considered the lepton number as the fourth color, which was long then an ad hoc assumption, whereas here, we give a good physical reason for such a quadruplet. It is based on the spinor helicity of an intrinsic spin one-half in connection with the Pauli–Lubański operator, which is the second Casimir operator of the Lorentz group.

However, the unification scheme employing $SU(4)$ requires 15 gauge fields, which are sometimes called leptoquarks, to link leptons and quarks in the multiplet. There is no observational evidence for the related gauge bosons, and therefore, it is desirable to reduce their number. Such a reduction is offered by the use of the less complex $SO(4)$ instead of $SU(4)$ in the spirit of Occam's razor. We suggest installing the $SO(4)$ symmetry, which requires nine fewer gauge fields than $SU(4)$. The mathematical reason is that the spinor helicity multiplet involves a real operator acting in the Euclidian space spanned by the four simplest possible and real eigenfunctions $\phi_1^\dagger = (1,0,0,0)$, $\phi_2^\dagger = (0,1,0,0)$, $\phi_3^\dagger = (0,0,1,0)$ and $\phi_4^\dagger = (0,0,0,1)$. The six purely imaginary matrices of the $SO(4)$ generators are quoted in the Appendix A. We may assemble them into two three-vectors and name them $\mathbf{M} = (M_1, M_2, M_3)$ or $\mathbf{N} = (N_1, N_2, N_3)$. They obey the linked algebra

$$\mathbf{M} \times \mathbf{M} = \mathrm{i}\mathbf{M}, \;\; \mathbf{N} \times \mathbf{N} = \mathrm{i}\mathbf{M}, \;\; \mathbf{M} \times \mathbf{N} = \mathbf{N} \times \mathbf{M} = \mathrm{i}\mathbf{N}. \tag{44}$$

This algebra resembles the Lorentz algebra of Equation (1), aside from a minus sign. Of course, the group $SO(4)$ includes $SO(3)$ as a subgroup. Using it instead of $SU(3)$ means that the symmetry of the strong SM interactions would be considerably simplified and would not require eight gluons, but just three new gauge bosons. Admittedly, this is a rather speculative new scheme that needs further scientific investigation.

8. Summary and Conclusions

In this paper, we outlined a new route to the extended Dirac equation and its symmetries on the basis of the four-vector representation of the Lorentz group (LG). It turned out that the $SU(2)$ symmetry emerges naturally via the linearization of the first Casimir operator of the LG and is intimately connected with the isospin. This is an outcome of the chiral nature of the LG. Similarly, the intrinsic isospin related to the second Casimir operator can be linearized by means of the spinor helicity formalism, which yields two independent singlet and triplet fermion states. We suggest interpreting them as being related to the symmetry $U(1)$ and the lepton or to the symmetry $SU(3)$ and the three colored quarks. These results indicate the very different origins and physical natures of these basic symmetries of the SM, but they do not genuinely support the idea of grand unification. However, when combining them in the product group $SU(4) = SU(3) \otimes U(1)$, and then by combining all groups into $SU(2) \otimes SU(4)$, one gets a combined symmetry scheme that seems to support unification by the group $SU(8)$. It is found that the smaller $SO(4)$ group, instead of $SU(4)$, also seems appropriate for achieving unification, and it offers the advantage that it simplifies the theory and reduces the number of gauge fields required.

Concerning the basic wave Equation (40) based on the second Casimir operator, we stress again that the space-time differential operator P_μ, the algebraic operators Λ^μ of the Clifford algebra related to the Lorentz group, and, finally, the operators $H_{0,1}$ related to the $SO(4)$ symmetry of the intrinsic spin $\mathbf{I}$ do all commute with each other, and therefore, their sequence when operating on the state spinor Φ does not matter. Its full dimension is $8 \times 2 \times (2l+1)$ according to Equation (40). At this point, we are just dealing with a Klein–Gordon equation for each of the many components of Φ. However, when we linearize that equation with respect to P_μ, we have to use the matrix representations of the algebraic operators. They are then connected through tensor multiplication, where their sequence does matter.

We have two options for constructing the resulting extended Dirac equation. Either the intrinsic spin operator acts first from the left on Φ and is followed the space-time operators, or vice versa. In the first case, the field equation looks like a single Dirac equation with many internal degrees of freedom; in the second case, it looks like a multiplet of Dirac spinors assembled in an N-tuple of the $SU(N)$ symmetry group in matrix representation. The latter case gives the standard picture of the Yang–Mills theory. The mathematical connections between the two approaches were extensively discussed in [18]. The way that is is advantageous needs to be worked out and may depend on the symmetry group involved. The analysis of this problem is beyond the scope of this paper.

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Appendix A. Lorentz Group, SO(4), and the Weyl Basis

Appendix A.1. Lorentz Group Matrices

In this subsection, we compose some of the relevant matrices of the key physical parameters. For the generators of the Lorentz group, we have the following 4×4 matrices. For the component matrices of $\mathbf{J}$, we obtain

$$J_\mathrm{x} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -\mathrm{i} \\ 0 & 0 & \mathrm{i} & 0 \end{pmatrix}, J_\mathrm{y} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \mathrm{i} \\ 0 & 0 & 0 & 0 \\ 0 & -\mathrm{i} & 0 & 0 \end{pmatrix}, J_\mathrm{z} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -\mathrm{i} & 0 \\ 0 & \mathrm{i} & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}. \quad \text{(A1)}$$

The component matrices of $\mathbf{K}$ are also quoted here:

$$K_\mathrm{x} = \begin{pmatrix} 0 & -\mathrm{i} & 0 & 0 \\ -\mathrm{i} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, K_\mathrm{y} = \begin{pmatrix} 0 & 0 & -\mathrm{i} & 0 \\ 0 & 0 & 0 & 0 \\ -\mathrm{i} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, K_\mathrm{z} = \begin{pmatrix} 0 & 0 & 0 & -\mathrm{i} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -\mathrm{i} & 0 & 0 & 0 \end{pmatrix}. \quad \text{(A2)}$$

By using the above matrix vectors, with the definition $\mathbf{K} \pm \mathbf{J} = \mathrm{i}\boldsymbol{\lambda}^{\pm}$, we obtain three new and real lambda 4×4 matrices. In terms of these matrices, we can rewrite the spatial Lambda matrix vector in the form

$$\lambda_{\mathrm{x}}^{\pm} = \begin{pmatrix} 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & \mp 1 \\ 0 & 0 & \pm 1 & 0 \end{pmatrix}, \lambda_{\mathrm{y}}^{\pm} = \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & \pm 1 \\ -1 & 0 & 0 & 0 \\ 0 & \mp 1 & 0 & 0 \end{pmatrix}, \lambda_{\mathrm{z}}^{\pm} = \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & \mp 1 & 0 \\ 0 & \pm 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}. \tag{A3}$$

Here, we first quote the Lambda matrices with negative superscripts in 2×2 matrix block form as

$$\lambda_{\mathrm{x}}^{-} = \begin{pmatrix} -\lambda_1 & 0 \\ 0 & -\lambda_2 \end{pmatrix}, \ \lambda_{\mathrm{y}}^{-} = \begin{pmatrix} 0 & -1_2 \\ -\lambda_0 & 0 \end{pmatrix}, \ \lambda_{\mathrm{z}}^{-} = \begin{pmatrix} 0 & \lambda_2 \\ -\lambda_1 & 0 \end{pmatrix}. \tag{A4}$$

Similarly, for the lambdas with positive superscripts, we obtain the following results:

$$\lambda_{\mathrm{x}}^{+} = \begin{pmatrix} -\lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}, \ \lambda_{\mathrm{y}}^{+} = \begin{pmatrix} 0 & -\lambda_0 \\ -1_2 & 0 \end{pmatrix}, \ \lambda_{\mathrm{z}}^{+} = \begin{pmatrix} 0 & -\lambda_1 \\ -\lambda_2 & 0 \end{pmatrix}. \tag{A5}$$

Moreover, in this way, the Delta matrix can also be written as

$$\Delta = \begin{pmatrix} \lambda_0 & 0 \\ 0 & -1_2 \end{pmatrix}. \tag{A6}$$

Finally, we quote here the spin matrices for the Sigmas with negative superscripts:

$$\Sigma_{\mathrm{x}}^{-} = \mathrm{i}\begin{pmatrix} -\lambda_2 & 0 \\ 0 & \lambda_2 \end{pmatrix}, \ \Sigma_{\mathrm{y}}^{-} = \mathrm{i}\begin{pmatrix} 0 & 1_2 \\ -1_2 & 0 \end{pmatrix}, \ \Sigma_{\mathrm{z}}^{-} = \mathrm{i}\begin{pmatrix} 0 & -\lambda_2 \\ -\lambda_2 & 0 \end{pmatrix}. \tag{A7}$$

Similarly, for the Sigmas with positive superscripts, we obtain the following result:

$$\Sigma_{\mathrm{x}}^{+} = \mathrm{i}\begin{pmatrix} \lambda_2 & 0 \\ 0 & \lambda_2 \end{pmatrix}, \ \Sigma_{\mathrm{y}}^{+} = \mathrm{i}\begin{pmatrix} 0 & -\lambda_0 \\ \lambda_0 & 0 \end{pmatrix}, \ \Sigma_{\mathrm{z}}^{+} = \mathrm{i}\begin{pmatrix} 0 & -\lambda_1 \\ \lambda_1 & 0 \end{pmatrix}. \tag{A8}$$

Appendix A.2. $SO(4)$ Group Matrices

The symmetry group $SO(4)$ describes the possible rotations around the four axes of the real Euclidian space of four dimensions, which is the space of the spinor helicity that has four real orthogonal eigenfunctions spanning that space. The associated unitary and purely imaginary matrices with zero traces can be written as follows:

$$M_1 = \mathrm{i}\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, M_2 = \mathrm{i}\begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, M_3 = \mathrm{i}\begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}. \tag{A9}$$

Together, they form the $SO(3)$ subgroup of $SO(4)$ and obey the angular momentum algebra. The three remaining matrices link the fourth dimension with the three other ones and read

$$N_1 = \mathrm{i}\begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}, N_2 = \mathrm{i}\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}, N_3 = \mathrm{i}\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix}. \tag{A10}$$

Any matrix of the SO(4) group can be represented by a linear combination of M_j and N_j with j running from 1 to 3. Therefore, any element G of the SO(4) Lie group can be written as

$$G = \exp(\mathrm{i}\sum_{j=1}^{3} M_j\alpha_j + \mathrm{i}\sum_{j=1}^{3} N_j\beta_j). \tag{A11}$$

Thus, G is real and represents the general phase factor involving six real numbers $\alpha_{1,2,3}$ and $\beta_{1,2,3}$ corresponding to two sets of rotation angles.

Appendix A.3. The Extended Dirac Equation on the Weyl Basis

In this subsection of the appendix, we will transform the extended Dirac equation from the Dirac into the Weyl basis. For that end, we rewrite the Lambda matrices and the Delta matrix and quote them in 2×2 block form, which is convenient for algebraic manipulations. They are quoted in the previous subsection of the appendix. Equations (12), (25), and (28) correspond to the extended Dirac equation on the Dirac basis. With a similarity transformation, one can readily change to the Weyl basis. With the help of the unitary ($V^{-1} = V^{\dagger}$) transformation

$$V = \frac{1}{\sqrt{2}}(1 + \mathrm{i}\Lambda_5) = \frac{1}{\sqrt{2}}\begin{pmatrix} 1_4 & -\Delta \\ \Delta & 1_4 \end{pmatrix}, \tag{A12}$$

one retains (25) for $\mathbf{\Lambda}$, but transforms Λ_0, which now reads

$$\Lambda_0 = \begin{pmatrix} 0 & \Delta \\ \Delta & 0 \end{pmatrix}. \tag{A13}$$

The Weyl basis is particularly convenient in the case of a vanishing mass m. On the Weyl basis, one obtains the new extended Dirac equation in the form

$$\left(\mathrm{i}\begin{pmatrix} 0 & \Delta \\ \Delta & 0 \end{pmatrix}\frac{\partial}{\partial t} - \begin{pmatrix} 0 & \boldsymbol{\lambda}^- \\ \boldsymbol{\lambda}^+ & 0 \end{pmatrix} \cdot \frac{\partial}{\partial \mathbf{x}}\right)\Psi = m\Psi. \tag{A14}$$

By squaring this equation, one retains the Klein–Gordon equation for each component of the spinor field Ψ. Inserting the standard plane-wave solution, one obtains the dispersion relation

$$\Delta^2 E^2 - (\boldsymbol{\lambda}^+ \cdot \mathbf{p})(\boldsymbol{\lambda}^- \cdot \mathbf{p}) = m^2 1_4, \tag{A15}$$

which is equivalent to the initial ones of Equations (7) and (9), if we use the fact that $\Delta^2 = 1_4$ and exploit the metric property (26). This yields $(\boldsymbol{\lambda}^+ \cdot \mathbf{p})(\boldsymbol{\lambda}^- \cdot \mathbf{p}) = (\boldsymbol{\lambda}^- \cdot \mathbf{p})(\boldsymbol{\lambda}^+ \cdot \mathbf{p}) = \mathbf{p}^2 1_4$. One can rewrite Equation (A14) in terms of separate equations for the two components of the spinor field $\Psi^{\dagger} = (\Psi_+^{\dagger}, \Psi_-^{\dagger})$. This yields

$$\begin{array}{l} \mathrm{i}\left(\Delta\frac{\partial}{\partial t} + \mathrm{i}\boldsymbol{\lambda}^- \cdot \frac{\partial}{\partial \mathbf{x}}\right)\Psi_- = m\Psi_+ \\ \mathrm{i}\left(\Delta\frac{\partial}{\partial t} + \mathrm{i}\boldsymbol{\lambda}^+ \cdot \frac{\partial}{\partial \mathbf{x}}\right)\Psi_+ = m\Psi_- \end{array}. \tag{A16}$$

Let us define $\mathbf{\Sigma} = \mathrm{i}\Delta\boldsymbol{\lambda}^-$. It obeys $\Sigma_x\Sigma_y\Sigma_z = \mathrm{i}1_4$ and, thus, the angular momentum algebra $\mathbf{\Sigma} \times \mathbf{\Sigma} = 2\mathrm{i}\mathbf{\Sigma} = 0$. It can be expressed in terms of tensor products of the three Pauli matrices in the form

$$\mathbf{\Sigma} = (1_2 \otimes \sigma_y, \sigma_y \otimes \sigma_z, \sigma_y \otimes \sigma_x). \tag{A17}$$

Close inspection of Equations (A17) and (A8) shows that $\mathbf{\Sigma}$ is, in fact, identical to the previous $\mathbf{\Sigma}^+$ in the spin operator (30). Similarly, we find that $-\mathrm{i}\boldsymbol{\lambda}^+\Delta$ is identical to the previous $\mathbf{\Sigma}^-$ in the spin operator. We now redefine the fields as $\tilde{\Psi}_- = \Psi_-$ and $\tilde{\Psi}_+ = \Delta\Psi_+$ and introduce the contravariant Sigma matrices $\Sigma_{\pm}^{\mu} = (1_4, \pm\mathbf{\Sigma})$. By using the fact that

$\Delta\lambda^{\pm} + \lambda^{\mp}\Delta = 0$, we can combine the two equations (A16) into a single concise one, which reads

$$\Sigma^{\mu}_{\pm} i\partial_{\mu}\tilde{\Psi}_{\mp} = m\tilde{\Psi}_{\pm}. \quad \text{(A18)}$$

For a massless fermion $(m = 0)$, the two Weyl fields decouple into independent left- and right-chiral fields with four degrees of freedom (particle/antiparticle and isospin doublets), and their dispersion relation is obtained from the requirement for a solution to exist, which yields

$$\det(\Sigma^{\mu}_{\pm} p_{\mu}) = (E^2 - \mathbf{p}^2)^2 = 0. \quad \text{(A19)}$$

The twofold degeneration corresponds to the two doublets involved. With this result, we conclude the subsection on the Weyl equations as derived from the extended Dirac equation.

References

1. Kaku, M. *Quantum Field Theory, A Modern Introduction*; Oxford University Press: New York, NY, USA, 1993.
2. Schwartz, M.D. *Quantum Field Theory and the Standard Model*; Cambridge University Press: Cambridge, UK, 2014.
3. Yang, C.N.; Mills, R.L. Conservation of isotopic spin and isotopic gauge invariance. *Phys. Rev.* **1954**, *96*, 191. [CrossRef]
4. Weinberg, S. A model of leptons. *Phys. Rev. Lett.* **1967**, *19*, 21. [CrossRef]
5. Gell-Mann, M. A schematic model of baryons and mesons. *Phys. Letts.* **1964**, *8*, 214. . [CrossRef]
6. Marsch, E.; Narita, Y. On isospin and flavour of leptons and quarks. *Eur. Phys. J. Plus* **2020**, *137*, 1353. [CrossRef]
7. Marsch, E.; Narita, Y. Lorentz invariance and the spinor helicity formalism yield the U(1) and SU(3) symmetry. *Eur. Phys. J. Plus* **2022**, *137*, 818. [CrossRef]
8. Hestenes, D. Real spinor fields. *J. Math. Phys.* **1967**, *8*, 798–808. [CrossRef]
9. Wigner, E. On unitary representations of the inhomogeneous Lorentz group. *Ann. Math. Second. Ser.* **1939**, *40*, 149. [CrossRef]
10. Bargman, V.; Wigner, E. Group theoretical discussion of relativistic wave equations. *Proc. Natl. Acad. Sci. USA* **1948**, *34*, 211. [CrossRef] [PubMed]
11. Joos, H. Zur Darstellungstheorie der inhomogenen Lorentzgruppe als Grundlage quantenmechanischer Kinematik. *Fortschritte Phys.* **1962**, *10*, 65. [CrossRef]
12. Jackson, J.D. *Classical Electrodynamics*; John Wiley and Sons, Inc.: New York, NY, USA, 2015.
13. Marsch, E.; Narita, Y. Dirac equation based on the vector representation of the Lorentz group. *Eur. Phys. J. Plus* **2020**, *135*, 782. [CrossRef]
14. Dirac, P.A.M. The quantum theory of the electron. *Proc. R. Soc. Lond. Ser. A Math. Phys. Sci.* **1928**, *117*, 610–624. [CrossRef]
15. Marsch, E. Fermion colour and flavour originating from multiple representations of the Lorentz group and Clifford algebra. *Phys. Sci. Intern. J.* **2019**, *23*, 1–13. [CrossRef]
16. Coleman, S.; Mandula, J. All possible symmetries of the S matrix. *Phys. Rev.* **1967**, *159*, 1251. [CrossRef]
17. Dirac, P.A.M. Relativistic wave equations. *Proc. R. Soc. Lond.* **1936**, *155*, 447–459. [CrossRef]
18. Marsch, E.; Narita, Y. Connecting in the Dirac equation the Clifford algebra of Lorentz invariance with the Lie algebra of $SU(N)$. *Symmetry* **2021**, *13*, 475. [CrossRef]
19. Marsch, E.; Narita, Y. Fermion unification model based on the intrinsic SU(8) symmetry of a generalized Dirac equation. *Front. Phys.* **2015**, *3*, 82. [CrossRef]
20. Marsch, E.; Narita, Y. Fundamental fermion interactions via vector bosons of unified SU(2) $\otimes$ SU(4) gauge fields. *Front. Phys.* **2016**, *4*, 5. [CrossRef]
21. Fritzsch, H.; Minkowski, P. Unified interactions of leptons and hadrons. *Ann. Phys.* **1975**, *93*, 193. [CrossRef]
22. Fritzsch, H. *Quantenfeldtheorie–Wie Man Beschreibt, Was Die Welt im Innersten Zusammenhält*; Springer Spektrum: Heidelberg, Germany, 2015.
23. Pati, J.C.; Salam, A. Unified lepton-hadron symmetry and a gauge theory of the basic interactions. *Phys. Rev. D* **1973**, *8*, 1240–1261. [CrossRef]
24. Pati, J.C.; Salam, A. Lepton number as the fourth "color". *Phys. Rev. D* **1974**, *10*, 275–289. [CrossRef]