COMPOSITE PARTICLE COLLISIONS IN THE NON-RELATIVISTIC THREE-BODY THEORY†

By

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1. INTRODUCTION

The composite particle collision problem is evidently of great importance in many fields of quantum physics. In the last eight years or so this problem has been studied with increasing interest, in particular for the case of three nonrelativistic elementary particles, i.e., for the scattering of an elementary particle by a two-particle bound-state (for instance, a nucleon by a deuteron).

Most of the recent investigations have been initiated by the mathematical work of Faddeev [1], [2] on the one hand, and by the practical success of three-body calculations in the separable potential model on the other hand. (Such calculations were performed, in particular, by Mitra [3] and by the group: Aaron, Amado and Yam [5], [4] . It was Lovelace [6],[7] who first applied the separable potential model to the Faddeev equations. Thereby many aspects of the problem became rather transparent.

From a general point of view the three-body problem re-

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P. Urban (ed.), *Particle Physics*

presents a first non-trivial (but still sufficiently simple) example of strong interaction dynamics. For, incoming and outgoing particles of complex structure can be studied which during the collision time may be rearranged to various other bound-states and resonances. A detailed study of the problem should, therefore, lead to an improvement of nuclear reaction theories, but it might also serve as a guide to a deeper understanding of some typical structures of elementary particle physics. In fact, a great part of the present interest in this field is induced by this hope. But let me say that in the latter respect the results are not very convincing up to now.

2. THE SINGLE-CHANNEL CASE

Before going to the three-body theory we have to recall some aspects of the single-channel problem. Here one studies the collision of two elementary particles with relative momenta $\vec{p}$ and $\vec{p}'$ (before and after the collision):

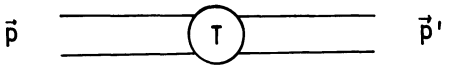


Fig. 1

What we need for the description of such a process is, of course, the scattering amplitude $T(\vec{p}',\vec{p};\, E)$, which is given by the integral equation

$$T(\vec{p}',\vec{p};E) = V(\vec{p}',\vec{p}) - \int d^3k\; V(\vec{p}',\vec{k})\frac{1}{\frac{k^2}{2\mu} - E - io}T(\vec{k},\vec{p};E) \qquad (2.1)$$

known as the Lippmann-Schwinger (L-S) equation for the T-matrix. (The "-io" in the denominator of the free

Green function indicates how we have to integrate around the pole at $k^2/2\mu = E$.) In operator form this equation simply reads (see also Fig. 2) :

$$T(E) = V - V\, G_o(E+io)\, T(E) \tag{2.2}$$

with

$$\langle \vec{p}' | T(E) | \vec{p} \rangle = T(\vec{p}', p; E) \tag{2.3}$$

and with the "free Green function"

$$G_o(E + io) = \frac{1}{H_o - E - io} \tag{2.4}$$

Fig. 2

The solution of the above L-S equation leads to the cross section

$$\frac{d\sigma}{d\Omega} = (2\pi)^4\, \mu^2 |T(\vec{p}', \vec{p}; E)|^2 \quad . \tag{2.5}$$

Due to energy conservation we have the ("on shell") condition

$$\frac{p'^2}{2\mu} = p^2/2\mu = E \quad . \tag{2.6}$$

This means, that for the calculation of $d\sigma/d\Omega$ we only need the scattering amplitude "on-the-energy-shell", while the L-S equation yields an "off-shell" extension of this physical amplitude (this follows easily from the fact that in the L-S equation (2.1) we have to integrate over all values of $\vec{k}$, without any restriction of the form $k^2=p^2=2\mu E$). In other words, from the above equations (2.1) or (2.2) we find the full operator T, not only the

special matrix elements which alone determine $d\sigma/d\Omega$. Indeed the L-S equation allows to calculate T(E) from the potential V, but conversely the potential V is given by the same equation if T(E) is known (we have only to interchange in (2.2) the role of T and V). Consequently the off-shell amplitude contains the full information on the potential.

Note that we have not discussed more conventional methods for the determination of the scattering amplitude as, for instance, the solution of the Schrödinger equation with the correct asymptotic condition (plane wave plus outgoing spherical wave). Such elementary techniques are too restricted for our general program.

3. THE THREE-BODY PROBLEM

In the three-body case we now proceed in complete analogy to the two-particle problem. We again introduce transition amplitudes. Let us consider, for instance, the following process

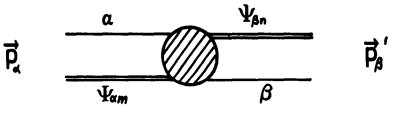


Fig. 3

Here, evidently, the transition amplitude $T_{\beta n,\alpha m}(\vec{p}_\beta', \vec{p}_\alpha)$ has to be labelled by

(i) an index $\alpha(\beta)$ which indicates the unbound particle ($\alpha = 1,2,3$) in the initial (final) state,

(ii) an index m(n) which collectively denotes all the quantum numbers of the two-particle bound-state,

(iii) the relative momentum $\vec{p}_\alpha(\vec{p}_\beta')$ of the colliding (composite) particles. (Translation invariance

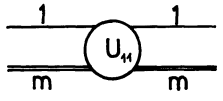
allows us to drop the total momentum).

It can be shown [8] that this amplitude is given again as a matrix element of a transition operator $U_{\beta\alpha}$:

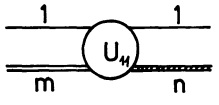
$$T_{\beta n,\alpha m}(\vec{p}'_\beta,\vec{p}_\alpha) = \langle \vec{p}'_\beta | \langle \psi_{\beta n} | U_{\beta\alpha} | \psi_{\alpha m} \rangle | \vec{p}_\alpha \rangle \quad (3.1)$$

This representation shows some general aspects of our problem:

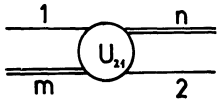
(i) To get the various amplitudes $T_{\beta n,\alpha m}$ we need several operators $U_{\beta\alpha}$. For example



elastic scattering



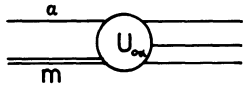
"inelastic" scattering



rearrangement process

Fig. 4

Moreover an operator $U_{0\alpha}$ can also be introduced which leads to the break-up amplitude:



break-up process.

(ii) $U_{\beta\alpha}$ is, of course, an operator in the three-particle space - in accordance with the fact that we study a three-(elementary) particle problem [9]. What we really need for the calculation of the cross sections are, however, the $U_{\beta\alpha}$ sandwiched between the two-particle bound states,

$$\langle \psi_{\beta n} | U_{\beta\alpha} | \psi_{\alpha m} \rangle \qquad (3.2)$$

i.e., effective two-particle operators which still act on the relative momenta of the colliding two particles. In more intuitive approaches one actually starts from equations for such "two-particle" amplitudes with effective potentials introduced phenomenologically. An exact theory has to be based, of course, on equations for the original three-particle operators $U_{\beta\alpha}$. Such equations turn out to be rather complicated (due to the multiplicity of variables). But we will show that it is possible to reduce the original three-particle equations to effective two-particle ones, which are as simple as in the intuitive approach but with "potentials" being now defined precisely by the original interactions.

4. DIFFICULTIES WITH THREE-BODY EQUATIONS

As discussed above we have now to write down integral equations for the operators $U_{\beta\alpha}$. In analogy to the two-particle case we find (for $\beta = \alpha$)

$$U_{\alpha\alpha} = \bar{V}_\alpha - \bar{V}_\alpha G_\alpha U_{\alpha\alpha} \quad . \qquad (4.1)$$

$\bar{V}_\alpha$ is the interaction of particle α with the two particles in the bound system, i.e., the sum of potentials depicted in Fig. 5:

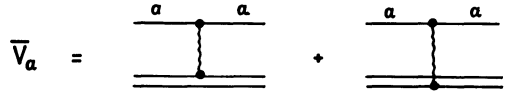


Fig. 5

Let us first consider the iteration of the two- and

three-particle equations. From (2.2) it follows

$$T = V - VG_oV + VG_oVG_oV - \ldots , \qquad (4.2)$$

i.e., the Born series which in diagrams looks like:

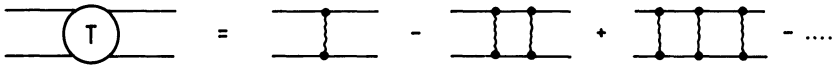


Fig. 6

Correspondingly we have in the three-particle case the following picture :

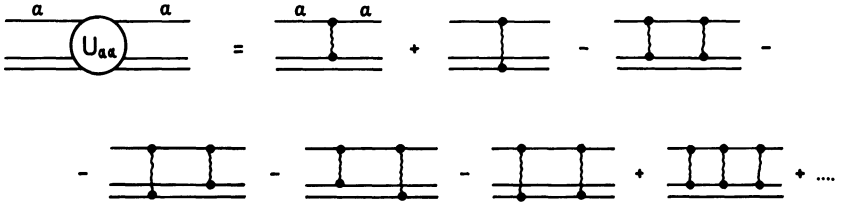


Fig. 7

We see, that the latter expansion contains partial series which coincide with the above two-particle Born series, apart from a third noninteracting particle: see, e.g., the terms 1, 3 and 7 in this expansion. Now it is well known that the Born series diverges for energies, which are in the neighbourhood of bound-states or resonances. Correspondingly the subsystem series of the three-particle problem will in general diverge since our interactions are strong enough to lead to two-particle bound states or resonances [10]. Consequently, if we work with a pure Born approximation of (4.1) we ignore the dominant structures of the subsystems.

In the two-particle case this difficulty is at once removed by application of the Fredholm theory (or of nu-

merical treatments) which works even if iterative methods fail. This, however, remains not valid for the above three-particle equations. More precisely, the kernel $\bar{V}_\alpha G_\alpha$ does not fulfill the Fredholm condition. For, due to its disconnected parts, corresponding to one noninteracting particle in the three-body system (compare Fig. 5), it contains momentum conservation δ-functions which prevent the kernel from being of the Hilbert-Schmidt type.

This observation led Faddeev to look for equations which cure the latter shortcoming. Equations of this type will be studied in the following. But before ending the present discussion let us given a final comment. It is often claimed that we are forced to work with Faddeev's equations, since the above ones were not unique. This false statement originated by carrying over to the operator equations an argument due to Faddeev which only holds for the L-S equations for the scattering states.

5. FADDEEV EQUATIONS

To overcome the difficulties mentioned, Faddeev proposed a system of integral equations [2] of the form [11]

$$U_{\beta\alpha} = -(1-\delta_{\beta\alpha})G_0^{-1} - \sum_{\gamma\neq\beta} T_\gamma G_0 U_{\gamma\alpha} \,. \qquad (5.1)$$

Let us summarize some typical properties of these equations.

(i) They represent a system of equations by which all the transition operators $U_{\beta\alpha}$ are treated simultaneously.

(ii) Eq. (5.1) contains in its kernel the two-particle scattering amplitudes T_γ instead of the potentials. Since we are now in a three-body problem we only have to label it by an index γ, in order to indi-

cate the two-particle system where it acts. This is demonstrated in Fig. 8.

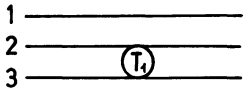


Fig. 8

It has to be stressed, however, that T_γ is not the two-particle amplitude on-the-energy-shell but the full two-particle operator (the off-shell amplitude!) studied in Section 2. Recalling the discussion given there we see that the complete information on the potential is contained and needed in the Faddeev equations - as in other three-particle equations. It will be shown that it is of great technical importance, that equations of the symmetric form (5.1) can be written down, which only implicitly (via T_γ) contain the potentials. But we have to note that this property does not imply that Faddeev's theory makes the knowledge of the potentials avoidable. Moreover, since the three-body problem requires the full knowledge of the potentials, it consequently allows to test various assumptions on it, which can not be decided solely by the two-particle data.

(iii) Further properties of the Faddeev equations become transparent by iterating them. For U_{11}, e.g. we arrive at

$$U_{11} = T_2 + T_3 - T_2 G_o T_3 - T_3 G_o T_2 + \dots , \tag{5.2}$$

that is, at a series shown in the following figure:

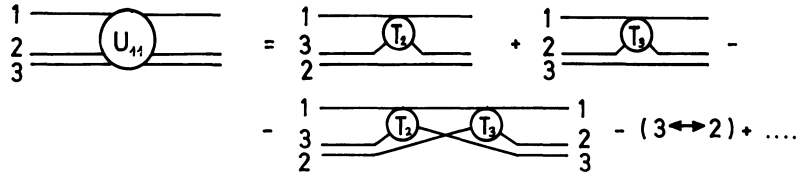


Fig. 9

Comparison with the expansion of (4.1), depicted in Fig. 7, shows that in the Faddeev equations the original subsystem series are summed [12] into the T_γ . This means that bound-states or resonances in the subsystems lead no longer to divergent series. We, therefore, expect better convergence of the expansion (5.2).

In this connection it is interesting to note that (5.2) represents Watson's multiple scattering series [13], proposed already in 1953. In particular it is closely related to Glauber's theory [14] which works so well in elementary particle - nucleus collision processes.

(iv) Despite of the fact that the divergence of the subsystem series is cured by the Faddeev equations, it is not true that the multiple scattering expansion is always convergent. In the neighbourhood of three-particle bound-states and resonances this series diverges, too. But now a second advantage of these equations turns up. Fig. 9 clearly shows that beginning with the second order of iteration the Faddeev equations only yield connected terms. This means that standard methods, as the Fredholm theory and, therefore, conventional numerical treatments, can be applied to them [2]. In other words, we are now in the same position as in single-channel scattering theory, where the Fredholm theory works even if perturbation theory fails. It was exactly this property which led to the statement that by Faddeev's theory the three-particle problem was completely solved.

Of course, such a statement is true. In practice,however, the equations (5.1) are too complicated. This is due to the fact that they are still two-dimensional after angular momentum decomposition. I.e., the typical difficulty of multiparticle problems, namely the high multiplicity of variables is not cured in Faddeev's theory. Thus, the formal advantage of this method is practically compensated by its difficult structure in detail.

6. SUBSYSTEM PROPERTIES AND REDUCTION OF THE THREE-PARTICLE PROBLEM

In this situation we ask: what is the real advantage of the Faddeev equations, if we intend to develop a practical composite particle theory? In fact, it turns out that neither its iteration properties nor the connectedness of the squared kernel are decisive. But, as already discussed, it is of considerable importance that instead of the potentials the two-particle scattering amplitudes T_γ occur. This allows to incorporate the dominant subsystem structures (bound-states and resonances) explicitly.

In the non-relativistic case we know (and in the relativistic case we have some opinions) how T_γ behaves near a bound-state energy $E = E_{\gamma r}$. Here we have a pole of the form

$$G_0 T_\gamma(E) G_0 \sim - |\psi_{\gamma r}\rangle \frac{1}{E_{\gamma r}-E} \langle\psi_{\gamma r}| =$$

$$= - |\psi_{\gamma r}\rangle \, t_{\gamma r}(E) \langle\psi_{\gamma r}| \quad . \qquad (6.1)$$

Approximating in the Faddeev equations the T_γ by a sum of bound-state (or resonance) poles,

$$U_{\beta\alpha} = -(1-\delta_{\beta\alpha})G_o^{-1} + \sum_\gamma -(1-\delta_{\beta\gamma})G_o^{-1} \underbrace{G_o T_\gamma G_o}_{-\sum_r |\psi_{\gamma r}> t_{\gamma r} <\psi_{\gamma r}|} U_{\gamma\alpha} \quad , \quad (6.2)$$

and sandwiching the whole equation between the bound-states $<\psi_{\beta n}|$ and $|\psi_{\alpha m}>$ we get

$$<\psi_{\beta n}|U_{\beta\alpha}|\psi_{\alpha m}> = -(1-\delta_{\beta\alpha})<\psi_{\beta n}|G_o^{-1}|\psi_{\alpha m}> - \quad (6.3)$$

$$- \sum_{\gamma,r} -(1-\delta_{\beta\gamma})<\psi_{\beta n}|G_o^{-1}|\psi_{\gamma r}> t_{\gamma r} <\psi_{\gamma r}|U_{\gamma\alpha}|\psi_{\alpha m}>.$$

We thus arrived at an equation for the effective two-particle operators $T_{\beta n,\alpha m}$, introduced in Section 3. In fact, (6.3) reads

$$T_{\beta n,\alpha m} = V_{\beta n,\alpha m} - \sum_{\gamma,r} V_{\beta n,\gamma r} t_{\gamma r} T_{\gamma r,\alpha m} \quad , \quad (6.4)$$

or in matrix notation

$$T = V - V\, G_o\, T \quad . \quad (6.5)$$

This means that, at least in the pole approximation, we found an effective two-particle Lippmann-Schwinger type equation (compare (2.2)). According to (6.3) and (6.4) the "potential" is given by

$$V_{\beta n,\alpha m} = -(1-\delta_{\beta\alpha})<\psi_{\beta n}|G_o^{-1}|\psi_{\alpha m}> =$$

$$= -(1-\delta_{\beta\alpha})<\psi_{\beta n}|G_o^{-1}\, G_o G_o^{-1}|\psi_{\alpha m}> =$$

$$= -(1-\delta_{\beta\alpha})<\beta n|G_o|\alpha m> \quad . \quad (6.6)$$

This expression represents an exchange potential of the form

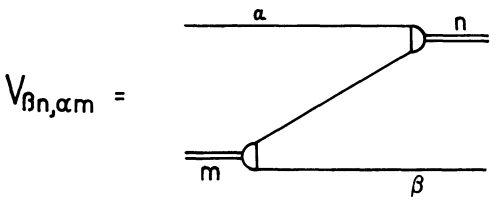


Fig. 10

The whole equation (6.4) has, therefore, the structure (compare also Fig. 2) :

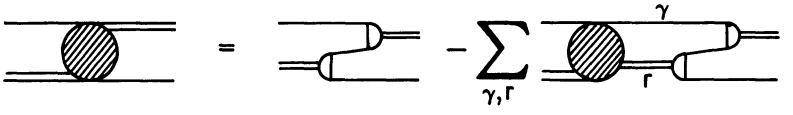


Fig. 11

By explicit incorporation of the dominant subsystem properties we succeeded in deriving effective two-particle equations which are practical as well as physically interpretable (compare the discussion in Section 3). We should stress once more that the possibility of incorporating directly the two-particle structures, and the resulting reduction to effective two-particle equations, is rather typical for Faddeev's theory and, in our opinion, its essential advantage. It was Lovelace [6] , who first emphasized this point and already arrived at equations of the form (6.4).

7. GENERALIZATION

In the preceding section we have shown, how the three-particle problem reduces to an effective two-particle

theory in the pole approximation of the subsystem operators. This result is not only important for practical applications (after angular momentum decomposition we have one-dimensional equations, as in the genuine two-body case), it is also rather tempting from the physical point of view. For, as discussed in Section 3, such a structure is expected, a priori, in our problem where, in fact, two (composite) particles collide. We therefore expect, and we will show in the following, that our reduction procedure can also be performed without any approximation. For this purpose let us split T_γ according to

$$T_\gamma = \sum_r T^{pole}_{\gamma,r} + T'_\gamma \quad . \tag{7.1}$$

Here $T^{pole}_{\gamma,r}$ is given by (6.1):

$$T^{pole}_{\gamma,r}(E) = - G_o^{-1} |\psi_{\gamma r}\rangle \, t_{\gamma r}(E) \langle\psi_{\gamma r}| G_o^{-1} = \tag{7.2}$$

$$= - |\gamma r\rangle \, t_{\gamma r}(E) \langle \gamma r| \quad .$$

We assume that in the sum $\sum_{\gamma,r} T^{pole}_{\gamma,r}$ all bound-state poles (and, more generally, all resonance poles) are contained [15]. In other words, T'_γ is assumed to represent a "small" non-polar rest term. By a generalization [16] of the techniques used in Section 6 we derive again an equation of the form (6.4):

$$T_{\beta n,\alpha m} = V_{\beta n,\alpha m} - \sum_{\gamma,r} V_{\beta n,\gamma r} \, t_{\gamma r} \, T_{\gamma r,\alpha m} \, . \tag{7.3}$$

Here, $T_{\beta n,\alpha m}$ is given as in (6.4) or (6.3), representing exactly the effective two-particle amplitude:

$$T_{\beta n,\alpha m} = \langle \psi_{\beta n} | U_{\beta\alpha} | \psi_{\alpha m} \rangle \quad . \tag{7.4}$$

According to (3.1) this is sufficient for the calculation of the considered processes. The only difference between

the present result and the approximate one of Section 6 is, that the "potentials" have now a more general form. They are given (similar to the amplitudes (7.4)) by

$$V_{\beta n,\alpha m} = \langle \psi_{\beta n} | U'_{\beta\alpha} | \psi_{\alpha m} \rangle \quad . \tag{7.5}$$

$U'_{\beta\alpha}$ obeys the same Faddeev-type equation (5.1) as $U_{\beta\alpha}$, but with the small rest terms T'_γ instead of the full amplitudes T_γ. We, therefore, expect that a determination of $U'_{\beta\alpha}$ by iteration is justified. I.e., the Watson series with T_γ replaced by T'_γ should well converge.

In lowest order we reproduce the result of the preceding section. The first order approximation in T'_γ is shown, for $\beta \neq \alpha$, in Fig. 12.

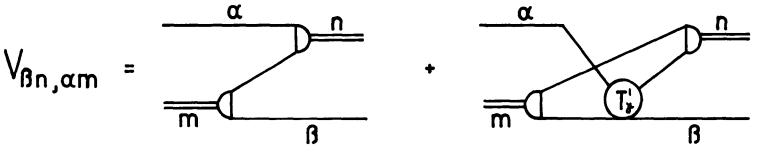


Fig. 12

For $\beta = \alpha$ no contribution exists in the pure pole approximation (compare Eq. (6.6)), i.e., only exchange diagrams occur for $T'_\gamma = 0$. But now we have the first order diagrams (compare Fig. 9):

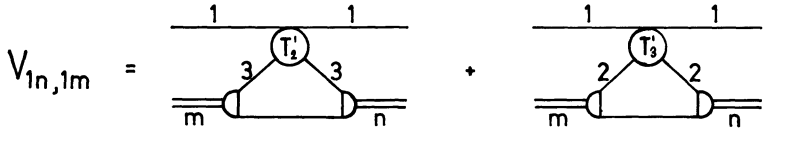


Fig. 13

Before ending this section we should note, that the equations (5.1) are not the original ones of Faddeev. Having

their typical structure (connectedness, appearance of the T_γ instead of the potentials) they are written down for the operators $U_{\beta\alpha}$ which are more closely related to physically observable quantities than the operators introduced by Faddeev. In fact, these equations have been introduced by Alt, Grassberger and myself [11] in order to derive the exact effective two-particle theory given here [17].

8. BREAK-UP PROCESSES

The techniques which we have discussed for the case of two (composite) particles in the initial and in the final state, are also applicable to processes with three elementary particles in the final state (break-up). In low order of iteration in T'_γ we find expressions of a structure as shown in the following diagrams:

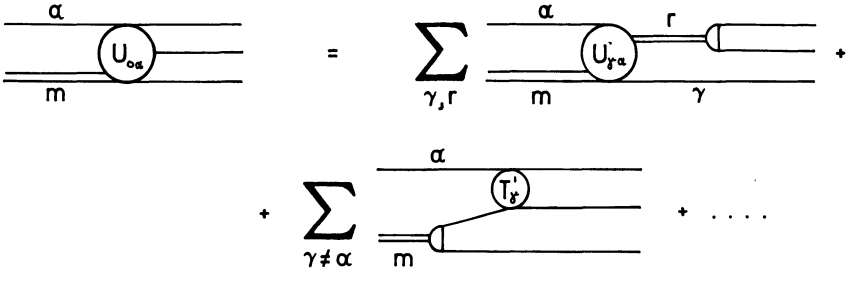


Fig. 14

We see that the result is not an integral equation. In fact, once we have calculated the occurring "two-particle" amplitudes $T_{\gamma r,\alpha m}$ (beyond the three-particle threshold!) by means of the integral equations (6.4) or (7.3), all expressions shown in Fig. 14 are calculable.

The first term which corresponds to the pure pole

approximation represents, of course, the isobar model (final state interaction). This term was already derived by Lovelace [6].

The second diagram has the form of the impulse approximation (spectator model). Of course, the isobar term is only large, if the relative energy of two of the produced particles is near to the energy of the intermediate bound-states or resonances $|\psi_{\gamma r}>$. Going away from the corresponding pole the spectator term becomes more and more important. I.e., its incorporation by means of the above exact theory is certainly not only a correction. Thus, we are in the position to test the applicability of the mentioned models in certain energy regions. Moreover, those regions can be studied where both mechanisms contribute.

9. ADDITIONAL COMMENTS

Some remarks on the above treatment might be helpful:
a) What we have used for our derivation is only the special way in which the Faddeev equations allow to consider the subsystem properties, but not any connectedness behaviour (Fredholm condition) of the Faddeev kernel. This observation is contrary to claims, often made with respect to the approximate treatment of Section 6.
b) The result (7.3) with the potentials (7.5) is exact. It is also practical, at least for the short range potentials of strong interaction theories, because in such cases we have only few bound-states or resonances. That is, few pole terms are sufficient to arrive at a small rest T'_γ .
c) Up to now, we nearly exclusively argued by means of physical intuition. However, the statements of b) can be

formulated also in a more mathematical language: for short-range potentials it turns out that the Schmidt norm of $T_\gamma G_o$, read as an operator of the two-particle space, exists [18], [19]. Then, from functional analysis we know that this operator can be well approximated by a finite number of separable terms of the form $T_\gamma^{pole} G_o$. I.e., $T'_\gamma G_o$ becomes indeed sufficiently small to allow iteration [20]. We stress once more, that it is not any Hilbert-Schmidt (Fredholm) property of the whole (squared) Faddeev kernel which has been used. In this connection it should be noted that the subsystem operators $V_\gamma G_o$, occurring in the three-particle L-S equation, are also of the Hilbert-Schmidt type, (if they are read in the two-particle space). This gives a first hint, that the above general method is not so intimately related to Faddeev's theory (compare in this respect the final discussion in Section 10). We do not intend to go into further mathematical details. But we should be aware that the real application of the described methods requires some experience in functional analysis techniques. In particular, splittings of the form (7.1) with more refined expressions for T_γ^{pole} than the one given in (7.2) can thereby be found.

d) Since (7. 5) is an exact expression for the effective potential $V_{\beta n, \alpha m}$ we are not necessarily forced to determine it by perturbation theory. A variational method has been applied by Carew and Rosenberg [21] to these expressions which seems to be promising. Generalizing and simplifying their proposal it can be shown that the "potentials" are given by low-order iterative expressions and additional variational terms, all of simple structure. (This will be studied in detail in a publication which is under preparation).

e) It was the essential point of the method described

in Section 7, that the original multiparticle equations are reduced to two-particle ones. This goal is also achieved in Feshbach's projection operator formalism [22]. It is interesting that his method, as applied to the (in-) elastic case, turns out to be a special case of the above theory [23]. In fact, the projection on to certain channel eigenfunctions corresponds to explicitly introducing bound-state poles in T_γ. However, the generalization to rearrangement and break-up channels, which is rather cumbersome in Feshbach's theory, requires no additional effort in our Faddeev-induced result which incorporates all channels simultaneously. Furthermore, this is accomplished in a symmetrical way, i.e., the inclusion of the Pauli principle - a complicated problem in Feshbach's theory - is achieved at once.

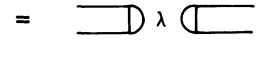
f) Finally we note that the validity of such an important model as the distorted wave Born approximation can be studied quite well in the above exact "two-particle" theory [24].

10. SEPARABLE POTENTIALS

We should say some words about the separable potential model. Instead of our original interaction V, which we assume (at the moment) to lead only to one two-body bound-state $|\psi\rangle$, we consider the separable potential [25] (as in Section 2 we drop the index γ denoting the respective two-particle subsystem):

$$V^S = \frac{V|\psi\rangle\langle\psi|V}{\langle\psi|V|\psi\rangle} = |\chi\rangle\ \lambda\ \langle\chi| \quad = \tag{10.1}$$

$$= \quad \lambda$$



By the replacement $V \to V^S$, we arrive at an explicitly

solvable model. This is most easily seen in diagrams. Instead of the original expansion

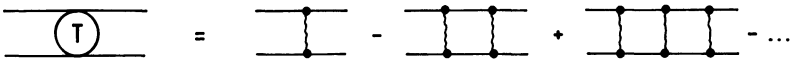
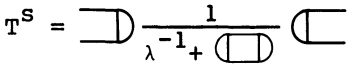


Fig. 15

we have now

Fig. 16

Fig. 16 represents a geometrical series of chain diagrams [26] (note that $\langle\chi|G_0|\chi\rangle$ is not an operator!) which can be summed into



$$= V|\psi\rangle \frac{1}{\langle\psi|V|\psi\rangle + \langle\psi|VG_0V|\psi\rangle} \langle\psi| V \,. \qquad (10.2)$$

We see that the result is again of a separable form. The most interesting point, however, is that the explicit expression (10.2) has the same pole structure (corresponding to the bound-state $|\psi\rangle$) as the original amplitude T. Inserting T^S in the Faddeev equations, effective two-particle equations are derived by the same method as demonstrated in Section 6 for the pure pole approximation.

Such equations have been found already by Mitra [3], [27] without any reference to Faddeev's theory. But as emphasized by Lovelace [6], Faddeev's theory shows rather suggestively why the separable potential model is a good approximation. For, replacing V by V^S we do not change

the dominant subsystem pole structure of the problem and this should be essential for a good approximation.

Generalizing this approximate method we can again derive exact "two-particle" equations, if we work with the splitting [28]

$$V = V^S + V' \tag{10.3}$$

The effective potentials are determined by series which only contain the "small" nonseparable rest terms V' (compare the analogous results of Section 7).

As in the pure separable model the derivation can also be performed without reference to Faddeev's theory [29]. This shows transparently that a reduction procedure as described in Section 7 is quite independent of the special version of the original three-body equations. Since, furthermore, the results are exact in any case we are not even forced to use Faddeev's equations for plausibility arguments, as done by Lovelace and as discussed above. Moreover, by the non-Faddeev approach we find at once a particularly simple form of the "potentials". For $\beta \neq \alpha$, e.g., we have [29]

$$V_{\beta n,\alpha m} = - \langle \chi_{\beta n} | \frac{1}{H_o + \sum_\gamma V'_\gamma - E - i0} | \chi_{\alpha m} \rangle \ . \tag{10.4}$$

11. NUMERICAL RESULTS

Many calculations of the three-nucleon problem have been performed in the separable model with various degrees of sophistication in the choice of the separable potentials (the parameters are adjusted to yield the deuteron binding energy and the low energy scattering

data of the two-nucleon system). The results obtained are in good agreement with experiment [30],[31]. This is consistent with the main conclusion of the preceding sections, namely that it is sufficient for a good description of the three-particle problem if the dominant subsystem properties, i.e., the essential structures of the internal dynamics, are suitably taken into account- that is just what separable potentials allow to do.

In detail, however,deviations are expected and, in fact, appear. In order to get some feeling for the effect of the non-separable rest parts $V'_\gamma = V_\gamma - V^S_\gamma$ (see Equ.(10.3)) we have studied a model problem of three identical spinless particles interacting via Yukawa potentials (with coupling constant = g).

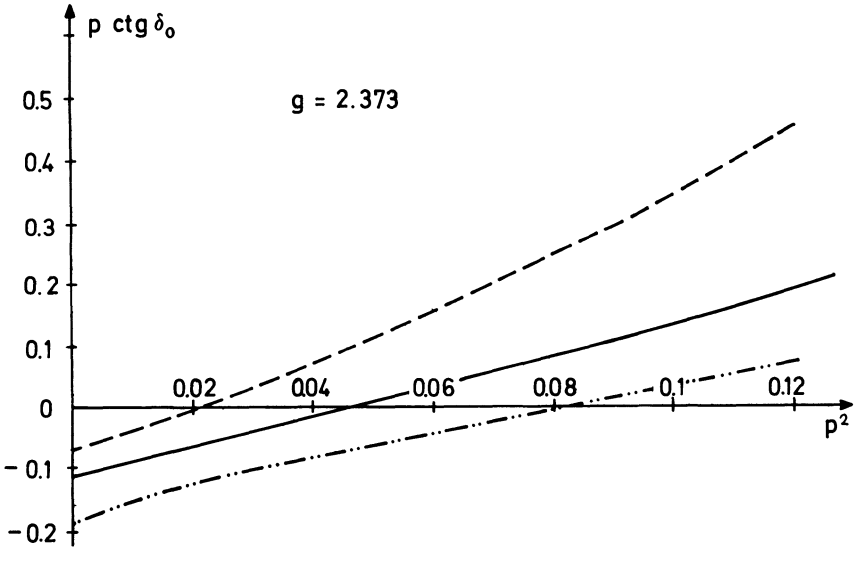


Fig. 17

The dashed line in Fig. 17 shows $p.ctg\delta_o$ (δ_o=s-wave phase shift) calculated in zeroth order (separable potential approximation) of our general iteration scheme described in Sections 7 and 10. The first order result in V'_γ is indicated by the solid line. Since it can be shown that the latter curve gives a rigorous upper bound for the exact values, we conclude that the first order correction represents a considerable step in the right direction [32]. Further improvement is expected by adding the variational terms mentioned in Section 9 d). Hereby the discrepancy between our results and nearly exact ones of Ball and Wong [33] (the lowest curve in Fig. 17 : -··—··—··—··) should still diminish. This would provide us with a practical tool for calculating in a sufficiently accurate way the effective potentials (and consequently the whole collision problem) in more realistic cases, too. Such methods are, of course, of decisive importance if we want to get any answer on the detailed form of the two-nucleon interaction.

Coulomb effects can be taken into account correctly in our formalism. We only have to incorporate the Coulomb potentials in the nonseparable rest terms V'_γ. This permits, e.g., to calculate the corresponding corrections in the proton-deuteron process compared with the neutron-deuteron case. Particularly simple expressions are obtained for separable nuclear potentials. Then, besides Coulomb corrected proton-proton form factors and propagators $t_{\gamma r}$, we have effective potentials which are given, without any approximation, by expressions of the form shown in Figs. 12 and 13: T'_γ is now the pure Coulomb T-matrix (compare Section 5 of the reference given in [11] and the discussion in [34]).

This method has been used by Alessandrini and collaborators [34] to calculate the difference, ΔE, of the

binding energies of triton and He^3 due to Coulomb effects. Two choices of the separable nucleon-nucleon potential led to $\Delta E = 0.65$ and 0.81 MeV, while the experimental value is $\Delta E_{exp} = 0.764$ MeV. In view of a possibly existing charge asymmetry (which could not be excluded by variational treatments [35] an extension of the calculations to more realistic nuclear interactions would be of fundamental importance.

12. THE N-BODY CASE

The n-body collision problem has been treated by several authors in analogy to Faddeev's approach. That is, equations with connected kernels were derived which are, therefore, solvable by standard methods - at least in principle [36]. However these derivations consist in purely algebraic transformations of the full n-particle operators. This means that they lead to equations of high dimension which are far beyond any practical applicability (recall that already in the three-body case a direct solution of the Faddeev equations is nearly impossible due to their high dimension).

On the other hand manageable effective two-particle equations can be derived also in the general case [37], [29] if, step by step, the essential subsystem structures are taken into account [38] by the methods described in Sections 6 to 10. In fact, the same treatment by which the three-body problem was reduced to an effective two-particle one (namely the explicit incorporation of the two-body poles) allowes to reduce the four-body problem to a formal three-particle theory (we already know how to handle this!). By extracting again the "two-particle" subsystem poles, which now, in fact, correspond to three-

body bound-states and resonances of the original problem, we end up with a system of effective two-particle equations of the form shown in Fig. 18 [39].

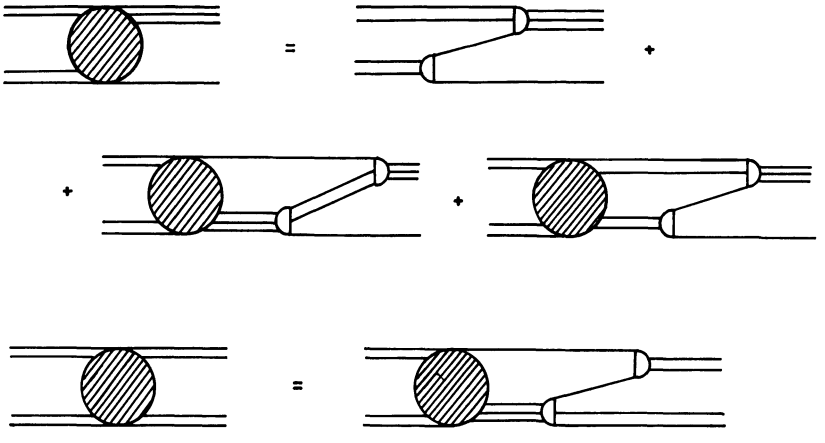


Fig. 18

These equations allow to calculate, among others, the differential cross section for the process deuteron + + deuteron → proton + triton [40]. The results obtained for the energy E_{Lab} = 13.8 MeV are compared with experimental values in Fig. 19.

We note, that in the calculation no open parameters exist which could be varied in order to reproduce the experimental data. Only the low energy parameters of the two-body interactions are taken from experiment, as discussed in Section 11 in connection with the three-body problem. The occurring transition potentials, shown in Fig. 18, have been determined carefully while, as a first crude attempt, we only used a K-matrix Born approximation of our final "two-particle" equation instead of solving the coupled integral equations. This is, probably, responsible for most of the deviations seen in Fig. 19. We

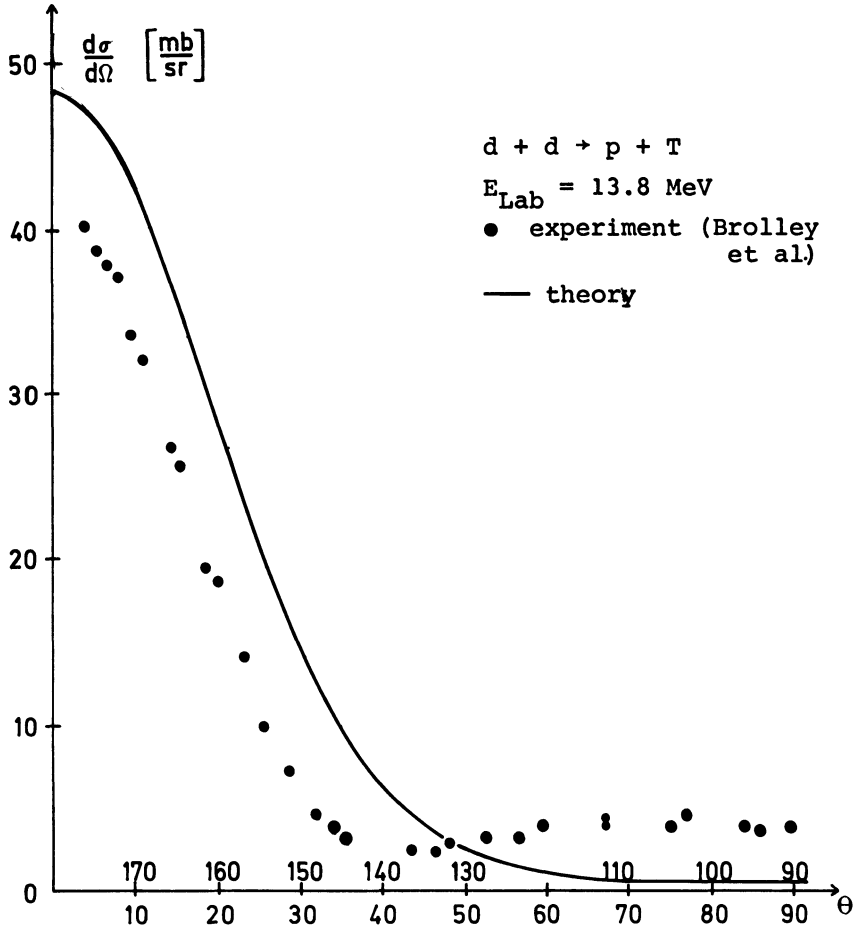


Fig. 19

stress, however, that the final equations, being one dimensional as in the genuine two-particle case, are amenable to modern computer possibilities. Thus, more computational effort along these lines seems to be suggested by the above results.

13. CONCLUSIONS

We have described a method which reduces the three-body collision problem (and even the n-bodycase) to two-particle equations with effective potentials. That is, exact equations are obtained of the form as expected, a priori, by physical intuition. From the general point of view this treatment shows how perturbation theory can be applied to strong interaction multiparticle problems, namely, by an iterative determination of the effective potentials. As in any perturbation theory we are, thereby, in the position to separate main effects from higher order corrections. Dominant structures and, therefore, the basic mechanisms of the processes can be studied in such an approach.

As the starting point we used Faddeev's equations. While they were not necessary for our derivations they made many arguments very suggestive. We have already mentioned, that one could also try to solve these equations directly, at least as soon as sufficiently fast computers become available. However, by putting such complicated (high dimensional) relations into a computer - which at the end (hopefully) yields the right numbers - we would learn less about the internal dynamics of multiparticle systems than in the approach discussed here.

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3. A. N. MITRA, Nucl. Phys. 32, 529 (1962); Phys. Rev. 139, B1472 (1965).

Compare also: A. N. MITRA and V. S. BHASIN, Phys. Rev. 131, 1265 (1963).

4. R. D. AMADO, Phys. Rev. 132, 486 (1963); 141, 902 (1966);

R. AARON, R. D. AMADO and Y. Y.YAM, Phys. Rev. 136, B650 (1964); 140, B1291 (1965); Phys. Rev. Lett. 13, 574 (1964).

5. Compare also: P. E. SHANLEY and R. AARON, Annals of Physics 44,363 (1967) where many relevant references are given.

6. C. LOVELACE, Phys. Rev. 135, B1225 (1964).

7. See also: A. C. PHILLIPS, Phys. Rev. 142, 984 (1966); 145, 733 (1966).

8. H. EKSTEIN, Phys. Rev. 101, 880 (1956);

Compare also [6] and Section 2 of the reference given in[11].

9. $U_{\beta\alpha}$ can be defined explicitly. Instead of doing this we give in the following the integral equations which it fulfills.

10. R. AARON, R. D. AMADO and B. W. LEE, Phys. Rev. 121, 319 (1961).

11. The Faddeev-type equations (5.1) were given by E. O. ALT, P. GRASSBERGER and W. SANDHAS, Nucl. Phys. B2, 167 (1967).

Compare in this respect the discussion at the end of Section 7, and footnote [17].

12. Faddeev originally constructed his equations by summing up the subsystem series [1].

13. K. M. WATSON, Phys. Rev. 89, 575 (1953).

14. R. J. GLAUBER, in: High Energy Physics and Nuclear Structure,ed.: G.ALEXANDER (North Holland, Amsterdam (1967)).

15. Instead of $|\gamma r\rangle = -G_0^{-1}|\psi_{\gamma r}\rangle$ more general expressions for the "form factors" $|\gamma r\rangle$ can also be used.

16. We apply, in fact, a suitable version of the quasiparticle method developed for the two-body case by

S. WEINBERG: Phys. Rev. 130, 776 (1963); 131, 440 (1963);
see also K. MEETZ, J. Math. Phys. 3, 690 (1962).
It should be noted that Weinberg himself proposed an extension of this treatment to the three-(n-)body-case (Phys. Rev. 133, B232 (1964)). However, his very complicated version seems to be beyond any practical applicability.

17. Faddeev-type equations for transition operators $U_{\beta\alpha}^{(\pm)}$ which are different from our $U_{\beta\alpha}$ have previously been introduced by Lovelace who proposed many of the discussed ideas. Since his equations, however, contain the T_γ and the potentials V_γ it is even cumbersome to apply the approximation of Section 6. For the performance of the general treatment of Section 7 the replacement of his relations by the equations (5.1) was decisive.

18. C. LOVELACE, in: Strong Interactions and High Energy Physics, ed.: R. G. MOORHOUSE (Oliver and Boyd, London 1964).
S. WEINBERG, Phys. Rev. 131, 440 (1963).

19. To be precise: this property holds for complex energies. Some mathematical effort is required in going to the real axis.

20. In the Faddeev equations $T_\gamma G_o$ or $T'_\gamma G_o$ is a two-particle operator acting in the three-particle space. The above considerations on the Schmidt norms are only valid for genuine two-particle operators. But they lead to similar statements with respect to the operator norms of $T_\gamma G_o$ as read in the three-particle space. Such properties are sufficient for our argumentation.

21. J. CAREW and L. ROSENBERG, preprint.

22. H. FESHBACH, Annals of Phys. 5, 357 (1958); 19, 287 (1962).

23. P. GRASSBERGER and W. SANDHAS, Zeitschr. f. Physik

220, 29 (1969).

24. E. O. ALT, P. GRASSBERGER and W. SANDHAS, Bonn preprint 2-37, See also the relevant references given there and in [5].
25. Y. YAMAGUCHI, Phys. Rev. 95, 1628 (1954).
26. Recall that solvable field theoretical models, as, e.g., the Zachariasen model (F. ZACHARIASEN, Phys. Rev. 121, 1851 (1961), W. THIRRING, Nuovo Cimento 23,1064 (1962)) have usually such a structure.
27. An equivalent field theoretical (Lee-type) model has been developed by R. D. AMADO [4].
28. The separation (10.3) leads to a splitting of T of the structure (7.1) [16].
29. P. GRASSBERGER and W. SANDHAS, Zeitschr. f. Physik 217, 9 (1968).
30. Compare the relevant papers cited in [3], [4], [7].
31. For further references see H. P. NOYES and H. FIELDELDEY, in: Three-Particle Scattering in Quantum Mechanics, ed.: J. GILLESPIE and J. NUTTALL (Benjamin, New York 1968).
32. The details are given in: E. ALT, P. GRASSBERGER and W. SANDHAS, Bonn-preprint 2-55 (1969), where also the three-body bound-state problem is studied.
33. J. S. BALL and D. Y. WONG, Phys. Rev. 169, 1362 (1968) Compare also: T. A. OSBORN, SLAC-report 79 (1967).
34. V. A. ALESSANDRINI, H. FANCHIOTTI and C. A. GARCIA, Phys. Rev. 170, 935 (1968).
35. Note, that in contrast to the above results, variational calculations usually yield too small values of ΔE. Compare e.g., the discussion of the whole problem in [31], Section V. 1, where further methods and results are reviewed.
36. Compare the references 2 - 7 given in [37] and furthermore:

R. G. NEWTON, J. Math. Phys. 8, 851 (1967);
R. OMNES, Phys. Rev. 165, 1265 (1968).

37. P. GRASSBERGER and W. SANDHAS, Nucl. Phys. B2, 181 (1967).

38. Applying such treatments, it is not necessary to start by first writing integral equations with connected kernels. This is in contrast to claims often made (compare also the discussion of this point in [29] and in Section 9).

39. More general expressions for the "potential" occur if nonpolar rest terms are retained in the subsystems (compare the generalizations of the treatment of Section 6 given in Section 7).

40. E. O. ALT, P. GRASSBERGER and W. SANDHAS, Bonn-preprint 2-48(1968).