Analytic Continuation Methods in Nuclear Reaction Theory and Indirect Approaches in Nuclear Astrophysics

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Abstract

General properties of nuclear vertex constants and asymptotic normalization coefficients as well as the methods of their determination are discussed. Selected problems of nuclear astrophysics are outlined. A relation between asymptotic normalization coefficients (ANCs) and astrophysical nuclear reactions is elucidated. An analytic continuation of the effective range expansion is applied to the $\alpha + d$ system which is of interest for nuclear astrophysics.

Keywords: *Analytic continuation; vertex constants; lithium; nuclear astrophysics*

1 Introduction

Asymptotic normalization coefficients (ANCs) determine the asymptotics of nuclear wave functions in binary channels. ANCs are proportional to nuclear vertex constants (NVC) which determine the virtual processes $a \to b + c$ and are related directly to the residue in energy of the elastic bc scattering amplitude at the pole corresponding to the bound state a [1]. NVCs and ANCs are fundamental nuclear characteristics. They are used actively in analyses of nuclear reactions within various approaches. NVCs and ANCs extracted from one process can be used for the prediction of characteristics of other processes. Comparing empirical values of NVCs and ANCs with theoretical ones enables one to evaluate the quality of a model.

The ANC for the channel $a \to b + c$ determines a probability of the configuration $b + c$ in nucleus a at distances greater than the radius of nuclear interaction. Thus ANCs arise naturally in the expressions for cross sections of nuclear reactions between charged particles at low energies, in particular, of astrophysical nuclear reactions. Nuclear reactions in stars and stellar explosions are responsible for the synthesis of chemical elements. Note that due to the Coulomb barrier cross sections at astrophysical energies are so small that their direct measurement in laboratories is very difficult, or even impossible. Hence knowing ANCs allows one to obtain additional and important information on astrophysical nuclear reactions.

ANC values could be determined from microscopic calculations, however such calculations are rather tedious. The theoretical results should be matched to the empirical ones obtained from data on scattering and reactions. One of the promising methods to extract ANCs is the analytic continuation of bc-scattering data to a pole of a scattering amplitude corresponding to a bound state a lying in the unphysical region of negative energies. The most effective way of realization of that procedure is the analytic continuation of the effective range function.

The present paper deals with various problems related to the methods of analytic continuation of scattering data and to utilization of these methods for obtaining an information on astrophysical nuclear reactions. The plan of the paper is as follows. We discuss the general properties of NVCs and ANCs and methods of their determination

in Section 2. Section 3 is dedicated to selected problems of nuclear astrophysics including indirect methods of obtaining information on astrophysical nuclear reactions. The analytic continuation of the effective range expansion is applied in Section 4 to the $\alpha + d$ system which is of interest for nuclear astrophysics.

The system of units with $\hbar = c = 1$ is used throughout the paper.

2 ANCs and NVCs

2.1 Definition and properties of ANCs and NVCs

ANC $C_{abc}(LS)$ for the $a \to b + c$ channel is defined as a coefficient in the asymptotics of the radial overlap integral of the wave functions of a, b, and c nuclei [1]

$$I_{abc}(LS; r)|_{r\to\infty} \to C_{abc}(LS) W_{-\eta_b, L+1/2}(2\kappa r)/r. \tag{1}$$

Here r is the distance between b and c, L and S are the channel orbital angular momentum and the channel spin, $\kappa^2 = 2\mu\epsilon$, $\epsilon = m_b + m_c - m_a$, m_i is the mass of particle i, $\mu = m_b m_c/m_a$, $\eta_b = Z_b Z_c e^2 \mu/\kappa$ is the Sommerfeld parameter for the bound state and $W_{m,n}(z)$ is the Whittaker function.

Note that the asymptotical form (1) has been rigorously proved only for the simplest case when the composite system a consists of two elementary constituents. In that case the form (1) follows directly from the Schrödinger equation. For three- and more particle systems the asymptotics of overlap integrals may differ from (1) ('anomalous asymptotics') [2, 3].

NVC $G_{abc}(LS)$ is the on-shell matrix element of the virtual $a \to b + c$ process in the given partial-wave state LS. It is related to the amplitude of elastic bc scattering

$$\operatorname{res} \langle LS|M^{J_a}|LS\rangle = (-1)^L G^2_{abc}(LS). \tag{2}$$

G_{abc} and C_{abc} are interrelated:

$$G_{abc}(LS) = -(\pi N_{bc}/\mu^2)^{1/2} L!/\Gamma(L + 1 + \eta_b)\, C_{abc}(LS). \tag{3}$$

N_{bc} arises due to the identity of nucleons. Its value depends on the way of antisymmetrization of wave functions: $1 \le N_{bc} \le (A_b + A_c)!/(A_b! A_c!)$, A_i being the mass number of the nucleus i. N_{bc} is often included into C_{abc}.

2.2 Methods of determination of ANCs and NVCs

1) Microscopic calculations of ANCs and NVCs are very tedious. Working in the configuration representation, one should make calculations in the asymptotical region where wave functions decrease exponentially entailing a low accuracy of the results. Using the momentum representation needs analytic continuation to imaginary values of momenta that is non-trivial. To author's knowledge, there are only two *ab initio* calculations of ANCs for nuclei with $A > 3$ [4, 5].

2) Theoretical results should be matched to the empirical ones obtained from analyses of scattering and reactions. There are various methods of extracting ANCs and NVCs from experimental data.

2a) Analysis of data on transfer reactions. If the pole diagram corresponding to the transfer of particle c contributes to the amplitude of the $a(x, y)b$ reaction, the differential cross section $\sigma(z)$ of this reaction possesses the 2nd order pole at $z = z_0$ ($z = \cos\theta$, θ is the c.m. scattering angle, $|z_0| > 1$) (Fig. 1). If one extrapolates the experimental values of $(z - z_0)^2\, \sigma(z)$ to the pole position, one immediately obtains the value of $|G_{abc} G_{yxc}|^2$.[2]

[2] Account of the Coulomb interaction in the vertices of the pole diagram turns a pole to a branch point.

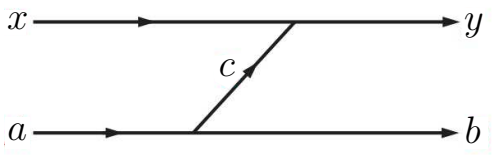

Figure 1: A pole diagram for a transfer reaction.

2b) Extrapolation in energy E of the partial-wave amplitude of elastic bc scattering (obtained by the phase-shift analysis) to the pole corresponding to the bound state a.

Note that the problem of using continuum-state data to obtain information on bound-state characteristics is non-trivial. It is written in the well-known monograph [6]: "It is impossible to obtain information on bound states from characteristics of scattering processes, as a matter of principle".

This assertion is based on the inverse scattering theory which states that to restore a local potential one needs to know: i) phase shifts $\delta_L(E)$ for some arbitrary L in the whole region $0 \le E < \infty$ and ii) $2N_L$ parameters characterizing N_L bound states for a given L [7]. One can use N_L binding energies and N_L ANCs as those parameters. Thus, if the system possesses bound states, knowing $\delta_L(E)$ is not sufficient to restore unambiguously a potential describing the system. Instead one gets an infinite set of so-called phase equivalent potentials (PEP) which lead to identical scattering phase shifts $\delta_L(E)$ but to different properties of the bound states for a given L.

There are various methods of constructing PEPs, e. g., Bargmann potentials [6] or the supersymmetric transformation [8]. In particular, the supersymmetric transformation can be used to construct a PEP which differs from the initial potential by any modification of the bound spectrum. A bound state can be added or suppressed; its binding energy and/or the ANC can be modified.

Hence within the formal potential approach with arbitrary potentials and without any additional conditions, it is impossible to determine unambiguously characteristics of bound states knowing only $\delta_L(E)$. A way to resolve this ambiguity problem is to use a natural requirement that amplitudes of processes are analytic functions of their kinematic variables. The analyticity property follows from a fundamental microcausality principle.

Using the analyticity and knowing the partial wave bc scattering amplitude $f_L(E)$ on some segment of the real positive semiaxis, one can continue analytically $f_L(E)$ to the unphysical region $E < 0$ and obtain both the position of the pole $E = -\epsilon < 0$ and the residue of $f_L(E)$ at that pole, that is, the NVC and ANC. (Note that we discuss here a principal side of the problem and not a practical realization of analytic continuation.)

Thus, in the case of potential scattering, knowing ϵ, ANC C_{abc}, and $f_L(E)$ at $0 \le E < \infty$, one can construct unambiguously a local potential $V(r)$ using methods of the inverse scattering theory [7]. As a result, a unique 'analytic' potential would be selected out of the set of PEPs, which leads to the needed analytic properties of the scattering amplitude. This potential describes all bound and continuum states of the given system.

Now let us discuss why the characteristics of a bound state obtained by the direct analytic continuation of $f_L(E)$ from $E \ge 0$ to $E < 0$ may differ from the characteristics found by solving the bound state problem with the potential which describes correctly $f_L(E)$ at $E \ge 0$. To be specific, we will consider the case $L = 0$ (the index 0 will be omitted). According to [6, Chapter 12], one can continue analytically the amplitude $f(k)$ to the band $|\mathrm{Im}\, k| < \alpha$ in the k plane if the potential $V(r)$ satisfies the condition

$$\int_0^\infty |V(r)| e^{2\alpha r} r dr < \infty, \quad \alpha > 0. \tag{4}$$

In the potential scattering theory the procedure of analytic continuation is based on the expression for the partial-wave amplitude in the form

$$f_L(k) = -\frac{\mu}{2\pi}\int_0^\infty \varphi_L(kr)V(r)\psi_L(kr)dr, \tag{5}$$

where φ_L and ψ_L are the plane wave and the exact wave function for given L, respectively. Upon the continuation of k from the positive semiaxis to the complex plane, the integrand in the r.h.s. of (5) develops terms proportional to $e^{2|\mathrm{Im}\,k|}V(r)$ causing a divergence of the integral in (5) at the upper limit if $V(r)$ decreases insufficiently fast for $r \to \infty$. In this case, the condition (4) ceases to hold and the analytic continuation of the amplitude with the aid of expression (5) becomes impossible. However, there is a possibility to perform an analytic continuation by a different method. This can be done, for example, if an expression for $f_L(E)$ in the region $E \geq 0$ is known or if $f_L(E)$ for $E \geq 0$ can be approximated quite accurately by some analytic expression.

Consider a trivial example

$$f(z) = \int_0^\infty e^{(a-z)t}dt. \tag{6}$$

$f(z)$ is defined initially only for $\mathrm{Re}\,z > \mathrm{Re}\,a$ since the integral diverges if this inequality is violated. On the other hand, the integration can be performed explicitly: $f(z) = 1/(z-a)$. This expression defines a function analytic on the entire complex z plane with a pole at $z = a$.

Let us consider an instructive example of the Bargmann-type potential specified in [6, Chapter 14] as

$$V_d(r) = -\frac{\kappa}{\mu}\frac{d}{dr}\left[\sinh(br)\frac{g_d(\kappa,r)}{g_d(\kappa+b,r)-g_d(\kappa-b,r)}\right], \tag{7}$$

where $g_d(x,r) = x^{-1}[e^{-\kappa r} + d\sinh(xr)]$.

In the S wave, this potential has one bound state, its binding energy $\epsilon = \kappa^2/2\mu$. The respective normalized radial wave function has the form:

$$\varphi_d = 2\sqrt{\frac{\kappa d}{b^2-\kappa^2}}\,\frac{\sinh(br)}{g_d(\kappa+b,r)-g_d(\kappa-b,r)}, \quad b > \kappa. \tag{8}$$

For the potential (7), the effective-range approximation coincides with the exact solution. In this case, the S wave phase shift is determined by the equation

$$k\cot\delta = -\kappa b/(b+\kappa) + k^2/(b+\kappa). \tag{9}$$

The S wave scattering amplitude has the form:

$$f(k) = \frac{e^{2i\delta}-1}{2ik} = \frac{1}{k\cot\delta - ik} = \frac{b+\kappa}{-b\kappa + k^2 - i(b+\kappa)k}. \tag{10}$$

As follows from (10), $f(k)$ is independent of the parameter d; that is, expression (7) determines a family of phase-equivalent potentials differing by the value of d. The amplitude $f(k)$ in (10) can be analytically continued to the region of imaginary k where it has a pole at $k = i\kappa$. Expressing the vertex constant G and the asymptotic normalization coefficient C in terms of the residue of $f(k)$ at this pole, one obtains:

$$G = \left[\frac{2\pi\kappa(b+\kappa)}{\mu^2(b-\kappa)}\right]^{1/2}, \qquad C = \left[\frac{2\kappa(b+\kappa)}{b-\kappa}\right]^{1/2}. \tag{11}$$

On the other hand, a d-dependent expression for the asymptotic normalization coefficient can be obtained directly from (8). Specifically, one has:

$$C_d = \left[\frac{4\kappa(b+\kappa)}{d(b-\kappa)}\right]^{1/2}. \tag{12}$$

One can see that only at $d = 2$ does C_d given by (12) coincide with C given by (11). The fact that the value of $d = 2$ stands out becomes understandable upon examining the asymptotic behavior of the potential $V_d(r)$ for $r \to \infty$. It can be shown that this asymptotic behavior is given by

$$V_d(r) = \begin{cases} -V_1 e^{-2\kappa r}, & d \neq 2 \\ -V_2 e^{-2br}, & d = 2. \end{cases} \tag{13}$$

Since $b > \kappa$, the analyticity condition (4) for $V_d(r)$ is satisfied at $d = 2$ but is violated for all $d \neq 2$. Thus, an analytic continuation of the amplitude $f(k)$ to the region of imaginary values of k makes it possible to select among the set of phase-equivalent potentials $V_d(r)$, the only 'analytic' potential which corresponds to $d = 2$, and to find the relevant correct values of the ANC C. As for phase-equivalent potentials that are obtained by means of supersymmetry transformations, they develop at the origin a singularity of the $1/r^2$ type (see [8]) and, hence, do not satisfy the analyticity condition (4) just in the same way as the potential (7) does not satisfy it for $d \neq 2$.

So far, we have addressed a problem of a pure potential scattering of structureless particles. For practical purposes, including applications in astrophysics, the case of composite particles, first of all, nuclei, is of greater importance. Complex nuclei are the subject of many-body theory. An attempt at describing elastic nucleon-nucleus or nucleus-nucleus scattering within a two-body potential problem would lead to a complex-valued optical potential that is in general nonlocal and energy and angular-momentum dependent. Nevertheless, both the bound-state energy (which is usually known from experimental data) and the respective NVC and ANC can in principle be found as before by performing an analytic continuation of the partial-wave amplitude $f_L(E)$ to the region of negative values of E (imaginary values of k). This continuation may be realized in various ways. For example, $G_{^6\mathrm{Li}\alpha d}$ and $C_{^6\mathrm{Li}\alpha d}$ for the S wave state of the $\alpha + d$ system were found by two methods in Ref. [9]. Within the first method, an analytic Padé approximation of the scattering function $k \cot \delta$ obtained for $E > 0$ from the experimental phase shifts for the $d\,^4$He scattering was analytically continued to the region $E < 0$, the parameters of the respective Padé approximants being determined by means of the χ^2 minimization. Within the second method, an effective two-body $d\alpha$ potential $V_{d\alpha}(r)$ describing the same $d\,^4$He phase shifts was constructed using the harmonic oscillator basis and the χ^2 minimization. The next step of this method involved deriving the two-body $(d + \alpha)$ wave function for the ^{6}Li bound state in the potential $V_{d\alpha}(r)$ and determining the respective ANC $C_{^6\mathrm{Li}\alpha d}$. The two methods yielded rather close values of $C_{^6\mathrm{Li}\alpha d}$. Since the potential $V_{d\alpha}(r)$ was a finite sum of harmonic-oscillator wave functions, it obviously satisfied the necessary analyticity condition (4).

It should be noted that, in a general case, when b or c (or both) are composite systems, the ANC C_{abc} refers to the overlap integral $I_{abc}(r)$ which is normalized to the spectroscopic factor S_{abc} rather than to unity. If, however, the ANC is found by solving the bound-state problem for nucleus A on the basis of two-body potential V_{bc} fitted to the bc phase shifts, the respective two-body wave function should be normalized to unity. It would be incorrect to normalize this function to the independently determined spectroscopic factor as was done, for example, in Ref. [10] for the $^3\mathrm{He} + \alpha$ system.

2.3 Inference

1. Using the fundamental analyticity property of scattering amplitudes and analytic continuation methods allows one to obtain information on characteristics of nuclear bound states (including ANCs) from the phase shift data. Thus the ambiguity related to the existence of phase-equivalent potentials is removed.

2. The most efficient method of analytic continuation is the analytic approximation of the experimental values of $k \cot \delta$.

3. If the continuation is performed by fitting a two-body potential, one should use a potential which decreases rapidly enough at $r \to \infty$. One should set the spectroscopic factor equal to 1.

3 Selected problems of nuclear astrophysics

3.1 Introduction

Nuclear reactions in stars and stellar explosions are responsible for ongoing synthesis of chemical elements. Nuclear physics plays an important role as it determines the signatures of isotopic and elemental abundances found in spectra of stars, novae, supernovae, and X-ray bursts.

The rapid neutron capture process (r-process) is responsible for existence of about a half of stable nuclei heavier than iron. Capture cross sections for most of nuclei involved are hard if just impossible to measure in the laboratory and indirect experimental approaches have to be employed to gather the relevant nuclear structure information. The same concerns (p, γ) and (p, α) reactions.

Quantities used in nucleosynthesis calculations are reaction rates. A thermonuclear reaction rate is a function of density of interacting nuclei, their relative velocity and the reaction cross section. Extrapolation procedures are often needed to derive cross sections in the energy or temperature region of astrophysical relevance. While non-resonant cross sections can be extrapolated rather well to the low-energy region, the presence of continuum or sub-threshold resonances can complicate these extrapolations.

As an example of an important astrophysical reaction one may mention $^7\mathrm{Be}(p, \gamma)^8\mathrm{B}$ which plays a major role for the production of high energy neutrinos from the β-decay of ^{8}B. These neutrinos come directly from the center of the Sun and are ideal probes of the Sun structure. The reaction $^{12}\mathrm{C}(\alpha, \gamma)^{16}\mathrm{O}$ is extremely relevant for the fate of massive stars. It determines if the remnant of a supernova explosion becomes a black hole or a neutron star. These two reactions are two examples only of a large number of reactions not known yet with an accuracy needed for astrophysics.

3.2 Thermonuclear cross sections and reaction rates

The number r of reactions between a target j and a projectile k per unit volume and time can be expressed as $r = \sigma v n_j n_k$ or, more generally, as

$$r_{jk} = \int \sigma v d^3 n_j d^3 n_k. \tag{14}$$

Here σ is the cross section, v is the relative velocity, n_j and n_k are number densities. For nuclei j and k in an astrophysical plasma obeying a Maxwell–Boltzmann distribution,

$$d^3 n_j = n_j \left(\frac{m_j}{2\pi k T}\right)^{3/2} \exp\left(\frac{m_j v_j^2}{2kT}\right) d^3 v_j, \tag{15}$$

k is the Boltzmann constant and T is the absolute temperature. Using (15), one can rewrite (14) as

$$r_{jk} = \langle \sigma v \rangle_{jk}, \quad \langle \sigma v \rangle_{jk} = \left(\frac{8}{\pi \mu_{jk}}\right)^{1/2} (kT)^{-3/2} \int_0^\infty E \sigma(E) \exp\left(-\frac{E}{kT}\right) dE, \tag{16}$$

where $\langle \sigma v \rangle_{jk}$ is an average over the temperature distribution.

3.3 Charged particles

Experimentally, it is more convenient to work with an astrophysical S factor:

$$S(E) = E\sigma(E)e^{2\pi\eta}, \quad \eta = Z_j Z_k e^2/v. \tag{17}$$

Eq. (16) can be written as

$$\langle\sigma v\rangle_{jk} = \left(\frac{8}{\pi\mu_{jk}}\right)^{1/2} (kT)^{-3/2} \int_0^\infty S(E) \exp\left(-\frac{E}{kT} - \frac{b}{E^{1/2}}\right) dE, \quad b = 2\pi\eta E^{1/2}. \tag{18}$$

If one assumes that $S(E)$ is a constant, the integrand in (18) is maximal at the Gamow energy $E_0 = (bkT/2)^{2/3}$.

Measurements of cross sections at low energies are difficult and their extrapolation from higher energies can be complicated by presence of unknown resonances.

3.4 Nuclear reactions at the Sun

The Sun belongs to the main-sequence stars which energy is governed by the *pp*- and CNO-cycles (Figs. 2, 3).

According to the Standard Sun model, 99% of the Sun energy is generated by the *pp*-cycle (see Fig. 4), an ultimate result of this cycle is the transmutation of 4 protons into helium

$$4p \to {}^4\mathrm{He} + 2e^+ + 2\nu_e. \tag{19}$$

The explosive nuclear burning in astrophysical environments produces short-lived exotic nuclei which in turn can play a role of targets in subsequent reactions. In addition, it involves a very large number of stable nuclei still not fully explored in experiments. Thus, it is necessary to be able to predict reaction cross sections and thermonuclear rates with the aid of theoretical models, moreover, a direct cross section measurement is often not possible with existing experimental techniques. For getting a reliable result obtained by extrapolation down to the stellar energies of the cross sections measured at the lowest possible energies in the laboratory, such extrapolations should have as strong theoretical foundation as possible. The theory is even more mandatory when excited or unstable nuclei are involved in the entrance channel.

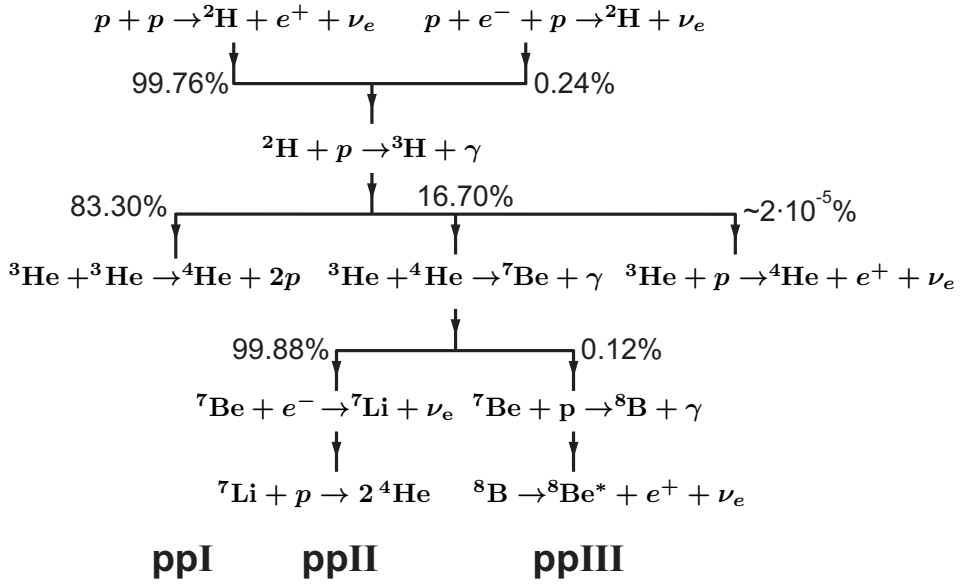

Figure 2: The *pp*-cycle.

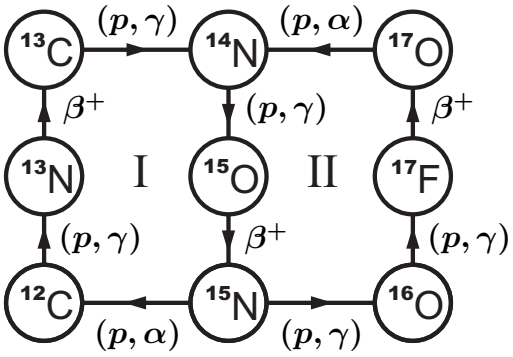

Figure 3: The CNO-cycle.

3.5 Nuclear reaction models

1. Potential models assume that physically important degrees of freedom are the relative motions between structureless nuclei in the entrance and exit channels. Interaction between them is described by an optical potential (usually of the Woods–Saxon form). DWBA is used practically for all astrophysical nuclear reactions. The only microscopic information is introduced in terms of spectroscopic factors and parameters of the optical potential. A deficiency of these models is that the optical parameters cannot be determined unambiguously.

2. In microscopic models, nucleons are grouped into clusters and completely antisymmetrized relative wave functions between various clusters are determined by solving the Schrödinger equation for a many-body Hamiltonian with an effective nucleon-nucleon interaction. Typical cluster models are based on the Resonating Group Method (RGM) or the Generator Coordinate Method (GCM). They result in a complicate set of coupled integro-differential equations. Modern nuclear shell-model calculations, such as the Monte Carlo shell model, or the no-core shell model, are able to provide the wave functions for light nuclei. However so far they cannot describe scattering wave functions with a sufficient accuracy.

Theoretical results for the astrophysical S-factor for the $^7\mathrm{Be}(p,\gamma)^8\mathrm{B}$ reaction are shown in Fig. 5. The dashed line corresponds to the no-core shell model and the dotted line to RGM. Experimental data are taken from 8 different papers. It is evident that

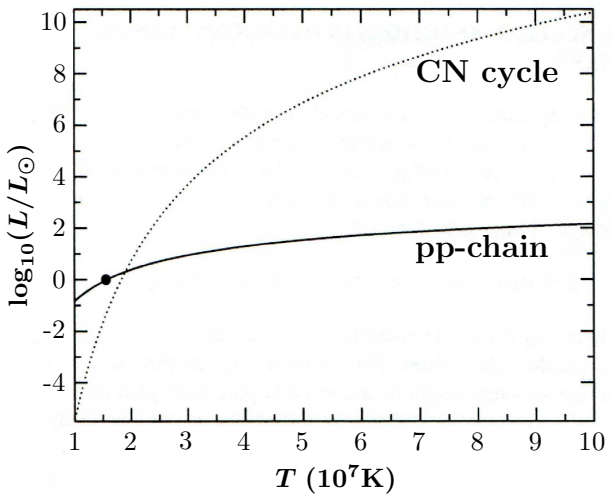

Figure 4: Relative energy release in stars as a function of temperature. The dotted line corresponds to the Sun.

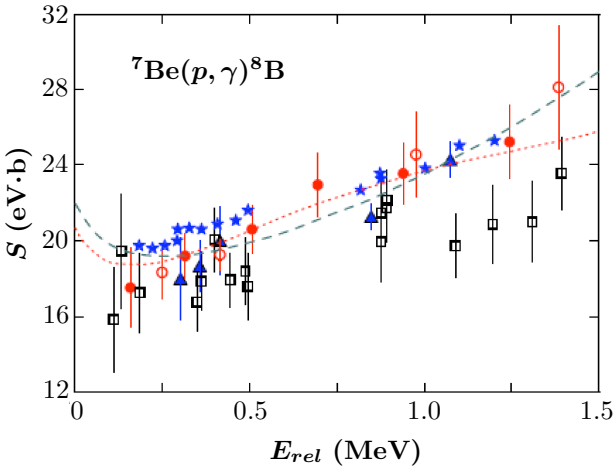

Figure 5: Comparison of theoretical and experimental results for the $^7\mathrm{Be}(p,\gamma)^8\mathrm{B}$ reaction. Dashed line — no-core shell model, dotted line — RGM. See Ref. [11] for details.

both theory and experiment need improvement for this important reaction.

3. Field theories adopt a completely independent approach to nuclear physics calculations which does not use the concept of nuclear potentials. The basic method of field theories is to start with a Lagrangian for the fields which is used to construct Feynman diagrams that are utilized for practical calculations. Effective field theory (EFT) bypasses complications of quantum chromodynamics (QCD) using an expansion over a small parameter determined as a ratio of short-range and long-range (or 'light' and 'heavy') scales. Practically, for the NN interaction, this parameter is conventionally defined as

$$p = \frac{(1/a, B, k)}{\Lambda}, \tag{20}$$

where for the 'light' scale one uses either $1/a$ (a is the NN scattering length), or a typical binding energy B, or a typical nucleon momentum k. The 'heavy' scale is determined by the pion mass: $\Lambda \sim m_\pi \sim 140$ MeV.

The reaction rates dominated by the contributions from a few resonant or bound states, are often extrapolated to energies of astrophysical interest in terms of R-matrix fits. The appeal of these methods rests on the fact that analytical expressions can be derived from underlying formal reaction theories allowing for a rather simple parameterization of the data. However, the relation between the parameters of the R-matrix model and the experimental data is quite indirect.

A large fraction of the reactions of interest proceed through compound systems that exhibit high enough level densities to provide a reliable description of the reaction mechanism by means of statistical methods. A theoretical treatment of nuclear reactions leading to formation and decay of compound nuclei was developed by Ewing and Weisskopf based on two ideas: (a) the compound nucleus formation independence hypothesis as proposed by Niels Bohr, and (b) the reciprocity theorem, or time-reversal properties of the underlying Hamiltonian. This allows one to relate capture and decay cross sections.

3.6 Effects of electron screening

The form of the astrophysical S factor given in Eq. (17) assumes that the electric charges of nuclei are 'bare' charges. However, this is the case neither at very low laboratory energies, nor in stellar environments. In stars, the bare Coulomb interaction between nuclei is screened by the electrons in the plasma surrounding them. If one

measures reaction rates in the laboratory using atomic targets (always), the atomic electrons provide screening as well.

1. Stellar electron screening

Coulomb interaction between two charges in a neutral plasma can be written as

$$V(r) = \frac{Z_1 Z_2 e^2}{r} \exp\left(-\frac{r}{R_D}\right), \tag{21}$$

where R_D is the Debye radius, i. e., the scale over which mobile charge carriers in the neutral medium screen out electric fields. In the weak screening approximation

$$V(r) \approx \frac{Z_1 Z_2 e^2}{r}\left(1 - \frac{r}{R_D}\right) = V_b(r) + U_0, \quad U_0 = -\frac{Z_1 Z_2 e^2}{R_D}. \tag{22}$$

As a result, the reaction velocity increases:

$$\langle \sigma v \rangle_{\text{screened}} = f \langle \sigma v \rangle_{\text{bare}}, \quad f = \exp(|U_0|/kT). \tag{23}$$

2. Atomic electron screening

The laboratory screening can be evaluated in the adiabatic approximation assuming that the electron velocities in the target are much larger than the velocity of the relative motion between the projectile and the target nucleus. In this case, the electronic cloud at each instant time t adjusts to the ground state of a 'molecule' consisting of two nuclei separated by a time-dependent distance $R(t)$. Since the closest approach distance between the nuclei is much smaller than typical atomic cloud sizes, the binding energy of the electrons will be given by the ground-state energy B of the $Z_p + Z_t$ atom. Energy conservation implies that the relative energy between the nuclei increases by

$$U_e = B(Z_p + Z_t) - B(Z_t). \tag{24}$$

U_e is the screening potential. This energy increment enhances the fusion (tunneling) probability. Supposing that U_e/E is small and using (17) one gets

$$\sigma(E + U_e) = \exp\left[\pi\eta(E)\frac{U_e}{E}\right]\sigma(E). \tag{25}$$

The values of U_e needed to reproduce the experimental data are systematically larger than the theoretical ones by a factor of 2 (see Fig. 6).

3.7 Indirect methods of obtaining information on astrophysical nuclear reactions

1. Trojan horse method

The Trojan horse (TH) method [12, 13] is an efficient indirect method of determining cross sections of astrophysical binary reactions by measuring cross sections of reactions with three particles in the final state. Suppose we are interested in the $A + x \to B + y$ reaction at low (astrophysical) energies, and direct measurements are not possible due to the Coulomb barrier. Consider the reaction $1 + A \to 3 + B + y$ where $1 = 3 + x$. The particle 1 is the Trojan horse which includes the particle x.

Consider the quasifree mechanism (Fig. 7). At low momentum transferred from 1 to 3, this mechanism may provide a dominant contribution (or at least determine angular and energy dependencies). The respective differential cross section is of the form:

$$\sigma_{3\text{diff}}(A + 1 \to B + y + 3) = \text{KF}\,\psi^2(1 \to 3 + x)\tilde{\sigma}_{2\text{diff}}(A + x \to B + y). \tag{26}$$

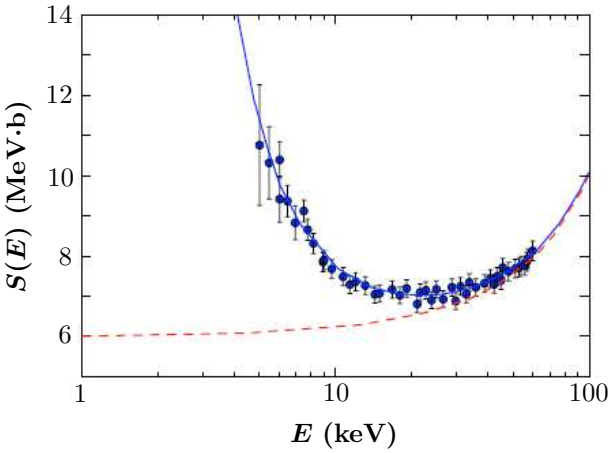

Figure 6: S factor of the ^{3}He$(d,p)^4$He reaction. Dashed curve — bare nuclei, solid curve — screened nuclei with $U_e = 219$ eV (theory gives $U_e = 119$ eV). See Ref. [11] for details.

Here KF is a known kinematical factor, ψ is the wave function of particle 1 in the $3+x$ channel, and $\tilde{\sigma}_{2\text{diff}}$ is a modified differential cross section of the binary reaction of interest. If KF and ψ are known, $\tilde{\sigma}_{2\text{diff}}$ can be extracted from $\sigma_{3\text{diff}}$.

As a typical example, one sets $1 = d$, $x = p$, and $3 = n$.

$\tilde{\sigma}_2$ differs from the free cross section σ_2 by particle x being virtual (off-shell), that is $\tilde{\sigma}_2$ describes the $A + x \to B + y$ process half-off-shell.

Using the energy and momentum conservation laws at the vertices of the diagram of Fig. 7, one can show that the relative momentum k of particles A and x in the initial state of the reaction $A + x \to B + y$ remains non-zero as the relative kinetic energy $E_{Ax} \to 0$. Hence the Coulomb barrier factor $e^{-2\pi\eta_i}$ does not appear in the expression for $\tilde{\sigma}_2$, and it remains finite at $E_{Ax} \to 0$. A qualitative explanation is that at the moment of interaction with particle A, the particle x has already penetrated through the Coulomb barrier in the initial state as a part of particle 1.

Note that the initial energy E_{A1} should be chosen large enough so that the reaction can be measured. A proper choice of E_{By} and the use of Eq. (26) and energy conservation in the $A+x \to B+y$ vertex makes it possible to find $\tilde{\sigma}_2$ at $E_{Ax} \approx 0$ and to obtain finally the desired $\sigma_2(E)$ and $S(E)$ at $E_{Ax} \approx 0$ by multiplying $\tilde{\sigma}_2$ by the Coulomb penetration factor. Practically, the absolute value of $S(E)$ is found by the normalization to direct measurements at higher energies when the penetration factor $e^{-2\pi\eta_i} \approx 1$.

By comparing the cross section thus obtained with the laboratory one at lower energies one can obtain an information on the electron screening effects. These effects which are essential at very low energies, are accounted by multiplication of the reaction cross section on the 'bare' nucleus by a factor $\exp(\pi\eta U_e/E)$ that results in the increase of the cross section. The TH cross section is free from the screening effects, and its

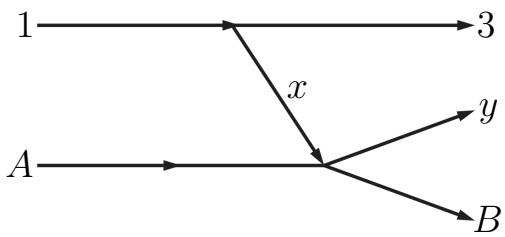

Figure 7: Quasifree mechanism of the $A + 1 \to B + y + 3$ reaction.

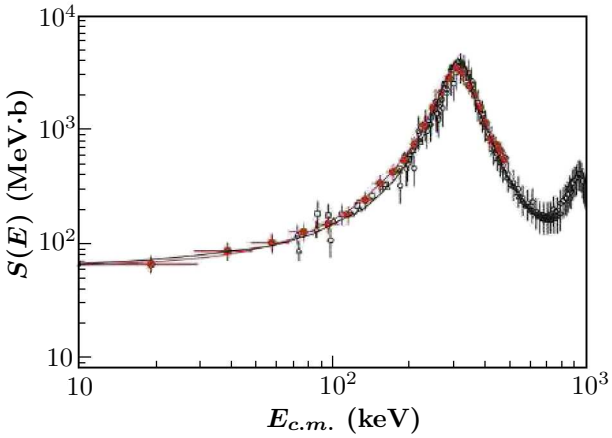

Figure 8: S factor for the $^{15}\mathrm{N}(p,\alpha)^{12}\mathrm{C}$ reaction obtained by the TH method using the $^{15}\mathrm{N}(d,n\alpha)^{12}\mathrm{C}$ reaction at $E_d = 60$ MeV (filled dots). Open dots are the direct data. The line corresponds to the Breit–Wigner fit. See Ref. [11] for details.

comparison with the directly measured cross section allows one to obtain information on U_e.

An example of using the TH method is shown in Fig. 8. Other examples of astrophysical reactions for which $S(0)$ has been found by the TH method (C.Spitaleri, A.M.Mukhamedzhanov *et al.*, INFN-LNS, Catania, Italy) are

$^7\mathrm{Li} + p \to \alpha + \alpha$ (from $d + {}^7\mathrm{Li} \to \alpha + \alpha + n$) ($x = p$, $1 = d$).
$^6\mathrm{Li} + d \to \alpha + \alpha$ (from $^6\mathrm{Li} + {}^6\mathrm{Li} \to \alpha + \alpha + \alpha$) ($x = d$, $1 = {}^6\mathrm{Li}$).
$^6\mathrm{Li} + p \to \alpha + {}^3\mathrm{He}$ (from $d + {}^6\mathrm{Li} \to \alpha + {}^3\mathrm{He} + n$) ($x = p$, $1 = d$).
$^{11}\mathrm{B} + p \to {}^8\mathrm{Be} + \alpha$ (from $d + {}^{11}\mathrm{B} \to {}^8\mathrm{Be} + \alpha + n$) ($x = p$, $1 = d$).

2. Coulomb dissociation method

In this method, the use is made of experimental data on a dissociation of a fast nucleus a in the Coulomb field of a heavy nucleus A (e. g. lead): $a+A \to b+c+A$. The cross section of this process induced by a high energy virtual photon could be related to the photoeffect cross section ($\gamma+a \to b+c$), which by the time reversal is related to the sought-for cross section of the inverse process of the radiative capture $b+c \to \gamma+a$. The strong interaction effects could be reduced if one performs the measurements at low scattering angles when the electromagnetic interaction dominates over the nuclear one.

3. Method of asymptotical normalization coefficients (ANC)

The ANC method [14] allows one to determine $S(E \approx 0)$ for radiative capture reactions using their peripheral character due to the Coulomb (or centrifugal) barrier. The cross section for a non-resonant radiative-capture reaction $b(c,\gamma)a$ at zero relative energy depends only on the long-distance behavior of the $b+c$ wave function (and on the overlap of that extended wave function with that of a). The detailed short-range behavior of the scattering state $b + c$ or the bound state a is not relevant to the reaction mechanism. At large distances the overlap integral of the wave functions of b, c, and a is determined by the corresponding ANC (see (1)).

The ANC needed for the $b(c,\gamma)a$ reaction may be found from another nuclear reaction which mechanism includes the $b+c \to a$ vertex. Usually ANCs are determined from peripheral transfer reactions using the DWBA. The particle energies in the initial and final states can be large enough.

The test of the method has been performed by comparing the experimental data for the $^{16}\mathrm{O}(^3\mathrm{He},d)^{17}\mathrm{F}$ and $^{16}\mathrm{O}(p,\gamma)^{17}\mathrm{F}$ reactions. The ANC method was used for many radiative capture reactions. In particular, the $^{10}\mathrm{B}(^7\mathrm{Be},^8\mathrm{B})^9\mathrm{Be}$ and $^{14}\mathrm{N}(^7\mathrm{Be},^8\mathrm{B})^{13}\mathrm{C}$

Table 1: Summary of updates to S values and derivatives for CNO reactions. The table is taken from review [15].

Reaction	Cycle	$S(0)$ keV b	$S'(0)$ b	$S''(0)$ keV^{-1} b
$^{12}\mathrm{C}(p,\gamma)^{13}\mathrm{N}$	I	1.34 ± 0.21	2.6×10^{-3}	8.3×10^{-5}
$^{13}\mathrm{C}(p,\gamma)^{14}\mathrm{N}$	I	7.6 ± 1.0	-7.83×10^{-3}	7.29×10^{-4}
		7.0 ± 1.5		
$^{14}\mathrm{N}(p,\gamma)^{15}\mathrm{O}$	I	1.66 ± 0.12	-3.3×10^{-3}	4.4×10^{-5}
$^{15}\mathrm{N}(p,\alpha_0)^{12}\mathrm{C}$	I	$(7.3 \pm 0.5) \times 10^4$	351	11
$^{15}\mathrm{N}(p,\gamma)^{16}\mathrm{O}$	II	36 ± 6		
		64 ± 6		
		29.8 ± 5.4		
$^{16}\mathrm{O}(p,\gamma)^{17}\mathrm{F}$	II	10.6 ± 0.8	-0.054	
$^{17}\mathrm{O}(p,\alpha)^{14}\mathrm{N}$	II		Resonances	
$^{17}\mathrm{O}(p,\gamma)^{18}\mathrm{F}$	III	6.2 ± 3.1	1.6×10^{-3}	-3.4×10^{-7}
$^{18}\mathrm{O}(p,\alpha)^{15}\mathrm{N}$	III		Resonances	
$^{18}\mathrm{O}(p,\gamma)^{19}\mathrm{F}$	IV	15.7 ± 2.1	3.4×10^{-4}	-2.4×10^{-6}

reactions were used to obtain the S factor $S(0)$ for an important process $^7\mathrm{Be}(p,\gamma)^8\mathrm{B}$. Other examples of using the ANC method to calculate the $S(E=0)$ for radiative capture processes are

$^4\mathrm{He}(d,\gamma)^6\mathrm{Li}$, $^4\mathrm{He}(^3\mathrm{He},\gamma)^7\mathrm{Be}$, $^{7,9}\mathrm{Be}(p,\gamma)^{8,10}\mathrm{B}$, $^8\mathrm{B}(p,\gamma)^9\mathrm{C}$, $^{11,13}\mathrm{C}(p,\gamma)^{12,14}\mathrm{N}$, $^{12-14}\mathrm{N}(p,\gamma)^{13-15}\mathrm{O}$, $^{17}\mathrm{F}(p,\gamma)^{18}\mathrm{Ne}$, $^{20}\mathrm{Ne}(p,\gamma)^{21}\mathrm{Na}$.

The sensitivity of the extracted cross section to the parameters of the optical potential used in the DWBA, has been also tested.

Nowadays astrophysical factors $S(E \approx 0)$ for numerous astrophysical reactions and their derivatives with respect to energy are determined by various methods (see Table 1). However such data are not available for many important processes, and the accuracy of available data should be improved.

3.8 Nuclear experiments using beams of rare (unstable) isotopes

Unstable nuclei take part in many astrophysical nuclear processes (r-process, rp-process). Experiments using beams of such nuclei are performed actively nowadays. Two main mechanisms of formation and separation of exotic nuclei are:

1. Beams of short-lived nuclei are formed in a thin target and are separated in-flight;

2. Exotic nuclei are formed and stopped in a thick target and then are extracted and accelerated anew (on-line).

Several examples of important astrophysical processes with unstable nuclei measured recently are ($T_{1/2}$ is shown in brackets):

$^7\mathrm{Be}(53\,\mathrm{d})(p,\gamma)^8\mathrm{B}$; $^{13}\mathrm{N}(10\,\mathrm{m})(p,\gamma)^{14}\mathrm{O}$; $^{19}\mathrm{Ne}(17\,\mathrm{s})(p,\gamma)^{20}\mathrm{Na}$; $^{15}\mathrm{O}(122\,\mathrm{s})(\alpha,\gamma)^{19}\mathrm{Ne}$; $^{18}\mathrm{F}(110\,\mathrm{m})(p,\alpha)^{15}\mathrm{O}$; $^{14}\mathrm{O}(71\,\mathrm{s})(\alpha,p)^{17}\mathrm{F}$.

Along with cross section measurements, measuring of unstable nucleus masses is an important goal when dealing with radioactive beams. There are two main methods of mass determination: by energy release in reaction and by deflection of ions in electromagnetic fields.

Recently a considerable progress has been achieved in experimental nuclear astrophysics and in developing theoretical methods of describing astrophysical processes. The further progress in this field is related both with creation of the next generation of installations (GSI/FAIR in Germany and FRIB in the USA) and with modernization of existing installations (GANIL in France and TRIUMF in Canada).

4 Analytic continuation of effective range expansion for $\alpha + d$ system

One of the most widespread methods of obtaining information on bound states from scattering data is an analytic continuation in energy of data on the partial wave amplitude of elastic bc scattering to the pole corresponding to the bound state a. The most efficient way of realization of this procedure is the analytic continuation of the effective range (ER) function $K_L(k^2)$. This method was used successfully in several works (see Refs. [9,16–19]). In these works, NVCs and ANCs were determined for the processes $^6\mathrm{Li} \to \alpha + d$ [9], $^2\mathrm{He} \to p + p$, $^3\mathrm{He} \to p + d$, $^8\mathrm{Be} \to \alpha + \alpha$ [16], $^5\mathrm{He}(^5\mathrm{Li}) \to n(p) + \alpha$ [17], $^{17}\mathrm{O}(^{17}\mathrm{F}) \to n(p) + {}^{16}\mathrm{O}$, $^{16}\mathrm{O} \to {}^{12}\mathrm{C} + \alpha$ [18] and $^7\mathrm{Li}(^7\mathrm{Be}) \to \alpha + t(^3\mathrm{He})$ [19].

All cited works treated a one-channel elastic scattering. However, a description of scattering of particles with nonzero spins usually demands accounting for the channel coupling even in the absence of inelastic channels. The most typical situation induced by tensor forces is the case of two coupled channels, 1 and 2, with the same J^π but different L (L_1 and $L_2 = L_1 + 2$). A generalization of the ER expansion to the case of two coupled channels and its utilization for determination of ANCs and NVCs was considered in Refs. [20, 21] using the np scattering as an example. The formalism developed in Refs. [20, 21] can be applied to any two-channel nuclear system for which the results of the phase-shift analysis are known. One of similar important systems is $^6\mathrm{Li}$ in the $\alpha + d$ channel. The ANC values for this system determine the cross section of the radiative capture $^4\mathrm{He}(d,\gamma)^6\mathrm{Li}$ which is the main source of $^6\mathrm{Li}$ formation in the Big Bang model. Direct measurement of this process at astrophysical energies is impossible due to the smallness of the cross section. Available data on the values of NVCs and ANCs for the $^6\mathrm{Li} \to \alpha + d$ channel ($L = 0; 2$) are characterized by a large spread, especially by the spread of the D state constants G_2 and C_2.

In the work [22], the NVCs and ANCs for $^6\mathrm{Li} \to \alpha + d$ are obtained by analytic continuation of the two-channel ER expansion. Several sets of $d\alpha$ scattering phase shifts are used as an input.

1. The energy-dependent phase-shift analysis of Ref. [23] neglecting the coupling of $L = 0$ and $L = 2$ channels (set 1).

2. The energy-independent phase-shift analysis of Ref. [24] accounting for the channel coupling (set 2).

3. Faddeev calculations neglecting the Coulomb interaction [9] (set 3).

Combining sets 1 and 2 results in $C_0 = 2.3$–2.4 fm$^{-1/2}$. Set 3 gives $C_0 = 2.0$ fm$^{-1/2}$. A low accuracy of phase-shift analysis at low energies and simplicity of Faddeev equations used make it impossible to obtain an accurate value of the ANC for $L = 2$: $C_2 = 0.02$–0.07 fm$^{-1/2}$. The sign of C_2 (relative to C_0) appears to be positive.

The method developed in Refs. [20, 21] and utilized in Ref. [22] considers elastic channels only. On the other hand, low-lying inelastic thresholds may modify the ER expansion. The simplest way to allow for an inelastic channel at $E = E_0$, is to include in the ER expansion an additional term which is complex at $E > E_0$. The form of this term should provide the correct analytic behavior of scattering amplitudes at the threshold $E = E_0$. According to the general theory of singularities of Feynman diagrams, a singular part of a scattering amplitude near a threshold

behaves as $(E-E_0)^{(3n-5)/2}$ for even n and as $(E-E_0)^{(3n-5)/2}\ln(E-E_0)$ for odd n, where $n=2,3,4,...$ is a number of intermediate particles at the threshold.

The work on accounting for inelastic channels in the ER expansion for αd scattering is in progress.

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