Counting Nambu-Goldstone Modes of Higher-Form Global Symmetries

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We discuss the counting of Nambu-Goldstone (NG) modes associated with the spontaneous breaking of higher-form global symmetries. Effective field theories of NG modes are developed based on symmetry-breaking patterns, using a generalized coset construction for higher-form symmetries. We derive a formula of the number of gapless NG modes, which involves expectation values of the commutators of conserved charges, possibly of different degrees.

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Introduction.—Spontaneous symmetry breaking (SSB) is a common thread of modern physics running through various fields from condensed-matter to high-energy physics. When continuous symmetries are spontaneously broken, gapless excitations appear, and they are called the Nambu-Goldstone (NG) modes [1–3] (see Ref. [4] for a recent review). Because of their gapless nature, they dominate the low-energy physics. In relativistic systems, there is a one-to-one correspondence between a broken generator and a gapless mode [5]. However, the Lorentz symmetry is not present in many physically interesting situations, especially in condensed-matter systems. In the absence of Lorentz invariance, the one-to-one correspondence no longer holds [8–11], and the number of NG modes N_{NG} can be smaller than the number of broken symmetry generators N_{BS} [12–17]:

$$N_{\mathrm{NG}} = N_{\mathrm{BS}} - \frac{1}{2}\mathrm{rank}\rho_{ab}, \tag{1}$$

where ρ_{ab} is the matrix of the expectation value of charge commutators, $\rho_{ab} \propto \langle [iQ_a, Q_b] \rangle$.

Recently, those symmetries are understood as a part of a wider class of symmetries, called higher-form symmetries [18] (see also [19–31]). The defining feature of higher-form symmetries is that charged objects under those symmetries are extended: The charged objects for a p-form symmetry are p dimensional. From this point of view, an ordinary symmetry corresponds to a 0-form symmetry, whose charged objects are pointlike. This concept provides us with a unifying perspective and has been used in describing topological orders [18,20–22,32–35], formulating the magnetohydrodynamics [36–41], and realizing a new class [18,42,43] of symmetry-protected topological phases [44]. Similarly to the ordinary symmetries, when a continuous higher-form symmetry is spontaneously broken, gapless excitations appear [18,45]. In fact, photons are understood as the NG bosons associated with the spontaneous breaking of a U(1) 1-form symmetry [18,46].

The question we would like to address in this Letter is, how many gapless NG modes appear when the higher-form symmetries are spontaneously broken? To answer this, we develop effective field theories by extending the coset construction [47,48], which is widely used for ordinary symmetries. In the case of higher-form symmetries, the counting of the number of physical NG modes becomes involved because of the gauge degrees of freedom. We derive a generalized counting formula of gapless NG modes for systems that are not necessarily Lorentz invariant, which reduces to Eq. (1) when all the symmetries are 0-form symmetries.

SSB and coset construction.—We consider the spontaneous breaking of continuous internal symmetries that can include higher-form symmetries in D-dimensional Minkowski spacetime, $\mathbb{R}^{1,D-1}$. We assume that the translational symmetry is unbroken. We label each of the broken generators by A, that generates a p_A-form symmetry. For $p_A \geq 1$, the symmetry is always Abelian [18]. Let us denote the charged object for the p_A-form symmetry by $W_A(C_{p_A})$, which is supported on a closed p_A-dimensional submanifold

C_{p_A}. In the case of a U(1) symmetry, the action of the symmetry is

$$W_A(C_{p_A}) \mapsto e^{i\alpha} W_A(C_{p_A}). \quad (2)$$

The spontaneous breaking of the symmetry A is diagnosed by Coulomb or perimeter behavior of the vacuum expectation value,

$$\langle W_A(C_{p_A})\rangle \sim 1 \quad \text{or} \quad \langle W_A(C_{p_A})\rangle \sim e^{-T\text{perimeter}[C_{p_A}]}, \quad (3)$$

where perimeter$[C_{p_A}]$ is the p_A-dimensional volume of C_{p_A} and T is the tension.

We would like to construct the low-energy effective field theory based on the symmetry-breaking patterns of higher-form symmetries. The coset construction is a powerful technique to write down the action of the effective theory systematically for ordinary symmetries [47–50] and also for spacetime symmetries [6,7,51–53]. The basic building block for this method is the Maurer-Cartan 1-form. To apply the coset construction to higher-form symmetries, we need the corresponding object. For a p_A-form symmetry breaking, we define a generalized Maurer-Cartan form f_A, which is a (p_A+1)-form, by

$$e^{i\int_X f_A} \coloneqq W_A^\dagger(C_{p_A}) W_A(C'_{p_A}), \quad (4)$$

where $W_A(C_{p_A}) = \exp(i\int_{C_{p_A}} a_A)$ is a generalization of coset variables, C_{p_A} and C'_{p_A} are p_A-dimensional submanifolds, and X is a (p_A+1)-dimensional subspace such that $\partial X = C_{p_A} \cup (-C'_{p_A})$ with $-C'_{p_A}$ being the orientation reversal of C'_{p_A}. For a 0-form symmetry, the left-hand side of Eq. (4) should be understood as a path-ordered product. The generalized Maurer-Cartan form f_A should satisfy the flatness condition $df_A = 0$, since the right-hand side of Eq. (4) is independent of the choice of the interpolating manifold X. The corresponding equation for a 0-form symmetry is the Maurer-Cartan equation. Since f_A is a closed (p_A+1)-form, it can be regarded as the conserved current of the dual $(D-p_A-2)$-form symmetry, which is emergent in the symmetry-broken phase. The coset variable has a gauge redundancy under $a_A \to a_A + d\theta_A$, where θ_A is a (p_A-1)-form gauge parameter, because a_A is integrated over a closed subspace C_{p_A}. A symmetry transformation shifts the field a_A by a flat connection, $a_A \mapsto a_A + \lambda_A$ [54]. Using the generalized Maurer-Cartan form as a building block, we can obtain effective actions by writing down the terms consistent with the spacetime symmetry of the systems of interest. More details on this construction will be given elsewhere [55].

Generalized counting rule.—We here present the generalized counting rule of gapless NG modes associated with spontaneously broken higher-form symmetries. In Lorentz-invariant systems, the number of gapless modes for a broken generator A of a p_A-form symmetry is given by [56,57]

$$\mathcal{N}_{D,A} = {}_{D-2}C_{p_A}. \quad (5)$$

Here, D is the spacetime dimensions and ${}_nC_m$ denotes the binomial coefficient where n and m are integers. In the absence of Lorentz invariance, the number can be reduced. In this Letter, we show that the total number of gapless NG modes is given by

$$N_{\rm NG} = \sum_A \mathcal{N}_{D,A} - \frac{1}{2}\text{rank}M_{\alpha\beta}, \quad (6)$$

where the summation is over all the broken generators [58]. Here, $M_{\alpha\beta}$ is an antisymmetric matrix, whose definition will be given below. Let us first define a quantity proportional to the expectation values of charge commutators:

$$M_{AB}(V_A, V_B) \coloneqq \frac{\langle [iQ_A^{[p_A]}(V_A), Q_B^{[p_B]}(V_B)]\rangle}{\text{vol}[V_A \cap V_B]}. \quad (7)$$

Here, we denote the conserved charge associated with the generator A as $Q_A^{[p_A]}(V_A)$, which is supported on a $(D-p_A-1)$-dimensional subspace V_A located in the spatial slice Σ, and vol$[V]$ indicates the volume of a subspace V. To account for the independent choices of subspaces $V_{A,B}$, we shall consider the following matrix:

$$M_{AB}^{i_1\cdots i_{p_A}, j_1\cdots j_{p_B}} \coloneqq \lim_{\substack{V_{i_1\cdots i_{p_A}}\to\infty \\ V_{j_1\cdots j_{p_B}}\to\infty}} M_{AB}(V_{i_1\cdots i_{p_A}}, V_{j_1\cdots j_{p_B}}), \quad (8)$$

where indices of $V_{i_1\cdots i_{p_A}}$ specify a $(D-p_A-1)$-dimensional plane placed inside the $(D-1)$-dimensional spatial manifold Σ. The indices $i_1, \cdots$ take only spatial ones and are ordered, $i_1 < i_2 < \cdots < i_{p_A}$. For example, in the case of a 1-form symmetry in 4-spacetime dimensions, i.e., $D=4$ and $p_A=1$, the symmetry generator is a two-dimensional plane and V_i represents the plane perpendicular to the i axis for $i = x, y, z$. In taking the limit, we first take $V_{i_1\cdots i_{p_A}} \to \infty$ and then take $V_{j_1\cdots j_{p_B}} \to \infty$. We collectively denote the indices using the Greek letters as $\alpha \coloneqq [A, (i_1, \ldots, i_{p_A})]$. With this notation, the matrix (8) is denoted as $M_{\alpha\beta}$, that appears in the formula (6). $M_{\alpha\beta}$ is a real antisymmetric [59] square matrix of dimension $\sum_A {}_{D-1}C_{p_A}$. Its rank is always even, and $(\text{rank}M_{\alpha\beta})/2$ is an integer. For a higher-form symmetry, a generator is supported on a closed subspace of dimension less than $D-1$. When we place them in the spatial slice Σ, we have to specify how to place it, which is the role of the indices $(i_1, \ldots, i_{p_A})$. Thus, the matrix $M_{\alpha\beta}$ contains the information of the expectation values of charge commutators, including how to place them, and that determines the number of gapless modes.

The formula (6) is a natural generalization of the one for 0-form symmetries [12–14]. When all the symmetries are 0-form symmetries, $p_A = 0$ for all A, then $\sum_A \mathcal{N}_{D,A} = \sum_A 1 = N_{\mathrm{BS}}$, and all the symmetry generators are supported on a $(D-1)$-dimensional subspace, $Q_A^{[0]}(V_{D-1})$. Then, the formula (6) reduces to the counting rule (1) for 0-form symmetries.

Let us make a comment on the dispersion relations. For 0-form symmetries, the NG modes are classified into type A and B, and they typically have linear and quadratic dispersion relations, respectively. When higher-form symmetries are involved, it is still possible to classify the modes into type A and B, but it is not clear if we can associate this to the behavior of dispersion relations in general. For example, whether the dispersion is linear or quadratic may depend on the direction of the propagation, as can be seen in an example discussed later [see Eq. (14)].

Examples.—To illustrate the coset construction and the counting rule, let us confirm that the photons are NG modes associated with a U(1) 1-form symmetry [61], which we denote by $\mathrm{U}(1)_e^{[1]}$ [65]. When $\mathrm{U}(1)_e^{[1]}$ is spontaneously broken, the coset variable is defined on a loop C and can parametrized as $W(C) \coloneqq e^{i\int_C a}$, which is nothing but the Wilson loop. Since it is defined on a loop, the field a has a gauge redundancy, $a \mapsto a + d\theta_0$, with a U(1) 0-form parameter $e^{i\theta_0}$. Therefore, we can identify a as a U(1) 1-form gauge field. The $\mathrm{U}(1)_e^{[1]}$ transformation of a is locally given by $a \mapsto a + \lambda_1$ with a flat 1-form λ_1. The generalized Maurer-Cartan form f is defined through $e^{i\int_S f} \coloneqq W^\dagger(C)W(C')$, where C and C' are loops and S is a two-dimensional surface with $\partial S = C' \cup (-C)$. We can identify f as $f = da$, and the flatness condition $df = 0$ is nothing but the Bianchi identity. In the Lorentz-invariant case, the gauge-invariant term with the lowest number of derivatives is given by

$$-\frac{1}{2e^2} f \wedge \star f, \tag{9}$$

where e is a coupling constant and $\star$ is the Hodge star operator. This is nothing but the Maxwell theory. There are ${}_{D-2}C_1$ gapless excitations in D spacetime dimensions (two photons for $D = 4$), which is consistent with the formula (6).

As a second example, let us consider photons in the presence of the gradient of the θ angle in $(3+1)$ dimensions [66], whose Lagrangian is given by

$$-\frac{1}{2e^2} f \wedge \star f + \frac{c}{2}\theta f \wedge f, \tag{10}$$

where c is a constant. The theory has $\mathrm{U}(1)_e^{[1]}$ symmetry as in the case of the Maxwell theory, and the corresponding conserved charge is identified as $Q_e^{[1]}(S) = (1/e^2)\int_S \star f - c\int_S \theta f$. This symmetry is spontaneously broken, and $Q_e^{[1]}(S)$ is a broken generator. Because of the presence of the nonvanishing $d\theta$, the charge $Q_e^{[1]}(S)$ becomes noncommutative:

$$\langle [iQ_e^{[1]}(S_1), Q_e^{[1]}(S_2)] \rangle \propto \int_{S_1 \cap S_2} d\theta \neq 0, \tag{11}$$

where $S_1, S_2 \subset \Sigma$ are two-dimensional surfaces located inside the spatial manifold Σ. Let us organize the charges as a vector, $Q_i = [Q_e^{[1]}(S_x), Q_e^{[1]}(S_y), Q_e^{[1]}(S_z)]^T$, where S_i is the surface perpendicular to the i axis. The charge commutator matrix is given by

$$M_{ij} = \langle [iQ_i, Q_j] \rangle \propto \epsilon_{ijk}\partial_k \theta, \tag{12}$$

where the gradient $\partial_k \theta$ is constant. The rank of this matrix is 2, and the number of gapless NG modes via the formula (6) is

$$N_{\mathrm{NG}} = 2 - \frac{1}{2}\mathrm{rank} M_{ij} = 1. \tag{13}$$

This coincides with the result obtained by explicitly solving the equations of motion (EOM). There are one gapless and one gapped modes [66], whose gap squared is given by $\omega^2(k=0) = C^2$. Here, $C \coloneqq |\boldsymbol{C}|$ is the length of the vector $\boldsymbol{C} \coloneqq -ce^2\nabla\theta$. The behavior of the dispersion relation depends on the direction of the momentum $\boldsymbol{k}$. When the angle of $\boldsymbol{k}$ and $\boldsymbol{C}$ is ϕ, the dispersion relation at small $k = |\boldsymbol{k}|$ is (see Supplemental Material [65])

$$\omega^2(k) = \begin{cases} (\sin^2\phi)k^2 + \frac{\cos^4\phi}{C^2}k^4 + O(k^6), \\ C^2 + (2 - \sin^2\phi)k^2 + O(k^4). \end{cases} \tag{14}$$

The dispersion relation of the gapless mode is quadratic, $\omega \sim k^2$, only when $\sin\phi = 0$ ($\boldsymbol{k}$ is parallel to $\boldsymbol{C}$) and is linear, $\omega \sim k$, for other angles.

A similar interpretation is possible for the system of neutral pions in the presence of background magnetic field [67,68], where the commutators of 0-form and 1-form symmetry generators acquire expectation values.

Derivation.—Let us now give the derivation of the counting formula (6). For $p_A \geq 1$, the higher-form symmetry is Abelian, so f_A can be written as da_A. For a 0-form non-Abelian symmetry, f_A includes nonlinear terms that represent the interaction between NG modes. For the counting of the modes, it suffices to consider the linear order. Thus, we can also express the Maurer-Cartan form for 0-form symmetry as $f_A = da_A$, and we include indices of the Lie algebra in A.

For Lorentz-invariant systems, the kinetic terms are written, to the lowest order in derivatives and fields, as

$$\mathcal{L}_0(da_A) = -\frac{1}{2}F^2_{AB} f_A \wedge \star f_B = -\frac{1}{2}F^2_{AB} da_A \wedge \star da_B, \tag{15}$$

which is invariant under $a_A \mapsto a_A + \lambda_A$. Here, $F^2_{AB} = F_{CA}F_{CB}$ is a positive-definite symmetric matrix, and F_{AB} corresponds to the decay-constant matrix. The Lagrangian can also have topological terms of the form $G_{AB} f_A \wedge f_B$, where G_{AB} are flat coefficients. Those terms do not contribute to the EOM, so they can be dropped. Since we can diagonalize F^2_{AB}, it suffices to count the modes when one p_A-form symmetry is spontaneously broken. The EOM and the flatness condition [69] read

$$d^\dagger f_A = 0, \qquad df_A = 0, \tag{16}$$

where $d^\dagger$ is the codifferential. One can immediately see that the field strength satisfies $\Delta f_A = 0$, where $\Delta \coloneqq dd^\dagger + d^\dagger d$ is the Hodge Laplacian. In the momentum space, the Hodge Laplacian is given by $-\omega^2 + k^2$. Each component of the field strength satisfies the wave equation, but not all of them are independent. The Bianchi identity and EOM have ${}_{D-2}C_{p_A+1}$ and ${}_{D-2}C_{p_A-1}$ relations without time derivatives, and these are the constraints. The number of physical modes is given by the number of components of the field strength, ${}_DC_{p_A+1}$, minus the number of constraints. Since the physical modes consist of pairs of these degrees of freedom, we find

$$\mathcal{N}_{D,p_A} = \frac{1}{2}({}_DC_{p_A+1} - {}_{D-2}C_{p_A+1} - {}_{D-2}C_{p_A-1}) = {}_{D-2}C_{p_A}. \tag{17}$$

Thus, the number of gapless modes in the Lorentz-invariant system is given by the sum of this number over each generator:

$$N_{\text{gapless}} = \sum_A \mathcal{N}_{D,p_A}. \tag{18}$$

Now let us discuss the case without Lorentz invariance. The breaking of the Lorentz invariance means that there are special directions in spacetime. We still assume that the translational symmetry is not broken. A key observation here is that the absence of the Lorentz symmetry can be represented as the presence of external objects placed in the spacetime. This allows us to write additional terms with one derivative:

$$\mathcal{L}_1(a_A, f_A) \coloneqq \frac{1}{2} f_A \wedge a_B \wedge \Omega_{AB}, \tag{19}$$

where Ω_{AB} is a flat d-form with $d \coloneqq D - p_A - p_B - 1$. The components of Ω_{AB} satisfy $\Omega_{AB} = -(-)^{p_A p_B}\Omega_{BA}$ so that they are not a total derivative. Those terms (19) are invariant under symmetry transformations only up to a total derivative and are reminiscent of the Wess-Zumino term [70,71]. We assume here that the external defect is located *spatially*, by which we mean that Ω_{AB} does not involve dx^0. Unless both A and B are both 0-form symmetries, the external defect Ω_{AB} explicitly breaks the spatial rotation symmetry, which can lead to direction-dependent dispersion relations, as is seen in Eq. (14). The terms with two derivatives can also be more generic compared to the relativistic case. Although such a deformation leads to more generic dispersion relations, it does not change the number of constraints, so the number of physical gapless modes is unaffected. Thus, for the purpose of the counting of the modes, it suffices to use the following Lagrangian:

$$\mathcal{L} = \mathcal{L}_0 + \mathcal{L}_1, \tag{20}$$

where $\mathcal{L}_0$ is given by Eq. (15).

We can identify the existence of the terms (19) with the nonvanishing expectation values of charge commutators. The Noether current associated with the generator A is given by

$$\star J_A = -F^2_{AB}\star f_B + a_B \wedge \Omega_{AB}. \tag{21}$$

The symmetry generator is obtained by integrating the current over a submanifold $V_{D-p_A-1} \subset \Sigma$:

$$Q_A^{[p_A]}(V_{D-p_A-1}) = \int_{V_{D-p_A-1}} \star J_A. \tag{22}$$

Since the action is quadratic, the current commutator can be readily computed using the canonical commutation relations or the Ward-Takahashi identity as

$$\begin{aligned} &\langle [iQ_A^{[p_A]}(V_{D-p_A-1}), Q_B^{[p_B]}(V_{D-p_B-1})] \rangle \\ &\quad\propto \int_{V_{D-p_A-1}\cap V_{D-p_B-1}} \Omega_{AB}. \end{aligned} \tag{23}$$

Now we are ready to discuss the number of physical gapless modes. The EOM of the Lagrangian (20) are written in the form of current conservation, $d\star J_A = 0$. Let us write this with gauge-invariant variables $\tilde{f}_A = F_{AB} f_B$ as

$$d\star \tilde{f}_A = \tilde{f}_B \wedge \tilde{\Omega}_{AB}, \tag{24}$$

where $\tilde{\Omega}_{AB} \coloneqq F_{AC}\Omega_{CD}F_{BD}$. If we write out Eq. (24) explicitly with indices,

$$\begin{aligned} &-\partial^\alpha(\tilde{f}_A)_{\mu_1\cdots\mu_{p_A}\alpha} \\ &\quad= \frac{\epsilon_{\mu_1\cdots\mu_{p_A}}{}^{\nu_1\cdots\nu_{p_B+1}\rho_1\cdots\rho_d}}{(p_B+1)!d!}(\tilde{\Omega}_{AB})_{\rho_1\cdots\rho_d}(\tilde{f}_B)_{\nu_1\cdots\nu_{p_B+1}}. \end{aligned} \tag{25}$$

In the relativistic case, we have $\sum_A {}_{D-2}C_{p_A}$ gapless modes, but the presence of Ω_{AB} makes a part of them gapped. We shall here count the number of *gapped* modes. For this purpose, we write Eq. (25) in the energy or momentum space and take $\boldsymbol{k} = \boldsymbol{0}$ and $\omega \neq 0$:

$$-i\omega(\tilde{f}_A)_{i_1\cdots i_{p_A}0} = \epsilon_{i_1\cdots i_{p_A}}{}^{j_1\cdots j_{p_B}0k_1\cdots k_d}(\tilde{\Omega}_{AB})_{k_1\cdots k_d}(\tilde{f}_B)_{j_1\cdots j_{p_B}0}, \tag{26}$$

where the Roman indices $i_1, \ldots, j_1, \ldots, k_1, \ldots$ are spatial ones. The sum of the indices is implicitly taken such that they satisfy $k_1 < k_2 < \cdots < k_d$, and $j_1 < j_2 < \cdots < j_{p_B}$, which removes the redundancy in the sum. Let us introduce a matrix notation

$$\tilde{M}_{AB}^{i_1\cdots i_{p_A},j_1\cdots j_{p_B}} = \epsilon_{i_1\cdots i_{p_A}}{}^{j_1\cdots j_{p_B}0k_1\cdots k_d}(\tilde{\Omega}_{AB})_{k_1\cdots k_d}. \tag{27}$$

Then we have

$$-i\omega(\tilde{f}_A)_{i_1\cdots i_{p_A}0} = \tilde{M}_{AB}^{i_1\cdots i_{p_A},j_1\cdots j_{p_B}}(\tilde{f}_B)_{j_1\cdots j_{p_B}0}. \tag{28}$$

By using the shorthand notation $\alpha \coloneqq [A, (i_1, \ldots, i_{p_A})]$, it can be further written as $-i\omega\tilde{f}_\alpha = \tilde{M}_{\alpha\beta}\tilde{f}_\beta$. The matrix $\tilde{M}_{\alpha\beta}$ is antisymmetric, and it can be always transformed to the following form by an orthogonal matrix O:

$$O\tilde{M}O^T = \begin{pmatrix} i\sigma_2\lambda_1 & & & \\ & \ddots & & \\ & & i\sigma_2\lambda_m & \\ & & & \boldsymbol{0} \end{pmatrix}, \tag{29}$$

where σ_2 is the second Pauli matrix, $m \coloneqq (\mathrm{rank}\tilde{M}_{\alpha\beta})/2$, and $\lambda_i \neq 0$ for $i = 1, \ldots, m$. Each 2×2 sector in the upper-left part gives a gapped mode. For example,

$$-i\omega\begin{pmatrix} \tilde{f}_{\alpha=1} \\ \tilde{f}_{\alpha=2} \end{pmatrix} = \begin{pmatrix} 0 & \lambda_1 \\ -\lambda_1 & 0 \end{pmatrix}\begin{pmatrix} \tilde{f}_{\alpha=1} \\ \tilde{f}_{\alpha=2} \end{pmatrix} \tag{30}$$

represents a gapped excitation whose gap is given by $\omega^2 = \lambda_1^2$. Thus, the number of gapped modes is given by

$$N_{\text{gapped}} = \frac{1}{2}\mathrm{rank}\tilde{M}_{\alpha\beta} = \frac{1}{2}\mathrm{rank}M_{\alpha\beta}, \tag{31}$$

where $M_{\alpha\beta}$ is defined similarly to $\tilde{M}_{\alpha\beta}$ by replacing $\tilde{\Omega}_{AB}$ with Ω_{AB} in Eq. (27) and we used the fact that F_{AB} is invertible. The matrix $M_{\alpha\beta}$ is nothing but the matrix (8). Since the presence of $\mathcal{L}_1$ does not change the number of physical degrees of freedom, the number of *gapless* modes in the absence of Lorentz invariance is given by Eq. (18) minus Eq. (31), which is the formula (6).

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