Letters in Mathematical Physics (2025) 115:47
https://doi.org/10.1007/s11005-025-01935-y

New partial trace inequalities and distillability of Werner states

Pablo Costa Rico[1,2]

Received: 21 November 2024 / Revised: 19 March 2025 / Accepted: 14 April 2025

Abstract
One of the oldest problems in quantum information theory is to study if there exists a state with negative partial transpose which is undistillable [1]. This problem has been open for almost 30 years, and still no one has been able to give a complete answer to it. This work presents a new strategy to try to solve this problem by translating the distillability condition on the family of Werner states into a problem of partial trace inequalities, this is the aim of our first main result. As a consequence, we obtain a new bound for the 2-distillability of Werner states, which does not depend on the dimension of the system. On the other hand, our second main result provides new partial trace inequalities for bipartite systems, connecting some of them also with the separability of Werner states. Throughout this work, we also present numerous partial trace inequalities, which are valid for many families of matrices.

Keywords Werner states · Distillability · Partial trace · Bound entanglement · Trace inequalities

1 Introduction

The theory of quantum entanglement, introduced in [14] in 1935 by Einstein, Podolsky and Rosen, has been one of the central topics of debate and progress in the last century in quantum mechanics. However, many questions remain to be solved in this field, and in this work, we will discuss one of them, the famous problem stated in [1]: Study if there exists a state with negative partial transpose which is undistillable. A quantum state $\rho \in L(\mathscr{H})$, where $\mathscr{H} = \mathscr{H}_1 \otimes \mathscr{H}_2$, is a positive semidefinite matrix with $\operatorname{tr} \rho = 1$. A state is called separable if it can be written as a convex sum of tensor products of positive semidefinite matrices. Otherwise, it is called entangled.

✉ Pablo Costa Rico
pablo.costa@tum.de

1 Department of Mathematics, Technische Universität München, Garching, Germany

2 Munich Center for Quantum Science and Technology (MCQST), Munich, Germany

Published online: 04 May 2025

In this paper, we will focus on the fundamental property called distillability: Suppose that we have two parties, call them Alice and Bob, who share n-copies of the same state $\rho \in L(\mathbb{C}^d \otimes \mathbb{C}^d)$, $\rho \geq 0$, $\operatorname{tr} \rho = 1$, and that both perform a local operation obtaining a new state of the form

$$\rho' = \frac{A \otimes B\rho^{\otimes n} A^* \otimes B^*}{\operatorname{tr}[A \otimes B\rho^{\otimes n} A^* \otimes B^*]}, \tag{1}$$

with $A, B : (\mathbb{C}^d)^{\otimes n} \to \mathbb{C}^2$, and where "*" denotes the adjoint matrix. If it is possible to find a pair of operations (A, B) such that the resulting state ρ' is entangled, it is said that ρ is n-distillable (see e.g., [22]) . If, on the other hand, for any pair of operations (A, B) the state ρ' is always separable, we say that ρ is n-undistillable. If for every $n \in \mathbb{N}$, ρ is n-undistillable, then ρ is called simply undistillable, otherwise it is distillable. An alternative definition is that ρ is n-undistillable if, for every Schmidt rank 2 vector $v \in (\mathbb{C}^d \otimes \mathbb{C}^d)^{\otimes n}$,

$$\langle v, \left(\rho^{T_1}\right)^{\otimes n} v\rangle \geq 0, \tag{2}$$

where T_1 denotes the partial transposition and with the Schmidt rank defined as the minimum number of terms needed to express a quantum state as a sum of tensor product states, see [26, 28] or [13].

In [20], it was shown that it is enough to reduce the distillability problem to the family of Werner states defined as (see e.g., [28] or [34])

$$\rho_\alpha = \frac{\mathbb{1} + \alpha F}{d^2 + \alpha d}, \tag{3}$$

where $\alpha \in [-1, 1]$ and F is the flip operator acting on tensor products as $F(x \otimes y) = y \otimes x$, for $x, y \in \mathbb{C}^d \otimes \mathbb{C}^d$. The reason for that, is that every state with a non-positive partial transpose can be mapped onto a Werner state with a non-positive partial transpose via the twirling map

$$\mathscr{T}(\rho) = \int_{U(d)} (U \otimes U)\rho(U \otimes U)^* dU, \tag{4}$$

where dU denotes the Haar measure on the unitary group of $d \times d$ matrices $U(d)$. This twirling operator is in particular a form of local operations and classical communication (LOCC) since it consists of a convex combination of local unitary operators. Therefore, the existence of undistillable states with non-positive partial transpose can be decided just by focusing on Werner states. This family of states satisfy the following properties:

1. ρ_α is separable $\Leftrightarrow$ ρ_α has positive partial transpose $\Leftrightarrow$ $\alpha \geq -\frac{1}{d}$.
2. For $n = 1$ in 1, ρ_α is 1-undistillable $\Leftrightarrow$ $\alpha \geq -\frac{1}{2}$.

Moreover, in [23] it is conjectured that this family might contain a subfamily of states which are undistillable but with non-positive partial transpose. In the last decades, there have been many different approaches to this problem, some of them leading to

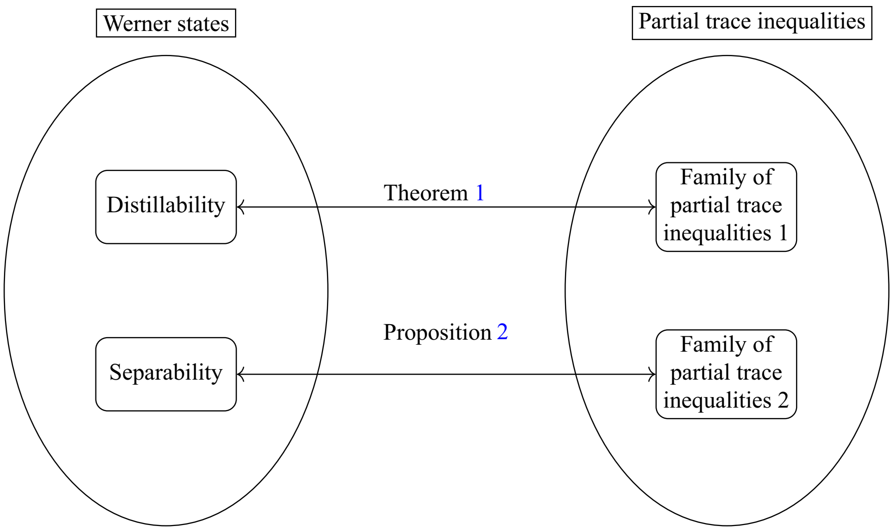

Fig. 1 Connection between partial trace inequalities and properties of Werner states

particular results that have been proved for the distillability of the Werner states , for example in [6, 7, 26, 28], but still the problem remains open. The conjecture on this family of Werner states is the following:

Conjecture 1 *Let* $\alpha \in [-1, 1]$*,* $d \geq 2$*. A Werner state* $\rho_\alpha \in L(\mathbb{C}^d \otimes \mathbb{C}^d)$*, is n-undistillable for every* $n \in \mathbb{N}$ *if, and only if* $\alpha \geq -\frac{1}{2}$*.*

A positive answer to Conjecture 1 would solve then the problem of finding a state with negative partial transpose and is undistillable.

1.1 Summary of main results and structure of this work

In this work, we provide a new characterization for the Conjecture 1 in terms of partial trace inequalities for the 2-norm, which depend on the parameter α associated with the Werner states 3. This is the goal of the first main result, Theorem 1, which is presented in section 3. Moreover in Proposition 2, we also show the connection between the separability and another family of partial trace inequalities. This connection is showed in Fig. 1.

In section 4, we will study how the quadratic forms associated to state inversion operators studied in e.g., [15, 16] or [25] are related with the distillability and separability properties of Werner states and tensor product of Werner states. We will exploit the correspondence shown in the previous figure to obtain new results on both partial trace inequalities and properties of Werner states. In Proposition 2, we will use that ρ_α is separable for $\alpha \geq -\frac{1}{d}$ to prove partial trace inequalities in n-partite systems for arbitrary matrices.

Theorem 2 is our second main result, where we present the partial trace inequalities that we have been able to prove for bipartite systems. The spirit will be then the opposite as before, that is, we will try to prove partial trace inequalities to obtain information on the 2-distillability properties of Werner states. In Theorem 2 appear four partial trace inequalities: One in the family 1 related to the distillability, the second one related to the family 2, and two others more concerning the distillability of the tensor product of two Werner states with different sign in the parameter α (see Remark 6). In particular, we show that both $\rho_{\frac{1}{2}} \otimes \rho_{-\frac{1}{2}}$ and $\rho_{-\frac{1}{2}} \otimes \rho_{\frac{1}{2}}$ are 1-distillable. The proof of Theorem 2 is presented in Section 7, due to its length.

Another remarkable result is Corollary 1 in section 3. There, we prove that for $\alpha \geq -\frac{1}{4}$, the Werner states ρ_α are 2-undistillable for every dimension $d \geq 2$. This becomes relevant for $d \geq 5$, since then for $\alpha \in (-\frac{1}{d}, -\frac{1}{4}]$ this implies that the states ρ_α are 2-undistillable and entangled.

In section 5, we discuss about partial trace inequalities in tripartite systems, and prove a particular case of the quadratic form associated to the 3-distillabilty. Finally in section 6, we present numerical results showing the existence of general families of partial trace inequalities for all the Schatten p-norms.

2 Preliminaries

Let $\mathscr{H}$ be a finite-dimensional Hilbert space. We will denote the set of bounded linear operators in $\mathscr{H}$ by $L(\mathscr{H})$. For $T \in L(\mathscr{H})$ the Schatten p-norms are defined for $p > 0$ as

$$\|T\|_p = \left(\operatorname{tr} |T|^p\right)^{\frac{1}{p}}, \tag{5}$$

where $|T| = \sqrt{TT^*}$. In particular, for $p = 2$, this norm comes from an inner product in $L(\mathscr{H})$ called Hilbert-Schmidt product defined as

$$\langle T, S\rangle = \operatorname{tr}(T^* S), \tag{6}$$

for $T, S \in L(\mathscr{H})$. The case $p = \infty$ corresponds with the operator norm. In the particular case where $\mathscr{H} = \mathscr{H}_1 \otimes \mathscr{H}_2$, one can define the partial trace operator valued functions for $T \in L(\mathscr{H}_1 \otimes \mathscr{H}_2)$, $T = \sum_{i=1}^{n} T_i^1 \otimes T_i^2$ as

$$\operatorname{tr}_1 T = \sum_{i=1}^{n} \operatorname{tr}(T_i^1) T_i^2, \qquad \operatorname{tr}_2 T = \sum_{i=1}^{n} \operatorname{tr}(T_i^2) T_i^1, \tag{7}$$

which are independent from the choice of decomposition in tensor products. The following inequalities show some well-known bounds for the norms 1 and 2

$$\|T\|_2 \leq \|T\|_1 \leq \sqrt{r}\|T\|_2, \tag{8}$$

$$\|T\|_2^2 \geq \frac{1}{r}|\operatorname{tr} T|^2, \tag{9}$$

$$\|\operatorname{tr}_i T\|_1 \leq \|T\|_1 \tag{10}$$

for $i = 1, 2$ and where $r = \text{rank}(T)$. For 10 see e.g., [31].

Remark 1 Many times in the literature in quantum mechanics, the bra-ket notation is used for normalized vectors. In this work, however, we will use this notation to denote arbitrary rank 1 matrices, i.e., matrices of the form $|v\rangle\langle w|$ where $v, w \in \mathscr{H}$, but not vectors or associated functionals. There will be only one exception for this in the proof of Proposition 5, where it simplifies the notation, and it is also indicated there.

Given a Hilbert space $\mathscr{H}$, the symmetric and antisymmetric subspaces for two copies of $\mathscr{H}$ are defined as

$$\mathscr{H}_+ = \{v \in \mathscr{H} \otimes \mathscr{H} : Fv = v\}, \quad \mathscr{H}_- = \{v \in \mathscr{H} \otimes \mathscr{H} : Fv = -v\}, \tag{11}$$

respectively, where F is the flip operator. The respective orthogonal projections are given by

$$P_+ = \frac{\mathbb{1} + F}{2}, \quad P_- = \frac{\mathbb{1} - F}{2}. \tag{12}$$

For $v, w \in \mathscr{H}$, define the symmetric product $\odot : \mathscr{H} \otimes \mathscr{H} \to \mathscr{H}_+$ and the antisymmetric product $\wedge : \mathscr{H} \otimes \mathscr{H} \to \mathscr{H}_-$

$$v \odot w = v \otimes w + w \otimes v, \quad v \wedge w = v \otimes w - w \otimes v. \tag{13}$$

Finally, the bosonic and fermionic creation operators acting on $w \in \mathscr{H}$ are

$$a_+^*(v)w = \sqrt{2}P_+(v \otimes w) = \frac{1}{\sqrt{2}}(v \odot w), \quad a_-^*(v)w = \sqrt{2}P_-(v \otimes w) = \frac{1}{\sqrt{2}}(v \wedge w), \tag{14}$$

respectively, for $v \in \mathscr{H}$, and the bosonic and fermionic annihilation operators on $\varphi \in \mathscr{H} \otimes \mathscr{H}$ are just

$$a_+(v)(\varphi) = \sqrt{2}\langle v, P_+\varphi\rangle_1, \quad a_-(v)(\varphi) = \sqrt{2}\langle v, P_-\varphi\rangle_1, \tag{15}$$

where $\langle\ ,\ \rangle_1 : \mathscr{H} \times \mathscr{H}^{\otimes 2} \to \mathscr{H}$ is the partial inner product in the first argument, i.e., the sesquilinear extension of

$$(v, \varphi_1 \otimes \varphi_2) \mapsto \langle v, \varphi_1\rangle\varphi_2. \tag{16}$$

See [4] for a more general definition of the creation and annihilation operators in the Fock space.

3 Distillability of Werner states

In recent times, on the way to solving the Werner states' distillability problem, several equivalent problems have been proposed in order to approach it with different strategies. A first example is the one formulated in [28] for the $\mathbb{C}^4 \otimes \mathbb{C}^4$ system, where the 2-distillability problem is equivalent to show that for matrices $A, B \in L(\mathbb{C}^4)$, with

$\operatorname{tr} A = \operatorname{tr} B = 0$ and $\|A\|_2^2 + \|B\|_2^2 = \frac{1}{2}$, the two largest singular values squared of the Kronecker sum $A \otimes \mathbb{1} + \mathbb{1} \otimes B$ are upper bounded by $\frac{1}{2}$. A second one is provided in [11] and it relates the undistillability with the existence of a completely positive map, which is not completely co-positive but 2-copositive in all its tensor n-th tensor power, see also [23]. Our first main result presents a new characterization in terms of partial trace inequalities.

Theorem 1 *Let $\mathscr{H}$ be a finite-dimensional Hilbert space that can be decomposed as $\mathscr{H} = \mathscr{H}_1 \otimes \ldots \otimes \mathscr{H}_n$, with $\dim(\mathscr{H}_i) = d_i$, and define for $\alpha \in \mathbb{R}$ the quadratic form*

$$q^{(n)}(\alpha, C) = \sum_{J \in P(\{1,2,\ldots,n\})} \alpha^{|J|} \| \operatorname{tr}_J C\|_2^2, \tag{17}$$

where $P(X)$ is the power set of X and we denote $tr_{\emptyset} = \mathbb{1}$. Then, ρ_α is n-distillable if and only if there exists a matrix $C \in L((\mathbb{C}^d)^{\otimes n})$ with $\operatorname{rank} C \leq 2$ such that $q^{(n)}(\alpha, C) < 0$, with $\alpha \in [-1, 1]$.

Proof Suppose that ρ_α is n-copies distillable, i.e, there exists A, B such that $\rho' \in L(\mathbb{C}^2 \otimes \mathbb{C}^2)$ in 1 is entangled. Since the Hilbert space for ρ' is $\mathbb{C}^2 \otimes \mathbb{C}^2$, this implies (see [21]) that $(\rho')^{T_1} \ngeq 0$, so there exists an element $\psi \in \mathbb{C}^2 \otimes \mathbb{C}^2$ such that

$$\langle \psi, (\rho')^{T_1} \psi \rangle < 0. \tag{18}$$

Let $V \in L(\mathbb{C}^2)$ such that $\psi = (\mathbb{1} \otimes V^*)\Omega$, where Ω is the maximally entangled state and denote by P_Ω the orthogonal projection onto Ω, and by $F^{\mathbb{C}^2}$ the flip operator in $\mathbb{C}^2 \otimes \mathbb{C}^2$, which satisfy the relation $P_\Omega^{T_1} = \frac{1}{2} F^{\mathbb{C}^2}$. We can then write

$$\begin{aligned}
\langle \psi, (\rho')^{T_1} \psi \rangle &= \operatorname{tr}[P_\Omega(\mathbb{1} \otimes V)(\rho')^{T_1}(\mathbb{1} \otimes V)^*] && \text{(19a)}\\
&= \operatorname{tr}[P_\Omega^{T_1}(\mathbb{1} \otimes V)(\rho')(\mathbb{1} \otimes V)^*] && \text{(19b)}\\
&= \frac{1}{2} \operatorname{tr}[F^{\mathbb{C}^2}(\mathbb{1} \otimes V)(\rho')(\mathbb{1} \otimes V)^*] && \text{(19c)}\\
&\backsim \operatorname{tr}[F^{\mathbb{C}^2}(A \otimes VB)(\rho_{A_1B_1} \otimes \ldots \otimes \rho_{A_nB_n})(A \otimes VB)^*], && \text{(19d)}
\end{aligned}$$

where $\backsim$ means up to the normalization factor. Defining $D = VB$ and using the cyclical property of the trace,

$$\operatorname{tr}\left[(\rho_{A_1B_1} \otimes \ldots \otimes \rho_{A_nB_n})(A \otimes D)^* F^{\mathbb{C}^2} (A \otimes D)\right] < 0. \tag{20}$$

Let $\tilde{F}$ be the linear extension of the operator which acts on the tensor product as $\tilde{F}(x \otimes y) = y \otimes x$, $x, y \in (\mathbb{C}^d)^{\otimes n}$. Then,

$$(A \otimes D)^* F^{\mathbb{C}^2}(A \otimes D)(x \otimes y) = (A^* D y) \otimes (D^* A x) = (A^* D \otimes D^* A)(y \otimes x),$$

so we obtain the relation

$$(A \otimes D)^* F^{\mathbb{C}^2} (A \otimes D) = (A^* D \otimes D^* A)\tilde{F}. \tag{21}$$

Now, let $C = A^* D$. On the one hand, this matrix satisfies that

$$\operatorname{rank}(C) = \operatorname{rank}(A^* B V) \le \min\{\operatorname{rank}(V), \operatorname{rank}(A), \operatorname{rank}(B)\} \le 2.$$

On the other hand, denote by $P(\{1, 2, \ldots, n\})$ the power set of $\{1, 2, \ldots, n\}$ and define for $J \in P(\{1, 2, \ldots, n\})$, $F_{A_J B_J}$ to be the flip operator swapping the systems A_j and B_j for every $j \in J$. Then if J^C is the complementary set of J in $P(\{1, 2, \ldots, n\})$, combining 19d and 21,

$$\langle \psi, (\rho')^{T_1} \psi \rangle \backsim \operatorname{tr}\left[\left(\mathbb{1} + \alpha F_{A_1 B_1}\right) \otimes \ldots \otimes \left(\mathbb{1} + \alpha F_{A_n B_n}\right) (C \otimes C^*)\tilde{F}\right] \tag{22a}$$

$$= \sum_{J \in P(\{1,2,\ldots,n\})} \alpha^{|J|} \operatorname{tr}\left[F_{A_J B_J} (C \otimes C^*)\tilde{F}\right] \tag{22b}$$

$$= \sum_{J \in P(\{1,2,\ldots,n\})} \alpha^{|J|} \operatorname{tr}\left[(C \otimes C^*) F_{A_{J^C} B_{J^C}}\right] \tag{22c}$$

$$= \sum_{J \in P(\{1,2,\ldots,n\})} \alpha^{|J|} \operatorname{tr}_{A_{J^C} B_{J^C}} \left[\operatorname{tr}_{A_J B_J} (C \otimes C^*) F_{A_{J^C} B_{J^C}}\right] \tag{22d}$$

$$= \sum_{J \in P(\{1,2,\ldots,n\})} \alpha^{|J|} \operatorname{tr}_{A_{J^C} B_{J^C}} \left[(\operatorname{tr}_J C \otimes \operatorname{tr}_J C^*) F_{A_{J^C} B_{J^C}}\right] \tag{22e}$$

$$= \sum_{J \in P(\{1,2,\ldots,n\})} \alpha^{|J|} \| \operatorname{tr}_J C \|_2^2, \tag{22f}$$

where in the last equation we used the "swap trick" $\operatorname{tr}_{A_{J^C} B_{J^C}} [(X \otimes Y) F_{A_{J^C} B_{J^C}}] = \operatorname{tr}_{A_{J^C} B_{J^C}} [XY]$.

Conversely, suppose that there exists a matrix $C \in L((\mathbb{C}^d)^{\otimes n})$ with rank lower or equal than 2 such that $q^{(n)}(\alpha, C) < 0$. By the previous argument this implies that

$$\operatorname{tr}[\rho_{A_1 B_1} \otimes \ldots \otimes \rho_{A_n B_n} (C \otimes C^*)\tilde{F}] < 0. \tag{23}$$

Decompose $C = |v_1\rangle\langle w_1| + |v_2\rangle\langle w_2|$, and notice that

$$(C \otimes C^* \tilde{F})^{\tilde{T}_1} = |\psi_C\rangle\langle\psi_C|, \tag{24}$$

where $\psi_C = v_1 \otimes w_1 + v_2 \otimes w_2$ and following the notation for the flip operator $\tilde{F}$, we denote $(\,\cdot\,)^{\tilde{T}_1}$ the partial transposition for $(\mathbb{C}^d)^{\otimes n} \otimes (\mathbb{C}^d)^{\otimes n}$. Then,

$$q^{(n)}(\alpha, C) = \langle \psi_C, (\mathbb{1} + \alpha F_{A_1 B_1})^{T_1} \otimes \ldots \otimes (\mathbb{1} + \alpha F_{A_n B_n})^{T_1} \psi_C \rangle < 0, \tag{25}$$

and hence, ρ_α is n-copies distillable using the characterization given by the equation 2. □

For the particular case $n = 1$, the quadratic form 17 is given by

$$q^{(1)}(\alpha, C) = \|C\|_2^2 + \alpha |\operatorname{tr} C|^2, \tag{26}$$

which is positive for every matrix of rank r if $\alpha \geq -\frac{1}{r}$, by the inequality 9. In particular for $r = 2$, we get the expected boundary value $\alpha = -\frac{1}{2}$ for the 1-distillability. This observation on the rank together with numerics performed on the positivity of these quadratic forms, lead us to the following conjecture on the positivity of the form $q^{(n)}$.

Conjecture 2 *Let $C \in L((\mathbb{C}^d)^{\otimes n})$ be a matrix with* $\operatorname{rank} C = r$. *Then, $q^{(n)}(\alpha, C) \geq 0$, for every $\alpha \geq -\frac{1}{r}$.*

Notice that proving Conjecture 2 for $r = 1$ and $r = 2$, proves Conjecture 1, due to Theorem 1. For the rank 1 case, we will show in the next section that we have positivity of 17 for $\alpha = -1$. To prove that $\alpha = -1$ is indeed the boundary (as established in Conjecture 2), we now look at what happens to e.g., $q^{(2)}$ for the 2-distillability, for values $\alpha < -1$. Take $u, v, w \in \mathbb{C}^d$ three normalized vectors with $v \perp w$, and define the matrix $C = |u\rangle\langle u| \otimes |v\rangle\langle w|$. Then,

$$q^{(2)}(-1 - \varepsilon, C) = 1 - (1 + \varepsilon) = -\varepsilon. \tag{27}$$

Since the 1-distillability of Werner states for $\alpha \in (-1, -\frac{1}{2})$ implies its n-distillability for $n \geq 2$, then by Theorem 1 there exists a matrix C with rank 2 such that $q^{(n)}(\alpha, C) < 0$. An explicit example of the saturation of the form $q^{(n)}$ (and hence an alternative proof of the previous statement) is shown in Appendix A for n even. For the particular case of the 2-distillability, using Theorem 1, we can find some Werner states that are not PPT and 2-undistillable for any dimension $d \geq 5$.

Corollary 1 *If $\alpha \geq -\frac{1}{2r}$, then $q^{(2)}(\alpha, C) \geq 0$, for every $C \in L(\mathscr{H}_1 \otimes \mathscr{H}_2)$ with rank r. As a consequence, ρ_α is not 2-distillable for $\alpha \geq -\frac{1}{4}$.*

Proof For $\alpha \geq 0$ the result is clear, so assume that $\alpha < 0$. We bound from below the quadratic form 17 using inequalities 8 and 10

$$\begin{aligned} q^{(2)}(\alpha, C) &= \|C\|_2^2 + \alpha \left[\|\operatorname{tr}_2 C\|_2^2 + \|\operatorname{tr}_1 C\|_2^2 \right] + \alpha^2 |\operatorname{tr} C|^2 \\ &\geq \|C\|_2^2 + \alpha \left[\|\operatorname{tr}_2 C\|_1^2 + \|\operatorname{tr}_1 C\|_1^2 \right] + \alpha^2 |\operatorname{tr} C|^2 \\ &\geq \|C\|_2^2 + 2\alpha \|C\|_1^2 + \alpha^2 |\operatorname{tr} C|^2 \\ &\geq \left(\frac{1}{r} + 2\alpha \right) \|C\|_1^2 + \alpha^2 |\operatorname{tr} C|^2. \end{aligned} \tag{28}$$

Thus, if $\alpha \geq -\frac{1}{2r}$. we get $q^{(2)}(\alpha, C) \geq 0$. □

For positive matrices, the a priori boundary value $\alpha = -\frac{1}{r}$ can actually be improved, since this value does not depend necessarily on the rank (see [15] or [30]), but as we have seen, this changes for the general case. We will discuss in the next section that for higher ranks, the boundary value for α might not be $\alpha = -\frac{1}{r}$ anymore, since the dimension of the systems also plays an important role.

4 Partial trace inequalities

In order to prove the positivity of the quadratic form 17 for rank one matrices for the value $\alpha = -1$, we state the following result, which also shows the underlying structure of these quadratic forms.

Proposition 1 *For $\mathscr{H} = \mathscr{H}_1 \otimes \mathscr{H}_2$, the form $q^{(2)}(-1, C)$ is positive for every rank 1 matrix $C \in L(\mathscr{H})$.*

Proof Let $v, w \in \mathscr{H} = \mathscr{H}_1 \otimes \mathscr{H}_2$, and write

$$|v\rangle\langle w| = \sum_{i,j=1}^{n} |v_i^1\rangle\langle w_j^1| \otimes |v_i^2\rangle\langle w_j^2|,$$

where $n = \max\{\dim \mathscr{H}_1, \dim \mathscr{H}_2\}$. Note that we can make this assumption by completing the vector with fewer elements with zeros. Now, we compute all the norms

$$\begin{aligned} \||v\rangle\langle w|\|_2^2 &= \sum_{i,j,k,l=1}^{n} \langle w_j^1, w_l^1\rangle\langle v_k^1, v_i^1\rangle\langle w_j^2, w_l^2\rangle\langle v_k^2, v_i^2\rangle \\ &= \sum_{i,j,k,l=1}^{n} \langle v_k^1 \otimes w_j^1, v_i^1 \otimes w_l^1\rangle\langle v_k^2 \otimes w_j^2, v_i^2 \otimes w_l^2\rangle \end{aligned} \tag{29a}$$

$$\begin{aligned} \| \mathrm{tr}_1 |v\rangle\langle w|\|_2^2 &= \sum_{i,j,k,l=1}^{n} \langle w_j^1, v_i^1\rangle\langle v_k^1, w_l^1\rangle\langle w_j^2, w_l^2\rangle\langle v_k^2, v_i^2\rangle \\ &= \sum_{i,j,k,l=1}^{n} \langle v_k^1 \otimes w_j^1, w_l^1 \otimes v_i^1\rangle\langle v_k^2 \otimes w_j^2, v_i^2 \otimes w_l^2\rangle \end{aligned} \tag{29b}$$

$$\begin{aligned} \| \mathrm{tr}_2 |v\rangle\langle w|\|_2^2 &= \sum_{i,j,k,l=1}^{n} \langle w_j^1, w_l^1\rangle\langle v_k^1, v_i^1\rangle\langle w_j^2, v_i^2\rangle\langle v_k^2, w_l^2\rangle \\ &= \sum_{i,j,k,l=1}^{n} \langle v_k^1 \otimes w_j^1, v_i^1 \otimes w_l^1\rangle\langle v_k^2 \otimes w_j^2, w_l^2 \otimes v_i^2\rangle \end{aligned} \tag{29c}$$

$$\begin{aligned} |\mathrm{tr} |v\rangle\langle w||^2 &= \sum_{i,j,k,l=1}^{n} \langle w_j^1, v_i^1\rangle\langle v_k^1, w_l^1\rangle\langle w_j^2, v_i^2\rangle\langle v_k^2, w_l^2\rangle \\ &= \sum_{i,j,k,l=1}^{n} \langle v_k^1 \otimes w_j^1, w_l^1 \otimes v_i^1\rangle\langle v_k^2 \otimes w_j^2, w_l^2 \otimes v_i^2\rangle, \end{aligned} \tag{29d}$$

and using $(\mathbb{1}-F)^2 = 2(\mathbb{1}-F)$,

$$\frac{1}{4}\left\|\sum_{k,j=1}^{n}(v_k^1\wedge w_j^1)\otimes(v_k^2\wedge w_j^2)\right\|^2 = \tag{30a}$$

$$= \frac{1}{4}\sum_{i,j,k,l=1}^{n}\left\langle v_k^1\otimes w_j^1, (\mathbb{1}-F)^2(v_i^1\otimes w_l^1)\right\rangle\left\langle v_k^2\otimes w_j^2, (\mathbb{1}-F)^2(v_i^2\otimes w_l^2)\right\rangle \tag{30b}$$

$$= \sum_{i,j,k,l=1}^{n}\left\langle v_k^1\otimes w_j^1, v_i^1\otimes w_l^1 - w_l^1\otimes v_i^1\right\rangle\left\langle v_k^2\otimes w_j^2, v_i^2\otimes w_l^2 - w_l^2\otimes v_i^2\right\rangle \tag{30c}$$

$$= \||v\rangle\langle w|\|_2^2 - \|\operatorname{tr}_1|v\rangle\langle w|\|_2^2 - \|\operatorname{tr}_2|v\rangle\langle w|\|_2^2 + |\operatorname{tr}|v\rangle\langle w||^2 \tag{30d}$$

$$= q^{(2)}(-1, |v\rangle\langle w|). \tag{30e}$$

□

Remark 2 Notice that these forms can be written in a shorter way. For example, $q^{(2)}(-1, |v\rangle\langle w|)$ can be written as follows:

$$\begin{aligned} q^{(2)}(-1, |v\rangle\langle w|) &= \frac{1}{4}\langle(\mathbb{1}-F)\otimes(\mathbb{1}-F)F_{23}v\otimes w, (\mathbb{1}-F)\otimes(\mathbb{1}-F)F_{23}v\otimes w\rangle \\ &= \langle v\otimes w, (\mathbb{1}-F_{13})(\mathbb{1}-F_{24})v\otimes w\rangle, \end{aligned} \tag{31}$$

where F_{ij} is the operator that flips the components i and j and $F = F_{13}F_{24}$. Similarly the rest.

Remark 3 In a similar way it can be checked that for $\mathscr{H} = \mathscr{H}_1\otimes\ldots\otimes\mathscr{H}_n$ and

$$C = |v\rangle\langle w| = \sum_{i,j=1}^{k}|v_i^1\rangle\langle w_j^1|\otimes\ldots\otimes|v_i^n\rangle\langle w_j^n|,$$

with $v, w \in \mathscr{H}$ and $k = \max\{\dim\mathscr{H}_1, \ldots, \dim\mathscr{H}_n\}$, then

$$q^{(n)}(-1, |v\rangle\langle w|) = \frac{1}{2^n}\left\|\sum_{i,j=1}^{k}(v_i^1\wedge w_j^1)\otimes\ldots\otimes(v_i^n\wedge w_j^n)\right\|^2 \geq 0. \tag{32}$$

Changing the antisymmetrizations "∧" by symmetrizations "⊙" in 32, it is possible to generate different rank 1 inequalities. In fact, given $n \geq 2$, there are 2^n combinations of symmetrizations and antisymmetrizations, and each one has associated one quadratic form. These symmetries coincide with the ones introduced in [30], and motivate us to introduce the following definition.

Definition 1 Let $\mathscr{H} = \mathscr{H}_1\otimes\ldots\otimes\mathscr{H}_n$ and $v \in \{0,1\}^n$, $v = (v_k)_{k=1}^n$, then we define

$$q_v(\alpha, C) = \sum_{J\in P(\{1,2,\ldots,n\})}\alpha^{|J|}(-1)^{(|J|+\sum_{k\in J}v_k)}\|\operatorname{tr}_J C\|_2^2 \tag{33}$$

where $P(X)$ is the power set of X and where we denote $\operatorname{tr}_\emptyset = \mathbb{1}$.

Remark 4 Notice that when $v_i = 0$ for $1 \le i \le n$, that corresponds with a symmetrization in the tensor factor i in 32. Conversely, when $v_i = 1$, then the tensor factor i in 32 corresponds with an antisymmetrization.

This definition corresponds with the quadratic form associated to a subfamily of the universal state inversions, which have been studied in e.g., [15, 16] or [25]. Notice that the vector $v_0 = (1, 1, \dots, 1)$, has the associated quadratic form $q_{v_0} = q^{(n)}$. For $n = 3$, for example, the different classes of forms are

$$\begin{aligned} q_{(1,1,1)}(\alpha, C) = \|C\|_2^2 + \alpha(\| \operatorname{tr}_1 C\|_2^2 + \| \operatorname{tr}_2 C\|_2^2 + \| \operatorname{tr}_3 C\|_2^2) \\ + \alpha^2(\| \operatorname{tr}_{12} C\|_2^2 + \| \operatorname{tr}_{13} C\|_2^2 + \| \operatorname{tr}_{23} C\|_2^2) + \alpha^3 |\operatorname{tr} C|^2, \end{aligned} \quad (34)$$

$$\begin{aligned} q_{(0,1,1)}(\beta, C) = \|C\|_2^2 + \beta(-\| \operatorname{tr}_1 C\|_2^2 + \| \operatorname{tr}_2 C\|_2^2 + \| \operatorname{tr}_3 C\|_2^2) \\ + \beta^2(-\| \operatorname{tr}_{12} C\|_2^2 - \| \operatorname{tr}_{13} C\|_2^2 + \| \operatorname{tr}_{23} C\|_2^2) - \beta^3 |\operatorname{tr} C|^2, \end{aligned} \quad (35)$$

$$\begin{aligned} q_{(0,0,1)}(\gamma, C) = \|C\|_2^2 + \gamma(-\| \operatorname{tr}_1 C\|_2^2 - \| \operatorname{tr}_2 C\|_2^2 + \| \operatorname{tr}_3 C\|_2^2) \\ + \gamma^2(\| \operatorname{tr}_{12} C\|_2^2 - \| \operatorname{tr}_{13} C\|_2^2 - \| \operatorname{tr}_{23} C\|_2^2) + \gamma^3 |\operatorname{tr} C|^2. \end{aligned} \quad (36)$$

By choosing the position of the symmetrizations, one can find 2 forms more like 36, and another 2 more like 35. At this point we are ready to introduce the main conjecture of this work.

Conjecture 3 *Let $\mathscr{H}$ be a finite-dimensional Hilbert space that can be decomposed as $\mathscr{H} = \mathscr{H}_1 \otimes \dots \otimes \mathscr{H}_n$, with* $\dim(\mathscr{H}_i) = d_i \ge 2$. *Then, for every $C \in L(\mathscr{H})$ with* $\operatorname{rank}(C) = r$ *and every $v \in \{0, 1\}^n$, $q_v(\alpha, C) \ge 0$ for*

$$|\alpha| \le \alpha_{opt} = \frac{1}{\min\{r, \max\{d_1, \dots, d_n\}\}}. \quad (37)$$

At this point, we recall that one of the quadratic forms in 33 was originally motivated by the distillability of Werner states. In the proof of the next result, it can be seen how the upper bound for α in 37 in terms of the dimension is connected to the separability of the Werner states.

Proposition 2 *For $|\alpha| \le \frac{1}{\max\{d_1, \dots, d_n\}}$, the Conjecture 3 holds.*

Proof Let $\mathscr{H} = \mathscr{H}_1 \otimes \dots \otimes \mathscr{H}_n$ with $\dim \mathscr{H}_i = d_i$ and $d = \max_i\{d_i\}$. Let $v \in \{0, 1\}^n$ with associated quadratic form q_v, so q_v can be written as similarly as we did in the proof of Theorem 1

$$q_v(\alpha, C) = \operatorname{tr}[(\mathbb{1} \pm \alpha F_{A_1 B_1}) \otimes \dots \otimes (\mathbb{1} \pm \alpha F_{A_n B_n})(C \otimes C^*)\tilde{F}], \quad (38)$$

with the corresponding choice of signs, and where $\tilde{F}$ is the flip operator in $(\mathbb{C}^d)^{\otimes n} \otimes (\mathbb{C}^d)^{\otimes n}$. Now, decompose as in Theorem 1, $C = \sum_{i=1}^r |v_i\rangle\langle w_i|$ and obtain again

$$(C \otimes C^* \tilde{F})^{\tilde{T}_1} = |\psi_C\rangle\langle\psi_C|, \tag{39}$$

where $\psi_C = \sum_{i=1}^r v_i \otimes w_i$. This allow us to write

$$q_v(\alpha, C) = \langle\psi_C, (\mathbb{1} \pm \alpha F_{A_1 B_1})^{T_1} \otimes \ldots \otimes (\mathbb{1} \pm \alpha F_{A_n B_n})^{T_1} \psi_C\rangle. \tag{40}$$

Since for $|\alpha| \leq \frac{1}{d}$ the Werner states are separable [34], in particular they are PPT and we conclude that $q_v(\alpha, C) \geq 0$ for $|\alpha| \leq \frac{1}{d}$. Finally, the result holds by considering the embedding of $L(\mathscr{H})$ in $L((\mathbb{C}^d)^{\otimes n})$. □

Notice that for positive matrices, the previous Proposition was already proven in e.g., [16], and here present an alternative argument that extends this inequality to general matrices using the separability of Werner states. For the bounds where only the dimension of the Hilbert spaces appear, we can reduce any quadratic form with any vector to the case of the subvector containing all the 1's, i.e., it is sufficient to prove the result for the vectors of 1's only. For example, for $v = (1, 0)$ and $\alpha = -\frac{1}{d}$, $d = \max\{d_1, d_2\}$, the correspondent quadratic form is

$$q_{(1,0)}\left(-\frac{1}{d}, C\right) = \|C\|_2^2 - \frac{1}{d}\|\operatorname{tr}_1 C\|_2^2 + \frac{1}{d}\|\operatorname{tr}_2 C\|_2^2 - \frac{1}{d^2}|\operatorname{tr} C|^2. \tag{41}$$

We modify the quadratic form $q_{(1)}$ and define

$$q_{(1)}^{\{\mathscr{H}_1\}}\left(-\frac{1}{d}, C\right) = \|C\|_2^2 - \frac{1}{d}\|\operatorname{tr}_1 C\|_2^2, \tag{42}$$

where the upper index in q denotes that we take partial trace on the first system instead of the full trace. This form is positive by [31] and also

$$q_{(1)}^{\{\mathscr{H}_1\}}\left(-\frac{1}{d}, \operatorname{tr}_2 C\right) = \|\operatorname{tr}_2 C\|_2^2 - \frac{1}{d}|\operatorname{tr} C|^2 \geq 0. \tag{43}$$

As a consequence, we can write

$$q_{(1,0)}\left(-\frac{1}{d}, C\right) = q_{(1)}^{\{\mathscr{H}_1\}}\left(-\frac{1}{d}, C\right) + q_{(1)}^{\{\mathscr{H}_1\}}\left(-\frac{1}{d}, \operatorname{tr}_2 C\right). \tag{44}$$

However, such a decomposition is no longer valid for the rank, because partial traces do not preserve the rank in general.

Next Theorem shows a version, in terms of partial trace inequalities, of the progress that we have done in Conjecture 3 for $n = 2$. The proof is presented in section 7.

Theorem 2 *Let $C \in L(\mathscr{H}_1 \otimes \mathscr{H}_2)$, $r = \operatorname{rank}(C)$ and $d = \max\{d_1, d_2\}$, then the following inequalities hold*

1.

$$\left| \| \operatorname{tr}_1 C\|_2^2 - \| \operatorname{tr}_2 C\|_2^2 \right| \le \min\{r, d\}\|C\|_2^2 - \frac{1}{\min\{r, d\}}|\operatorname{tr}(C)|^2. \tag{45}$$

2.

$$\| \operatorname{tr}_1 C\|_2^2 + \| \operatorname{tr}_2 C\|_2^2 \le d\|C\|_2^2 + \frac{1}{d}|\operatorname{tr}(C)|^2. \tag{46}$$

If, in addition, C can be written as $C = C_1 + C_2$ with $\operatorname{rank}(C_1) = 1$ and C_2 normal such that the vectors spanning the range of C_1 and C_1^* are orthogonal to all eigenvectors of C_2, then

$$\| \operatorname{tr}_1 C\|_2^2 + \| \operatorname{tr}_2 C\|_2^2 \le r\|C\|_2^2 + \frac{1}{r}|\operatorname{tr}(C)|^2. \tag{47}$$

Remark 5 For the particular case $n = 2$, $\operatorname{rank}(C) = 2$, the condition $q^{(2)}\left(\alpha = -\frac{1}{2}, C\right) \ge 0$ in 17 can be rewritten as

$$\| \operatorname{tr}_1 C\|_2^2 + \| \operatorname{tr}_2 C\|_2^2 \le 2\|C\|_2^2 + \frac{1}{2}|\operatorname{tr} C|^2, \tag{48}$$

so 47 is the generalization inequality for a rank r matrix. Since we cannot prove 47 for a general rank 2 matrix yet, the problem of the 2-distillabilty remains open.

Remark 6 Inequality 45 for rank 1 and rank 2 shows (following the reasoning of Theorem 1) that for every $A, B : (\mathbb{C}^d)^{\otimes 2} \to \mathbb{C}^2$ and $\psi_C \in \mathbb{C}^2 \otimes \mathbb{C}^2$, the Werner states ρ_α satisfy:

$$\left\langle \psi_C, (A \otimes B)\left(\rho_{\frac{1}{2}} \otimes \rho_{-\frac{1}{2}}\right)^{T_1} (A \otimes B)^* \psi_C \right\rangle \ge 0, \tag{49}$$

and

$$\left\langle \psi_C, (A \otimes B)\left(\rho_{-\frac{1}{2}} \otimes \rho_{\frac{1}{2}}\right)^{T_1} (A \otimes B)^* \psi_C \right\rangle \ge 0, \tag{50}$$

i.e., $\rho_{\frac{1}{2}} \otimes \rho_{-\frac{1}{2}}$ and $\rho_{-\frac{1}{2}} \otimes \rho_{\frac{1}{2}}$ are 1-distillable in $L((\mathbb{C}^d)^{\otimes 2})$. Thus, a positive answer to Conjecture 3 for rank 1 and rank 2 matrices would not only provide a proof of the distillability of Werner states, but would also show the distillability properties of tensor product of Werner states.

5 3-distillability and tripartite systems inequalities

Once we have studied 2-distillability in depth, in this section we begin a small approach to the 3-distillability, i.e., the problem of showing the positivity of $q^{(3)}\left(-\frac{1}{2}, C\right)$ for rank 2 matrices C. This problem turns out to be more challenging as the 2-distillability case, and for the particular case of a rank 2 matrix we can only show positivity for self-adjoint matrices with one positive and one negative eigenvalue.

Proposition 3 *If $C \in L(\mathscr{H}_1 \otimes \mathscr{H}_2 \otimes \mathscr{H}_3)$ is a self-adjoint rank 2 matrix with one positive and one negative eigenvalue, then $q^{(3)}\left(-\frac{1}{2}, C\right) \ge 0$.*

Proof Consider the inversion of a pure state

$$\begin{aligned}Q_a^{(3),r} &= |a\rangle\langle a| - \frac{1}{r}\left(\mathbb{1}_{d_1}\otimes \mathrm{tr}_1(|a\rangle\langle a|) + \mathbb{1}_{d_2}\otimes \mathrm{tr}_2(|a\rangle\langle a|) + \mathrm{tr}_3(|a\rangle\langle a|)\otimes \mathbb{1}_{d_3}\right)\\ &+\frac{1}{r^2}\left(\mathbb{1}_{d_1d_2}\otimes \mathrm{tr}_{12}(|a\rangle\langle a|) + \mathbb{1}_{d_1d_3}\otimes \mathrm{tr}_{13}(|a\rangle\langle a|) + \mathrm{tr}_{23}(|a\rangle\langle a|)\otimes \mathbb{1}_{d_2d_3}\right)\\ &-\frac{1}{r^3}\|a\|^2\mathbb{1}_{d_1d_2d_3}. \end{aligned} \tag{51}$$

for $a \in \mathscr{H}$. By denoting $\tilde{Q}_a^{(3),r} = P_a^{\perp} Q_a^{(3),r} P_a^{\perp}$, where $P_a^{\perp}$ is again the projection on $\ker(|a\rangle\langle a|)$, we can upper bound this operator

$$\tilde{Q}_a^{(3),r} \leq \frac{1}{r^2}\left(1-\frac{1}{r}\right). \tag{52}$$

To prove this, notice that by direct computation

$$\langle \mathrm{tr}_I(|a\rangle\langle a|), \mathrm{tr}_I(|x\rangle\langle x|)\rangle = \|\mathrm{tr}_{I^C}(|a\rangle\langle x|)\|_2^2, \tag{53}$$

for any partition I, I^C of $\{1,\ldots,n\}$ (in this case $n=3$) so we get

$$\langle x, -\tilde{Q}_a^{(3),r}x\rangle = \frac{1}{r^2}q^{(3)}(-1, |a\rangle\langle x|) - \frac{1}{r^2}\left(1-\frac{1}{r}\right)\|a\|^2\|x\|^2 \tag{54a}$$

$$+\frac{1}{r}\left(1-\frac{1}{r}\right)(\|\mathrm{tr}_{12}(|a\rangle\langle x|)\|^2 + \|\mathrm{tr}_{13}(|a\rangle\langle x|)\|^2 + \|\mathrm{tr}_{23}(|a\rangle\langle x|)\|^2) \tag{54b}$$

$$\geq -\frac{1}{r^2}\left(1-\frac{1}{r}\right)\|a\|^2\|x\|^2, \tag{54c}$$

since $q^{(3)}(-1, |a\rangle\langle x|) \geq 0$ by Proposition 1.

Let C be a self-adjoint matrix with spectral decomposition $C = |v_1\rangle\langle v_1| - |v_2\rangle\langle v_2|$, with $v_1 \perp v_2$, but the vectors are not normalized.

$$\begin{aligned}q^{(3)}\left(-\frac{1}{2}, C\right) &= \sum_{i=1}^{2}\Bigg[\left(1-\frac{1}{8}\right)\|v_i\|_2^4 - \frac{1}{2}\left(\|\mathrm{tr}_1\,|v_i\rangle\langle v_i|\|_2^2 + \|\mathrm{tr}_2\,|v_i\rangle\langle v_i|\|_2^2 + \|\mathrm{tr}_3\,|v_i\rangle\langle v_i|\|_2^2\right)\\ &+\frac{1}{4}\left(\|\mathrm{tr}_{12}\,|v_i\rangle\langle v_i|\|_2^2 + \|\mathrm{tr}_{13}\,|v_i\rangle\langle v_i|\|_2^2 + \|\mathrm{tr}_{23}\,|v_i\rangle\langle v_i|\|_2^2\right)\Bigg] - 2\langle v_1, \tilde{Q}_{v_2}^{(3)}v_1\rangle. \end{aligned} \tag{55}$$

Finally, using 52 with $r=2$ together with 53, we can write

$$\begin{aligned}q^{(3)}\left(-\frac{1}{2}, C\right) &= \frac{1}{4}\sum_{i=1}^{2}\sum_{j=1}^{3}\left(\|v_i\|_2^2 - \|\mathrm{tr}_j\,|v_i\rangle\langle v_i|\|_2^2\right) + \frac{1}{8}(\|v_1\|_2^2 - \|v_2\|_2^2)^2 \\ &\geq 0.\end{aligned} \tag{56}$$

□

We also study the other two families of inequalities in tripartite systems 35 and 36 for the case that the parameter depends on the rank of the matrix. Proving the positivity for $\beta, \gamma = -\frac{1}{2}$ for arbitrary rank 1 and rank 2 matrices, would give us similar conclusions to the ones obtained in Remark 6. Here, we will show them only for positive semidefinite matrices. We will need first the following result.

Lemma 1 *Let $C \in L(\mathscr{H}_1 \otimes \mathscr{H}_2)$ be a positive semidefinite matrix with $r = \operatorname{rank}(C)$. Then, $\| \operatorname{tr}_1 C\|_2^2 \geq \frac{1}{r}\| \operatorname{tr}_2 C\|_2^2$.*

Proof Write the spectral decomposition of $C = \sum_{i=1}^{r} |v_i\rangle\langle v_i|$ and by 53 we get that

$$\| \operatorname{tr}_1(|v_i\rangle\langle v_i|)\|_2 = \| \operatorname{tr}_2(|v_i\rangle\langle v_i|)\|_2, \tag{57}$$

for every $1 \leq i \leq r$. Thus, by 53 again we can express everything in terms of the partial trace over the second system

$$\begin{aligned} &\| \operatorname{tr}_1 C\|_2^2 - \frac{1}{r}\| \operatorname{tr}_2 C\|_2^2 \\ &\geq \left(1 - \frac{1}{r}\right) \sum_{i=1}^{r} \| \operatorname{tr}_2(|v_i\rangle\langle v_i|)\|_2^2 + 2 \sum_{\substack{i,j=1\\ i>j}}^{r} \| \operatorname{tr}_2(|v_i\rangle\langle v_j|)\|_2^2 \\ &-\frac{2}{r} \sum_{\substack{i,j=1\\ i>j}}^{r} \| \operatorname{tr}_2(|v_i\rangle\langle v_i|)\|_2 \| \operatorname{tr}_2(|v_j\rangle\langle v_j|)\|_2, \end{aligned} \tag{58}$$

To conclude, define the polynomial

$$p_r(x) = (r-1)\sum_{i=1}^{r} x_i^2 - 2\sum_{\substack{i,j=1\\ i>j}}^{d_1} x_i x_j = \sum_{\substack{i,j=1\\ i>j}}^{r} (x_i - x_j)^2 \geq 0, \tag{59}$$

so we get

$$\begin{aligned} \| \operatorname{tr}_1 C\|_2^2 - \frac{1}{r}\| \operatorname{tr}_2 C\|_2^2 \geq \frac{1}{r} p_r\left(\| \operatorname{tr}_2(|v_1\rangle\langle v_1|)\|_2, \ldots, \| \operatorname{tr}_2(|v_r\rangle\langle v_r|)\|_2\right) \\ + 2 \sum_{\substack{i,j=1\\ i>j}}^{r} \| \operatorname{tr}_2(|v_i\rangle\langle v_j|)\|_2^2 \qquad (60a) \\ \geq 0. \qquad (60b) \end{aligned}$$

□

Proposition 4 *The conjecture over the class of forms of 35 and 36 for $\beta, \gamma = -\frac{1}{r}$, holds for positive matrices.*

Proof We will start with the proof of the family of inequalities 35. We will just prove 35, the other 2 are analogous. In this case the associated linear operator is

$$\begin{aligned} P_a^{(3),r} &= |a\rangle\langle a| + \frac{1}{r}\left(\mathbb{1}_{d_1} \otimes \mathrm{tr}_1(|a\rangle\langle a|) - \mathbb{1}_{d_2} \otimes \mathrm{tr}_2(|a\rangle\langle a|) - \mathrm{tr}_3(|a\rangle\langle a|) \otimes \mathbb{1}_{d_3}\right) \\ &\quad + \frac{1}{r^2}\left(-\mathbb{1}_{d_1 d_2} \otimes \mathrm{tr}_{12}(|a\rangle\langle a|) - \mathbb{1}_{d_1 d_3} \otimes \mathrm{tr}_{13}(|a\rangle\langle a|)\right. \\ &\quad \left. + \mathrm{tr}_{23}(|a\rangle\langle a|) \otimes \mathbb{1}_{d_2 d_3}\right) + \\ &\quad + \frac{1}{r^3}\|a\|^2 \mathbb{1}_{d_1 d_2 d_3}. \end{aligned} \tag{61}$$

This operator is bounded from below by

$$P_a^{(3),r} \geq \frac{1-r^2}{r^3}\|a\|^2, \tag{62}$$

on $\ker(|a\rangle\langle a|)$ by computing the expectation value

$$\begin{aligned} \langle x, P_a^{(3),r} x\rangle &= \frac{1}{r^2} p_{-1}(|a\rangle\langle x|) + \frac{1-r^2}{r^3}\|a\|^2\|x\|^2 + \left(1 - \frac{1}{r^2}\right)|\langle a, x\rangle|^2 \\ &\quad + \frac{1}{r}\left(1 - \frac{1}{r}\right)(\|a\|^2\|x\|^2 - \|\mathrm{tr}_{12}(|a\rangle\langle x|)\|^2 \\ &\quad - \|\mathrm{tr}_{13}(|a\rangle\langle x|)\|^2 + \|\mathrm{tr}_{23}(|a\rangle\langle x|)\|^2) \geq 0, \end{aligned} \tag{63}$$

since

$$\begin{aligned} &\|a\|^2\|x\|^2 - \|\mathrm{tr}_{12}(|a\rangle\langle x|)\|^2 - \|\mathrm{tr}_{13}(|a\rangle\langle x|)\|^2 + \|\mathrm{tr}_{23}(|a\rangle\langle x|)\|^2 = \\ &\quad = \langle a \otimes x, (1 - F_{14}F_{25})(1 - F_{14}F_{36}) a \otimes x\rangle \geq 0. \end{aligned} \tag{64}$$

Let C be a positive matrix with rank r and write its (non-normalized) spectral decomposition $C = \sum_{i=1}^r |v_i\rangle\langle v_i|$, then using 53 we can write

$$\begin{aligned} q_{(0,1,1)}^{(3)}\left(-\frac{1}{r}, C\right) &= \left(1 - \frac{1}{r} - \frac{1}{r^2} + \frac{1}{r^3}\right)\sum_{i=1}^r \|v_i\|^4 + 2\sum_{\substack{i,j=1 \\ i>j}}^r \langle v_j, P_{v_i}^{(3),r} v_j\rangle \\ &+ \frac{1}{r}\left(1 + \frac{1}{r}\right)\sum_{i=1}^r \left(\|v_i\|^4 - \|\mathrm{tr}_1(|v_i\rangle\langle v_i|)\|^2 - \|\mathrm{tr}_2(|v_i\rangle\langle v_i|)\|^2 + \|\mathrm{tr}_3(|v_i\rangle\langle v_i|)\|^2\right). \end{aligned} \tag{65}$$

To conclude, the first to terms in 65 are positive since they are lower bounded by the expression

$$\frac{r^2-1}{r^3}\left((r-1)\sum_{i=1}^{r}\|v_i\|^4-2\sum_{\substack{i,j=1\\i>j}}^{r}\|v_i\|^2\|v_j\|^2\right)\geq 0, \tag{66}$$

and the last one is equal to

$$\frac{1}{r}\left(1+\frac{1}{r}\right)\sum_{i=1}^{r}\langle v_i\otimes v_i,(1-F_{14})(1-F_{25})(1+F_{36})v_i\otimes v_i\rangle\geq 0, \tag{67}$$

which concludes the first part. For 36, the proof follows then by Lemma 1 writing

$$\begin{aligned}q_{(0,0,1)}\left(-\frac{1}{r},C\right)&=\left(\|C\|_2^2-\frac{1}{r}\|\operatorname{tr}_3 C\|_2^2+\frac{1}{r^2}\|\operatorname{tr}_{12} C\|_2^2-\frac{1}{r^3}|\operatorname{tr} C|^2\right)\\&+\frac{1}{r}\left(\|\operatorname{tr}_1 C\|_2^2-\frac{1}{r}\|\operatorname{tr}_{23} C\|_2^2\right)+\frac{1}{r}\left(\|\operatorname{tr}_2 C\|_2^2-\frac{1}{r}\|\operatorname{tr}_{13} C\|_2^2\right)\geq 0.\end{aligned} \tag{68}$$

□

Remark 7 Proposition 4 shows that the positivity of the quadratic forms associated with the state inversions might also depend on the rank and not only on the dimension. Similar considerations might also hold for the state inversion maps.

6 Inequalities for p-Schatten norms

Finally, in this last section, we set a more general function than 33, which will also depend on the norm and the exponent as follows:

Definition 2 Let $\mathscr{H}=\mathscr{H}_1\otimes\ldots\otimes\mathscr{H}_n$ and $v\in\{0,1\}^n$. Define for $p\geq 1$, $\gamma\geq 1$, $\alpha\in\mathbb{R}$, $C\in L(\mathscr{H})$ and $v\in\{0,1\}^n$ the map

$$q_v(p,\gamma,\alpha,C)=\sum_{J\in P(\{1,2,\ldots,n\})}\alpha^{|J|}(-1)^{(|J|+\sum_{k\in J}v_k)}\|\operatorname{tr}_J C\|_p^{\gamma}. \tag{69}$$

The objective is to study wether it is possible to extend Conjecture 3 to inequalities depending on $p,\gamma,r=\operatorname{rank}(C)$ or $d=\max\{d_1,\ldots,d_n\}$, instead of only depending on the rank r and d, i.e, we want to study if it is possible to introduce a function $\alpha_v(p,\gamma,r,d)$ that provides tight bounds for the positivity of 69. For the particular case of $p=2$ and $\gamma=2$, $\alpha_v(2,2,r,d)=\alpha_{opt}$ given by 37.

For $v=(1)$, the bound with the dimension was actually studied in [31], where it was proved that

$$\|\operatorname{tr}_i C\|_p\leq d^{\frac{p-1}{p}}\|C\|_p, \tag{70}$$

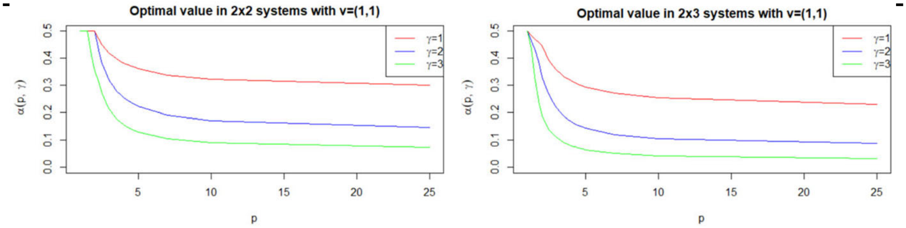

Fig. 2 Optimal values for $v = (1, 1)$ for different values of p and γ in $\mathbb{R}^2 \otimes \mathbb{R}^2$ and $\mathbb{R}^2 \otimes \mathbb{R}^3$

and is also possible to obtain a rank bound using the Hölder inequality for the Schatten p-norms

$$\| \operatorname{tr}_i C\|_p \leq \| \operatorname{tr}_i C\|_1 \leq \|C\|_1 \leq \|\mathbb{1}_r\|_{p'}\|C\|_p = r^{\frac{p-1}{p}}\|C\|_p, \tag{71}$$

where p' is the dual Hölder index of p satisfying $\frac{1}{p} + \frac{1}{p'} = 1$. Thus,

$$\alpha_{(1)}(p, \gamma, r, d) = \frac{1}{\min\left\{r^{\gamma\frac{p-1}{p}}, d^{\gamma\frac{p-1}{p}}\right\}} \tag{72}$$

is a tight bound satisfying

$$\|C\|_p^\gamma - \alpha_{(1)}(p, \gamma, r, d)|\operatorname{tr} C|^\gamma \geq 0. \tag{73}$$

For the vector $v = (1, 1)$ it was proved in [2] that for any state ρ, $p > 1$ and $\gamma = 1, p$

$$\|\rho\|_p^\gamma - \|\operatorname{tr}_1 \rho\|_p^\gamma - \|\operatorname{tr}_2 \rho\|_p^\gamma + |\operatorname{tr} \rho|^\gamma \geq 0, \tag{74}$$

i.e., $\alpha_{(1,1)}(p, \gamma, r, d) = 1$, for $\gamma = 1, p$ when restricted to states. This was used to show the subadditivity of the Tsallis entropy. However, for fixed dimensions and different values of p and γ and a general full-rank matrix C, (Fig. 2) shows the evolution of the quantity $\alpha_v(p, \gamma)$ which is the quantity analogous to α_{opt} introduced in 37 but now dependent on p and γ when r and d are fixed, in the systems $\mathbb{R}^2 \otimes \mathbb{R}^2$ and $\mathbb{R}^2 \otimes \mathbb{R}^3$, which seems to have continuous dependence with respect to p and γ.

This shows that there are large families of partial trace inequalities that remain to be studied that generalize the ones of Conjecture 3, when we also take into account the p-norms and the exponents γ.

7 Proof of Theorem 2

First of all, by Proposition 2, inequality 46 and the bound with the dimension of the Hilbert spaces of 45 are proved, so only the bounds with the ranks have to be proved. We will divide the proof into two parts: first, we will prove 45 and then 47. Before proving 45, we will need an auxiliary result that allow us to bound the difference of partial traces in a tight way. For this purpose, we will make use of the creation and annihilation operators introduced in 14, 15.

Proposition 5 *Let $v, w \in \mathscr{H} = \mathscr{H}_1 \otimes \mathscr{H}_2$, $d_1 = \dim \mathscr{H}_1$, $d_2 = \dim \mathscr{H}_2$, then*

$$\mathbb{1}_{d_1} \otimes \mathrm{tr}_1(|v\rangle\langle w|) = \frac{1}{2}[a_+(w)F_{24}a_+^*(v) + a_-(w)F_{24}a_-^*(v)]|_{\mathscr{H}}\,, \tag{75}$$

$$\mathrm{tr}_2(|v\rangle\langle w|) \otimes \mathbb{1}_{d_2} = \frac{1}{2}[a_+(w)F_{24}a_+^*(v) - a_-(w)F_{24}a_-^*(v)]|_{\mathscr{H}}\,, \tag{76}$$

where F_{24} is the flip operator that exchanges components 2 and 4.

Proof Without loss of generality, we can assume that the vectors are normalized. In order to simplify this proof, we will use here the bra-ket notation understanding $|v\rangle$ as a vector (not necessarily normalized) with associated functional $\langle v|$. Write $|v\rangle = \sum_{i=1}^{n} |v_i^1\rangle|v_i^2\rangle$ and $|w\rangle = \sum_{j=1}^{n} |w_j^1\rangle|w_j^2\rangle$, where we can assume again that n is the same. We will prove

$$\mathbb{1}_{d_1} \otimes \mathrm{tr}_1(|v\rangle\langle w|) - \mathrm{tr}_2(|v\rangle\langle w|) \otimes \mathbb{1}_{d_2} = a_-(\langle w|)F_{24}a_-^*(|v\rangle)|_{\mathscr{H}}\,, \tag{77}$$

and

$$\mathbb{1}_{d_1} \otimes \mathrm{tr}_1(|v\rangle\langle w|) + \mathrm{tr}_2(|v\rangle\langle w|) \otimes \mathbb{1}_{d_2} = a_+(\langle w|)F_{24}a_+^*(|v\rangle)|_{\mathscr{H}}\,. \tag{78}$$

For the first one, let $|x\rangle = \sum_{k=1}^{n} |x_k^1\rangle|x_k^2\rangle$, then

$$\left[\mathbb{1}_{d_1} \otimes \mathrm{tr}_1(|v\rangle\langle w|) - \mathrm{tr}_2(|v\rangle\langle w|) \otimes \mathbb{1}_{d_2}\right]|x\rangle \tag{79a}$$

$$= \sum_{i,j,k=1}^{n} \langle w_j^1, v_i^1\rangle\langle w_j^2, x_k^2\rangle|x_k^1\rangle|v_i^2\rangle - \langle w_j^2, v_i^2\rangle\langle w_j^1, x_k^1\rangle|v_i^1\rangle|x_k^2\rangle \tag{79b}$$

$$= \sum_{i,j,k=1}^{n} \left(\langle w_j^1|\langle w_j^2| \otimes \mathbb{1}_{d_1} \otimes \mathbb{1}_{d_2}\right)\left(|v_i^1\rangle|x_k^2\rangle|x_k^1\rangle|v_i^2\rangle - |x_k^1\rangle|v_i^2\rangle|v_i^1\rangle|x_k^2\rangle\right) \tag{79c}$$

$$= \left(\langle w| \otimes \mathbb{1}_{d_1} \otimes \mathbb{1}_{d_2}\right) F_{24}(|v\rangle|x\rangle - |x\rangle|v\rangle) \tag{79d}$$

$$= \left(\langle w| \otimes \mathbb{1}_{d_1} \otimes \mathbb{1}_{d_2}\right) F_{24}(\mathbb{1} - F)(|v\rangle|x\rangle), \tag{79e}$$

where $F = F_{13}F_{24}$. To conclude the proof, we use that $(1 - F)^2 = 2(1 - F)$ together with the fact that $[F_{24}, F] = 0$, and we can write

$$\left[\mathbb{1}_{d_1} \otimes \mathrm{tr}_1(|v\rangle\langle w|) - \mathrm{tr}_2(|v\rangle\langle w|) \otimes \mathbb{1}_{d_2}\right]|x\rangle = \tag{80a}$$

$$= 2\left(\langle w| \otimes \mathbb{1}_{d_1} \otimes \mathbb{1}_{d_2}\right) P_- F_{24} P_-(|v\rangle|x\rangle) \tag{80b}$$

$$= a_-(\langle w|)F_{24}a_-^*(|v\rangle)|x\rangle, \tag{80c}$$

using the definition of fermionic creation and annihilation operators restricted to one copy of the space given in 14 and 15. The inequality 78 is analogous. □

Due to linearity, this result can be extended to any $C \in L(\mathscr{H})$, resulting in the operator $\mathbb{1}_{d_1} \otimes \operatorname{tr}_1(C) - \operatorname{tr}_2(C) \otimes \mathbb{1}_{d_2}$ having a "fermionic character", while that the operator $\mathbb{1}_{d_1} \otimes \operatorname{tr}_1(C) + \operatorname{tr}_2(C) \otimes \mathbb{1}_{d_2}$ has a "bosonic character". From the fermionic one, we can obtain the following result:

Corollary 2 *For any matrix $C \in L(\mathscr{H}_1 \otimes \mathscr{H}_2)$ with rank r,*

$$\|\mathbb{1}_{d_1} \otimes \operatorname{tr}_1(C) - \operatorname{tr}_2(C) \otimes \mathbb{1}_{d_2}\|_\infty \leq \sum_{i=1}^{r} \sigma_i = \|C\|_1, \tag{81}$$

where $\{\sigma_i\}_{i=1}^r$ is the set of singular values of C. In particular, for $v, w \in \mathscr{H}$, then

$$\|\mathbb{1}_{d_1} \otimes \operatorname{tr}_1(|v\rangle\langle w|) - \operatorname{tr}_2(|v\rangle\langle w|) \otimes \mathbb{1}_{d_2}\|_\infty \leq \|v\|\|w\|. \tag{82}$$

This result follows from the previous Proposition, the singular value decomposition, and the fact that $\|a_-(v)\|_\infty = \|a_-^*(v)\|_\infty = \|v\|$ (see [4]).

Proof of inequality 45 To show inequality 45, we need to show that both $q_{(1,0)}\left(-\frac{1}{r}, C\right), q_{(0,1)}\left(-\frac{1}{r}, C\right) \geq 0$. We will only show the first, since the other is analogous, i.e., we will prove

$$q_{(1,0)}\left(-\frac{1}{r}, C\right) = \|C\|_2^2 - \frac{1}{r}\|\operatorname{tr}_1 C\|_2^2 + \frac{1}{r}\|\operatorname{tr}_2 C\|_2^2 - \frac{1}{r^2}|\operatorname{tr} C|^2 \geq 0. \tag{83}$$

Using the singular value decomposition of C, we can write

$$C = \sum_{i=1}^{r} |v_i\rangle\langle w_i|, \tag{84}$$

where $\{v_i\}_{i=1}^r$ and $\{w_i\}_{i=1}^r$ are orthogonal systems of $\mathscr{H} = \mathscr{H}_1 \otimes \mathscr{H}_2$ (note that the vectors are not normalized, since they absorb the singular value). Write for every $1 \leq i \leq r$,

$$\begin{aligned} q_{(1,0)}\left(-\frac{1}{r}, |v_i\rangle\langle w_i|\right) &= \frac{1}{r} q_{(1,0)}\left(-1, |v_i\rangle\langle w_i|\right) + \left(1 - \frac{1}{r}\right)\|v_i\|^2\|w_i\|^2 \\ &+ \frac{r-1}{r^2}|\langle v_i, w_i\rangle|^2, \end{aligned} \tag{85}$$

where

$$q_{(1,0)}\left(-1, |v_i\rangle\langle w_i|\right) = \langle v_i \otimes w_i, (\mathbb{1} - F_{13})(\mathbb{1} + F_{24}) v_i \otimes w_i\rangle \geq 0, \tag{86}$$

analogously to Remark 2. Then,

$$q_{(1,0)}\left(-\frac{1}{r}, C\right) = \frac{1}{r}\sum_{i=1}^{r} q_{(1,0)}\left(-1, |v_i\rangle\langle w_i|\right) + \left(1-\frac{1}{r}\right)\sum_{i=1}^{r}\|v_i\|^2\|w_i\|^2 + \frac{r-1}{r^2}\sum_{i=1}^{r}|\langle v_i, w_i\rangle|^2 \tag{87a}$$

$$+\frac{2}{r}Re\sum_{\substack{i,j=1\\ i>j}}^{r}\Big[-\langle \mathrm{tr}_1(|v_i\rangle\langle w_i|), \mathrm{tr}_1(|v_j\rangle\langle w_j|)\rangle + \langle \mathrm{tr}_2(|v_i\rangle\langle w_i|), \mathrm{tr}_2(|v_j\rangle\langle w_j|)\rangle - \frac{1}{r}\langle v_i, w_i\rangle\langle w_j, v_j\rangle\Big]. \tag{87b}$$

Now, the key idea is to use the norms and the absolute value of the inner products in 87a to bound the term 87b. We proceed as follows:

$$\frac{2}{r}\left|Re\sum_{\substack{i,j=1\\ i>j}}^{r}\left[\langle -\mathrm{tr}_1(|v_i\rangle\langle w_i|), \mathrm{tr}_1(|v_j\rangle\langle w_j|)\rangle + \langle \mathrm{tr}_2(|v_i\rangle\langle w_i|), \mathrm{tr}_2(|v_j\rangle\langle w_j|)\rangle\right]\right| = \tag{88a}$$

$$\leq \frac{2}{r}\sum_{\substack{i,j=1\\ i>j}}^{r}\left|\langle |v_i\rangle\langle w_i|, -\mathbb{1}_{d_1}\otimes \mathrm{tr}_1(|v_j\rangle\langle w_j|) + \mathrm{tr}_2(|v_j\rangle\langle w_j|)\otimes \mathbb{1}_{d_2}\rangle\right| \tag{88b}$$

$$\leq \frac{2}{r}\sum_{\substack{i,j=1\\ i>j}}^{r}\|v_i\|\|w_i\|\|\mathbb{1}_{d_1}\otimes \mathrm{tr}_1(|v_j\rangle\langle w_j|) - \mathrm{tr}_2(|v_j\rangle\langle w_j|)\otimes \mathbb{1}_{d_2}\|_\infty \tag{88c}$$

$$\leq \frac{2}{r}\sum_{\substack{i,j=1\\ i>j}}^{r}\|v_i\|\|w_i\|\|v_j\|\|w_j\|, \tag{88d}$$

where we used Corollary 2. Finally, considering again the polynomial 59 given by

$$p_r(x) = (r-1)\sum_{i=1}^{r}x_i^2 - 2\sum_{\substack{i,j=1\\ i>j}}^{d_1}x_i x_j = \sum_{\substack{i,j=1\\ i>j}}^{r}(x_i - x_j)^2 \geq 0,$$

we obtain

$$q_{(1,0)}\left(-\frac{1}{r}, C\right) \geq \frac{1}{r}\sum_{i=1}^{r} q_{(1,0)}\left(-1, |v_i\rangle\langle w_i|\right) + \frac{1}{r}p_r(\|v_1\|\|w_1\|, \ldots, \|v_r\|\|w_r\|)$$

$$+\frac{1}{r^2}p_r(|\langle v_1, w_1\rangle|, \ldots, |\langle v_r, w_r\rangle|), \tag{89}$$

which is positive, so the result follows. □

In order to show 47, one could think of using the bosonic creation and annihilation operators as we did in the first part. However, this technique does not seem to work, since these are bounded by the square root of the number operator ([4]), but we will prove it for matrices of the form sum of a rank 1 plus a normal matrix making use of a different strategy. Before going into the proof of 47, consider first the following inversion of a pure state

$$Q_a^r = |a\rangle\langle a| - \frac{1}{r}\left(\mathbb{1}_{d_1}\otimes \mathrm{tr}_1(|a\rangle\langle a|) + \mathrm{tr}_2(|a\rangle\langle a|)\otimes \mathbb{1}_{d_2}\right) + \frac{1}{r^2}\mathrm{tr}(|a\rangle\langle a|)\mathbb{1}_{d_1d_2}, \tag{90}$$

which is self-adjoint, for $a \in \mathscr{H}$. We make use again of 53. i.e.,

$$\langle \mathrm{tr}_1(|a\rangle\langle a|), \mathrm{tr}_1(|x\rangle\langle x|)\rangle = \|\,\mathrm{tr}_2(|a\rangle\langle x|)\|_2^2,$$

for every $x \in \mathscr{H}$. We can obtain then a bound for the spectral radius on $\ker(|a\rangle\langle a|)$ as follows: Let $x \in \ker(|a\rangle\langle a|)$, then

$$\langle x, Q_a^r x\rangle = \frac{1}{r^2}\|a\|^2\|x\|^2 + |\langle a, x\rangle|^2 - \frac{1}{r}\|\,\mathrm{tr}_1(|a\rangle\langle x|)\|_2^2 - \frac{1}{r}\|\,\mathrm{tr}_2(|a\rangle\langle x|)\|_2^2 \tag{91}$$

$$= \frac{1}{r}q^{(2)}(-1, |a\rangle\langle x|) - \frac{1}{r}\left(1-\frac{1}{r}\right)\|a\|^2\|x\|^2, \tag{92}$$

and conversely

$$\left\langle x, \left(\frac{1}{r^2}\|a\|^2 - Q_a^r\right)x\right\rangle = \frac{1}{r}\|\,\mathrm{tr}_1(|a\rangle\langle x|)\|_2^2 + \frac{1}{r}\|\,\mathrm{tr}_2(|a\rangle\langle x|)\|_2^2 \geq 0, \tag{93}$$

so

$$-\frac{1}{r}\left(1-\frac{1}{r}\right)\|a\|^2 \leq Q_a^r \leq \frac{1}{r^2}\|a\|^2 \tag{94}$$

on $\ker(|a\rangle\langle a|)$. In particular, if we denote $P_{a^\perp}$ the projection onto $\ker(|a\rangle\langle a|)$, then

$$\tilde{Q}_a^r = P_{a^\perp}Q_a^r P_{a^\perp} \tag{95}$$

is self-adjoint and $\|\tilde{Q}_a^r\|_\infty \leq \frac{1}{r}\left(1-\frac{1}{r}\right)\|a\|^2$, for $r \geq 2$.

Proof of inequality 47 The case $r = 1$ was proven in Proposition 1, so we can assume that $r \geq 2$. Suppose that $C = C_1 + C_2$ with $C_1 = |v_1\rangle\langle w_1|$ such that both v_1, w_1 are orthogonal to all eigenvectors of the normal matrix C_2. Using the spectral decomposition in C_2 we can write

$$C = |v_1\rangle\langle w_1| + \sum_{i=2}^{r} \varepsilon_i |v_i\rangle\langle v_i|, \tag{96}$$

where $\varepsilon_i \in \mathbb{C}$, and we can assume that $|\varepsilon_i| = 1$ for every i, and the vectors v_i and w_1 are not normalized for $1 \le i \le r$. Our objective is to show that for this choice of C,

$$q_{(1,1)}\left(-\frac{1}{r}, C\right) = q^{(2)}\left(-\frac{1}{r}, C\right) = \|C\|_2^2 - \frac{1}{r}\|\operatorname{tr}_1 C\|_2^2 - \frac{1}{r}\|\operatorname{tr}_2 C\|_2^2 + \frac{1}{r^2}|\operatorname{tr} C|^2 \ge 0. \tag{97}$$

Similarly as we did in the proof of inequality 45, we write the quadratic form acting on each rank one matrix

$$\begin{aligned} q_{(1,1)}\left(-\frac{1}{r}, |v_1\rangle\langle w_1|\right) &= \frac{1}{r} q_{(1,1)}\left(-1, |v_1\rangle\langle w_1|\right) + \left(1 - \frac{1}{r}\right)\|v_1\|^2\|w_1\|^2 \\ &\quad - \frac{r-1}{r^2}|\langle v_1, w_1\rangle|^2, \end{aligned} \tag{98}$$

$$q_{(1,1)}\left(-\frac{1}{r}, |v_i\rangle\langle v_i|\right) = \frac{1}{r} q_{(1,1)}\left(-1, |v_i\rangle\langle w_i|\right) + \left(1 - \frac{1}{r}\right)^2 \|v_i\|^4, \tag{99}$$

for every $2 \le i \le r$, so we obtain

$$\begin{aligned} q_{(1,1)}\left(-\frac{1}{r}, C\right) &= \frac{1}{r}\left[q_{(1,1)}(-1, |v_1\rangle\langle w_1|) + \sum_{i=2}^{r} q_{(1,1)}(-1, |v_i\rangle\langle v_i|)\right] \\ &\quad + \left(1 - \frac{1}{r}\right)\|v_1\|^2\|w_1\|^2 - \frac{r-1}{r^2}|\langle v_1, w_1\rangle|^2 \\ &\quad + \left(1 - \frac{1}{r}\right)^2 \sum_{i=2}^{r}\|v_i\|^4 + 2\sum_{i=2}^{r} Re\left[\varepsilon_i \langle v_1, Q_{v_i}^r w_1\rangle\right] \\ &\quad + 2\sum_{\substack{i,j=2 \\ i>j}}^{r} Re\left[\varepsilon_i \overline{\varepsilon_j} \langle v_j, Q_{v_i}^r v_j\rangle\right], \end{aligned} \tag{100}$$

where $q_{(1,1)}(-1, |v_1\rangle\langle w_1|), q_{(1,1)}(-1, |v_i\rangle\langle v_i|) \ge 0$ due to Proposition 1 (or due to Remark 2) for every $2 \le i \le r$. Use now the bound of the operator 95

$$2\sum_{i=2}^{r} Re\left[\varepsilon_i \langle v_1, \tilde{Q}_{v_i}^r w_1\rangle\right] \ge -2\frac{1}{r}\left(1 - \frac{1}{r}\right)\sum_{i=1}^{r}\|v_1\|\|w_1\|\|v_i\|^2, \tag{101}$$

and

$$2\sum_{\substack{i,j=2\\i>j}}^{r} Re\left[\varepsilon_i\overline{\varepsilon_j}\langle v_j, \tilde{Q}^r_{v_i} v_j\rangle\right] \geq -2\sum_{\substack{i,j=2\\i>j}}^{r}\frac{1}{r}\left(1-\frac{1}{r}\right)\|v_i\|^2\|v_j\|^2. \tag{102}$$

Thus,

$$q_{(1,1)}\left(-\frac{1}{r}, C\right) \geq \left(1-\frac{1}{r}\right)\|v_1\|^2\|w_1\|^2 - \frac{r-1}{r^2}|\langle v_1, w_1\rangle|^2 + \left(1-\frac{1}{r}\right)^2\sum_{i=2}^{r}\|v_i\|^4 \tag{103a}$$

$$-2\frac{1}{r}\left(1-\frac{1}{r}\right)\sum_{i=1}^{r}\|v_1\|\|w_1\|\|v_i\|^2 - 2\sum_{\substack{i,j=2\\i>j}}^{r}\frac{1}{r}\left(1-\frac{1}{r}\right)\|v_i\|^2\|v_j\|^2 \tag{103b}$$

$$= \left(1-\frac{1}{r}\right)^2\left[\|v_1\|^2\|w_1\|^2 + \sum_{i=2}^{r}\|v_i\|^4\right] + \frac{r-1}{r^2}\left(\|v_1\|^2\|w_1\|^2 - |\langle v_1, w_1\rangle|^2\right) \tag{103c}$$

$$-2\frac{1}{r}\left(1-\frac{1}{r}\right)\sum_{i=1}^{r}\|v_1\|\|w_1\|\|v_i\|^2 - 2\sum_{\substack{i,j=2\\i>j}}^{r}\frac{1}{r}\left(1-\frac{1}{r}\right)\|v_i\|^2\|v_j\|^2 \tag{103d}$$

$$= \frac{1}{r}\left(1-\frac{1}{r}\right)p_r(\|v_1\|\|w_1\|, \|v_2\|^2, \ldots, \|v_r\|^2) + \frac{r-1}{r^2}\left(\|v_1\|^2\|w_1\|^2 - |\langle v_1, w_1\rangle|^2\right) \tag{103e}$$

$$\geq 0, \tag{103f}$$

where p_r is the polynomial 59. □

A Example of a matrices saturating the form $q^{(n)}$

In this appendix, we present a family of matrices that violate the positivity of the form $q^{(n)}$, for n even , $\alpha = -\frac{1}{2} - \varepsilon$, $\varepsilon > 0$ and a rank 2 matrix C. Let n even and consider

$$C = |v_1 \otimes \ldots \otimes v_n\rangle\langle v_1 \otimes \ldots \otimes v_n| - |w_1 \otimes \ldots \otimes w_n\rangle\langle w_1 \otimes \ldots \otimes w_n|, \tag{104}$$

with $\|v_i\| = \|w_i\| = 1$ and $v_i \perp w_i$ for every $1 \leq i \leq n$. In this case, for $J \in P(\{1, \ldots, n\})$,

$$\|\operatorname{tr}_J C\|_2^2 = \begin{cases} 0 \text{ if } J = \{1, \ldots, n\} \\ 2 \text{ otherwise} \end{cases} \tag{105}$$

so for every $\varepsilon > 0$,

$$q^{(n)}\left(-\frac{1}{2} - \varepsilon, C\right) = 2\sum_{J\in P(\{1,\ldots,n\})\setminus\{1,\ldots,n\}}\left(-\frac{1}{2} - \varepsilon\right)^{|J|} \tag{106a}$$

$$= 2\sum_{k=0}^{n-1}\binom{n}{k}\left(-\frac{1}{2}-\varepsilon\right)^k \tag{106b}$$

$$= 2\sum_{k=0}^{n-1}\sum_{m=0}^{k}\binom{n}{k}\binom{k}{m}\frac{(-1)^k}{2^{k-m}}\varepsilon^m \tag{106c}$$

$$= 2\sum_{m=0}^{n-1}\left[\sum_{k=m}^{n-1}\binom{n}{k}\binom{k}{m}\frac{(-1)^k}{2^{k-m}}\right]\varepsilon^m, \tag{106d}$$

where in the last step we have permuted the order of the summations. By the following Lemma, all the even powers of ε vanish and the odd are negatives. Moreover it goes to zero when $\varepsilon \to 0^+$. Consequently, $q^{(n)}\left(-\frac{1}{2}-\varepsilon, C\right) < 0$, for every $\varepsilon > 0$ and ρ_α is not n-distillable for $\alpha \in \left[-1, -\frac{1}{2}\right)$, n even.

Lemma 2 *If $n \in \mathbb{N}$ is even and $m < n$, then*

$$\sum_{k=m}^{n-1}\frac{(-1)^k}{2^{k-m}}\binom{n}{k}\binom{k}{m}\begin{cases} = 0 & \text{if } m = 0 \text{ or } m \text{ is even} \\ < 0 & \text{if } m \text{ is odd}\end{cases} \tag{107}$$

Proof The proof follows from the identity

$$\sum_{k=m}^{n} x^k\binom{n}{k}\binom{k}{m} = \binom{n}{m}x^m(1+x)^{n-m}. \tag{108}$$

Evaluating in $x = -\frac{1}{2}$

$$\begin{aligned}\sum_{k=m}^{n-1}\frac{(-1)^k}{2^{k-m}}\binom{n}{k}\binom{k}{m} &= 2^m\sum_{k=m}^{n}\left(-\frac{1}{2}\right)^k\binom{n}{k}\binom{k}{m} - \frac{(-1)^n}{2^{n-m}}\binom{n}{m} \\ &= \binom{n}{m}\left(\frac{1}{2}\right)^{n-m}\left((-1)^m - (-1)^n\right),\end{aligned} \tag{109}$$

and the result holds. □

Acknowledgements I would like to thank Prof. Michael M. Wolf for his fruitful discussions and ideas regarding this project, also thank to Dr. Felix Huber for his help with the bibliography and finally to all reviewers, who gave me a great feedback. Moreover, I want to thank the funding by the Deutsche Forschungsgemeinschaft (DFG, German Research Foundation) under Germany's Excellence Strategy-EXC2111-390814868.

Funding Open Access funding enabled and organized by Projekt DEAL.

Data availability Data sharing is not applicable to this article as no datasets were generated or analyzed during the current study.

Declarations

Conflict of interest The author has no conflict of interests related to this publication.

References

1. https://oqp.iqoqi.oeaw.ac.at/undistillability-implies-ppt
2. Audenaert, K.: Subadditivity of q-entropies for q>1. J. Math. Phys. **48**(8), 083507 (2007). https://doi.org/10.1063/1.2771542
3. Berta, M., Tomamichel, M.: Entanglement monogamy via multivariate trace inequalities. Commun. Math. Phys. **405**(2), 29 (2024). https://doi.org/10.1007/s00220-023-04920-5
4. Bratteli, O., Robinson, D.W.: Operator algebras and quantum statistical mechanics II: equilibrium states models in quantum statistical mechanics. Springer Science & Business Media (2013)
5. Butterley, P., Sudbery, A., Szulc, J.: Compatibility of subsystem states. Found. Phys. **36**(1), 83–101 (2006). https://doi.org/10.1007/s10701-005-9006-z
6. Chen, L., Chen, Y.: Rank-three bipartite entangled states are distillable. Phys. Rev. A **78**(2), 022318 (2008). https://doi.org/10.1103/physreva.78.022318
7. Chen, L.X., Đ oković, D. Ž.: Distillability and PPT entanglement of low-rank quantum states. J. Phys. A **44**(28), 285303 (2011). https://doi.org/10.1088/1751-8113/44/28/285303
8. Chen, L.X., He, H., Shi, X., Zhao, L.: Proving the distillability problem of two-copy 4×4 Werner states for monomial matrices. Quantum Inf. Process. **20**(4), 157 (2021). https://doi.org/10.1007/s11128-021-03098-w
9. Choi, D.: Inequalities related to partial transpose and partial trace. Linear Algebra Appl. **516**, 1–7 (2017). https://doi.org/10.1016/j.laa.2016.11.027
10. Clarisse, L.: Characterization of distillability of entanglement in terms of positive maps. Phys. Rev. A (2005). https://doi.org/10.1103/physreva.71.032332
11. DiVincenzo, D.P., Shor, P.W., Smolin, J.A., Terhal, B.M., Thapliyal, A.V.: Evidence for bound entangled states with negative partial transpose. Phys. Rev. A **61**(6), 062312 (2000). https://doi.org/10.1103/physreva.61.062312
12. Đ oković, D. Ž.: On two-distillable Werner states. Entropy **18**(6), 216 (2016). https://doi.org/10.3390/e18060216
13. Dür, W., Cirac, J.I., Lewenstein, M., Bruß, D.: Distillability and partial transposition in bipartite systems. Phys. Rev. A **61**(6), 062313 (2000). https://doi.org/10.1103/physreva.61.062313
14. Einstein, A., Podolsky, B., Rosen, N.: Can Quantum-Mechanical description of physical reality be considered complete? Phys. Rev. **47**(10), 777–780 (1935). https://doi.org/10.1103/physrev.47.777
15. Eltschka, C., Huber, F., Gühne, O., Siewert, J.: Exponentially many entanglement and correlation constraints for multipartite quantum states. Phys. Rev. **98**(5), 052317 (2018). https://doi.org/10.1103/physreva.98.052317
16. Eltschka, C., Siewert, J.: Maximum N-body correlations do not in general imply genuine multipartite entanglement. Quantum **4**, 229 (2020). https://doi.org/10.22331/q-2020-02-10-229
17. Fu, X., Lau, P., Tam, T.: Inequalities on partial traces of positive semidefinite block matrices. Can. Math. Bull. **64**(4), 964–969 (2020). https://doi.org/10.4153/s0008439520000971
18. Hall, W.B.: Multipartite reduction criteria for separability. Phys. Rev. A **72**(2), 022311 (2005). https://doi.org/10.1103/physreva.72.022311
19. Horn, R.A., Johnson, C.R.: Matrix analysis. Cambridge University Press (1985)

20. Horodecki, M., Horodecki, P.: Reduction criterion of separability and limits for a class of distillation protocols. Phys. Rev. A **59**(6), 4206–4216 (1999). https://doi.org/10.1103/physreva.59.4206
21. Horodecki, M., Horodecki, P., Horodecki, R.: Separability of mixed states: necessary and sufficient conditions. Phys. Lett. A **223**(1–2), 1–8 (1996). https://doi.org/10.1016/s0375-9601(96)00706-2
22. Horodecki, M., Horodecki, P., Horodecki, R.: Mixed-state entanglement and distillation: Is there a *Bound* entanglement in nature? Phys. Rev. Lett. **80**(24), 5239–5242 (1998). https://doi.org/10.1103/physrevlett.80.5239
23. Horodecki, P., Rudnicki, Ł, Życzkowski, K.: Five Open Problems in Quantum Information Theory. PRX Quantum **3**(1), 010101 (2022). https://doi.org/10.1103/prxquantum.3.010101
24. Huber, F.: Positive maps and trace polynomials from the symmetric group. J. Math. Phys. **10**(1063/5), 0028856 (2021)
25. Lewenstein, M., Augusiak, R., Chruściński, D., Rana, S., Samsonowicz, J.: Sufficient separability criteria and linear maps. Phys. Review. **93**(4), 042335 (2016). https://doi.org/10.1103/physreva.93.042335
26. Lewenstein, M., Bruss, D., Cirac, J.I., Kraus, B., Kus, M., Samsonowicz, J., Sanpera, A., Tarrach, R.: Separability and distillability in composite quantum systems-a primer. J. Modern Opt. **47**(14–15), 2481–2499 (2000). https://doi.org/10.1080/09500340008232176
27. Li, Y., Liu, W., Huang, Y.: A new matrix inequality involving partial traces. Operators Matrices **3**, 1189–1199 (2021). https://doi.org/10.7153/oam-2021-15-75
28. Pankowski, Ł, Piani, M., Horodecki, M., Horodecki, P.: A few steps more towards NPT bound entanglement. IEEE Trans. Inf. Theor. **56**(8), 4085–4100 (2010). https://doi.org/10.1109/tit.2010.2050810
29. Qian, L., Chen, L., Chu, D., Shen, Y.: A matrix inequality for entanglement distillation problem. Linear Algebra Appl. **616**, 139–177 (2021). https://doi.org/10.1016/j.laa.2021.01.006
30. Rains, E.M.: Polynomial invariants of quantum codes. IEEE Trans. Inf. Theor. **46**(1), 54–59 (2000). https://doi.org/10.1109/18.817508
31. Rastegin, A.E.: Relations for certain symmetric norms and anti-norms before and after partial trace. J. Stat. Phys. **148**(6), 1040–1053 (2012). https://doi.org/10.1007/s10955-012-0569-8
32. Rico, A., Huber, F.: Entanglement detection with trace polynomials. Phys. Revi. Lett. **132**(7), 070202 (2024). https://doi.org/10.1103/physrevlett.132.070202
33. Sutter, D., Berta, M., Tomamichel, M.: Multivariate trace inequalities. Commun. Math. Phys. **352**(1), 37–58 (2016). https://doi.org/10.1007/s00220-016-2778-5
34. Vianna, R.O., Doherty, A.C.: Distillability of Werner states using entanglement witnesses and robust semidefinite programs. Phys. Rev. A **74**(5), 052306 (2006). https://doi.org/10.1103/physreva.74.052306
35. Werner, R.F.: Quantum states with Einstein-Podolsky-Rosen correlations admitting a hidden-variable model. Phys. Rev. A Gen. Phys. **40**(8), 4277–4281 (1989). https://doi.org/10.1103/physreva.40.4277