Journal of Physics: Conference Series **2866** (2024) 012082 doi:10.1088/1742-6596/2866/1/012082

On the Equivalence of Gravitational Waves Formulations in Teleparallel Gravity and General Relativity

Andre Saputra[1] and Bintoro Anang Subagyo[2]
Department of Physics, Institut Teknologi Sepuluh Nopember
Kampus ITS Sukolilo, Surabaya, 60111, Indonesia
Email: [1]18.andresaputra@gmail.com, [2]bintoro.subagyo@its.ac.id

Abstract. Gravitational waves have become a vital way to test our current understanding of gravity. In this work, we explore the formulations of gravitational waves in two distinct theories of gravity: the Teleparallel Equivalent of General Relativity (TEGR), a subclass of Teleparallel Gravity (TG), and General Relativity (GR). By linearizing the gravitational field equations in each theory, we demonstrate that the formulations of gravitational waves in TEGR and GR are equivalent. This equivalence suggests that the teleparallel formalism can effectively describe gravitational waves, providing a novel perspective on the topic and potentially broadening the scope of gravitational wave research.

1. Introduction

For the last century, General Relativity (GR) has revolutionized our understanding of gravity and the universe on the large scale. However, recent astronomical and cosmological observations suggest that GR may not offer a complete framework for explaining the fundamental nature of gravity and the universe as a whole [1]. Modified gravity has emerged as one of the alternative models to explain phenomena such as the accelerating expansion of the universe [2], the nature of dark matter [3], and inflation in the early universe [4]. To validate these theories, it is crucial to contrast their predictions against a broad range of tests conducted by various methods and bound by astronomical and cosmological measurements [5].

A modified gravity theory proposed by Albert Einstein shortly after the introduction of GR is distant parallelism [6], which now serves as the foundation of Teleparallel Gravity (TG) [7]. In TG, the tetrad field is used as the primary variable to describe the geometrical aspects of spacetime, instead of the metric used in GR [8]. In TG, the gravitational field is characterized by torsion and/or non-metricity, in contrast to GR, where the gravitational field is characterized by curvature [9]. One subclass of TG is the Teleparallel Equivalent of General Relativity (TEGR), which provides an alternative geometrical formulation to GR. TEGR employs the Weitzenböck connection, which is both metric-compatible and curvature-free, unlike the Levi-Civita connection used in GR, which is both metric-compatible and torsion-free [10].

From an astrophysical perspective, the first direct detection of gravitational waves has provided a novel way to test various theories of gravity, confirming predictions made by Einstein a century ago [11]. Furthermore, detections of gravitational waves from various astrophysical sources offer a unique opportunity to test modified gravity theories under strong gravitational field conditions. For instance,

Content from this work may be used under the terms of the Creative Commons Attribution 4.0 licence. Any further distribution of this work must maintain attribution to the author(s) and the title of the work, journal citation and DOI.
Published under licence by IOP Publishing Ltd

Journal of Physics: Conference Series **2866** (2024) 012082 doi:10.1088/1742-6596/2866/1/012082

the detection of GW170817, the first direct observation of a neutron star-black hole merger, imposed constraints on many modified gravity theories based on the observed properties of the gravitational waves [12,13].

Given the significance of gravitational waves in testing existing theories of gravity, our work explores the formulations of gravitational waves in both GR and TEGR. This involves linearizing the gravitational field equations in each theory and applying specific gauge conditions for the study of gravitational waves. We then compare the linearized field equations to demonstrate the equivalence between the formulations in each theory. Our analysis provides a new perspective by utilizing the teleparallel formalism to explore the nature of gravitational waves.

2. Method

2.1. Theoretical Foundation

The tetrad is the primary variable in TG and the study of differential geometry. Tetrad ${e^a}_\mu$ transforms the coordinate basis dx^μ of cotangent space to an orthonormal basis of the cotangent space on a manifold. The inverse of tetrad, ${E_a}^\mu$, transforms the coordinate basis $\partial/\partial x^\mu$ of the tangent space to orthonormal basis of the tangent space on a manifold [14].

$$e^a = {e^a}_\mu dx^\mu \qquad E_a = {E_a}^\mu \frac{\partial}{\partial x^\mu} \ . \tag{1}$$

Through the relationships below, the metric $g_{\mu\nu}$, its inverse $g^{\mu\nu}$, and determinants g can be related to tetrad ${e^a}_\mu$, its inverse ${E_a}^\mu$, and its determinant e,

$$g_{\mu\nu} = \eta_{ab}{e^a}_\mu {e^b}_\nu \ , \qquad g^{\mu\nu} = \eta^{ab}{E_a}^\mu {E_b}^\nu \ , \qquad \sqrt{-g} = \mathrm{e}, \tag{2}$$

where η_{ab} is the Minkowski metric in the tangent space, and η^{ab} is its inverse,

$$\eta_{ab} = \mathrm{diag}(-1,1,1,1) \ . \tag{3}$$

The tetrad and its inverse also obey the orthogonality conditions

$${e^a}_\mu {E_a}^\nu = \delta^\nu_\mu \ , \qquad {e^a}_\mu {E_b}^\mu = \delta^a_b \ . \tag{4}$$

We use a convention where Greek letters $(\mu, \nu, \dots)$ represent spacetime indices, while Latin letters $(a, b, \dots)$ represent Lorentz indices for Minkowski spacetime [15]. Latin letters $(i, j, \dots)$ are used for the space components of spacetime indices, as commonly known. We can utilize ${e^a}_\mu$ and ${E_a}^\mu$ to transform Lorentz indices to spacetime indices and vice versa. Additionally, η_{ab} and η^{ab} can be utilized to raise and lower Lorentz indices. As commonly known in metric formalism, $g_{\mu\nu}$ and $g^{\mu\nu}$ can be utilized to raise and lower spacetime indices. We can establish the relationship between affine connection ${\hat{\Gamma}^\lambda}_{\mu\nu}$ and affine spin connection ${\hat{\omega}^a}_{b\mu}$, also known as the tetrad postulate

$$\widehat{\mathfrak{D}}_\mu {e^a}_\nu \equiv \partial_\mu {e^a}_\nu + {\hat{\omega}^a}_{b\mu} {e^b}_\nu = {\hat{\Gamma}^\lambda}_{\mu\nu} {e^a}_\lambda \ , \tag{5}$$

where $\widehat{\mathfrak{D}}_\mu {e^a}_\nu$ is the Fock-Ivanenko derivative of tetrad ${e^a}_\mu$. We also use the convention from [15] to classify the possible affine connections. General affine connection ${\hat{\Gamma}^\lambda}_{\mu\nu}$ has no specific constraints on spacetime geometry. Levi-Civita affine connection ${\mathring{\Gamma}^\lambda}_{\mu\nu}$ is used to define metric-compatible, torsion-free spacetime in GR.

$${\mathring{\Gamma}^\lambda}_{\mu\nu} = \frac{1}{2} g^{\lambda\sigma}(\partial_\mu g_{\sigma\nu} + \partial_\nu g_{\sigma\mu} - \partial_\sigma g_{\mu\nu}) \tag{6}$$

Moreover, teleparallel affine connection $\Gamma^{\lambda}{}_{\mu\nu}$ is used in TEGR to define metric-compatible, curvature-free spacetime. Quantities associated with a specific affine connection are differentiated analogously to the differentiation process of that particular affine connection. From the affine connection, we can derive the torsion tensor $\widehat{T}^{\lambda}{}_{\mu\nu}$ and Riemann (curvature) tensor $\widehat{R}^{\alpha}{}_{\beta\mu\nu}$ using Cartan's structure equations,

$$\widehat{T}^{\lambda}{}_{\mu\nu} = E_a{}^{\lambda}\left(\partial_\mu e^a{}_\nu - \partial_\nu e^a{}_\mu + \widehat{\omega}^a{}_{b\mu} e^b{}_\nu - \widehat{\omega}^a{}_{b\nu} e^b{}_\mu\right) , \quad (7)$$

$$\widehat{R}^{\alpha}{}_{\beta\mu\nu} = E_a{}^{\alpha} e^b{}_\beta\left(\partial_\mu \widehat{\omega}^a{}_{b\nu} - \partial_\nu \widehat{\omega}^a{}_{b\mu} + \widehat{\omega}^a{}_{c\mu}\,\widehat{\omega}^c{}_{b\nu} - \widehat{\omega}^a{}_{c\nu}\,\widehat{\omega}^c{}_{b\mu}\right) . \quad (8)$$

In TEGR, we utilize the Weitzenböck gauge, $\omega^a{}_{b\mu} = 0$, thereby simplifying Eq. (5) to [16]

$$\Gamma^{\lambda}{}_{\mu\nu} = E_a{}^{\lambda}\partial_\mu e^a{}_\nu \, . \quad (9)$$

When we apply Weitzenböck gauge in teleparallel affine connection, we get Weitzenböck affine connection. Therefore, the torsion tensor and curvature tensor in Weitzenböck affine connection reduces to [17]

$$T^{\lambda}{}_{\mu\nu} = E_a{}^{\lambda}\left(\partial_\mu e^a{}_\nu - \partial_\nu e^a{}_\mu\right) \qquad R^{\alpha}{}_{\beta\mu\nu} = 0 \, . \quad (10)$$

We can also define the contortion tensor from the torsion tensor,

$$\widehat{K}^{\lambda}{}_{\mu\nu} = \frac{1}{2}\left(\widehat{T}_\nu{}^{\lambda}{}_\mu + \widehat{T}_\mu{}^{\lambda}{}_\nu + \widehat{T}^{\lambda}{}_{\mu\nu}\right) . \quad (11)$$

With the definition of contortion tensor in Eq. (11), Riemann tensor in Levi-Civita affine connection can be expressed in terms of teleparallel quantities as

$$\mathring{R}^{\alpha}{}_{\beta\mu\nu} = K^{\alpha}{}_{\nu\gamma} K^{\gamma}{}_{\mu\beta} - K^{\alpha}{}_{\mu\gamma} K^{\gamma}{}_{\nu\beta} + \mathring{\nabla}_\nu K^{\alpha}{}_{\mu\beta} - \mathring{\nabla}_\mu K^{\alpha}{}_{\nu\beta} \, . \quad (12)$$

To formulate the field equations in TEGR, we can define the torsion vector (T_μ and T^μ), torsion scalar T, and the superpotential tensor $S^{\mu\nu\gamma}$ as follows [16].

$$T_\mu = T^{\alpha}{}_{\mu\alpha} \qquad T^\mu = T^{\alpha\mu}{}_{\alpha} \quad (13)$$

$$S^{\mu\nu\gamma} = \frac{1}{2}\left(T^{\mu\nu\gamma} + T^{\gamma\nu\mu} + T^{\nu\mu\gamma}\right) + g^{\mu\nu}T^{\gamma} - g^{\mu\gamma}T^{\nu} \qquad T = \frac{1}{2}T_{\mu\nu\gamma}S^{\mu\nu\gamma} \quad (14)$$

2.2. Teleparallel equivalent of general relativity (TEGR) and its relation to general relativity (GR)

To derive the field equations in TEGR, we introduce the action of TEGR as follows:

$$S_{TEGR} = -\frac{c^4}{16\pi G}\int d^4x\, eT + \int d^4x\, e\mathcal{L}_m \, . \quad (15)$$

Here, $\mathcal{L}_m$ is the matter Lagrangian, c is the speed of light, and G is the Newton's gravitational constant. In comparison, the Einstein-Hilbert action is used to derive the field equations in GR,

$$S_{EH} = \frac{c^4}{16\pi G}\int d^4x\, \sqrt{-g}\mathring{R} + \int d^4x\, \sqrt{-g}\mathcal{L}_m \, , \quad (16)$$

where $\mathring{R}$ is the Ricci scalar defined in Levi-Civita affine connection.

$$\mathring{R} = g^{\mu\nu}\mathring{R}_{\mu\nu} = g^{\mu\nu}\mathring{R}^{\alpha}{}_{\mu\alpha\nu} \quad (17)$$

$\mathring{R}_{\mu\nu}$ is the Ricci tensor defined in Levi-Civita affine connection. By substituting the definition of the Riemann tensor in Eq. (12) to Eq. (17), we can represent $\mathring{R}$ in terms of teleparallel quantities as

$$\mathring{R} = -\left(T + 2\mathring{\nabla}_\mu T^\mu\right) = -\left(T + B_T\right) \, , \quad (18)$$

where B_T is the boundary term for TEGR [15]. This boundary term vanishes when we use the principle of least action for TEGR, yielding the same field equations obtained when the same principle is applied to the Einstein-Hilbert action in GR. This condition establishes equivalence between TEGR and GR. By applying the principle of least action to Eq. (15), we obtain the expression for the gravitational field equations in TEGR as

$$\frac{1}{e}\partial_\mu(eS_a{}^{\kappa\mu}) - T^{\mu\nu}{}_a S_{\mu\nu}{}^\kappa + \frac{1}{2}E_a{}^\kappa T = \frac{8\pi G}{c^4}\Theta_a{}^\kappa \ , \tag{19}$$

where $\Theta_a{}^\kappa$ is the stress-energy expressed in terms of tetrad. Meanwhile, the Einstein field equations in GR can be obtained by applying the principle of least action to Eq. (16),

$$\mathring{G}_{\mu\nu} \equiv \mathring{R}_{\mu\nu} - \frac{1}{2}g_{\mu\nu}\mathring{R} = \frac{8\pi G}{c^4}\Theta_{\mu\nu} \ , \tag{20}$$

where $\mathring{G}_{\mu\nu}$ is the Einstein tensor and $\Theta_{\mu\nu}$ is the stress-energy tensor.

3. Result and Discussion

In the theoretical study of gravitational waves, several assumptions are made. The first assumption is weak and static gravitational field (Newtonian limit). The second assumption is the test particle is non-relativistic, i.e. $v \ll c$. Additional assumptions are established according to each theory.

3.1. Gravitational waves formulation in GR

The first additional assumption in the linearized Einstein field equations is that the metric utilized in the analysis corresponds to the weak-field metric, which is expanded around the flat spacetime,

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu} \ , \tag{21}$$

where $\eta_{\mu\nu}$ is Minkowski metric in spacetime indices and $h_{\mu\nu}$ denotes the small perturbation on the metric. For the inverse metric, we have

$$g^{\mu\nu} = \eta^{\mu\nu} - h^{\mu\nu} \ . \tag{22}$$

In linearized field equations in GR, we use $\eta_{\mu\nu}$ and $\eta^{\mu\nu}$ to raise and lower spacetime indices. The second additional assumption is the small perturbation on the metric considered in the analysis is only on the first order, $\mathcal{O}(h_{\mu\nu})$. To satisfy this condition, the necessary requirements are as follows. For the metric perturbation, we have

$$|h_{\mu\nu}| \ll 1 \qquad\qquad |\partial_\rho h_{\mu\nu}| \ll 1 \ . \tag{23}$$

By applying the metric and its inverse from Eqs. (21) and (22) to Eq. (20), we can linearize the Einstein field equations, resulting in the following expression,

$$\Box h_{\mu\nu} + \eta_{\mu\nu}\partial^\alpha\partial^\beta h_{\alpha\beta} + \partial_\mu\partial_\nu h - \eta_{\mu\nu}\Box h - \partial^\sigma\partial_\mu h_{\sigma\nu} - \partial^\sigma\partial_\nu h_{\sigma\mu} = -\frac{16\pi G}{c^4}\Theta_{\mu\nu} \ , \tag{24}$$

where $\Box = \eta_{\rho\sigma}\partial^\rho\partial^\sigma$ is the usual d'Alembert operator. To simplify Eq. (24), we introduce the trace-reverse of $h_{\mu\nu}$, which is defined as

$$\bar{h}_{\mu\nu} \equiv h_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}h^\alpha{}_\alpha \ . \tag{25}$$

We also utilize the harmonic gauge, also known as the Lorenz gauge,

$$\partial_\nu \bar{h}^{\mu\nu} = \partial^\nu \bar{h}_{\mu\nu} = 0 \ . \tag{26}$$

Based on Eqs. (25)- (26), Eq. (24) simplifies to

$$\Box\bar{h}_{\mu\nu} = -\frac{16\pi G}{c^4}\Theta_{\mu\nu}\,. \tag{27}$$

To simplify the solution of Eq. (27), we introduce a transverse-traceless (TT) gauge in the form of

$$h_{0\mu} = h_{\mu 0} = 0\,, \qquad h^i{}_i = 0\,, \qquad \partial^j h_{ij} = 0\,. \tag{28}$$

This gauge reduces 10 degrees of freedom of the metric perturbation to just 2 degrees of freedom. Physically, this corresponds to the two polarization states of gravitational waves: plus and cross polarizations. Furthermore, the trace of the metric perturbation becomes zero, which results in the relation $\bar{h}_{\mu\nu} = h_{\mu\nu}$ [18]. Consequently, Eq. (27) can be expressed as

$$\Box h_{ij} = -\frac{16\pi G}{c^4}\Theta_{ij}\,. \tag{29}$$

3.2. Gravitational waves formulation in TEGR

The first additional assumption of the linearization of gravitational field equations in TEGR is that the tetrad used in the analysis is the weak-field tetrad in the form of

$$e^a{}_\mu = \delta^a{}_\mu + B^a{}_\mu\,, \tag{30}$$

where $\delta^a{}_\mu$ is a flat diagonal background tetrad and $B^a{}_\mu$ denotes the small perturbation on the tetrad.

$$\delta^a{}_\mu = \mathrm{diag}(1,1,1,1) \tag{31}$$

The inverse of Eq. (30) can be expressed in the form of

$$E_a{}^\mu = \delta_a{}^\mu + B_a{}^\mu\,, \tag{32}$$

where $\delta_a{}^\mu = \delta^a{}_\mu$ and $B_a{}^\mu$ is the inverse of $B^a{}_\mu$. In linearized field equations in TEGR, we use $\delta^a{}_\mu$ and its inverse $\delta_a{}^\mu$ to transform Lorentz indices to spacetime indices and vice versa. Additionally, η_{ab} and η^{ab} are also utilized to raise and lower Lorentz indices, while $\eta_{\mu\nu}$ and $\eta^{\mu\nu}$ serve the same roles as in GR. By utilizing Eq. (2), we can establish the relationship between the tetrad perturbation and metric perturbation as

$$h_{\mu\nu} = \eta_{ab}\delta^a{}_\mu B^b{}_\nu + \eta_{ab}\delta^b{}_\nu B^a{}_\mu = B_{\mu\nu} + B_{\nu\mu}\,. \tag{33}$$

Furthermore, we can establish the relationship between the Minkowksi metric in tangent space and the Minkowksi metric in the manifold as

$$\eta_{\mu\nu} = \eta_{ab}\delta^a{}_\mu\delta^b{}_\nu \tag{34}$$

The second additional assumption is the small perturbation on the tetrad considered in the analysis is only on the first order, $\mathcal{O}(B^a{}_\mu)$. To satisfy this condition, the necessary requirements are as follows. For the tetrad perturbation, we have

$$|B^a{}_\mu| \ll 1 \qquad |\partial_a B^a{}_\mu| \ll 1 \tag{35}$$

This assumption is related with the corresponding assumption in GR, as shown by Eqs. (23) and (33).

Based on the tetrad expansion in Eq. (30) and (32), it follows from Eq. (35) that $e \approx 1$. Furthermore, several teleparallel quantities—including the torsion tensor, torsion vector, and superpotential tensor—are defined using the first-order tetrad perturbation. Meanwhile, other quantities such as torsion scalar and the term $T^{\mu\nu}{}_a S_{\mu\nu}{}^\kappa$ in Eq. (19) come from the second-order tetrad perturbation, which result from the product of two first-order tetrad perturbations. Therefore, based on Eq. (35), these quantities can be approximated as zero. Therefore, the linearization becomes the following expression

$$\begin{aligned}\frac{1}{2}\eta_{ab}\partial^{\kappa}\partial^{\sigma}B^{b}{}_{\sigma}-\frac{1}{2}\eta_{ab}\eta^{\rho\kappa}\partial^{\sigma}\partial_{\sigma}B^{b}{}_{\rho}+\frac{1}{2}\delta_{a}{}^{\sigma}\partial^{\kappa}\partial_{b}B^{b}{}_{\sigma}-\frac{1}{2}\delta_{a}{}^{\sigma}\eta^{\rho\kappa}\partial_{\sigma}\partial_{b}B^{b}{}_{\rho}\\ -\frac{1}{2}\delta_{b}{}^{\kappa}\delta_{a}{}^{\sigma}\partial^{\rho}\partial_{\rho}B^{b}{}_{\sigma}+\frac{1}{2}\delta_{b}{}^{\kappa}\delta_{a}{}^{\sigma}\partial_{\sigma}\partial^{\rho}B^{b}{}_{\rho}+\delta_{a}{}^{\kappa}\partial^{\rho}\partial_{\rho}(\delta_{c}{}^{\alpha}B^{c}{}_{\alpha})\\ -\delta_{a}{}^{\kappa}\partial^{\rho}\partial_{c}B^{c}{}_{\rho}-\partial_{a}\partial^{\kappa}(\delta_{c}{}^{\alpha}B^{c}{}_{\alpha})+\eta^{\rho\kappa}\partial_{a}\partial_{c}B^{c}=\frac{8\pi G}{c^{4}}\Theta_{a}{}^{\kappa}\ .\end{aligned} \tag{36}$$

To simplify Eq. (36), we can impose the symmetric properties of $h_{\mu\nu}$ to the tetrad perturbation so that the tetrad perturbation in spacetime indices is symmetric in its both indices, i.e. $B_{\mu\nu}=B_{\nu\mu}$. Consequently, Eq. (33) becomes

$$h_{\mu\nu}=2\eta_{ab}\delta^{a}{}_{\mu}B^{b}{}_{\nu}=2B_{\mu\nu}\ . \tag{37}$$

In addition, we can also employ the TT gauge from GR by adapting it from the metric form to the tetrad form using Eq. (37). Therefore, TT gauge in tetrad is expressed as

$$B_{\mu 0}=B_{0\mu}=0\ ,\qquad \delta_{b}{}^{\nu}B^{b}{}_{\nu}=0\ ,\qquad \partial_{b}B^{b}{}_{\nu}=0\ ,\qquad \partial^{\mu}B^{a}{}_{\mu}=0\ . \tag{38}$$

By multiplying Eq. (36) by $\delta^{a}{}_{\mu}\eta_{\kappa\nu}$, and using the gauge from Eq. (38), we obtain the following expression.

$$\begin{aligned}\eta_{\kappa\nu}\delta^{a}{}_{\mu}\eta_{ab}\eta^{\rho\kappa}\Box B^{b}{}_{\rho}+\eta_{\kappa\nu}\delta^{a}{}_{\mu}\delta_{b}{}^{\kappa}\delta_{a}{}^{\sigma}\Box B^{b}{}_{\sigma}&=-\frac{16\pi G}{c^{4}}e^{a}{}_{\mu}\eta_{\kappa\nu}\Theta_{a}{}^{\kappa}\\ \Box(B_{ij}+B_{ji})&=-\frac{16\pi G}{c^{4}}\Theta_{ij}\\ \Box h_{ij}&=-\frac{16\pi G}{c^{4}}\Theta_{ij}\end{aligned} \tag{39}$$

The result from Eqs. (39) and (29) demonstrates that the formulations of gravitational waves in GR is equivalent to those in TEGR. Therefore, all the solutions for the gravitational waves obtained in GR can be also applied to TEGR, at least in the first-order perturbation of the metric or tetrad.

4. Conclusion

This study investigates the equivalence of gravitational wave formulations in GR and TEGR. By linearizing the gravitational field equations for each theory, we demonstrate that despite the differences in the underlying spacetime geometry, both GR and TEGR produce equivalent wave-like perturbations.

In GR, gravity is characterized by the curvature of spacetime, and the field equations are derived using the Levi-Civita connection, which is metric-compatible and torsion-free. On the other hand, TEGR employs the Weitzenböck connection, which is metric-compatible and curvature-free, and describes gravity through torsion. Our analysis shows that these differing geometrical frameworks do not affect the fundamental nature of gravitational waves, as both formulations converge to the same linearized field equations in the weak-field limit. This equivalence highlights that torsion can be used as an alternative to curvature in describing gravitational phenomena, offering a fresh perspective on the geometrical nature of gravity. These results also suggest that the teleparallel formalism is not only a valid approach but also a potentially advantageous one for exploring the theoretical aspects of gravitational waves. This insight is crucial as it broadens the scope of theoretical frameworks that can be employed in gravitational wave research, potentially leading to new discoveries and deeper understanding.

Future research could expand on this foundation by exploring other aspects of gravitational wave modeling within the teleparallel framework. For instance, the Post-Newtonian (PN) formalism, commonly used in GR to describe the motion of bodies in weak gravitational fields, could be adapted for TG. Such an adaptation would provide further validation of TG in various gravitational scenarios. Additionally, Numerical Relativity (NR), which involves solving the field equations using numerical methods, could be applied to TG to simulate different gravitational wave phenomena, such as the ringdown phase of binary black hole mergers or extreme-mass-ratio inspirals (EMRIs). More advanced modeling techniques for gravitational waves in TG need to be developed. By integrating these advanced modeling techniques with observational data, we can rigorously test the predictions of TG and compare them with those of GR, potentially uncovering subtle differences that could lead to breakthroughs in our understanding of gravity.

In conclusion, our study demonstrates the equivalence of gravitational wave formulations in GR and TEGR, providing a new approach to studying gravitational waves through the teleparallel formalism. This equivalence opens up new avenues for research and highlights the robustness of our theoretical understanding of gravitational waves. As future detectors come online and more data becomes available, the insights gained from this study will be invaluable in pushing the boundaries of gravitational wave astronomy and our understanding of the fundamental nature of gravity.

5. Acknowledgments
This work was supported by the Ministry of Education, Culture, Research, and Technology of Indonesia and ITS DRPM (Direktorat Riset dan Pengabdian kepada Masyarakat) under Grant No. 1152/PKS/ITS/2024.

References
[1] Clifton T, Ferreira P G, Padilla A and Skordis C 2012 Modified gravity and cosmology *Phys. Rep.* **513** 1–189
[2] Joyce A, Lombriser L and Schmidt F 2016 Dark Energy Versus Modified Gravity *Annu. Rev. Nucl. Part. Sci.* **66** 95–122
[3] Calmet X and Kuntz I 2017 What is modified gravity and how to differentiate it from particle dark matter? *Eur. Phys. J. C* **77** 132
[4] Nojiri S, Odintsov S D and Oikonomou V K 2017 Modified gravity theories on a nutshell: Inflation, bounce and late-time evolution *Phys. Rep.* **692** 1–104
[5] Shankaranarayanan S and Johnson J P 2022 Modified theories of gravity: Why, how and what? *Gen. Relativ. Gravit.* **54** 44
[6] Einstein A 1928 Riemann-Geometrie mit Aufrechterhaltung des Begriffes des Fernparallelismus *Preuss. Akad. Wiss. Phys-Math Kl. Sitzungsberichte* 217–21
[7] De Andrade V C, Guillen L C T and Pereira J G 2002 Teleparallel Gravity: An Overview *The Ninth Marcel Grossmann Meeting* (The University of Rome 'La Sapienza': World Scientific Publishing Company) pp 1022–3
[8] Krššák M, van den Hoogen R J, Pereira J G, Böhmer C G and Coley A A 2019 Teleparallel theories of gravity: illuminating a fully invariant approach *Class. Quantum Gravity* **36** 183001
[9] Hohmann M 2023 Teleparallel Gravity *Modified and Quantum Gravity* (Springer) pp 145–98
[10] Maluf J W 2013 The teleparallel equivalent of general relativity *Ann. Phys.* **525** 339–57
[11] Einstein A 1916 Die Grundlage der allgemeinen Relativitätstheorie *Ann. Phys.* **354** 769–822
[12] Jana S and Mohanty S 2019 Constraints on f(R) theories of gravity from GW170817 *Phys. Rev. D* **99** 044056
[13] Karimi S and Karami K 2020 f(G) gravity after GW170817 *Astrophys. Space Sci.* **365** 26
[14] Eguchi T, Gilkey P B and Hanson A J 1980 Gravitation, gauge theories and differential geometry *Phys. Rep.* **66** 213–393
[15] Bahamonde S, Dialektopoulos K F, Escamilla-Rivera C, Farrugia G, Gakis V, Hendry M, Hohmann M, Levi Said J, Mifsud J and Di Valentino E 2023 Teleparallel gravity: from theory to cosmology *Rep. Prog. Phys.* **86** 026901
[16] Golovnev A, Koivisto T and Sandstad M 2017 On the covariance of teleparallel gravity theories *Class. Quantum Gravity* **34** 145013
[17] Lucas T G, Obukhov Y N and Pereira J G 2009 Regularizing role of teleparallelism *Phys. Rev. D* **80** 064043
[18] Maggiore M 2007 *Gravitational Waves: Volume 1: Theory and Experiments* (Oxford University Press)