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Bianchi-I spacetimes in loop quantum cosmology: physics of singularity resolution

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Abstract. Quantum geometry effects in loop quantum cosmology are known to resolve various singularities in the isotropic and anisotropic models. Previous investigations have shown that singularity resolution is tied to boundedness of physical quantities such as the energy density and the expansion and shear scalars in the spacetime. Using effective dynamics, we show that these do not necessarily lead to a generic boundedness of spacetime curvature in the Bianchi-I model and find the potential possibilities where the curvature invariants can diverge. These include cases where the pressure of the matter content diverges at a finite volume and energy density. By analyzing the strength of singularities, we show that such pressure singularities are weak. All known types of strong singularities of the classical theory are resolved. This analysis brings us closer to generalize the results on generic resolution of singularities in the isotropic models in loop quantum cosmology, to the anisotropic spacetimes.

1. Introduction

Bianchi-I spacetimes are one of the simplest spacetimes which admit a non-vanishing Weyl curvature. The presence of anisotropies, which lead to the Weyl part of the spacetime curvature, generate a very rich structure of singularities in the classical theory. These singularities have been widely studied in GR (see for eg. [1]). Depending on the behavior of different scale factors, singularities in Bianchi-I spacetimes have been classified as isotropic (point-like) singularities, barrel, cigar and pancake singularities [2, 3]. The occurrence of these singularities depend on the choice of matter component, and whether it has a vanishing (such as a perfect fluid) or non-vanishing anisotropic stress (such as in presence of magnetic fields). For all the singularities studied so far in the classical theory, at least one of the scale factors in the metric vanishes. At these singularities, directional Hubble rates, energy density, expansion and shear scalars diverge as the physical volume vanishes, which results in various curvature invariants becoming infinite. Analysis of the geodesic equations show that geodesic evolution breaks down at these singuilarities [4]. In terms of the conditions of Tipler [5] and Królak [6], these singularities are strong curvature type, which lead to a complete destruction of arbitrary detectors falling in the singularities. Existence of these singularities signals that the limit of the applicability of GR has been reached. It has been hoped that a quantum theory of gravity would lead to the resolution of space-like singularities. Our goal, here is to understand the way quantum properties of spacetime as understood in loop quantum cosmology (LQC) lead to the resolution of above singularities in the Bianchi-I spacetime sourced with a fluid with a vanishing anisotropic stress.

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LQC is a quantization of homogeneous spacetimes based on the techniques of loop quantum gravity (see Ref. [7] for an extensive review). In contrast to earlier attempts to quantize cosmological and symmetry reduced spacetimes, such as in Wheeler-DeWitt theory, a full quantization of several spacetimes has been accomplished and detailed physics has been analyzed in recent years. These include spatially flat isotropic models [8, 9, 10, 11, 12, 13, 14], spatially curved models [15, 16, 17, 18] and anisotropic models [19, 20, 21, 22]. Detailed physics of these models has been analyzed using full quantum evolution (see for eg. [9, 10, 16]) as well as effective Hamiltonian method (see Sec V of ref. [7] and references there in) using embedding method approach [23, 24, 25]. One of the main results of these investigations is that the cosmological singularities are resolved and replaced by a bounce in the Planck regime.[1] The occurrence of bounce (or multiple bounces in case of anisotropic spacetimes) is tied to the existence of upper bound on expansion scalar (θ) of the geodesics [27, 28, 29] which arises due to the underlying quantum geometric effects. In the isotropic models, these bounces can also be understood as arising due to an upper bound on the energy density (ρ) (see for eg. [30, 31, 10, 26]). However, for anisotropic models, due to a non-vanishing contribution by the Weyl curvature, this correspondence becomes invalid and one has to carefully include the role of shear scalar σ^2, which also turns out to be universally bounded for Bianchi-I model [32, 33] and under certain conditions for Bianchi-II and Bianchi-IX models [28].

The boundedness of ρ, θ and σ^2 signals positively for the generic resolution of singularities. Indeed, in the spatially flat isotropic models in LQC, using effective dynamics, one finds that the boundedness of expansion scalar leads to a resolution of all strong singularities and the spacetime is geodesically complete [27]. Investigations on the spatially curved models strongly indicate that these results are valid also for $k = \pm 1$ models [34]. These studies find that even though the ρ and θ are bounded in isotropic LQC, the spacetime curvature can still be unbounded (see [36] for the first phenomenological example of this kind), however in all such cases the singularity turns out be weak and is thus harmless [27]. Further, geodesics can be extended beyond such events [27]. In this work, we analyze the issue of boundedness of curvature invariants and the strength of the singularities in Bianchi-I model. Our analysis is based on the work in Ref. [29]. In the following we first briefly review the dynamics of Bianchi-I model in classical theory and the effective Hamiltonian approach and show the way bounds on ρ, θ and σ^2 arise in LQC. This is followed by an analysis of curvature invariants and necessary and sufficient conditions for singularities to be strong are investigated. From these conditions, we show that all known strong singularities of Bianchi-I model (all these occur at a vanishing physical volume) in classical theory are resolved. We also briefly comment on the conditions which lead to weak singularities. Details of the consequences for geodesic evolution, and a generic lack of strong singularities for perfect fluids and more general fluids are discussed in Ref. [29].

2. Dynamical equations in Bianchi-I spacetimes

We consider a homogeneous Bianchi-I anisotropic spacetime with a spacetime manifold $\mathbb{R}^4$. Due to its non-compact topology, one needs to introduce a fiducial cell – a regulator, to define the symplectic structure. We denote this cell by $\mathcal{V}$ with a fiducial volume $V_o = l_1 l_2 l_3$ with respect to the fiducial metric $\mathring{q}_{ab}$ endowed on the spatial manifold, and l_i refer to the coordinate lengths of the each side of the fiducial cell.

Imposing the underlying symmetries of a homogeneous spacetime, the phase space variables – the matrix valued Ashtekar-Barbero connection A_a^i and triad E_i^a, can be expressed in a simple form: symmetry reduced connection c_i and and triads p_i respectively (where $i = 1, 2, 3$) [35, 19]. These satisfy $\{c_i, p_j\} = 8\pi G\gamma\delta_{ij}$, where $\gamma = 0.2375$ is the Barbero-Immirzi parameter. The

[1] In the case of a spatially flat model sourced with a massless scalar field, it has been shown that the bounce occurs for *all* the states in a dense subspace of the physical Hilbert space [26].

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triads are kinematically related to the directional scale factors a_i in the spacetime metric of the orthogonal Bianchi-I spacetime given by

$$ds^2 = -N^2\,dt^2 + a_1^2\,dx^2 + a_2^2 dy^2 + a_3^2 dz^2 \ , \tag{1}$$

as

$$|p_1| = l_2 l_3\, a_2 a_3, \quad |p_2| = l_1 l_3\, a_1 a_3, \quad |p_3| = l_2 l_3\, a_2 a_3 \tag{2}$$

where the modulus sign arises because of orientation of the triads. Without any loss of generality, in the following we restrict to the positive orientation of the triads.

2.1. *Classical dynamics*

The Hamiltonian constraint for the Bianchi-I spacetime in classical GR, when expressed in terms of c_i and p_i takes the following form for $N = 1$:

$$\mathcal{H}_{\rm cl} = -\frac{N}{8\pi G\gamma^2 V}(c_1 p_1\, c_2 p_2 + c_3 p_3\, c_1 p_1 + c_2 p_2\, c_3 p_3) + \mathcal{H}_{\rm matt} \tag{3}$$

where $\mathcal{H}_{\rm matt}$ is the matter Hamiltonian. The vanishing of the Hamiltonian constraint: $\mathcal{H}_{\rm cl} \approx 0$ lead us to the physical solutions, and also to a relation between the energy density ρ and the directional Hubble rates H_i (defined as $H_i = \dot{a}_i/a_i$, where a dot denotes the derivative with respect to proper time):

$$\rho = \frac{1}{8\pi G}(H_1 H_2 + H_2 H_3 + H_3 H_1) \ . \tag{4}$$

Equations of motions for the phase space variables are obtained by solving Hamilton's equations:

$$\dot{p}_i = \{p_i, \mathcal{H}_{\rm cl}\} = -8\pi G\gamma\, \frac{\partial \mathcal{H}}{\partial c_i} \tag{5}$$

and

$$\dot{c}_i = \{c_i, \mathcal{H}_{\rm cl}\} = 8\pi G\gamma\, \frac{\partial \mathcal{H}}{\partial p_i} \ . \tag{6}$$

Using (3) and (5) we obtain the relation between connection components and the time derivative of scale factors,

$$c_i = \gamma l_i \dot{a}_i = \gamma l_i\, H_i a_i \ . \tag{7}$$

Dynamical equations for the matter components can be obtained in a similar way. It is interesting to note that c_i and p_i satisfy the following relation if one considers matter with a vanishing anisotropic stress, i.e. when $\rho(p_1, p_2, p_3) = \rho(V)$,

$$p_i c_i - p_j c_j = \gamma V_o \alpha_{ij} \tag{8}$$

where α_{ij} is a constant antisymmetric matrix [35]. Using (7), in the above equation, one obtains: $V(H_i - H_j) = \gamma V_o \alpha_{ij}$. This in turn implies that Σ^2, a measure of the anisotropic shear defined as $\Sigma^2 = \sigma^2 a^6/6$ is a constant of motion in GR. Here $a = (a_1 a_2 a_3)^{1/3} = V^{1/3}$ is the mean scale factor and σ^2 is the shear scalar $\sigma^2 \equiv \sigma_{ab}\sigma^{ab}$, where

$$\sigma_{ij} = H_{ij} - \frac{1}{3}(H_1 + H_2 + H_3)\delta_{ij} \ , \tag{9}$$

H_{ij} is the expansion matrix and θ is the expansion scalar $\theta = (H_1 + H_2 + H_3)/3$. The expression for the shear scalar in terms of the Hubble rates is given by

$$\begin{aligned} \sigma^2 &= \sum_{i=1}^{3}(H_i - \theta)^2 = \frac{1}{3}\left((H_1 - H_2)^2 + (H_2 - H_3)^2 + (H_3 - H_1)^2\right) \\ &= \frac{1}{3a^6}\left(\alpha_{12}^2 + \alpha_{23}^2 + \alpha_{31}^2\right) \ . \end{aligned} \tag{10}$$

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In the classical evolution if the mean scale factor vanishes or the directional Hubble rates diverge (in a finite time), then energy density, expansion and shear scalar also diverge. Due to their divergence, the geodesic equations break down and a singularity is reached in a finite time. We will later see that such a divergence also leads to a strong singularity where all in-falling detectors, irrespective of their construction, are destroyed by the tidal forces.

2.2. Effective dynamics

The effective Hamiltonian of the Bianchi-I model with lapse $N = 1$ can be written as:

$$\mathcal{H}_{\rm eff} = -\ \frac{1}{8\pi G\gamma^2 V}\left(\frac{\sin(\bar\mu_1 c_1)}{\bar\mu_1}\frac{\sin(\bar\mu_2 c_2)}{\bar\mu_2} p_1 p_2 + \text{cyclic terms}\right) \ +\ \mathcal{H}_{\rm matt} \tag{11}$$

where

$$\bar\mu_1 = \lambda\sqrt{\frac{p_1}{p_2 p_3}}, \quad \bar\mu_2 = \lambda\sqrt{\frac{p_2}{p_1 p_3}}, \quad \text{and } \bar\mu_3 = \lambda\sqrt{\frac{p_3}{p_1 p_2}}\ . \tag{12}$$

Here $\lambda^2 = 4\sqrt{3}\pi\gamma l_{\rm Pl}^2$ arises from the underlying quantum geometry. It is the minimum area of the closed loop over which holonomies of connection c_i are computed [37].

Using the above Hamiltonian, the Hamilton's equations yield

$$\dot p_1 = \frac{p_1}{\gamma\lambda}\cos(\bar\mu_1 c_1)\ (\sin(\bar\mu_2 c_2) + \sin(\bar\mu_2 c_2))\,, \tag{13}$$

$$\begin{aligned} \dot c_1 \ =\ & \frac{1}{2p_1\gamma\lambda}\Bigg[c_2 p_2\cos(\bar\mu_2 c_2)(\sin(\bar\mu_1 c_1) + \sin(\bar\mu_3 c_3)) +\ c_3 p_3 \cos(\bar\mu_3 c_3)(\sin(\bar\mu_1 c_1) + \sin(\bar\mu_2 c_2)) \\ & -\ c_1 p_1\cos(\bar\mu_1 c_1)(\sin(\bar\mu_2 c_2) + \sin(\bar\mu_3 c_3)) -\ \bar\mu_1 p_2 p_3\Big[\sin(\bar\mu_2 c_2)\sin(\bar\mu_3 c_3) \\ & + \sin(\bar\mu_1 c_1)\sin(\bar\mu_2 c_2) + \sin(\bar\mu_3 c_3)\sin(\bar\mu_1 c_1)\Big]\Bigg] \ +\ 8\pi G\gamma\sqrt{\frac{p_2 p_3}{p_1}}\left(\frac{\rho}{2} + p_1\frac{\partial\rho}{\partial p_1}\right)\,, \end{aligned} \tag{14}$$

and similarly for the other components of the symmetry reduced connection and triad. Using these equations, it is straightforward to obtain the expressions for directional Hubble rates in LQC, such as

$$H_i = \frac{\dot a_i}{a_i} = \frac{1}{2\gamma\lambda}\left(\sin(\bar\mu_i c_i - \bar\mu_j c_j) + \sin(\bar\mu_i c_i - \bar\mu_k c_k) + \sin(\bar\mu_j c_j + \bar\mu_k c_k)\right) \quad \text{where} \quad i \neq j \neq k\ . \tag{15}$$

Using this equation, one finds that $c_i \neq \gamma\dot a_i$. Thus, the relation between connection components and the time derivative of the scale factors is not as simple as in GR. As in the classical theory, if one considers matter with a vanishing anisotropic stress, $(p_i c_i - p_j c_j)$ is a constant in LQC. However, unlike in GR, this does not translate to the invariance of Σ^2 in the quantum evolution, since $p_i c_i - p_j c_j \neq V(H_i - H_j)$.

The behavior of the expansion and shear scalars in LQC is qualitatively different than in GR. Using (15), in the definitions of the expansion and shear scalars we get [32, 28]

$$\theta = \frac{1}{2\gamma\lambda}\left(\sin\left(\bar\mu_1 c_1 + \bar\mu_2 c_2\right) + \sin\left(\bar\mu_2 c_2 + \bar\mu_3 c_3\right) + \sin\left(\bar\mu_1 c_1 + \bar\mu_3 c_3\right)\right) \tag{16}$$

and

$$\sigma^2 = \frac{1}{3\gamma^2\lambda^2}\Bigg[(\cos(\bar\mu_3 c_3)(\sin(\bar\mu_1 c_1)+\sin(\bar\mu_2 c_2)) - \cos(\bar\mu_2 c_2)(\sin(\bar\mu_1 c_1)+\sin(\bar\mu_3 c_3)))^2$$
$$+ (\cos(\bar\mu_3 c_3)(\sin(\bar\mu_1 c_1)+\sin(\bar\mu_2 c_2)) - \cos(\bar\mu_1 c_1)(\sin(\bar\mu_2 c_2)+\sin(\bar\mu_3 c_3)))^2$$
$$+ (\cos(\bar\mu_2 c_2)(\sin(\bar\mu_1 c_1)+\sin(\bar\mu_3 c_3)) - \cos(\bar\mu_1 c_1)(\sin(\bar\mu_2 c_2)+\sin(\bar\mu_3 c_3)))^2\Bigg] .$$

Thus, we find that in the effective dynamics, expansion and shear scalars can be expressed as a combination of trignometric functions, and are bounded. Their maximum values are given by [28],

$$\theta_{\max} = \frac{3}{2\gamma\lambda} \approx \frac{2.77786}{l_{\rm Pl}} . \quad (17)$$

and

$$\sigma^2{}_{\max} = \frac{10.125}{3\gamma^2\lambda^2} = \frac{11.56995}{l_{\rm Pl}^2} . \quad (18)$$

Similarly, it is straightforward to show that the energy density ρ is also a bounded function in LQC. To show this, one solves for $\mathcal{H}_{\rm eff} \approx 0$ by using (12), which yields

$$\rho = \frac{1}{8\pi G\gamma^2\lambda^2}\,(\sin(\bar\mu_1 c_1)\,\sin(\bar\mu_2 c_2) + \text{cyclic terms}) . \quad (19)$$

The energy density has a maxima given by

$$\rho \leq \rho_{\max}, \quad \rho_{\max} = \frac{3}{8\pi G\lambda^2} \approx 0.41\rho_{\rm Pl} . \quad (20)$$

It is interesting to note that the upper bound for the energy density in the Bianchi-I anisotropic model is equal to maximum of ρ in the loop quantization of the spatially flat isotropic model[10].

The boundedness of Hubble rates, expansion and shear scalars and energy density in the loop quantization of Bianchi-I model is a direct consequence of the underlying quantum geoemtry. In the limit, where $\lambda \to 0$, above expressions lead to their classical analogs which are unbounded. These results show that classical singularities which are accompanied by a divergence of above physical quantities are avoided in LQC. If initial data leading to such singularities in the classical theory are evolved using effective dynamics in LQC, then physical solutions agree with their classical counterparts when spacetime curvature is small compared to the Planck scale. However, as the curvature grows, departures from GR become significant. As the Planck regime is approached, directional Hubble rates, given by eq.(15), attain their maximum values. In the subsequent evolution, Hubble rates vanish and the scale factors bounce or recollapse depending on whether they were initially contracting or expanding. The evolution of individual scale factors and Hubble rates is analogous to the dynamical evolution in the loop quantization of the spatially flat isotropic model. There in the backward evolution of an expanding branch, universe enters a regime of superinflation ($\dot H > 0$) where the isotropic Hubble rate attains a maximum, before a bounce occurs at $H = 0$. In the isotropic case, energy density attains its maximum values. In contrast, at bounces of directional scale factors in the Bianchi-I spacetime, ρ, θ and σ^2 do not necessarily attain their maximum values [32, 28].

As an example, we illustrate the way singularity resolution occurs for the case of Ekpyrotic/Cyclic model in the Bianchi-I spacetime. The Ekpyrotic/Cyclic model aims to provide a non-inflation based alternative explanation of the origin of structures in our universe, however it suffers from the past singularity problem which does not allow viability of cyclic evolution. Though the past singularity problem can be resolved in the effective dynamics of the isotropic LQC, the modified dynamics in isotropic LQC does not allow cyclic evolution for Ekpyrotic potential [31]. Interestingly, these problems can be successfully overcome in the Bianchi-I model (see Ref. [32] for details). In Fig. 1, we show the results from a simulation for this model using effective dynamical equations obtained in this section, which were analyzed in Ref. [32]. From the plots, we see that the mean scale factor ($a = (a_1 a_2 a_3)^{1/3}$) bounces and the shear scalar remains bounded during the evolution, resulting in that the singularity is resolved.

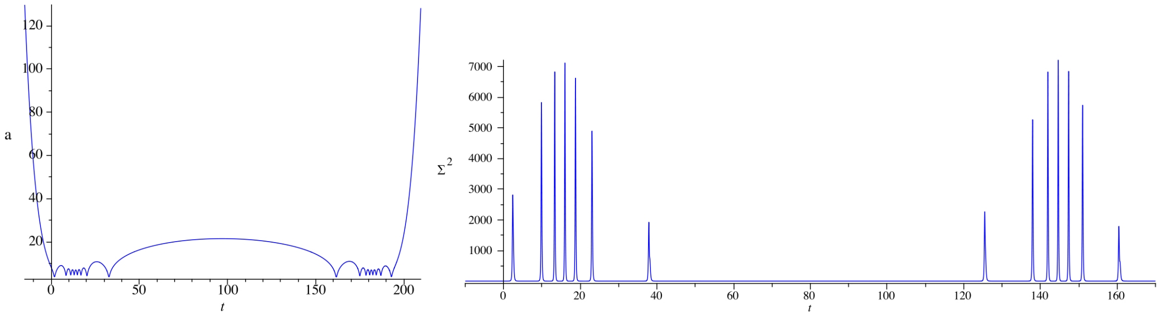

Figure 1. Plots of the non-singular evolution of the mean scale factor and Σ^2 are shown for an evolution with Ekpyrotic/Cyclic potential in Bianchi-I LQC. Multiple bounces of the mean scale factor occur and Σ^2 attains a constant value in the classical limit.

3. Behavior of curvature invariants and the absence of strong singularities

Dynamical equations obtained from the effective Hamiltonian (11) show that energy density and expansion and shear scalars are bounded by universal values in the loop quantization of Bianchi-I spacetime [32, 33, 28]. Boundedness of these quantities plays a crucial role in the resolution of various singularities in the Bianchi-I model. However, it is important to note that these do not necessarily imply that the spacetime curvature is *always* bounded in LQC. This is straightforward to understand from the expressions of the curvature invariants. For simplicity, we here discuss only the Ricci and the Kretschmann scalars, which for the metric (1), are given by

$$R = 2\left(H_1H_2 + H_2H_3 + H_3H_1 + \sum_{i=1}^{3}\frac{\ddot{a}_i}{a_i}\right) , \tag{21}$$

and

$$K = 4\left(H_1^2H_2^2 + H_1^2H_3^2 + H_2^2H_3^2 + \sum_{i=1}^{3}\frac{\ddot{a}_i^2}{a_i}\right) . \tag{22}$$

Other curvature scalars can be written in a similar way in terms of the directional Hubble rates and second order time derivatives of the scale factors. The pertinent question is whether curvature invariants are bounded functions as is the case for H_i, ρ, θ and σ^2 in the loop quantization of Bianchi-I model. To answer this, we have to analyze the second order time derivatives of scale factors. Using the Hamilton's equations for p_i and c_i, a straightforward calculation yields [29],

$$\frac{\ddot{a}_1}{a_1} = \frac{1}{2\gamma\lambda}\Bigg[\cos(\bar{\mu}_1 c_1 - \bar{\mu}_2 c_2)\frac{\mathrm{d}}{\mathrm{d}t}(\bar{\mu}_1 c_1 - \bar{\mu}_2 c_2) + \cos(\bar{\mu}_1 c_1 - \bar{\mu}_3 c_3)\frac{\mathrm{d}}{\mathrm{d}t}(\bar{\mu}_1 c_1 - \bar{\mu}_3 c_3) \tag{23}$$
$$+ \cos(\bar{\mu}_2 c_2 + \bar{\mu}_3 c_3)\frac{\mathrm{d}}{\mathrm{d}t}(\bar{\mu}_2 c_2 + \bar{\mu}_3 c_3) + \frac{1}{2\gamma\lambda}\Big(\sin(\bar{\mu}_1 c_1 - \bar{\mu}_2 c_2) + \sin(\bar{\mu}_1 c_1 - \bar{\mu}_3 c_3) +$$
$$+ \sin(\bar{\mu}_2 c_2 + \bar{\mu}_3 c_3)\Big)^2\Bigg] \; .$$

Equations for $\ddot{a}_2$ and $\ddot{a}_3$ can be obtained by cyclic permutations of the terms in the above equation. If we assume matter has a vanishing anisotropic stress, then we obtain

$$\frac{\ddot{a}_1}{a_1} = -4\pi G(\rho + P) + \frac{1}{4V}\Bigg(6(\cos(\bar{\mu}_1 c_1 - \bar{\mu}_2 c_2)\kappa_{21} + \cos(\bar{\mu}_1 c_1 - \bar{\mu}_3 c_3)\kappa_{31})H + (\kappa_{12} + \kappa_{13})\frac{\dot{p}_1}{p_1}$$
$$+\kappa_{23}\left(\frac{\dot{p}_2}{p_2} - \frac{\dot{p}_3}{p_3}\right)\Bigg) + \frac{1}{4\gamma^2\lambda^2}\left(\sin(\bar{\mu}_1 c_1 - \bar{\mu}_2 c_2) + \sin(\bar{\mu}_1 c_1 - \bar{\mu}_3 c_3) + \sin(\bar{\mu}_2 c_2 + \bar{\mu}_3 c_3)\right)^2 \; .$$

Substituting eqs. (15) and (23), and similar equations for the time derivatives of a_2 and a_3 in the expression for the Ricci scalar (21), we obtain for matter with a vanishing anisotropic stress [29]:

$$R = -24\pi G(\rho + P) + \frac{1}{V}\left(\frac{\dot{p}_1}{p_1}(\kappa_{12} + \kappa_{13}) + \frac{\dot{p}_2}{p_2}(\kappa_{23} + \kappa_{21}) + \frac{\dot{p}_3}{p_3}(\kappa_{31} + \kappa_{32})\right)$$
$$+\frac{1}{2\gamma^2\lambda^2}\Bigg[3 + \cos^2(\bar{\mu}_1 c_1)\left(\sin^2(\bar{\mu}_3 c_3) + 4\sin(\bar{\mu}_2 c_2)\sin(\bar{\mu}_3 c_3) - \cos(2\bar{\mu}_2 c_2)\right)$$
$$+ \cos^2(\bar{\mu}_2 c_2)\left(\sin^2(\bar{\mu}_1 c_1) + 4\sin(\bar{\mu}_1 c_1)\sin(\bar{\mu}_3 c_3) - \cos(2\bar{\mu}_3 c_3)\right)$$
$$+ \cos^2(\bar{\mu}_3 c_3)\left(\sin^2(\bar{\mu}_2 c_2) + 4\sin(\bar{\mu}_1 c_1)\sin(\bar{\mu}_2 c_2) - \cos(2\bar{\mu}_1 c_1)\right)$$
$$- \left(\sin^2(\bar{\mu}_1 c_1)\sin^2(\bar{\mu}_2 c_2) + \sin^2(\bar{\mu}_1 c_1)\sin^2(\bar{\mu}_3 c_3) + \sin^2(\bar{\mu}_2 c_2)\sin^2(\bar{\mu}_3 c_3)\right)\Bigg] \; . \tag{24}$$

Similarly, the expression for the Kretschmann scalar turns out to be [29]

$$K = \frac{\left(\sin(\bar{\mu}_1 c_1 - \bar{\mu}_2 c_2) + \sin(\bar{\mu}_1 c_1 - \bar{\mu}_3 c_3) + \sin(\bar{\mu}_2 c_2 + \bar{\mu}_3 c_3)\right)^2}{2\gamma^4\lambda^4}\{(\cos(\bar{\mu}_1 c_1 - \bar{\mu}_2 c_2 - 2\bar{\mu}_3 c_3)$$
$$+2 + \cos(\bar{\mu}_1 c_1 - \bar{\mu}_2 c_2 + 2\bar{\mu}_3 c_3))\sin^2\left(\tfrac{\bar{\mu}_1 c_1 + \bar{\mu}_2 c_2}{2}\right)$$
$$+ \sin^2(\bar{\mu}_1 c_1 - \bar{\mu}_2 c_2) + 2(\cos(\bar{\mu}_2 c_2) - \cos(\bar{\mu}_1 c_1))\sin(\bar{\mu}_1 c_1 - \bar{\mu}_3 c_3)\sin(\bar{\mu}_3 c_3)\}$$
$$+ \frac{1}{4\gamma^2\lambda^2 V^2}\Bigg[\frac{\dot{p}_1}{p_1}(\kappa_{12} + \kappa_{13}) + \kappa_{23}\left(\frac{\dot{p}_2}{p_2} - \frac{\dot{p}_3}{p_3}\right) + 6H(\kappa_{21}\cos(\bar{\mu}_1 c_1 - \bar{\mu}_2 c_2)$$
$$+\kappa_{31}\cos(\bar{\mu}_1 c_1 - \bar{\mu}_3 c_3)) - 16\pi G(\rho + P)V\Bigg]\Bigg[\gamma\lambda\,\{6H\gamma\lambda(\kappa_{21}\cos(\bar{\mu}_1 c_1 - \bar{\mu}_2 c_2)$$
$$+\kappa_{31}\cos(\bar{\mu}_1 c_1 - \bar{\mu}_3 c_3)) + \kappa_{23}(\cos(\bar{\mu}_2 c_2)(\sin(\bar{\mu}_1 c_1) + \sin(\bar{\mu}_3 c_3))$$
$$- \cos(\bar{\mu}_3 c_3)(\sin(\bar{\mu}_1 c_1) + \sin(\bar{\mu}_2 c_2))) + (\kappa_{12} + \kappa_{13})\cos(\bar{\mu}_1 c_1)(\sin(\bar{\mu}_2 c_2) + \sin(\bar{\mu}_3 c_3))\}$$
$$+2V\left(-8\pi G\gamma^2\lambda^2(\rho + P) + \left(\sin(\bar{\mu}_1 c_1 - \bar{\mu}_2 c_2) + \sin(\bar{\mu}_1 c_1 - \bar{\mu}_3 c_3) + \sin(\bar{\mu}_2 c_2 + \bar{\mu}_3 c_3)\right)^2\right)\Bigg]$$
$$+\text{cyclic terms} \; . \tag{25}$$

Unlike the case of H_i, ρ, θ and σ^2, these expressions are not bounded functions. There are only two situations in which the curvature invariants can diverge in the effective dynamics of the loop quantization of the Bianchi-I model. The first case arises if the physical volume vanishes in the physical evolution, and the second arises when the pressure of the matter content becomes infinite. Both of these scenarios must arise at a finite value of ρ, θ and σ^2, to be physically realized in the evolution. It is to be noted that all known types of singularities in the Bianchi-I model in GR, fail to satisfy the requirements for the first scenario to occur. In all known cases of singular evolution, curvature invariants diverge at a vanishing volume along with a divergence of ρ, θ and σ^2. All such events will have curvature invariants bounded in LQC, and the singularity will be resolved by the effective dynamics [29]. On the other hand, the second case has an analog in the loop quantization of the isotropic model, where the curvature scalars can diverge when pressure $P \to \pm\infty$ with ρ remaining finite. This implies a divergence in the equation of state, which can indeed occur with an appropriate choice of matter. Unlike the second scenario, there is no analog of the first possibility in the isotropic case [27].

The key question is whether the cases where curvature invariants can diverge in LQC, imply a singular evolution. To answer this question, one has to analyze the behavior of geodesics and examine the strength of singularities. Conditions to determine the strength of the singularities have been formulated in the seminal works of Tipler [5] and Królak [6], which involve integrals of the curvature components over the geodesics. For the anisotropic spacetimes, a singularity occurring at the affine parameter $\tau = \tau_e$ is strong according to Królak, if

$$\int_0^\tau \mathrm{d}\tau' R_{ab}\, u^a u^b \tag{26}$$

or

$$\int_0^\tau \mathrm{d}\tau' \left(\int_0^{\tau''} \mathrm{d}\tau'' |C_{abcd}\, u^b u^d| \right)^2 \tag{27}$$

diverges as $\tau \to \tau_e$. Here $u^a = \mathrm{d}x^a/\mathrm{d}\tau$. If the above integrals are finite, then the singularity is weak. To determine whether a singularity is strong according to Tipler's criteria one considers an additional integral in the above equations. Due to this difference in conditions, it is possible that a singularity which is strong à la Królak may turn out to be weak à la Tipler. In terms of these conditions, a detector is inevitably destroyed by a strong singularity but survives a weak singularity. Indeed it is quite possible, that an event where curvature invariants diverge may not lead to a break down of geodesics and the singularity may be weak. An example of such an event in GR, occurs for the the case of a sudden singularity which occurs due to a divergence in pressure at a finite volume and energy density [38]. Even in LQC, such an event is curvature divergent but the singularity turns out to be weak, in spatially flat [27] as well as spatially curved models [34].

For simplicity, we analyze here the details of the condition (26), and refer the reader to Ref. [29] for analysis of the integral over Weyl curvature components. The integrand in this case turns out to be,

$$\begin{aligned} R_{ab}\, u^a u^b \;=\; & \frac{k_x^2}{a_1^2} \left(H_1 H_2 + H_1 H_3 - \frac{\ddot{a}_2}{a_2} - \frac{\ddot{a}_3}{a_3} \right) + \frac{k_y^2}{a_2^2} \left(H_1 H_2 + H_2 H_3 - \frac{\ddot{a}_1}{a_1} - \frac{\ddot{a}_3}{a_3} \right) \\ & + \frac{k_z^2}{a_3^2} \left(H_3 H_1 + H_2 H_3 - \frac{\ddot{a}_1}{a_1} - \frac{\ddot{a}_2}{a_2} \right) \end{aligned} \tag{28}$$

where k_x, k_y and k_z are constants. Substituting above expression in Królak's condition for Ricci tensor, we find that due to an integral over the affine parameter, any divergence in the terms

$\ddot{a}_i/a_i$ does not translate to a divergence of the integral (26). However, if in a physical evolution, any of the scale factors vanishes at a finite value of the affine parameter, then the integral (26) can diverge. Whether or not the singularity in this case would be strong, will depend on the detailed behavior of the dynamical solutions which allow this singularity.

We are now ready to answer the questions we posed: Do curvature invariant diverging events in Bianchi-I LQC imply a strong singularity? The answer turns out to be the following (see Ref. [29] for details):

(i) If the divergence of curvature invariants is caused by the divergence in pressure of the matter component at non-vanishing scale factors and finite value of ρ, θ and σ^2, then the singularity is weak. Analysis of the geodesic equations show that they remain well defined and geodesics can be extended beyond such curvature invariant diverging events. Thus, a pressure singularity in the Bianchi-I model in LQC has very similar characteristics as in the isotropic model. In both cases, such curvature invariant diverging events turns out to be a weak singularity and is harmless.

(ii) If effective dynamics in LQC allows a vanishing of the scale factor, and hence physical volume in a finite time, at a *finite* value of ρ, θ and σ^2, then the singularity can be strong. Geodesic evolution in this case breaks down. However, it is not known, whether such a curvature divergent event is allowed by the effective dynamics. To our knowledge, an event with divergent curvature invariants and vanishing volume with a finite ρ, θ and σ^2 is not known even in GR. It will be an interesting exercise to discover such a physical solution and understand whether the allowed singularity is strong or weak.

4. Summary

In the present analysis, we investigated the effective dynamics of LQC for the Bianchi-I spacetime for matter with a vanishing anisotropic stress. A quantization of this spacetime with a massless scalar field has been recently performed in Ref. [19] (see also Ref. [20]), which demonstrated that the quantum Hamiltonian constraint is non-singular. Using effective dynamics, physics of this spacetime has been analyzed for different forms of matter [35, 32, 33, 28, 39, 40] which show the occurrence of bounces of the directional scale factors which lead to avoidance of classical singularities. Here we summarized some of the main points of the results obtained in Ref. [29], where we show that boundedness of ρ, θ and σ^2 does not imply a bound on the curvature scalars. The latter can diverge in the effective dynamics of loop quantization of Bianchi-I model. Such curvature invariant diverging events exist even for isotropic LQC. These were earlier proved by the author to be harmless as they correspond to weak singularities and geodesics can be continued beyond them [27]. The situation in Bianchi-I model turns out to be similar. We find that divergence in spacetime curvature, when occurring due to a divergence in pressure of the matter content at non-zero values of the scale factors lead neither to strong singularity nor the breakdown of geodesics. Further, all known types of strong singularities of the classical theory in the Bianchi-I spacetime are shown to be avoided.

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5. References

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