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3 New Way of Second Quantized Theory of Fermions With Either Clifford or Grassmann Coordinates and *Spin-Charge-Family* Theory ⋆

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Abstract. Fermions with the internal degrees of freedom described in Clifford space carry in any dimension a half integer spin. There are two kinds of spins in Clifford space. The spin-charge-family theory ([1–7,9,8], and the references therein), assuming even $d = (13+1)$, uses one kind of spins to describe in $d = (3+1)$ spins and charges of quarks and leptons and antiquarks and antileptons, while the other kind is used to describe families.
In this work the new way of second quantization, suggested by the spin-charge-family theory, is presented. It is shown that the creation and annihilation operators of 1-fermion states, written as products of nilpotents and projectors of an odd Clifford character, fulfill the anticommutation relations as required in the second quantization procedure for fermions, what means that 1-fermion states are in Clifford space already second quantized, and that the creation operators for n-fermion second quantized vectors are products of one fermion creation operators, operating on the empty vacuum state. There is no need in this theory for the negative energy states fulfilled with fermions.
It is demonstrated that also in Grassmann space there exist the creation and annihilation operators of an odd Grassmann character, generating "fermions", which fulfill as well the anticommutation relations for fermions, representing correspondingly the second quantized 1-"fermion" states. However, while the internal spins determined by the generators of the Lorentz group of the Clifford objects of both kinds are half integer, the internal spins determined by the Grassmann objects are integer. Grassmann space offers no families.
We discuss the new second quantization procedure of the fields in both spaces. For the Grassmann case we present the action, the basic states, the solution of the "Weyl" equation for free massless "fermions" and the discrete symmetry operators. A short overview of the achievements of the spin-charge-family theory is done, and open problems of this theory still waiting to be solved are presented. We compare the Grassmann and the Clifford case in order to better understand to how many open questions in physics of elementary fermion and boson fields and in cosmology the spin-charge-family theory is able to answer.

Povzetek. Cliffordova algebra ponudi v vseh dimenzijah dva neodvisna vektorska prostora za opis fermionov. Teorija spinov-nabojev-družin ([1–7,9,8], in reference v teh člankih), ki predpostavi da ima prostor-čas $d \geq (13+1)$ dimenzij, uporabi eno vrsto spina za opis spina in nabojev karkov in leptonov in antikvarkov in antileptonov, drugo vrsto spina pa

⋆ Talk presented by N.S. Mankoč Borštnik

za opis družin.
Avtorica teorije spinov-nabojev-družin je dokazala, da vektorji, ki so lastni vektorji Cartanove podalgebre Lorentzove algebre in so produkt lihega stevila Cliffordovih operatorjev, izpolnjujejo vse lastnosti fermionov v drugi kvantizaciji. To pomeni, da opis fermionov v Cliffordovi algebri razloži Diracove postulate za drugo kvantizacijo fermionov. Kreacijski in anihilacijski operatorji, ki določajo v tej drugi kvantizaciji 1-fermionska stanja, zadostijo antikomutacijskim relacijam za drugo kvantizacijo fermionov, če jih zapišemo kot produkt niloptentov in projektorjev lihega števila Cliffordovih operatorjev. Kreacijski operatorji za n fermionska stanja so v tej drugi kvantizaciji produkti enofermionskih kreacijskih operatorjev, ki delujejo na praznem vakuumskem stanju. V tej teoriji ni potrebe po negativnih energijskih stanjih zapolnjenih s fermioni.
Avtorja postavita zahtevo, ki ohrani le enega od obeh vektorskih prostorov, druga vrsta operatorjev pa poveže neodvisne nerazcepne upodobitve Lorentzove algebre v tem prostoru in jim " podeli" kvantno število "družin ". Tako omogoči Cliffordova algebra opis ne le spinov in nabojev kvarkov in leptonov in antikavarkov in antileptonov, ampak tudi njihovih družin.

Članek pokaže, da tudi Grassmannova algebra ponudi kreacijske in anihilacijske operatorje, ki zadoščajo antikomutacijskim relacijam za 1 fermionska stanja. Vendar so spini teh vektorjev celoštevilski. Grassmannov prostor ne ponudi družin.
Akcija in enečbe gibanja, ko so v Cliffordovi algebri poznani, za Grassmannov algebro pa članek predlaga akcijo in diskretne operatorje. Za obe algebri ponudi rešitve ustrezne "Weylove" enačbe za proste "fermione" brez mase in jih komentira. Avtorja ponudita tudi kratek pregled dosežkov teorije spinov-nabojev-družin in njenih odprtih problemov. Primerjava Grassmannovega in Cliffordovega primera osvetli mnoga odprta vprašanja fizike osnovnih fermionov in bozonov ter kozmologije.

Keywords: Second quantization of fermion fields in Clifford and in Grassmann space, Spinor representations in Clifford and in Grassmannspace, Kaluza-Klein-like theories, Discrete symmetries, Higher dimensional spaces, Beyond the standard model

3.1 Introduction

More than 50 years ago the *standard model* offered an elegant new step in understanding elementary fermion and boson fields by postulating:
i. Massless family members of coloured quarks and colouress leptons, the left handed members as the weak charged doublets and the weak chargeless right hand members, the left handed quarks distinguishing in the hyper charge from the left handed leptons, each right handed member having a different hyper charge. All fermion charges are in the fundamental representation of the corresponding groups. Antifermions carry the corresponding anticharges and opposite handedness. The existence of massless families to each family member is as well postulated. There is no right handed neutrino, since it would carry none of the so far observed charges, and correspondingly there is also no left handed antineutrino.
ii. The existence of the massless vector gauge fields to the observed charges of quarks and leptons, carrying charges in the corresponding adjoint representations.

iii. The existence of a massive scalar Higgs, gaining at some step of the expanding universe the nonzero vacuum expectation value, causing masses of fermions and heavy bosons and the Yukawa couplings. The Higgs carry a half integer weak and hyper charge.
iv. Fermion and boson fields can be (second) quantized.

The *standard model* assumptions have in the literature several explanations, mostly with many new not explained assumptions. The most successful seem to be the grand unifying theories [12–28], if postulating in addition the family group and the corresponding gauge scalar fields.

The *spin-charge-family* theory, the project of N.S.M.B. [1–7,9,8,10], is offering the explanation for all the assumptions of the *standard model*, unifying not only charges, but also charges and spins and families, explaining the appearance of families, of the vector gauge fields, of the scalar field and the Yukawa couplings, offering the explanation for the matter-antimatter asymmetry, making several predictions. This theory also offers the explanation for the appearance of creation and annihilation operators, fulfilling the anticommutation relations for fermions,which in the Dirac theory [67] is just assumed.

The *spin-charge-family* theory is a kind of the Kaluza-Klein like theories [29–36,8] due to the assumption that in $d \geq 5$ (in the *spin-charge-family* theory $d \geq (13+1)$) fermions interact with the gravity only. Correspondingly this theory shares with the Kaluza-Klein like theories their weak points, at least: **a.** Not yet solved the quantization problem of the gravitational field. **b.** Breaking spontaneously the starting symmetry, which would at low energies manifest the observed almost massless fermions [30]. Concerning this second point we proved on the toy model of $d = (5+1)$ that the break of symmetry can lead to (almost) massless fermions [68–70]. It remains to study how does appear the spontaneous breaking of the starting symmetry in $d = (13+1)$, first with the appearance of the condensate of two right handed neutrinos, Table 3.3, Ref. [4], and then when scalar fields with space index $(7, 8)$ obtain nonzero vacuum expectation values. (This second point is common to all the unifying theories.)

Since in $d = (3+1)$-dimensional space — at low energies — the gauge gravitational fields manifest as the observed vector gauge fields [5], which can be quantized in the usual way, quantization procedure of gravity can wait to be made. The author is in mean time trying to find out (together with the collaborators) how far can the *spin-charge-family* theory — starting in $d = (13+1)$-dimensional space with a simple and "elegant" action, Eq. (3.1) — reproduce in $d = (3+1)$ the observed properties of quarks and leptons [3–7,9,8,10], the observed gauge fields, the assumed scalar field, the appearance of the dark matter and of the matter-antimatter asymmetry, as well as the other open questions, connecting elementary fermion and boson fields and cosmology. The work done so far seems promising.

Let us in what follows and in Subsect. 3.1.1 overview shortly the starting assumptions and so far achievements of the *spin-charge-family* theory, and discuss as well open problems.

The recognition that there are in Grassmann space two kinds of the Clifford algebra objects [2] (γ^a and $\tilde{\gamma}^a$) enables that the *spin-charge-family* theory is explaining the origin of families [47–49,1,2], Table 3.1.

The assumption made in the *spin-charge-family* theory that the dimension of space is $\geq (13+1)$ enables the explanation for by the *standard model* assumed spins and charges of quarks and leptons [71,72], explaining as well the miraculous cancellation of triangle anomalies [8,9,4] in the *standard model*, however, without relating handedness and charges "by hand" as needed in $SO(10)$ [37–39].

Since there are in $SO(13+1)$ additional quantum numbers to those assumed by the *standard model*, the theory predicts that right handed neutrinos and left handed antineutrinos, carrying nonzero additional quantum numbers — τ^{23} and τ^4 instead of Y in the *standard model* ($Y = (\tau^{23} + \tau^4)$ in the *spin-charge-family* theory as presented in Table 3.6 and in Eqs. (3.111, 3.112, 3.113, 3.114)) — are regular members of families of quarks and leptons [71,72,3,9]. This prediction is common also to $SO(10)$ [37–39].

In the *spin-charge-family* theory spins and charges are described by the superposition of S^{ab} $(= \frac{i}{4}(\gamma^a\gamma^b - \gamma^b\gamma^a)$, Eq. (3.2)), with γ^a belonging to the first kind of the Clifford algebra objects and with $S^{mn}, (m,n) = (0,1,2,3)$, describing spins and handedness of quarks and leptons (Eq. (3.111)), and $S^{st}, (s,t) = (5,6,\cdots,14)$, describing their charges, Table 3.6, Eqs. (3.112, 3.113) and Refs. [2,47,49,72].

Family quantum numbers are determined by the second kind of the Clifford algebra objects, by the superposition of $\tilde{S}^{ab}$ $(= \frac{i}{4}\tilde{\gamma}^a\tilde{\gamma}^b - \tilde{\gamma}^b\tilde{\gamma}^a))$, Eq. (3.2), Table 3.1 [2,48].

The vector gauge fields, assumed in the *standard model* as the gauge fields of the corresponding fermion charges, are in the *spin-charge-family* theory explainable as the superposition of the gauge fields of the generators of the Lorentz transformations S^{st} ($S^{st}\,\omega_{stm}, (s,t) = (5,6,\cdots,14)$, Eqs. (3.1, 3.9, 3.111)), with the vector index $m = (0,1,2,3)$, Eq. (3.10), Ref. [5].

In the *standard model* the scalar fields appear as the Higgs scalar and the Yukawa couplings by the assumption. In the *spin-charge-family* theory both kinds of the gauge fields, $\sum_{s',t'} c^{s't'}\,\omega_{s't's}$, which are the gauge fields of S^{st} with $(s',t') = (5,6,7,8)$, and $\sum_{a,b} \tilde{c}^{ab}\,\tilde{\omega}_{abs}$, which are the gauge fields of $\tilde{S}^{ab}$, with $(a,b) = (0,1,\cdots,8)$, both with the scalar index $s = (7,8)$, manifesting properties of the Higss scalar (by carrying weak and hyper charges in the "fundamental representation"), define masses of quarks and leptons and of heavy bosons, Eq. (3.10), Refs. [72,9,3].
These scalar fields determine in the *spin-charge-family* theory masses of the two groups of four families [51,53–56,3,9]. The lower group predicts the existence of the fourth family of quarks and leptons, coupled to the observed three families [51,53,56,54,70]. From the symmetry of the mass matrices predicted the 4×4 mixing matrix of quarks [56] appear to be in better agreement with the experiments than if only three families are assumed [40].

The lowest family of the upper four families offers the explanation for the existence of the dark matter [54,61].

There are additional scalar fields in the *spin-charge-family* theory [4], having the scalar space index $t \in (9,10,\ldots,14)$. They carry colour charges in the "fundamental" representations, cause transitions of antileptons and antiquarks into quarks and back, enabling the decay of baryons. These scalar fields are offering in the presence of the right handed neutrino condensate, Table 3.3, Ref. [4], which breaks the $\mathcal{CP}$ symmetry, the answer to the question about the matter-antimatter asymmetry in the universe [4].

Authors of this paper proved on the toy model of $d = (5+1)$ that breaking the symmetry in Kaluza-Klein theories can lead to massless fermions [68–70]. The authors determine as well the discrete symmetries operators in observable dimensions $d = (3+1)$ for any d, Eqs. (3.94), Ref. [65].

The breaking of the starting symmetry $SO(13+1)$ is in the *spin-charge-family* theory triggered by the appearance of the condensate (Table 3.3) of the right handed neutrinos [4] and, like in the *standard model*, by the nonzero vacuum expectation values of the scalar fields with the space index $s = (7, 8)$.

In this paper it is demonstrated that the odd products of nilpotents and projectors, which are the "egienfunctions" of the Cartan subalgebra of the Lorentz algebra in Clifford space, and which solve the Weyl equations for free massless fermions, fulfill together with the corresponding Hermitian conjugated annihilation operators the anti-commutation relations as needed in the second quantized fermion fields [50]. No assumption of the Dirac kind about the creation and annihilation operators is needed.

The *spin-charge-family* theory has many common points with other unifying theories ([12–17,29–36] and other references), and because of that and because of the fact that by starting with the very simple action, Eq. (3.1), the theory is able to offer explanations for so many observed phenomena, built into assumptions of the *standard model*(s) of the elementary boson and fermion fields and also of cosmology, and also in other unifying theories, it might be that it is the right next step beyond the standard models.

The achievements of the *spin-charge-family* theory are discussed in more details in Subsect. 3.1.1. There also problems waiting to be solved are presented.

Let us present a very simple starting action of the *spin-charge-family* theory of N.S.M.B., in which massless fermions in $d = (13+1)$-dimensional space interact with massless bosons, that is only with gravity — the vielbeins $f^{\alpha}{}_{a}$ (the gauge fields of moments p_a) and the two kinds of the spin connections ($\omega_{ab\alpha}$ and $\tilde{\omega}_{ab\alpha}$, the gauge fields of the two kinds of the Clifford algebra objects γ^a and $\tilde{\gamma}^a$, respectively).

$$\begin{aligned}\mathcal{A} = & \int d^d x \; E \; \frac{1}{2} \, (\bar{\psi} \, \gamma^a p_{0a} \psi) + \text{h.c.} + \\ & \int d^d x \; E \; (\alpha \, R + \tilde{\alpha} \, \tilde{R}) \, , \end{aligned} \tag{3.1}$$

with $p_{0a} = f^{\alpha}{}_{a} p_{0\alpha} + \frac{1}{2E} \{p_{\alpha}, E f^{\alpha}{}_{a}\}_{-}$, $p_{0\alpha} = p_{\alpha} - \frac{1}{2} S^{ab} \omega_{ab\alpha} - \frac{1}{2} \tilde{S}^{ab} \tilde{\omega}_{ab\alpha}$ and $R = \frac{1}{2} \{f^{\alpha[a} f^{\beta b]} \; (\omega_{ab\alpha,\beta} - \omega_{ca\alpha} \, \omega^{c}{}_{b\beta})\} + \text{h.c.}$, $\tilde{R} = \frac{1}{2} \{f^{\alpha[a} f^{\beta b]} \; (\tilde{\omega}_{ab\alpha,\beta} - \tilde{\omega}_{ca\alpha} \, \tilde{\omega}^{c}{}_{b\beta})\} + \text{h.c.}$. Here [1] $f^{\alpha[a} f^{\beta b]} = f^{\alpha a} f^{\beta b} - f^{\alpha b} f^{\beta a}$.

[1] $f^{\alpha}{}_{a}$ are inverted vielbeins to $e^{a}{}_{\alpha}$ with the properties $e^{a}{}_{\alpha} f^{\alpha}{}_{b} = \delta^{a}{}_{b}$, $e^{a}{}_{\alpha} f^{\beta}{}_{a} = \delta^{\beta}_{\alpha}$, $E = \det(e^{a}{}_{\alpha})$. Latin indices $a, b, .., m, n, .., s, t, ..$ denote a tangent space (a flat index), while

The two kinds of the Clifford algebra objects, γ^a and $\tilde{\gamma}^a$, Eq. (3.2), anticommute and determine the infinitesimal generators of the Lorentz transformations in the internal space of fermions — S^{ab} for $SO(13,1)$, arranging states into representations (Table 3.6), and $\tilde{S}^{ab}$ for $\widetilde{SO}(13,1)$, arranging states into families (Table 3.1). Eq. (3.69) relates these two internal dgrees of freedom, keeping the relations of Eq. (3.2) unchanged.

$$\begin{aligned}
&\{\gamma^a, \gamma^b\}_+ = 2\eta^{ab} = \{\tilde{\gamma}^a, \tilde{\gamma}^b\}_+ \,, \\
&\{\gamma^a, \tilde{\gamma}^b\}_+ = 0 \,, \\
&S^{ab} = \frac{i}{4}(\gamma^a \gamma^b - \gamma^b \gamma^a) \,, \\
&\tilde{S}^{ab} = \frac{i}{4}(\tilde{\gamma}^a \tilde{\gamma}^b - \tilde{\gamma}^b \tilde{\gamma}^a) \,.
\end{aligned} \tag{3.2}$$

The generators S^{ab} are used in the *spin-charge-family* theory to determine spins and charges of spinors of any family, Table 3.6, another kind, $\tilde{S}^{ab}$, determines the family quantum numbers, Table 3.1. These two degrees of freedom are connected by the requirement, presented in Eq. (3.69).

The scalar curvatures R and $\tilde{R}$ determine dynamics of the gauge fields — the spin connections and the vielbeins — manifesting in $d = (3+1)$ as all the known vector gauge fields, as well as the scalar fields [5], which offer the explanation for the appearance of the Higgs and the Yukawa couplings, of the ordinary matter-antimatter asymmetry [4] and the dark matter [54], provided that the symmetry breaks from the starting $SO(13,1)$ to $SO(3,1) \times SU(3) \times U(1)$.

In this paper we start to study the possibility that fermions are described in Grassmann space, in order to better understand how far can the simple starting action, Eq. (3.1), of the *spin-charge-family* theory agree with the at low energies observed properties of fermions and bosons.

We demonstrate in this paper that besides Clifford space also Grassmann space offers the description of the internal degrees of freedom of fermions in the second quantized procedure. In both cases there exist the creation and annihilation operators, which fulfill the anticommutation relations required for fermions, Eqs. (3.54, 3.81). But while the internal spins determined by the generators of the Lorentz group of the Clifford objects S^{ab} and $\tilde{S}^{ab}$ — we repeat here that in the *spin-charge-family* theory S^{ab} determine the spin degrees of freedom and $\tilde{S}^{ab}$ the family degrees of freedom — are half integer, the internal spin determined by $\mathbf{S}^{ab}$ (expressible with $S^{ab} + \tilde{S}^{ab}$) is integer.

Correspondingly Clifford space offers according to the *spin-charge family* theory the description of spins, charges and families, all in the fundamental representations of the subgroups of the Lorentz group $SO(d-1,1)$, while Grassmann space offers spins and charges in the adjoint representations of the subgroups

Greek indices $\alpha, \beta, .., \mu, \nu, ..\sigma, \tau, ..$ denote an Einstein index (a curved index). Letters from the beginning of both the alphabets indicate a general index ($a, b, c, ..$ and $\alpha, \beta, \gamma, ..$), from the middle of both the alphabets the observed dimensions $0, 1, 2, 3$ ($m, n, ..$ and $\mu, \nu, ..$), indexes from the bottom of the alphabets indicate the compactified dimensions ($s, t, ..$ and $\sigma, \tau, ..$). We assume the signature $\eta^{ab} = diag\{1, -1, -1, \cdots, -1\}$.

of the Lorentz group $SO(d-1,1)$ and no family degrees of freedom. Fermions with integer spins would lead to completely different nucleons, nuclei, atoms, molecules, matter than the so far observed ones.

Let us make a short introduction into the Grassmann space as well.

In Grassmann space the infinitesimal generators of the Lorentz transformations $\mathbf{S}^{ab}$ are expressible with anticommuting coordinates θ^a and their conjugate momenta $p^{\theta a} = i\frac{\partial}{\partial \theta_a}$ [2],

$$\begin{aligned} \{\theta^a, \theta^b\}_+ &= 0\,, \quad \{p^{\theta a}, p^{\theta b}\}_+ = 0\,, \quad \{p^{\theta a}, \theta^b\}_+ = i\eta^{ab}\,, \\ \mathbf{S}^{ab} &= \theta^a p^{\theta b} - \theta^b p^{\theta a}\,. \end{aligned} \tag{3.3}$$

Taking into account that γ^a and $\tilde{\gamma}^a$, expressible in terms of θ^a and their conjugate momenta $p^{\theta a}$, anticommute [2],

$$\gamma^a = (\theta^a - i p^{\theta a})\,, \quad \tilde{\gamma}^a = i\,(\theta^a + i p^{\theta a})\,, \tag{3.4}$$

one recognizes

$$\mathbf{S}^{ab} = S^{ab} + \tilde{S}^{ab}\,, \tag{3.5}$$

from where one concludes, if taking into account Eq. (3.1), that in the Grassmann case the covariant momenta $p_{0\alpha}$ are

$$p_{0\alpha} = p_\alpha - \frac{1}{2}\mathbf{S}^{ab}\Omega_{ab\alpha}\,, \tag{3.6}$$

with $\Omega_{ab\alpha}$ as the only kind of the connection fields (instead of the two kinds in the Clifford case — $\omega_{ab\alpha}$, which is the gauge fields of S^{ab} and $\tilde{\omega}_{ab\alpha}$, which is the gauge fields of $\tilde{S}^{ab}$).

Let us point out that Eq. (3.69) relates the two anticommuting degrees of freedom, $\{\gamma^a, \tilde{\gamma}^b\}_+ = 0$, making a choice of γ^a to detremine the internal degrees of freedom in Clifford space, while keeping all the relation of Eq. (3.2) unchanged.

It follows for $\mathbf{S}^{ab}$

$$\begin{aligned} \{\mathbf{S}^{ab}, \mathbf{S}^{cd}\}_- &= i\{\mathbf{S}^{ad}\eta^{bc} + \mathbf{S}^{bc}\eta^{ad} - \mathbf{S}^{ac}\eta^{bd} - \mathbf{S}^{bd}\eta^{ac}\}\,, \\ \mathbf{S}^{ab\dagger} &= \eta^{aa}\eta^{bb}\mathbf{S}^{ab}\,. \end{aligned} \tag{3.7}$$

The same relations are true also if $\mathbf{S}^{ab}$ is replaced with either S^{ab} or $\tilde{S}^{ab}$. These infinitesimal generators of the Lorentz group — the two kinds of the Clifford operators and the Grassmann operators — operate as follows

$$\begin{aligned} \{S^{ab}, \gamma^e\}_- &= -i\,(\eta^{ae}\,\gamma^b - \eta^{be}\,\gamma^a)\,, \\ \{\tilde{S}^{ab}, \tilde{\gamma}^e\}_- &= -i\,(\eta^{ae}\,\tilde{\gamma}^b - \eta^{be}\,\tilde{\gamma}^a)\,, \\ \{S^{ab}, \tilde{S}^{cd}\}_- &= 0\,, \\ \{\mathbf{S}^{ab}, \theta^e\}_- &= -i\,(\eta^{ae}\,\theta^b - \eta^{be}\,\theta^a)\,, \\ \{\mathbf{S}^{ab}, p^{\theta e}\}_- &= -i\,(\eta^{ae}\,p^{\theta b} - \eta^{be}\,p^{\theta a})\,, \\ \{\mathbf{M}^{ab}, A^{d\ldots e\ldots g}\}_- &= -i\,(\eta^{ae}\,A^{d\ldots b\ldots g} - \eta^{be}\,A^{d\ldots a\ldots g})\,, \end{aligned} \tag{3.8}$$

where $\mathbf{M}^{ab}$ are defined in the Clifford case by the sum of L^{ab} plus either S^{ab} (if γ^a's are chosen to describe the basis, otherwise $\tilde{S}^{ab}$ replace S^{ab}), while in the Grassmann case $\mathbf{M}^{ab}$ is $L^{ab} + \mathbf{S}^{ab}$ (which is, Eq. (3.5), $\mathbf{M}^{ab} = L^{ab} + S^{ab} + \tilde{S}^{ab}$).

In Sect. 3.2 the actions and norms for free massless fermions, with the internal degrees of freedom described in Clifford and in Grassmann space in d-dimensional spaces are presented. The discrete symmetry operators in d-dimensional space — Clifford and Grassmann — and their manifestation in $d = (3+1)$-dimensional space are presented in Subsect. 3.3.3 of Sect. 3.3. While the action and the discrete symmetry operators in Clifford space are known from before [9,65], the action in Grassmann space as well as the discrete symmetry operators are here assumed by N.S.M.B..

The new way of second quantization of fermion fields in both spaces is discussed in Sect. 3.3. We treat in both spaces only massless free particles. Sect. 3.4 presents what we learn from this work.

This work is a part of the project of both authors, which includes the *fermionization* procedure of boson fields (or the *bosonization* procedure of fermion fields), discussed in Refs. [42,43,45] for any dimension d (by the authors of this contribution, while one of them, H.B.F.N. [44], has succeeded with another author to do the *fermionization* for $d = (1+1)$), and which would hopefully also help to understand a little better the content and dynamics of our universe.

3.1.1 Comments on the achievements of the *spin-charge-family* theory so far and the open questions to be solved

Let us illustrate the achievements of the *spin-charge-family* theory, presented in the introduction, by adding some comments.

I. In the action, Eq. (3.1), fermions carry in $d = (13+1)$ two kinds of spins — no charges and interact with gravity only — with the vielbeins $f^{\alpha}{}_{a}$ and the two kinds of the spin connection fields, the gauge fields of S^{ab} — $\omega_{ab\alpha}$ — and the gauge fields of $\tilde{S}^{ab}$ — $\tilde{\omega}_{ab\alpha}$.

One can formally rewrite the fermion part of the action so that it manifests in $d = (3+1)$ the free massless fermion part (first line in Eq. (3.9)), the interaction of fermions with the vector gauge fields (the second line in Eq. (3.9)), the interaction of fermions with the scalar fields (the third line in Eq. (3.9)), and the rest.

$$\begin{aligned}\mathcal{L}_f = &\sum_{m} \bar{\psi}\gamma^m p_m \psi \\ &- \sum_{A,i} \bar{\psi}\,\gamma^m \tau^{Ai} A^{Ai}_m \,\psi + \\ &+ \sum_{s=7,8} \bar{\psi}\gamma^s p_{0s}\,\psi \\ &+ \sum_{t=5,6,9,\dots,14} \bar{\psi}\gamma^t p_{0t}\,\psi\,,\end{aligned} \tag{3.9}$$

with $\tau^{Ai} = \sum_{st} c_{st}{}^{Ai} S^{st}$, $(s,t) = (5,6,\cdots,13,14)$, which are generators of the subgroups of $SO(13,1)$, determining charges of fermions, Eq. (3.112, 3.113, 3.114), with A^{Ai}_{m}, which are the corresponding superposition of ω_{stm} ([4,9] and the references therein), $p_{0s} = p_s - \frac{1}{2}S^{s's"}\omega_{s's"s} - \frac{1}{2}\tilde{S}^{ab}\tilde{\omega}_{abs}$ and $p_{0t} = p_t - \frac{1}{2}S^{t't"}\omega_{t't"t} - \frac{1}{2}\tilde{S}^{ab}\tilde{\omega}_{abt}$, while $m \in (0,1,2,3)$, $s \in (7,8)$, $(s', s") \in (5,6,7,8)$, (a,b) (appearing in $\tilde{S}^{ab}$) run within $(0,1,2,3)$ and $(5,6,7,8)$, $t \in (5,6,9,\ldots,13,14)$, $(t', t") \in (5,6,7,8)$ and $\in (9,10,\ldots,14)$.

I.i The spinor function ψ represents all the family members, $2^{\frac{d}{2}-1} = 64$ for $d = 13+1$, of all the $2^{\frac{7+1}{2}-1} = 8$ families, including fermions and antifermions. Tables 3.6 and 3.1 represent the creation operators for the states of one family and the creation operators for the eight families, respectively. The rest of families are assumed to have very large masses as discussed and proved for a toy model in Ref. [68–70,73]. The creation operators operate on a vacuum state, Eq. (3.79).

I. A. The Clifford object γ^a are in the *spin-charge-family* theory used to determine from the point of view of $d = (3+1)$ spins and all the charges of fermions.

I. A.i. $d = (13+1)$-dimensional space offers $2^{\frac{d}{2}-1} = 64$ members of $SO(13,1)$. In Table 3.6 the properties of quarks and leptons and antiquarks and antileptons, forming 64 members, are presented from the point of view of subgroups of $SO(13,1)$ breaking first into $SO(7,1) \times SU(3) \times U(1)$, keeping connection between handedness and the two $SU(2)_{I,II}$ charges, and further to — $SU(2)_R \times SU(2)_L \times SU(2)_I \times SU(2)_{II} \times SU(3) \times U(1)$ — representing in $d = (3+1)$ the spin and handedness, the weak charge τ^{13} of $SU(2)_I$, the second τ^{23} of $SU(2)_{II}$, the colour charge τ^{33} and τ^{38} of $SU(3)$ and τ^4 of $U(1)$ *for quarks and leptons and for antiquarks and antileptons*.
Cartan subalgebra has $\frac{d}{2} = 7$ members, the *standard model* assumes one commuting operator less.

I. A.ii. Due to the additional commuting operator (the member of the Cartan subalgebra of S^{ab}) in the *spin-charge-family* theory, the neutrinos become a regular members of quarks and leptons, with masses determined by the interaction with the scalar fields as all the rest of family members [51,53–56,3,9] (in Eq. (3.9) the interaction of fermions with the scalar fields is contained in the third line). This is the case also in $SO(10)$ theories [12–15]. The difference in the *spin-charge-family* theory is, that spin and handedness are correlated with charges, while in $SO(10)$ this is not the case (and must be correlated by "hand"). This fact is discussed in details in Ref. [8].

Let us point out that colour chargeless leptons and quarks of any of the three colours have completely the same $SO(7,1)$ part. Quarks and leptons distinguish only in the $SU(3) \times U(1)$ part.

I. B. The second Clifford object $\tilde{\gamma}^a$ offers the explanation for the existence of families.

I. B.i. There are twice four families of quarks and leptons in the *spin-charge-family* theory ([3] and the references therein) after the appearance of the condensate of the two right handed neutrinos, presented in Table 3.3, Ref. [4]. Since we have not really shown yet how this dynamically happens (we did this so far only for the toy model [68–70]), this remains as an open problem. All eight families obtain masses when the scalar gauge fields with the space index (7,8) — third line in Eq. (3.9) — gain nonzero vacuum expectation values at the electroweak phase transition. Table 3.1 represents in the left column eight families of creation operators of $\hat{u}^{c1\dagger}_{R}$ — the first member in Table 3.6 — and of chargeless $\hat{\nu}^{\dagger}_{R}$ — the 25^{th} member in Table 3.6. ($S^{11\,12}$, for example, transforms $\hat{u}^{ci\dagger}_{R}$ into $\hat{\nu}^{\dagger}_{R}$ and opposite).

I. B.ii. The eight-plets separate into two groups of four families: One group contains doublets with respect to $\vec{\tilde{N}}_{R}$ and $\vec{\tilde{\tau}}^{2}$, these families are singlets with respect to $\vec{\tilde{N}}_{L}$ and $\vec{\tilde{\tau}}^{1}$. Another group of families contains doublets with respect to $\vec{\tilde{N}}_{L}$ and $\vec{\tilde{\tau}}^{1}$, these families are singlets with respect to $\vec{\tilde{N}}_{R}$ and $\vec{\tilde{\tau}}^{2}$. Mass matrices of both groups manifest correspondingly, when the scalar fields — the gauge fields of $(\vec{\tilde{N}}_{R}, \vec{\tilde{\tau}}^{2}, U(1))$ and $(\vec{\tilde{N}}_{L}, \vec{\tilde{\tau}}^{1}, U(1))$ — obtain nonzero vacuum expectation values. Correspondingly both groups manifest $SU(2) \times SU(2) \times U(1)$ symmetry, with the same $U(1)$ and two different $SU(2)_{(L,R)} \times SU(2)_{(I,II)}$ symmetries, Ref. [57].

To the lower four families the observed three families of quarks and leptons contribute [51–53,55,56,58]. By the *spin-charge-family* theory predicted $SU(2) \times SU(2) \times U(1)$ symmetry of mass matrices, which limits the number of free parameters of mass matrices, the properties of the fourth family could be predicted by fitting free parameters to the experimental data. However, the accuracy of the so far measured 3×3 mixing (sub)matrices are even for quarks far from the required precision, which would enable prediction of masses of the fourth family members [55,56]. We predict for the assumed masses of the fourth family of quarks the corresponding matrix elements. Calculations show [56] that the larger the masses of the fourth family — up to 6 TeV seems to be allowed by experiments [40] — the smaller the unwanted mixing elements which could cause the flavour-changing neutral currents and the better is agreement with the experimental data, which require, due to the observations in Refs. [40,41], that there should be the fourth family due to the nonunitarity of the 3×3 so far measured mixing matrix for quarks and that the 4×4 mixing matrix elements should have the properties: $V_{u_1 d_4} > V_{u_1 d_3}$, $V_{u_2 d_4} < V_{u_1 d_4}$, and $V_{u_3 d_4} < V_{u_1 d_4}$. Here $u_i, d_i, i = 1,2,3,4$ represent u, c, t, u_4 and d, s, b, d_4 quarks.

The lowest of the upper four families is, as evaluated in Refs. [54,61], the candidate, which can explain (or at least can contribute to) the appearance of the dark matter in the universe. Comparing the results from following the fifth family members in the expanding universe with the astrophysical observations of dark matter and the direct measurements of the dark matter, the predicted masses of the fifth family quarks would be 10^2 TeV $< m_{q_5} c^2 < 4 \cdot 10^2$ TeV, and the scattering cross section σ for the fifth family neutron at least $10^{-6}\times$ smaller than the cross section for the first family neutron. These values change if the fifth family neutron is not the only source of the dark matter.

The fifth family would correspondingly manifest completely different "nuclear force" than the members of the lower four families [54], leading to different atoms and molecules, if they would success to form a matter in the expanding universe.

II. The gauge fields — the vielbeins, $f^{\alpha}{}_{a}$, and the two kinds of the spin connection fields, $\omega_{ab\alpha}$ and $\tilde{\omega}_{ab\alpha}$, of Eq. (3.1), appearing in the $2^{nd}, 3^{rd}$ and 4^{th} lines in Eq. (3.9) — manifest in $d = (3+1)$ as the vector gauge fields of $\vec{\tau}^{3}$,

					$\tilde{\tau}^{13}$	$\tilde{\tau}^{23}$	$\tilde{N}^3_L$	$\tilde{N}^3_R$	$\tilde{\tau}^4$
I	$\hat{u}^{c1\dagger}_{R\,1}$	$\stackrel{03}{(+i)}\stackrel{12}{[+]}\,\vert\stackrel{56}{[+]}\stackrel{78}{(+)}\,\Vert\stackrel{9\,10}{(+)}\;\stackrel{11\,12}{(-)}\;\stackrel{13\,14}{(-)}$	$\hat{\nu}^{\dagger}_{R\,1}$	$\stackrel{03}{(+i)}\stackrel{12}{[+]}\,\vert\stackrel{56}{[+]}\stackrel{78}{(+)}\,\Vert\stackrel{9\,10}{(+)}\;\stackrel{11\,12}{[+]}\;\stackrel{13\,14}{[+]}$	$-\frac{1}{2}$	0	$-\frac{1}{2}$	0	$-\frac{1}{2}$
I	$\hat{u}^{c1\dagger}_{R\,2}$	$\stackrel{03}{[+i]}\stackrel{12}{(+)}\,\vert\stackrel{56}{[+]}\stackrel{78}{(+)}\,\Vert\stackrel{9\,10}{(+)}\;\stackrel{11\,12}{(-)}\;\stackrel{13\,14}{(-)}$	$\hat{\nu}^{\dagger}_{R\,2}$	$\stackrel{03}{[+i]}\stackrel{12}{(+)}\,\vert\stackrel{56}{[+]}\stackrel{78}{(+)}\,\Vert\stackrel{9\,10}{(+)}\;\stackrel{11\,12}{[+]}\;\stackrel{13\,14}{[+]}$	$-\frac{1}{2}$	0	$\frac{1}{2}$	0	$-\frac{1}{2}$
I	$\hat{u}^{c1\dagger}_{R\,3}$	$\stackrel{03}{(+i)}\stackrel{12}{[+]}\,\vert\stackrel{56}{(+)}\stackrel{78}{[+]}\,\Vert\stackrel{9\,10}{(+)}\;\stackrel{11\,12}{(-)}\;\stackrel{13\,14}{(-)}$	$\hat{\nu}^{\dagger}_{R\,3}$	$\stackrel{03}{(+i)}\stackrel{12}{[+]}\,\vert\stackrel{56}{(+)}\stackrel{78}{[+]}\,\Vert\stackrel{9\,10}{(+)}\;\stackrel{11\,12}{[+]}\;\stackrel{13\,14}{[+]}$	$\frac{1}{2}$	0	$-\frac{1}{2}$	0	$-\frac{1}{2}$
I	$\hat{u}^{c1\dagger}_{R\,4}$	$\stackrel{03}{[+i]}\stackrel{12}{(+)}\,\vert\stackrel{56}{(+)}\stackrel{78}{[+]}\,\Vert\stackrel{9\,10}{(+)}\;\stackrel{11\,12}{(-)}\;\stackrel{13\,14}{(-)}$	$\hat{\nu}^{\dagger}_{R\,4}$	$\stackrel{03}{[+i]}\stackrel{12}{(+)}\,\vert\stackrel{56}{(+)}\stackrel{78}{[+]}\,\Vert\stackrel{9\,10}{(+)}\;\stackrel{11\,12}{[+]}\;\stackrel{13\,14}{[+]}$	$\frac{1}{2}$	0	$\frac{1}{2}$	0	$-\frac{1}{2}$
II	$\hat{u}^{c1\dagger}_{R\,5}$	$\stackrel{03}{[+i]}\stackrel{12}{[+]}\,\vert\stackrel{56}{[+]}\stackrel{78}{[+]}\,\Vert\stackrel{9\,10}{(+)}\;\stackrel{11\,12}{(-)}\;\stackrel{13\,14}{(-)}$	$\hat{\nu}^{\dagger}_{R\,5}$	$\stackrel{03}{[+i]}\stackrel{12}{[+]}\,\vert\stackrel{56}{[+]}\stackrel{78}{[+]}\,\Vert\stackrel{9\,10}{(+)}\;\stackrel{11\,12}{[+]}\;\stackrel{13\,14}{[+]}$	0	$-\frac{1}{2}$	0	$-\frac{1}{2}$	$-\frac{1}{2}$
II	$\hat{u}^{c1\dagger}_{R\,6}$	$\stackrel{03}{(+i)}\stackrel{12}{(+)}\,\vert\stackrel{56}{[+]}\stackrel{78}{[+]}\,\Vert\stackrel{9\,10}{(+)}\;\stackrel{11\,12}{(-)}\;\stackrel{13\,14}{(-)}$	$\hat{\nu}^{\dagger}_{R\,6}$	$\stackrel{03}{(+i)}\stackrel{12}{(+)}\,\vert\stackrel{56}{[+]}\stackrel{78}{[+]}\,\Vert\stackrel{9\,10}{(+)}\;\stackrel{11\,12}{[+]}\;\stackrel{13\,14}{[+]}$	0	$-\frac{1}{2}$	0	$\frac{1}{2}$	$-\frac{1}{2}$
II	$\hat{u}^{c1\dagger}_{R\,7}$	$\stackrel{03}{[+i]}\stackrel{12}{[+]}\,\vert\stackrel{56}{(+)}\stackrel{78}{(+)}\,\Vert\stackrel{9\,10}{(+)}\;\stackrel{11\,12}{(-)}\;\stackrel{13\,14}{(-)}$	$\hat{\nu}^{\dagger}_{R\,7}$	$\stackrel{03}{[+i]}\stackrel{12}{[+]}\,\vert\stackrel{56}{(+)}\stackrel{78}{(+)}\,\Vert\stackrel{9\,10}{(+)}\;\stackrel{11\,12}{[+]}\;\stackrel{13\,14}{[+]}$	0	$\frac{1}{2}$	0	$-\frac{1}{2}$	$-\frac{1}{2}$
II	$\hat{u}^{c1\dagger}_{R\,8}$	$\stackrel{03}{(+i)}\stackrel{12}{(+)}\,\vert\stackrel{56}{(+)}\stackrel{78}{(+)}\,\Vert\stackrel{9\,10}{(+)}\;\stackrel{11\,12}{(-)}\;\stackrel{13\,14}{(-)}$	$\hat{\nu}^{\dagger}_{R\,8}$	$\stackrel{03}{(+i)}\stackrel{12}{(+)}\,\vert\stackrel{56}{(+)}\stackrel{78}{(+)}\,\Vert\stackrel{9\,10}{(+)}\;\stackrel{11\,12}{[+]}\;\stackrel{13\,14}{[+]}$	0	$\frac{1}{2}$	0	$\frac{1}{2}$	$-\frac{1}{3}$

Table 3.1. Eight families of creation operators of $\hat{u}^{c1\dagger}_R$ — the right handed u-quark with spin $\frac{1}{2}$ and the colour charge ($\tau^{33} = 1/2$, $\tau^{38} = 1/(2\sqrt{3})$), appearing in the first line of Table 3.6 — and of the colourless right handed neutrino $\hat{\nu}^{\dagger}_R$ — of spin $\frac{1}{2}$, appearing in the 25^{th} line of Table 3.6 — are presented in the left and in the right column, respectively. Table is taken from [9]. Families belong to two groups of four families, one (I) is a doublet with respect to ($\vec{\tilde{N}}_L$ and $\vec{\tilde{\tau}}^{(1)}$) and a singlet with respect to ($\vec{\tilde{N}}_R$ and $\vec{\tilde{\tau}}^{(2)}$), the other (II) is a singlet with respect to ($\vec{\tilde{N}}_L$ and $\vec{\tilde{\tau}}^{(1)}$) and a doublet with with respect to ($\vec{\tilde{N}}_R$ and $\vec{\tilde{\tau}}^{(2)}$), Eq. (3.111). All the families follow from the starting one by the application of the operators ($\tilde{N}^{\pm}_{R,L}$, $\tilde{\tau}^{(2,1)\pm}$), Eq. (3.129). The generators ($N^{\pm}_{R,L}$, $\tau^{(2,1)\pm}$) (Eq. (3.129)) transform $\hat{u}^{\dagger}_{1R}$ to all the members of one family of the same colour. The same generators transform equivalently the right handed neutrino $\hat{\nu}^{\dagger}_{1R}$ to all the colourless members of the same family.

Eq .(3.113), τ^4, Eq. (3.113), $\vec{\tau}^1$, Eq. (3.112), and $\vec{\tau}^2$, Eq. (3.112), if the space index is $m = (0,1,2,3)$ (2^{nd} line in Eq. (3.9)), as well as the scalar gauge fields, if the space index is $s \geq 5$ (3^{rd} and 4^{th} line in Eq. (3.9)), of the same operators as in the vector gauge fields case, Ref. [5].

Only if there are no fermion present, then both, $\omega_{ab\alpha}$ and $\tilde{\omega}_{ab\alpha}$, are uniquely expressed by vielbeins, Ref. ([9], Eq. (C9)).

$$
\begin{aligned}
\omega_{ab\alpha} = \tilde{\omega}_{ab\alpha} = &-\frac{1}{2E}\Big\{ e_{e\alpha} e_{b\gamma}\, \partial_\beta (E f^{\gamma[e} f^{\beta}{}_{a]}) + e_{e\alpha} e_{a\gamma}\, \partial_\beta (E f^{\gamma}{}_{[b} f^{\beta e]}) \\
&\qquad - e_{e\alpha} e^{e}{}_{\gamma}\, \partial_\beta \left(E f^{\gamma}{}_{[a} f^{\beta}{}_{b]}\right) \Big\} \\
&- \frac{1}{d-2}\Big\{ e_{a\alpha} \frac{1}{E} e^{d}{}_{\gamma} \partial_\beta \left(E f^{\gamma}{}_{[d} f^{\beta}{}_{b]}\right) \\
&\qquad - e_{b\alpha} \frac{1}{E} e^{d}{}_{\gamma} \partial_\beta \left(E f^{\gamma}{}_{[d} f^{\beta}{}_{a]}\right) \Big\}\,.
\end{aligned}
\tag{3.10}
$$

II. A. It is proven in Ref. [5] that the vector (as well as the scalar gauge fields) can indeed be expressed with the spin connections (rather than with the vielbeins),

$$
A^{Ai}_{m} = \sum_{s,t} c^{Ai}{}_{st}\, \omega^{st}{}_{m}\,, \tag{3.11}
$$

demonstrating the symmetry of space with $(s,t) \geq 5$, making the *spin-charge-family* theory transparent and correspondingly "elegant", so that it is easier to recognize that the origin of charges of the observed fermions, vector gauge fields, Higgs's scalar and Yukawa couplings might really be in $(d-4)$ space.

In the presence of the condensate, Table 3.3, of the right handed neutrinos, all the vector gauge fields and the scalar gauge fields, which interact with the condensate, gain masses. Only the weak ($SU(2)_I$), the colour ($SU(3)$) and the hyper ($U(1)$, $Y = \tau^4 + \tau^{23}$) gauge fields, which do not interact with the condensate, remain massless.

II. A.i. The weak vector gauge fields $\vec{A}^1_m$, the gauge field of $SU(2)_I$, and $\vec{A}^2_m$, the gauge fields of $SU(2)_{II}$, are the superposition of gauge fields $\omega_{s't's}$ (Ref. [9], Eqs. (8,9,10)),

$$
\begin{aligned}
\vec{A}^1_m &= (\omega_{58m} - \omega_{67m}, \omega_{57m} + \omega_{68m}, \omega_{56m} - \omega_{78m})\,, \\
\vec{A}^2_m &= (\omega_{58m} + \omega_{67m}, \omega_{57m} - \omega_{68m}, \omega_{56m} + \omega_{78m})\,.
\end{aligned}
\tag{3.12}
$$

Taking into account Eq. (3.113) one easily finds the colour vector gauge field expressed with ω_{stm}. $\vec{A}^2_m$ get masses by interaction with the condensate.

In Ref. [5], Eqs. (24-25), the reader can find Lagrange density for the $R^{(d-4)}$ part of the gravity field R, Eq.(3.1), expressed by the vector gauge fields $\vec{A}^A_m$.

II. B. The scalar gauge fields are the superposition of either $\omega_{s't's}$, with $(s',t',s) = (5,6,\cdots,14)$, Ref. [5], or $\tilde{\omega}_{abs}$, with $(a,b) = (0,1,\cdots,8)$ and $(s) = (5,6,7,8)$, Refs. [4,7,9], the fourth line in Eq. (3.9).

Both kinds of scalar fields with $s = (7,8)$ contribute to the masses of the two groups of four families. Scalar fields $\omega_{s't's}$, with $(s',t') = (5,6,\cdots,14)$, $s = (9,10,\cdots,14)$ contribute to matter-antimatter asymmetry and to proton decay [4].

II. B.i. In the *spin-charge-family* theory the scalar fields with the space index $s = (7,8)$ carry with respect to this space index the weak charge and the hyper charge $(\mp\frac{1}{2}, \pm\frac{1}{2})$, respectively, independent of whether they are superposition of $\omega_{s't's}$ or of $\tilde{\omega}_{abs}$, $s = (7,8)$, Refs. [9,3,4].

There are twice two triplets, the superposition of $\tilde{\omega}_{abs}$, Eqs. (3.111, 3.112) with S^{ab} replaced by $\tilde{S}^{ab}$, the gauge scalar fields of either the group $\widetilde{SU}(2)_{\widetilde{SO}(3,1)_L} \times \widetilde{SU}(2)_I$ or of the group $\widetilde{SU}(2)_{\widetilde{SO}(3,1)_R} \times \widetilde{SU}(2)_{II}$, the first two triplets interacting with one group of four families, the second two triplets interacting with another group of four families, both groups presented in Table 3.1. There are also three singlets, the gauge scalar fields of $= (Q, Q', Y')$, Eq. (3.114), which are the superposition of $\omega_{s't's}$ and interact with members of all the eight families of Table 3.1 [7,9,3,4].

Let us use a common notation A^{Ai}_s for all the scalar fields, independently of whether they originate in $\tilde{\omega}_{abs}$ or ω_{abs}, $s = (7,8)$,

$$A^{Ai}_s \in (\, A^{Q}_s\,, A^{Q'}_s\,, A^{Y'}_s\,, \vec{\tilde{A}}^{\tilde{1}}_s\,, \vec{\tilde{A}}^{\tilde{N}_{\tilde{L}}}_s\,, \vec{\tilde{A}}^{\tilde{2}}_s\,, \vec{\tilde{A}}^{\tilde{N}_{\tilde{R}}}_s\,)\,,$$
$$\tau^{Ai} \supset (Q,\, Q',\, Y',\, \vec{\tilde{\tau}}^{1},\, \vec{\tilde{N}}_L,\, \vec{\tilde{\tau}}^{2},\, \vec{\tilde{N}}_R)\,. \tag{3.13}$$

Here τ^{Ai} represent the operators of the groups the gauge scalar fields of which are A^{Ai}_s.

Let us rewrite the third line in Eq. (3.9) as follows, Ref. ([9], Eqs. (18-19)).

$$\sum_{s=(7,8),A,i} \bar{\psi}\,\gamma^s\,(-\tau^{Ai}\,A^{Ai}_s\,)\,\psi =$$
$$\sum_{A,i} -\bar{\psi}\{\stackrel{78}{(+)}\;\tau^{Ai}\,(A^{Ai}_7 - i\,A^{Ai}_8) + \stackrel{78}{(-)}\;(\tau^{Ai}\,(A^{Ai}_7 + i\,A^{Ai}_8)\}\psi\,,$$
$$\stackrel{78}{(\pm)} = \frac{1}{2}\,(\gamma^7 \pm i\,\gamma^8\,)\,, \quad A^{Ai}_{\stackrel{78}{(\pm)}} := (A^{Ai}_7 \mp i\,A^{Ai}_8)\,, \tag{3.14}$$

with the summation over A, i performed, since A^{Ai}_s represent the scalar fields (A^{Q}_s, $A^{Q'}_s$, $A^{Y'}_s$, $\tilde{A}^{\tilde{4}}_s$, $\vec{\tilde{A}}^{\tilde{1}}_s$, $\vec{\tilde{A}}^{\tilde{2}}_s$, $\vec{\tilde{A}}^{\tilde{N}_{\tilde{R}}}_s$ and $\vec{\tilde{A}}^{\tilde{N}_{\tilde{L}}}_s$). In the low energy regime the momentum p_s, $s = (7,8)$ can be neglected.

Taking into account that $\tau^{13} = \frac{1}{2}(S^{56} - S^{78})$, $Y = (\tau^{23} + \tau^{4})$, $\tau^{23} = \frac{1}{2}(S^{56} + S^{78})$, while $\tau^4 = -\frac{1}{3}(S^{9\,10} + S^{11\,12} + S^{13\,14})$, and $S^{ab}A_c = i(A^a\delta^b_c - A^b\delta^a_c)$, one finds

$$\tau^{13}\,(A^{Ai}_7\; \mp i\,A^{Ai}_8) = \pm\,\frac{1}{2}\,(A^{Ai}_7\; \mp i\,A^{Ai}_8)\,,$$
$$Y\,(A^{Ai}_7\; \mp i\,A^{Ai}_8) = \mp\,\frac{1}{2}\,(A^{Ai}_7\; \mp i\,A^{Ai}_8)\,,$$
$$Q\,(A^{Ai}_7\; \mp i\,A^{Ai}_8) = 0\,. \tag{3.15}$$

This are quantum numbers of the by the *standard model* assumed Higgs. These scalar gauge fields with the space index $(7,8)$, gaining nonzero vacuum expectation values (by assumption as in the *standard model* so far), cause the electroweak break, breaking the weak and the hyper charge, explaining the appearance of in the *standard model* assumed Higgs and the Yukawa couplings, predicting the existence of several scalars — two triplets and three singlets, which couple to the lower four families, making them massive and giving masses to weak bosons.

These scalar fields manifest the $SU(2)\times SU(2)\times U(1)$ symmetry, which reduces the number of free parameters in mass matrices of quarks and leptons, enabling predictions of properties of the four families [55–57].

II. B.ii. The scalar fields with the space index $s = (9, 10, \cdots, 14)$, presented in Table 3.2, carry triplet or antitriplet colour charges and the "spinor" charge equal to twice the quark or antiquark "spinor" charge, and the fractional hyper and electromagnetic charge.

They carry in addition the quantum numbers of the adjoint representations originating in S^{ab} or in $\tilde{S}^{ab}$. (Although carrying the colour charge of one of the triplet or antitriplet quantum numbers, these fields can not be interpreted as superpartners of the quarks, since they do not have quantum numbers as required by, let say, the $N = 1$ supersymmetry. The hyper charges and the electromagnetic charges are namely not those required by the supersymmetric partners to the family members.)

Let us have a look what do the scalar fields, appearing in the fourth line of Eq. (3.9) and in the seventh line of Table 3.2, do when applying on the left handed members of the Weyl representation presented on Table 3.6, containing quarks and leptons and antiquarks and antileptons [71,72,65].

Fig. 3.1 presents the creation of proton due to the interaction of quarks and leptons with these scalar fields. One can read on this Fig. 3.1 all the quantum numbers of a positron (57^{th} line of Table 3.6), an antiquark (43^{rd} line of Table 3.6), and a quark (9^{th} line of Table 3.6), as well as of the scalar field $A^{2\boxminus}_{\substack{9\,10\\(+)}}$, seventh line of Table 3.2, involved in the proton birth. The opposite transition at low energies would make the proton decay.

After the appearance of the condensate of the two right handed neutrinos, Table 3.3, the discrete symmetry $\mathbb{C}_N\mathcal{P}_N$ is obviously broken. In the expanding universe, fulfilling the Sakharov request for appropriate non-thermal equilibrium, the triplet scalars from Table 3.2 have a chance to explain the matter-antimatter asymmetry in the universe [4].

III. The *spin-charge-family* theory suggests two kinds of phase transitions — two kinds of breaking symmetries: The appearance of the condensate and the nonzero vacuum expectation values of the scalar fields with the space index $s = (7, 8)$.

III. A. Table 3.3 represents the properties of the condensate of the two right handed neutrinos $\hat{\nu}^{\dagger}_{R8}$ — Table 3.1 — of spin up and spin down, breaking the discrete $\mathbb{C}_N\mathcal{P}_N$ symmetry Subsect. 3.3.3, [4,65].

Due to the interaction with the condensate of Table 3.3 the gauge vector fields of $\vec{\tau}^2$ and τ^4 become massive. The colour vector gauge fields of $\vec{\tau}^3$, the weak vector gauge fields of $\vec{\tau}^1$ and the hyper vector gauge field of Y do not interact with the condensate (the corresponding quantum numbers of the condensate are zero) and correspondingly remain massless, the gravity in $d = (3 + 1)$, which is the gauge field of S^{mn} and p_m, remains massless as well.

field	prop.	τ^4	τ^{13}	τ^{23}	(τ^{33}, τ^{38})	Y	Q	$\tilde{\tau}^4$	$\tilde{\tau}^{13}$	$\tilde{\tau}^{23}$	$\tilde{N}^3_L$	$\tilde{N}^3_R$
$A^{1\boxplus}_{9\,10}(\oplus)$	scalar	$\oplus\,\frac{1}{3}$	$\boxplus\,1$	0	$(\oplus\,\frac{1}{2}, \oplus\,\frac{1}{2\sqrt{3}})$	$\oplus\,\frac{1}{3}$	$\oplus\,\frac{1}{3}+\boxplus\,1$	0	0	0	0	0
$A^{13}_{9\,10}(\oplus)$	scalar	$\oplus\,\frac{1}{3}$	0	0	$(\oplus\,\frac{1}{2}, \oplus\,\frac{1}{2\sqrt{3}})$	$\oplus\,\frac{1}{3}$	$\oplus\,\frac{1}{3}$	0	0	0	0	0
$A^{1\boxplus}_{11\,12}(\oplus)$	scalar	$\oplus\,\frac{1}{3}$	$\boxplus\,1$	0	$(\oplus\,\frac{1}{2}, \oplus\,\frac{1}{2\sqrt{3}})$	$\oplus\,\frac{1}{3}$	$\oplus\,\frac{1}{3}+\boxplus\,1$	0	0	0	0	0
$A^{13}_{11\,12}(\oplus)$	scalar	$\oplus\,\frac{1}{3}$	0	0	$(\oplus\,\frac{1}{2}, \oplus\,\frac{1}{2\sqrt{3}})$	$\oplus\,\frac{1}{3}$	$\oplus\,\frac{1}{3}$	0	0	0	0	0
$A^{1\boxplus}_{13\,14}(\oplus)$	scalar	$\oplus\,\frac{1}{3}$	$\boxplus\,1$	0	$(0, \oplus\,\frac{1}{\sqrt{3}})$	$\oplus\,\frac{1}{3}$	$\oplus\,\frac{1}{3}+\boxplus\,1$	0	0	0	0	0
$A^{13}_{13\,14}(\oplus)$	scalar	$\oplus\,\frac{1}{3}$	0	0	$(0, \oplus\,\frac{1}{\sqrt{3}})$	$\oplus\,\frac{1}{3}$	$\oplus\,\frac{1}{3}$	0	0	0	0	0
$A^{2\boxplus}_{9\,10}(\oplus)$	scalar	$\oplus\,\frac{1}{3}$	0	$\boxplus\,1$	$(\oplus\,\frac{1}{2}, \oplus\,\frac{1}{2\sqrt{3}})$	$\oplus\,\frac{1}{3}+\boxplus\,1$	$\oplus\,\frac{1}{3}+\boxplus\,1$	0	0	0	0	0
$A^{23}_{9\,10}(\oplus)$ ...	scalar	$\oplus\,\frac{1}{3}$	0	0	$(\oplus\,\frac{1}{2}, \oplus\,\frac{1}{2\sqrt{3}})$	$\oplus\,\frac{1}{3}$	$\oplus\,\frac{1}{3}$	0	0	0	0	0
$\tilde{A}^{1\boxplus}_{9\,10}(\oplus)$	scalar	$\oplus\,\frac{1}{3}$	0	0	$(\oplus\,\frac{1}{2}, \oplus\,\frac{1}{2\sqrt{3}})$	$\oplus\,\frac{1}{3}$	$\oplus\,\frac{1}{3}$	0	$\boxplus\,1$	0	0	0
$\tilde{A}^{13}_{9\,10}(\oplus)$ ...	scalar	$\oplus\,\frac{1}{3}$	0	0	$(\oplus\,\frac{1}{2}, \oplus\,\frac{1}{2\sqrt{3}})$	$\oplus\,\frac{1}{3}$	$\oplus\,\frac{1}{3}$	0	0	0	0	0
$\tilde{A}^{2\boxplus}_{9\,10}(\oplus)$	scalar	$\oplus\,\frac{1}{3}$	0	0	$(\oplus\,\frac{1}{2}, \oplus\,\frac{1}{2\sqrt{3}})$	$\oplus\,\frac{1}{3}$	$\oplus\,\frac{1}{3}$	0	0	$\boxplus\,1$	0	0
$\tilde{A}^{23}_{9\,10}(\oplus)$ ...	scalar	$\oplus\,\frac{1}{3}$	0	0	$(\oplus\,\frac{1}{2}, \oplus\,\frac{1}{2\sqrt{3}})$	$\oplus\,\frac{1}{3}$	$\oplus\,\frac{1}{3}$	0	0	0	0	0
$\tilde{A}^{N_L\boxplus}_{9\,10}(\oplus)$	scalar	$\oplus\,\frac{1}{3}$	0	0	$(\oplus\,\frac{1}{2}, \oplus\,\frac{1}{2\sqrt{3}})$	$\oplus\,\frac{1}{3}$	$\oplus\,\frac{1}{3}$	0	0	0	$\boxplus\,1$	0
$\tilde{A}^{N_L3}_{9\,10}(\oplus)$ ...	scalar	$\oplus\,\frac{1}{3}$	0	0	$(\oplus\,\frac{1}{2}, \oplus\,\frac{1}{2\sqrt{3}})$	$\oplus\,\frac{1}{3}$	$\oplus\,\frac{1}{3}$	0	0	0	0	0
$\tilde{A}^{N_R\boxplus}_{9\,10}(\oplus)$	scalar	$\oplus\,\frac{1}{3}$	0	0	$(\oplus\,\frac{1}{2}, \oplus\,\frac{1}{2\sqrt{3}})$	$\oplus\,\frac{1}{3}$	$\oplus\,\frac{1}{3}$	0	0	0	0	$\boxplus\,1$
$\tilde{A}^{N_R3}_{9\,10}(\oplus)$ ...	scalar	$\oplus\,\frac{1}{3}$	0	0	$(\oplus\,\frac{1}{2}, \oplus\,\frac{1}{2\sqrt{3}})$	$\oplus\,\frac{1}{3}$	$\oplus\,\frac{1}{3}$	0	0	0	0	0
$A^{3i}_{9\,10}(\oplus)$ ...	scalar	$\oplus\,\frac{1}{3}$	0	0	$(\boxplus\,1+\oplus\,\frac{1}{2}, \oplus\,\frac{1}{2\sqrt{3}})$	$\oplus\,\frac{1}{3}$	$\oplus\,\frac{1}{3}$	0	0	0	0	0
$A^{4}_{9\,10}(\oplus)$ ...	scalar	$\oplus\,\frac{1}{3}$	0	0	$(\oplus\,\frac{1}{2}, \oplus\,\frac{1}{2\sqrt{3}})$	$\oplus\,\frac{1}{3}$	$\oplus\,\frac{1}{3}$	0	0	0	0	0

Table 3.2. Quantum numbers of the scalar gauge fields carrying the space index $t = (9, 10, \cdots, 14)$, appearing in the fourth line of Eq. (3.9), are presented. To the colour charge of all these scalar fields the space degrees of freedom — the space index — contribute one of the triplets or antitriplet values. These scalars are with respect to the two $SU(2)$ charges, ($\vec{\tau}^1$ and $\vec{\tau}^2$), and the two $\widetilde{SU}(2)$ charges, ($\vec{\tilde{\tau}}^1$ and $\vec{\tilde{\tau}}^2$), triplets (that is in the adjoint representations of the corresponding groups), and they all carry twice the "spinor" number (τ^4) of the quarks or antiquarks. The quantum numbers of the two vector gauge fields, the colour and the $U(1)_{II}$ ones, are added. These Table is taken from Ref. [4], Table I. We invite the reader to visit Ref. [4] for more details.

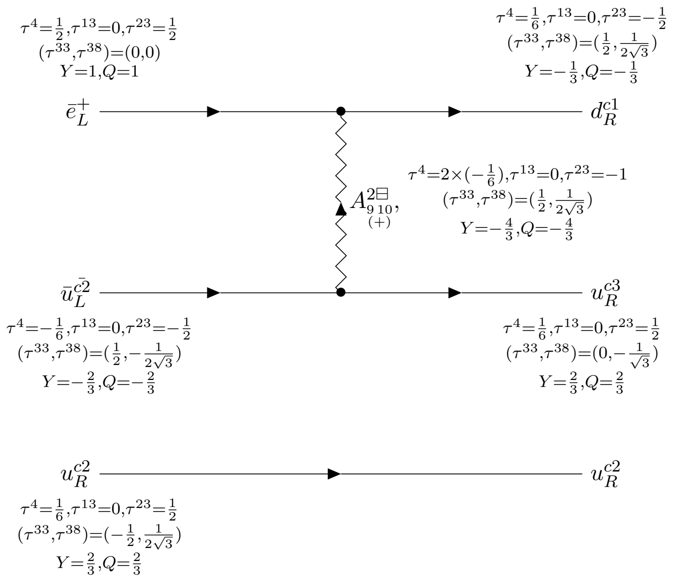

Fig. 3.1. The birth of a "right handed proton" out of a positron $\bar{e}^{+}_{L}$, antiquark $\bar{u}^{\bar{c2}}_{L}$ and quark (spectator) u^{c2}_{R}. The family quantum number can be any.

state	S^{03}	S^{12}	τ^{13}	τ^{23}	τ^{4}	Y	Q	$\tilde{\tau}^{13}$	$\tilde{\tau}^{23}$	$\tilde{\tau}^{4}$	$\tilde{Y}$	$\tilde{Q}$	$\tilde{N}^{3}_{L}$	$\tilde{N}^{3}_{R}$
$(\lvert\nu^{\mathbf{VIII}}_{\mathbf{1R}}>_{\mathbf{1}}\ \lvert\nu^{\mathbf{VIII}}_{\mathbf{2R}}>_{\mathbf{2}})$	0	0	0	1	−1	0	0	0	1	−1	0	0	0	1
$(\lvert\nu^{VIII}_{1R}>_{1}\ \lvert e^{VIII}_{2R}>_{2})$	0	0	0	0	−1	−1	−1	0	1	−1	0	0	0	1
$(\lvert e^{VIII}_{1R}>_{1}\ \lvert e^{VIII}_{2R}>_{2})$	0	0	0	−1	−1	−2	−2	0	1	−1	0	0	0	1

Table 3.3. The condensate of the two right handed neutrinos ν_R, with the quantum numbers of the VIIIth family, coupled to spin zero and belonging to a triplet with respect to the generators τ^{2i}, is presented, together with its two partners. The condensate carries $\vec{\tau}^{1}=0$, $\tau^{23}=1$, $\tau^{4}=-1$ and $Q=0=Y$. The triplet carries $\tilde{\tau}^{4}=-1$, $\tilde{\tau}^{23}=1$ and $\tilde{N}^{3}_{R}=1$, $\tilde{N}^{3}_{L}=0$, $\tilde{Y}=0$, $\tilde{Q}=0$. The family quantum numbers of quarks and leptons are presented in Table 3.1.

Due to nonzero family quantum numbers of the condensate the corresponding scalar gauge fields become massive. The condensate gives masses to all the scalars from Table 3.2, either because they couple to the condensate due to τ^{4} or $\tilde{\tau}^{4}$ or τ^{23} or $\tilde{\tau}^{23}$ quantum numbers. It gives masses also to all the scalar fields with $s\in(5,6,7,8)$, since they couple to the condensate due to the nonzero τ^{23}. The scalar fields with the quantum numbers of the upper four families couple in addition through their family quantum numbers.

III. B. The electroweak phase transition is caused by the nonzero vacuum expectation values of twice two triplets and three singlet scalars, giving masses to the lower fourth families — two of twice two triplets and three singlets — and to the upper four families — another two triplets and the same three singlets.

IV. Predictions of the *spin-charge-family* theory so far.

IV. A. The *spin-charge-family* theory predicts the fourth family to the observed three to be observed at the LHC [53]. By predicting symmetry of mass matrices (in all orders of loop corrections [57]) the theory enables for accurate enough measured mixing matrices of the 3×3 submatrices (the sensitivity of the fitting procedure on masses of the so far measured quarks and leptons is much smaller [55,56]), and due to other measured properties of quarks and leptons [40], to predict the properties of the 4×4 mixing matrices and to explain correspondingly the origin of Higgs and Yukawa couplings. The 4×4 mixing matrix elements for quarks are predicted to have the properties: $V_{u_1 d_4} > V_{u_1 d_3}$, $V_{u_2 d_4} < V_{u_1 d_4}$, and $V_{u_3 d_4} < V_{u_1 d_4}$, here $u_i, d_i, i = 1,2,3$ represent u, c, t, u_4 and d, s, b, d_4 quarks.

The theory explains [58] why the fourth family has not yet been observed, which is the main argument against the existence of four families [59,60] among experts in high energy physics.

IV. B. The theory predicts the existence of several scalar fields — there are two triplets and three singlets which determine masses of the lower four families [9,7,3,6] — some of which will be observed in the near future measurements.

IV. C. The theory predicts the second group of four families, the stable one of these four families contributing to the dark matter [54]. The nuclear force among these baryons differs a lot from the so far observed nuclear force [54,61].

IV. D. The masses of quarks and leptons are, according to these two groups of four families, spread from 10^{-3} eV to few TeV — at least 12 orders of magnitude for the first four families — and from 100 TeV to 10^{13} TeV — at least 11 orders of magnitude for the second four families, offering the explanation for the hierarchy problem. (The mass matrices of the two groups of mass matrices are very closed to the democratic ones [55,56]).

IV. E. The *spin-charge-family* theory predicts the masses of the dark matter baryons [54].

IV. F. The *spin-charge-family* theory predicts the scalar fields which contribute to the matter-antimatter asymmetry in the universe [4] and correspondingly also to the proton decay.

V. The *spin-charge-family* theory has (so far) several open problems, although it is also true that the more work is done, the more solutions of the open problems follow.

V. A. In the *spin-charge-family* the vector and scalar gauge fields originate in gravity as the two kinds of the spin connection fields and the vielbeins. In the low energy region these vector and scalar gauge fields can be quantized in the usual way [5]. Yet the quantization of gravity remains as an open problem when the energies rise up to 10^{16} GeV and above.

V. B. The dimension of space time — $13 + 1$ — remains as an open problem: Why $d = (13 + 1)$, why not ∞? (Only 0 and ∞ need no explanation.) How has the universe come to $d = (13 + 1)$ [77]?

V. C. Breaking the symmetry with the appearance of the condensate [4], which lead to observable properties of fermion and boson fields, explaining all the assumptions of the *standard models*, needs to be studied as a dynamical appearance of the condensate in the expanding universe.

V. D. It should be demonstrated dynamically how do the scalar fields gain nonzero vacuum expectation values, leading to the effective fields as assumed for the Higgs. The demonstrations, made in Refs. [68–70] for the toy model in $d = (5 + 1)$ must be done also for $d = (13 + 1)$.

V. F. The coupling constants of the gauge and scalar fields in the low energy regime should be evaluated when starting with the simple action of Eq. (3.1) in $d = (13 + 1)$, with only one (or already with two) coupling constants.

V. G. There are additional open problems which we already see and either solve, like the one treated in this paper about the internal degrees of freedom of fermions in Clifford and Grassmann space and the new way of second quantization procedure, which explains the usual way of second quantization, or they wait to be solved, like the lepton number non conservation in the *spin-charge-family* theory. And there are open problems which we do not see yet or which we could better understand if learning more from all the trials to understand the evolution of the universe and the creation of hadrons of all kinds in the literature.

3.2 Fermions in Grassmann and in Clifford space

In the literature the Clifford algebra is frequently discussed as an useful tool to describe internal degrees of freedom of fermions [62–64]. In the *spin-charge-family* theory Clifford space is used to describe all the internal degrees of fermions — quarks and leptons with their families included [1,2,9].

In this paper we demonstrate that the Clifford algebra offers an elegant and transparent way to better understand fermions properties: In even dimensional spaces — we make a choice of $d = 2(2n + 1)$, $n = 3$ — the creation operators of an odd Clifford character can be defined (they are superposition of odd numbers of the Clifford algebra objects (γ^a's or $\tilde{\gamma}^a$'s, Eq. (3.2)), each of them is a product of $\frac{d}{2}$ nilpotents and projectors, Eq. (3.27, 3.70) [47,48], so that they are the eigenvectors of twice all the $\frac{d}{2}$ members of the two kinds of the Cartan subalgebras of the Lorentz algebra — S^{ab} and $\tilde{S}^{ab}$ — with the half integer eigenvalues, Eq. (3.72). These creation operators, Eq. (3.76), and their Hermitian conjugated partners — the annihilation operators, Eq. (3.77) — fulfill on the vacuum state, Eq. (3.79), the anti commutation relations required for fermions, Eq. 3.81.
The superposition of these creation operators solve for a particular momentum p^a the equation of motions for free massless fermions, Eq. (3.36), determining in $d = (3 + 1)$ spins, handedness, charges and family quantum numbers. Again they fulfill on the vacuum state, Eq. (3.79), together with their Hermitian conjugated annihilation operators, the anti commutation relations required for fermions, Eq. (3.83). Correspondingly the *creation and annihilation operators are indeed defined with the first quantized fermion fields already*.

We demonstrate in this paper that there exist also in Grassmann space of anticommuting coordinates, Eq. (3.3), the eigenvectors of the Cartan commuting subalgebra of the Lorentz algebra $\mathbf{S}^{ab}$, Eq. (3.3, 3.21), the $\frac{d}{2}$ products of which form creation operators, Eq. (3.51), and which fulfill together with their Hermitian

conjugated partners the annihilation operators, Eq. (3.18), as well the anticommutation relations required for fermions, Eq. (3.54). However, the *eigenvalues of the Cartan subalgebra are in this case integer*.
Also in the Grassmann case the superposition of these creation operators solve for a particular momentum p^a the equation of motions for free massless fermions, presented in Eq. (3.43), determining in $d = (3 + 1)$ spins, handedness and charges. There are no families in this case.

For both cases, Clifford and Grassmann, we present the proofs for the above statements and illustrate the properties of fermions of both kinds on a few examples.

3.2.1 Actions and equation of motion in Clifford and in Grassmann space

We define in $d = ((d - 1) + 1)$-dimensional space states with integer spin — in Grassmann space — and states with half integer spin — in Clifford space — proving that norms in both spaces can be determined by the integral in Grassmann space, Eqs. (3.32, 3.33), since the Clifford algebra objects are expressible with the Grassmann algebra objects, Eq. (3.4) [2]. When reformulating the vacuum in the Clifford case, Eq. (3.79), half integer spinors presentation in Clifford space become more elegant, that is easier to recognize properties of fermions.

We present as well actions in both cases, Grassmann, Eq. (3.41), and Clifford, Eq. (3.36), leading to the equations of motion (in the Clifford case the Weyl equation is known for a long time, in the Grassmann case it is present for the first time by N.S.M.B.). We compare Euler-Lagrange equations in both cases to compare properties of Grassmann "fermions" with the Clifford fermions.

a. *Fields with the integer spin in Grassmann space*

A point in d-dimensional Grassmann space of anticommuting coordinates θ^a, $(a = 0, 1, 2, 3, 5, \dots, d)$, is determined by a vector $\{\theta^a\} = (\theta^0, \theta^1, \theta^2, \theta^3, \theta^5, \dots, \theta^d)$. A linear vector space over the coordinate Grassmann space has correspondingly the dimension 2^d, due to the fact that $(\theta^{a_i})^2 = 0$ for any $a_i \in (0, 1, 2, 3, 5, \dots, d)$.

Correspondingly are fields in Grassmann space expressible in terms of the Grassmann algebra objects

$$\mathbf{B} = \sum_{k=0}^{d} a_{a_1 a_2 \dots a_k}\, \theta^{a_1} \theta^{a_2} \dots \theta^{a_k} |\phi_{og} >, \quad a_i \leq a_{i+1}\,, \tag{3.16}$$

where $|\phi_{og} >$ is the vacuum state, here assumed to be $|\phi_{og} >= |1 >$, so that $\frac{\partial}{\partial \theta^a} |\phi_{og} >= 0$ for any θ^a. The *Kalb-Ramond* boson fields $a_{a_1 a_2 \dots a_k}$ are antisymmetric with respect to the permutation of indexes, since the Grassmann coordinates anticommute $\{\theta^a, \theta^b\}_+ = 0$, Eq. (3.3).

[2] Observations in this paper might help also when fermionizing boson fields or bosonizing fermion fields [42].

The left derivative $\frac{\partial}{\partial \theta_a}$ on vectors of the space of monomials $\mathbf{B}(\theta)$ is defined as follows

$$\frac{\partial}{\partial \theta_a}\,\mathbf{B}(\theta) = \frac{\partial \mathbf{B}(\theta)}{\partial \theta_a}\,,$$
$$\left\{\frac{\partial}{\partial \theta_a}, \frac{\partial}{\partial \theta_b}\right\}_{+}\,\mathbf{B} = 0\,,\ \text{for all}\ \mathbf{B}\,. \tag{3.17}$$

The commutation relations are for $p^{\theta a} = i\frac{\partial}{\partial \theta_a}$ defined in Eq. (3.3), where the metric tensor η^{ab} ($= diag(1, -1, -1, \ldots, -1)$) lowers the indexes of a vector $\{\theta^a\}$: $\theta_a = \eta_{ab}\,\theta^b$ (the same metric tensor lowers the indexes of the ordinary vector x^a of commuting coordinates).

Defining [3]

$$(\theta^a)^\dagger = \frac{\partial}{\partial \theta_a}\eta^{aa} = -i\,p^{\theta a}\eta^{aa}\,, \tag{3.18}$$

it follows

$$(\frac{\partial}{\partial \theta_a})^\dagger = \eta^{aa}\,\theta^a\,, \quad (p^{\theta a})^\dagger = -i\eta^{aa}\theta^a\,. \tag{3.19}$$

Making a choice for the complex properties of θ^a, and correspondingly of $\frac{\partial}{\partial \theta_a}$, as follows

$$\{\theta^a\}^* = (\theta^0, \theta^1, -\theta^2, \theta^3, -\theta^5, \theta^6, ..., -\theta^{d-1}, \theta^d)\,,$$
$$\{\frac{\partial}{\partial \theta_a}\}^* = (\frac{\partial}{\partial \theta_0}, \frac{\partial}{\partial \theta_1}, -\frac{\partial}{\partial \theta_2}, \frac{\partial}{\partial \theta_3}, -\frac{\partial}{\partial \theta_5}, \frac{\partial}{\partial \theta_6}, ..., -\frac{\partial}{\partial \theta_{d-1}}, \frac{\partial}{\partial \theta_d})\,, \tag{3.20}$$

it follows for the two Clifford algebra objects $\gamma^a = (\theta^a + \frac{\partial}{\partial \theta_a})$, and $\tilde{\gamma}^a = i(\theta^a - \frac{\partial}{\partial \theta_a})$, Eq. (3.4), that γ^a is real if θ^a is real, and γ^a is imaginary if θ^a is imaginary, while $\tilde{\gamma}^a$ is imaginary when θ^a is real and $\tilde{\gamma}^a$ is real if θ^a is imaginary, just as it is required in Eq. (3.26).

Applying the operator $\mathbf{S}^{ab}$ of Eq. (3.3) on the "states" $\frac{1}{\sqrt{2}}(\theta^a + \frac{\eta^{aa}}{ik}\theta^b)$, $a \neq b$, and $\frac{1}{\sqrt{2}}\,(1 + \frac{i}{k}\theta^a\theta^b)$, $a \neq b$, it follows

$$\mathbf{S}^{ab}\,\frac{1}{\sqrt{2}}\,(\theta^a + \frac{\eta^{aa}}{ik}\theta^b) = k\,\frac{1}{\sqrt{2}}(\theta^a + \frac{\eta^{aa}}{ik}\theta^b)\,,$$
$$\mathbf{S}^{ab}\,\frac{1}{\sqrt{2}}\,(1 + \frac{i}{k}\theta^a\theta^b) = 0\,, \tag{3.21}$$

$k^2 = \eta^{aa}\eta^{bb}$.

We define here the commuting objects γ^a_G, which will be helpful when looking for the appropriate action for Grassmann fermions, Eq. (3.41). These operators will be needed also when looking for the definition of appropriate discrete symmetry operators in the Grassmann case. Following the definition of the discrete symmetry

[3] In Ref. [2] the definition of $\theta^{a\dagger}$ was differently chosen. Correspondingly also the scalar product needed a (slightly) different weight function in Eq. (3.32).

operators in the Clifford algebra case [65] in $((d-1)+1)$ space-time and in $(3+1)$ space-time, the discrete symmetry operators ($\mathcal{C}_{\mathcal{G}}$, $\mathcal{T}_{\mathcal{G}}$, $\mathcal{P}_{\mathcal{G}}$) in $((d-1)+1)$ and ($\mathcal{C}_{\mathcal{NG}}$, $\mathcal{T}_{\mathcal{NG}}$, $\mathcal{P}_{\mathcal{NG}}$) in $(3+1)$ will be defined in Subsct. 3.3.3, respectively.

$$\gamma^a_G = (1 - 2\theta^a \eta^{aa} \frac{\partial}{\partial \theta_a}) = -i\eta^{aa}\gamma^a \tilde{\gamma}^a\,, \quad \{\gamma^a_G, \gamma^b_G\}_- = 0\,. \tag{3.22}$$

Index ^{a} is not the Lorentz index in the usual sense. γ^a_G are commuting operators for all (a, b). They are real and Hermitian.

$$\gamma^{a\dagger}_G = \gamma^a_G\,, \quad (\gamma^a_G)^* = \gamma^a_G\,. \tag{3.23}$$

Correspondingly it follows: $\gamma^{a\dagger}_G \gamma^a_G = I$, $\gamma^a_G \gamma^a_G = I$. I represents the unit operator.

By introducing [2] the generators of the infinitesimal Lorentz transformations in Grassmann space, as presented in Eq. (3.3), and making use of the Cartan subalgebra of commuting operators, Eq. (3.110), the basic states in Grassmann space can be arranged into representations of the eigenstates of the Cartan subalgebra operators, Eq. (3.21), Ref. [2,46]. All these states have integer spins (k is $\pm i$ or ± 1). The starting state in d-dimensional space, for example, with the eigenvalues of the Cartan subalgebra equal to either i or 1 is: $(\theta^0 - \theta^3)(\theta^1 + i\theta^2)(\theta^5 + i\theta^6)\cdots(\theta^{d-1} + i\theta^d)|\phi_{og}>$, with $|\phi_{og}> = |1\ >$, Eq. (3.21). All the states of the representation, which starts with this state, follow by the application of those $\mathbf{S}^{ab}$, which do not belong to the Cartan subalgebra of the Lorentz algebra. $\mathbf{S}^{01}$, for example, transforms this starting state into $(\theta^0\theta^3 + i\theta^1 i\theta^2)(\theta^5 + i\theta^6)\cdots(\theta^{d-1} + i\theta^d)|\phi_{og}>$, while ($\mathbf{S}^{01} - i\mathbf{S}^{02}$) transforms this state into $(\theta^0 + \theta^3)(\theta^1 - i\theta^2)(\theta^5 + i\theta^6)\cdots(\theta^{d-1} + i\theta^d)|\phi_{og}>$.

b. *Fields with the half integer spin in Clifford space*

Let us present as well the properties of the fermion fields with the half integer spin, expressed by the Clifford algebra objects γ^a's ([1,2,9,3,5,4,47] and the references therein)

$$\mathbf{F} = \sum_{k=0}^{d} a_{a_1 a_2 \ldots a_k} \gamma^{a_1}\gamma^{a_2}\ldots\gamma^{a_k}|\psi_{oc}>\,, \quad a_i \le a_{i+1}\,, \tag{3.24}$$

where $|\psi_{oc}>$ is the vacuum state. The *Kalb-Ramond* fields $a_{a_1 a_2 \ldots a_k}$ are again in general boson fields, which are antisymmetric with respect to the permutation of indexes, since the Clifford objects have the anticommutation relations, Eq. (3.2), $\{\gamma^a, \gamma^b\}_+ = 2\eta^{ab}$. The linear vector space over the Clifford coordinate space has, as in the Grassmann case, the dimension 2^d, due to the fact that $(\gamma^{a_i})^2 = \eta^{a_i a_i}$ for any $a_i \in (0, 1, 2, 3, 5, \ldots, d)$.

As written in Eq. (3.4), γ^a are expressible in terms of the Grassmann coordinates and their conjugate momenta, as $\gamma^a = (\theta^a - i\,p^{\theta a})$, and $\tilde{\gamma}^a = i\,(\theta^a + i\,p^{\theta a})$, with the anticommutation relation of Eq. (3.2), $\{\gamma, \gamma^b\}_+ = 2\eta^{ab} = \{\tilde{\gamma}^a, \tilde{\gamma}^b\}_+$, $\{\gamma^a, \tilde{\gamma}^b\}_+ = 0$. Taking into account Eqs. (3.18, 3.19, 3.4) one finds

$$\begin{aligned}
&(\gamma^a)^\dagger = \gamma^a \eta^{aa}\,, \quad (\tilde{\gamma}^a)^\dagger = \tilde{\gamma}^a \eta^{aa}\,,\\
&\gamma^a \gamma^a = \eta^{aa}\,, \quad \gamma^a (\gamma^a)^\dagger = I\,, \quad \tilde{\gamma}^a \tilde{\gamma}^a = \eta^{aa}\,, \quad \tilde{\gamma}^a (\tilde{\gamma}^a)^\dagger = I\,,
\end{aligned} \tag{3.25}$$

where I represents the unit operator. Making a choice for the θ^a properties as presented in Eq. (3.20), it follows for the Clifford objects

$$\begin{aligned}
\{\gamma^a\}^* &= (\gamma^0, \gamma^1, -\gamma^2, \gamma^3, -\gamma^5, \gamma^6, ..., -\gamma^{d-1}, \gamma^d)\,, \\
\{\tilde{\gamma}^a\}^* &= (-\tilde{\gamma}^0, -\tilde{\gamma}^1, \tilde{\gamma}^2, -\tilde{\gamma}^3, \tilde{\gamma}^5, -\tilde{\gamma}^6, ..., \tilde{\gamma}^{d-1}, -\tilde{\gamma}^d)\,,
\end{aligned} \tag{3.26}$$

Applying the operators S^{ab} and $\tilde{S}^{ab}$, Eq. (3.2), on $\frac{1}{2}(\gamma^a + \frac{\eta^{aa}}{ik}\gamma^b)$ and on $\frac{1}{2}(1 + \frac{i}{k}\gamma^a\gamma^b)$, and taking into account the relation of Eq. (3.69), one obtains

$$\begin{aligned}
S^{ab}\frac{1}{2}(\gamma^a + \frac{\eta^{aa}}{ik}\gamma^b) &= \frac{k}{2}\frac{1}{2}(\gamma^a + \frac{\eta^{aa}}{ik}\gamma^b)\,, \\
\tilde{S}^{ab}\frac{1}{2}(\gamma^a + \frac{\eta^{aa}}{ik}\gamma^b) &= \frac{k}{2}\frac{1}{2}(\gamma^a + \frac{\eta^{aa}}{ik}\gamma^b)\,, \\
S^{ab}\frac{1}{2}(1 + \frac{i}{k}\gamma^a\gamma^b) &= \frac{k}{2}\frac{1}{2}(1 + \frac{i}{k}\gamma^a\gamma^b) \\
\tilde{S}^{ab}\frac{1}{2}(1 + \frac{i}{k}\gamma^a\gamma^b) &= -\frac{k}{2}\frac{1}{2}(1 + \frac{i}{k}\gamma^a\gamma^b)\,.
\end{aligned} \tag{3.27}$$

One could make a choice of $\tilde{\gamma}^a$ instead of γ^a and change correspondingly the relations in Eqs. (3.69, 3.27).

All the three choices for the linear vector space — spanned over either the Grassmann θ^a's, or over the vector space of γ^a's, or over the vector space of $\tilde{\gamma}^a$'s — have the dimension 2^d. More about the meaning of these degrees of freedom in any of theses cases can be found in Ref. [11].

Let us point out here that θ^a's and $\frac{\partial}{\partial\theta_a}$'s (each of them has 2^d degrees of freedom) are expressible with γ^a's and $\tilde{\gamma}^a$'s (with 2^d degrees of freedom each) and opposite. Since $\{\gamma^a, \tilde{\gamma}^b\}_+ = 0$, γ^a's and $\tilde{\gamma}^a$'s form independent degrees of freedom. We should therefore allow also $\tilde{\gamma}^a$'s to form the vector space.

We can express Grassmann coordinates θ^a and momenta $p^{\theta a} = i\frac{\partial}{\partial\theta_a}$ in terms of γ^a and $\tilde{\gamma}^a$ as well [4]

$$\begin{aligned}
\theta^a &= \frac{1}{2}(\gamma^a - i\tilde{\gamma}^a)\,, \\
\frac{\partial}{\partial\theta_a} &= \frac{1}{2}\,(\gamma^a + i\tilde{\gamma}^a)\,,
\end{aligned} \tag{3.28}$$

with $\frac{\partial}{\partial\theta_b}\theta^a|1> = \eta^{ab}|1>$.

Requiring that the application of $\tilde{\gamma}^a$'s on γ^a's are determined by Eq. (3.69), the $\tilde{\gamma}^a$'s part is sacrificed [11]. The two possibilities are no longer acceptable: γ^a's are chosen to span the basis, while $\tilde{\gamma}^a$'s become operators which determine the family quantum numbers. From Eqs. (3.28, 3.69) follows that $\frac{\partial}{\partial\theta_b}\theta^a = 0$ and $\theta^a = \gamma$. All the relations of Eq. (3.2) remain valid, while the space of $\tilde{\gamma}^a$'s is sacrificed and the Grassmann space has lost $\frac{\partial}{\partial\theta_b}$, the Hermitian conjugated partner of θ^a.

(Of course, we can still replace γ^a by $\tilde{\gamma}^a$, if we change correspondingly the vacuum state $|\psi_{oc}>$ and relation in Eq. (3.69)).

[4] In Ref. [76] the author suggested in Eq. (47) a choice of superposition of γ^a and $\tilde{\gamma}^a$, which resembles the choice of one of the authors (N.S.M.B.) in Ref. [2] and both authors in Ref. [47,48] and in present article.

The vacuum state $|\phi_{og}>=|1>$ must after Eq. (3.69) be transformed into $|\psi_{oc}>$ with the property [2,7,9]

$$< \psi_{oc}|\gamma^a|\psi_{oc}> = 0\,, \quad \tilde{\gamma}^a|\psi_{oc}>= i\gamma^a|\psi_{oc}>\,, \quad \tilde{\gamma}^a\gamma^b|\psi_{oc}>= -i\gamma^b\gamma^a|\psi_{oc}>\,,$$
$$\tilde{\gamma}^a\tilde{\gamma}^b|\psi_{oc}>|_{a\neq b} = -\gamma^a\gamma^b|\psi_{oc}>\,, \quad \tilde{\gamma}^a\tilde{\gamma}^b|\psi_{oc}>|_{a=b} = \eta^{ab}|\psi_{oc}>\,. \tag{3.29}$$

This is in agreement with the requirement

$$\begin{aligned}\gamma^a\,\mathbf{F}(\gamma)\,|\psi_{oc}>:= &(\,a_0\,\gamma^a + a_{a_1}\,\gamma^a\gamma^{a_1} + a_{a_1a_2}\,\gamma^a\gamma^{a_1}\gamma^{a_2} + \dots + \\ &a_{a_1\cdots a_d}\,\gamma^a\gamma^{a_1}\cdots\gamma^{a_d}\,)\,|\psi_{oc}>\,,\\ \tilde{\gamma}^a\,\mathbf{F}(\gamma)\,|\psi_{oc}>:= &(\,i\,a_0\gamma^a - i\,a_{a_1}\gamma^{a_1}\,\gamma^a + i\,a_{a_1a_2}\gamma^{a_1}\gamma^{a_2}\,\gamma^a + \dots + \\ &i\,(-1)^d\,a_{a_1\cdots a_d}\gamma^{a_1}\cdots\gamma^{a_d}\,\gamma^a\,)\,|\psi_{oc}>\,.\end{aligned} \tag{3.30}$$

The basic states in Clifford space can be arranged in representations, in which any state is the eigenstate of the Cartan subalgebra operators of Eq. (3.110). The state, for example, in d-dimensional space with the eigenvalues of $S^{03}, S^{12}, S^{56}, \dots, S^{d-1\,d}$ and of $\tilde{S}^{03}, \tilde{S}^{12}, \tilde{S}^{56}, \dots, \tilde{S}^{d-1\,d}$ equal to $\frac{1}{2}(i, 1, 1, \dots, 1)$ is $(\gamma^0 - \gamma^3)(\gamma^1 + i\gamma^2)(\gamma^5 + i\gamma^6)\cdots(\gamma^{d-1} + i\gamma^d)$. The states of one representation follow from the starting state by the application of S^{ab}, which do not belong to the Cartan subalgebra operators, while $\tilde{S}^{ab}$, which operate on family quantum numbers, cause jumps from the starting family to the new one.

Norms of vectors in Grassmann and Clifford space Let us look for the norm of vectors in Grassmann space, $\mathbf{B} = \sum_{k=0}^{d} a_{a_1a_2\dots a_k}\,\theta^{a_1}\theta^{a_2}\dots\theta^{a_k}|\phi_{og}>$, and in Clifford space, $\mathbf{F} = \sum_{k=0}^{d} a_{a_1a_2}\dots a_k\,\gamma^{a_1}\gamma^{a_2}\dots\gamma^{a_k}|\psi_{oc}>$, where $|\phi_{og}>$ and $|\psi_{oc}>$ are the vacuum states in the Grassmann and Clifford case, respectively. In what follows we refer to Ref. [2].

a. *Norms of Grassmann vectors*

Let us define the integral over the Grassmann space [2] of two functions of the Grassmann coordinates $<\mathbf{B}|\theta><\mathbf{C}|\theta>$, $<\mathbf{B}|\theta>=<\theta|\mathbf{B}>^\dagger$, by requiring

$$\{d\theta^a, \theta^b\}_+ = 0\,, \quad \int d\theta^a = 0\,, \quad \int d\theta^a\theta^a = 1\,, \quad \int d^d\theta\;\theta^0\theta^1\cdots\theta^d = 1\,,$$
$$d^d\theta = d\theta^d\dots d\theta^0\,, \quad \omega = \Pi_{k=0}^{d}\left(\frac{\partial}{\partial\theta_k} + \theta^k\right), \tag{3.31}$$

with $\frac{\partial}{\partial\theta_a}\theta^c = \eta^{ac}$. We shall use the weight function $\omega = \Pi_{k=0}^{d}(\frac{\partial}{\partial\theta_k} + \theta^k)$ to define the scalar product $<\mathbf{B}|\mathbf{C}>$

$$<\mathbf{B}|\mathbf{C}> = \int d^{d-1}x d^d\theta^a\;\omega <\mathbf{B}|\theta><\theta|\mathbf{C}>= \sum_{k=0}^{d}\int d^{d-1}x\,b^*_{b_1\dots b_k}c_{b_1\dots b_k}\,, \tag{3.32}$$

where, according to Eq. (3.18), it follows:

$$<\mathbf{B}|\theta>= \sum_{p=0}^{d}(-i)^p\;a^*_{a_1\dots a_p}p^{\theta a_p}\,\eta^{a_pa_p}\cdots p^{\theta a_1}\,\eta^{a_1a_1}.$$

The vacuum state is chosen to be $|\phi_{og}>=|1>$, as assumed in Eq. (3.16).

The norm $<\mathbf{B}|\mathbf{B}>$ is correspondingly always nonnegative. Let us notice that the choice of the Hermitian conjugated value of θ^a is $\frac{\partial}{\partial\theta^a}$ ($(\theta^a)^\dagger = \eta^{aa}\frac{\partial}{\partial\theta_a}$, Eq. (3.18)) makes that we easily evaluate in any d the scalar product

$$<\phi_{og}|(\frac{\partial}{\partial\theta^d}\frac{\partial}{\partial\theta^{d-1}}\frac{\partial}{\partial\theta^{d-2}}\cdots\frac{\partial}{\partial\theta^1}\frac{\partial}{\partial\theta^0})(\theta^0\theta^1\cdots\theta^{d-2}\theta^{d-1}\theta^d)|\phi_{og}>=1$$

for $|\phi_{og}>=|1>$ (without integration over coordinate space of θ^a's).

b. *Norms of Clifford vectors*

To evaluate norms in the Clifford space for vectors of Eq. (3.24) we can use as well Eqs. (3.31, 3.32), if expressing γ^a in terms of θ^a and $p^{\theta a}$: $<(\theta^a - ip^{\theta a})|\mathbf{F}>$. In this case $|\psi_{oc}>=|\phi_{og}>=|1>$. It follows

$$<\mathbf{F}|\mathbf{G}> = \int d^{d-1}x d^d\theta^a \;\omega <\mathbf{F}|\gamma><\gamma|\mathbf{G}>= \sum_{k=0}^{d}\int d^{d-1}x\, a^*_{a_1\ldots a_k} b_{b_1\ldots b_k}\,. \tag{3.33}$$

To simplify the evaluation we use instead [3,47] in the Clifford case the vacuum state $|\psi_{oc}>$, Eq. (3.79), which is the product of projectors, Eq. (3.70). It takes care of the orthogonality of states (like if we would evaluate the integration in Grassmann space).

Correspondingly we can write

$$\int d^d\theta^a \;\omega(a_{a_1a_2\ldots a_k}\gamma^{a_1}\gamma^{a_2}\ldots\gamma^{a_k})^\dagger(a_{a_1a_2\ldots a_k}\gamma^{a_1}\gamma^{a_2}\ldots\gamma^{a_k}) = a^*_{a_1a_2\ldots a_k}\,a_{a_1a_2\ldots a_k}\,. \tag{3.34}$$

The norm of each scalar term in the sum of $\mathbf{F}$ is nonnegative.

Actions in Grassmann and Clifford space We construct an action for free massless fermion in which the internal degrees of freedom is described: **i.** in Grassmann space, **ii.** in Clifford space. In the first case the internal degrees of freedom manifest integer spins, in the second case the half integer spin.

While the action in Clifford space is well known since long [67], the action in Grassmann space will be defined here (by N.S.M.B.). In both cases we present an action for free massless fermions in $((d-1)+1)$ space [5]. States in Grassmann space as well as states in Clifford space will be arranged to be the eigenstates of

[5] In $d=(3+1)$ space masses of fermions are in the *spin-charge-family* theory in the Clifford case caused by the interaction of fermions with scalar gauge fields with the space index $(7,8)$, that is the vielbeins and the spin connections of two kinds — the gauge scalar fields of S^{ab} and of $\tilde{S}^{ab}$. We expect that masses of "fermions" appear also in the Grassmann case due to the interaction of fermions with scalar gauge fields with the space index $(7,8)$, but in this case due to the vielbeins and the spin connection of one kind only — the gauge field of $\mathbf{S}^{ab}$

the Cartan subalgebra — with respect to $\mathbf{S}^{ab}$ in Grassmann space and with respect to S^{ab} and $\tilde{S}^{ab}$ in Clifford space, Eq. (3.110), and orthogonal and normalized with respect to Eq. (3.31) [6].

In both spaces the requirement that states are obtained by the application of creation operators on the vacuum state — in the Grassmann case $\hat{b}^{\theta^k \dagger}_i$ on $|1>$, Eq. (3.58), obeying together with the $\hat{b}^{\theta^k}_i$ the anti commutation relations of Eq. (3.54) on the vacuum state $|\phi_{og}>=|1>$, and in the Clifford case $\hat{b}^{\alpha\dagger}_i$, Eq.(3.76), obeying together with the $\hat{b}^{\beta}_i$ the equivalent anticommutation relations of Eq. (3.81) on the vacuum states $|\psi_{oc}>$, Eq. (3.79) — reduces the number of states, in Clifford space more than in Grassmann space. But while in Clifford space all physically applicable states are reachable by either S^{ab} (defining family members quantum numbers) or by $\tilde{S}^{ab}$ (defining family quantum numbers), the states in Grassmann space, belonging to different representations with respect to the Lorentz generators, seem not to be connected.

a. *Action in Clifford space*

In Clifford space the action for a free massless fermion must be Lorentz invariant

$$\mathcal{A} = \int d^d x \, \frac{1}{2} \, (\psi^{\dagger}\gamma^0 \, \gamma^a p_a \psi) + h.c. \,, \tag{3.35}$$

$p_a = i\,\frac{\partial}{\partial x^a}$, leading to the equations of motion

$$\gamma^a p_a |\psi> = 0 \,, \tag{3.36}$$

which fulfill also the Klein-Gordon equation

$$\gamma^a p_a \gamma^b p_b |\psi> = p^a p_a |\psi> = 0 \,, \tag{3.37}$$

for each of the basic states $\hat{b}^{\alpha\dagger}_i |\psi_{0c}> = |\psi^{\alpha}_i>$. γ^0 appears in the action since we pay attention that

$$\begin{aligned} S^{ab\dagger}\,\gamma^0 &= \gamma^0\, S^{ab}\,, \quad S^{\dagger}\gamma^0 = \gamma^0 S^{-1}\,, \\ S &= e^{-\frac{i}{2}\omega_{ab}(S^{ab}+L^{ab})}\,. \end{aligned} \tag{3.38}$$

The Lagrange density, Eq. (3.35),

$$\mathcal{L}_C = \frac{1}{2}\{\psi^{\dagger}\,\gamma^0\,\gamma^a\,\hat{p}_a\psi - \hat{p}_a\psi^{\dagger}\,\gamma^0\,\gamma^a\,\psi\}\,, \tag{3.39}$$

leads to

$$\begin{aligned} \frac{\partial \mathcal{L}_C}{\partial \psi^{\dagger}} - \hat{p}_a\,\frac{\partial \mathcal{L}_C}{\partial \hat{p}_a \psi^{\dagger}} &= 0 = \gamma^0 \gamma^a\,\hat{p}_a\,\psi\,, \\ \frac{\partial \mathcal{L}_C}{\partial \psi} - \hat{p}_a\,\frac{\partial \mathcal{L}_C}{\partial (\hat{p}_a \psi)} &= 0 = -\hat{p}_a\,\psi^{\dagger}\gamma^0\,\gamma^a\,. \end{aligned} \tag{3.40}$$

[6] In the Clifford case the states can be orthogonalized also with respect to Eq. (3.79), while taking into account Eq. (3.71).

All the states, belonging to different values of the Cartan subalgebra — they differ at least in one value of either the set of S^{ab} or the set of $\tilde{S}^{ab}$, Eq. (3.110) — are orthogonal according to the scalar product, defined as the integral over the Grassmann coordinates, Eq. (3.31), for a chosen vacuum state $|1>$. Correspondingly the states generated by the creation operators, Eq. (3.76), on the vacuum state, Eq. (3.79), are orthogonal as well.

b. *Action in Grassmann space*

We define here the action in Grassmann space, for which we require — similarly as in the Clifford case — that the action for a free massless fermion is Lorentz invariant

$$\mathcal{A}_G = \int d^d x \, d^d\theta \; \omega \{\phi^\dagger (1 - 2\theta^0 \frac{\partial}{\partial \theta^0}) \, \frac{1}{2} \, \theta^a p_a \phi\} + h.c.. \tag{3.41}$$

We use the integral over θ^a coordinates with the weight function ω from Eq. (3.31, 3.32). Requiring the Lorentz invariance we add after $\phi^\dagger$ the operator γ^0_G ($\gamma^a_G = (1 - 2\theta^a \frac{\partial}{\partial \theta^a})$), which takes care of the Lorentz invariance. Namely

$$\begin{aligned} \mathbf{S}^{ab\dagger} \, (1 - 2\theta^0 \frac{\partial}{\partial \theta^0}) &= (1 - 2\theta^0 \frac{\partial}{\partial \theta^0}) \, \mathbf{S}^{ab} \,, \\ \mathbf{S}^\dagger \, (1 - 2\theta^0 \frac{\partial}{\partial \theta^0}) &= (1 - 2\theta^0 \frac{\partial}{\partial \theta^0}) \, \mathbf{S}^{-1} \,, \\ \mathbf{S} &= e^{-\frac{i}{2}\omega_{ab}(L^{ab} + \mathbf{S}^{ab})} \,, \end{aligned} \tag{3.42}$$

while θ^a, $\frac{\partial}{\partial \theta_a}$ and p^a transform as Lorentz vectors. The equations of motion follow from the action, Eq. (3.41),

$$\begin{aligned} \frac{1}{2} \gamma^0_G \, (\theta^a - \frac{\partial}{\partial \theta^a}) \, p_a \, |\phi> \, &= 0 \,, \\ \gamma^0_G = (1 - 2\theta^0 \frac{\partial}{\partial \theta^0}) \quad &, \end{aligned} \tag{3.43}$$

as well as the Klein-Gordon equation, $\gamma^0_G \, (\theta^a - \frac{\partial}{\partial \theta^a}) \, p_a \, \gamma^0_G \, (\theta^b - \frac{\partial}{\partial \theta^a}) \, p_b \, |\phi> = 0$, leading to

$$\{\theta^a p_a, \frac{\partial}{\partial \theta_b} \, p_b\}_+ \, = p^a p_a = 0 \,. \tag{3.44}$$

From the Lagrange density, presented in Eq. (3.41), using Eqs. (3.18, 3.19, 3.28) ($\gamma^0_G = -i\eta^{aa}\gamma^a\tilde{\gamma}^a$, $(\theta^a - \frac{\partial}{\partial \theta_a}) = -i\tilde{\gamma}^a$) it follows, up to the surface term,

$$\begin{aligned} \mathcal{L}_G \; &= -i\frac{1}{2}\phi^\dagger \, \gamma^0_G \, \tilde{\gamma}^a \, (\hat{p}_a \phi) \\ &= -i\frac{1}{4}\{\phi^\dagger \, \gamma^0_G \, \tilde{\gamma}^a \, \hat{p}_a \phi \, - \hat{p}_a \phi^\dagger \, \gamma^0_G \, \tilde{\gamma}^a \, \phi\}. \end{aligned} \tag{3.45}$$

One correspondingly finds

$$\begin{aligned} \frac{\partial \mathcal{L}_G}{\partial \phi^\dagger} - \hat{p}_a \, \frac{\partial \mathcal{L}_G}{\partial \hat{p}_a \phi^\dagger} &= 0 = \frac{-i}{2} \gamma^0_G \, \tilde{\gamma}^a \, \hat{p}_a \, \phi \,, \\ \frac{\partial \mathcal{L}_G}{\partial \phi} - \hat{p}_a \, \frac{\partial \mathcal{L}_G}{\partial (\hat{p}_a \phi)} &= 0 = \frac{i}{2} \hat{p}_a \, \phi^\dagger \gamma^0_G \, \tilde{\gamma}^a \,, \end{aligned} \tag{3.46}$$

The solutions of these equations are presented in Eq. (3.98).

We shall see that, if one identifies the creation operators in both spaces with the products of odd numbers of either θ^a — in the Grassmann case — or γ^a — in the Clifford case — and the annihilation operators as the Hermitian conjugated operators of the creation operators, the creation and annihilation operators fulfill the anticommutation relations, required for fermions. The internal parts of states are then defined by the application of the creation operators on the vacuum state.

But while the Clifford subalgebra defines states with the half integer "eigenvalues" of the Cartan subalgebra operators of the corresponding Lorentz algebra, the Grassmann algebra defines states with the integer "eigenvalues" of the Cartan subalgebra operators of the corresponding Lorentz algebra.

3.3 Second quantization of Grassmann and Clifford vectors

It is proven in this section that solutions of the Weyl equations — following from the Hermitian and Lorentz invariant actions for free massless fermions, using to describe their internal degrees of freedom either Clifford space, Eqs. (3.35, 3.36), or Grassmann space, Eq. (3.41, 3.43), — can be represented as creation operators, operating on the appropriate vacuum state. The corresponding Hermitian conjugated operators, taken as their annihilation partners, fulfill together with the creation operators, if both are of an odd either Clifford or Grassmann character, the anticommutation relations required for fermions.

Correspondingly there is no need to assume the anticommutation relations as done in the Dirac theory [67,74,75], since the creation and annihilation operators of an odd either Clifford or Grasmmann character by themselves fulfill the anticommutation relations for fermions without postulating them.

Creation operators in both spaces determine the Hilbert space of n fermions for any integer n and have all the properties of the corresponding Slater determinants, if we recognize that a product of two creation operators of two different moments in the ordinary space (p_k, p_l) — $\hat{b}^{\alpha\dagger}_{i\,p_k} \cdot \hat{b}^{\beta\dagger}_{j\,p_l}$, applying on the vacuum state $|\psi_{oc}>$, are zero if and only if $i = j$, $\alpha = \beta$ and $p_k = p_l$. In the Grassmann case $\hat{b}^{\alpha\dagger}_{i\,p_k} \cdot \hat{b}^{\beta\dagger}_{j\,p_l}$ is replaced by $\hat{b}^{\theta\dagger}_{i\,p_k} \cdot \hat{b}^{\theta\dagger}_{j\,p_l}$ and the vacuum $|\psi_{oc}>$ by $|\psi_{og}>$.

Let us point out that fermions with the internal degrees of freedom described in Clifford space manifests half integer spins, while "fermions" with the internal degrees of freedom described in Grassmann space demonstrate integer spins.

We pay attention in this paper on $d = 2(2n+1)$-dimensional spaces, arranging all the vectors to be "eigenvectors" of the Cartan subalgebra operators of S^{ab} and $\tilde{S}^{ab}$ in the Clifford case and of $\mathbf{S}^{ab}$ in the Grassmann case, Eqs. (3.110, 3.2, 3.3).

In d-dimensional spaces the linear vector space, spanned over either the Clifford coordinates γ^a's or the Grassmann coordinates θ^a's, has the dimension 2^d. One can in both cases represent the vector space as 2^d operators, which — when applied on the vacuum state — create 2^d vectors. Half of these operators have an odd and half an even either Clifford (with respect to odd or even products of γ^a's) or Grassmann (with respect to odd or even products of θ^s's) character.

In the Clifford case there are in the group of an odd Clifford character two groups of operators: each member of one group has its Hermitian conjugated partner in another group. One of the two groups can be therefore chosen to represent the creation operators, the other to represent the corresponding annihilation operators. Each of the two groups has 2^{d-2} members.

Each of the two Clifford odd groups, one with 2^{d-2} creation the other with 2^{d-2} annihilation operators, further divides into $2^{\frac{d}{2}-1}$ subgroups with $2^{\frac{d}{2}-1}$ members. All the $2^{\frac{d}{2}-1}$ members of one particular subgroup are related by the operators S^{ab}, while $\tilde{S}^{ab}$ transform each member of this subgroup of particular family into the same member of one of $2^{\frac{d}{2}-1}$ families.

In the group of the Clifford even operators there are again two groups, each with $2^{\frac{d}{2}-1} \cdot 2^{\frac{d}{2}-1}$ operators related by either S^{ab} or by $\tilde{S}^{ab}$. Within each of the group there are $2^{\frac{d}{2}-1}$ subgroups with $2^{\frac{d}{2}-1}$ members, related by the application of S^{ab}, while $\tilde{S}^{ab}$ transform each member of a particular subgroup into the same member — with respect to the operators S^{ab} — of another subgroup with again $2^{\frac{d}{2}-1}$ members.

These two groups are not related by the Hermitian conjugation as in the case of odd Clifford objects. In each of the two groups of an even Clifford character there are $2^{\frac{d}{2}-1}$ self adjoint operators. The rest of $2^{\frac{d}{2}-1} \cdot (2^{\frac{d}{2}-1} - 1)$ Clifford even operators have the Hermitian conjugated partners within the same group.

$\tilde{\gamma}^a \gamma^a$ transform $2^{\frac{d}{2}-1}$ self adjoint operators of one Clifford even group into $2^{\frac{d}{2}-1}$ self adjoint operators of another Clifford even group, while $\tilde{\gamma}^a \gamma^a$ transform the rest of this group — that is $2^{\frac{d}{2}-1} \cdot (2^{\frac{d}{2}-1} - 1)$ operators, having the Hermitian conjugated partners within the same subgroup — into $2^{\frac{d}{2}-1} \times (2^{\frac{d}{2}-1} - 1)$ operators of another Clifford even group, having again the Hermitian conjugated partners within the same subgroup.

Any odd Clifford member of the assumed (chosen to be) creation operators gives, when applied on one (only one) of the even self adjoint operators of only one of the two groups with $(2^{\frac{d}{2}-1})^2$ members, a nonzero contribution, which is the same creation operator back. It gives nonzero contribution also on one (only one) of the rest $2^{\frac{d}{2}-1} \cdot (2^{\frac{d}{2}-1} - 1)$ operators of the same group to which also the self adjoint operator belong, transforming it to one of creation operators, belonging to another family of the creation operators. On all the others Clifford even objects this creation operator gives zero.

The annihilation operators manifest, when applied on the Clifford even objects, equivalent properties as creation operators.

Let $\hat{b}^{\alpha\dagger}_i$ be the creation operator of an odd Clifford character, α denoting the subgroup with a particular value of the Cartan subalgebra of $\tilde{S}^{ab}$ (family) and with i denoting a particular member of a family α. To all the members of particular α one and only one of the selfadjoint operators of an even Clifford character corresponds, which, when any of these members applies on it, gives the same creation operator back.

$(\hat{b}^{\alpha\dagger}_i)^\dagger = \hat{b}^{\alpha}_i$, denoting the corresponding annihilation operator of an odd Clifford character, gives zero when applied on the selfadjoint operators on which $\hat{b}^{\alpha\dagger}_i$ gives nonzero contribution.

We choose the superposition of these selfadjoint operators to determine the vacuum state in the Clifford case, Eq. (3.79).

All the members of the odd Clifford character, half of them creation operators and half of them annihilation operators, fulfill the anticommutation relations, required for fermions. Correspondingly there are only $2^{\frac{d}{2}-1} \cdot 2^{\frac{d}{2}-1}$ creation operators, determining $2^{\frac{d}{2}-1}$ families with $2^{\frac{d}{2}-1}$ family members each, which when applied on the superposition of selfadjoint operators of one group of Clifford even operators, create fermion states. These creation operators determine n fermions Hilbert space.

In the Grassmann case there are two kinds of operators, θ^a and $\frac{\partial}{\partial\theta^a}$, Hermitian conjugated to each other, Eqs. (3.18, 3.19). If θ^a represent the creation operators, then $\frac{\partial}{\partial\theta^a}$ are the corresponding annihilation operators. Not having the Hermitian conjugated partner with the property that when applying on $|\psi_{og}>=|!>$ gives zero, the identity (I) can not belong either to creation or to annihilation operators.

In $d = 2(2n+1)$-dimensional Grassmann spaces there are correspondingly $2^d - 1$ creation operators. The largest two representations have together $\frac{d!}{\frac{d}{2}!\frac{d}{2}!}$ creation operators and the same number of annihilation operators of an odd Grassmann character, Eq. (3.59), chosen to be eigenstates of the Cartan subalgebra, Eq. (3.110), of $\mathbf{S}^{ab}$. All the irreducible representations of the Grassmann case are decoupled. The application of the creation operators, which are products of $\frac{d}{2}$ θ^a's, on the identity (I) gives them back, while the annihilation operators applied on I give zero.

The $\frac{d!}{\frac{d}{2}!\frac{d}{2}!}$ creation operators split into two by the generators of the Lorentz transformations $\mathbf{S}^{ab}$ unconnected groups, each with $\frac{1}{2}$ $\frac{d!}{\frac{d}{2}!\frac{d}{2}!}$ members.

We introduce common notation for the Clifford and Grassmann case to simplify the discussion: Let $\hat{b}_i^{\alpha\dagger}$ be the creation operator of an odd Grassmann character with $\alpha = (1,2)$ denoting one of the two (by $\mathbf{S}^{ab}$ unconnected) the largest subgroups and let i denotes one of the $\frac{1}{2}$ $\frac{d!}{\frac{d}{2}!\frac{d}{2}!}$ members related among themselves by $\mathbf{S}^{ab}$. We make a choice of the vacuum state in the Grassmann case to be $|\psi_{o\alpha}>=|\ 1>$.

All members of two groups of $\frac{1}{2}$ $\frac{d!}{\frac{d}{2}!\frac{d}{2}!}$ number of creation operators of an odd Grassmann character, and their Hermitian conjugated partners, fulfill the anticommutation relations, required for fermions.

The number of vectors in the Hilbert space of n-fermions depends for a chosen momentum p_k^a on the number of the creation operators, creating a particular fermion in the Clifford case or a particular "fermion" in the Grassmann case.

There are for each p_k^a in the odd Clifford case $2^{\frac{d}{2}-1} \cdot 2^{\frac{d}{2}-1}$ and in the odd Grassmann case (for the two the largest representations) $\frac{d!}{\frac{d}{2}!\frac{d}{2}!}$ creation operators $\hat{\mathbf{b}}_{i\,p_k}^{\alpha\dagger}$ of an odd character — either Clifford odd character, with $\alpha = (1,\cdots,2^{\frac{d}{2}-1}), i = (1,\cdots,2^{\frac{d}{2}-1})$, or Grassmann odd character, with $\alpha = (1,2), i = (1,\cdots,\frac{d!}{\frac{d}{2}!\frac{d}{2}!})$), creating the corresponding single particle states, when applied on the vacuum states $|\psi_o>$ — in the Clifford case is the vacuum state $|\psi_{oc}>$, the superposition of all selfadjoint operators, on which an odd $\hat{\mathbf{b}}_{p_k}^{\alpha i\dagger}$ gives a nonzero contribution, and in the Grassmann case the vacuum state is $|\psi_{og}>=|\ 1>$.

Let the zero fermion state for any p_k^a in either Clifford or Grassmann space, be written as $|\psi_o>:=|0^{\alpha=1}_{i=1\,p_1}, 0^{\alpha=1}_{i=2\,p_1}, 0^{\alpha=1}_{i=3\,p_1}, \ldots, 0^{\alpha=1}_{i_{max}\,p_1}, \ldots, 0^{\alpha=\alpha_{max}}_{i_{max}\,p_1}, \ldots, 0^{\alpha=1}_{i=1\,p_2},$

$0^{\alpha=1}_{i=2\,p_2}, 0^{\alpha 1}_{i=3\,p_2}, \ldots, 0^{\alpha=1}_{i_{max}\,p_2}, \ldots, 0^{\alpha=\alpha_{max}}_{i_{max}\,p_k}, \cdots, \cdots |\psi_o>$, with $|\psi_o>= (|\psi_{oc}>, |\ 1>$), in the Clifford and the Grassmann case, respectively, and $\alpha_{max} = (2^{\frac{d}{2}-1}, 2)$ and $i_{max} = (2^{\frac{d}{2}-1}, \frac{1}{2}\frac{d!}{\frac{d}{2}!\frac{d}{2}!})$, again in the Clifford and the Grassmann case, respectively. Then the vector space with n fermions in the Clifford case or n "fermions" in the Grassmann case, for any n looks like

$$\begin{aligned}
\hat{\mathbf{b}}^{\alpha\dagger}_{i\,p_k} |\psi_o> = & |0^{\alpha=1}_{i=1\,p_1}, 0^{\alpha=1}_{i=2\,p_1}, \ldots, 0^{\alpha=\alpha_{max}}_{i_{max}\,p_1}, \ldots, 1^{\alpha}_{i\,p_k} \cdots, |\psi_o> \\
& \ldots \\
& \text{there are } \alpha_{max}\cdot i_{max} \text{ such } 1-\text{fermion states for each } p_k\,, \\
& \hat{\mathbf{b}}^{\alpha\dagger}_{i\,p_k}\, \hat{\mathbf{b}}^{\alpha\dagger}_{j\,p_k} |\psi_o>, \\
& \ldots \\
& \ldots \\
& \Pi_{\alpha=1,\alpha_{max}} \Pi_{i=1,i_{max}} \hat{\mathbf{b}}^{\alpha\dagger}_{i\,p_k} |\psi_o>,\,, \\
& \ldots \\
& \Pi_{\alpha=1,\alpha_{max}} \Pi_{i=1,i_{max}} \hat{\mathbf{b}}^{\alpha\dagger}_{i\,p_l} |\psi_o>,\,, \\
& \ldots \\
& \text{there are } 2^{\alpha_{max}\cdot i_{max}} \text{ Slater determinants of fermions for each } p_k\,, \\
& \ldots
\end{aligned} \tag{3.47}$$

$\alpha_{max} = (2^{\frac{d}{2}-1}, 2)$ and $i_{max} = (2^{\frac{d}{2}-1}, \frac{1}{2}\frac{d!}{\frac{d}{2}!\frac{d}{2}!})$ in the Clifford and Grassmann case, respectively.

One sees that

$$\begin{aligned}
& \hat{\mathbf{b}}^{\alpha\dagger}_{i\,p_k}\, \hat{\mathbf{b}}^{\beta\dagger}_{j\,p_l} |0^{\alpha=1}_{i=1\,p_1}, 0^{\alpha=1}_{i=2\,p_1}, \ldots, \\
& \quad 1^{\alpha'}_{i'p_{k'}}, \ldots, 0^{\alpha'''}_{i'''=1\,p_{k'''}}, \ldots, 1^{\alpha^{iv}}_{i^{iv}p_{k^{iv}}}, \ldots, 1^{\beta'}_{j'\,p_{l'}}, \ldots, |\psi_o>= \\
& -\hat{\mathbf{b}}^{\beta\dagger}_{j\,p_l}\, \hat{\mathbf{b}}^{\alpha\dagger}_{i\,p_k} |0^{\alpha=1}_{i=1\,p_1}, 0^{\alpha=1}_{i=2\,p_1}, \ldots, 1^{\alpha'}_{i'p_{k'}}, \ldots, 0^{\alpha'''}_{i'''=1\,p_{k'''}}, \ldots 1^{\alpha^{iv}}_{i^{iv}p_{k^{iv}}}, \ldots, \\
& \qquad 1^{\beta'}_{j'\,p_{l'}}, \ldots, |\psi_o>,
\end{aligned} \tag{3.48}$$

and is zero only if any of the occupied states is the same as one (or both) of the two states determined by $\hat{\mathbf{b}}^{\alpha\dagger}_{i\,p_k}$ or $\hat{\mathbf{b}}^{\beta\dagger}_{j\,p_l}$ applied on $|\psi_o>$ [7].

[7] Each single particle state caries its own internal space, described by a creation operator with a superposition of an odd number of γ^a_i's, and its own coordinate space, described by x^a_i's (or p^a_i). The creation operators of any two pairs of particles therefore anti-commute. Correspondingly the two states of two particles must distinguish in either internal space or in the coordinate space, as it follows from Eq. (3.86). The property of the creation operators $\hat{b}^{\alpha\dagger}_{spi}\hat{b}^{\alpha'\dagger}_{s'p'j}$ applying on the n-particle state $|1^{\alpha}_{sp1}, 1^{\alpha'}_{s'p'2}, 1^{\alpha"}_{s"p"3}, \ldots, 0^{\alpha"'}_{s"'p_i\,i}, \ldots, 0^{\alpha^{iv}}_{s^{iv}p_j\,j}, \ldots, >$,presented in Eq. (3.86), can be as well described by (superposition of) Slater determinants of single particle states. Let us add that the vacuum state, having the sum of the spins of both kinds of operators, S^{ab} and $\tilde{S}^{ab}$, equal to zero and therefore neutral, remains neutral also when filled with fermions of all the spins, S^{ab} and $\tilde{S}^{ab}$.

One fermion states are either in Clifford or in Grassmann space already second quantized, since in both cases they fulfill the anticommutation relations required for fermions, Eqs. (3.66, 3.87).

All together there are $2^{2^{d-2}}$ Slater determinants for a chosen p_k in the Clifford case and $2^{\frac{d!}{\frac{d}{2}!\frac{d}{2}!}}$ Slater determinants for a chosen p_k in the Grassmann case (if only the two largest group of odd irreducible representations are taken into account, if we take all odd representations into account we have $2^{2^{d-1}}$ Slater determinants), p_k has a continuously changing value, $p^0 = (0, \infty)$, $-\infty \leq p^l \leq \infty$, $l = (1, 2, 3, 5, \cdot, d)$.

It can be concluded that there are only second quantized states, since the anticommuting creation and annihilation operators, creating a Clifford fermion or Grassmann "fermion" states, determine all the properties of the n-particle Hilbert space for any n.

We shall as well recognize that no Dirac sea is needed either in the Clifford or in the Grassmann case, since the same Lorentz representation includes in both cases fermions and antifermions.

We discuss in the subsections the second quantization procedure in both spaces, Clifford and Grassmann, when dimension of the space-time is larger than four. We demonstrate that if the dynamics manifests only in $d = (3 + 1)$, that is when momentum is different from zero only in $d = (3 + 1)$, $p^a = (p^0, p^1, p^2, p^3, 0, 0, \cdots, 0)$ — what happens at low energies after the break of Lorentz symmetries in $d \geq 5$ — spins in $d \geq 5$ manifest as charges in $d = (3 + 1)$.

While the Clifford case offers the explanation for all the properties of observable fermions (after sacrificing the space of $\tilde{\gamma}^a$'s), the Grassmann case, having difficulties in describing energy within the usual second quantized procedure, as long as the Lorentz invariance in internal space is unbroken, leads to unobserved "fermions" with integer spins.

Let us point out that states in Grassmann space as well as states in Clifford space are organized to be — within each of the two spaces — orthogonal and normalized with respect to Eq. (3.31, 3.32, 3.33). All the states in each of spaces are chosen to be eigenstates of the Cartan subalgebra — with respect to $\mathbf{S}^{ab}$ in Grassmann space, Eqs. (3.3, 3.5, 3.110), and with respect to S^{ab} and $\tilde{S}^{ab}$, Eq. (3.2), in Clifford space, Eq. (3.110).

We pay attention in this paper almost only to spaces with $d = 2(2n + 1)$ [8].

3.3.1 Second quantization in Grassmann space

There are 2^d states in Grassmann space, orthogonal to each other with respect to Eqs. (3.31, 3.32). To any coordinate there exists the conjugate momentum. We pay attentionin what follows mostly to spaces with $d = 2(2n + 1)$. The states, which

[8] The main reason that we treat here mostly $d = 2(2n + 1)$ spaces is that one Weyl representation, expressed by the product of the Clifford algebra objects, manifests in $d = (1 + 3)$ all the observed properties of quarks and leptons, if $d \geq 2(2n + 1), n = 3$, and that the breaks of the starting symmetry down to $d = (3 + 1)$ can lead to massless fermions [68,69].

contribute in the second quantization procedure and manifest anticommutation relations required for fermions, are Grassmann odd products of eigenstates of the Cartan subalgebra, Eq. (3.110), of the Lorentz algebra. In $d = 2(2n+1)$ spaces there are two Grassmann odd irreducible representations of the Lorentz algebra with the largest number of members, divided into two separated groups of $\frac{1}{2}\frac{d!}{\frac{d}{2}!\frac{d}{2}!}$ members, Eq. (3.59). (All states of one group are reachable from a starting state by the application of $\mathbf{S}^{ab}$.) Any Grassmann odd state can be written as a creation operator, operating on the vacuum state, while the Hermitian conjugated creation operator is the corresponding annihilation operator. Creation and annihilation operators of an odd Grassmann character fulfill the anticommutation relations of Eq. (3.50, 3.54). Let us see how it works.

If $\hat{b}^{\theta\dagger}_i$ is a creation operator, which creates a state in the Grassmann space when operating on a vacuum state $|\psi_{og}>$ and $\hat{b}^{\theta}_i = (\hat{b}^{\theta\dagger}_i)^\dagger$ is the corresponding annihilation operator, then for a set of creation operators $\hat{b}^{\theta\dagger}_i$ and the corresponding annihilation operators $\hat{b}^{\theta}_i$ it must be

$$\begin{aligned} \hat{b}^{\theta}_i|\phi_{og}> &= 0\,,\\ \hat{b}^{\theta\dagger}_i|\phi_{og}> &\neq 0\,. \end{aligned} \tag{3.49}$$

We first pay attention on only the internal degrees of freedom of the Grassmann "fermions": the spin in any dimension $d = 2(2n+1)$, n is a positive integer.

Choosing $\hat{b}^{\theta\dagger}_a = \theta^a$, then it follows that $(\hat{b}^{\theta\dagger}_a)^\dagger = \frac{\partial}{\partial\theta^a}$, Eqs. (3.18, 3.19). One correspondingly finds

$$\begin{aligned} \hat{b}^{\theta\dagger}_a = \theta^a\,, \quad \hat{b}^{\theta}_a &= \frac{\partial}{\partial\theta^a}\,,\\ \{\hat{b}^{\theta}_a, \hat{b}^{\theta\dagger}_b\}_+|\phi_{og}> &= \delta_{ab}|\phi_{og}>\,,\\ \{\hat{b}^{\theta}_a, \hat{b}^{\theta}_b\}_+|\phi_{og}> &= 0\,,\\ \{\hat{b}^{\theta\dagger}_a, \hat{b}^{\theta\dagger}_b\}_+|\phi_{og}> &= 0\,,\\ \hat{b}^{\dagger\theta}_a\,|\phi_{og}> &= \theta^a|\phi_{og}>\,,\\ \hat{b}^{\theta}_a\,|\phi_{og}> &= 0\,. \end{aligned} \tag{3.50}$$

The vacuum state $|\phi_{og}>$ will in this case be chosen as $|\phi_{og}>=|\,1\,>$.

The number operator $\hat{N}^{\theta}_a = \hat{b}^{\theta\dagger}_a\hat{b}^{\theta}_a$ has the property, due to thefirst line in Eq. (3.49) and the second line in Eq. (3.50), that $(\hat{N}^{\theta}_a)^2 = \hat{N}^{\theta}_a$, with the eigenvalue 0 or 1.

The identity I ($I^\dagger = I$) can not be taken as a creation operator, since its annihilation partner does not fulfill Eq. (3.49). The identity is obviously selfadjoint operator determining the vacuum state $|\phi_{og}>=|\,1\,>$.

We can use the superposition of products of θ^a's as creation operators and the corresponding superposition of products of $\frac{\partial}{\partial\theta_a}$'s as the corresponding annihilation operators, provided that they fulfill the requirements for the creation and annihilation operators, Eq. (3.54), with the vacuum state $|\phi_{og}>=|1\,>$. In general they would not. Only an odd number of θ^a in any superposition would have the required anticommutation properties.

To construct creation operators it is convenient to take products of such superposition of vectors θ^a and θ^b that each factor is the "eigenstate" of one of the Cartan subalgebra members of the Lorentz algebra (3.110). Let us start in $d = 2(2n+1)$ with the creation operator, which is a product of $\frac{d}{2}$ "eigenstates" of an odd Grassmann character of the Cartan subalgebra S^{ab} $\frac{1}{\sqrt{2}}(\theta^a + \frac{\eta^{aa}}{ik}\theta^b) = k\frac{1}{\sqrt{2}}(\theta^a + \frac{\eta^{aa}}{ik}\theta^b)$, Eq. (3.21). Then the corresponding annihilation is a product of $\frac{d}{2}$ of the corresponding factors $\frac{1}{\sqrt{2}}(\frac{\partial}{\partial\theta^a} + \frac{\eta^{aa}}{-ik}\frac{\partial}{\partial\theta_b})$, In both cases (a, b) belong to $(0,3), (1,2), (5,6), \cdots, (d-1, d)$.

Let us in $d = 2(2n+1)$, n is a positive integer, start with the state

$$
\begin{aligned}
|\phi^1_1> &= (\frac{1}{\sqrt{2}})^{\frac{d}{2}} (\theta^0 - \theta^3)(\theta^1 + i\theta^2)(\theta^5 + i\theta^6)\cdots(\theta^{d-1} + i\theta^d)\hat{b}^{\theta 1}_1 |1>, \\
&= \hat{b}^{\theta 1\dagger}_1 |1>, \quad \text{with} \\
\hat{b}^{\theta 1\dagger}_1 &:= (\frac{1}{\sqrt{2}})^{\frac{d}{2}} (\theta^0 - \theta^3)(\theta^1 + i\theta^2)(\theta^5 + i\theta^6)\cdots(\theta^{d-1} + i\theta^d). \qquad (3.51)
\end{aligned}
$$

One finds for the eigenvalues of the Cartan subalgebra operators, Eq. (3.110), the values $(+i, +1, +1, \cdots +1)$.

The rest of states, belonging to the same Lorentz irreducible representation, follow from the starting state by the application of the operators $\mathbf{S}^{cf}$, which do not belong to the Cartan subalgebra operators.

One can find creation and annihilation operators for $d = 4n$ in App. 3.5.

i. We proposed in Eq. (3.51) the starting creation operator $\hat{b}^{\theta 1\dagger}_1$, the upper index indicates one of the two groups, the lower index indicates the starting member. By taking into account Eqs. (3.18, 3.19) the starting creation operator and its annihilation partner are for $d = 2(2n+1)$ equal to

$$
\begin{aligned}
\hat{b}^{\theta 1\dagger}_1 &= (\frac{1}{\sqrt{2}})^{\frac{d}{2}} (\theta^0 - \theta^3)(\theta^1 + i\theta^2)(\theta^5 + i\theta^6)\cdots(\theta^{d-1} + i\theta^d), \\
\hat{b}^{\theta 1}_1 &= (\frac{1}{\sqrt{2}})^{\frac{d}{2}} (\frac{\partial}{\partial\theta^{d-1}} - i\frac{\partial}{\partial\theta^d})\cdots(\frac{\partial}{\partial\theta^0} - \frac{\partial}{\partial\theta^3}), \\
&\text{for } d = 2(2n+1). \qquad (3.52)
\end{aligned}
$$

The rest of creation operators belonging to this group (group 1) in $d = 2(2n+1)$ follow by the application of operators $\mathbf{S}^{ef}$. The corresponding annihilation operators are the Hermitian conjugated partners of the corresponding of creation operators. For $d = 2(2n+1)$ one finds by the application of $\mathbf{S}^{01}$ another creation operator and the corresponding annihilation operator as follows

$$
\begin{aligned}
\hat{b}^{\theta 1\dagger}_2 &= (\frac{1}{\sqrt{2}})^{\frac{d}{2}-1} (\theta^0\theta^3 + i\theta^1\theta^2)(\theta^5 + i\theta^6)\cdots(\theta^{d-1} + i\theta^d), \\
\hat{b}^{\theta 1}_2 &= (\frac{1}{\sqrt{2}})^{\frac{d}{2}-1} (\frac{\partial}{\partial\theta^{d-1}} - i\frac{\partial}{\partial\theta^d})\cdots(\frac{\partial}{\partial\theta^3}\frac{\partial}{\partial\theta^0} - i\frac{\partial}{\partial\theta^2}\frac{\partial}{\partial\theta^1}), \\
&\text{in general}: \\
\hat{b}^{\theta 1\dagger}_i &\propto \mathbf{S}^{ab}\cdots\mathbf{S}^{ef}\hat{b}^{\theta 1\dagger}_1, \\
\hat{b}^{\theta 1}_i &= (\hat{b}^{\theta 1\dagger}_i)^\dagger. \qquad (3.53)
\end{aligned}
$$

It was taken into account in the above equation that any $\mathbf{S}^{ac}$ ($a \neq c$), which does not belong to the Cartan subalgebra, Eq.(3.110), transforms $(\frac{1}{\sqrt{2}})^2(\theta^a + i\theta^b)(\theta^c + i\theta^d)$ ($a \neq c$ and $a \neq d$, $b \neq c$ and $b \neq d$, $\eta^{aa} = \eta^{bb}$) into $\frac{1}{\sqrt{2}}(\theta^a\theta^b + \theta^c\theta^d)$. The states are normalized and the simplest phases are assumed. One evaluates that either $\mathbf{S}^{ab}$ or $\mathbf{S}^{cd}$, applied on $(\theta^a\theta^b \pm \theta^c\theta^d)$, gives zero. The vacuum state is in all these cases $|\,1\,>$.

All the creation operators of an odd Grassmann character — the Grassmann even S^{ac} does not change the oddness of the creation operators and neither do the Hermitian conjugation — fulfill the anticommutation relations

$$
\begin{aligned}
\{\hat{b}^{\theta k}_i, \hat{b}^{\theta l\dagger}_j\}_+ |\phi_{og}> &= \delta_{ij}\, \delta_{kl}\, |\phi_{og}>\,,\\
\{\hat{b}^{\theta k}_i, \hat{b}^{\theta l}_j\}_+ |\phi_{og}> &= 0\, |\phi_{og}>\,,\\
\{\hat{b}^{\theta k\dagger}_i, \hat{b}^{\theta l\dagger}_j\}_+ |\phi_{og}> &= 0\, |\phi_{og}>\,,\\
\hat{b}^{\theta k\dagger}_i |\phi_{og}> &= |\phi^k_i>\,,\\
\hat{b}^{\theta k}_j |\phi_{og}> &= 0\, |\phi_{og}>\,,\\
(k, l) &= (1, 2)\,.
\end{aligned}
\tag{3.54}
$$

Since there is another group of states, presented in Eq. (3.56), not reachable from the starting state by $\mathbf{S}^{ab}$, we denote, to generalize the notation, creation operator with $\hat{b}^{\theta k\dagger}_i$ and the annihilationoperator with $\hat{b}^{\theta k}_i$.

It is not difficult to see that states included into one representation, which started with $\hat{b}^{\theta 1\dagger}_i\,|1\,>$ as presented in Eq. (3.52) for $d = (2n+1)2$ *have the properties, required by* Eq. (3.54) for $k = 1$:

i.a. In any d-dimensional space the product $\frac{\partial}{\partial\theta^{a_1}} \cdots \frac{\partial}{\partial\theta^{a_k}}$, with all different a_i (if all or some of them are equal, then this is trivially true since $(\frac{\partial}{\partial\theta^a})^2 = 0$), if applied on the vacuum $|1\,>$, is equal to zero. Correspondingly the second equation and the fifth equation of Eq. (3.54) are fulfilled.

i.b. In any d-dimensional space the product of different θ^as — $\theta^{a_1}\theta^{a_2}\cdots\theta^{a_k}$ with all different θ^a's ($a_i \neq a_j$ for all a_i and a_j) — applied on the vacuum $|\,1\,>$, is different from zero. Since all the θ's, appearing in Eqs. (3.52, 3.53), are different, forming orthogonal and normalized states, the fourth equation of Eq. (3.54) is fulfilled.

i.c. The third equation of Eq. (3.54) is fulfilled provided that there is an odd number of θ^s in the expression for a creation operator. Then, when in the anticommutation relation different θ^a's appear (like in the case of $d = 6$ $\{\theta^0\theta^3\theta^5, \theta^1\theta^2\theta^6\}_+$), such a contribution gives zero. When two or several equal θ's appear in the anticommutation relation, the contribution is zero (since $(\theta^a)^2 = 0$).

i.d. Also for the first equation in Eq. (3.54) it is not difficult to show that it is fulfilled only for a particular creation operator and its Hermitian conjugated partner: Let us show this for $d = (3+1)$ and the creation operator $\frac{1}{\sqrt{2}}(\theta^0 - \theta^3)\,\theta^1\theta^2$ and its Hermitian conjugate (annihilation) operator: $\frac{1}{\sqrt{2}}\{\frac{\partial}{\partial\theta^2}\,\frac{\partial}{\partial\theta^1}\,(\frac{\partial}{\partial\theta^0} - \frac{\partial}{\partial\theta^3}), \frac{1}{\sqrt{2}}(\theta^0 - \theta^3)\,\theta^1\theta^2\}_+$. Applying $(\frac{\partial}{\partial\theta^0} - \frac{\partial}{\partial\theta^3})$ on $(\theta^0 - \theta^3)$ gives two, while $\frac{\partial}{\partial\theta^2}\,\frac{\partial}{\partial\theta^1}$ applied on $\theta^1\theta^2$ gives one.

i.e. If we define the number operator $N^{\theta k}_i$ as follows

$$\hat{N}^{\theta k}_i = \hat{b}^{\theta k\dagger}_i \hat{b}^{\theta k}_i\,, \qquad (3.55)$$

it follows, taking into account the third equation of Eq. (3.54), that $(\hat{N}^{\theta k}_i)^2 = \hat{b}^{\theta k\dagger}_i \hat{b}^{\theta k}_i \hat{b}^{\theta k\dagger}_i \hat{b}^{\theta k}_i = \hat{N}^{\theta k}_i$, requiring that the eigenvalue of this operator $\hat{N}^{\theta k}_i$ on the state $\hat{b}^{\theta k\dagger}_i |\phi^{k'}_{i'} >$ is 0 or 1.

ii. There is one additional irreducible representation of creation and annihilation operators in $d = 2(2n+1)$, which follows from the starting state

$$\begin{aligned}
&|\phi^2_1 > = \hat{b}^{\theta 2\dagger}_{01} |1>\,,\\
&\hat{b}^{\theta 2\dagger}_{01} := (\frac{1}{\sqrt{2}})^{\frac{1}{2}} (\theta^0 + \theta^3)(\theta^1 + i\theta^2)(\theta^5 + i\theta^6)\cdots(\theta^{d-3} + i\theta^{d-2})(\theta^{d-1} + i\theta^d)\,,\\
&\quad\text{for } d = 2(2n+1)\,. \qquad (3.56)
\end{aligned}$$

This state can not be obtained from the previous group of states, presented in Eqs. (3.52, 3.53) by the application of $\mathbf{S}^{ef}$, since each $\mathbf{S}^{ef}$ changes an even number of factors, never an odd one. All the other states of this new group of states follow from the starting one by the application of $\mathbf{S}^{ef}$. The corresponding creation and annihilation operators are

$$\begin{aligned}
\hat{b}^{\theta 2\dagger}_1 &= (\frac{1}{\sqrt{2}})^{\frac{d}{2}} (\theta^0 + \theta^3)(\theta^1 + i\theta^2)(\theta^5 + i\theta^6)\cdots(\theta^{d-1} + i\theta^d)\,,\\
\hat{b}^{\theta 2}_1 &= (\frac{1}{\sqrt{2}})^{\frac{d}{2}} (\frac{\partial}{\partial\theta^{d-1}} - i\frac{\partial}{\partial\theta^d})\cdots(\frac{\partial}{\partial\theta^0} + \frac{\partial}{\partial\theta^3})\,,\\
&\text{for } d = 2(2n+1)\,. \qquad (3.57)
\end{aligned}$$

The corresponding annihilation operators follow by the Hermitian conjugation of the creation operators.

$$\begin{aligned}
\hat{b}^{\theta 2\dagger}_i &\propto \mathbf{S}^{ab}\cdots\mathbf{S}^{ef}\hat{b}^{\theta 2\dagger}_1\,,\\
\hat{b}^{\theta 2}_i &= (\hat{b}^{\theta 2\dagger}_i)^\dagger\,. \qquad (3.58)
\end{aligned}$$

Also all these creation and annihilation operators fulfill the requirements for the creation and annihilation operators, presented in Eq. (3.54), due to the same reasons as in the first case.

It is true also in this case, as stated below Eq. (3.55), that $\hat{N}^{\theta k}_i$ applied on the state $|\phi^k_i >$ gives 0 or 1, due to the fact that $(\hat{N}^{\theta k}_i)^2 = \hat{N}^{\theta k}_i$. Thus the basic states, determined by the application of creation operators of Eqs. (3.53, 3.58) on the vacuum state $|1>$ have the properties required for fermions.

Let us now count the number of states in each of the two groups presented in Eqs. (3.53, 3.58).

There are in $(d = 2)$ two creation $((\theta^0 \mp \theta^1$, for $\eta^{ab} = diag(1,-1))$ and correspondingly two annihilation operators $(\frac{\partial}{\partial\theta^0} \mp \frac{\partial}{\partial\theta^1})$, each belonging to its own group with respect to the Lorentz transformation operators, both fulfilling Eq. (3.54).

It is not difficult to see that the number of all creation operators of an odd Grassmann character in $d = 2(2n+1)$-dimensional space, with all γ^a's included is equal to $\frac{d!}{\frac{d}{2}!\frac{d}{2}!}$.

We namely ask: In how many ways can one put on $\frac{d}{2}$ places d different θ^a's. And the answer is — the central binomial coefficient for $x^{\frac{d}{2}}\, 1^{\frac{d}{2}}$ — with all x different. This is just $\frac{d!}{\frac{d}{2}!\frac{d}{2}!}$. But we have counted all the states with an odd Grassmann character, while we know that these states belong to two different groups of representations with respect to the Lorentz group.

Correspondingly one concludes: *There are two groups of states in* $d = 2(2n+1)$ *with an odd Grassmann character with all* θ^a*'s included, each of these two groups has*

$$\frac{1}{2}\,\frac{d!}{\frac{d}{2}!\frac{d}{2}!} \tag{3.59}$$

members.

In $d = 2$ we have two groups with one state, which have an odd Grassmann character, in $d = 6$ we have two groups of 10 states, in $d = 10$ we have two groups of 126 states with an odd Grassmann characters. And so on. All together there are 2^{d-1} the states of an odd Grassmann character.

Correspondingly we have in $d = 2(2n+1)$-dimensional spaces two groups of creation operators of the kind presented in Eqs. (3.53, 3.58), each kind with $\frac{1}{2}\,\frac{d!}{\frac{d}{2}!\frac{d}{2}!}$ members, creating states with an odd Grassmann character and the same number of annihilation operators. Creation and annihilation operators fulfill anticommutation relations presented in Eq. (3.54).

The rest of creation operators [and the corresponding annihilation operators] with the opposite Grassmann character than the ones studied so far — like $\theta^0\theta^1$ $[\frac{\partial}{\partial\theta^1}\frac{\partial}{\partial\theta^0}]$ in $d = (1+1)$ $(\theta^0 \mp \theta^3)(\theta^1 \pm i\theta^2)$ $[(\frac{\partial}{\partial\theta^1} \mp i\frac{\partial}{\partial\theta^2})(\frac{\partial}{\partial\theta^0} \mp \frac{\partial}{\partial\theta^3}]$, $\theta^0\theta^3\theta^1\theta^2$ $[\frac{\partial}{\partial\theta^2}\frac{\partial}{\partial\theta^1}\frac{\partial}{\partial\theta^3}\frac{\partial}{\partial\theta^0}]$ in $d = (3+1)$, do not fulfill the anticommutation relations required for fermions in Eq. (3.54), with $\hat{b}^{\theta 1}_i$ and $\hat{b}^{\theta 1\dagger}_i$ replaced by $\hat{b}^{\theta k}_i$ and $\hat{b}^{\theta k\dagger}_i$, $k = (1,2)$ and correspondingly with $\{\hat{b}^{\theta k}_i, \hat{b}^{\theta l\dagger}_j\}|\phi_{og}>= \delta_{kl}\,\delta_{ij}|\phi_{og}>$, $(k,l) = (1,2), (i,j)$ running from $(1,\ldots,\frac{1}{2}\,\frac{d!}{\frac{d}{2}!\frac{d}{2}!})$.

All the states $|\phi^k_i>$, $k = (1,2)$, generated by the creation operators, Eqs. (3.54, 3.58), on the vacuum state $|\phi_{og}>$ $(= |1\;>)$ are the eigenstates of the Cartan subalgebra operators and are orthogonal and normalized with respect to the norm of Eq. (3.31)

$$\begin{aligned} &< \phi^k_i|\phi^{k'}_j > = \delta_{ij}\,\delta^{kk'}\,, \\ &(k,k') = (1,2)\,,\; (i,j) = (1,2,\ldots,\frac{1}{2}\frac{d!}{\frac{d}{2}!\frac{d}{2}!})\,. \end{aligned} \tag{3.60}$$

All these basic states describing the internal degrees of freedom can be used to solve Eq. (3.43) for free massless "fermions", with the part in ordinary space proportional to $e^{-ip^a x_a}$. The eigenstates of Eq. (3.43) are superposition of the basic states $|\phi^k_i >$ with coefficients depending on momentum p^a, $a = (0, 1, 2, 3, 5, \ldots, d)$

$$\begin{aligned}
\hat{b}^{\theta k\dagger}_{sp} &= \sum_i c^k{}_{spi}\, \hat{b}^{\theta k\dagger}_i ,\\
|\phi^k_{sp} > &= \hat{b}^{\theta k\dagger}_{sp}\, |\phi_{og} > ,\\
|\phi^k_{sp} > &= \sum_i c^k{}_{spi}\, |\phi^k_i > ,
\end{aligned} \tag{3.61}$$

s represents different solutions of the equations of motion, and, since they are orthogonalized, they fulfill the relation $< \phi^k_{sp}|\phi^{k'}_{s'p'} >= \delta_{kk'}\, \delta_{ss'}\, \delta^{pp'}$, where we assumed the discretization of momenta.

The corresponding creation operators, creating the basic states describing free massless "fermions" — $\hat{b}^{\theta k\dagger}_{sp}$ — are superposition of creation operators $\hat{b}^{\theta k\dagger}_i$, $\hat{b}^{\theta k\dagger}_{sp} = \sum_i c^k{}_{spi} \hat{b}^{\theta k\dagger}_i$ and fulfill together with the corresponding annihilation operators $\hat{b}^{\theta k}_{sp} = (\hat{b}^{\theta k\dagger}_{sp})^\dagger$ the relations

$$\begin{aligned}
\{\hat{b}^{\theta k}_{sp}, \hat{b}^{\theta k'\dagger}_{s'p'}\}_+ |\phi_{og} > &= \delta^k_{k'}\, \delta_{ss'} \delta_{pp'}\, |\phi_{og} > ,\\
\{\hat{b}^{\theta k}_{sp}, \hat{b}^{\theta k'}_{s'p'}\}_+ |\phi_{og} > &= 0\, |\phi_{og} > ,\\
\{\hat{b}^{\theta k\dagger}_{sp}, \hat{b}^{\theta k'\dagger}_{s'p'}\}_+ |\phi_{og} > &= 0\, |\phi_{og} > ,\\
\hat{b}^{\theta k}_{sp} |\phi_{og} > &= 0\, |\phi_{og} > ,\\
\hat{b}^{\theta k\dagger}_{sp} |\phi_{og} > &= |\phi^k_{sp} > ,\\
|\phi_{og} > &= |1 > \; .
\end{aligned} \tag{3.62}$$

Again index $k = (1, 2)$ in ($\hat{b}^{\theta 1}_{sp}$, $\hat{b}^{\theta 1\dagger}_{sp}$) ($\hat{b}^{\theta 2}_i$, $\hat{b}^{\theta \dagger 2}_i$) denotes creation and annihilation operators of one of the two groups of states describing the internal space of "fermions", reachable by $\mathbf{S^{ab}}$, and $\hat{b}^{\theta k\dagger}_{sp}$ creates the state for a particular momentum in ordinary space p^a, solving Eq. (3.43).

The number operator for a "fermion" state $|\phi^k_{sp} >$ is now

$$\begin{aligned}
\hat{N}^{\theta k}_{sp} &= \hat{b}^{\theta k\dagger}_{sp} \hat{b}^{\theta k}_{sp} ,\\
(\hat{N}^{\theta k}_{sp})^2 &= \hat{N}^{\theta k}_{sp} ,
\end{aligned} \tag{3.63}$$

with the eigenvalues 0 or 1, since the states of a chosen discretized p^a are orthogonal. Correspondingly each state can be occupied or empty. If $|1^{\theta k}_{s_1 p_1}, 1^{\theta k}_{s_2 p_2}, 1^{\theta k}_{s_3 p_3}, \ldots, 0^{\theta k}_{s_k p_k}, \ldots, 0^{\theta k}_{s_l p_l}, \ldots, >$ is a n particle state of "fermions" (and "antifermions"), where 1 denotes the occupied state and 0 the unoccupied state, then it follows, for example, due to the third line in Eq. (3.62), that

$$\begin{aligned}
&\hat{b}^{\theta k\dagger}_{s_i p_i} \hat{b}^{\theta k\dagger}_{s_j p_j} |1^{\theta k}_{s_1 p_1}, 1^{\theta k}_{s_2 p_2}, 1^{\theta k}_{s_3 p_3}, \ldots, 0^{\theta k}_{s_i p_i}, \ldots, 0^{\theta k}_{s_j p_j}, \ldots, >=\\
&-\hat{b}^{\theta k\dagger}_{s_j p_j} \hat{b}^{\theta k\dagger}_{s_i p_i} |1^{\theta k}_{s_1 p_1}, 1^{\theta k}_{s_2 p_2}, 1^{\theta k}_{s_3 p_3}, \ldots, 0^{\theta k}_{s_i p_i}, \ldots, 0^{\theta k}_{s_j p_j}, \ldots, > \; .
\end{aligned} \tag{3.64}$$

Any n "fermion" state is therefore a product of n creation operators $\hat{b}^{\theta k\dagger}_{i p_k}$ as presented in Eq. (3.47).

The number operator for "fermions" in the n-particle state of Eq. (3.64) is correspondingly

$$\hat{N}^{\theta} = \sum_{k, s_i p_i} \hat{N}^{\theta k}_{s_i p_i} \tag{3.65}$$

When coefficients $c^k{}_{spi}$ depend also on coordinates x^a (for free "fermions" $\mathbf{c}^k{}_{spi}(x) = c^k{}_{spi} \cdot e^{-ip_a x^a}$), it follows for $p^a p_a = 0$

$$\hat{\mathbf{b}}^{\theta k\dagger}_s(x^0, \vec{x}) = \sum_i \int \frac{d^{d-1}p}{(2\pi)^{d-1}} \, \mathbf{c}^k{}_{spi}(x)\, \hat{b}^{\theta k\dagger}_i \,.$$

$$\begin{aligned}
&\{\hat{\mathbf{b}}^{\theta k}_s(x^0,\vec{x}),\, \hat{\mathbf{b}}^{\theta k'\dagger}_{s'}(x^0,\vec{y})\}_+ |\psi_{oc}> = \delta^{kk'}\,\delta^{ss'}\delta^{d-1}(\vec{x}-\vec{y})\,|\psi_{oc}>\,,\\
&\{\hat{\mathbf{b}}^{\theta k\dagger}_s(x^0,\vec{x}),\, \hat{\mathbf{b}}^{\theta k'\dagger}_{s'}(x^0,\vec{y})\}_+ |\psi_{oc}> = 0\,, \quad \{\hat{\mathbf{b}}^{\theta k}_s(x^0,\vec{x}),\, \hat{\mathbf{b}}^{\theta k'\dagger}_{s'}(x^0,\vec{y})\}_+ |\psi_{oc}>= 0\,.
\end{aligned} \tag{3.66}$$

It is discussed in the subsection 3.3.3 how do discrete symmetry operators in the Grassmann case take care of "fermion" and "antifermion" states.

Let us now take into account Eq. (3.45) with

$$\mathcal{L}_G = \frac{1}{4}\{\hat{\phi}^\dagger \gamma^0_G \tilde{\gamma}^a (\hat{p}_a \phi) - (\hat{p}_a \phi^\dagger)\gamma^0_G \tilde{\gamma}^a \phi\}.$$

The Euler-Lagrange equations lead to $-i\frac{1}{2}\gamma^0_G\tilde{\gamma}^a\hat{p}_a\phi = 0$ and $i\frac{1}{2}\hat{p}_a\phi^\dagger\gamma^0_G\tilde{\gamma}^a = 0$.

Let us find the Hamilton function for a second quantized field: $\hat{\phi}(x^0,\vec{y})$, generated by one of the creation operators $\hat{\mathbf{b}}^{\theta\dagger}_s$ on the vacuum state $|\phi_{og}>$,

$$\begin{aligned}
\Pi_{\hat{\phi}} &= \frac{\partial \mathcal{L}_G}{\partial(\hat{p}_0\hat{\phi})} = \frac{1}{4}\,\hat{\phi}^\dagger\gamma^0_G\tilde{\gamma}^0\,, \quad \Pi_{\hat{\phi}^\dagger} = \frac{\partial \mathcal{L}_G}{\partial(\hat{p}_0\hat{\phi}^\dagger)} = -\frac{1}{4}\,\gamma^0_G\tilde{\gamma}^0\hat{\phi}\,,\\
\mathcal{H}_G &= \Pi_\phi\,(\hat{p}_0\hat{\phi}) + (\hat{p}_0\hat{\phi}^\dagger)\Pi_{\hat{\phi}^\dagger} - \mathcal{L}_G\,,\\
&= \frac{i}{4}\left[\hat{\phi}^\dagger\gamma^0_G\tilde{\gamma}^i(\hat{p}_i\hat{\phi}) - (\hat{p}_i\hat{\psi}^\dagger)\gamma^0_G\tilde{\gamma}^i\hat{\phi}\right],\\
H_G &= \int d^{d-1}x\,\mathcal{H}_G\,.
\end{aligned} \tag{3.67}$$

A vector $\hat{\phi}$ depends on $k = (I, II)$ and on spins (what in $d = (3+1)$ manifests as spins and charges).

Hamilton function is obviously an odd Grassmann object and *does not define the energy of the system*. However, if assuming the relation: $\frac{i}{2}\gamma^0 p_0\,\hat{\phi}^k(x^0,\vec{x}) = \{\hat{\phi}^k(x^0,\vec{x}),\, H_G\}_-$, one still ends up with the equations of motion, Eq. (3.45). One namely obtains

$$\gamma^0\hat{p_0}\hat{\phi}^k(t,\vec{x}) = \left\{\hat{\phi}^k(t,\vec{x})\,,\, H_G\right\}_- = -\gamma^0_G\tilde{\gamma}^i\hat{p}_i\hat{\phi}^k(t,\vec{x})\,, \tag{3.68}$$

what might help to find the procedure to define the energy for the interacting "Grassmann fermions". One must at this point either give up with the Grassmann "fermions" with the integer spins or find a consistent unconventional way to define the energy, like the one suggested in Eq. (3.68).

3.3.2 Second quantization in Clifford space

In Grassmann space the requirement that products of "eigenstates" of the Cartan subalgebra operators form the creation and annihilation operators, obeying the relations of Eq. (3.54), reduces the number of creation operators and correspondingly the number of states from 2^d (allowed for "eigenstates" of the Cartan subalgebra operators) to two isolated groups of $\frac{1}{2}\frac{d!}{\frac{d}{2}!\frac{d}{2}!}$ creation operators. (There are no generators of the Lorentz transformations $\mathbf{S}^{ab}$ that would connect both groups of states and correspondingly there are no families.)

Let us study what happens, when, let say, γ^a's are used to create the basis and correspondingly also to create the creation and annihilation operators. Here we briefly follow Ref. [50].

Let us point out that γ^a is expressible with θ^a and its derivative ($\gamma^a = (\theta^a + \frac{\partial}{\partial \theta_a})$), Eq. (3.4), and that we again require that creation (annihilation) operators create (annihilate) states, which are "eigenstates" (Eq. (3.72)) of the Cartan subalgebra operators, Eq. (3.110). Then the application of $\tilde{\gamma}^a$ on any Clifford algebra object $A(\gamma^a)$, (determined by γ^a's), can be evaluated as follows, Eq. (3.29, 3.30),

$$(\tilde{\gamma}^a A = i(-)^{(A)} A\gamma^a)|\psi_{oc}>, \tag{3.69}$$

where $(-)^{(A)} = -1$, if A is an odd Clifford algebra object and $(-)^{(A)} = 1$, if A is an even Clifford algebra object, while $|\psi_{oc}>$ is the vacuum state, replacing the vacuum state in the Grassmann case $|\psi_{og}>= |1>$ with the one of Eq. (3.79), in accordance with the relation of Eqs. (3.4, 3.32, 3.31), Refs. [50,10]. We could as well make a choice of $\tilde{\gamma}^a = i(\theta^a - \frac{\partial}{\partial\theta_a})$ instead of γ^a's to create the basic states, exchanging correspondingly the role of γ^a and $\tilde{\gamma}^a$ [9]).

Making a choice of the Cartan subalgebra "eigenstates" of S^{ab}, Eq. (3.27), one defines nilpotents $\stackrel{ab}{(k)}$ and projectors $\stackrel{ab}{[k]}$

$$\stackrel{ab}{(k)}:= \frac{1}{2}(\gamma^a + \frac{\eta^{aa}}{ik}\gamma^b)\,, \quad \stackrel{ab}{(k)}{}^2 = 0\,,$$
$$\stackrel{ab}{[k]}:= \frac{1}{2}(1 + \frac{i}{k}\gamma^a\gamma^b)\,, \quad \stackrel{ab}{[k]}{}^2 = \stackrel{ab}{[k]}\,, \tag{3.70}$$

where $k^2 = \eta^{aa}\eta^{bb}$. Recognizing that the Hermitian conjugate values of $\stackrel{ab}{(k)}$ and $\stackrel{ab}{[k]}$ are

$$\stackrel{ab}{(k)}{}^\dagger = \eta^{aa} \stackrel{ab}{(-k)}, \quad \stackrel{ab}{[k]}{}^\dagger = \stackrel{ab}{[k]}\,, \tag{3.71}$$

while the corresponding "eigenvalues" of S^{ab} and $\tilde{S}^{ab}$ on nilpotents and projectors, Eq. (3.27), are

$$S^{ab} \stackrel{ab}{(k)} = \frac{k}{2} \stackrel{ab}{(k)}\,, \quad S^{ab} \stackrel{ab}{[k]} = \frac{k}{2} \stackrel{ab}{[k]}\,,$$
$$\tilde{S}^{ab} \stackrel{ab}{(k)} = \frac{k}{2} \stackrel{ab}{(k)}\,, \quad \tilde{S}^{ab} \stackrel{ab}{[k]} = -\frac{k}{2} \stackrel{ab}{[k]}\,, \tag{3.72}$$

[9] In the case that we would choose $\tilde{\gamma}^a$'s instead of γ^a's, Eq.(3.4), the role of $\tilde{\gamma}^a$ and γ^a should be then correspondingly exchanged in Eq. (3.69).

we find for $d = 2(2n+1)$ that from the starting state made as a product of an odd number of only nilpotents

$$
\begin{aligned}
|\psi^1_1> &= \hat{b}^{1\dagger}_1 |\psi_{oc}>\,,\\
\hat{b}^{1\dagger}_1 &:= \stackrel{03}{(+i)}\stackrel{12}{(+)}\stackrel{35}{(+)}\cdots \stackrel{d-3\, d-2}{(+)}\; \stackrel{d-1\, d}{(+)}\,,\\
\hat{b}^{1}_1 &= (\hat{b}^{1\dagger}_1)^\dagger = \stackrel{d-1\, d}{(-)}\; \stackrel{d-3\, d-2}{(-)} \cdots \stackrel{35}{(-)}\stackrel{12}{(-)}\stackrel{01}{(-i)}\,,
\end{aligned}
\tag{3.73}
$$

having correspondingly an odd Clifford character, all other states of the same Lorentz representation, there are $2^{\frac{d}{2}-1}$ members, follow by the application of S^{cd} (which do not belong to the Cartan subalgebra) on the starting state [10], Eq. (3.110), ($S^{cd}\, |\psi^1_1 >= |\psi^1_i >$).

$$
\begin{aligned}
\hat{b}^{1\dagger}_i &\propto S^{ab} \dots S^{ef} \hat{b}^{1\dagger}_1\,, \quad |\psi^1_i >= S^{ab} .. S^{ef} |\psi^1_1 >\,,\\
\hat{b}^{1}_i &\propto \hat{b}^{1}_1 S^{ef} \dots S^{ab}\,,
\end{aligned}
\tag{3.74}
$$

with $S^{ab\dagger} = \eta^{aa}\eta^{bb} S^{ab}$. We make a choice of the proportionality factors so that the corresponding states $|\psi^1_i >= \hat{b}^{1\dagger}_i |\psi_{oc}>$ are normalized [50,10].

The operators $\tilde{S}^{cd}$, which belong to the Cartan subalgebra of $\tilde{S}^{ab}$, Eq. (3.110), generate "eigenstates" of the Cartan subalgebra operators ($\tilde{S}^{03}, \tilde{S}^{12}, \tilde{S}^{56}, \cdots, \tilde{S}^{d-1\, d}$), with the eigenvalues which determine the "family" quantum numbers. There are $2^{\frac{d}{2}-1}$ families. From the starting new member with a different "family" quantum number the whole Lorentz representation of family members with this "family" quantum number follows by the application of S^{ef}: $S^{ab} \cdots S^{ef}\, \tilde{S}^{cd} |\psi^1_1 >= |\psi^\alpha_i >$. All states of one Lorentz representation of any particular "family" quantum number have an odd Clifford character, since neither S^{cd} nor $\tilde{S}^{cd}$ — both of an even Clifford character — can change the odd character of the starting state.

Any vector $|\psi^\alpha_i >$ follows from the starting vector, Eqs. (3.73), by the application of either $\tilde{S}^{ef}$, which change the family quantum number, or S^{gh}, which change the family member quantum number of a particular family or with the corresponding product of S^{ef} and $\tilde{S}^{ef}$

$$
|\psi^\alpha_i > \propto \tilde{S}^{ab} \cdots \tilde{S}^{ef} |\psi^1_i > \propto \tilde{S}^{ab} \cdots \tilde{S}^{ef} S^{mn} \cdots S^{pr} |\psi^1_1 > \,. \tag{3.75}
$$

Again, α denotes "family" quantum numbers, i denotes family member quantum number. Correspondingly we define $\hat{b}^{\alpha\dagger}_i$ (up to a constant) to be

$$
\begin{aligned}
\hat{b}^{\alpha\dagger}_i &\propto \tilde{S}^{ab} \cdots \tilde{S}^{ef} S^{mn} \cdots S^{pr} \hat{b}^{1\dagger}_1\\
&\propto S^{mn} \cdots S^{pr} \hat{b}^{1\dagger}_1 S^{ab} \cdots S^{ef}\,.
\end{aligned}
\tag{3.76}
$$

This last expression follows due to the property of the Clifford object $\tilde{\gamma}^a$ and correspondingly of $\tilde{S}^{ab}$, presented in Eqs. (3.69, 3.120).

We accordingly have for an annihilation operator $\hat{b}^\alpha_i (= (\hat{b}^{\alpha\dagger}_i)^\dagger)$

$$
\hat{b}^\alpha_i = (\hat{b}^{\alpha\dagger}_i)^\dagger \propto S^{ef} \cdots S^{ab} \hat{b}^1_1 S^{pr} \cdots S^{mn}\,. \tag{3.77}
$$

[10] The smallest number of all the generators S^{ac}, which do not belong to the Cartan subalgebra, Eq. (3.110), needed to create from the starting state all the other members, is $2^{\frac{d}{2}-1} - 1$. This is true for both even dimensional spaces – $2(2n+1)$ and $4n$.

The proportionality factor ought to be chosen so that the corresponding states $|\psi^{\alpha}_{i}>=\hat{b}^{\alpha\dagger}_{i}|\psi_{oc}>$ are normalized when the vacuum state $|\psi_{oc}>$ is normalized, $<\psi_{oc}|\psi_{oc}>=1$, while all the states belonging to the physically acceptable states, like $\stackrel{03}{[+i]}\stackrel{12}{[+]}\stackrel{56}{[-]}\stackrel{78}{[-]}\cdots\stackrel{d-3\;d-2}{(+)}\;\stackrel{d-1\;d}{(+)}\;|\psi_{oc}>$, must not give zero for either $d=2(2n+1)$ or for $d=4n$. We also want that states, obtained by the application of ether S^{cd} or $\tilde{S}^{cd}$ or both, are orthogonal. To make a choice of the vacuum it is needed to know the relations of Eq. (3.116). It must be

$$
\begin{aligned}
&<\psi_{oc}|\cdots\stackrel{ab}{(k)}^{\dagger}\cdots|\cdots\stackrel{ab}{(k')}\cdots|\psi_{oc}> = \delta_{kk'}\,,\\
&<\psi_{oc}|\cdots\stackrel{ab}{[k]}^{\dagger}\cdots|\cdots\stackrel{ab}{[k']}\cdots|\psi_{oc}> = \delta_{kk'}\,,\\
&<\psi_{oc}|\cdots\stackrel{ab}{[k]}^{\dagger}\cdots|\cdots\stackrel{ab}{(k')}\cdots|\psi_{oc}> = 0\,.
\end{aligned}
\tag{3.78}
$$

We must choose the vacuum state in a way that fulfills the above requirements as well as the requirements $\hat{b}^{\beta\dagger}_{i}|\psi_{oc}>\neq 0$ and $\hat{b}^{\beta}_{i}|\psi_{oc}>=0$ for all members i of any family β. Since any $\tilde{S}^{eg}$ changes $\stackrel{ef}{(+)}\stackrel{gh}{(+)}$ into $\stackrel{ef}{[+]}\stackrel{gh}{[+]}$ and $\stackrel{ab}{[+]}^{\dagger}=\stackrel{ab}{[+]}$, while $\stackrel{ab}{(+)}^{\dagger}\stackrel{ab}{(+)}=\stackrel{ab}{[-]}$, the vacuum state $|\psi_{oc}>$ must be

$$
\begin{aligned}
|\psi_{oc}>=&\\
&\stackrel{03}{[-i]}\stackrel{12}{[-]}\stackrel{56}{[-]}\cdots\stackrel{d-1\;d}{[-]}+\stackrel{03}{[+i]}\stackrel{12}{[+]}\stackrel{56}{[-]}\cdots\stackrel{d-1\;d}{[-]}+\stackrel{03}{[+i]}\stackrel{12}{[-]}\stackrel{56}{[+]}\cdots\stackrel{d-1\;d}{[-]}+\cdots|0>\,,\\
&\text{for } d=2(2n+1)\,,
\end{aligned}
\tag{3.79}
$$

n is a positive integer. There are $2^{\frac{d}{2}-1}$ summands, since we can start with the vacuum state $\stackrel{03}{[-i]}\stackrel{12}{[-]}\stackrel{56}{[-]}\cdots\stackrel{d-1\;d}{[-]}\;|1>$, which fulfills the requirement for $\hat{b}^{1\dagger}_{1}|\psi_{oc}>\neq 0$ and $\hat{b}^{1}_{1}|\psi_{oc}>=0$, and then we must step by step replace all possible pairs of $\stackrel{ab}{[-]}\cdots\stackrel{ef}{[-]}$ in the starting part $\stackrel{03}{[-i]}\stackrel{12}{[-]}\stackrel{35}{[-]}\cdots\stackrel{d-1\;d}{[-]}$ into $\stackrel{ab}{[+]}\cdots\stackrel{ef}{[+]}$ and include new terms into the vacuum state so that the last $(2n+1)$ summands have for $d=2(2n+1)$ case, n is a positive integer, only one factor $[-]$ and all the rest $[+]$, each $[-]$ at different position [11].

This vacuum has all the spins, either with respect to S^{ab} or with respect to $\tilde{S}^{ab}$, equal to zero.

The vacuum state has then the normalization factor $1/\sqrt{2^{d/2-1}}$,
while there is

$$
2^{\frac{d}{2}-1}\,2^{\frac{d}{2}-1}
\tag{3.80}
$$

[11] The choice of Eq. (3.79) for the vacuum state is not unique. If one would multiply any of summands by a number β_{α}, where α represents the α-th family, and then multiply each of $2^{\frac{d}{2}-1}$ members of creation operators belonging to this family $\hat{b}^{\alpha\dagger}_{i}$ by $\sqrt{\beta_{\alpha}}$ and the corresponding annihilation operator $\hat{b}^{\alpha}_{i}$ by $\sqrt{\beta^{*}_{\alpha}}$, β^{*}_{α} is the complex conjugated value of β_{α}, it would still be true that $\hat{b}^{\alpha}_{i}\,\hat{b}^{\beta\dagger}_{j}=\delta^{\alpha\beta}\delta_{ij}$ times the corresponding summand of the vacuum back.

number of creation operators, defining the orthonormalized states when applying on the vacuum state of Eqs. (3.79) and the same number of annihilation operators, which are Hermitian conjugated to creation operators. Again, operators $\tilde{S}^{ab}$ connect members of different families, operators S^{ab} generate all the members of one family.

Paying attention on only internal degrees of freedom, that is on the spin, the creation and annihilation operators must fulfill the relations

$$\begin{aligned}
\{\hat{b}^{\alpha}_{i}, \hat{b}^{\alpha'\dagger}_{j}\}_{+}|\psi_{oc}> &= \delta^{\alpha\alpha'}\, \delta_{ij}|\psi_{oc}>\,,\\
\{\hat{b}^{\alpha}_{i}, \hat{b}^{\alpha'}_{j}\}_{+}|\psi_{oc}> &= 0\; |\psi_{oc}>\,,\\
\{\hat{b}^{\alpha\dagger}_{i}, \hat{b}^{\alpha'\dagger}_{j}\}_{+}|\psi_{oc}> &= 0\; |\psi_{oc}>\,,\\
\hat{b}^{\alpha}_{i}|\psi_{oc}> &= 0\; |\psi_{oc}>\,,\\
\hat{b}^{\alpha\dagger}_{i}|\psi_{oc}> &= |\psi^{\alpha}_{i}>\,,
\end{aligned} \tag{3.81}$$

with (i, j) determining family members quantum numbers and (α, α') denoting "family" quantum numbers.

Only Clifford odd objects fulfill the relations of Eq. (3.81), since the odd Clifford objects anti-commute (like: $\{(\gamma^0 - \gamma^3), (\gamma^1 + i\gamma^2)\}_{+} = 0$), while the Clifford even objects commute (like: $\{(1 - \gamma^0\gamma^3), (1 - i\gamma^1\gamma^2)\}_{+} = 2\,(1 - \gamma^0\gamma^3)(1 - i\gamma^1\gamma^2)$).

The reader can find the detailed proofs for the above statements, for either $d = 2(2n + 1)$ or $d = 4n$, in Refs. [50,10].

Let us again, like in the Grassmann case, Eq. (3.62), look for the creation (and their annihilation operators) which, when applied on the vacuum state, Eq. (3.79), solve the equation of motion, Eq. (3.36). The solution for each momentum p^a_k, $a = (1, \ldots, d)$, for discretized values of momenta, is a superposition of $\hat{b}^{\alpha\dagger}_{i}$,

$$\hat{b}^{\alpha\dagger}_{s\,p_k} = \sum_i c^{\alpha}{}_{si}(p_k)\, \hat{b}^{\alpha\dagger}_{i}\,, \tag{3.82}$$

applied on the vacuum state, Eq. (3.79). Since $\hat{b}^{\alpha\dagger}_{i}$ and $\hat{b}^{\alpha}_{j}$ fulfill the relations of Eq. (3.81) and, if the states for two different momenta are orthogonalized, it follows

$$\begin{aligned}
\{\hat{b}^{\alpha}_{s\,p_k}, \hat{b}^{\alpha'\dagger}_{s'\,p_l}\}_{+}|\psi_{oc}> &= \delta^{\alpha\alpha'}\, \delta_{ss'}\, \delta_{p_k\,p_l}\,|\psi_{oc}>\,,\\
\{\hat{b}^{\alpha}_{s\,p_k}, \hat{b}^{\alpha'}_{s'\,p_l}\}_{+}|\psi_{oc}> &= 0\; |\psi_{oc}>\,,\\
\{\hat{b}^{\alpha\dagger}_{s\,p_k}, \hat{b}^{\alpha'\dagger}_{s'\,p_l}\}_{+}|\psi_{oc}> &= 0\; |\psi_{oc}>\,,\\
\hat{b}^{\alpha}_{s\,p_k}|\psi_{oc}> &= 0\; |\psi_{oc}>\,,\\
\hat{b}^{\beta\dagger}_{s\,p_k}|\psi_{oc}> &= |\psi^{\alpha}_{s\,p_k}>\,,
\end{aligned} \tag{3.83}$$

with the vacuum state $|\psi_{oc}>$ defined in Eq. (3.79), with s denoting the corresponding solution of equations of motion and for a discretized momentum space.

The number operator of a particular solution s, a particular momentum p_k and a particular "family" α,

$$\hat{N}^{\alpha}_{sp_k} = \hat{b}^{\alpha\dagger}_{sp_k}\, \hat{b}^{\alpha}_{sp_k}\,, \quad (\hat{N}^{\alpha}_{sp_k})^2 = \hat{N}^{\alpha}_{sp_k}\,, \tag{3.84}$$

has the eigenvalues 1 or 0.

The number of fermions in the n-particle state of Eq. (3.86) is correspondingly

$$\hat{N} = \sum_{\alpha, s, p_k} \hat{N}^{\alpha}_{s p_k} . \tag{3.85}$$

For a n fermion and antifermion state, Eqs. (3.47, 3.48) in the introduction to Sect. 3.3, $|1^{\alpha=1}_{s=1\,p_1}, 1^{\alpha=1}_{s=2\,p_1}, 1^{\alpha=1}_{s=3\,p_1}, \dots, 0^{\alpha}_{s\,p_i}, \dots, 0^{\alpha}_{s^{iv} p_j}, \dots, >$ it follows, for example, due to the third line in Eq. (3.83), that

$$\begin{aligned} &\hat{b}^{\alpha' \dagger}_{s' p_i}\, \hat{b}^{\alpha" \dagger}_{s" p_j} |1^{\alpha=1}_{s=1\,p_1}, 1^{\alpha=1}_{s=2\,p_1}, 1^{\alpha=1}_{s=3\,p_1}, \dots, 0^{\alpha}_{s\,p_i}, \dots, 0^{\alpha}_{s^{iv} p_j}, \dots, >= \\ -\;& \hat{b}^{\alpha" \dagger}_{s" p_j}\, \hat{b}^{\alpha' \dagger}_{s' p_i} |1^{\alpha=1}_{s=1\,p_1}, 1^{\alpha=1}_{s=2\,p_1}, 1^{\alpha=1}_{s=3\,p_1}, \dots, 0^{\alpha}_{s\,p_i}, \dots, 0^{\alpha}_{s^{iv} p_j}, \dots, >, \end{aligned} \tag{3.86}$$

where 1 denotes the occupied state and 0 the unoccupied state, and $|1^{\alpha=1}_{s=1\,p_1} >= \hat{b}^{\alpha=1 \dagger}_{s=1 p_1} |\psi oc>$.

Eq. (3.86, 3.47) demonstrates properties of Slater determinants. One fermion state is obviously second quantized by construction.

Two states with n_1 and n_2 fermions each, defined by $\hat{A}^{a \dagger}$ as n_1 products of $\hat{b}^{\alpha \dagger}_{s p_i}$ (which distinguish among themselves in at least one of the properties (α, s, p_i)) and by $\hat{A}^{b \dagger}$ as n_2 products of $\hat{b}^{\beta \dagger}_{s' p_j}$ (which distinguish among themselves in at least one of the properties (α', s', p_j)), applying on $|\psi_{oc}>$, must distinguish in either internal space or in the coordinate space, as it follows from Eq. (3.86), that the product of $\hat{A}^{a \dagger}$ and $\hat{A}^{b \dagger}$ applying on $|\psi_{oc}>$ would give a state with $(n_1 + n_2)$ fermions.

Let us add that the vacuum state, having the sum of the spins of both kinds of operators, S^{ab} and $\tilde{S}^{ab}$, equal to zero and therefore neutral, remains neutral also when filled with fermions of all the spins, S^{ab} and $\tilde{S}^{ab}$.

When coefficients $c^{\alpha}{}_{si}(p_k)$ depend also on coordinates x^a (for free fermions $\mathbf{c}^{\alpha}{}_{si}(p_k, x) = c^{\alpha}{}_{si}(p_k) \cdot e^{-i p_a x^a}$), it follows for $p^a p_a = 0$,

$$\hat{\mathbf{b}}^{\alpha \dagger}_{s}(x^0, \vec{x}) = \sum_i \int \frac{d^{d-1}p}{(2\pi)^{d-1}}\, \mathbf{c}^{\alpha}{}_{si}(p_k, x)\, \hat{b}^{\alpha \dagger}_{i} .$$

$$\begin{aligned} &\{\hat{\mathbf{b}}^{\alpha}_{s}(x^0, \vec{x}),\, \hat{\mathbf{b}}^{\alpha' \dagger}_{s'}(x^0, \vec{y})\}_+ |\psi_{oc}> = \delta^{\alpha \alpha'}\, \delta_{s s'}\, \delta^{d-1}(\vec{x} - \vec{y})\, |\psi_{oc}>, \\ &\{\hat{\mathbf{b}}^{\alpha \dagger}_{s}(x^0, \vec{x}),\, \hat{\mathbf{b}}^{\alpha' \dagger}_{s'}(x^0, \vec{y})\}_+ |\psi_{oc}> = 0, \quad \{\hat{\mathbf{b}}^{\alpha}_{s}(x^0, \vec{x}),\, \hat{\mathbf{b}}^{\alpha'}_{s'}(x^0, \vec{y})\}_+ |\psi_{oc}>= 0 . \end{aligned} \tag{3.87}$$

Let us now take into account Eq. (3.35) with

$$\mathcal{L}_C = \frac{1}{2}\{\hat{\psi}^{\dagger} \gamma^0 \gamma^a (\hat{p}_a \psi) - (\hat{p}_a \psi^{\dagger}) \gamma^0 \gamma^a \psi\}.$$

The Euler-Lagrange equations lead to $\gamma^0 \gamma^a \hat{p}_a \psi = 0$ and $-\hat{p}_a \phi^{\dagger} \gamma^0 \gamma^a = 0$.

Let us look for the Hamilton function for fermions determined by one of the creation operators, like $\hat{\psi}^{\alpha}_{s}(x^0, \vec{x}) = \hat{\mathbf{b}}^{\alpha \dagger}_{s}(x^0, \vec{x})] |\psi_{oc}>$, which is already the second quantized state.

For a vector $\hat{\psi}$ and $\hat{\psi}^{\dagger}$ it therefore follows

$$\Pi_{\hat{\psi}} = \frac{\partial \mathcal{L}_C}{\partial(\hat{p}_0 \hat{\psi})} = \frac{1}{2}\,\hat{\psi}^{\dagger}\,, \quad \Pi_{\hat{\psi}^{\dagger}} = \frac{\partial \mathcal{L}_C}{\partial(\hat{p}_0 \hat{\psi}^{\dagger})} = \frac{1}{2}\,\hat{\psi}\,,$$
$$\mathcal{H}_C = \Pi_{\psi}\,(\hat{p}_0 \hat{\psi}) + (\hat{p}_0 \hat{\psi}^{\dagger})\Pi_{\hat{\psi}^{\dagger}} \; - \mathcal{L}_C\,,$$
$$= -\frac{1}{2}\left[\hat{\psi}^{\dagger}\,\gamma^0\,\gamma^i\,(\hat{p}_i \hat{\psi}) - (\hat{p}_i \hat{\psi}^{\dagger})\gamma^0\gamma^i\hat{\psi}\,\right],$$
$$H_C = \int d^{d-1}x\,\mathcal{H}_C\,, \tag{3.88}$$

Correspondingly one finds for a component $\hat{\psi}^{\alpha}_{s}(x^0, \vec{x})$ [74], $\vec{x}$ is a vector in $(d-1)$-dimensional coordinate space,

$$\hat{p_0}\hat{\psi}^{\alpha}_{s}\,(x^0,\vec{x}) = \left\{\,\hat{\psi}^{\alpha}_{s}\,(x^0,\vec{x})\,,\,H_C\right\}_{-}$$
$$= \left\{\,\hat{\psi}^{\alpha}_{s}(x^0,\vec{x})\,,\,\int d^{d-1}x'\,\sum_{\alpha',s'}\,\hat{\psi}^{\alpha'\dagger}_{s'}\,(x^0,\vec{x'})\,\gamma^0\gamma^i\,(\hat{p'}_i\hat{\psi}^{\alpha'}_{s'}(x^0,\vec{x'}))\,\right\}_{-}$$
$$= \int d^{d-1}x'\,\sum_{\alpha's'}\,\left\{\,\hat{\psi}^{\alpha}_{s}(x^0,\vec{x})\,,\hat{\psi}^{\alpha'\dagger}_{s'}(x^0,\vec{x'})\,\right\}_{+}\gamma^0\gamma^i\,(\hat{p'}_i\hat{\psi}^{\alpha'}_{\mathbf{s}'}\,(x^0,\vec{x'}))$$
$$= -\gamma^0\gamma^i\,(\hat{p}_i\hat{\psi}^{\alpha}_{\mathbf{s}}(x^0,\vec{x}))\,. \tag{3.89}$$

(We took into account that $\gamma^0\gamma^i$ transforms $\hat{\psi}^{\alpha'}_{s'}\,(x^0,\vec{x'})$ into $\sum_{s''} c^{\alpha'}{}_{s's''}\,\hat{\psi}^{\alpha'}_{s''}(x^0,\vec{x'})$, which anticommute with $\hat{\psi}^{\alpha}_{s}(x^0,\vec{x})$ (Eq. (3.87)), we also assumed that states, obtained when operators operate on a vacuum state, do not contribute to the surface term. Integrating per partes and dropping the surface term simplifies H_C into $-\int\sum d^{d-1}x'\,\hat{\psi}^{\alpha'\dagger}_{s'}(x^0,\vec{x'})\,\gamma^0\gamma^i\,(\hat{p'}_i\hat{\psi}^{\alpha'}_{s'}(x^0,\vec{x'}))$.) The obtained equations of motion agree with the ones from Eqs. (3.39, 3.40). Correspondingly the energy of the n-fermion state of free massless fermions created by $\hat{\mathbf{b}}^{\alpha\dagger}_{s}$ on the vacuum state $|\psi_{oc}>$ all with zero momentum p_o (solving the Weyl equation Eqs. (3.36,3.40)) is equal to $E = \sum_{\alpha s}\hat{N}^{\alpha}_{s}p_0$. The current is correspondingly $\hat{\mathbf{j}}^{a} = \hat{\psi}^{\alpha\dagger}_{\mathbf{s}}\gamma^0\gamma^a\hat{\psi}^{\alpha}_{\mathbf{s}}$.

The observed fermions — quarks and leptons — manifest their properties obviously in $d = (3+1)$. The internal space in $d = (3+1)$ can therefore be used to describe the spin and handedness of massless fermions, in the *spin-charge-family* theory also families, while the internal space in $d \geq 5$ can be used to describe charges of fermions, contributing in the *spin-charge-family* theory as well to families.

One family representation contains in $d = 2(2n+1), n = 3$, $2^{\frac{d}{2}-1} = 64$ members, described by the creation and annihilation operators fulfilling the anticommutation relations of Eq. (3.81), explaining from the point of view of $d = (3+1)$ spins, handedness and charges of the observed quarks and leptons and antiquarks and antileptons. Correspondingly there is no need for the negative energy "Dirac sea".

We discuss below discrete symmetry operators for both cases, the Clifford one and the Grassmann one, in d and in observable dimension $d = (3+1)$. In Subsect. 3.3.4 we present a few examples.

3.3.3 Discrete symmetries in Grassmann space and in Clifford space in d and in d = (3 + 1) part of the space

We have treated so far free massless fermions in Grassmann and in Clifford space. The fermion "nature" of states are in both spaces demonstrated by the fact that the corresponding creation and annihilation operators fulfill the anticommutation relations of Eq. (3.62) in Grassmann case and of Eq. (3.83) in Clifford space. Fermions — in both spaces — are in superposition of eigenstates of the Cartan subalgebra operators of $\mathbf{S}^{ab}$ in the Grassmann case, in the Clifford case they are in superposition of the Cartan subalgebra operators of S^{ab} as well as of $\tilde{S}^{ab}$.

We distinguish in d-dimensional space two kinds of discrete symmetry $\mathcal{C}, \mathcal{P}$ and $\mathcal{T}$ operators with respect to the internal space in which the fermion properties are described.

In the Clifford case we have [65]

$$\begin{aligned}
\mathcal{C}_{\mathcal{H}} &= \prod_{\gamma^a \in \Im} \gamma^a \; K\,, \\
\mathcal{T}_{\mathcal{H}} &= \gamma^0 \prod_{\gamma^a \in \Re} \gamma^a \; K\, I_{x^0}\,, \\
\mathcal{P}^{(d-1)}_{\mathcal{H}} &= \gamma^0\, I_{\vec{x}}\,, \\
I_x x^a &= -x^a\,, \quad I_{x^0} x^a = (-x^0, \vec{x})\,, \quad I_{\vec{x}} \vec{x} = -\vec{x}\,, \\
I_{\vec{x}_3} x^a &= (x^0, -x^1, -x^2, -x^3, x^5, x^6, \dots, x^d)\,.
\end{aligned} \tag{3.90}$$

The product $\prod \gamma^a$ is meant in the ascending order in γ^a, K stands for complex conjugation.

In the Grassmann case we correspondingly define

$$\begin{aligned}
\mathcal{C}_{G} &= \prod_{\gamma^a_G \in \Im \gamma^a} \gamma^a_G \, K\,, \\
\mathcal{T}_{G} &= \gamma^0_G \prod_{\gamma^a_G \in \Re \gamma^a} \gamma^a_G \, K\, I_{x^0}\,, \\
\mathcal{P}^{(d-1)}_{G} &= \gamma^0_G\, I_{\vec{x}}\,,
\end{aligned} \tag{3.91}$$

γ^a_G is defined in Eq. (3.22) as $\gamma^a_G = (1 - 2\theta^a \eta^{aa} \frac{\partial}{\partial \theta_a})$, while $I_x x^a = -x^a\,, I_{x^0} x^a = (-x^0, \vec{x})\,, I_{\vec{x}} \vec{x} = -\vec{x}\,, I_{\vec{x}_3} x^a = (x^0, -x^1, -x^2, -x^3, x^5, x^6, \dots, x^d)$, like in the Clifford case. Let be noticed, that since γ^a_G $(= -i\eta^{aa}\gamma^a \tilde{\gamma}^a)$ is always real as we see in Eq. (3.28) [12]. Since γ^a is either real or imaginary, Eq. (3.22), we use in Eq. (3.91) γ^a to make a choice of appropriate γ^a_G. In what follows we shall use the notation as in Eq. (3.91).

Let us define in the Clifford case and in the Grassmann case the operator "emptying" [13]. The operation "emptying$_{NH}$" after the charge conjugation $\mathcal{C}_{\mathcal{H}}$ in

[12] If we choose a real θ^a, then γ^a is real and $\tilde{\gamma}^a$ imaginary, if θ^a is imaginary, then γ^a is imaginary and $\tilde{\gamma}^a$ real, as is demonstrated in Eq. (3.28).

[13] The operator "emptying" empties the "Dirac sea" of negative energies [65], although in the *spin-charge-family* theory is no need for the "Dirac sea" of negative energies, as

the Clifford case [65,7,9] (arxiv:1312.1541) and "emptying$_{NG}$" after the charge conjugation $\mathcal{C}_G$ in the Grassmann case, namely transforms the positive energy fermions into positive energy antifermions in both cases, solving Eq. (3.36) in the Clifford case, and Eq. (3.43) in the Grassmann case.

$$\text{"emptying}_{NH}\text{"} = \prod_{\Re\gamma^a} \gamma^a \, K \quad \text{in Clifford space}\,,$$
$$\text{"emptying}_{NG}\text{"} = \prod_{\Re\gamma^a} \gamma^a_G \, K \quad \text{in Grassmann space}\,. \tag{3.92}$$

Then the anti-particle state creation operator to the corresponding particle state creation operator can be obtained by the application of

$$\mathbb{C}_{\mathcal{H},\mathcal{G}} = \text{"emptying}_{NH,NG}\text{"} \cdot \mathcal{C}_{\mathcal{H},\mathcal{G}} \tag{3.93}$$

$\mathbb{C}_{\mathcal{H}}$ and $\mathbb{C}_G$, with indexes $_{\mathcal{H}}$ and $_{NH}$ denoting the Clifford case and with $_{\mathcal{G}}$ and $_{NG}$ denoting the Grassman case, on the creation operator for a particle state, or opposite. Let us remind the reader that in the *spin-charge-family* theory, using the Clifford algebra, the family members of each family include fermions and antifermions — quarks and leptons and antiquarks and antileptons. This is the case also for Grassmann fermions and antifermions, but in this casethere are instead of families two by $\mathbf{S}^{ab}$ unconnected representations.

Ref. [65] proposes in the Clifford case the following discrete symmetry operators, manifesting dynamics in $d = (3+1)$

$$\begin{aligned}
\mathcal{C}_{\mathcal{N}} &= \prod_{\Im\gamma^m, m=0}^{3} \gamma^m \;\; \Gamma^{(3+1)} \, K \, I_{x^6,x^8,\dots,x^d}\,,\\
\mathcal{T}_{\mathcal{N}} &= \prod_{\Re\gamma^m, m=1}^{3} \gamma^m \;\; \Gamma^{(3+1)} \, K \, I_{x^0} \, I_{x^5,x^7,\dots,x^{d-1}}\,,\\
\mathcal{P}^{(d-1)}_{\mathcal{N}} &= \gamma^0 \, \Gamma^{(3+1)} \, \Gamma^{(d)} \, I_{\vec{x}_3}\,,\\
\mathbb{C}_{\mathcal{N}} &= \gamma^0\gamma^5\gamma^7 \cdots \gamma^{d-1} I_{\vec{x}_3} \, I_{x^6,x^8,\dots,x^d}\\
\mathcal{C}_{\mathcal{N}}\mathcal{P}^{(d-1)}_{\mathcal{N}} &= \gamma^0 \, \gamma^2 \, I_{\vec{x}_3} K \, I_{x^6,x^8,\dots,x^d}\,,\\
\mathbb{C}_{\mathcal{N}}\mathcal{P}^{(d-1)}_{\mathcal{N}} &= \gamma^0 \, \gamma^5 \, \cdots \gamma^{d-1} \, I_{\vec{x}_3} \, I_{x^6,x^8,\dots,x^d}\,.
\end{aligned} \tag{3.94}$$

In the Grassmann case we use the Grassmann even, Hermitian and real operators γ^a_G, Eq. (3.22), to determine discrete symmetries in $((d-1)+1)$ space (as presented in Eq. (3.91)) and in $d = (3+1)$ space. In $(3+1)$ space we proceed —

we discussed already in the introduction of Sect. 3.3, for either Clifford or Grassmann fermions. The operation of "emptying$_{NH}$" after the charge conjugation $\mathcal{C}_{\mathcal{H}}$ in the Clifford case, which transforms the state put on the top of the Clifford "Dirac sea" into the corresponding negative energy state, namely creates the anti-particle state to the starting particle state, each anti-particle state, put on the top of the "Dirac sea", solving the Weyl equation in the Clifford case, Eq. (3.36).

in analogy with the operators in the Clifford case [65] — as follows

$$
\begin{aligned}
\mathcal{C}_{NG} &= \prod_{\gamma^m_G \in \Re\gamma^m} \gamma^m_G \, K \, I_{x^6 x^8 \dots x^d} \,, \\
\mathcal{T}_{NG} &= \gamma^0_G \prod_{\gamma^m_G \in \Im\gamma^m} \gamma^m_G \, K \, I_{x^0} I_{x^5 x^7 \dots x^{d-1}} \,, \\
\mathcal{P}^{(d-1)}_{NG} &= \gamma^0_G \prod_{s=5}^{d} \gamma^s_G I_{\vec{x}} \,, \\
\mathbb{C}_{NG} &= \prod_{\gamma^s_G \in \Re\gamma^s} \gamma^s_G \,, I_{x^6 x^8 \dots x^d} \,, \\
\mathcal{C}_{NG}\mathcal{P}^{(d-1)}_{NG} &= \gamma^0_G \, \gamma^2_G \, K \, I_{\vec{x}_3} \, I_{x^6 x^8 \dots x^d} \,, \\
\mathbb{C}_{NG}\mathcal{P}^{(d-1)}_{NG} &= \gamma^0_G \prod_{\gamma^s_G \in \Im\gamma^s, s=5}^{d} \gamma^s_G \, I_{\vec{x}_3} \, I_{x^6 x^8 \dots x^d} \,, \\
\mathbb{C}_{NG}\mathcal{T}_{NG}\mathcal{P}^{(d-1)}_{NG} &= \prod_{\gamma^s_G \in \Im\gamma^a} \gamma^a_G \, I_x K \,.
\end{aligned} \tag{3.95}
$$

3.3.4 Examples of massless fermion and antifermion states in Clifford and in Grassmann space

Let us illustrate solutions for free fermion states, represented by the creation operators applied on the vacuum states for the Clifford and the Grassmann case in $((d-1)+1)$-dimensional space, representing indeed the contribution of a one fermion second quantized state in the Fock space of any number of fermions. We analyze states in both cases from the point of view $d = (3+1)$-dimensional space, with the momentum in ordinary space $p^a = (p^0, p^1, p^2, p^3, 0, \cdots, 0)$, so that the charges "seen" in $d = (3+1)$ are determined by the generators of the Lorentz transformations in the internal space — $S^{st}, (s,t) = (5, 6, 7, \cdots, d)$ in the Clifford case and $\mathbf{S}^{st}, (s,t) = (5, 6, 7, \cdots, d)$ in the Grassmann case. In the Clifford case we discuss one family in details (let be reminded that the generators S^{ab} connect all the members belonging to one family, while $\tilde{S}^{ab}$ transform a particular member of one family into the same member of another family), commenting also on the appearance of families (all the families are reachable by $\tilde{S}^{ab}$) and present them briefly. In the Grassmann case different representations can not be reached by the generators of the Lorentz representations $\mathbf{S}^{ab}$. The discrete symmetry operators are in the Clifford case presented in Eq. (3.94), and in the Grassmann case in Eq. (3.95).

We start with examples in $d = (5+1)$-dimensional space, with charges determined by $S^{st}, (s,t) = (5,6)$ in the Clifford case and $\mathbf{S}^{st}, (s,t) = (5,6)$ in the Grassmann case.

The dimension $(13+1)$, used in the *spin-charge-family* theory to describe quarks and lepton as well the gauge fields and scalar fields, offers to free fermions at low energies additional charges, what explains observable properties of quarks and leptons. We present the creation operators creating all the states of one family members in Clifford space. The family members creation operators are reachable by S^{ab}. All the families are reachable from the starting family by $\tilde{S}^{ab}$ in the case of Clifford odd representations. In the case of the Clifford even representations there are $\tilde{S}^{ab}$ and $\gamma^a \tilde{\gamma}^a$, which take care of all irreducible representations.

In Ref. [50,68–70] $(d = 5+1)$-dimensional space is studied as a toy model to manifest that the break of symmetry from the higher dimensional space to the $(3+1)$-dimensional space *can* lead to massless fermions. Fermions were described in Clifford space. Here we briefly follow these references, and Refs. [65,66], adding new observations.

The first study of Grassmann case can be found in Ref. [46].

Clifford fermions and antifermions Let us start with the examples in the Clifford case. To make discussions transparent let us first treat the $d = (5+1)$ case. The $d = (13+1)$ case is not so easy to present in particular when also families are treated.

Clifford case in $d = (5+1)$**:**

In Table 3.4 the basic creation operators $\hat{b}^{\alpha\dagger}_{i=(ch,s)}$ and their annihilation partners $\hat{b}^{\alpha}_{i=(ch,s)}$ in $d = (5+1)$ are presented for all four $(2^{\frac{d}{2}-1})$ families $\alpha = (I, II, III, IV)$. Index i is devided into s, determining spin and into ch to point out that S^{56} represents the charge from the point of view of $d = (3+1)$, having two values, $+\frac{1}{2}$ and $-\frac{1}{2}$. The vacuum state, Eq. (3.79), is the sum of selfajoint operators ($\stackrel{03}{[-i]}\stackrel{12}{[-]}|\stackrel{56}{[-]}$, $\stackrel{03}{[+i]}\stackrel{12}{[+]}|\stackrel{56}{[-]}$, $\stackrel{03}{[+i]}\stackrel{12}{[-]}|\stackrel{56}{[+]}$, and $\stackrel{03}{[-i]}\stackrel{12}{[+]}|\stackrel{56}{[+]}$), needed that the first, second, third and fourth family creation operators, respectively, applying on the vacuum state, give nonzero value.

There are superposition of the basic creation operators — $\hat{b}^{\alpha\dagger}_{i=(ch,s)}$ — which solve, applied on the vacuum state, the Weyl equation Eq. (3.36). Let us make the choice of $p^a = (p^0, p^1, p^2, p^3, 0, \cdots, 0)$ to see how the spin in $d = (5, 6)$ manifest charges in $d = (3+1)$.

$$
\begin{aligned}
p^a &= (p^0, p^1, p^2, p^3, 0, \cdots, 0)\,, \\
\hat{b}^{\alpha\dagger}_{(ch,sol)}(p)|\psi_{oc}> &= \sum_{s} c^{\alpha\, i=(ch,s)}{}_{(ch,sol)}(p)\, \hat{b}^{\alpha\dagger}_{i=(ch,s)} e^{-ip_a x^a}|\psi_{oc}>\,, \quad (3.96)
\end{aligned}
$$

where index $_{(ch,sol)}$, represents charges and different solutions, respectively, of the Weyl equation for massless free fermions.

We present in Eq. (3.97) the creation operators, the superposition of the first family members, presented in Table 3.4, which solve the Weyl equation, Eq. (3.36), for $p^a = (p^0, p^1, p^2, p^3, 0, 0)$. The corresponding annihilation operators follow by the Hermitian conjugation of the creation operators.

There are two fermion solutions with the charge $\frac{1}{2}$ and two antifermion solutions with the charge $-\frac{1}{2}$, both having the positive energy. The first two creation operators are related by the time reversal operator ($\mathcal{T}_{\mathcal{N}} = \gamma^1 \gamma^3 K I_{x^0} I_{x^5, x^7, \cdots, x^{d-1}}$), while the second two follow from the first two by the application of $\mathbb{C}_{\mathcal{N}} \mathcal{P}^{(d-1)}_{\mathcal{N}}$ $= \gamma^0 \gamma^5 \cdots \gamma^{d-1} I_{\vec{x}_3} I_{x^6, x^8, \ldots, x^d}$, both are presented in Eq. (3.94).

family α	$i=(ch,s)$	$\hat{b}^{\alpha\dagger}_{ch,s}$	$\hat{b}^{\alpha}_{ch,s}$	S^{03}	S^{12}	S^{56}	Γ^{3+1}	$\tilde{S}^{03}$	$\tilde{S}^{12}$	$\tilde{S}^{56}$
I	$(\frac{1}{2},\frac{1}{2})$	$\stackrel{03}{(+i)}\stackrel{12}{(+)}\,|\stackrel{56}{(+)}$	$(-)\stackrel{56}{(-)}\,|(-)\stackrel{12}{(-)}\stackrel{03}{(-i)}$	$\frac{i}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	1	$\frac{i}{2}$	$\frac{1}{2}$	$\frac{1}{2}$
I	$(\frac{1}{2},-\frac{1}{2})$	$\stackrel{03}{[-i]}\stackrel{12}{[-]}\,|\stackrel{56}{(+)}$	$(-)\stackrel{56}{(-)}\,|\stackrel{12}{[-]}\stackrel{03}{[-i]}$	$-\frac{i}{2}$	$-\frac{1}{2}$	$\frac{1}{2}$	1	$\frac{i}{2}$	$\frac{1}{2}$	$\frac{1}{2}$
I	$(-\frac{1}{2},\frac{1}{2})$	$\stackrel{03}{[-i]}\stackrel{12}{(+)}\,|\stackrel{56}{[-]}$	$\stackrel{56}{[-]}\,|(-)\stackrel{12}{(-)}\stackrel{03}{[-i]}$	$-\frac{i}{2}$	$\frac{1}{2}$	$-\frac{1}{2}$	-1	$\frac{i}{2}$	$\frac{1}{2}$	$\frac{1}{2}$
I	$(-\frac{1}{2},-\frac{1}{2})$	$\stackrel{03}{(+i)}\stackrel{12}{[-]}\,|\stackrel{56}{[-]}$	$\stackrel{56}{[-]}\,|\stackrel{12}{[-]}\stackrel{03}{(-i)}$	$\frac{i}{2}$	$-\frac{1}{2}$	$-\frac{1}{2}$	-1	$\frac{i}{2}$	$\frac{1}{2}$	$\frac{1}{2}$
II	$(\frac{1}{2},\frac{1}{2})$	$\stackrel{03}{[+i]}\stackrel{12}{[+]}\,|\stackrel{56}{(+)}$	$(-)\stackrel{56}{(-)}\,|\stackrel{12}{[+]}\stackrel{03}{[+i]}$	$\frac{i}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	1	$-\frac{i}{2}$	$-\frac{1}{2}$	$\frac{1}{2}$
II	$(\frac{1}{2},-\frac{1}{2})$	$\stackrel{03}{(-i)}\stackrel{12}{(-)}\,|\stackrel{56}{(+)}$	$(-)\stackrel{56}{(-)}\,|(-)\stackrel{12}{(+)}\stackrel{03}{(+i)}$	$-\frac{i}{2}$	$-\frac{1}{2}$	$\frac{1}{2}$	1	$-\frac{i}{2}$	$-\frac{1}{2}$	$\frac{1}{2}$
II	$(-\frac{1}{2},\frac{1}{2})$	$\stackrel{03}{(-i)}\stackrel{12}{[+]}\,|\stackrel{56}{[-]}$	$\stackrel{56}{[-]}\,|\stackrel{12}{[+]}\stackrel{03}{(+i)}$	$-\frac{i}{2}$	$\frac{1}{2}$	$-\frac{1}{2}$	-1	$-\frac{i}{2}$	$-\frac{1}{2}$	$\frac{1}{2}$
II	$(-\frac{1}{2},-\frac{1}{2})$	$\stackrel{03}{[+i]}\stackrel{12}{(-)}\,|\stackrel{56}{[-]}$	$\stackrel{56}{[-]}\,|(-)\stackrel{12}{(+)}\stackrel{03}{[+i]}$	$\frac{i}{2}$	$-\frac{1}{2}$	$-\frac{1}{2}$	-1	$-\frac{i}{2}$	$-\frac{1}{2}$	$\frac{1}{2}$
III	$(\frac{1}{2},\frac{1}{2})$	$\stackrel{03}{[+i]}\stackrel{12}{(+)}\,|\stackrel{56}{[+]}$	$\stackrel{56}{[+]}\,|(-)\stackrel{12}{(-)}\stackrel{03}{[+i]}$	$\frac{i}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	1	$-\frac{i}{2}$	$\frac{1}{2}$	$-\frac{1}{2}$
III	$(\frac{1}{2},-\frac{1}{2})$	$\stackrel{03}{(-i)}\stackrel{12}{[-]}\,|\stackrel{56}{[+]}$	$\stackrel{56}{[+]}\,|\stackrel{12}{[-]}\stackrel{03}{(+i)}$	$-\frac{i}{2}$	$-\frac{1}{2}$	$\frac{1}{2}$	1	$-\frac{i}{2}$	$\frac{1}{2}$	$-\frac{1}{2}$
III	$(-\frac{1}{2},\frac{1}{2})$	$\stackrel{03}{(-i)}\stackrel{12}{(+)}\,|\stackrel{56}{(-)}$	$(-)\stackrel{56}{(+)}\,|(-)\stackrel{12}{(-)}\stackrel{03}{(+i)}$	$-\frac{i}{2}$	$\frac{1}{2}$	$-\frac{1}{2}$	-1	$-\frac{i}{2}$	$\frac{1}{2}$	$-\frac{1}{2}$
III	$(-\frac{1}{2},-\frac{1}{2})$	$\stackrel{03}{[+i]}\stackrel{12}{[-]}\,|\stackrel{56}{(-)}$	$(-)\stackrel{56}{(+)}\,|\stackrel{12}{[-]}\stackrel{03}{[+i]}$	$\frac{i}{2}$	$-\frac{1}{2}$	$-\frac{1}{2}$	-1	$-\frac{i}{2}$	$\frac{1}{2}$	$-\frac{1}{2}$
IV	$(\frac{1}{2},\frac{1}{2})$	$\stackrel{03}{(+i)}\stackrel{12}{[+]}\,|\stackrel{56}{[+]}$	$\stackrel{56}{[+]}\,|\stackrel{12}{[+]}\stackrel{03}{(-i)}$	$\frac{i}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	1	$\frac{i}{2}$	$-\frac{1}{2}$	$-\frac{1}{2}$
IV	$(\frac{1}{2},-\frac{1}{2})$	$\stackrel{03}{[-i]}\stackrel{12}{(-)}\,|\stackrel{56}{[+]}$	$\stackrel{56}{[+]}\,|(-)\stackrel{12}{(+)}\stackrel{03}{[-]}$	$-\frac{i}{2}$	$-\frac{1}{2}$	$\frac{1}{2}$	1	$\frac{i}{2}$	$-\frac{1}{2}$	$-\frac{1}{2}$
IV	$(-\frac{1}{2},\frac{1}{2})$	$\stackrel{03}{[-i]}\stackrel{12}{[+]}\,|\stackrel{56}{(-)}$	$(-)\stackrel{56}{(+)}\,|\stackrel{12}{[+]}\stackrel{03}{[-i]}$	$-\frac{i}{2}$	$\frac{1}{2}$	$-\frac{1}{2}$	-1	$\frac{i}{2}$	$-\frac{1}{2}$	$-\frac{1}{2}$
IV	$(-\frac{1}{2},-\frac{1}{2})$	$\stackrel{03}{(+i)}\stackrel{12}{(-)}\,|\stackrel{56}{(-)}$	$(-)\stackrel{56}{(+)}\,|(-)\stackrel{12}{(+)}\stackrel{03}{(-i)}$	$\frac{i}{2}$	$-\frac{1}{2}$	$-\frac{1}{2}$	-1	$\frac{i}{2}$	$-\frac{1}{2}$	$-\frac{1}{2}$

Table 3.4. The basic creation operators — $\hat{b}^{\alpha\dagger}_{i=(ch,s)}$, ch (charge) and s (spin) explain the index i — and their annihilation partners — $\hat{b}^{\alpha}_{i=(ch,s)}$ — are presented for the $d=(5+1)$-dimensional case. The basic creation operators are the products of nilpotents and projectors, which are the "eigenstates" of the Cartan subalgebra generators, (S^{03}, S^{12}, S^{56}), ($\tilde{S}^{03}$, $\tilde{S}^{12}$, $\tilde{S}^{56}$), presented in Eq. (3.110). Operators $\hat{b}^{\dagger}_{ch,s}$ and $\hat{b}_{ch,s}$ fulfill the commutaion relations of Eq. (3.81).

$$
\begin{aligned}
&\text{Creation operators for the fermion states in Clifford space for } d=(5+1)\\
p^0 &= |p^0|\,,\\
\hat{b}^{I_1\dagger}_{\frac{1}{2},\frac{1}{2}}(\vec{p}) &= \beta\left(\stackrel{03}{(+i)}\stackrel{12}{(+)}\,|\stackrel{56}{(+)} + \frac{p^1+ip^2}{|p^0|+|p^3|}\stackrel{03}{[-i]}\stackrel{12}{[-]}\,|\stackrel{56}{(+)}\right) e^{-i(|p^0|x^0-\vec{p}\cdot\vec{x})}\,,\\
\hat{b}^{I_2\dagger}_{\frac{1}{2},-\frac{1}{2}}(\vec{p}) &= \beta^*\left(\stackrel{03}{[-i]}\stackrel{12}{[-]}\,|\stackrel{56}{(+)} - \frac{p^1-ip^2}{|p^0|+|p^3|}\stackrel{03}{(+i)}\stackrel{12}{(+)}\,|\stackrel{56}{(+)}\right) e^{-i(|p^0|x^0+\vec{p}\cdot\vec{x})}\,,\\
&\text{Creation operators for the antifermion states in Clifford space for } d=(5+1)\\
p^0 &= |p^0|\,,\\
\hat{b}^{I_3\dagger}_{-\frac{1}{2},\frac{1}{2}}(\vec{p}) &= -\beta\left(\stackrel{03}{[-i]}\stackrel{12}{(+)}\,|\stackrel{56}{[-]} + \frac{p^1+ip^2}{|p^0|+|p^3|}\stackrel{03}{(+i)}\stackrel{12}{[-]}\,|\stackrel{56}{[-]}\right) e^{-i(|p^0|x^0+\vec{p}\cdot\vec{x})}\,,\\
\hat{b}^{I_4\dagger}_{-\frac{1}{2},-\frac{1}{2}}(\vec{p}) &= -\beta^*\left(\stackrel{03}{(+i)}\stackrel{12}{[-]}\,|\stackrel{56}{[-]} - \frac{p^1-ip^2}{|p^0|+|p^3|}\stackrel{03}{[-i]}\stackrel{12}{(+)}\,|\stackrel{56}{[-]}\right) e^{-i(|p^0|x^0-\vec{p}\cdot\vec{x})}\,,
\end{aligned}
\tag{3.97}
$$

Index $_{i=(1,2,3,4)}$ counts the solutions, while $\beta^*\beta = \frac{|p^0|+|p^3|}{2|p^0|}$ takes care that the corresponding states are normalized. All the states are correspondingly orthogonalized. The coefficients $c^{\alpha\, i=(ch,s)}{}_{(ch,sol)}(p)$ can be read from the solutions. The solutions have the definite handedness and orientation of the spin with respect to the momentum: $\hat{b}^{I_1\dagger}_{\frac{1}{2},\frac{1}{2}}$ defines the state with $\Gamma^{(3+1)}=1$ and the spin and momentum both up, $\hat{b}^{I_2\dagger}_{\frac{1}{2},-\frac{1}{2}}$ defines the state with $\Gamma^{(3+1)}=1$ and with spin and momentum both down, $\hat{b}^{I_3\dagger}_{-\frac{1}{2},\frac{1}{2}}$ defines the state with $\Gamma^{(3+1)}=-1$ and the spin up and the momentum down, $\hat{b}^{I_4 \dagger}_{-\frac{1}{2},-\frac{1}{2}}$ defines the state with $\Gamma^{(3+1)} = -1$, the spin down and the momentum up.

The same indexes — $c^{\alpha\, i=(ch,s)}{}_{(ch,sol)}(p)$ — define the solution of the Weyl equation also for for the rest three families presented in Table 3.4.

The phases of creation operators are in agreement with the application of discrete symmetry operators $\mathbb{C}_{\mathcal{N}} \cdot \mathcal{P}_{\mathcal{N}}$, and $\mathcal{T}_{\mathcal{N}}$.

Let us point out that the scalar fields, interacting with fermions (in the *spin-charge-family* theory [[4,3] and the references cited therein] the scalar fields origin in the spin connection fields — ω_{abc}, the gauge fields of S^{ab}, and $\tilde{\omega}_{abc}$, the gauge fields of $\tilde{S}^{ab}$, appearing in Eq. (3.1) — with the space indexes $c \geq 5$) can make massless fermions massive [68,69,73,66]. In this case the creation operators (and correspondingly the annihilation operators) start to be superposition of basic operators of different charges ch as well :

$$(\hat{b}^{\alpha \dagger}_{sol'}(p) = \sum_{ch,sol} c^{\alpha, ch, sol}{}_{ch, sol'}(p)\, \hat{b}^{\alpha\dagger}_{ch,sol} e^{-ip_a x^a})\, |\psi_{oc}> .$$

In this case the solutions of the corresponding equations of motion, presented in Eq. (3.97) for massless states, become superposition of different charges and different families.

For $p^m = (0,0,0), m = (1,2,3)$ and one massive family only [66] the creation operators for the basic states (usually used in text books [74,75] for massive states) are presented at Table 3.5. The creation operators, presented in Table 3.5, define

family α	$\hat{b}^{\alpha\dagger}_{s,m}$	$\tilde{S}^{12}$
1	$\frac{1}{\sqrt{2}}\,(\stackrel{03}{(+i)}\stackrel{12}{(+)}\stackrel{56}{(+)} + \frac{m}{m_+}\,\stackrel{03}{[-i]}\stackrel{12}{(+)}\stackrel{56}{[-]})$	$\frac{1}{2}$
2	$\frac{1}{\sqrt{2}}\,(\stackrel{03}{[-i]}\stackrel{12}{[-]}\stackrel{56}{(+)} + \frac{m}{m_+}\,\stackrel{03}{(+i)}\stackrel{12}{[-]}\stackrel{56}{[-]})$	$-\frac{1}{2}$

Table 3.5. The basic creation operators — $\hat{b}^{\alpha\dagger}_{s,m}$ — for massive states, the first with spin up and the second with spin down, are presented. $\hat{b}^{\alpha\dagger}_{s,m}\, e^{-imx^0}$, $s = \pm\frac{1}{2}$, solve the equations of motion $\{p_0 + \gamma^0(\stackrel{56}{(+)}\; m_+ + \stackrel{56}{(-)}\; m_-)\}\, \hat{b}^{\alpha\dagger}_{s,m}\, e^{-imx^0} = 0$, for the two positive energy states, (1,2), (one with spin up and the other with spin down). $m^2 = m_+ m_-$, $m_+ = -m_-$, $(p_0)^2 = m^2$, $p_{0a} = -\frac{1}{2} S^{cd} \omega_{cda}$ is assumed to be real [66].

orthonormal states when applied on the vacuum state and fulfill, together with the annihilation operators, the anticommutation relations presented in Eq. (3.83).

Clifford case in $d = (13+1)$:

There are $2^{\frac{14}{2}-1} = 64$ creation operators for family members of one family, all reachable from the starting one by S^{ab}. They are presented in Table 3.6, analyzed so that the internal degrees of freedom manifest in $d = (3+1)$ quantum numbers of the observed quarks and leptons. Applied on the vacuum state $|\psi_{oc}>$ they form in the *spin-charge-family* theory 64 basic states *for quarks and leptons and anti-quarks and anti-leptons* for each family. In the *spin-charge-family* theory there are two times four families — $2^{\frac{8}{2}-1}$ — getting masses after the two triplet scalar fields, the superposition of $\tilde{\omega}_{abc}$, $(a,b)=(0,1,\cdots,8)$ and three singlet scalar fields, the superposition of ω_{abc}, $(a,b)=(5,6)$ or $(7,8)$ or $(9,\cdots 14)$, while $c=(5,6,7,8)$ for all these scalar fields, get nonzero vacuum expectation values at low energies [9,3,4,6,7].

Table 3.1 represents the creation operators creating 8 families of $\hat{u}_R^{c1\dagger}$ and of $\hat{\nu}_R^{\dagger}$. All the family members of each of these families follow by the application of S^{ab}.

All the rest of families not included in these eight families get in the spin-charge-family theory masses by the interaction with the condensate [9,3,4,6,7].

To the lower four families the three so far observed families of quarks and leptons belong.

i		$^{a}\hat{b}_i^{\dagger}$ (Anti)octet, $\Gamma^{(7+1)}=(-1)\,1$, $\Gamma^{(6)}=(1)\,-1$ of (anti)quarks and (anti)leptons	$\Gamma^{(3+1)}$	S^{12}	τ^{13}	τ^{23}	τ^{33}	τ^{38}	τ^{4}	Y	Q
1	$\hat{u}_R^{c1\dagger}$	$\stackrel{03}{(+i)}\stackrel{12}{[+]}\mid\stackrel{56}{[+]}\stackrel{78}{(+)}\mid\mid\stackrel{9\,10}{(+)}\stackrel{11\,12}{[-]}\stackrel{13\,14}{[-]}$	1	$\frac{1}{2}$	0	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2\sqrt{3}}$	$\frac{1}{6}$	$\frac{2}{3}$	$\frac{2}{3}$
2	$\hat{u}_R^{c1\dagger}$	$\stackrel{03}{[-i]}\stackrel{12}{(-)}\mid\stackrel{56}{[+]}\stackrel{78}{(+)}\mid\mid\stackrel{9\,10}{(+)}\stackrel{11\,12}{[-]}\stackrel{13\,14}{[-]}$	1	$-\frac{1}{2}$	0	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2\sqrt{3}}$	$\frac{1}{6}$	$\frac{2}{3}$	$\frac{2}{3}$
3	$\hat{d}_R^{c1\dagger}$	$\stackrel{03}{(+i)}\stackrel{12}{[+]}\mid\stackrel{56}{(-)}\stackrel{78}{[-]}\mid\mid\stackrel{9\,10}{(+)}\stackrel{11\,12}{[-]}\stackrel{13\,14}{[-]}$	1	$\frac{1}{2}$	0	$-\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2\sqrt{3}}$	$\frac{1}{6}$	$-\frac{1}{3}$	$-\frac{1}{3}$
4	$\hat{d}_R^{c1\dagger}$	$\stackrel{03}{[-i]}\stackrel{12}{(-)}\mid\stackrel{56}{(-)}\stackrel{78}{[-]}\mid\mid\stackrel{9\,10}{(+)}\stackrel{11\,12}{[-]}\stackrel{13\,14}{[-]}$	1	$-\frac{1}{2}$	0	$-\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2\sqrt{3}}$	$\frac{1}{6}$	$-\frac{1}{3}$	$-\frac{1}{3}$
5	$\hat{d}_L^{c1\dagger}$	$\stackrel{03}{[-i]}\stackrel{12}{[+]}\mid\stackrel{56}{(-)}\stackrel{78}{(+)}\mid\mid\stackrel{9\,10}{(+)}\stackrel{11\,12}{[-]}\stackrel{13\,14}{[-]}$	-1	$\frac{1}{2}$	$-\frac{1}{2}$	0	$\frac{1}{2}$	$\frac{1}{2\sqrt{3}}$	$\frac{1}{6}$	$\frac{1}{6}$	$-\frac{1}{3}$
6	$\hat{d}_L^{c1\dagger}$	$\stackrel{03}{(+i)}\stackrel{12}{(-)}\mid\stackrel{56}{(-)}\stackrel{78}{(+)}\mid\mid\stackrel{9\,10}{(+)}\stackrel{11\,12}{[-]}\stackrel{13\,14}{[-]}$	-1	$-\frac{1}{2}$	$-\frac{1}{2}$	0	$\frac{1}{2}$	$\frac{1}{2\sqrt{3}}$	$\frac{1}{6}$	$\frac{1}{6}$	$-\frac{1}{3}$
7	$\hat{u}_L^{c1\dagger}$	$\stackrel{03}{[-i]}\stackrel{12}{[+]}\mid\stackrel{56}{[+]}\stackrel{78}{[-]}\mid\mid\stackrel{9\,10}{(+)}\stackrel{11\,12}{[-]}\stackrel{13\,14}{[-]}$	-1	$\frac{1}{2}$	$\frac{1}{2}$	0	$\frac{1}{2}$	$\frac{1}{2\sqrt{3}}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{2}{3}$
8	$\hat{u}_L^{c1\dagger}$	$\stackrel{03}{(+i)}\stackrel{12}{(-)}\mid\stackrel{56}{[+]}\stackrel{78}{[-]}\mid\mid\stackrel{9\,10}{(+)}\stackrel{11\,12}{[-]}\stackrel{13\,14}{[-]}$	-1	$-\frac{1}{2}$	$\frac{1}{2}$	0	$\frac{1}{2}$	$\frac{1}{2\sqrt{3}}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{2}{3}$
9	$\hat{u}_R^{c2\dagger}$	$\stackrel{03}{(+i)}\stackrel{12}{[+]}\mid\stackrel{56}{[+]}\stackrel{78}{(+)}\mid\mid\stackrel{9\,10}{[-]}\stackrel{11\,12}{(+)}\stackrel{13\,14}{[-]}$	1	$\frac{1}{2}$	0	$\frac{1}{2}$	$-\frac{1}{2}$	$\frac{1}{2\sqrt{3}}$	$\frac{1}{6}$	$\frac{2}{3}$	$\frac{2}{3}$
10	$\hat{u}_R^{c2\dagger}$	$\stackrel{03}{[-i]}\stackrel{12}{(-)}\mid\stackrel{56}{[+]}\stackrel{78}{(+)}\mid\mid\stackrel{9\,10}{[-]}\stackrel{11\,12}{(+)}\stackrel{13\,14}{[-]}$	1	$-\frac{1}{2}$	0	$\frac{1}{2}$	$-\frac{1}{2}$	$\frac{1}{2\sqrt{3}}$	$\frac{1}{6}$	$\frac{2}{3}$	$\frac{2}{3}$
11	$\hat{d}_R^{c2\dagger}$	$\stackrel{03}{(+i)}\stackrel{12}{[+]}\mid\stackrel{56}{(-)}\stackrel{78}{[-]}\mid\mid\stackrel{9\,10}{[-]}\stackrel{11\,12}{(+)}\stackrel{13\,14}{[-]}$	1	$\frac{1}{2}$	0	$-\frac{1}{2}$	$-\frac{1}{2}$	$\frac{1}{2\sqrt{3}}$	$\frac{1}{6}$	$-\frac{1}{3}$	$-\frac{1}{3}$
12	$\hat{d}_R^{c2\dagger}$	$\stackrel{03}{[-i]}\stackrel{12}{(-)}\mid\stackrel{56}{(-)}\stackrel{78}{[-]}\mid\mid\stackrel{9\,10}{[-]}\stackrel{11\,12}{(+)}\stackrel{13\,14}{[-]}$	1	$-\frac{1}{2}$	0	$-\frac{1}{2}$	$-\frac{1}{2}$	$\frac{1}{2\sqrt{3}}$	$\frac{1}{6}$	$-\frac{1}{3}$	$-\frac{1}{3}$
13	$\hat{d}_L^{c2\dagger}$	$\stackrel{03}{[-i]}\stackrel{12}{[+]}\mid\stackrel{56}{(-)}\stackrel{78}{(+)}\mid\mid\stackrel{9\,10}{[-]}\stackrel{11\,12}{(+)}\stackrel{13\,14}{[-]}$	-1	$\frac{1}{2}$	$-\frac{1}{2}$	0	$-\frac{1}{2}$	$\frac{1}{2\sqrt{3}}$	$\frac{1}{6}$	$\frac{1}{6}$	$-\frac{1}{3}$
14	$\hat{d}_L^{c2\dagger}$	$\stackrel{03}{(+i)}\stackrel{12}{(-)}\mid\stackrel{56}{(-)}\stackrel{78}{(+)}\mid\mid\stackrel{9\,10}{[-]}\stackrel{11\,12}{(+)}\stackrel{13\,14}{[-]}$	-1	$-\frac{1}{2}$	$-\frac{1}{2}$	0	$-\frac{1}{2}$	$\frac{1}{2\sqrt{3}}$	$\frac{1}{6}$	$\frac{1}{6}$	$-\frac{1}{3}$
15	$\hat{u}_L^{c2\dagger}$	$\stackrel{03}{[-i]}\stackrel{12}{[+]}\mid\stackrel{56}{[+]}\stackrel{78}{[-]}\mid\mid\stackrel{9\,10}{[-]}\stackrel{11\,12}{(+)}\stackrel{13\,14}{[-]}$	-1	$\frac{1}{2}$	$\frac{1}{2}$	0	$-\frac{1}{2}$	$\frac{1}{2\sqrt{3}}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{2}{3}$
16	$\hat{u}_L^{c2\dagger}$	$\stackrel{03}{(+i)}\stackrel{12}{(-)}\mid\stackrel{56}{[+]}\stackrel{78}{[-]}\mid\mid\stackrel{9\,10}{[-]}\stackrel{11\,12}{(+)}\stackrel{13\,14}{[-]}$	-1	$-\frac{1}{2}$	$\frac{1}{2}$	0	$-\frac{1}{2}$	$\frac{1}{2\sqrt{3}}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{2}{3}$
17	$\hat{u}_R^{c3\dagger}$	$\stackrel{03}{(+i)}\stackrel{12}{[+]}\mid\stackrel{56}{[+]}\stackrel{78}{(+)}\mid\mid\stackrel{9\,10}{[-]}\stackrel{11\,12}{[-]}\stackrel{13\,14}{(+)}$	1	$\frac{1}{2}$	0	$\frac{1}{2}$	0	$-\frac{1}{\sqrt{3}}$	$\frac{1}{6}$	$\frac{2}{3}$	$\frac{2}{3}$
18	$\hat{u}_R^{c3\dagger}$	$\stackrel{03}{[-i]}\stackrel{12}{(-)}\mid\stackrel{56}{[+]}\stackrel{78}{(+)}\mid\mid\stackrel{9\,10}{[-]}\stackrel{11\,12}{[-]}\stackrel{13\,14}{(+)}$	1	$-\frac{1}{2}$	0	$\frac{1}{2}$	0	$-\frac{1}{\sqrt{3}}$	$\frac{1}{6}$	$\frac{2}{3}$	$\frac{2}{3}$
19	$\hat{d}_R^{c3\dagger}$	$\stackrel{03}{(+i)}\stackrel{12}{[+]}\mid\stackrel{56}{(-)}\stackrel{78}{[-]}\mid\mid\stackrel{9\,10}{[-]}\stackrel{11\,12}{[-]}\stackrel{13\,14}{(+)}$	1	$\frac{1}{2}$	0	$-\frac{1}{2}$	0	$-\frac{1}{\sqrt{3}}$	$\frac{1}{6}$	$-\frac{1}{3}$	$-\frac{1}{3}$
20	$\hat{d}_R^{c3\dagger}$	$\stackrel{03}{[-i]}\stackrel{12}{(-)}\mid\stackrel{56}{(-)}\stackrel{78}{[-]}\mid\mid\stackrel{9\,10}{[-]}\stackrel{11\,12}{[-]}\stackrel{13\,14}{(+)}$	1	$-\frac{1}{2}$	0	$-\frac{1}{2}$	0	$-\frac{1}{\sqrt{3}}$	$\frac{1}{6}$	$-\frac{1}{3}$	$-\frac{1}{3}$
21	$\hat{d}_L^{c3\dagger}$	$\stackrel{03}{[-i]}\stackrel{12}{[+]}\mid\stackrel{56}{(-)}\stackrel{78}{(+)}\mid\mid\stackrel{9\,10}{[-]}\stackrel{11\,12}{[-]}\stackrel{13\,14}{(+)}$	-1	$\frac{1}{2}$	$-\frac{1}{2}$	0	0	$-\frac{1}{\sqrt{3}}$	$\frac{1}{6}$	$\frac{1}{6}$	$-\frac{1}{3}$
22	$\hat{d}_L^{c3\dagger}$	$\stackrel{03}{(+i)}\stackrel{12}{(-)}\mid\stackrel{56}{(-)}\stackrel{78}{(+)}\mid\mid\stackrel{9\,10}{[-]}\stackrel{11\,12}{[-]}\stackrel{13\,14}{(+)}$	-1	$-\frac{1}{2}$	$-\frac{1}{2}$	0	0	$-\frac{1}{\sqrt{3}}$	$\frac{1}{6}$	$\frac{1}{6}$	$-\frac{1}{3}$
23	$\hat{u}_L^{c3\dagger}$	$\stackrel{03}{[-i]}\stackrel{12}{[+]}\mid\stackrel{56}{[+]}\stackrel{78}{[-]}\mid\mid\stackrel{9\,10}{[-]}\stackrel{11\,12}{[-]}\stackrel{13\,14}{(+)}$	-1	$\frac{1}{2}$	$\frac{1}{2}$	0	0	$-\frac{1}{\sqrt{3}}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{2}{3}$
24	$\hat{u}_L^{c3\dagger}$	$\stackrel{03}{(+i)}\stackrel{12}{(-)}\mid\stackrel{56}{[+]}\stackrel{78}{[-]}\mid\mid\stackrel{9\,10}{[-]}\stackrel{11\,12}{[-]}\stackrel{13\,14}{(+)}$	-1	$-\frac{1}{2}$	$\frac{1}{2}$	0	0	$-\frac{1}{\sqrt{3}}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{2}{3}$
25	$\hat{\nu}_R^{\dagger}$	$\stackrel{03}{(+i)}\stackrel{12}{[+]}\mid\stackrel{56}{[+]}\stackrel{78}{(+)}\mid\mid\stackrel{9\,10}{(+)}\stackrel{11\,12}{(+)}\stackrel{13\,14}{(+)}$	1	$\frac{1}{2}$	0	$\frac{1}{2}$	0	0	$-\frac{1}{2}$	0	0

Continued on next page

i		$^a\hat{b}^\dagger_i$	$\Gamma^{(3+1)}$	S^{12}	τ^{13}	τ^{23}	τ^{33}	τ^{38}	τ^4	Y	Q
		(Anti)octet, $\Gamma^{(7+1)} = (-1)\,1$, $\Gamma^{(6)} = (1)\,-1$ of (anti)quarks and (anti)leptons									
26	$\hat{\nu}^\dagger_R$	$\stackrel{03}{[-i]}\stackrel{12}{(-)}\vert\stackrel{56}{[+]}\stackrel{78}{(+)}\Vert\stackrel{9\,10}{(+)}\stackrel{11\,12}{(+)}\stackrel{13\,14}{(+)}$	1	$-\frac{1}{2}$	0	$\frac{1}{2}$	0	0	$-\frac{1}{2}$	0	0
27	$\hat{e}^\dagger_R$	$\stackrel{03}{(+i)}\stackrel{12}{[+]}\vert\stackrel{56}{(-)}\stackrel{78}{[-]}\Vert\stackrel{9\,10}{(+)}\stackrel{11\,12}{(+)}\stackrel{13\,14}{(+)}$	1	$\frac{1}{2}$	0	$-\frac{1}{2}$	0	0	$-\frac{1}{2}$	-1	-1
28	$\hat{e}^\dagger_R$	$\stackrel{03}{[-i]}\stackrel{12}{(-)}\vert\stackrel{56}{(-)}\stackrel{78}{[-]}\Vert\stackrel{9\,10}{(+)}\stackrel{11\,12}{(+)}\stackrel{13\,14}{(+)}$	1	$-\frac{1}{2}$	0	$-\frac{1}{2}$	0	0	$-\frac{1}{2}$	-1	-1
29	$\hat{e}^\dagger_L$	$\stackrel{03}{[-i]}\stackrel{12}{[+]}\vert\stackrel{56}{(-)}\stackrel{78}{(+)}\Vert\stackrel{9\,10}{(+)}\stackrel{11\,12}{(+)}\stackrel{13\,14}{(+)}$	-1	$\frac{1}{2}$	$-\frac{1}{2}$	0	0	0	$-\frac{1}{2}$	$-\frac{1}{2}$	-1
30	$\hat{e}^\dagger_L$	$\stackrel{03}{(+i)}\stackrel{12}{(-)}\vert\stackrel{56}{(-)}\stackrel{78}{(+)}\Vert\stackrel{9\,10}{(+)}\stackrel{11\,12}{(+)}\stackrel{13\,14}{(+)}$	-1	$-\frac{1}{2}$	$-\frac{1}{2}$	0	0	0	$-\frac{1}{2}$	$-\frac{1}{2}$	-1
31	$\hat{\nu}^\dagger_L$	$\stackrel{03}{[-i]}\stackrel{12}{[+]}\vert\stackrel{56}{[+]}\stackrel{78}{[-]}\Vert\stackrel{9\,10}{(+)}\stackrel{11\,12}{(+)}\stackrel{13\,14}{(+)}$	-1	$\frac{1}{2}$	$\frac{1}{2}$	0	0	0	$-\frac{1}{2}$	$-\frac{1}{2}$	0
32	$\hat{\nu}^\dagger_L$	$\stackrel{03}{(+i)}\stackrel{12}{(-)}\vert\stackrel{56}{[+]}\stackrel{78}{[-]}\Vert\stackrel{9\,10}{(+)}\stackrel{11\,12}{(+)}\stackrel{13\,14}{(+)}$	-1	$-\frac{1}{2}$	$\frac{1}{2}$	0	0	0	$-\frac{1}{2}$	$-\frac{1}{2}$	0
33	$\hat{d}^{\bar{c}1\dagger}_L$	$\stackrel{03}{[-i]}\stackrel{12}{[+]}\vert\stackrel{56}{[+]}\stackrel{78}{(+)}\Vert\stackrel{9\,10}{[-]}\stackrel{11\,12}{(+)}\stackrel{13\,14}{(+)}$	-1	$\frac{1}{2}$	0	$\frac{1}{2}$	$-\frac{1}{2}$	$-\frac{1}{2\sqrt{3}}$	$-\frac{1}{6}$	$\frac{1}{3}$	$\frac{1}{3}$
34	$\hat{d}^{\bar{c}1\dagger}_L$	$\stackrel{03}{(+i)}\stackrel{12}{(-)}\vert\stackrel{56}{[+]}\stackrel{78}{(+)}\Vert\stackrel{9\,10}{[-]}\stackrel{11\,12}{(+)}\stackrel{13\,14}{(+)}$	-1	$-\frac{1}{2}$	0	$\frac{1}{2}$	$-\frac{1}{2}$	$-\frac{1}{2\sqrt{3}}$	$-\frac{1}{6}$	$\frac{1}{3}$	$\frac{1}{3}$
35	$\hat{u}^{\bar{c}1\dagger}_L$	$\stackrel{03}{[-i]}\stackrel{12}{[+]}\vert\stackrel{56}{(-)}\stackrel{78}{[-]}\Vert\stackrel{9\,10}{[-]}\stackrel{11\,12}{(+)}\stackrel{13\,14}{(+)}$	-1	$\frac{1}{2}$	0	$-\frac{1}{2}$	$-\frac{1}{2}$	$-\frac{1}{2\sqrt{3}}$	$-\frac{1}{6}$	$-\frac{2}{3}$	$-\frac{2}{3}$
36	$\hat{u}^{\bar{c}1\dagger}_L$	$\stackrel{03}{(+i)}\stackrel{12}{(-)}\vert\stackrel{56}{(-)}\stackrel{78}{[-]}\Vert\stackrel{9\,10}{[-]}\stackrel{11\,12}{(+)}\stackrel{13\,14}{(+)}$	-1	$-\frac{1}{2}$	0	$-\frac{1}{2}$	$-\frac{1}{2}$	$-\frac{1}{2\sqrt{3}}$	$-\frac{1}{6}$	$-\frac{2}{3}$	$-\frac{2}{3}$
37	$\hat{d}^{\bar{c}1\dagger}_R$	$\stackrel{03}{(+i)}\stackrel{12}{[+]}\vert\stackrel{56}{[+]}\stackrel{78}{[-]}\Vert\stackrel{9\,10}{[-]}\stackrel{11\,12}{(+)}\stackrel{13\,14}{(+)}$	1	$\frac{1}{2}$	$\frac{1}{2}$	0	$-\frac{1}{2}$	$-\frac{1}{2\sqrt{3}}$	$-\frac{1}{6}$	$-\frac{1}{6}$	$\frac{1}{3}$
38	$\hat{d}^{\bar{c}1\dagger}_R$	$\stackrel{03}{[-i]}\stackrel{12}{(-)}\vert\stackrel{56}{[+]}\stackrel{78}{[-]}\Vert\stackrel{9\,10}{[-]}\stackrel{11\,12}{(+)}\stackrel{13\,14}{(+)}$	1	$-\frac{1}{2}$	$\frac{1}{2}$	0	$-\frac{1}{2}$	$-\frac{1}{2\sqrt{3}}$	$-\frac{1}{6}$	$-\frac{1}{6}$	$\frac{1}{3}$
39	$\hat{u}^{\bar{c}1\dagger}_R$	$\stackrel{03}{(+i)}\stackrel{12}{[+]}\vert\stackrel{56}{(-)}\stackrel{78}{(+)}\Vert\stackrel{9\,10}{[-]}\stackrel{11\,12}{(+)}\stackrel{13\,14}{(+)}$	1	$\frac{1}{2}$	$-\frac{1}{2}$	0	$-\frac{1}{2}$	$-\frac{1}{2\sqrt{3}}$	$-\frac{1}{6}$	$-\frac{1}{6}$	$-\frac{2}{3}$
40	$\hat{u}^{\bar{c}1\dagger}_R$	$\stackrel{03}{[-i]}\stackrel{12}{(-)}\vert\stackrel{56}{(-)}\stackrel{78}{(+)}\Vert\stackrel{9\,10}{[-]}\stackrel{11\,12}{(+)}\stackrel{13\,14}{(+)}$	1	$-\frac{1}{2}$	$-\frac{1}{2}$	0	$-\frac{1}{2}$	$-\frac{1}{2\sqrt{3}}$	$-\frac{1}{6}$	$-\frac{1}{6}$	$-\frac{2}{3}$
41	$\hat{d}^{\bar{c}2\dagger}_L$	$\stackrel{03}{[-i]}\stackrel{12}{[+]}\vert\stackrel{56}{[+]}\stackrel{78}{(+)}\Vert\stackrel{9\,10}{(+)}\stackrel{11\,12}{[-]}\stackrel{13\,14}{(+)}$	-1	$\frac{1}{2}$	0	$\frac{1}{2}$	$\frac{1}{2}$	$-\frac{1}{2\sqrt{3}}$	$-\frac{1}{6}$	$\frac{1}{3}$	$\frac{1}{3}$
42	$\hat{d}^{\bar{c}2\dagger}_L$	$\stackrel{03}{(+i)}\stackrel{12}{(-)}\vert\stackrel{56}{[+]}\stackrel{78}{(+)}\Vert\stackrel{9\,10}{(+)}\stackrel{11\,12}{[-]}\stackrel{13\,14}{(+)}$	-1	$-\frac{1}{2}$	0	$\frac{1}{2}$	$\frac{1}{2}$	$-\frac{1}{2\sqrt{3}}$	$-\frac{1}{6}$	$\frac{1}{3}$	$\frac{1}{3}$
43	$\hat{u}^{\bar{c}2\dagger}_L$	$\stackrel{03}{[-i]}\stackrel{12}{[+]}\vert\stackrel{56}{(-)}\stackrel{78}{[-]}\Vert\stackrel{9\,10}{(+)}\stackrel{11\,12}{[-]}\stackrel{13\,14}{(+)}$	-1	$\frac{1}{2}$	0	$-\frac{1}{2}$	$\frac{1}{2}$	$-\frac{1}{2\sqrt{3}}$	$-\frac{1}{6}$	$-\frac{2}{3}$	$-\frac{2}{3}$
44	$\hat{u}^{\bar{c}2\dagger}_L$	$\stackrel{03}{(+i)}\stackrel{12}{(-)}\vert\stackrel{56}{(-)}\stackrel{78}{[-]}\Vert\stackrel{9\,10}{(+)}\stackrel{11\,12}{[-]}\stackrel{13\,14}{(+)}$	-1	$-\frac{1}{2}$	0	$-\frac{1}{2}$	$\frac{1}{2}$	$-\frac{1}{2\sqrt{3}}$	$-\frac{1}{6}$	$-\frac{2}{3}$	$-\frac{2}{3}$
45	$\hat{d}^{\bar{c}2\dagger}_R$	$\stackrel{03}{(+i)}\stackrel{12}{[+]}\vert\stackrel{56}{[+]}\stackrel{78}{[-]}\Vert\stackrel{9\,10}{(+)}\stackrel{11\,12}{[-]}\stackrel{13\,14}{(+)}$	1	$\frac{1}{2}$	$\frac{1}{2}$	0	$\frac{1}{2}$	$-\frac{1}{2\sqrt{3}}$	$-\frac{1}{6}$	$-\frac{1}{6}$	$\frac{1}{3}$
46	$\hat{d}^{\bar{c}2\dagger}_R$	$\stackrel{03}{[-i]}\stackrel{12}{(-)}\vert\stackrel{56}{[+]}\stackrel{78}{[-]}\Vert\stackrel{9\,10}{(+)}\stackrel{11\,12}{[-]}\stackrel{13\,14}{(+)}$	1	$-\frac{1}{2}$	$\frac{1}{2}$	0	$\frac{1}{2}$	$-\frac{1}{2\sqrt{3}}$	$-\frac{1}{6}$	$-\frac{1}{6}$	$\frac{1}{3}$
47	$\hat{u}^{\bar{c}2\dagger}_R$	$\stackrel{03}{(+i)}\stackrel{12}{[+]}\vert\stackrel{56}{(-)}\stackrel{78}{(+)}\Vert\stackrel{9\,10}{(+)}\stackrel{11\,12}{[-]}\stackrel{13\,14}{(+)}$	1	$\frac{1}{2}$	$-\frac{1}{2}$	0	$\frac{1}{2}$	$-\frac{1}{2\sqrt{3}}$	$-\frac{1}{6}$	$-\frac{1}{6}$	$-\frac{2}{3}$
48	$\hat{u}^{\bar{c}2\dagger}_R$	$\stackrel{03}{[-i]}\stackrel{12}{(-)}\vert\stackrel{56}{(-)}\stackrel{78}{(+)}\Vert\stackrel{9\,10}{(+)}\stackrel{11\,12}{[-]}\stackrel{13\,14}{(+)}$	1	$-\frac{1}{2}$	$-\frac{1}{2}$	0	$\frac{1}{2}$	$-\frac{1}{2\sqrt{3}}$	$-\frac{1}{6}$	$-\frac{1}{6}$	$-\frac{2}{3}$
49	$\hat{d}^{\bar{c}3\dagger}_L$	$\stackrel{03}{[-i]}\stackrel{12}{[+]}\vert\stackrel{56}{[+]}\stackrel{78}{(+)}\Vert\stackrel{9\,10}{(+)}\stackrel{11\,12}{(+)}\stackrel{13\,14}{[-]}$	-1	$\frac{1}{2}$	0	$\frac{1}{2}$	0	$\frac{1}{\sqrt{3}}$	$-\frac{1}{6}$	$\frac{1}{3}$	$\frac{1}{3}$
50	$\hat{d}^{\bar{c}3\dagger}_L$	$\stackrel{03}{(+i)}\stackrel{12}{(-)}\vert\stackrel{56}{[+]}\stackrel{78}{(+)}\Vert\stackrel{9\,10}{(+)}\stackrel{11\,12}{(+)}\stackrel{13\,14}{[-]}$	-1	$-\frac{1}{2}$	0	$\frac{1}{2}$	0	$\frac{1}{\sqrt{3}}$	$-\frac{1}{6}$	$\frac{1}{3}$	$\frac{1}{3}$
51	$\hat{u}^{\bar{c}3\dagger}_L$	$\stackrel{03}{[-i]}\stackrel{12}{[+]}\vert\stackrel{56}{(-)}\stackrel{78}{[-]}\Vert\stackrel{9\,10}{(+)}\stackrel{11\,12}{(+)}\stackrel{13\,14}{[-]}$	-1	$\frac{1}{2}$	0	$-\frac{1}{2}$	0	$\frac{1}{\sqrt{3}}$	$-\frac{1}{6}$	$-\frac{2}{3}$	$-\frac{2}{3}$
52	$\hat{u}^{\bar{c}3\dagger}_L$	$\stackrel{03}{(+i)}\stackrel{12}{(-)}\vert\stackrel{56}{(-)}\stackrel{78}{[-]}\Vert\stackrel{9\,10}{(+)}\stackrel{11\,12}{(+)}\stackrel{13\,14}{[-]}$	-1	$-\frac{1}{2}$	0	$-\frac{1}{2}$	0	$\frac{1}{\sqrt{3}}$	$-\frac{1}{6}$	$-\frac{2}{3}$	$-\frac{2}{3}$
53	$\hat{d}^{\bar{c}3\dagger}_R$	$\stackrel{03}{(+i)}\stackrel{12}{[+]}\vert\stackrel{56}{[+]}\stackrel{78}{[-]}\Vert\stackrel{9\,10}{(+)}\stackrel{11\,12}{(+)}\stackrel{13\,14}{[-]}$	1	$\frac{1}{2}$	$\frac{1}{2}$	0	0	$\frac{1}{\sqrt{3}}$	$-\frac{1}{6}$	$-\frac{1}{6}$	$\frac{1}{3}$
54	$\hat{d}^{\bar{c}3\dagger}_R$	$\stackrel{03}{[-i]}\stackrel{12}{(-)}\vert\stackrel{56}{[+]}\stackrel{78}{[-]}\Vert\stackrel{9\,10}{(+)}\stackrel{11\,12}{(+)}\stackrel{13\,14}{[-]}$	1	$-\frac{1}{2}$	$\frac{1}{2}$	0	0	$\frac{1}{\sqrt{3}}$	$-\frac{1}{6}$	$-\frac{1}{6}$	$\frac{1}{3}$
55	$\hat{u}^{\bar{c}3\dagger}_R$	$\stackrel{03}{(+i)}\stackrel{12}{[+]}\vert\stackrel{56}{(-)}\stackrel{78}{(+)}\Vert\stackrel{9\,10}{(+)}\stackrel{11\,12}{(+)}\stackrel{13\,14}{[-]}$	1	$\frac{1}{2}$	$-\frac{1}{2}$	0	0	$\frac{1}{\sqrt{3}}$	$-\frac{1}{6}$	$-\frac{1}{6}$	$-\frac{2}{3}$
56	$\hat{u}^{\bar{c}3\dagger}_R$	$\stackrel{03}{[-i]}\stackrel{12}{(-)}\vert\stackrel{56}{(-)}\stackrel{78}{(+)}\Vert\stackrel{9\,10}{(+)}\stackrel{11\,12}{(+)}\stackrel{13\,14}{[-]}$	1	$-\frac{1}{2}$	$-\frac{1}{2}$	0	0	$\frac{1}{\sqrt{3}}$	$-\frac{1}{6}$	$-\frac{1}{6}$	$-\frac{2}{3}$
57	$\hat{\bar{e}}^\dagger_L$	$\stackrel{03}{[-i]}\stackrel{12}{[+]}\vert\stackrel{56}{[+]}\stackrel{78}{(+)}\Vert\stackrel{9\,10}{[-]}\stackrel{11\,12}{[-]}\stackrel{13\,14}{[-]}$	-1	$\frac{1}{2}$	0	$\frac{1}{2}$	0	0	$\frac{1}{2}$	1	1
58	$\hat{\bar{e}}^\dagger_L$	$\stackrel{03}{(+i)}\stackrel{12}{(-)}\vert\stackrel{56}{[+]}\stackrel{78}{(+)}\Vert\stackrel{9\,10}{[-]}\stackrel{11\,12}{[-]}\stackrel{13\,14}{[-]}$	-1	$-\frac{1}{2}$	0	$\frac{1}{2}$	0	0	$\frac{1}{2}$	1	1
59	$\hat{\bar{\nu}}^\dagger_L$	$\stackrel{03}{[-i]}\stackrel{12}{[+]}\vert\stackrel{56}{(-)}\stackrel{78}{[-]}\Vert\stackrel{9\,10}{[-]}\stackrel{11\,12}{[-]}\stackrel{13\,14}{[-]}$	-1	$\frac{1}{2}$	0	$-\frac{1}{2}$	0	0	$\frac{1}{2}$	0	0
60	$\hat{\bar{\nu}}^\dagger_L$	$\stackrel{03}{(+i)}\stackrel{12}{(-)}\vert\stackrel{56}{(-)}\stackrel{78}{[-]}\Vert\stackrel{9\,10}{[-]}\stackrel{11\,12}{[-]}\stackrel{13\,14}{[-]}$	-1	$-\frac{1}{2}$	0	$-\frac{1}{2}$	0	0	$\frac{1}{2}$	0	0
61	$\hat{\bar{\nu}}^\dagger_R$	$\stackrel{03}{(+i)}\stackrel{12}{[+]}\vert\stackrel{56}{(-)}\stackrel{78}{(+)}\Vert\stackrel{9\,10}{[-]}\stackrel{11\,12}{[-]}\stackrel{13\,14}{[-]}$	1	$\frac{1}{2}$	$-\frac{1}{2}$	0	0	0	$\frac{1}{2}$	$\frac{1}{2}$	0
62	$\hat{\bar{\nu}}^\dagger_R$	$\stackrel{03}{[-i]}\stackrel{12}{(-)}\vert\stackrel{56}{(-)}\stackrel{78}{(+)}\Vert\stackrel{9\,10}{[-]}\stackrel{11\,12}{[-]}\stackrel{13\,14}{[-]}$	1	$-\frac{1}{2}$	$-\frac{1}{2}$	0	0	0	$\frac{1}{2}$	$\frac{1}{2}$	0
63	$\hat{\bar{e}}^\dagger_R$	$\stackrel{03}{(+i)}\stackrel{12}{[+]}\vert\stackrel{56}{[+]}\stackrel{78}{[-]}\Vert\stackrel{9\,10}{[-]}\stackrel{11\,12}{[-]}\stackrel{13\,14}{[-]}$	1	$\frac{1}{2}$	$\frac{1}{2}$	0	0	0	$\frac{1}{2}$	$\frac{1}{2}$	1
64	$\hat{\bar{e}}^\dagger_R$	$\stackrel{03}{[-i]}\stackrel{12}{(-)}\vert\stackrel{56}{[+]}\stackrel{78}{[-]}\Vert\stackrel{9\,10}{[-]}\stackrel{11\,12}{[-]}\stackrel{13\,14}{[-]}$	1	$-\frac{1}{2}$	$\frac{1}{2}$	0	0	0	$\frac{1}{2}$	$\frac{1}{2}$	1

Table 3.6. The left handed ($\Gamma^{(13,1)} = -1$), multiplet of creation operators of spinors — the members of the fundamental representation of the $SO(13,1)$ group, manifesting the subgroup $SO(7,1)$ of the colour charged quarks and anti-quarks and the colourless leptons and anti-leptons — is presented in the massless basis using the technique presented in App. 3.7. It represent the left handed ($\Gamma^{(3+1)} = -1$, App. 3.7) weak ($SU(2)_I$) charged ($\tau^{13} = \pm\frac{1}{2}$, ($\vec{\tau}^1 = \frac{1}{2}(S^{58} - S^{67}, S^{57} + S^{68}, S^{56} - S^{78})$) and $SU(2)_{II}$ chargeless ($\tau^{23} = 0$, $\vec{\tau}^2 = \frac{1}{2}(S^{58} +$

$S^{67}, S^{57} - S^{68}, S^{56} + S^{78}$)) quarks and leptons and the right handed ($\Gamma^{(3+1)} = 1$), weak ($SU(2)_I$) chargeless and $SU(2)_{II}$ charged ($\tau^{23} = \pm\frac{1}{2}$) quarks and leptons, both with the spin S^{12} up and down ($\pm\frac{1}{2}$, respectively). The creation operators of quarks distinguish from those of leptons only in the $SU(3) \times U(1)$ part: Quarks are triplets of three colours ($= (\tau^{33}, \tau^{38}) = [(\frac{1}{2}, \frac{1}{2\sqrt{3}}), (-\frac{1}{2}, \frac{1}{2\sqrt{3}}), (0, -\frac{1}{\sqrt{3}})]$, ($\vec{\tau}^3 = \frac{1}{2}(S^{9\,12} - S^{10\,11}, S^{9\,11} + S^{10\,12}, S^{9\,10} - S^{11\,12}, S^{9\,14} - S^{10\,13}, S^{9\,13} + S^{10\,14}, S^{11\,14} - S^{12\,13}, S^{11\,13} + S^{12\,14}, \frac{1}{\sqrt{3}}(S^{9\,10} + S^{11\,12} - 2S^{13\,14}$)), carrying the "fermion charge" ($\tau^4 = \frac{1}{6}, = -\frac{1}{3}(S^{9\,10} + S^{11\,12} + S^{13\,14})$. The colourless leptons carry the "fermion charge" ($\tau^4 = -\frac{1}{2}$). The same multiplet of creation operators represents also the left handed weak ($SU(2)_I$) chargeless and $SU(2)_{II}$ charged anti-quarks and anti-leptons and the right handed weak ($SU(2)_I$) charged and $SU(2)_{II}$ chargeless anti-quarks and anti-leptons. Anti-quarks distinguish from anti-leptons again only in the $SU(3) \times U(1)$ part: Anti-quarks are anti-triplets, carrying the "fermion charge" ($\tau^4 = -\frac{1}{6}$). The anti-colourless anti-leptons carry the "fermion charge" ($\tau^4 = \frac{1}{2}$). $Y = (\tau^{23} + \tau^4)$ is the hyper charge, the electromagnetic charge is $Q = (\tau^{13} + Y)$. The creation operators of opposite charges (anti-particle creation operators) are reachable from the particle ones besides by S^{ab} also by the application of the discrete symmetry operator $\mathbb{C}_{\mathcal{N}}\,\mathcal{P}_{\mathcal{N}}$, presented in Refs. [65,66]. The reader can find this Weyl representation also in Refs. [4,71,72,9] and in the references therein.

Table 3.6 represents in the *spin-charge-family* theory the basic creation operators for observed *quarks and leptons and anti-quarks and anti-leptons* for a particular family. Hermitian conjugation of the creation operators of Table 3.6 generates the corresponding annihilation operators, fulfilling together with the creation operators anticommutation relations for fermions of Eq. (3.81).

In observable dimension $d = (3+1)$ the $d = (13+1)$ case differs from $d = (5+1)$ case, Table 3.5, in a much reacher offer of charges. The kinematics of the fermion states in $d = (13+1)$, Table 3.6, in $d = (3+1)$ is, however, very similar to the one of Table 3.97.

The coefficients of the superposition of the basic creation operators — $\hat{b}^{\alpha\dagger}_i$ — which solve, applied on the vacuum state, the Weyl equation, Eq. (3.36), for the choice of $p^a = (p^0, p^1, p^2, p^3, 0, \cdots, 0)$, can be taken from Eq. (3.97). For the positive energy solution of spin $\frac{1}{2}$ one only has to replace $\stackrel{03}{(+i)}\stackrel{12}{(+)}\stackrel{56}{(+)}$ by $\hat{u}^{c1\dagger}_{R,1/2}$ with spin $\frac{1}{2}$ and $\stackrel{03}{[-i]}\stackrel{12}{[-]}\stackrel{56}{(+)}$ by $\hat{u}^{c1\dagger}_{R,-1/2}$ with spin $-\frac{1}{2}$. The coefficients, β and $\frac{p^1+ip^2}{|p^0|+|p^3|}$, remain the one of the case with $d = (5+1)$.

The operator $\mathcal{T}_{\mathcal{N}} = \gamma^1\gamma^3 K I_{x^0} I_{x^5, x^7, \cdots, x^{d-1}}$ transforms this superposition of creation operators, $\beta\,(\hat{u}^{c1\dagger}_{R,1/2} + \frac{p^1+ip^2}{|p^0|+|p^3|}\hat{u}^{c1\dagger}_{R,-1/2}) \cdot e^{-i(p^0x^0 - \vec{p}\cdot\vec{x})}$, into $\beta^*\,(\hat{u}^{c1\dagger}_{R,-1/2} - \frac{p^1-ip^2}{|p^0|+|p^3|}\hat{u}^{c1\dagger}_{R,1/2}) \cdot e^{-i(p^0x^0 + \vec{p}\cdot\vec{x})}$.

The operator $\mathbb{C}_{\mathcal{N}}\mathcal{P}^{(d-1)}_{\mathcal{N}} = \gamma^0\gamma^5\gamma^7 \cdots \gamma^{d-1} I_{\vec{x}_3} \cdots I_{x^6, x^8, \cdots, x^d}$ transforms the positive energy solution creation operator for u quark, $\beta\,(\hat{u}^{c1\dagger}_{R,1/2} + \frac{p^1+ip^2}{|p^0|+|p^3|}\,\hat{u}^{c1\dagger}_{R,-1/2}) \cdot e^{-i(p^0x^0 - \vec{p}\cdot\vec{x})}$, into the positive energy solution of anti-u quark, $-\beta\,(\hat{\bar{u}}^{\bar{c}1\dagger}_{L,1/2} + \frac{p^1+ip^2}{|p^0|+|p^3|}\,\hat{\bar{u}}^{\bar{c}1\dagger}_{L,-1/2}) \cdot e^{-i(p^0x^0 + \vec{p}\cdot\vec{x})}$.

One can proceed in the same way also for the $\hat{u}^{c1\dagger}_L$, $\hat{d}^{c1\dagger}_R$, and all the other quarks ^{ci}, as well as for leptons.

Spins in higher dimensional space manifest charges in $d = (3+1)$, Table 3.6, provided that the angular momentum in ordinary space at higher dimensions do not contribute, which is supposed to be the case at low energies. All the creation operators of any family and any family member, or the orthogonal superposition of them, together with their Hermitian conjugate annihilation operators fulfill the anticommutation relations of Eqs. (3.81, 3.82, 3.83).

The commuting part of the operators of S^{ab}, Eq. (3.110), determine in $d = (3+1)$ the handedness ($\Gamma^{(3+1)}) = -4i \cdot S^{03}S^{12}$)), the spin ($S^{12}$), the third component of the weak $SU(2)$ charge (τ^{13}), the third component of the second $SU(2)$ charge

(τ^{23}), the two components of the $SU(3)$ colour charge (τ^{33}, τ^{38}) and the "fermion charge" (τ^4, originating in $U(1)$ from $SO(6)$, which includes $SU(3) \times U(1)$). The hypercharge Y, which is in the *standard model* "guessed" from the experimental data, is in the *spin-charge-family* theory equal to ($\tau^4 + \tau^{23}$), while electromagnetic charge Q is, like in the *standard model*, equal to ($Y + \tau^{13}$).

One representation of creation operators with $2^{\frac{d}{2}-1}$ members includes all the left and the right handed coloured quarks and colourless leptons and left and right handed (anti coloured) antiquarks and (anti colourless) antileptons. The right handed neutrinos and the left handed antineutrinos, like all the other members of one Lorentz representation, carry the additional hypercharge (the additional superposition of τ^4 and τ^{23}) and are correspondingly not chargeless like in the *standard model*.

The sum of the charges, the sum of the spins and the sum of the handedness —properties defined with respect to $d = (3+1)$ — over all the members of one representation are equal to zero in any d, as it is the case of $d = (5+1)$. However, in the $d = (13+1)$ case this is true even within quarks and leptons separately and within antiquarks and antileptons separately. Let be repeated that this is so since the right handed neutrinos and the left handed antineutrinos are the regular members of one representation, as it is true for quarks and charged leptons. This can be checked in Table 3.6. Exclusion of the right handed neutrinos and left handed antineutrinos makes nonzero the sum of ($\Gamma^{(3+1)}$), τ^{23} and τ^4 over the spinor part separately and correspondingly also over the antispinor part. The whole representation has even in this case sums over all the quantum numbers of spins and charges equal to zero.

Grassmann "fermions" and "antifermions" Let us represent creation and annihilation operators in Grassmann space, like we did in the Clifford case.

In the Grassmann case the representations in $d = (13+1)$ space start to be very large and correspondingly almost uncontrollable, Eq. (3.59). We learn in the Clifford case that at the low energy regime, when we treat the equations of motion for free massless fermions with nonzero momentum only in $d = (3+1)$, the higher dimensional space contributes charges, which are reacher the larger is space, but kinematics in $d = (3+1)$ are in all such cases the same. We treat therefore only the $d = (5+1)$ case.

In Table 3.7 the basic creation operators for $d = (5+1)$ case, with Grassmann space used to describe internal degrees of freedom of "fermions" and "antifermions", are presented. "Fermions" carry in Grassmann space integer spins and charges in the adjoint representations.

There are two independent decuplets (unconnected by $\mathbf{S}^{ab}$).

Both decuplets [46] of creation operators are of an odd Grassmann character, representing the second quantized $n = 1$ "fermion" states, Eq. (3.54), which belong in general to n (any n) "fermion" states. There are, from the point of view of $d = (3+1)$ space, two triplets, one doublet and two singlets in each of the two decouplets.

In Subsect. 3.3.3 the discrete symmetry operators in Grassmann space are discussed, with the discrete symmetry operators for the case that "fermions"

I	i	decuplet of creation operators $\hat{b}_i^{\theta k\dagger}$	S^{03}	S^{12}	S^{56}
	1	$(\theta^0-\theta^3)(\theta^1+i\theta^2)(\theta^5+i\theta^6)$	i	1	1
	2	$(\theta^0\theta^3+i\theta^1\theta^2)(\theta^5+i\theta^6)$	0	0	1
	3	$(\theta^0+\theta^3)(\theta^1-i\theta^2)(\theta^5+i\theta^6)$	$-i$	-1	1
	4	$(\theta^0-\theta^3)(\theta^1-i\theta^2)(\theta^5-i\theta^6)$	i	-1	-1
	5	$(\theta^0\theta^3-i\theta^1\theta^2)(\theta^5-i\theta^6)$	0	0	-1
	6	$(\theta^0+\theta^3)(\theta^1+i\theta^2)(\theta^5-i\theta^6)$	$-i$	1	-1
	7	$(\theta^0-\theta^3)(\theta^1\theta^2+\theta^5\theta^6)$	i	0	0
	8	$(\theta^0+\theta^3)(\theta^1\theta^2-\theta^5\theta^6)$	$-i$	0	0
	9	$(\theta^0\theta^3+i\theta^5\theta^6)(\theta^1+i\theta^2)$	0	1	0
	10	$(\theta^0\theta^3-i\theta^5\theta^6)(\theta^1-i\theta^2)$	0	-1	0

II	i	decuplet of creation operators $\hat{b}_i^{\theta k\dagger}$	S^{03}	S^{12}	S^{56}
	1	$(\theta^0+\theta^3)(\theta^1+i\theta^2)(\theta^5+i\theta^6)$	$-i$	1	1
	2	$(\theta^0\theta^3-i\theta^1\theta^2)(\theta^5+i\theta^6)$	0	0	1
	3	$(\theta^0-\theta^3)(\theta^1-i\theta^2)(\theta^5+i\theta^6)$	i	-1	1
	4	$(\theta^0+\theta^3)(\theta^1-i\theta^2)(\theta^5-i\theta^6)$	$-i$	-1	-1
	5	$(\theta^0\theta^3+i\theta^1\theta^2)(\theta^5-i\theta^6)$	0	0	-1
	6	$(\theta^0-\theta^3)(\theta^1+i\theta^2)(\theta^5-i\theta^6)$	i	1	-1
	7	$(\theta^0+\theta^3)(\theta^1\theta^2+\theta^5\theta^6)$	$-i$	0	0
	8	$(\theta^0-\theta^3)(\theta^1\theta^2-\theta^5\theta^6)$	i	0	0
	9	$(\theta^0\theta^3-i\theta^5\theta^6)(\theta^1+i\theta^2)$	0	1	0
	10	$(\theta^0\theta^3+i\theta^5\theta^6)(\theta^1-i\theta^2)$	0	-1	0

Table 3.7. Two decuplets of the basic creation operators $\hat{b}_i^{\theta k\dagger}$, $k=(I,II)$, $i=(1,\dots,10)$, of the orthogonal group $SO(5,1)$ in Grassmann space are presented. The creation operators form "eigenstates" of the Cartan subalgebra, Eq. (3.110), ($\mathbf{S}^{03}, \mathbf{S}^{12}, \mathbf{S}^{56}$ for $SO(5,1)$) with integer spins and charges, defining "fermions" and "antifermions". The creation operators within each decuplet are reachable from any member by (a product of) $\mathbf{S}^{ab}$'s (which do not belong to the Cartan subalgebra). Creation operators $\hat{b}_i^{\theta k\dagger}$ and their Hermitian conjugated annihilation operators $\hat{b}_i^{\theta k}$ fulfill the anticommutation relations for fermions, Eq. (3.62). The product of the discrete symmetry operators $\mathbb{C}_{NG}$ and $\mathcal{P}_{NG}^{(d-1)}$, Eq. (3.95), ($\mathbb{C}_{NG}\mathcal{P}_{NG}^{(d-1)}$ $=\gamma^0_G\gamma^5_G\, I_{\vec{x}_3} I_{x^6}$ in $d=(5+1)$) transforms, for example, $\hat{b}_1^{\theta I\dagger}$ into $\hat{b}_6^{\theta I\dagger}$, $\hat{b}_2^{\theta I\dagger}$ $\hat{b}_5^{\theta I\dagger}$ and $\hat{b}_3^{\theta I\dagger}$ into $\hat{b}_4^{\theta I\dagger}$, transforming "fermions" with the charge 1 into "antifermions" with the charge -1.

manifest kinematics only in $d=(3+1)$-dimensional space, while the higher dimensions contribute charges, included.

Let us notice that the Grassmann even operator $\mathbb{C}_{NG}\mathcal{P}_{NG}^{(d-1)}$, Eq. (3.95), transforms the creation operator creating the positive energy particle state ($p^a = (|p^0|,0,0,|p^3|,0,0)$) with the charge 1, $\hat{b}_1^{\theta I\dagger}$, into the creation operator of the anti-particle state, $\hat{b}_6^{\theta I\dagger}$, with the positive energy $|p^0|$ and with $-|p^3|$ and with the charge -1 , for example. Correspondingly $\mathbb{C}_{NG}\mathcal{P}_{NG}^{(d-1)}$, Eq. (3.95), transforms the particle state $\hat{b}_3^{\theta I\dagger}$ with the positive energy into the anti-particle state $\hat{b}_4^{\theta I\dagger}$ with the positive energy. All these states belong to the same representation, the same decuplet.

In Eq. (3.98) the superposition of the creation operators of the two triplets of the first decuplet of creation operators — ($\hat{b}_1^{\theta I\dagger}, \hat{b}_2^{\theta I\dagger}, \hat{b}_3^{\theta I\dagger}$) — which solve Eq. (3.43) for free massless "fermions" in Grassmann space, with the space function $e^{-ip_a x^a}$, $p^a=(p^0,p^1,p^2,p^3,0,0)$, Eq. (3.66), is presented. Two indexes — (ch, s) — replace the index i, ch represents the charge, defined by $\mathbf{S}^{56}$, and s represents the spin, $\mathbf{S}^{12}$.

Creation operators for "fermion" states in Grassmann space for $d = (5+1)$

$$\begin{aligned}
p^0 &= |p^0|\,,\\
\hat{b}^{\theta 1\dagger}_{1,1}(\vec{p}) &= \beta\,\{(\frac{1}{\sqrt{2}})^3\,(\theta^0-\theta^3)(\theta^1+i\theta^2) - \frac{2(|p^0|-|p^3|)}{p^1-ip^2}\,(\frac{1}{\sqrt{2}})^2(\theta^0\theta^3+i\theta^1\theta^2)\\
&\quad -(\frac{p^1+ip^2}{|p^0|+|p^3|})^2\,(\frac{1}{\sqrt{2}})^3\,(\theta^0+\theta^3)(\theta^1-i\theta^2)\}\,(\theta^5+i\theta^6)\,e^{-i(|p^0|x^0-\vec{p}\cdot\vec{x})}\,,\\
\hat{b}^{\theta 2\dagger}_{1,-1}(\vec{p}) &= \beta^*\,\{(\frac{1}{\sqrt{2}})^3(\theta^0+\theta^3)(\theta^1-i\theta^2) - \frac{2(|p^0|-|p^3|)}{p^1+ip^2}\,(\frac{1}{\sqrt{2}})^2(\theta^0\theta^3+i\theta^1\theta^2)\\
&\quad -(\frac{p^1-ip^2}{|p^0|+|p^3|})^2\,(\frac{1}{\sqrt{2}})^3(\theta^0-\theta^3)(\theta^1+i\theta^2)\}\,(\theta^5+i\theta^6)\,e^{-i(|p^0|x^0+\vec{p}\cdot\vec{x})}\,,
\end{aligned}$$

Creation operators for "anti — fermion" states in Grassmann space for $d = (5+1)$

$$\begin{aligned}
p^0 &= |p^0|\,,\\
\hat{b}^{\theta 3\dagger}_{-1,1}(\vec{p}) &= \beta\,\{(\frac{1}{\sqrt{2}})^3(\theta^0+\theta^3)(\theta^1+i\theta^2) - \frac{2(|p^0|-|p^3|)}{p^1-ip^2}\,(\frac{1}{\sqrt{2}})^2(\theta^0\theta^3-i\theta^1\theta^2)\\
&\quad -(\frac{p^1+ip^2}{|p^0|+|p^3|})^2\,(\frac{1}{\sqrt{2}})^3(\theta^0-\theta^3)(\theta^1-i\theta^2)\}\,(\theta^5-i\theta^6)\,e^{-i(|p^0|x^0+\vec{p}\cdot\vec{x})}\,,\\
\hat{b}^{\theta 4\dagger}_{-1,-1}(\vec{p}) &= \beta^*\,\{(\frac{1}{\sqrt{2}})^3(\theta^0-\theta^3)(\theta^1-i\theta^2) - \frac{2(|p^0|-|p^3|)}{p^1+ip^2}\,(\frac{1}{\sqrt{2}})^2(\theta^0\theta^3-i\theta^1\theta^2)\\
&\quad -(\frac{p^1-ip^2}{|p^0|+|p^3|})^2(\frac{1}{\sqrt{2}})^3(\theta^0+\theta^3)(\theta^1+i\theta^2)\}\,(\theta^5-i\theta^6)\,e^{-i(|p^0|x^0-\vec{p}\cdot\vec{x})}\,,
\end{aligned} \tag{3.98}$$

Here $\beta^*\beta = \frac{(|p^0|+|p^3|)^2}{2(3(p^0)^2-(p^3)^2)}$. All the corresponding states are orthonormal.

The corresponding annihilation operators follow from the creation ones by taking into account Eq. (3.18). Let us write down, as an example, the annihilation operator partner to the creation operator $\hat{b}^{\theta 1\dagger}_{1,1}(\vec{p})$ from Eq. (3.98). Taking into account Eq. (3.18) (saying that $\theta^{a\dagger} = \eta^{aa}\frac{\partial}{\partial\theta_a} = \frac{\partial}{\partial\theta^a}$), it follows $\hat{b}^{\theta 1}_{1,1}(\vec{p}) = (\frac{1}{\sqrt{2}})^3\beta^*(\partial_{\theta^5}-i\partial_{\theta^6})\Big\{(\partial_{\theta^1}-i\partial_{\theta^2})(\partial_{\theta^0}-\partial_{\theta^3}) - \frac{2(|p|^0-|p^3|)}{p^1+ip^2}\sqrt{2}(\partial_{\theta^3}\partial_{\theta^0}-i\partial_{\theta^2}\partial_{\theta^1}) - (\frac{p^1-ip^2}{|p^0|+|p^3|})^2\,(\partial_{\theta^1}+i\partial_{\theta^2})(\partial_{\theta^0}+\partial_{\theta^3})\Big\}e^{i(|p^0|x^0-\vec{p}\cdot\vec{x})}$.

The creation and annihilation operators fulfill the anti-commutation relations of Eq. (3.62).

Creation operators $\hat{b}^{\theta k\dagger}_{ch,\,s}(\vec{p})\,e^{-i(p_m x^m)}$, $m = (0,\cdots,3)$, while $p^5 = 0 = p^6$, generate states, which solve the equation of motion $(\theta^a - \frac{\partial}{\partial\theta_a})\,p_a\,\phi^{\theta k}_{ch,\,s}(x^0,\vec{x}) = 0$, Eq. (3.43), [14].

Let be noticed that the second creation operator $\hat{b}^{\theta 2\dagger}_{1,-1}$ follows from the first one — $\hat{b}^{\theta 1\dagger}_{1,1}$ — by the application of the operator $\mathcal{T}_{\mathcal{NG}} = \gamma^1_G\,\gamma^3_G\,K\,I_{x^0}\,I_{x^5,x^7,\cdots,x^{d-1}}$, Eq. (3.95).

When applying on the first two creation operators of positive charge ($\hat{b}^{\theta 1\dagger}_{1,1}$, $\hat{b}^{\theta 2\dagger}_{1,-1}$), defining the "fermion" states of positive energy, the operator $\mathbb{C}_{\mathcal{NG}}\cdot\mathcal{P}^{(d-1)}_{\mathcal{NG}}(= \gamma^0_G\gamma^5_G\gamma^7_G\cdots\gamma^{d-1}_G\,I_{\vec{x}_3}\,I_{x^6,x^8,\ldots,x^d})$, the third and the fourth creation operators follow, defining the "antifermion" states of negative charge and positive energy ($\hat{b}^{\theta\dagger}_{-1,1}$, $\hat{b}^{\theta 4\dagger}_{-1,-1}$).

Solutions of the equation of motion of the second decouplet, and correspondingly the creation and annihilation operators, can be obtained in equivalent way.

[14] The equation $(\theta^a - \frac{\partial}{\partial\theta_a})\,p_a\,\phi(\theta,x) = 0$ can be rewritten into $-i\tilde{\gamma}^a\,p_a\,\phi = 0$, from where the equation $\{\tilde{\Gamma}^{(3+1)}\,\hat{p}^0 = 2(\tilde{S}^{23}\,\hat{p}^1 + \tilde{S}^{31}\,\hat{p}^2 + \tilde{S}^{12}\,\hat{p}^3)\}\,\phi(\theta,x)$ follows, leading to the same solutions as presented in Eq. (3.98). Similar relation appears also in the Clifford case.

We learned that states transform under the application of the discrete symmetry operators (defined in the Clifford case in Eq. (3.90) and Eq. (17) in Ref. [65], or Eq. (10) in Ref. [66], and in the Grassmann case in Eqs. (3.91, 3.95)), *equivalently in the Clifford and in the Grassmann case.*

3.3.5 What do we learn from the second quantization procedure in Grassmann and in Clifford space?

We proved that in both spaces, in Clifford space and in Grassmann space, the corresponding creation operators and their Hermitian conjugated annihilation operators of an odd (either Clifford or Grassmann) character fulfill the anticommutation relations as required for fermions, Eqs (3.83, 3.62), if operating on an appropriate vacuum state, representing in both spaces a $n = 1$ fermion space out of n, any n, fermion Hilbert space.

No postulated creation operators are needed as in ordinary second quantization procedure.

In Clifford space the creation operators are (after the requirement of Eq. (3.69)) products of odd numbers of γ^a's, arranged into nilpotents and projectors, Eq. (3.70), which are the "eigenstates" of the Cartan subalgebras of S^{ab}, Eq. (3.72), generating spins and charges, and of $\tilde{S}^{ab}$, generating families, Eqs. (3.2, 3.4). In Grassmann space they are products of θ^a, arranged in "eigenstates" of the Cartan subalgebra of $\mathbf{S}^{ab}$, Eq. (3.5, 3.52)).

While in the Grassmann case the vacuum state is simple, $|\phi_{og} >= |1\ >$, in the Clifford case the vacuum state is a sum of $2^{\frac{d}{2}-1}$ products of projectors, Eq. (3.79).

In $2(2n+1)$-dimensional spaces there are in the Clifford case in one representation $2^{\frac{d}{2}-1}$ creation operators. The whole representation is reachable from the (any) starting operator by products of S^{ab}, while products of $\tilde{S}^{ab}$ transform each of these creation operators into the creation operator of the same family member, but belonging to another family, Eq. (3.76). There are correspondingly $2^{\frac{d}{2}-1} \cdot 2^{\frac{d}{2}-1}$ creation operators, and correspondingly the same number of states, reachable by products of S^{ab}'s or $\tilde{S}^{ab}$'s or of both, S^{ab}'s and $\tilde{S}^{ab}$'s. Each state follows by the corresponding creation operator on the vacuum state and it is annihilated by its Hermitian conjugated operator, Eq.(3.71).

In $2(2n+1)$-dimensional spaces there are in the Grassmann case (before the requirement of Eq. (3.69)) two decoupled representations with all the θ^a's included into the representations, each with $\frac{1}{2}\,\frac{d!}{\frac{d}{2}!\frac{d}{2}!}$ creation operators, and correspondingly with the same number of states. Each state can be obtained by the corresponding creation operator operating on the vacuum state and any state is annihilated by the corresponding Hermitian conjugated creation operator. While all of $2^{\frac{d}{2}-1} \cdot 2^{\frac{d}{2}-1}$ states in Clifford space of an odd character are reachable from any of Clifford odd states by either products of S^{ab}'s or by products of $\tilde{S}^{ab}$'s or by products of both, and states of an even Clifford character by either products of S^{ab}'s or by products of $\tilde{S}^{ab}$'s or $\tilde{\gamma}^a\gamma^a$ or all of them, in Grassmann space all the irreducible representations are decoupled — no products of $\mathbf{S}^{ab}$'s transform states of one group into states of another groups.

The creation (annihilation) operators — which are superposition of the creation (annihilation) operators defining the eigenstates of the Cartan subalgebra in the internal space, fulfilling the relations of Eqs. (3.62, 3.83), respectively — form the eigenstates of the equations of motion for free massless "fermions" with integer spins and no families in the Grassmann case, Eqs. (3.43, 3.61), and for free massless fermions with half integer spins and families in the Clifford case, Eqs. (3.36, 3.82).

The number operators for the odd part of either Clifford or Grassmann case have the eigenvalues 0 or 1, Eqs. (3.55, 3.84).

One can as well define in both cases the Hamilton functions, which lead to the equations of motion in the Grassmann case, Eqs. (3.67, 3.68), and in the Clifford case, Eqs. (3.88, 3.89). While in the Clifford case the procedure to find the Hamilton function is the usual one, that is the known one, in the Grassmann case is not. It remains to understand better the Hamilton function in the Grassmann case.

Comparing solutions for free massless states in a toy model with $d = (5+1)$ from the point of view of $d = (3+1)$ (assuming that $p^a = (p^0, p^1, p^2, p^3, 0, \cdots, 0)$) for the Clifford case and for the Grassmann case, one observes several similarities. The main differences are: **i.** that spins and charges are in the Clifford case half integer while in the Grassmann case are integer, **ii.** that Clifford space offers, after the assumption of Eq. (3.69), the existence of families, while Grassmann space, before the assumption of Eq. (3.69), does not, and **iii.** that the requirement that the action is Lorentz invariant leads in Clifford space to well defined Hamilton function, while in the Grassmann case this point needs further study.

We can conclude: **a.** The — odd part of the — Clifford algebra presentation of the internal degrees of freedom of fermions offers the $n = 1$ second quantized fermion part of the n second quantized Hilbert space, offering the fermion creation and annihilation operators, fulfilling the required relations, explaining therefore the assumption of Dirac about introducing creation and annihilation operators in the second quantized fields.

b. The *spin-charge-family* theory of N.S.M.B., assuming $d \geq (13+1)$-dimensional space and the Clifford algebra to explain internal degrees of freedom of fermions, enables to justify the assumption of the usual second quantized procedure. The group theory alone, without connecting the internal degrees of freedom with the *Clifford objects for explaining spins, charges, and families*, can not do that.

c. Table 3.6 demonstrates that any family contains all the fermions and antifermions, what in the *spin-charge-family* theory means all the quarks and the antiquarks and leptons and anileptons, left and right handed. No Dirac sea of negative energy states is needed to explain the existence of antifermions. Correspondingly the vacuum state is simple, of an even Clifford character, with the sum of all the quantum numbers over the family members equal to zero.

d. The sum of all the quantum numbers within one family representation, but also separately within fermions and separately within antifermions within the same representation, is zero. Also the sum over family quantum numbers is zero.

e. In the Clifford case the operator $\mathbb{C}_{\mathcal{N}} \mathcal{P}^{(d-1)}_{\mathcal{N}}$, Eq. (3.94), transforms the fermion state into the anti-fermion state.

In the Grassmann case it is the operator $\mathbb{C}_{NG} \mathcal{P}^{(d-1)}_{NG}$, which transforms the Grassmann "fermion" into the "antifermion".

3.4 Conclusions

We have learned in the present study that both Clifford and Grassmann space offer 1-fermion second quantized part of vector space, with creation and annihilation operators — defined as an odd products of either Clifford or Grassmann eigenstates of the corresponding Cartan subalgebra operators in even dimensional space, Eq. (3.110) — fulfilling the desired anticommutation relations for fermions, Eqs. (3.62, 3.83). The corresponding number operators have the eigenvalues 0 or 1 in both cases. The fact that states, solving equations of motions, fulfill the desired anticommutation relations for second quantized fermions explains the second quantization postulates of Dirac.

Grassmann coordinates and Clifford coordinates offer the same degrees of freedom: Two times 2^d each. θ^a's and their Hermitian conjugated partners $\frac{\partial}{\partial\theta_a}$ are expressible with the two kinds of Clifford coordinates, γ^a's and $\tilde{\gamma}^a$'s — defining two independent spaces — and opposite. The vacuum states ought to be changed from $|1>$in the Grassmann case to the one presented in Eq. (3.79) for either γ^a's or $\tilde{\gamma}^a$'s. The Grassmann states carry integer spins, while Clifford states carry in both spaces half integer spins.

The requirement of Eq. (3.69) breaks the equivalence of both kinds of the Clifford coordinates and opens the possibility for the appearance of families. Clifford space, defined by the two kinds of objects, narrow now to only one of the two, determined by γ^a's, while $\tilde{\gamma}^a$'s take care of families. Correspondingly also in Grassmann space there remain only θ^a's, becoming γ^a's, while their Hermitian conjugated partners $\frac{\partial}{\partial\theta_a}$ no longer exist. Consequently, after the requirement of Eq. (3.69), the possibility of having integer spins "fermions" no longer exists.

The 1-fermion second quantized vector space has for a chosen momentum p^a_k in the Clifford case (after the requirement of Eq. (3.69)) $2^{\frac{d}{2}-1}\cdot 2^{\frac{d}{2}-1}$ members (that is $2^{\frac{d}{2}-1}$ families, each family having $2^{\frac{d}{2}-1}$ members), and in the Grassmann case (before the requirement of Eq. (3.69)), when all θ^a's contribute in forming a state, $\frac{d!}{\frac{d}{2}!\frac{d}{2}!}$ members in two decoupled representations.

In both spaces the members of one representation include fermions and antifermions and correspondingly there is no need for the Dirac sea of negative energies filled by fermions.

In both cases the creation and annihilation operators of different momentum p^a and the same internal part represent different creation operators.

The n (any n) second quantized vector space of fermions (or "fermions" in the Grassmann case) follows in both cases as products of n creation operators defining each one fermion states when applying on the corresponding vacuum state (in the Clifford case on $|\psi_{oc}>$, Eq. (3.79), in the Grassmann case $|\psi_{og}>=|\ 1>$), if the creation operators distinguish at least either in one of the quantum numbers of the corresponding Cartan subalgebra or in momentum p^a_k.

But while in the Clifford case states carry spin and charges from the point of view of $d=(3+1)$ in the fundamental representations of the Lorentz group carrying therefore half integer spins, states in the Grassmann case are in adjoint representations of the Lorentz group, carrying therefore integer spins.

We present in this paper as well the action (Eq. (3.41, 3.42)), describing free massless "fermions" with the internal degrees of freedom describable in Grassmann space. The action leads to the equations of motion (Eq. (3.43)), analogous to the Weyl equation in Clifford space (Eq. (3.36)), fulfilling as well the Klein-Gordon equation (Eq. (3.44)). We also present the discrete symmetry operators in the Grassmann case.

Since the Clifford objects γ^a and $\tilde{\gamma}^a$ are expressible with the Grassmann coordinates θ^a and their conjugate moments $\frac{\partial}{\partial \theta^a}$ — $\gamma^a = (\theta^a + \frac{\partial}{\partial \theta_a})$, $\tilde{\gamma}^a = i(\theta^a - \frac{\partial}{\partial \theta_a})$, Eq. (3.4) — either basic states in Grassmann space, Eq. (3.16), or basic states in Clifford space, Eq. (3.73), can be normalized with the same integral, Eq. (3.31, 3.32, 3.33).

To understand better the difference in the description of the fermion internal degrees of freedom either with Clifford algebra (after the requirement of Eq. (3.69)) or with Grassmann algebra (before the requirement of Eq. (3.69)), let us replace in the starting action of the *spin-charge-family* theory, Eq. (3.1), using the Clifford algebra (after the requirement of Eq. (3.69)) to describe fermion degrees of freedom, the covariant momentum $p_{0a} = f^{\alpha}{}_a\, p_{0\alpha}$, $p_{0\alpha} = p_\alpha - \frac{1}{2} S^{ab} \omega_{ab\alpha} - \frac{1}{2} \tilde{S}^{ab} \tilde{\omega}_{ab\alpha}$, with $p_{0\alpha} = p_\alpha - \frac{1}{2}\, \mathbf{S}^{ab} \Omega_{ab\alpha}$, where $\mathbf{S}^{ab} = S^{ab} + \tilde{S}^{ab}$, Eq. (3.5), and $\Omega_{ab\alpha}$ are the spin connection gauge fields of $\mathbf{S}^{ab}$ (which are the generators of the Lorentz transformations in Grassmann space), while $f^{\alpha}{}_a\, p_{0\alpha}$ replaces the ordinary momentum when massless objects start to interact with the gravitational field through the vielbeins and the spin connections. Let us add that it follows, if varying the action with respect to either $\omega_{ab\alpha}$ or $\tilde{\omega}_{ab\alpha}$ when no fermions are present, that both spin connections are uniquely determined by the vielbeins ([9,3,5] and references therein) and correspondingly in this particular case $\Omega_{ab\alpha} = \omega_{ab\alpha} = \tilde{\omega}_{ab\alpha}$.

The present study was stimulated by one of the author in order to better understand whether and to which extend the *spin-charge-family* theory offers the next step to both *standard models* — the one of the fermion and boson fields and the cosmological one. Correspondingly we present in Subsect. 3.1.1 of the introductory Sect. 3.1, the achievements so far of the *spin-charge-family* theory as well as the open problems of this theory, both suggested by the referees.

In shortly, the *spin-charge-family* theory (using Clifford objects to describe the internal space of fermions) offers, while starting with the simple action in $d \geq (13+1)$ with fermions interacting with gravity only (the vielbeins and the two kinds of the spin connection fields, the gauge fields of moments and the generators of the Lorentz transformations S^{ab} and $\tilde{S}^{ab}$, respectively), Eq. (3.1), the explanation for all the assumptions of the *standard model* — for quarks and leptons, antiquarks and antileptons, for fermion families, for the vector gauge fields, for the scalar Higgs and Yukawa couplings — explaining also the phenomena like the existence of the dark matter [54], of the matter-antimatter asymmetry [4], offering correspondingly the next step beyond both standard models — cosmological one and the one of the elementary fields, Sect. 3.1.1. This theory predicts the fourth family to the observed three, Sect. 3.1.1, and the new scalar fields, some of those which explains the properties of the observed Higgs and Yukawa couplings, Sect. 3.1.1, and which will be observed at the LHC and other experiments in the future. This theory predicts also the existence of the stable fifth family, manifesting the dark matter and with the "new nuclear" force among the hadrons of these much heavier families, Sect. 3.1.1.

To these achievements the present study adds the recognition that the creation operators for one fermion states are in Clifford space already second quantized, and that the creation operators for any n fermion second quantized vectors are products of one fermion creation operators, operating on the empty vacuum state. The *spin-charge-family* theory namely describes all the internal degrees of freedom of fermions in Clifford space — spins and charges.

There is in this theory no need for the existence of the negative energy states filled with fermions.

The most severe among the open problems of the *spin-charge-family* theory is the quantization of gravity gauge fields, although the *spin-charge-family* theory is explaining the phenomena in the low energy regime where all the vector and scalar gauge fields can be quantized in the known procedure. There are also other open problems, some of them needing only time to be solved, presented in Sect. 3.1.1.

The second quantization of "fermions" with the internal degrees of freedom described in Grassmann space might help to understand better the properties of scalars and vectors in the *spin-charge-family* theory.

Let us conclude with a question: Could "fermions" with integer spins and charges in adjoint representations be an acceptable possibility and no requirement of Eq. (3.69) needed?

3.5 APPENDIX: Creation and annihilation operators in Grassmann and Clifford space for $d = 4n$

We discuss in Subsect. 3.3 mainly cases with $d = 2(2n+1)$, since if assuming no conserved charges in the fundamental theory with fermions, which carry only spins and interact with only the gravity — as the *spin-charge-family* theory assumes — the dimensions $4n$, n is positive integer, as well as all odd dimensions, are excluded under the requirement of mass protection [77].

Let us nevertheless add in this appendix comments on the second quantization procedure in $d = 4n$ spaces.

i. Grassmann space

In Eq. (3.51) we define in Grassmann space a possible starting creation operator for $d = 2(2n+1)$ spaces. In $d = 4n$ we correspondingly start with the state

$$|\phi^1_1> = b^{\theta 1\dagger}_1 |1>\,,$$
$$b^{\theta 1\dagger}_1 = (\frac{1}{\sqrt{2}})^{\frac{d}{2}-1}\,(\theta^0 - \theta^3)(\theta^1 + i\theta^2)(\theta^5 + i\theta^6)\cdots(\theta^{d-3} + i\theta^{d-2})\theta^{d-1}\theta^d\,, \tag{3.99}$$

generated by the creation operator $b^{\theta 1\dagger}_1$, which is, as it ought to be — like in the $d = 2(2n+1)$ case — of an odd Grassmann character to fulfill the anticommutation relations for fermions, Eq. (3.62). Again the rest of states, belonging to the same

Lorentz representation, follow from the starting state by the application of the operators $\mathbf{S}^{cf}$, which do not belong to the Cartan subalgebra operators. Their annihilation partners follow by Hermitian conjugation.

One finds therefore for the (chosen) starting creation and the corresponding annihilation operator

$$\begin{aligned}
\hat{b}^{\theta 1\dagger}_1 &= (\frac{1}{\sqrt{2}})^{\frac{d}{2}-1}\,(\theta^0-\theta^3)(\theta^1+i\theta^2)(\theta^5+i\theta^6)\cdots(\theta^{d-3}+i\theta^{d-2})\theta^{d-1}\theta^d\,,\\
\hat{b}^{\theta 1}_1 &= (\frac{1}{\sqrt{2}})^{\frac{d}{2}-1}\,\frac{\partial}{\partial\theta^d}\frac{\partial}{\partial\theta^{d-1}}(\frac{\partial}{\partial\theta^{d-3}}-i\frac{\partial}{\partial\theta^{d-2}})\cdots(\frac{\partial}{\partial\theta^0}-\frac{\partial}{\partial\theta^3})\,,\\
d &= 4n\,.
\end{aligned} \tag{3.100}$$

The application of $\mathbf{S}^{01}$, for example, generates

$$\begin{aligned}
\hat{b}^{\theta 1\dagger}_2 &= (\frac{1}{\sqrt{2}})^{\frac{d}{2}-2}\,(\theta^0\theta^3+i\theta^1\theta^2)(\theta^5+i\theta^6)\cdots(\theta^{d-3}+i\theta^{d-2})\,\theta^{d-1}\theta^d\,,\\
\hat{b}^{\theta 1}_2 &= (\frac{1}{\sqrt{2}})^{\frac{d}{2}-2}\,\frac{\partial}{\partial\theta^d}\frac{\partial}{\partial\theta^{d-1}}\,(\frac{\partial}{\partial\theta^{d-3}}-i\frac{\partial}{\partial\theta^{d-2}})\cdots(\frac{\partial}{\partial\theta^3}\frac{\partial}{\partial\theta^0}-i\frac{\partial}{\partial\theta^2}\frac{\partial}{\partial\theta^1})\,.
\end{aligned} \tag{3.101}$$

There is the additional group of creation and annihilation operators in $d=4n$, which follows from the starting creation operator $\hat{b}^{\theta 2\dagger}_1$

$$\begin{aligned}
\hat{b}^{\theta 2\dagger}_1 &= (\frac{1}{\sqrt{2}})^{\frac{d}{2}-1}(\theta^0+\theta^3)(\theta^1+i\theta^2)(\theta^5+i\theta^6)\cdots(\theta^{d-3}+i\theta^{d-2})\,\theta^{d-1}\theta^d\,,\\
\hat{b}^{\theta 2}_1 &= (\hat{b}^{\theta 2\dagger}_1)^\dagger = (\frac{1}{\sqrt{2}})^{\frac{d}{2}-1}\,\frac{\partial}{\partial\theta^d}\frac{\partial}{\partial\theta^{d-1}}(\frac{\partial}{\partial\theta^{d-3}}-i\frac{\partial}{\partial\theta^{d-2}})\cdots(\frac{\partial}{\partial\theta^0}+\frac{\partial}{\partial\theta^3})\,,\\
&\text{for } d=4n\,.
\end{aligned} \tag{3.102}$$

All the rest of creation operators follow from the starting creation operator of each of the two groups by the (left) application of products of $\mathbf{S}^{ab}$

$$\begin{aligned}
\hat{b}^{\theta k\dagger}_i &\propto \mathbf{S}^{ab}\cdots\mathbf{S}^{ef}\hat{b}^{\theta k\dagger}_1\,,\\
\hat{b}^{\theta k}_i &= (\hat{b}^{\theta 2\dagger}_i)^\dagger\,,\quad k=1,2\,.
\end{aligned} \tag{3.103}$$

Only creation and annihilation operators with an odd Grassmann character, fulfill, applied on the vacuum state $|1>$, the anticommutation relations required for fermions, Eq. (3.54).

i. Clifford space

In Eq. (3.73) we define in Clifford space a possible starting creation operator for $d=2(2n+1)$ spaces. In $d=4n$ we correspondingly start with the state with an odd number of nilpotents and with one projector

$$\begin{aligned}
|\psi^1_1> &= \hat{b}^{1\dagger}_1|\psi_{oc}>\,,\\
\hat{b}^{1\dagger}_1 &:= \overset{03}{(+i)}\overset{12}{(+)}\overset{35}{(+)}\cdots\overset{d-3\;d-2}{(+)}\;\overset{d-1\;d}{[+]}\,,\\
\hat{b}^1_1 &= (\hat{b}^{1\dagger}_1)^\dagger = \overset{d-1\;d}{[+]}\;\overset{d-3\;d-2}{(-)}\cdots\overset{35}{(-)}\overset{12}{(-)}\overset{01}{(-i)}
\end{aligned} \tag{3.104}$$

All the other creation operators, creating all the members of the representation of this particular family, are obtainable by the application of products of S^{ab} on this creation operator from the left hand side. There are $2^{\frac{d}{2}-1}$ members of each family. All the other families follows from the starting one by the application of products of $\tilde{S}^{ab}$. There are $2^{\frac{d}{2}-1}$ families with $2^{\frac{d}{2}-1}$ members each.

A general creation operator in $d = 4n$ follows by the application of S^{ab} and $\tilde{S}^{ab}$ on the starting creation operator of Eq. (3.104) and the corresponding annihilation operator is its Hermitian conjugated value.

Correspondingly we define $\hat{b}^{\alpha\dagger}_i$ (up to a constant) to be

$$\begin{aligned}\hat{b}^{\alpha\dagger}_i &\propto \tilde{S}^{ab}\cdots\tilde{S}^{ef}S^{mn}\cdots S^{pr}\hat{b}^{1\dagger}_1\\ &\propto S^{mn}\cdots S^{pr}\hat{b}^{1\dagger}_1 S^{ab}\cdots S^{ef}\,,\\ \hat{b}^{\alpha}_i = (\hat{b}^{\alpha\dagger}_i)^\dagger &\propto S^{ef}\cdots S^{ab}\hat{b}^{1}_1 S^{pr}\cdots S^{mn}\,,\\ &d = 4n\,. \end{aligned} \tag{3.105}$$

These creation and annihilation operators — again of an odd Clifford character in $4n$ — fulfill the anticommutation relations of Eq. (3.83), if applied on the vacuum state of Eq. (3.79),

$$\begin{aligned}|\psi_{oc}> &= \overset{03}{[-i]}\overset{12}{[-]}\overset{35}{[-]}\cdots\overset{d-3\,d-2}{[-]}\overset{d-1\,d}{[+]} + \overset{03}{[+i]}\overset{12}{[+]}\overset{56}{[-]}\cdots\overset{d-3\,d-2}{[-]}\overset{d-1\,d}{[+]} + \cdots|1>\,,\\ &d = 4n,\end{aligned} \tag{3.106}$$

n is a positive integer. There are $2^{\frac{d}{2}-1}$ summands, since we step by step replace all possible pairs of $\overset{ab}{[-]}\cdots\overset{ef}{[-]}$ in the starting part $\overset{03}{[-i]}\overset{12}{[-]}\overset{35}{[-]}\cdots\overset{d-3\,d-2}{[-]}\overset{d-1\,d}{[+]}$ into $\overset{ab}{[+]}\cdots\overset{ef}{[+]}$ and include new terms into the vacuum state so that the last $2n+1$ summand has for $d = 4n$ also the factor $\overset{d-1\,d}{[+]}$ in the starting term $\overset{03}{[-i]}\overset{12}{[-]}\overset{35}{[-]}\cdots\overset{d-3\,d-2}{[-]}\overset{d-1\,d}{[+]}$ changed into $\overset{d-1\,d}{[-]}$. The vacuum state has then the normalization factor $1/\sqrt{2^{d/2-1}}$.

3.6 APPENDIX: Lorentz algebra and representations in Grassmann and Clifford space

The Lorentz transformations of vector components θ^a, γ^a, or $\tilde{\gamma}^a$ — usable for the description of the internal degrees of freedom of fermion fields obeying in the second quantization the anticommutation relations for fermions — and of vector components x^a, which are real (ordinary) commuting coordinates, $\theta'^a = \Lambda^a{}_b\,\theta^b$, $\gamma'^a = \Lambda^a{}_b\,\gamma^b$, $\tilde{\gamma}'^a = \Lambda^a{}_b\,\tilde{\gamma}^b$ and $x^a = \Lambda^a{}_b\,x^b$, leave forms $a_{a_1a_2\ldots a_i}\;\theta^{a_1}\theta^{a_2}\ldots\theta^{a_i}$, $a_{a_1a_2\ldots a_i}\;\gamma^{a_1}\gamma^{a_2}\ldots\gamma^{a_i}$, $a_{a_1a_2\ldots a_i}\;\tilde{\gamma}^{a_1}\tilde{\gamma}^{a_2}\ldots\tilde{\gamma}^{a_i}$ and $b_{a_1a_2\ldots a_i}\,x^{a_1}x^{a_2}\ldots x^{a_i}$, $i = (1,\ldots,d)$, invariant.

While $b_{a_1a_2\ldots a_i}$ $(=\eta_{a_1b_1}\eta_{a_2b_2}\cdots\eta_{a_ib_i}\,b^{b_1b_2\ldots b_i})$ is a symmetric tensor field, $a_{a_1a_2\ldots a_i}$ $(=\eta_{a_1b_1}\eta_{a_2b_2}\cdots\eta_{a_ib_i}\,a^{b_1b_2\ldots b_i})$ are antisymmetric *Kalb-Ramond* fields.

The requirements: $x'^a x'^b \eta_{ab} = x^c x^d \eta_{cd}$, $\theta'^a\theta'^b\varepsilon_{ab} = \theta^c\theta^d\varepsilon_{cd}$, $\gamma'^a\gamma'^b\varepsilon_{ab} = \gamma^c\gamma^d\varepsilon_{cd}$ and $\tilde{\gamma}'^a\tilde{\gamma}'^b\varepsilon_{ab} = \tilde{\gamma}^c\tilde{\gamma}^d\varepsilon_{cd}$ lead to $\Lambda^a{}_b\,\Lambda^c{}_d\,\eta_{ac} = \eta_{bd}$. Here η^{ab} (in our case $\eta^{ab} = diag(1,-1,-1,\dots,-1)$) is the metric tensor lowering the indexes of vectors ($\{x^a\} = \eta^{ab}x_b$, $\{\theta^a\} = \eta^{ab}\,\theta_b$, $\{\gamma^a\} = \eta^{ab}\,\gamma_b$ and $\{\tilde{\gamma}^a\} = \eta^{ab}\,\tilde{\gamma}_b$) and ε_{ab} is the antisymmetric tensor. An infinitesimal Lorentz transformation for the case with $\det\Lambda = 1$, $\Lambda^0{}_0 \geq 0$ can be written as $\Lambda^a{}_b = \delta^a_b + \omega^a{}_b$, where $\omega^a{}_b + \omega_b{}^a = 0$.

In Eqs. (3.4, 3.8) the commutation relations among the above objects are presented.

3.6.1 Lorentz properties of basic vectors

What follows is taken from Ref. [2] and Ref. [9], Appendix B.

Let us first repeat some properties of the anticommuting Grassmann and Clifford coordinates, taking into account Eqs. (3.3,3.4). An infinitesimal Lorentz transformation of the proper ortochronous Lorentz group is then

$$\begin{aligned}
\delta\theta^c &= -\frac{i}{2}\omega_{ab}\mathbf{S}^{ab}\theta^c = \omega^c{}_a\theta^a\,,\\
\delta\gamma^c &= -\frac{i}{2}\omega_{ab}S^{ab}\gamma^c = \omega^c{}_a\gamma^a\,,\\
\delta\tilde{\gamma}^c &= -\frac{i}{2}\omega_{ab}\tilde{S}^{ab}\tilde{\gamma}^c = \omega^c{}_a\tilde{\gamma}^a\,,\\
\delta x^c &= -\frac{i}{2}\omega_{ab}L^{ab}x^c = \omega^c{}_a x^a\,,
\end{aligned} \tag{3.107}$$

where ω_{ab} are parameters of a transformation and γ^a and $\tilde{\gamma}^a$ are expressible by θ^a and $\frac{\partial}{\partial\theta_a}$ in Eqs. (3.3,3.4).

Let us write the operator of finite Lorentz transformations as follows

$$\mathbf{S} = e^{-\frac{i}{2}\omega_{ab}(\mathbf{S}^{ab}+L^{ab})}\,, \tag{3.108}$$

$\mathbf{S}^{ab}$ have to be replaced by S^{ab} and $\tilde{S}^{ab}$ in the Clifford case. We see that the Grassmann θ^a and the ordinary x^a coordinates and the Clifford objects γ^a and $\tilde{\gamma}^a$ transform as vectors

$$\begin{aligned}
\theta'^c &= e^{-\frac{i}{2}\omega_{ab}(\mathbf{S}^{ab}+L^{ab})}\,\theta^c\,e^{\frac{i}{2}\,\omega_{ab}(\mathbf{S}^{ab}+L^{ab})}\\
&= \theta^c - \frac{i}{2}\omega_{ab}\{\mathbf{S}^{ab},\theta^c\}_- + \cdots = \theta^c + \omega^c{}_a\theta^a + \cdots = \Lambda^c{}_a\theta^a\,,\\
x'^c &= \Lambda^c{}_a x^a\,,\quad \gamma'^c = \Lambda^c{}_a\gamma^a\,,\quad \tilde{\gamma}'^c = \Lambda^c{}_a\tilde{\gamma}^a\,.
\end{aligned} \tag{3.109}$$

Correspondingly one finds that compositions like $\gamma^a p_a$ and $\tilde{\gamma}^a p_a$, here p_a are p^x_a $(= i\frac{\partial}{\partial x^a})$, transform as scalars (remaining invariants), while $S^{ab}\,\omega_{abc}$ and $\tilde{S}^{ab}\,\tilde{\omega}_{abc}$ transform as vectors. Objects like $R = \frac{1}{2}\,f^{\alpha[a}f^{\beta b]}\,(\omega_{ab\alpha,\beta} - \omega_{ca\alpha}\,\omega^c{}_{b\beta})$ and $\tilde{R} = \frac{1}{2}\,f^{\alpha[a}f^{\beta b]}\,(\tilde{\omega}_{ab\alpha,\beta} - \tilde{\omega}_{ca\alpha}\tilde{\omega}^c{}_{b\beta})$ from Eq. (3.1) transform with respect to the Lorentz transformations as scalars.

Making a choice of the Cartan subalgebra set of the algebra $\mathbf{S}^{ab}$, S^{ab} and $\tilde{S}^{ab}$, Eqs. (3.2, 3.5, 3.7),

$$\begin{aligned}
&\mathbf{S}^{03}, \mathbf{S}^{12}, \mathbf{S}^{56}, \cdots, \mathbf{S}^{d-1\;d}\,,\\
&S^{03}, S^{12}, S^{56}, \cdots, S^{d-1\;d}\,,\\
&\tilde{S}^{03}, \tilde{S}^{12}, \tilde{S}^{56}, \cdots, \tilde{S}^{d-1\;d}\,,
\end{aligned} \tag{3.110}$$

one can arrange the basic vectors so that they are eigenstates of the Cartan subalgebra, belonging to representations of $\mathbf{S}^{ab}$, or of S^{ab} and $\tilde{S}^{ab}$, with ab from Eq (3.110).

3.7 APPENDIX: Technique to generate spinor representations in terms of Clifford algebra objects

Here we briefly repeat the main points of the technique for generating spinor representations from Clifford algebra objects, following Ref. [2,47]. We advise the reader to look for details and proofs in these references. No requirements for the second quantization is taken into account.

We assume the objects γ^a, Eq. (3.4), which fulfill the Clifford algebra relations of Eq. (3.2), $\{\gamma^a, \gamma^b\}_+ = I \cdot 2\eta^{ab}$, for $a, b \in \{0, 1, 2, 3, 5, \cdots, d\}$, for any d, even or odd. I is the unit element in the Clifford algebra, while $\{\gamma^a, \gamma^b\}_\pm = \gamma^a\gamma^b \pm \gamma^b\gamma^a$.

The "Hermiticity" property for γ^a's and $\tilde{\gamma}^a$'s, Eq. (3.25), follows from Eq. (3.18), $\gamma^{a\dagger} = \eta^{aa}\gamma^a$ $\tilde{\gamma}^{a\dagger} = \eta^{aa}\tilde{\gamma}^a$, leading to $\gamma^{a\dagger}\gamma^a = I$, $\tilde{\gamma}^{a\dagger}\tilde{\gamma}^a = I$.

The Clifford algebra objects S^{ab} close the Lie algebra of the Lorentz group $\{S^{ab}, S^{cd}\}_- = i(\eta^{ad}S^{bc} + \eta^{bc}S^{ad} - \eta^{ac}S^{bd} - \eta^{bd}S^{ac})$, Eq. (3.7). One finds from Eq.(3.25) that $(S^{ab})^\dagger = \eta^{aa}\eta^{bb}S^{ab}$ and that $\{S^{ab}, S^{ac}\}_+ = \frac{1}{2}\eta^{aa}\eta^{bc}$.

Recognizing that two Clifford algebra objects (S^{ab}, S^{cd}) with all indexes different commute, we select (out of many possibilities) the Cartan subalgebra set of the algebra of the Lorentz group of Eq. (3.110)

Let us present the operators of subgroups of the $SO(13+1)$ group

$$\vec{N}_\pm (= \vec{N}_{(L,R)}) := \frac{1}{2}(S^{23} \pm iS^{01}, S^{31} \pm iS^{02}, S^{12} \pm iS^{03}), \tag{3.111}$$

$$\begin{aligned} \vec{\tau}^1 &:= \frac{1}{2}(S^{58} - S^{67},\, S^{57} + S^{68},\, S^{56} - S^{78}), \\ \vec{\tau}^2 &:= \frac{1}{2}(S^{58} + S^{67},\, S^{57} - S^{68},\, S^{56} + S^{78}), \end{aligned} \tag{3.112}$$

$$\begin{aligned} \vec{\tau}^3 := \; & \frac{1}{2}\{S^{9\,12} - S^{10\,11}, S^{9\,11} + S^{10\,12}, S^{9\,10} - S^{11\,12}, \\ & S^{9\,14} - S^{10\,13}, S^{9\,13} + S^{10\,14}, S^{11\,14} - S^{12\,13}, \\ & S^{11\,13} + S^{12\,14}, \frac{1}{\sqrt{3}}(S^{9\,10} + S^{11\,12} - 2S^{13\,14})\}, \\ \tau^4 : \; & = -\frac{1}{3}(S^{9\,10} + S^{11\,12} + S^{13\,14}). \end{aligned} \tag{3.113}$$

$$Y := \tau^4 + \tau^{23}, \quad Y' := -\tau^4 \tan^2\vartheta_2 + \tau^{23}, \quad Q := \tau^{13} + Y, \quad Q' := -Y \tan^2\vartheta_1 + \tau^{13}. \tag{3.114}$$

The equivalent expressions for the group $\widetilde{SO}(13, 1)$ follows from the above one, if replacing S^{ab} by $\tilde{S}^{ab}$.

To make the technique simple, we introduce the graphic representation, [47], Eq. (3.70),

$$
\begin{aligned}
\stackrel{ab}{(k)} &:= \frac{1}{2}(\gamma^a + \frac{\eta^{aa}}{ik}\gamma^b)\,,\\
\stackrel{ab}{[k]} &:= \frac{1}{2}(1 + \frac{i}{k}\gamma^a\gamma^b)\,,
\end{aligned}
$$

where $k^2 = \eta^{aa}\eta^{bb}$. One can easily check by taking into account the Clifford algebra relation (Eqs. (3.4, 3.18)) and the definition of S^{ab} (Eq. (3.2)) that if one multiplies from the left hand side by S^{ab} the Clifford algebra objects $\stackrel{ab}{(k)}$ and $\stackrel{ab}{[k]}$, it follows that, Eq. (3.72), $S^{ab}\stackrel{ab}{(k)}= \frac{1}{2}k\stackrel{ab}{(k)}$, $S^{ab}\stackrel{ab}{[k]}= \frac{1}{2}k\stackrel{ab}{[k]}$. This means that $\stackrel{ab}{(k)}$ and $\stackrel{ab}{[k]}$ acting from the left hand side on the vacuum state $|\psi_{oc}\rangle$), Eqs. (3.79, 3.106) for $d = 2(2n + 1)$ and $d = 4n$ respectively, are eigenvectors of S^{ab}.

We further find

$$
\begin{aligned}
\gamma^a \stackrel{ab}{(k)} &= \eta^{aa} \stackrel{ab}{[-k]},\\
\gamma^b \stackrel{ab}{(k)} &= -ik \stackrel{ab}{[-k]},\\
\gamma^a \stackrel{ab}{[k]} &= \stackrel{ab}{(-k)},\\
\gamma^b \stackrel{ab}{[k]} &= -ik\eta^{aa} \stackrel{ab}{(-k)}\;.
\end{aligned}
\tag{3.115}
$$

It follows that $S^{ac}\stackrel{ab}{(k)}\stackrel{cd}{(k)}= -\frac{i}{2}\eta^{aa}\eta^{cc}\stackrel{ab}{[-k]}\stackrel{cd}{[-k]}$, $S^{ac}\stackrel{ab}{[k]}\stackrel{cd}{[k]}= \frac{i}{2}\stackrel{ab}{(-k)}\stackrel{cd}{(-k)}$, $S^{ac}\stackrel{ab}{(k)}\stackrel{cd}{[k]}= -\frac{i}{2}\eta^{aa}\stackrel{ab}{[-k]}\stackrel{cd}{(-k)}$, $S^{ac}\stackrel{ab}{[k]}\stackrel{cd}{(k)}= \frac{i}{2}\eta^{cc}\stackrel{ab}{(-k)}\stackrel{cd}{[-k]}$.

It is useful to deduce the following relations

$$
\begin{array}{llll}
\stackrel{ab}{(k)}\stackrel{ab}{(k)} = 0\,, & \stackrel{ab}{(k)}\stackrel{ab}{(-k)}= \eta^{aa}\stackrel{ab}{[k]}\,, & \stackrel{ab}{(-k)}\stackrel{ab}{(k)}= \eta^{aa}\stackrel{ab}{[-k]}\,, & \stackrel{ab}{(-k)}\stackrel{ab}{(-k)}= 0\,,\\
\stackrel{ab}{[k]}\stackrel{ab}{[k]} = \stackrel{ab}{[k]}\,, & \stackrel{ab}{[k]}\stackrel{ab}{[-k]}= 0\,, & \stackrel{ab}{[-k]}\stackrel{ab}{[k]}= 0\,, & \stackrel{ab}{[-k]}\stackrel{ab}{[-k]}=\stackrel{ab}{[-k]}\,,\\
\stackrel{ab}{(k)}\stackrel{ab}{[k]} = 0\,, & \stackrel{ab}{[k]}\stackrel{ab}{(k)}=\stackrel{ab}{(k)}\,, & \stackrel{ab}{(-k)}\stackrel{ab}{[k]}=\stackrel{ab}{(-k)}\,, & \stackrel{ab}{(-k)}\stackrel{ab}{[-k]}= 0\,,\\
\stackrel{ab}{(k)}\stackrel{ab}{[-k]} = \stackrel{ab}{(k)}\,, & \stackrel{ab}{[k]}\stackrel{ab}{(-k)}= 0\,, & \stackrel{ab}{[-k]}\stackrel{ab}{(k)}= 0\,, & \stackrel{ab}{[-k]}\stackrel{ab}{(-k)}=\stackrel{ab}{(-k)}\;.
\end{array}
\tag{3.116}
$$

We recognize in the first equation of the first row and the first equation of the second row the demonstration of the nilpotent and the projector character of the Clifford algebra objects $\stackrel{ab}{(k)}$ and $\stackrel{ab}{[k]}$, respectively.

Whenever the Clifford algebra objects apply from the left hand side, they always transform $\stackrel{ab}{(k)}$ *to* $\stackrel{ab}{[-k]}$, *never to* $\stackrel{ab}{[k]}$, *and similarly* $\stackrel{ab}{[k]}$ *to* $\stackrel{ab}{(-k)}$, *never to* $\stackrel{ab}{(k)}$.

We define in Eq. (3.79, 3.106) the vacuum state $|\psi_{oc}>$ so that one finds

$$
\begin{aligned}
&< \stackrel{ab}{(k)}^{\dagger} \stackrel{ab}{(k)} >= 1\,, \\
&< \stackrel{ab}{[k]}^{\dagger} \stackrel{ab}{[k]} >= 1\,.
\end{aligned}
\tag{3.117}
$$

Taking the above equations into account it is easy to find a Weyl spinor irreducible representation for d-dimensional space, with d even or odd. (We advise the reader to see Ref. [2,47] in particular for d odd.)

For d even, we simply set the starting state as a product of $d/2$, let us say, only nilpotents $\stackrel{ab}{(k)}$ for $d = 2(2n+1)$, Eq. (3.73), or nilpotents and one projector, Eq. (3.104), for $d = 4n$, one for each S^{ab} of the Cartan subalgebra elements (Eq. (3.110)), applying it on the vacuum state, Eqs. (3.79, 3.106). Then the generators S^{ab}, which do not belong to the Cartan subalgebra, applied to the starting state from the left hand side, generate all the members of one Weyl spinor.

$$
\begin{aligned}
&\stackrel{0d}{(k_{0d})}\stackrel{12}{(k_{12})}\stackrel{35}{(k_{35})}\cdots\stackrel{d-1\;d-2}{(k_{d-1\;d-2})}\,|\psi_{oc}>\,,\\
&\stackrel{0d}{[-k_{0d}]}\stackrel{12}{[-k_{12}]}\stackrel{35}{(k_{35})}\cdots\stackrel{d-1\;d-2}{(k_{d-1\;d-2})}\,|\psi_{oc}>\,,\\
&\stackrel{0d}{[-k_{0d}]}\stackrel{12}{(k_{12})}\stackrel{35}{[-k_{35}]}\cdots\stackrel{d-1\;d-2}{(k_{d-1\;d-2})}\,|\psi_{oc}>\,,\\
&\vdots\\
&\stackrel{od}{(k_{0d})}\stackrel{12}{[-k_{12}]}\stackrel{35}{[-k_{35}]}\cdots\stackrel{d-1\;d-2}{[-k_{d-1\;d-2}]}\,|\psi_{oc}>\,,\\
&\text{for } d = 2(2n+1)\,,\quad n = \text{positive integer}\,.
\end{aligned}
\tag{3.118}
$$

$$
\begin{aligned}
&\stackrel{0d}{(k_{0d})}\stackrel{12}{(k_{12})}\stackrel{35}{(k_{35})}\cdots\stackrel{d-1\;d-2}{[k_{d-1\;d-2}]}\,|\psi_{oc}>\,,\\
&\stackrel{0d}{[-k_{0d}]}\stackrel{12}{[-k_{12}]}\stackrel{35}{(k_{35})}\cdots\stackrel{d-1\;d-2}{[k_{d-1\;d-2}]}\,|\psi_{oc}>\,,\\
&\stackrel{0d}{[-k_{0d}]}\stackrel{12}{(k_{12})}\stackrel{35}{[-k_{35}]}\cdots\stackrel{d-1\;d-2}{[k_{d-1\;d-2}]}\,|\psi_{oc}>\,,\\
&\vdots\\
&\stackrel{od}{(k_{0d})}\stackrel{12}{[-k_{12}]}\stackrel{35}{[-k_{35}]}\cdots\stackrel{d-1\;d-2}{[k_{d-1\;d-2}]}\,|\psi_{oc}>\,,\\
&\text{for } d = 4n\,,\quad n = \text{positive integer}\,.
\end{aligned}
\tag{3.119}
$$

3.7.1 Technique to generate "families" of spinor representations in terms of Clifford algebra objects

We found in this paper that for d even there are $2^{d/2-1}$ "family members" and $2^{d/2-1}$ "families" of spinors, which can be second quantized. (The reader is advised to see also Refs. [2,71,47,48,72,9].) We shall here pay attention on only even d.

One Weyl representation forms a left ideal with respect to the multiplication with the Clifford algebra objects. We proved in Refs. ([9,48], and the references therein) that there is the application of the Clifford algebra object from the right hand side, which generates "families" of spinors.

Right multiplication with the Clifford algebra objects namely transforms the state with the quantum numbers of one "family member" belonging to one "family" into the state of the same "family member" (into the same state with respect to the generators S^{ab} when the multiplication from the left hand side is performed) of another "family".

We defined in Ref.[2,48] the Clifford algebra objects $\tilde{\gamma}^a$'s as operations which operate formally from the left hand side (as γ^a's do) on any Clifford algebra object A as follows, Eq. (3.69),

$$\tilde{\gamma}^a A = i(-)^{(A)} A \gamma^a \,, \tag{3.120}$$

with $(-)^{(A)} = -1$, if A is an odd Clifford algebra object and $(-)^{(A)} = 1$, if A is an even Clifford algebra object.

Then it follows, in accordance with Eq. (3.4), that $\tilde{\gamma}^a$ obey the same Clifford algebra relation as γ^a.

$$(\tilde{\gamma}^a \tilde{\gamma}^b + \tilde{\gamma}^b \tilde{\gamma}^a) A = -ii((-)^{(A)})^2 A (\gamma^a \gamma^b + \gamma^b \gamma^a) = I \cdot 2.\eta^{ab} A \tag{3.121}$$

and that $\tilde{\gamma}^a$ and γ^a anticommute

$$(\tilde{\gamma}^a \gamma^b + \gamma^b \tilde{\gamma}^a) A = i(-)^{(A)} (-\gamma^b A \gamma^a + \gamma^b A \gamma^a) = 0 \,. \tag{3.122}$$

We may write

$$\{\tilde{\gamma}^a, \gamma^b\}_+ = 0, \quad \text{while} \quad \{\tilde{\gamma}^a, \tilde{\gamma}^b\}_+ = I \cdot 2\eta^{ab} \,. \tag{3.123}$$

One accordingly finds

$$\begin{aligned}
\tilde{\gamma}^a \stackrel{ab}{(k)} &: = -i \stackrel{ab}{(k)} \gamma^a = -i\eta^{aa} \stackrel{ab}{[k]} \,, \\
\tilde{\gamma}^b \stackrel{ab}{(k)} &: = -i \stackrel{ab}{(k)} \gamma^b = -k \stackrel{ab}{[k]} \,, \\
\tilde{\gamma}^a \stackrel{ab}{[k]} &: = \ i \stackrel{ab}{[k]} \gamma^a = \ i \stackrel{ab}{(k)} \,, \\
\tilde{\gamma}^b \stackrel{ab}{[k]} &: = \ i \stackrel{ab}{[k]} \gamma^b = -k\eta^{aa} \stackrel{ab}{(k)} \,.
\end{aligned} \tag{3.124}$$

If we define, Eq. (3.2),

$$\tilde{S}^{ab} = \frac{i}{4} [\tilde{\gamma}^a, \tilde{\gamma}^b] = \frac{i}{4} \{\tilde{\gamma}^a, \tilde{\gamma}^b\}_- = \frac{1}{4} (\tilde{\gamma}^a \tilde{\gamma}^b - \tilde{\gamma}^b \tilde{\gamma}^a) \,, \tag{3.125}$$

it follows

$$\tilde{S}^{ab} A = A \frac{1}{4} (\gamma^b \gamma^a - \gamma^a \gamma^b) \,, \tag{3.126}$$

manifesting accordingly that $\tilde{S}^{ab}$ fulfill the Lorentz algebra relation as S^{ab} do. Taking into account Eq. (3.69), we further find

$$\{\tilde{S}^{ab}, S^{ab}\}_- = 0 \,, \quad \{\tilde{S}^{ab}, \gamma^c\}_- = 0 \,, \quad \{S^{ab}, \tilde{\gamma}^c\}_- = 0 \,. \tag{3.127}$$

One also finds

$$
\begin{aligned}
&\{\tilde{S}^{ab}, \Gamma\}_- = 0\,, \quad \{\tilde{\gamma}^{a}, \Gamma\}_- = 0\,, \quad \{\tilde{S}^{ab}, \tilde{\Gamma}\}_- = 0\,, \quad \text{for } d \text{ even}\,,\\
&\Gamma^{(d)} := (i)^{d/2} \prod_{a} (\sqrt{\eta^{aa}}\gamma^{a})\,, \quad \text{if} \quad d = 2n\,,\\
&\tilde{\Gamma}^{(d)} := (i)^{d/2} \prod_{a} (\sqrt{\eta^{aa}}\tilde{\gamma}^{a})\,, \quad \text{if} \quad d = 2n\,,
\end{aligned} \tag{3.128}
$$

where handedness Γ ($\{\Gamma, S^{ab}\}_- = 0$) is a Casimir of the Lorentz group, which means that in d even transformation of one "family" into another with either $\tilde{S}^{ab}$ or $\tilde{\gamma}^{a}$ leaves handedness Γ unchanged.

We advise the reader to read [2] where the two kinds of Clifford algebra objects follow as two different superpositions of a Grassmann coordinate and its conjugate momentum.

Below some useful relations [71,72] are presented

$$
\begin{aligned}
&N^{\pm}_{+} = N^{1}_{+} \pm i\,N^{2}_{+} = -\stackrel{03}{(\mp i)}\stackrel{12}{(\pm)}\,, \quad N^{\pm}_{-} = N^{1}_{-} \pm i\,N^{2}_{-} = \stackrel{03}{(\pm i)}\stackrel{12}{(\pm)}\,,\\
&\tilde{N}^{\pm}_{+} = -\stackrel{03}{\tilde{(\mp i)}}\stackrel{12}{\tilde{(\pm)}}\,, \quad \tilde{N}^{\pm}_{-} = \stackrel{03}{\tilde{(\pm i)}}\stackrel{12}{\tilde{(\pm)}}\,,\\
&\tau^{1\pm} = (\mp)\,\stackrel{56}{(\pm)}\stackrel{78}{(\mp)}\,, \quad \tau^{2\mp} = (\mp)\,\stackrel{56}{(\mp)}\stackrel{78}{(\mp)}\,,\\
&\tilde{\tau}^{1\pm} = (\mp)\,\stackrel{56}{\tilde{(\pm)}}\stackrel{78}{\tilde{(\mp)}}\,, \quad \tilde{\tau}^{2\mp} = (\mp)\,\stackrel{56}{\tilde{(\mp)}}\stackrel{78}{\tilde{(\mp)}}\;.
\end{aligned} \tag{3.129}
$$

$$
\begin{aligned}
\tilde{S}^{ab}\stackrel{ab}{(k)} &= \frac{k}{2}\stackrel{ab}{(k)}\,,\\
\tilde{S}^{ab}\stackrel{ab}{[k]} &= -\frac{k}{2}\stackrel{ab}{[k]}\,,\\
\tilde{S}^{ac}\stackrel{ab}{(k)}\stackrel{cd}{(k)} &= \frac{i}{2}\eta^{aa}\eta^{cc}\stackrel{ab}{[k]}\stackrel{cd}{[k]}\,,\\
\tilde{S}^{ac}\stackrel{ab}{[k]}\stackrel{cd}{[k]} &= -\frac{i}{2}\stackrel{ab}{(k)}\stackrel{cd}{(k)}\,,\\
\tilde{S}^{ac}\stackrel{ab}{(k)}\stackrel{cd}{[k]} &= -\frac{i}{2}\eta^{aa}\stackrel{ab}{[k]}\stackrel{cd}{(k)}\,,\\
\tilde{S}^{ac}\stackrel{ab}{[k]}\stackrel{cd}{(k)} &= \frac{i}{2}\eta^{cc}\stackrel{ab}{(k)}\stackrel{cd}{[k]}\;.
\end{aligned} \tag{3.130}
$$

We transform the state of one "family" to the state of another "family" by the application of $\tilde{S}^{ac}$ (formally from the left hand side) on a state of the first "family" for a chosen a, c. To transform all the states of one "family" into states of another "family", we apply $\tilde{S}^{ac}$ to each state of the starting "family". It is, of course, sufficient to apply $\tilde{S}^{ac}$ to only one state of a "family" and then use generators of the Lorentz group (S^{ab}) to generate all the states of one Dirac spinor d-dimensional space.

One must notice that nilpotents $\stackrel{ab}{(k)}$ and projectors $\stackrel{ab}{[k]}$ are "eigenvectors"not only of the Cartan subalgebra S^{ab} but also of $\tilde{S}^{ab}$. Accordingly only $\tilde{S}^{ac}$, which do not carry the Cartan subalgebra indices, cause the transition from one "family" to another "family".

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