Energy Momentum Squared Gravity in Doneva-Jazadjiev anisotropic pressure model of non-rotating neutron stars

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Abstract. We implement the Doneva-Yazadjiev (DY) anisotropic model of neutron stars into Energy Momentum Squared Gravity (EMSG) theory. We have shown that the Tolmann-Oppenheimer-Volkoff (TOV) equation within this model can be expressed in the terms of effective pressure and energy density. The structure of neutron stars (NS) is then calculated by using Basic Standard Parameter equation of state with hyperons in the center. We found that the combined model is able to adjust the mass-radius relation of the NS. We also found that this model could affect boundary limit of EMSG parameter, α, in the term of stability.

1. Introduction

The Tolman-Oppenheimer-Volkoff (TOV) equations have been used as a standard to calculate the interior of neutron stars since 1939 [1]. These equations are expressed as the first derivative order of mass and energy density towards the radial directions. The standard TOV equation is derived from Einstein field equation solutions in the Schwarzchild geometry, which are widely used to calculate the interior profile of relativistic stars. Along with gravity, The Relativistic Mean Field (RMF) model is widely used to calculate Equation of State (EoS) for neutron stars. This theory is used by defining Lagrangian density from all particles inside the stars.

Meanwhile, the results of the neutron star mass-radius calculation are still much smaller than the latest data [2]. The presence of non-nucleonic and leptonic matter (e.g. hyperons) in the nucleus of neutron stars can be one of the responsible causes, as this matter has an important role in stiffening neutron stars.The standard description of NS assumed that the interior pressure is isotropic. However, it is also suggested the anisotropic behavior matter is an alternative to the standard isotropic matter which predicts a larger maximum mass than the standard neutron star. Neutron stars with anisotropic pressure models studied by Setiawan and Sulaksono have maximum mass predictions that are consistent with PSR J1614-2230 pulsar observations and PSR J0348+0432 [2]. Further review of the impact of anisotropic models with hyperons on mass-radius relations, moments of inertia, and tidal deformability found that the prediction of neutron star anisotropic radius was sensitive to Bowers-Liang anisotropic models; Horvat, et al; and Cosenza, et al [3].

On the other hand, from the gravity side, Einstein's general relativity is extended to explain the astrophysical phenomenon which still failed to be explained by standard general relativity.

For example, the reduction in cosmological constant values due to the expansion of the universe causes significant differences in neutron star parameters between the expansion and quantum field theory. Akarsu, et al use the theory of gravity modification of the square of momentum and energy tensors or are called EMSG (Energy Momentum Squared Gravity) by focusing on modification of Lagrangian material and non-linear analytic functions on scalar EMT (Energy Momentum Tensor), $T^2 : F(R, T) = R + \alpha T^2$. The energy-momentum tensor used by Akarsu et al is assumed to be in isotropic pressure. In this study, it is interesting if one of the isotropic models in the EMSG study [4] is substituted with one of the neutron star anisotropic models with hyperons [2].

2. Methods
We calculate the NS properties by deriving the TOV equation analytically. First, it is used in the pressure function with the definition of anisotropic Lagrangian as follows [5]:

$$\mathcal{L}_{\mathcal{M}}^{\mathcal{A}} = \frac{1}{3}(p + 2q) \tag{1}$$

The Lagrangian anisotropic form above is incorporated into the general form of energy-momentum tensors in equation (2):

$$T_{\mu\nu} = g_{\mu\nu}\mathcal{L}_{\mathcal{M}} - 2\frac{\partial \mathcal{L}_{\mathcal{M}}}{\partial g^{\mu\nu}}T_{\mu\nu} = \epsilon u_\mu u_\nu + pk_\mu k_\nu + q\left[g_{\mu\nu} + u_\mu u_\nu - k_\mu k_\nu\right]. \tag{2}$$

The result of the first step can specify $\theta_{\mu\nu}$ in equation (3),

$$\theta_{\mu\nu} = -2\mathcal{L}_{\mathcal{M}}\left(T_{\mu\nu} - \frac{1}{2}g_{\mu\nu}T\right) - TT_{\mu\nu} + 2T_\mu^\gamma T_{\nu\gamma} - 4T^{\sigma\epsilon}\frac{\partial^2 \mathcal{L}_{\mathcal{M}}}{\partial g^{\mu\nu}\partial g^{\sigma\epsilon}} \tag{3}$$

so that anisotropic EMSG EFE will be obtained with additional σ terms when compared to isotropic ones:

$$G_{\mu\nu} + \Lambda g_{\mu\nu} = \kappa T_{\mu\nu} + \kappa\alpha\left(g_{\mu\nu}T_{\sigma\epsilon}T^{\sigma\epsilon} - 2\theta_{\mu\nu}\right). \tag{4}$$

This EFE form must be reduced to equation (4) if σ is set zero. The TOV solution is analogously solved with a solution for isotropic EMSG, but it takes anisotropic pressure on the Einstein field equation which is also defined ineffective pressure and energy. After reproducing from the standard anisotropic TOV in equations (5) and (6):

$$\frac{dm}{dr} = 4\pi\epsilon r^2 \tag{5}$$

$$\frac{dp}{dr} = -G\frac{\epsilon m}{r^2}\left(1 + \frac{p}{\epsilon}\right)\frac{\left(1 + \frac{4\pi r^3 p}{m}\right)}{\left(1 - \frac{2Gm}{r}\right)} - \frac{2\sigma}{r} \tag{6}$$

into effective interaction for EMSG, the definition of σ can be explicitly entered into the TOV equation. Through algebra and sound speed definitions, we obtain the new TOV EMSG equation under DY anisotropic pressure conditions in the $\frac{dp}{dr}$ and $\frac{dm}{dr}$ functions, this model must also be reduced to the isotropic EMSG TOV equations :

$$\frac{dm}{dr} = 4\pi r^2\epsilon\left[1 + \alpha\epsilon\left(1 + 8\frac{P}{\epsilon} + 3\frac{P^2}{\epsilon^2}\right)\right], \tag{7}$$

$$\frac{dP}{dr} = -G\frac{m\epsilon}{r^2}\left(1+\frac{P}{\epsilon}\right)\left(1-2G\frac{m}{r}\right)^{-1}\left[1+\frac{4\pi r^3 p}{m}+\alpha\frac{4\pi r^3 \epsilon^2}{m}\left(1+3\frac{P^2}{\epsilon^2}\right)\right]\left[1+2\alpha\epsilon\left(1+3\frac{P}{\epsilon}\right)\right]$$

$$\left[1+2\alpha\epsilon\left(Cs^{-2}+3\frac{P}{\epsilon}\right)\right]^{-1}. \quad (8)$$

Second, numerical calculations are carried out by the 4th order Runge-Kutta method. This method is used to integrate pressure and mass parameters in the TOV equation starting from pressure and mass at the center of $(r=r_c\approx 0)$, from here the maximum pressure value $p=p_c$ and minimum mass $m=0$, then integration is carried to the surface so that the longer the pressure runs out $p\approx 0$, which at this point is defined as the radius of the star $r=R$ and $m(r)=M$.

3. Results and Discussion

3.1. Einstein Field Equation (EFE) for EMSG within Anisotropic Pressure

Lagrangian and EMT from anisotropic pressure from equations (3) and (1) are used to solve EFE. Lagrangian from equation (1) substituted into standard EMT in equation (2), then we get:

$$T_{\mu\nu} = \frac{1}{3}(p+2q)g_{\mu\nu} - 2\frac{\partial\mathcal{L}_M}{\partial g^{\mu\nu}}. \quad (9)$$

First order derivation of EMT defines:

$$\frac{\partial\mathcal{L}_M}{\partial g^{\mu\nu}} = \frac{\sigma}{2}\left(\frac{1}{3}g_{\mu\nu}-k_\mu k_\nu\right) - \frac{(\epsilon+q)}{2}u_\mu u_\nu \quad (10)$$

The $\theta_{\mu\nu}$ tensor on the equation (3) put into Einstein field on the equation (4). Then, all components of $\theta_{\mu\nu}$ calculated:

$$\frac{\partial^2\mathcal{L}_M}{\partial g^{\mu\nu}\partial g^{\sigma\epsilon}} = \frac{1}{6}(p-q) = \frac{\sigma}{6}, \quad (11)$$

$$T = T^\mu_\mu = g^{\mu\nu}T_{\mu\nu}, \quad (12)$$

$$T^A_{\mu\nu} = qg_{\mu\nu} + (\epsilon+q)u_\mu u_\nu + (p-q)k_\mu k_\nu, \quad (13)$$

$$T^A = g^{\mu\nu}T^A_{\mu\nu} = 3q+\sigma-\epsilon, \quad (14)$$

$$T^A_{\mu\nu}T^{\mu\nu}_A = \epsilon^2 + 3q^2 + \sigma(\sigma+2q), \quad (15)$$

with $u^\alpha u_\alpha = -1$ and $k^\alpha k_\alpha = 1$, so:

$$T^\alpha_\mu T_{\nu\alpha} = q^2 g_{\mu\nu} - \left(\epsilon^2-q^2\right)u_\mu u_\nu + \left(\sigma^2+2q\sigma\right)k_\mu k_\nu. \quad (16)$$

The accumulation of the above equations produces a tensor $\theta_{\mu\nu}$ as:

$$\theta_{\mu\nu} = -\frac{\sigma}{3}\left[(q+\epsilon\sigma)g_{\mu\nu} + \left(7(q+\epsilon)+\frac{3}{\sigma}\left(4q\epsilon+3q^2+\epsilon^2\right)\right)u_\mu u_\nu\right] - \frac{\sigma}{3}(\sigma+3(q-\epsilon))k_\mu k_\nu. \quad (17)$$

3.2. Correction Terms of Pressure and Energy Density caused by Anisotropic EMSG Model
From equation (17), we can determine EMSG tensor by:

$$\theta_{\mu\nu} = \theta_{\mu\nu}^{g}\left(\epsilon, q, \sigma\right) g_{\mu\nu} + \theta_{\mu\nu}^{\nu}\left(\epsilon, q, \sigma\right) u_{\mu} u_{\nu} + \theta_{\mu\nu}^{\kappa}\left(\epsilon, q, \sigma\right) k_{\mu} k_{\nu}, \tag{18}$$

with

$$\theta_{\mu\nu}^{g}\left(\epsilon, q, \sigma\right) = -\frac{\sigma q}{3} + \frac{\sigma^2}{3} - \frac{\sigma \epsilon}{3}, \tag{19}$$

$$\theta_{\mu\nu}^{\nu}\left(\epsilon, q, \sigma\right) = -\frac{7\sigma\epsilon}{3} - \frac{7\sigma q}{3} - 4q\epsilon - 3q^2 - \epsilon^2, \tag{20}$$

$$\theta_{\mu\nu}^{\kappa}\left(\epsilon, q, \sigma\right) = -\frac{\sigma^2}{3} - \sigma q + \sigma \epsilon. \tag{21}$$

The Einstein field equation used when $\Lambda = 0$ from equation (4) becomes a new form :

$$G_{\nu}^{\mu} = \kappa \left[q_{eff}\left(q, \epsilon, \sigma\right) g_{\nu}^{\mu} + \left[\epsilon_{eff}\left(q, \epsilon, \sigma\right) + q_{eff}\left(q, \epsilon, \sigma\right)\right] u^{\mu} u_{\nu} + \sigma_{eff}\left(q, \epsilon, \sigma\right) k^{\mu} k_{\nu}\right]. \tag{22}$$

If we define $p = q + \sigma$ and $p_{eff} = q_{eff} + \sigma_{eff}$, then a new form of effective radial pressure :

$$p_{eff} = p + \alpha \left(\epsilon^2 + 3p^2 - \frac{4\sigma p}{3} - \frac{2\sigma^2}{3} - \frac{4\sigma\epsilon}{3} \right) \tag{23}$$

First effective pressure derivation of r for equation (23) with $\epsilon = \epsilon\left(p\left(r\right)\right)$, becomes:

$$\frac{dp_{eff}}{dr} = C\frac{dp}{dr} + D\frac{d\sigma}{dr} \tag{24}$$

with

$$C = 1 + \alpha \left[2\epsilon \left(\frac{\partial \epsilon}{\partial p} \right) + 6p - \frac{4\sigma}{3} - \frac{4\sigma}{3} \left(\frac{\partial \epsilon}{\partial p} \right) \right], \tag{25}$$

$$D = -\alpha \left(\frac{4p}{3} + \frac{4\sigma}{3} + \frac{4\epsilon}{3} \right). \tag{26}$$

A new form for correction of energy density and anisotropic pressure is:

$$\epsilon_{eff} + p_{eff} = (\epsilon + p)\left[1 + \frac{2}{3}\alpha\left(3\epsilon + 9p - 8\sigma\right)\right], \tag{27}$$

$$\sigma_{eff} = \sigma \left[1 + \alpha \left(2p - 2\epsilon - \frac{4\sigma}{3} \right) \right], \tag{28}$$

$$\epsilon_{eff} = \epsilon + \alpha \left[\epsilon^2 - 4\sigma\left(\epsilon + p\right) + 8\epsilon p + 3p^2 + \frac{2\sigma^2}{3} \right]. \tag{29}$$

Equations (27-29) put in equation (23) and we can substitute σ as the Doneva-Jazadjiev model to get TOV solutions.

3.3. TOV Equation for Anisotropic DY EMSG Model

The first order of anisotropic derivation of r from equation (29) becomes,

$$\sigma_{DY} \equiv \gamma \left(\frac{2MG}{r} \right) p \tag{30}$$

$$\frac{d\sigma_{DY}}{dr} = 2\gamma G \left[\frac{p}{r} 4\pi r^2 \epsilon_{eff} + \frac{m}{r} \frac{dp}{dr} - \frac{mp}{r^2} \right] \tag{31}$$

$$\frac{d\sigma_{DY}}{dr} = C_3^{DY} \frac{dp}{dr} + C_4^{DY}. \tag{32}$$

The EMSG TOV equation is in anisotropic pressure with corrections:

$$\frac{dp_{eff}}{dr} = -\frac{G \left(\epsilon_{eff} + p_{eff} \right) \left(m + 4\pi r^3 p_{eff} \right)}{1 - \frac{2Gm}{r}} - \frac{2\sigma_{eff}}{r}, \tag{33}$$

Then, in the equation (34), the first derivative pressure correction for the DY model is defined by:

$$\frac{dp_{eff}}{dr} = \left[C + DC_3^{DY} \right] \frac{dp}{dr} + DC_4^{DY}. \tag{34}$$

So we get anisotropic EMSG TOV equations for the DY model as:

$$\frac{dP^{DY}}{dr} = -\frac{G \left(\epsilon_{eff}^{DY} + p_{eff}^{DY} \right) \left(m + 4\pi r^3 p_{eff}^{DY} \right) r^2 \left(1 - \frac{2Gm}{r} \right) + \frac{2\sigma_{eff}^{DY}}{r} + DC_4^{DY}}{C + DC_3^{DY}} \tag{35}$$

and

$$\frac{dm^{DY}}{dr} = 4\pi r^2 \epsilon_{eff}^{DY}. \tag{36}$$

Then, we also found that α boundaries, identified by ϵ_{eff} and p_{eff}, was changed from isotropic model (α_{EMSG}^I) to the new anisotropic DY model (α_{EMSG}^A) as follows:

$$\alpha_{EMSG}^I = \alpha \epsilon \left(1 + 8\frac{p}{\epsilon} + 3\frac{p^2}{\epsilon^2} \right) \tag{37}$$

$$\alpha_{EMSG}^A = \alpha \epsilon \left[\alpha_{EMSG}^I + \frac{2\sigma^2}{3\epsilon^2} - 4 \left(\frac{\sigma}{\epsilon} + \frac{\sigma p}{\epsilon^2} \right) \right]. \tag{38}$$

The stability condition must fulfill $\alpha_{EMSG}^A > -1$, for $\frac{dm}{dr} > 0$ and $\frac{dp}{dr} < 0$ of the radial directions:

$$\alpha_{EMSG}^A > -\frac{1}{\epsilon_c \left[\alpha_{EMSG}^I + \frac{2\sigma^2}{3\epsilon_c^2} - 4 \left(\frac{\sigma}{\epsilon_c} + \frac{\sigma p}{\epsilon_c^2} \right) \right]} \tag{39}$$

$$\alpha_{EMSG}^A > -\frac{1}{2\epsilon_c c_s^{-2} + 6p - \frac{4}{3}\sigma \left(1 + c_s^{-2} \right)}. \tag{40}$$

3.4. The Mass-Radius Relations of Neutron Stars

We solve equation (35-36) numerically by using fourth-order Runge-Kutta with BSP [3] parameter set EoS. The result is shown on figure 1 and 2. The mass value against the maximum radius of the neutron star is shown in figure 1 for variations in the anisotropic parameter γ and in figure 2 for variations in the value α as an EMSG parameter. From the graph figure 1, when the value $\alpha = 0,62 \times 10^{-38} cm^3/erg$ is set and the value γ is

varied, the maximum mass increases. There is an increase in the maximum mass compared to isotropic conditions around $0,75M_{sun}$ with maximum radii in the range of $11.5 - 11.75$ km. Analysis of Riley and Miller on PSRJ0030+051 measures the maximum mass of neutron stars $1,34 - 0,16 \leq m \leq 1,34 + 0,15M_{sun}$ with radii $12,71 - 1,19 \leq R \leq 12,71 + 1,14\ km$. And their analysis on PSR J0740+6620 shown that NS mass becomes $2,08 - 0,07 \leq m \leq 2,08 + 0,07Msun$ with radii $12,39 - 0,98 \leq R \leq 12,39 + 1,3\ km$ [5], [6], [7]. figure 2 graph shows the greater value of α with the maximum mass but the results of the α variation are far in line with PSR data J0740+6620 because the maximum mass produced does not exceed $2,1M_{sun}$. The α variations are also sensitive to the radius at the maximum mass of the star. At the negative α value, it gets radius 11.6 km, and mass of NS touches $2,1M_{sun}$. Meanwhile, when α increase to the positive value, radius of NS widens.
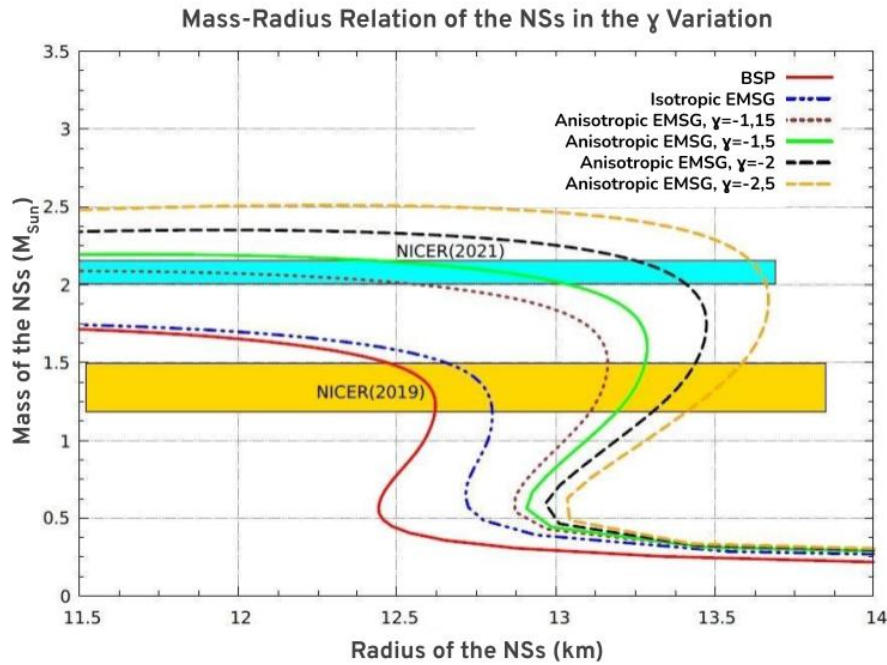

Figure 1. Radius-Mass relation from variations of γ at $\alpha = 0,62 \times 10^{-38}\ cm^3/erg$ on an anisotropic DY model + NICER constraints.

4. Conclusions

We have investigated anisotropic pressure of EMSG model on the neutron stars by using modified EMT to get EFE and TOV modified equations. These correction terms for parameters: $\sigma_{eff}, p_{eff}, q_{eff},$ and ϵ_{eff} dominated by non-linear pressure. Radius-mass relation can add neutron stars' mass also corresponds with NICER 2019 and 2021. Radius-mass relation on the γ variations at $\alpha = 0,62 \times 10^{-38}\ cm^3/erg$ includes all R_{maks} of both NICER 2019 and 2021 constraints, but it is inappropriate to predict the maximum mass. Afterward, m_{maks} in radius-mass relation for the α variable when $\gamma = -1,15$ is more compatible with NICER 2019 and 2021. The maximum NS mass on anisotropic EMSG DY Model is in range of $2,1 \leq m_{maks} \leq 2,5M_{sun}$. The mass increase is more influenced by the γ parameter and the reduction in R_{maks} is more influenced by the α parameter.
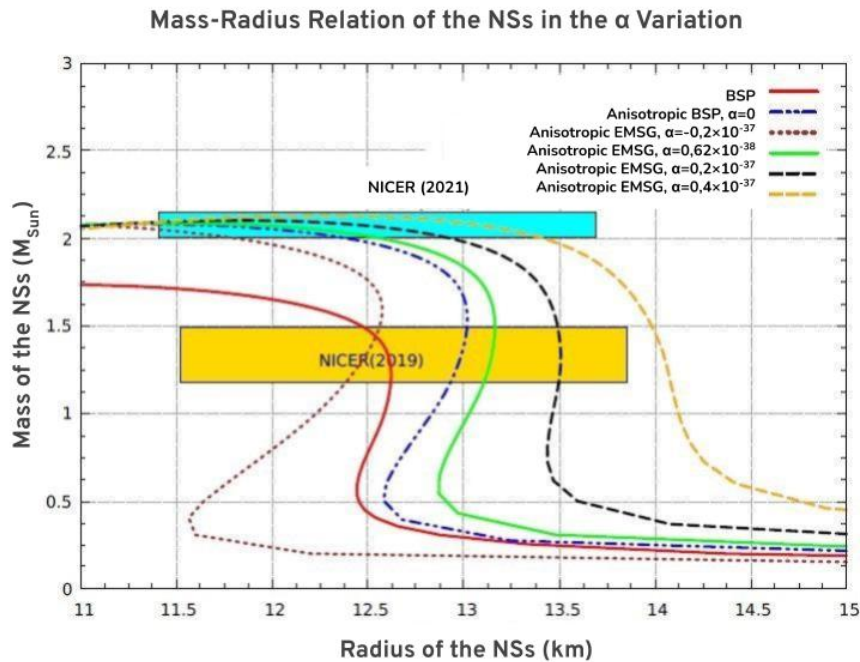
Figure 2. Radius-Mass relation from variations of α at $\gamma = -1,15$ on an anisotropic DY model + NICER constraints.

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