QUARK CONFINEMENT IN GAUGE THEORIES OF STRONG INTERACTIONS

Leonard Susskind†

Belfer Graduate School of Science, New York, N.Y., USA
and
Tel Aviv University, Tel Aviv, Israel

I. ABELIAN MODELS

a) Introduction

I will not begin by telling you all the reasons why you have to believe in quarks as hadron constituents[1]. Lets just suppose that I had and that we all believe hadrons are loosely bound collections of quarks*. Secondly I want you to suppose that nobody will ever discover a free isolated quark. We are then faced with the puzzling problem of explaining how finite forces conspire to confine quarks to the interior of hadrons.

To begin with we must realize that most of our intuitions, even our idea that a puzzle exists, come from our experience with weakly coupled quantum electrodynamics and its perturbative solution. We are often led astray into asking questions which make the phenomenon sound much more complicated then it really is. For example: What class of graphs is important to confine quarks? Or: Do the catastrophic infrared divergences of Yang Mills theory combine to screen quarks? We ought to understand that these questions do not really refer to the behaviour of the system but rather to the method of solution - perturbation theory about free fields. The reason that quark confinement seems so odd to us is because we start with all the wrong ideas about how (the correct strong interaction) field theory behaves and then attempt to perturb our way to an infinitely distant behaviour.

---

† Most of the work described in these lectures was carried out in collaboration with J. Kogut while the author was a visitor at Cornell University.

* Loosely bound in the sense that they behave almost freely at short distances.

In these lectures I will show you three examples of theories with confinement. In each case it is easy to see that quarks are confined although perturbation theory buries the obvious in a jungle of complicated graphs. The three examples share a key element, namely local gauge invariance. The importance of local gauge invariance is that it connects additive conserved charges to long range fields through Gauss's theorem. The most familiar case is the long range Coulomb field accompanying every isolated charge in electrodynamics. Similarly in the non-abelian color-gauge theory of quarks every state with a non-zero color must have a long range color-electric field in order to be gauge invariant. This includes all states with non-vanishing triality.

The quark confining mechanism does not directly deal with the quarks but rather with their long range color-electric fields. If the color-electric fields are confined so that the electric flux lines are prevented from radiating to infinity then the finite energy states must be color-neutral.

The three examples are Schwinger's one dimensional QED[2,3,4]; a semiclassical model based on unusual dielectric properties of the vacuum[5,6]; and a hamiltonian formulation of Wilson's lattice gauge theory[7,8,9].

b) The Schwinger Model

I will now make two approximations on the real problem. First I will replace the three colors of the 3-triplet model by a single abelian color called charge. Instead of three kinds of quarks (red, yellow, blue) I now have only one. The confinement mechanism will operate to eliminate all objects except neutral bosons.

Having agreed to approximate three by one I will apply the approximation again, this time on the number of space dimensions. The result of these approximations is the Schwinger model or QED in one dimension[2,3,4].

I am not going to derive the formal solution[3] to the model or give a rigorous demonstration of confinement. This is partly because you can find these things in the literature, but more importantly, I want to avoid the special features of one dimension which make the model solvable. In fact I will work in a gauge which is particularly inconvenient for exact solutions. The gauge is defined by setting the time component of the vector potential to zero

$$A_t = 0 \tag{I.1}$$

The space component of the vector potential will be called A. The gauge invariant field tensor has only one independent component, the electric field, which is given by

$$E = \frac{d}{dt} A \tag{I.2}$$

The hamiltonian is given by

$$H = \int dz \left\{ \psi^{+} \alpha \left(\partial_z + i g A \right) \psi + \frac{1}{2} \dot{A}^{2} \right\} \tag{I.3}$$

Properly speaking eq. (I.3) defines a class of gauges related by time independent gauge transformations

$$\begin{aligned} \psi &\longrightarrow e^{i g \Lambda(z)} \psi \\ A &\longrightarrow A + \partial_z \Lambda \end{aligned} \tag{I.4}$$

for time independent Λ. The hamiltonian in eq. (I.3) is of course gauge invariant for this class of gauge transformations. Furthermore, and this is important, <u>all physical states must be invariant under (I.4)</u>. The reason I have chosen this special class of gauges and restricted the gauge invariance to time independent Λ is because the restricted gauge transformations can be represented using unitary operators $U(\Lambda)$.

$$\begin{aligned} U \psi U^{-1} &= e^{i g \Lambda} \psi \\ U A U^{-1} &= A + \partial_z \Lambda \end{aligned} \tag{I.5}$$

Infinitesimal generators $G(\Lambda)$ can also be introduced for infinitesimal Λ

$$\begin{aligned} [G, \psi] &= i g \Lambda(z) \psi(z) \\ [G, A] &= \partial_z \Lambda \end{aligned} \tag{I.6}$$

When t-dependent gauge transformations are considered operators like U and G no longer exist and I don't know how to express the gauge invariance of the physical states. Gauge invariance under the restricted gauge group simply requires every physical state $|\rangle$ to satisfy

$$U(\Lambda)\,|\,\rangle = |\,\rangle$$
$$G(\Lambda)\,|\,\rangle = 0 \tag{I.7}$$

for all Λ.

Fortunately it is very easy to find $G(\Lambda)$. Using the canonical commutation relations

$$[A(z), \dot{A}(z')] = i\,\delta(z-z') = [A(z), E(z')]$$
$$[\psi(z), \rho(z')] = g\,\psi(z)\,\delta(z-z') \tag{I.8}$$

where $\rho = g\psi^\dagger\psi$, you can easily verify that

$$G = \int \Lambda(z)\left[\rho(z) - \frac{\partial E}{\partial z}\right] dz \tag{I.9}$$

We see that gauge invariance requires

$$\left\{\rho(z) - \frac{\partial E}{\partial z}\right\}|\,\rangle = 0 \tag{I.10}$$

This equation expresses the familiar fact that the charge density is the source of electric field. It is not an equation of motion but a constraint on the physical states. In the physical subspace it implies

$$E(z) = E(-\infty) + \int_{-\infty}^{z} \rho(z')\,dz' \tag{I.11}$$
$$= E(+\infty) - \int_{z}^{\infty} \rho(z')\,dz' \tag{I.12}$$

Lets suppose $E(\infty)$ or $E(-\infty)$ is non-zero. Then since H contains the term $\int E^2 dz$ it is evident that the energy will be infinite. To remain in the space of finite energy $E(\pm\infty)$ must be zero. But then the two expressions (I.11) and (I.12) will not be equal unless

$$\int_{-\infty}^{\infty} \rho(z')\, dz' = 0 \qquad \text{(I.13)}$$

What I have proved is that the finite energy, gauge invariant states have zero total charge. But I would be cheating if I told you that this proves charged particles don't exist. What I really want to show is that the finite energy, gauge invariant states do not contain well separated quarks and antiquarks.

For definiteness I will use eq. (I.11) for the electric field. We can picture a charge at z_0 as being the source of an electric field which vanishes for $z < z_0$. This is shown in Fig. 1

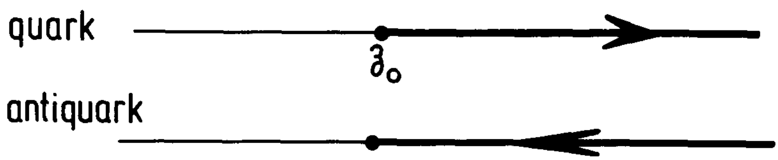

Fig. 1

I want to digress briefly to describe the objects in Fig. 1 by operators. Since the field ψ is not gauge invariant the state $\psi(z_0)|0\rangle$ is not a good description of a physical quark. To make a state which satisfies gauge invariance we have to do something to create the line of electric flux which must accompany the charge. Consider the operator

$$\sqcup(f) = \exp\left[i \int_{-\infty}^{\infty} A(z)\, f(z)\, dz\right] \qquad \text{(I.14)}$$

where f is a c-number function of position. Using the fact that A and E are canonical conjugates we find

$$\sqcup(f)\, E(z)\, \sqcup^{-1}(f) = E(z) + f(z) \qquad \text{(I.15)}$$

This means that $\sqcup$ acts on a state to shift the electric field by amount f. This is useful because we need to shift E by amount $g\,\Theta(z - z_0)$ when a quark is created at z_0. Therefore it makes sense to multiply $\psi(z_0)$ by the factor exp $ig \int_{z_0}^{\infty} A(z)\, dz$. The resulting gauge invariant operator can be used to create the physical states shown in Fig. 1. We define

$$\Psi(z) = \exp\left[ig\int_z^{\infty} A(z')\,dz'\right]\psi(z)$$

or (I.16)

$$\Psi(z) = U(z)\,\psi(z)$$

Exercise: Prove Ψ is gauge invariant. Show that the expectation value of E in the state $\Psi|0\rangle$ is $g\,\theta(z-z_0)$.

Now the reason why quarks are confined in this model is not because it costs an infinite energy to apply $\psi(z)$ to a state but rather because the factor $U(z)$ costs an infinite energy. This of course is due to the uniform electric field which fills space from z_0 to ∞.

Next lets consider a high energy $q\bar{q}$ pair which is produced, perhaps by a lepton annihilation, at the origin. My discussion is going to be at the impressionistic level so I suggest you look at Ref. 3 for formal arguments. The initial state is something like

$$|\,initial\rangle = \bar{\psi}(0)\,\psi(0)\,|0\rangle \tag{I.17}$$

Since the two operators ψ and $\bar{\psi}$ are evaluated at the same point the initial state is gauge invariant.

As the system evolves the quark pair will separate. If you forgot gauge invariance you might guess that the state at a later time is something like

$$|\,later\rangle = \bar{\psi}(-z)\,\psi(z)\,|0\rangle \tag{I.18}$$

But this is impossible because the state (I.18) is not gauge invariant and could not have been obtained if the system's evolution is governed by a gauge invariant hamiltonian.

A more correct guess is

$$\begin{aligned}|\,later\rangle &= \bar{\Psi}(-z)\,\Psi(z)\,|0\rangle \\ &= \bar{\psi}(-z)\,e^{ig\int_{-z}^{z} A\,dz'}\,\psi(z)\,|0\rangle \end{aligned} \tag{I.19}$$

which describes a $q\bar{q}$ pair at the ends of a line of electric flux. (See Fig. 2)

Fig. 2

Since the electric field is uniform between the quarks, the energy stored in the field is $\sim g^2|x|$ where $|x|$ is the distance separating the pair. Thus the quarks can separate to a distance proportional to their initial energy.

Of course the real state $|\text{later}\rangle$ is not really as simple as (I.19). Since ψ is a relativistic field the interaction between the quark field and the electric field can create pairs in the region between the original pair. For example $|\text{later}\rangle$ will have a piece like

$$\overline{\Psi}(-x)\ \Psi(x_1)\ \overline{\Psi}(x_2)\ \Psi(x)\ |0\rangle \qquad \text{(I.20)}$$

which looks like Fig. 3.

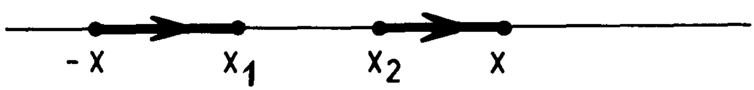

Fig. 3

However it is obvious that the evolution of the system can never lead to an isolated quark which is separated from compensating charges by more than a distance proportional to the initial energy. The exact solution of the Schwinger model shows that the real final state consists of a number of $q\bar{q}$ pairs, each with its connecting flux line and that the probability to find a quark at a distance $> \frac{1}{g}$ falls exponentially.

I have dwelled at length on this trivial model so that you would get a clear picture of the connection between gauge invariance, continuity of electric flux and confinement. Just in case it was not clear I will say it again: Gauge invariance requires every quark to be the end of a flux line with uniform energy density and every end to be a quark. Since every flux line has two ends (unless it is infinite and therefore infinitely heavy) quarks must occur in pairs. This idea will be repeated throughout the rest of these lectures.

c) Semiclassical Model

At first sight the situation in 3 space dimensions looks very unfavorable for confinement. Gauge invariance still requires the charges to be the sources of electric flux

$$\nabla \cdot E = \rho \tag{I.21}$$

but this time the flux lines have two more dimensions to spread out into. (See Fig. 4)

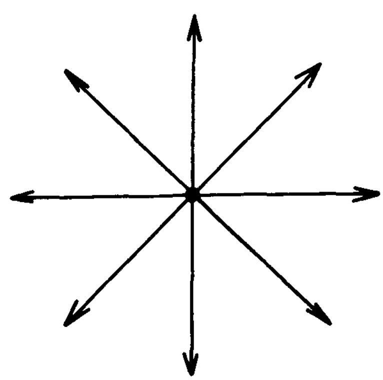

Fig. 4

If the field spreads with spherical symmetry then continuity of flux insures that it falls like

$$|E| \sim \frac{1}{r^2} \tag{I.22}$$

and the total energy is finite except for ultraviolet $(r \to 0)$ effects which are removed by renormalization.

I am going to describe a model, cooked up by Kogut and myself[5] and independently by 't Hooft[6] which forces the electric field to behave very differently from Fig. 4. The model is very unrealistic but it will help prepare you for the more ambitious model of lecture 3.

The model assumes that the vacuum is a dielectric medium with some unusual properties. I will begin by reminding you of the electrostatics of dielectrics.

The free charges (quarks) are sources of the Maxwell D field

$$\vec{\nabla} \cdot \vec{D} = \rho \tag{I.23}$$

The electric field E is curl free and is related to D by the dielectric permeability $\varepsilon(x)$

$$\vec{D}(x) = \varepsilon(x) \vec{E}(x) \tag{I.24}$$

In this model $\varepsilon(x)$ can take one of two values, namely zero and one at any point. The regions where $\varepsilon = 0$ will be called forbidden because the D field is excluded from such regions. Wherever $\varepsilon = 1$ the material is normal.

The energy consists of two terms, the first being electrostatic energy and the second being the internal energy stored in the dielectric. The electrostatic energy is

$$W_{e.s.} = \int d^3x \, \vec{D} \cdot \vec{E} = \int d^3x \, \frac{D^2}{\varepsilon} \tag{I.25}$$

From (I.25) it is evident that D is excluded from regions where $\varepsilon = 0$. The internal energy of the dielectric will be chosen so that the forbidden regions have less energy than the normal. Thus the ground state or vacuum is forbidden. We will write the internal energy as

$$W_\varepsilon = c \int \varepsilon(x) \, d^3x \tag{I.26}$$

remembering that ε has only 2 values. The total energy is

$$W = \int \frac{D^2}{\varepsilon} d^3x + c \int \varepsilon(x) \, d^3x \tag{I.27}$$

The model was invented so that the long range component of the D field would cost an infinite amount of energy. To see how the model works let's suppose the dielectric material fills a sphere of radius R and outside the sphere $\varepsilon = 0$. Suppose a charge g is placed at the origin. The D field then satisfies

$$\nabla \cdot \vec{D} = g \, \delta^3(x) \tag{I.28}$$

The first type of solution to try is a spherically symmetric distribution of flux

$$\vec{D} = g\,\frac{\hat{r}}{r^2} \tag{I.29}$$

Since any forbidden region with non-vanishing D costs infinite energy, the entire dielectric must be normal. The resulting energy is

$$W = \frac{4\pi g}{a} + \frac{4}{3}\pi R^3 c \tag{I.30}$$

In this formula a represents the size over which the charge is smeared. The first term is the electrostatic energy and the second term is the internal energy of the dielectric when the whole sphere is normal. The second term diverges as R^3 when the volume of the dielectric goes to infinity.

The energy can be lowered by allowing the electric flux to be distributed non-symmetrically. For example, suppose all of the flux is distributed over a solid angle Ω within which the dielectric is normal. The D field is given by

$$\vec{D} = \frac{4\pi}{\Omega}\, g\,\frac{\hat{r}}{r^2} \tag{I.31}$$

within the solid angle Ω and is zero outside. This time the total energy is

$$W = g\,\frac{16\pi^2}{\Omega a} + \frac{\Omega}{3} R^3 c \tag{I.32}$$

The electrostatic energy has increased because the field lines are squeezed but the internal energy is lowered. Since when $R \to \infty$ the internal energy dominates it always pays to decrease Ω .

The limiting form of field which lowers the energy to its absolute minimum is to allow all the flux to go through a long thin tube of normal material until it reaches the surface of the dielectric. (See Fig. 5)

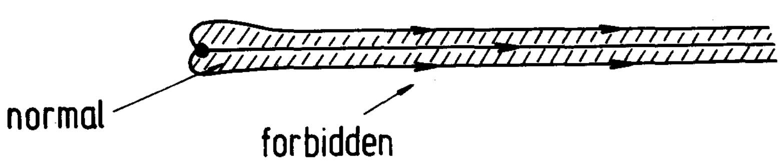

Fig. 5

The thickness of the tube is obtained by varying the energy per unit length with respect to the radius. If the radius is r the $\mathcal{D}$ field (which is parallel to the tube) is

$$\mathcal{D} = \frac{4g}{r^2} \tag{I.32}$$

and the electrostatic energy per unit length is

$$\frac{W_{es}}{L} = \frac{16\pi g^2}{r^2} \tag{I.33}$$

and the internal energy$/L$ is $\pi r^2 c$. Thus

$$\frac{W}{L} = \pi r^2 c + \frac{16\pi g^2}{r^2} \tag{I.34}$$

and the minimum occurs at

$$r^2 = \frac{4g}{\sqrt{c}} \tag{I.35}$$

Eq. (I.35) represents the thickness of the tube far from the charge. Near the charge the situation is more complicated. What is clear is that the minimum energy of an isolated charge grows linearly with R since, far from the charge, the energy per unit length is constant.

The remaining arguments now parallel the one dimensional case. The separation of quarks can only take place until the available energy is used up or until the tube breaks by pair production.

In the next two lectures I will show you how the nonlinearities of quantized Yang Mills theory can squeeze the electric flux into one dimensional tubes.

II. YANG MILLS IN ZERO DIMENSIONS

a) Gauge Invariance in Zero Space Dimensions

What is field theory in zero space dimensions? It is field theory in which there is only one or a finite number of points of space and therefore a finite number of degrees of freedom. For the free scalar field theory, the zero dimensional version is a single harmonic oscillator or a finite number of coupled oscillators. The first step in understanding a field theory is to understand its zero dimensional analog.

The second step involves a lattice of elementary zero dimensional systems with some form of coupling between the neighboring systems. If the lattice spacing is not too large a qualitative understanding of the large scale behaviour of the field theory is usually possible at this level. Of course the short distance behaviour is absent.

The final and most difficult step is allowing the lattice spacing to go to zero. Typically this involves renormalization of the parameters so that the low energy (long wave length) behaviour is prevented from varying as the spacing tends to zero.

In this lecture I am going to show you how to do step one. We will formulate Yang Mills theory for two spatial points (one point is too trivial). Then in lecture III we will do step 2 and show how quarks may be confined in the strongly coupled theory. Unfortunately the third step will have to wait until someone figures out how to do it.

We begin with a universe consisting of a pair of points 1 and 2 and a continuum of time. The presence of colored quarks on site 1 and 2 is described by fields $\psi_j(i)$ and $\psi_j^\dagger(i)$. Here i labels the 2 points 1 and 2 and j is the color index*. The field ψ may be represented in terms of fermion creation and annihilation operators for each site

$$\psi^\dagger(i) = a^+(i) + b^-(i) \tag{II.1}$$

where $a^+(i)$ $\left(b^-(i)\right)$ creates (annihilates) a quark (antiquark) at site i.

We will begin with a very simple hamiltonian which just assigns an energy μ to a quark

$$H = \mu \sum_i :\psi^\dagger(i)\,\psi(i): \tag{II.2}$$

In addition to global color rotations

$$\psi(i) \longrightarrow V\psi(i) \tag{II.3}$$

H is invariant under separate color rotations at sites 1 and 2

$$\psi(i) \longrightarrow V(i)\,\psi(i) \tag{II.4}$$

In equs. (II.3) and (II.4) the quantities V, $V(1)$ and $V(2)$ are any special unitary 2x2 matrices.

---

* For illustrative purposes the color group will be SU_2 instead of SU_3.

Transformations like (II.4) in which different color rotations may act at 1 and 2 are called local gauge transformations. They are symmetries of the hamiltonian in (II.2) since the degrees of freedom at 1 and 2 are completely uncoupled. But the lack of coupling is not necessary for local gauge invariance. For example the term

$$\psi^{\dagger}(1)\,\psi(1)\,\psi^{\dagger}(2)\,\psi(2) \tag{II.5}$$

couples sites 1 and 2 and is gauge invariant. The important feature of hamiltonians like (II.2) and (II.5) is that they do not transport quarks from one site to another.

To make H a little more interesting we can introduce terms which do transport quarks from 1 to 2. For example

$$i\left[\psi^{\dagger}(1)\,\psi(2) - \psi^{\dagger}(2)\,\psi(1)\right] \tag{II.6}$$

annihilates a quark at 2 and creates one at 1. This term is still globally color invariant but local gauge invariance is lost. This implies an absolute standard of comparison between color directions at 1 and 2.

I don't know of any mathematical principle which forbids such an absolute standard but it does seem to me to endow space with some extra machinery to keep track of the relative phases between 1 and 2. Let me make this machinery more explicit in the form of a matrix U which relates the two color reference frames. If the two frames are parallel then $U = 1$ and H is given by (II.6). Now let's imagine that the color frame at 2 was secretly rotated relative to 1. The relative rotation would be detected because the dynamics would now involve a nontrivial matrix U in the form

$$i\left[\psi^{\dagger}(1)\,U\,\psi(2) - \psi^{\dagger}(2)\,U^{-1}\psi(1)\right] \tag{II.7}$$

In the Yang Mills theory the relative rotation would be undetectable. The gauge invariance is restored by making the connecting matrix U a dynamical variable with time dependence, an equation of motion and quantum fluctuations. The new degree of freedom U belongs to neither site but jointly to the two sites, or better yet, to the space between the sites.

Since U is an SU_2 matrix connecting the color frames at sites 1 and 2 it can be written in the form

$$U = \exp \frac{i}{2} \tau \cdot B \qquad \text{(II.8)}$$

where τ are the three Pauli matrices. The two indices of U are associated with the two sites. Under a local gauge transformation U transforms as

$$U \to V(1)\, U\, V^{-1}(2) \qquad \text{(II.9)}$$

in order to keep the hamiltonian in (II.7) unchanged.

We will soon introduce gauge invariant terms into H which do not commute* with U. When this is done U will no longer be a static set of numbers but will become a full fledged quantum dynamical variable. We will no longer be able to transform U away by a color rotation at one site. And finally, although H permits processes in which quarks hop from site to site, the dynamics remains invariant under local gauge transformation.

b) Kinematics and Dynamics of U [9]

The real heart of non-abelian gauge mechanics is in the properties of the operators U . On what space of states do components of U act? What are the variable conjugate to U and what are the commutation relations? The answer to these questions in the simplified zero dimensional model will determine the principles of quantization of the infinitely richer lattice model of lecture III[9].

The system described by U has as its configuration space the set of all possible rotations in 3-dimensional color space (More exactly elements of the universal covering group SU_2). The elements of U are a particular set of coordinates in this space. There are many other possible ways to coordinitize this space. For example the Euler angles can be used to parametrize rotations. Or the matrices U may be written in terms of a vector potential B as in (II.8). A particularly useful family of coordinates is defined by the representation matrices for color spin j . These $(2j+1) \times (2j+1)$ matrices may be written

---

* At present U is a matrix in the 2x2 color space. The individual components of U will become operators in the quantum space of states. We are using the term commute in the latter sense.

$$U_j = \exp \frac{i}{2} \left[T_j \cdot B \right] \tag{II.10}$$

where T_j are the Pauli matrices for spin j. The U of eq. (II.7) is the special case $U_{1/2}$. Whenever U occurs without a subscript j it will be understood as $U_{1/2}$.

The symmetry group associated with local gauge invariance is $SU_2 \times SU_2$. The two SU_2 groups are the local gauge transformations at sites 1 and 2 and each has its own generators. The 3 generators at site (i) are called $E_\alpha(i)$ and have the commutation relations

$$\left[E_\alpha(i), E_\beta(j) \right] = i\, \varepsilon_{\alpha\beta\gamma}\, \delta_{ij}\, E_\gamma(i) \tag{II.11}$$

From the transformation laws (II.9) it follows that the E's and U's satisfy the commutation relations

$$\left[E_\alpha(1), U_j \right] = \frac{1}{2} (T_j)_\alpha U_j \qquad \text{(no sum on } j\text{)}$$

$$\left[E_\alpha(2), U_j \right] = - U_j \frac{T_j}{2} \tag{II.12}$$

Since $U_{1/2}$ completely determines an element of the rotation group, all the U_j are functions of $U_{1/2}$. Therefore the quantum conditions are completely specified by the relations (II.12) for $j = 1/2$. Furthermore the three sets of variables $E(1)$, $E(2)$ and U are not independent. In fact $E(2)$ is given in terms of $E(1)$ and U_1 by

$$E(2) = - U_1\, E(1) \tag{II.13}$$

This can be shown by substituting

$$\left(U_1 \right)_{\alpha\beta} = \frac{1}{2} \operatorname{Tr} U \tau_\alpha U^{-1} \tau_\beta$$

Then the second of eq. (II.12) follows from the first and (II.13). Eq. (II.13) says that the color vectors $E(1)$ and $E(2)$ are related by the rotation described by U. This observation will play a central role in our understanding of electric flux in Y. M. theory.

From (II.13) it follows that

$$E(2)^2 = E(1)^2 \tag{II.14}$$

In general states classified under the group $SU_2 \times SU_2$ are labelled by two total angular momenta $j(1)$ and $j(2)$ and two magnetic quantum numbers $m(1)$ and $m(2)$ such that

$$-j(i) \leq m(i) \leq j(i)$$

In the present case eq. (II.14) requires

$$j(1) = j(2) \equiv j \tag{II.15}$$

so that the states form $(2j+1)^2$ degenerate multiplets.

The conditions (II.11) - (II.15) can be realized on a space of states generated as follows. We begin with a "base" state $|0\rangle$ which is invariant under $SU_2 \times SU_2$. We then construct a $(2j+1)^2$ dimensional multiplet by acting with the $(2j+1)^2$ elements of U_j on $|0\rangle$. Thus we define a unique $|0\rangle$ such that

$$E(1)|0\rangle = E(2)|0\rangle = 0 \tag{II.16}$$

The $(2j+1)^2$ states forming the (j,j) representation of $SU_2 \times SU_2$ are given by

$$U_j|0\rangle \tag{II.17}$$

It is easy to prove that the states in (II.17) are eigenvectors of $E(1)^2 = E(2)^2$.

$$\begin{aligned} E(1)^2 U_j|0\rangle &= E_\alpha(1) E_\alpha(1) U_j|0\rangle \\ &= E_\alpha(1)[E_\alpha(1), U_j]|0\rangle \quad \text{(see eq. (II.16))} \\ &= E_\alpha(1)\frac{(T_j)_\alpha}{2} U_j|0\rangle \quad \text{(see eq. (II.12))} \\ &= \tfrac{1}{2}[E_\alpha(1), (T_j)_\alpha U_j]|0\rangle \\ &= \tfrac{1}{4}(T_j)_\alpha (T_j)_\alpha U_j|0\rangle \\ &= j(j+1) U_j|0\rangle \end{aligned} \tag{II.18}$$

It is also possible to generate the space of states using only the matrices U of the $1/2$ color representation. This is done by expressing U_j as a homogeneous polynomial of order $2j$ in the components of U and U^{-1}. This corresponds to the fact that any angular momentum can be built from spin $1/2$ systems. As an example we express U_1 in terms of $U_{1/2}$

$$(U_1)_{\alpha\beta} = \frac{1}{2} \mathrm{Tr}\left[U \tau_\alpha U^{-1} \tau_\beta \right] \qquad \text{(II.19)}$$

The matrices U are the zero dimensional analogs of

$$\exp\, i\, a\, g \frac{\tau}{2} \cdot A \qquad \text{(II.20)}$$

where a is the spatial distance between sites 1 and 2, g is the coupling constant and A is the vector potential. Similarly the generators $E(1)$ and $E(2)$ have analogs in the conventional Y. M. theory. The generators E are the non-abelian analogs of electric field. More precisely $E(1)$ ($E(2)$) is the electric field at site 1 (2) pointing toward 2 (1).

You should notice a certain formal similarity between the abelian and non abelian theories. In the abelian theory the operator $\exp[igA \cdot \varepsilon]$ acts to create an electric field along the direction $\vec{\varepsilon}$. In the non abelian theory $U_j = \exp \frac{1}{2} i a g\, T_j A$ creates a non abelian electric field with magnitude $E^2 = j(j+1)$. However in the abelian theory the electric flux adds linearly, in the Y. M. theory it adds like angular momentum. The interpretation of E as electric field will become clearer when it is shown that $\nabla \cdot E = \rho$ in the next lecture.

The total color carried by the system consists of the color carried by fermions plus the color carried by the gauge field U. The color carried by U is defined as the quantity which generates global color rotations of U. A global rotation rotates both frames equally

$$U \rightarrow V U V^{-1} \qquad \text{(II.21)}$$

and under an infinitesimal rotation about the color axis α

$$\delta U = [c(\alpha), U] = [\tau_\alpha U - U \tau_\alpha]/2 \qquad \text{(II.22)}$$

where $c(\alpha)$ is the total α component of color. From (II.12) it is evident that the color carried by the gauge field is $E_\alpha(1) + E_\alpha(2)$. The total color is then

$$\sum_{i=1}^{2} :\psi^\dagger(i) \frac{\tau_\alpha}{2} \psi(i): + E_\alpha(1) + E_\alpha(2) \qquad \text{(II.23)}$$

The color carried by the gauge field $E(1) + E(2)$ may be thought of as the zero dimensional analog of $\nabla \cdot E$.

Not all the states of the system are physical. As in 1-dimensional QED the constraint of local gauge invariance must be applied to the physical states. To derive these conditions we note that the local color rotation at site i is generated by

$$G^{\alpha}(i) = \frac{1}{2}\,\psi^{+}(i)\,\tau_{\alpha}\psi(i) + E_{\alpha}(i) \qquad \text{(II.24)}$$

The terms $\frac{1}{2}\,\psi^{+}(i)\,\tau\,\psi(i)$ rotate the quark fields while the E's rotate U. As in abelian theory the gauge constraints state that $G(i)$ annihilate any physical state

$$\left\{E(i) + \psi^{+}(i)\,\frac{\tau}{2}\,\psi(i)\right\}\,|\;\rangle = 0 \qquad \text{(II.25)}$$

When eq. (II.25) and (II.13) are combined an interesting physical picture emerges. We can visualize sites 1 and 2 as sources and sinks of electric field. This is shown in Fig. 6.

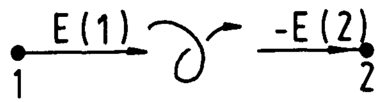

Fig. 6

Eq. (II.25) tells us that the total flux leaving site i is equal to the charge at that point. However in going from site 1 to 2 the electric flux undergoes a color rotation as indicated by eq. (II.13). This change in electric flux can be viewed as a source if we recall that the total color carried by the gauge field between 1 and 2 is

$$C_{\alpha}(\text{field}) = E_{\alpha}(1) + E_{\alpha}(2) \qquad \text{(II.26)}$$

The point which I will reemphasize in the Lecture III is that the color in the gauge field does not originate new lines of flux but rather it twists them in color space.

The construction of the physical space of states begins by defining a gauge invariant product state $|0\rangle$ by means of the relations

$$\begin{aligned} a^{-}(i)\,|0\rangle = b^{-}(i)\,|0\rangle = 0 \\ E(i)\,|0\rangle = 0 \end{aligned} \qquad \text{(II.27)}$$

where a^{-} (b^{-}) annihilates quarks (antiquarks) at site i. The next step is to define enough gauge invariant operators to generate the whole space when acting on $|0\rangle$. We will do this by considering products of $\psi(i)$, $\psi^{+}(i)$ and $U_{1/2}$. Let's first formulate a rule which will allow us to easily recognize gauge invariant operators. The rule is that if we focus attention on the indices associated with one site (or the other) the operator should form a scalar. This can only happen if all the indices at a given site are contracted among themselves. I'll give some examples. First the operator $\psi_i^{\dagger}(1)\,\psi_i(1)$ (here i is a color index) is gauge invariant because the contracted indices (i) belong to the same site. However $\psi^{\dagger}(1)\,\psi(2)$ is not gauge invariant. The list of gauge invariant operators which are necessary to create the full space of states is given by

$$
\begin{aligned}
&\psi^{\dagger}(1)\,\psi(1)\\
&\psi^{\dagger}(2)\,\psi(2)\\
&\psi^{\dagger}(1)\,U\,\psi(2)\\
&\psi^{\dagger}(2)\,U^{-1}\psi(1)
\end{aligned}
\qquad \text{(II.28)}
$$

The idea of local contraction of indices is rather trivial for the zero dimensional case but it will be very useful in constructing the gauge invariant operators in the more complex lattice theory.

Now lets examine the states that can be made by repeated application of the operations (II.28).

1) $\psi^{\dagger}(i)\,\psi(i)\,|0\rangle$. This is a $q\bar{q}$ pair in the color singlet state. Both particles are at site i .
2) $\psi^{\dagger}(1)\,\psi(1)\,\psi^{\dagger}(2)\,\psi(2)|0\rangle$. This is a color singlet pair at each site.
3) $\psi^{\dagger}(1)\,U\,\psi(2)$. This is a quark at (1) and an antiquark at (2). The operator U creates an electric flux satisfying (II.25).
4) $\{\psi^{\dagger}(1)\,U\,\psi(2)\}\{\psi^{\dagger}(2)\,U^{-1}\psi(1)\}\,|0\rangle$. This state is more complex than the others since it contains two superimposed electric fluxes.

I will illustrate the technique for adding flux by using a simple identity whose proof you can supply.

$$
\begin{aligned}
U_{ij}\,(U^{-1})_{kl} &= \tfrac{1}{2}\,\delta_{il}\,\delta_{jk} + \tfrac{1}{4}\left(\mathrm{Tr}\,U^{-1}\tau_\beta U \tau_\alpha\right)(\tau_\beta)_{il}\,(\tau_\alpha)_{jk}\\
&= \tfrac{1}{2}\,\delta_{il}\,\delta_{jk} + \tfrac{1}{2}\,\left(U_1\right)_{\beta\alpha}\,(\tau_\beta)_{il}\,(\tau_\alpha)_{jk}
\end{aligned}
\qquad \text{(II.29)}
$$

When (II.29) is substituted into the state (4) we get a superposition of states

$$\frac{1}{2}\,\psi^{+}(1)\,\psi(1)\;\psi^{+}(2)\,\psi(2)$$

$$+\frac{1}{2}\left[\psi^{+}(1)\,\tau_{\alpha}\,\psi(1)\right]\left(U_{1}\right)_{\alpha\beta}\left[\psi^{+}(2)\,\tau_{\beta}\,\psi(2)\right] \qquad \text{(II.30)}$$

The first term we have already talked about. The second represents a new object composed of a color-spin 1 pair at each site. The colored pairs are accompanied by a color-1 flux line created by U_1. This example illustrates how you must combine flux in a non-abelian theory.

In general the states 1-4 are not energy eigenvectors. If H is gauge invariant it will not lead out of this subspace but it may have transition elements within the subspace.

I will choose H to be as close as possible to a real covariant Y. M. hamiltonian. For this purpose we can write

$$U_{1/2} = \exp \frac{ig}{2}\,\tau\cdot A = 1 + \frac{ig}{2}\,\tau\cdot A + \ldots \qquad \text{(II.31)}$$

First consider $\psi^{+}(1)\,U\,\psi(2)$. Applying (II.31) gives

$$\psi^{+}(1)\,U\,\psi(2) = \psi^{+}(1)\,\psi(2) + ig\,\psi^{+}(1)\,\frac{\tau}{2}\,\psi(2)\cdot A$$

adding the h. c. gives

$$i\Big[\psi^{+}(1)\,\psi(2) - \psi^{+}(2)\,\psi(1)\;+$$

$$+\,ig\,A\cdot\Big(\psi^{+}(1)\,\frac{\tau}{2}\,\psi(2) + \psi^{+}(2)\,\frac{\tau}{2}\,\psi(1)\Big)\Big] \qquad \text{(II.32)}$$

These terms are analogs of the kinetic and interaction terms in a conventional gauge theory.

The next term represents the energy stored in the electric field. It is given by the gauge invariant operator

$$\frac{E(1)^2}{2} = \frac{E(2)^2}{2} \qquad \text{(II.33)}$$

You should compare these terms with eq. (I.3) to see how they are similar to ordinary terms in a gauge theory. The only terms which are not present in the zero dimensional model (and one dimensional models) are the magnetic energy. Magnetic fields do not occur for spatial dimensions < 2. In the lattice theory in 3 dimensions they will be included.

Exercise: Construct a Yang Mills theory for 4 points arranged in a square. Each corner has a ψ and each side a U. What is the significance of the operator

$$\mathrm{Tr}\; U(1)\, U(2)\, U(3)\, U(4) \qquad ?$$

What is the effect of adding this operator into the hamiltonian?

III. LATTICE YANG MILLS THEORY

a) Degrees of Freedom of a Lattice

The usual continuous coordinates (x, y, z) of space are replaced by a triplet of integers $(r_x, r_y, r_z) = (\vec{r})$. The points $(\vec{r})$ are called sites. At each site there are six lattice vectors $\hat{n}_x$, $\hat{n}_y$, $\hat{n}_z$, $\hat{n}_{-x}$, $\hat{n}_{-y}$, $\hat{n}_{-z}$ shown in Fig. 7 .

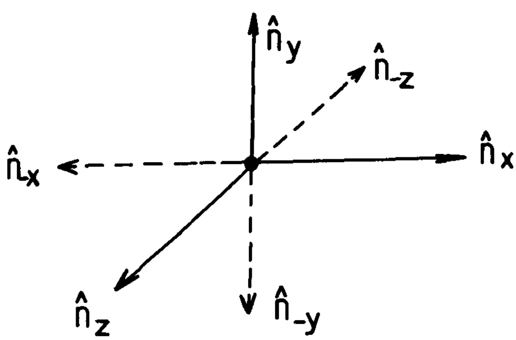

Fig. 7

In general sums over the lattice vectors will include all 6 directions.

The spaces between sites will be called links. The links will usually be considered to be directed and will be labelled by a site and a lattice vector. For example $(r, \hat{n})$ and $(r+\hat{n}, -\hat{n})$ describe the two directed links associated with the space between r and $r+\hat{n}$.

A 4 component fermion field $\psi(r)$ can be represented in terms of creation and annihilation operators for quarks and antiquarks at site (r)

$$\psi(r) = a_i^{+}(r)\, \xi_i + b_i^{-}(r)\, \overline{\xi}_i \qquad \text{(III.1)}$$

where $\xi_1 = \begin{pmatrix}1\\0\\0\\0\end{pmatrix}$, $\xi_2 = \begin{pmatrix}0\\1\\0\\0\end{pmatrix}$, $\xi_3 = \begin{pmatrix}0\\0\\1\\0\end{pmatrix}$, $\xi_4 = \begin{pmatrix}0\\0\\0\\1\end{pmatrix}$

in a representation in which

$$\gamma_0 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

Each link will carry a degree of freedom $U(r, \hat{n})$ to describe how color information is transported between neighboring sites. The two directed links associated with the same lattice space do not have independent degrees of freedom. In fact the two U's are inverses of one another

$$U(r, \hat{n}) = U^{-1}(r + \hat{n}, -\hat{n}) \tag{III.2}$$

Each link has two generators analogous to $E(1)$ and $E(2)$ in the last lecture. I will use a labelling scheme for the E's defined as follows. If we consider the link $(r, \hat{n})$ it has two ends, one at (r) and one at $(r+\hat{n})$. The two generators for the degree of freedom $U(r, \hat{n})$ at r and $r+\hat{n}$ will be called

$$E(r)\cdot\hat{n} \quad \text{and} \quad E(r+\hat{n})\cdot(-\hat{n}) \tag{III.3}$$

The notation indicates that the two generators represent electric flux in opposing directions as shown in Fig. 8.

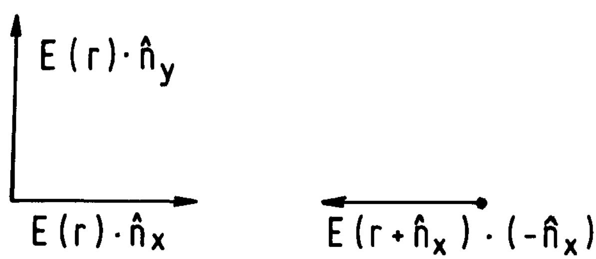

Fig. 8

Thus the generators $E\cdot\hat{n}$ are electric fluxes flowing outward from r in the direction $\hat{n}$. The commutation relations between the components of $E\cdot\hat{n}$ are the usual SU_2 commutation relations. Eq. (II.13) is replaced by

$$E(r+\hat{n})\cdot(-\hat{n}) = -U(r, \hat{n})\, E(r)\cdot\hat{n} \tag{III.4}$$

and (II.14) by

$$\left(E(r)\cdot\hat{n}\right)^2 = \left(E(r+\hat{n})\cdot(-\hat{n})\right)^2 \tag{III.5}$$

Evidently the source of electric flux on link $(r,\hat{n})$ is

$$c(r,\hat{n}) = E(r)\cdot\hat{n} + E(r+\hat{n})\cdot(-\hat{n}) \tag{III.6}$$

b) The Gauge Invariant Subspace

The physical constraint of gauge invariance again requires every quark to be a source of electric flux. The way to show this is to follow the same logic we used in one dimensional QED and zero dimension Y. M. theory - construct the local generator of gauge transformations and then set it to zero. The gauge transformation at site r acts on $\psi(r)$ and on the six gauge fields $U(r,\hat{n})$. Accordingly the generator is the sum of seven terms

$$G(r) = \psi^\dagger(r)\frac{\tau}{2}\psi(r) + \sum_{\hat{n}} E(r)\cdot\hat{n} \tag{III.7}$$

The physical subspace is then defined by

$$\left\{\psi^\dagger(r)\frac{\tau}{2}\psi(r) + \sum_{\hat{n}} E(r)\cdot\hat{n}\right\}|\ \rangle = 0 \tag{III.8}$$

The quantity $\sum_{\hat{n}} E(r)\cdot\hat{n}$ is the total flux diverging from the point r . Eq. (III.8) then gives the usual connection between the divergence of E and the charge density $\psi^\dagger\frac{\tau}{2}\psi$.

Unlike the abelian gauge field the Y. M. field is also a source. This can be seen from eq's. (III.4) and (III.6) which say that the field varies along a link by an amount equal to the color carried by that link. It is evident that the flux passing through a closed surface is the sum of the colors carried by the sites (quarks) and links (gauge field) enclosed.

We can now construct the physical space of states beginning with a vector $|0\rangle$ satisfying

$$a^-(r)|0\rangle = b^-(r)|0\rangle = 0$$

$$E(r)\cdot\hat{n}|0\rangle = 0 \qquad (\text{all } r \text{ and } \hat{n}) \tag{III.9}$$

Let's ignore the quarks and concentrate on the gauge invariant operators which can be built from the U's. The principle for forming gauge invariants is again the local contraction of indices. To form the general class of gauge invariant operators we first specify a closed oriented path of links Γ. The path may cover any link one or more times (see Fig. 9)

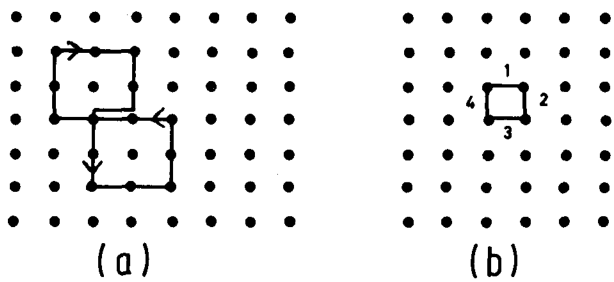

Fig. 9

Now beginning with an arbitrary link on the path, multiply (2x2 matrix multiplication) the U's in the order indicated by Γ. For example for the path shown in Fig. 9b the required product is

$$U(1)\,U(2)\,U(3)\,U(4) \tag{III.10}$$

There are still two open indices which are contracted by taking the trace. The resulting object is called $U(\Gamma)$. Since the indices in $U(\Gamma)$ are all locally contracted $U(\Gamma)$ is gauge invariant. It can be shown that the entire gauge invariant space is generated by repeated application of the $U(\Gamma)$ operators applied to $|0\rangle$.

The physical properties of $U(\Gamma)|0\rangle$ are very simple and interesting. First consider any link not on the path Γ. Since no U has acted to create electric field these links have no electric flux through them. The links which appear in Γ (suppose no link appears more than once) have an electric flux satisfying

$$E^2 = \frac{1}{2}\left(\frac{1}{2}+1\right) = \frac{3}{4} \tag{III.11}$$

Accordingly the closed curve Γ can be described as a closed line of electric flux. The fact that electric flux lines must form closed lines in the absence of quarks is of course the familiar idea of continuity of electric flux originally envisioned by Faraday. However there are two differences between lines of flux in ordinary electrodynamics and Y. M. theory on a lattice.

The first difference is due to the fact that the Y. M. field is its own source. However it is a particularly simple kind of source which according to (III.4) causes the electric field to undergo a color rotation between the two ends of a link. The important observation is that the color on a link is not a source or origin of a new flux line but rather it color-twists the flux lines.

The second difference is due to the fact that the color group is compact. This means that the generators $E\cdot\hat{n}$ are quantized in the sense that

$$\left(E\cdot\hat{n}\right)^2 = \frac{n}{2}\left(\frac{n}{2}+1\right) \quad , \quad n = 0, 1, 2, \dots \qquad \text{(III.12)}$$

The flux through a link can not be arbitrarily small. To see what this means we can compare the situation with conventional electrodynamics formulated on a spatial lattice. In this case the flux can be arbitrarily subdivided. The flux emanating from a charge can spread out so that the flux through a distant link goes as $1/r^2$. This contrasts sharply with the flux in a non abelian theory which comes in quantized units.

When quark fields are included the electric flux lines can begin and end on sites occupied by quarks. This is because the open indices of an expression like $U(1)\,U(2)\,U(3)\,\dots\,U(6)$ (see Fig. 10) can be contracted with quark field indices.

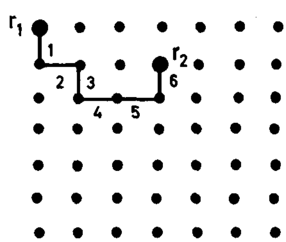

Fig. 10

For example we can form

$$\psi^{\dagger}(r_1)\,\sigma\,U(1)\dots U(6)\,\psi(r_2) \qquad \text{(III.13)}$$

where σ is an arbitrary Dirac matrix. In general the full set of gauge invariant functions of $\psi, \psi^{\dagger}$ and U will depend on the group describing color. If the group is SU_2 then we have operators like (III.13) as well as operators

$$\left(\psi^{c}(r)\right)^{\dagger} \sigma\, U(1) \ldots U(n)\, \psi(r_2) \tag{III.14}$$

where ψ^{c} is the charge conjugate to ψ. These operators describe quark pairs as opposed to quark antiquark pairs. If the color group is SU_3 the diquark operators like (III.14) are replaced by qqq operators. These are formed by considering a connected collection of links with the topology of a Y as in Fig. 11.

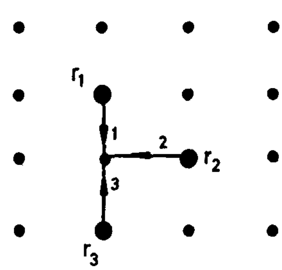

Fig. 11

The operator for Fig. 11 is

$$\psi_i^{\dagger}(r_1)\, U_{ij}(1)\, \psi_k^{\dagger}(r_2)\, U_{kl}(2)\, \psi_m^{\dagger}(r_3)\, U_{mn}(3)\, \varepsilon_{jln}$$

The entire space of states can be represented in terms of arbitrary products of closed flux-loops and open flux-lines with quark ends. However a more useful representation exists for cases in which a given link is covered more than once. In this case it is useful to combine the flux according to the rules of angular momentum addition. I will illustrate this for the example in Fig. 12.

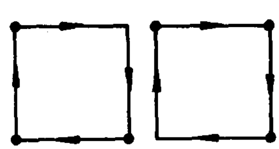

Fig. 12

The doubly covered link can be treated according to the method used in eq. (II.29). The resulting state is a linear superposition of two states in which that particular link carries electric flux of 0 and 1. This is shown in Fig. 13.

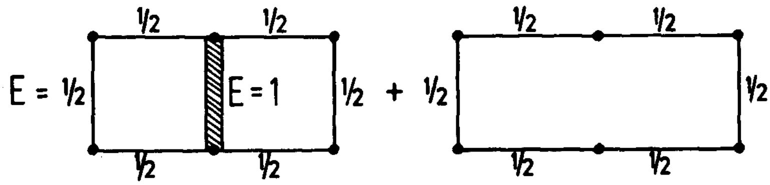

Fig. 13

This method may be generalized in order to introduce a representation of states in which flux may branch as in Fig. 14 as long as E_1 is

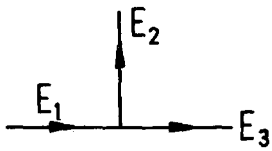

Fig. 14

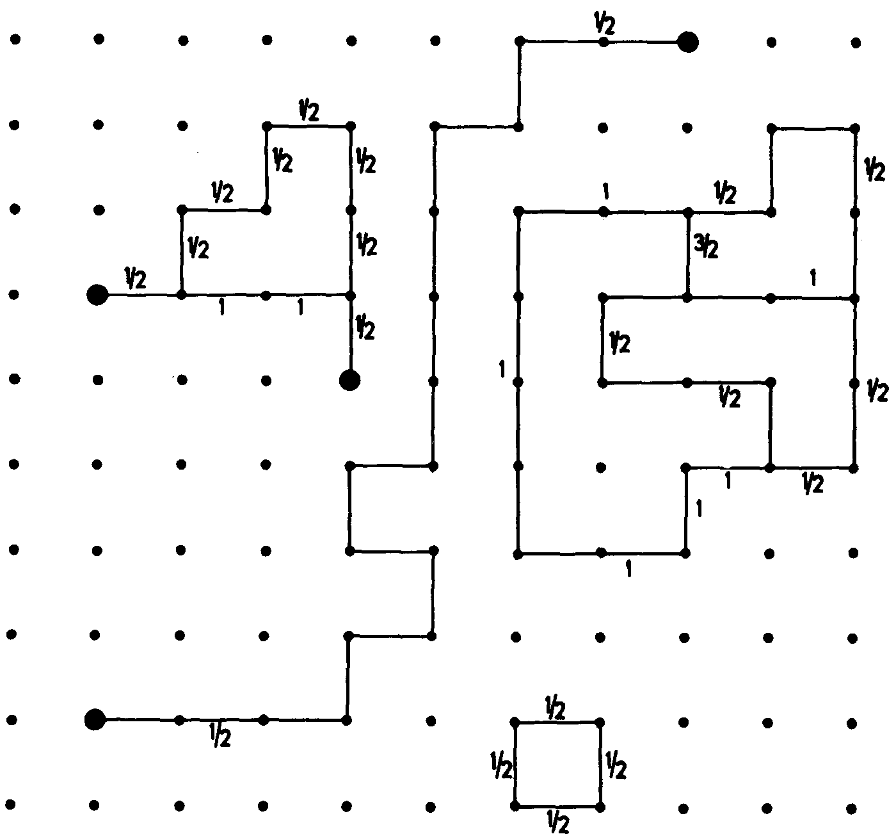

Fig. 15

included in the addition of E_2 and E_3 (added as angular momenta). A typical state is shown in Fig. 15.

Evidently we can characterize the space of states in terms of arbitrary branching strings or electric flux lines subject only to the constraint of flux continuity. All string ends must be quarks and all quarks must be string ends. This defines the kinematics of lattice Y. M. theory.

c) The Hamiltonian

The dynamics is defined by a gauge invariant hamiltonian whose matrix elements do not lead out of the gauge invariant subspace. In choosing H two principles, in addition to gauge invariance will guide us. The first is that in the limit of zero lattice spacing $(a \to 0)$, conventional Y. M. theory should be recovered. The second condition is that H shall be as local as possible. We will restrict our choice so that no links or sites are coupled if they are more than a single lattice space apart.

I will not do the algebra involved in taking the continuum limit but I will tell you how to do it yourself. You first define new variables $\chi(x)$, $\vec{A}(x)$, $\vec{\mathcal{E}}(x)$ by the equations

$$\psi(x) = a^{3/2}\chi(x)$$

$$E(r)\cdot\hat{n} = \frac{a^2}{g}\,\mathcal{E}(x)\cdot\hat{n}$$

$$U(r,\hat{n}) = \exp\left[iga\,A\cdot\hat{n}\,\frac{\tau}{2}\right] = 1 + iga\frac{\tau}{2}A\cdot\hat{n} + \cdots \qquad \text{(III.15)}$$

where a is the lattice spacing. Finite differences are replaced by derivatives

$$a^{-1}\left[f(r) - f(r-\hat{n})\right] = \hat{n}\cdot\partial f$$

and sums by integrals

$$\sum a^3 f(r) = \int d^3x\, f(x)$$

The hamiltonian will contain the following terms :

1) $$\frac{g^2}{2a}\sum_{r,\hat{n}}\left\{E(r)\cdot\hat{n}\right\}^2 \qquad \text{(III.16)}$$

In the limit $a\to 0$ this becomes the usual electrostatic energy

$$\frac{1}{2}\int d^3x\ \mathcal{E}^2(x)\ .$$

2) $$a^{-1}\sum_{r,\hat{n}}\left\{\psi^\dagger(r)\,\frac{\alpha\cdot\hat{n}}{i}\,U(r,\hat{n})\,\psi(r+\hat{n}) + \mu\,\bar{\psi}(r)\,\psi(r)\right\} \qquad \text{(III.17)}$$

when the continuum limit is taken this becomes the free quark hamiltonian plus the interaction energy (in the gauge $A_0 = 0$)

$$\int d^3x\left\{\bar{\chi}\ \gamma_i\,\partial_i\chi + ig\,\bar{\chi}\,\gamma_i A_i\,\frac{\tau}{2}\,\chi + \mu\,\bar{\chi}\chi\right\}$$

3) A term which has been absent in our simplified zero and one dimensional models is the magnetic energy. These terms are associated with elementary boxes or squares on the lattice. For each oriented unit square we include

$$\frac{4}{ag^2}\ \mathrm{Tr}\ UUUU \qquad \text{(III.18)}$$

In the limit $a\to 0$ this term becomes the usual magnetic energy

$$\frac{1}{2}\int d^3x\ \mathcal{H}(x)^2$$

where

$$\mathcal{H}_\alpha(x) = \nabla\times A_\alpha + g\,\varepsilon_{\alpha\beta\gamma}\,A_\beta\times A_\gamma$$

Each term in H has a particular significance for the stringlike flux lines. We shall study these terms in the order of their importance when the coupling g is large.
The most important term for $g \gg 1$ is the electric energy $\frac{g^2}{2a}\sum E^2$.
This term gives an energy

$$\frac{g^2}{2a}\left(\frac{n}{2}\right)\left(\frac{n}{2}+1\right) \qquad \text{(III.19)}$$

to every link carrying flux $E^2 = \frac{n}{2}\left(\frac{n}{2}+1\right)$. The vacuum of the strongly coupled limit is the state which minimizes this term. Therefore it is evident that the vacuum is the state $|0\rangle$ in which no flux lines are excited.

If we consider states in which the electric flux lines cover no link more than once then the electric energy gives each state an energy proportional to the total length of flux lines. This is the source of quark confinement in the strongly coupled Y. M. theory. For example if we consider a quark pair located at sites r_1 and r_2 then the minimum energy configuration of the gauge field will involve an electric flux line of minimal number of links. Therefore the energy will be stored on a straight line between the quarks and will grow linearly with their separation. Evidently the strongly coupled Y. M. theory is behaving exactly like the dielectric model of lecture I. If the lattice spacing is a then the minimum energy for a $q\bar{q}$ pair separated by distance D is

$$\frac{g^2}{2a}\left(\frac{1}{2}\right)\left(\frac{1}{2}+1\right)\frac{D}{a} = \frac{3g^2}{8a^2}D \tag{III.20}$$

Quarks will be confined if the electric energy dominates.

The electric energy is also instrumental in giving the pure gauge field excitations mass. For example consider the state

$$\operatorname{Tr} UUUU \,|0\rangle \tag{III.21}$$

in which a single box of electric flux is excited. The electric energy of this state is

$$H(\text{Box}) = 4\,\frac{g^2}{2a}\cdot\frac{3}{4} = \frac{3g^2}{2a}\,.$$

Since each electric flux configuration is an eigenvector of the electric energy, no propagation of signals through the lattice will take place until the other terms are included.

The next term in importance is

$$a^{-1}\sum \psi^\dagger(r)\,\frac{\alpha\cdot\hat{n}}{i}\,U(r,\hat{n})\,\psi(r+\hat{n}) + \mu\bar{\psi}\psi\,.$$

This term allows the fermions to propagate through the lattice. It describes a process in which a quark and antiquark are created or annihilated at two neighboring points. The factor U creates or cancels the flux between them. The sequence of events in Fig. 16 shows how fermions may move through the lattice.

Processes induced by this term allow an electric flux line to break as quarks separate. Of course this only occurs when a pair is produced as in our earlier examples. Finally this term causes the physical vacuum to have a fluctuating number of $q\bar{q}$ pairs.

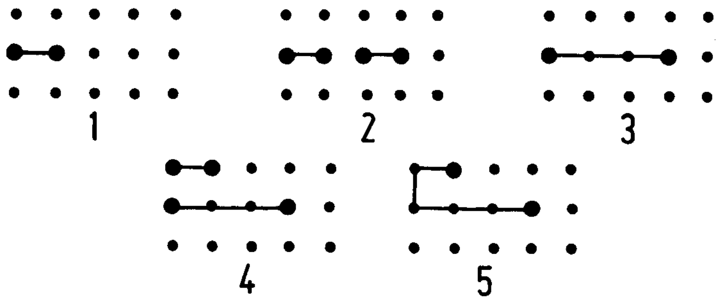

Fig. 16

The last term is the magnetic energy

$$\frac{4}{ag^2}\sum_{\text{Boxes}} \mathrm{Tr}\, UUUU$$

This term causes fluctuations in the position and structure of the string-like flux lines. For example consider a static $q\bar{q}$ pair with a straight electric flux line as in Fig. 17

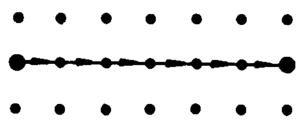

Fig. 17

Suppose we consider a box with a side in common with the flux line. If we apply $\mathrm{Tr}\, UUUU$ we create the superposition shown in Fig. 18 .

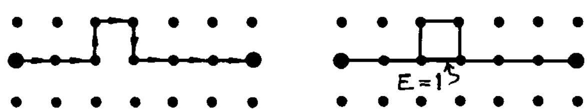

Fig. 18

In addition this term allows the vacuum to contain a fluctuating sea of closed flux lines.

If the fluctuations due to the magnetic energy become too large the quark confining mechanism can become undone. To get a rough idea of how this can happen we suppose the vacuum contains a dense sea of closed flux lines as in Fig. 19.

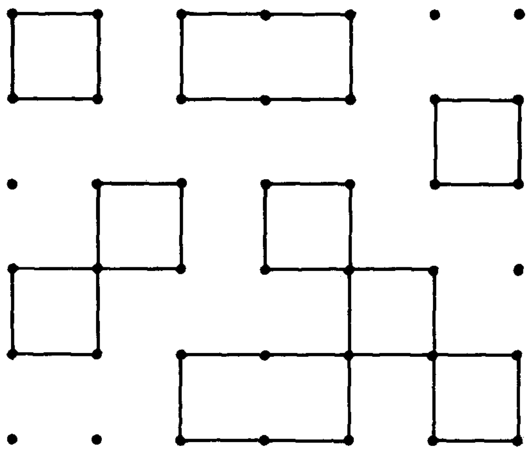

Fig. 19

Now suppose a quark is placed in the lattice as in Fig. 20. The flux due to the quark (which must go to ∞) is represented by the dark line.

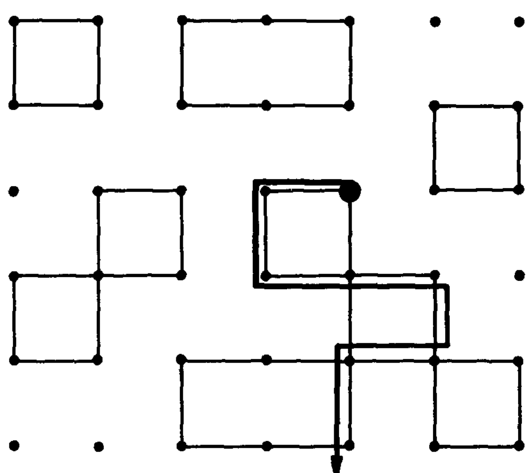

Fig. 20

As usual the doubly occupied links can be resolved into a coherent superposition with flux zero and flux one. Let us consider the particular contribution in which all these links carry $E = 0$. It is shown in Fig. 21.

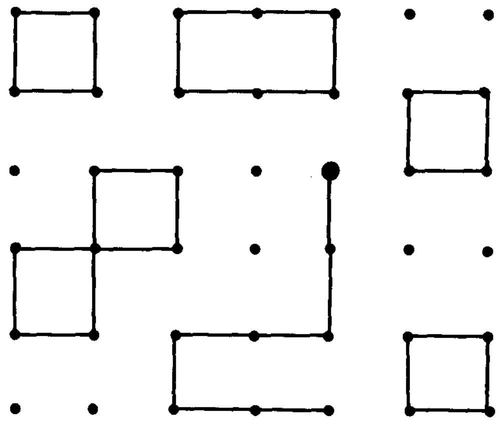

Fig. 21

Fig. 19 has 40 excited links but Fig. 21 has only 33. You see we have actually lowered the electric energy by adding a quark. However this obviously can not happen if the flux density of the vacuum is very low, i.e. if $g \gg 1$. The term

$$\sum \frac{4}{ag^2} \mathrm{Tr}\, UUUU$$

must have sufficient strength to fill the vacuum with a high density of flux loops.

The term $\mathrm{Tr}\, UUUU$ also causes motion through the lattice. The sequence in Fig. 22 shows how the gauge field excitations are caused to move through the lattice:

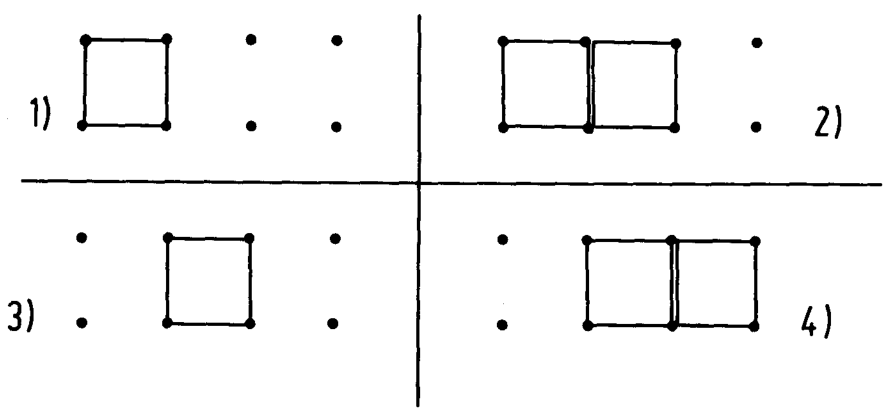

Fig. 23

d) Removing the Lattice and Infrared Slavery

The most difficult unanswered question posed by lattice Yang Mills theory concerns the removal of the lattice from the theory[10]. A proper discussion of this point is well beyond the range of these lectures and also this lecturer. Nevertheless I will try to give you a vague idea of how I think things should go. First of all we must realize that taking a to zero is far more delicate in quantum field theory than in classical theory. This is because the large scale behaviour of the theory is sensitive to a unless the bare parameters of the theory are continuously readjusted as $a \to 0$. This is the process of renormalization.

For example suppose that with lattice spacing a we use a coupling $g \gg 1$. The energy stored in a $q\bar{q}$ pair separated by distance $\mathcal{D}$ is

$$\frac{3\,g^2}{8\,a^2}\,\mathcal{D}$$

Now suppose we wish to represent the same physics (for large $\mathcal{D}$) by a new model in which the lattice spacing is $a/2$. In order to keep the energy unchanged we must use a new coupling constant which satisfies

$$(g')^2 = \frac{1}{2}g^2$$

Thus as the lattice spacing decreases the squared coupling constant must also decrease in order to keep the large scale physics unchanged.

The right theory probably requires

$$g^2 \sim \frac{1}{\log a}$$

as $a \to 0$. This was discovered by 't Hooft, Politzer and by Gross and Wilczek. This means that an accurate representation of continuum Yang Mills theory on a very fine lattice would require a very small coupling. However renormalization effects cause the effective coupling to increase with a until we (hopefully) reach a point where a is comparable to the hadron radius and $g > 1$. We can then apply the strong coupling methods outlined in this lecture.

If this view is correct then there is no "phase transition" between large g and small g so that no discontinuous effects occur as a and g^2 become small. Under these conditions quark confinement can be decided by examination of the large g limit.

In this regard I should mention the relation between the mechanism described here and the idea of infrared slavery[11]. The quark confining mechanism I've described begins with the idea that the "running" coupling constant is $\gg 1$ for large a and then provides a picture of how quarks are trapped by the electric field. It does not tell us why the coupling is large. On the other hand the infrared slavery ideas tell us why the coupling increases with a but fail to explain how a strong coupling confines quarks. The quark confining mechanism of lattice Y. M. theory and infrared slavery are not different mechanisms but are complementary aspects of the same thing.

ACKNOWLEDGMENTS

The ideas presented in these lectures are the combined efforts of John Kogut and myself. The lattice theory was inspired by Ken Wilson. I am very grateful to him for explaining many things about lattices and gauge theories to me.

REFERENCES

1. H. J. Lipkin, Physics Reports, Vol. 8c, Number 3, Aug. 1973
2. J. Schwinger, Theoretical Physics (International Atomic Energy Agency, Vienna, 1963) p. 89
3. Lowenstein and Swieca, Annals of Physics 68, 172 (1971)
4. A. Casher, J.Kogut and L.Susskind,Phys. Rev. Lett.31, 792 (1973)
5. J. Kogut and L. Susskind, Vacuum Polarization and the Absence of Free Quarks in 4 Dimensions, Phys. Rev. D (to appear)
6. G. 't Hooft, private communication
7. K. G. Wilson, C.L.N.S. 262 (Feb. 1974) to appear Phys. Rev. D
8. Lattice Gauge Theories have also been studied by Polyakov. Private communication from J. Bjorken
9. J. Kogut and L. Susskind, Hamiltonian Formulation of Wilson's Lattice Gauge Theories (to appear Phys. Rev. D)
10. The process of letting $a \rightarrow 0$ in a quantum field theory is called renormalization group. The most complete approach to the renormalization group as a computational tool is due to K. G. Wilson. See for example K. G. Wilson and J. Kogut, The Renormalization Group and the $\mathcal{E}$ Expansion, perhaps to appear in Physics Reports C
11. It is known that in Y. M. theory the running coupling constant can increase as the cutoff distance becomes large. G.'t Hooft, Marseille Conference on Gauge Theories, June 1972,H.D. Politzer, Phys. Rev. Lett. 30, 1346 (1973), D. J. Gross and F. Wilczek, Phys. Rev. Lett. 30, 1343 (1973). Speculations that this effect can account for quark confinement in some way have been made by 't Hooft, Weinberg, Georgi and Glashow and probably many more.