Null singularity on nonsingular spacetime

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Abstract. It is known that when coupled to nonlinear electrodynamics (NLED), Einstein's equations can have nonsingular black hole (BH) solutions. For such BH the absence of horizon poses no violation of cosmic censorship theorem. In this work we show that the horizonless nonsingular BH charged with NLED, viewed from photon's perspective, ironically is also responsible for the existence of singularity in its null geodesics. This is due to fact that the effective geometry gives additional conformal factor to the metric. For massive particles, on the other hand, the spacetime is still perfectly regular.

1. Introduction

A naked singularity can be produced by overcharging a black hole [1]. In Reissner-Nordstrom (RN) BH it is given by $q^2 > m^2$. This exposed singularity is rather disturbing since it raises questions on the conservation laws. Penrose conjectured that nature forbids the existence of such singularity. This is known as the cosmic censorship theorem [2]. In 1968, Bardeen proposed a black hole without singularity at its center [3]. In this BH, the absence of horizon poses no problem. However, it was not known what source could produce the metric function proposed by Bardeen.

In 2000, Ayon-Beato and Garcia (ABG) constructed a Lagrangian with magnetic monopole as the source that can produce the Bardeen metric [4]. Previously, Ayon-Beato and Garcia have constructed regular metric with electric source by employing Legendre Transformation [5] and [5]. However, Bronnikov proposed a no-go theorem that restricts the Lagrangian that can produce regular metric [7]. In 2018, Rodrigues and de Silva constructed an electric counterpart of the Lagrangian proposed by ABG [8]. Another regular metric was also proposed by [9] known as the Hayward metric.

Studying the characteristics of the photon geodesics could enlighten the search for regular black hole and give us clue about nonlinear electrodynamics. The solution of photon geodesics for of regular black hole model have been studied by [10], [11], [12], [13], [14]. In this work, we take the effect of nonlinear electrodynamics from [15] into account. For several nonlinear electrodynamics models the effect has been studied, for example by [16], [17].

2. Regular Spacetime

In this work, we study two types of regular metrics, the Bardeen spacetime and Hayward spacetime.

2.1. Bardeen Spacetime
The regular Bardeen spacetime is represented by [3]

$$f(r) = 1 - \frac{2mr^2}{(r^2+q^2)^{3/2}}. \tag{1}$$

This metric depends on mass m and q that can be interpreted as charge.

2.2. Modified Hayward Spacetime
The spacetime we will use is the modified Hayward spacetime from Kruglov, where the M is no longer a constant but a function of mass distribution $M(r)$ [18]

$$f(r) = 1 - \frac{2M(r)r^2}{r^3 + 2M(r)l^2}. \tag{2}$$

Where l is a constant corresponds to Hubble length. The metric is modified using $\rho = -\mathcal{L}$, where $\mathcal{L}$

$$\mathcal{L} = -\frac{F}{\cosh^2 \sqrt[4]{|\beta F|}}, \tag{3}$$

the constant mass M is now considered as mass distribution function $M(r)$,

$$M = M(r) = mo + \int_0^\infty \rho(r)r^2 dr - \int_r^\infty \rho(r)r^2 dr. \tag{4}$$

The magnetic mass is defined as

$$\begin{aligned} m_M &= \int_0^\infty \rho(r)r^2 dr - \int_r^\infty \rho(r)r^2 dr, \\ &= \frac{q^{\frac{3}{2}}}{2^{\frac{3}{4}}\beta^{\frac{1}{4}}}. \end{aligned} \tag{5}$$

Substituting the $M(r)$ to $f(r)$ the new metric function is obtained

$$f(r) = 1 - \frac{2\left(mo + \frac{q^{\frac{3}{2}}}{2^{\frac{3}{4}}\beta^{\frac{1}{4}}} - \frac{q^{\frac{3}{2}}}{2^{\frac{3}{4}}\beta^{\frac{1}{4}}}\tanh\left(\frac{\beta^{\frac{1}{4}}\sqrt{q}}{2^{\frac{1}{4}}r}\right)\right)r^2}{r^3 + 2\left(mo + \frac{q^{\frac{3}{2}}}{2^{\frac{3}{4}}\beta^{\frac{1}{4}}} - \frac{q^{\frac{3}{2}}}{2^{\frac{3}{4}}\beta^{\frac{1}{4}}}\tanh\left(\frac{\beta^{\frac{1}{4}}\sqrt{q}}{2^{\frac{1}{4}}r}\right)\right)l^2}. \tag{6}$$

3. Effective Geometry
In nonlinear electrodynamics, light does not move in the usual geometry but follows its own effective geometry [15]. The effective geometry has to obey the null condition,

$$0 = \left(\mathcal{L}_F g^{\mu\nu} - 4\mathcal{L}_{FF} F^\mu_\alpha F^{\alpha\nu}\right) k_\mu k_\nu, \tag{7}$$

where $\mathcal{L}_F$ and $\mathcal{L}_{FF}$ are the first and the second order derivation of the source Lagrangian, $F^{\mu\nu}$ is the Maxwell's field strength tensor and k^μ is the photon's propagation vector. The effective metric tensor can be defined as

$$g^{\mu\nu}_{eff} = g^{\mu\nu} - \frac{4\mathcal{L}_{FF}}{\mathcal{L}_F} F^\mu_\alpha F^{\alpha\nu}. \tag{8}$$

The effective geometry gives additional factor to our spacetime metric

$$ds^2_{eff(m)} = -f(r)dt^2 + f(r)^{-1}dr^2 + h_m(r)r^2d^2\Omega \tag{9}$$

for magnetic case and

$$ds^2_{eff(e)} = h_e(r)[-f(r)dt^2 + f(r)^{-1}dr^2] + r^2d^2\Omega \tag{10}$$

for electric case.
Where

$$h_m(r) = \left(1 + \frac{4\mathcal{L}_{FF}}{\mathcal{L}_F}\frac{q^2}{r^4}\right)^{-1} \tag{11}$$

and

$$h_e(r) = \left(1 - \frac{4L_{FF}}{L_F}\frac{q^2}{r^4}L_F^{-2}\right)^{-1} \tag{12}$$

3.1. Bardeen Spacetime with ABG Source

The Lagrangian proposed by ABG has magnetic monopole as its source, the non zero Maxwell tensor components are F_{23} and F_{32}, where $F_{23} = q\sin\theta$ and $F = \frac{q^2}{2r^4}$.

$$\mathcal{L}(F) = \frac{3}{sk^2q^2}\left(\frac{\sqrt{2q^2F}}{1+\sqrt{2q^2F}}\right)^{\frac{5}{2}}, \tag{13}$$

the $h(r)$ component for ABG source is

$$h_{ABG}(r) = \left(1 - \frac{2(6q^2 - r^2)}{(q^2 + r^2)}\right)^{-1}. \tag{14}$$

3.2. Bardeen Spacetime with Rodrigues-de Silva Source

Rodrigues and de Silva proposed a electric charge source that generates Bardeen spacetime [8]. Start from the definition of F^{10}

$$F^{10}(r) = \frac{q}{r^2}\mathcal{L}_F^{-1}(r). \tag{15}$$

Therefore, F equals to

$$F(r) = -\frac{225m^2q^2r^8}{2k^4(q^2+r^2)^7}. \tag{16}$$

$L_F(r)$ can be obtained as

$$\begin{aligned} \mathcal{L}(r) &= \frac{q^2m(6q^2 - 9r^2)}{k^2(r^2+q^2)^{\frac{7}{2}}} \\ \mathcal{L}_F(r) &= \frac{k^2(r^2+q^2)^{\frac{7}{2}}}{15mr^6}, \end{aligned} \tag{17}$$

To construct the $h(r)$, we need to obtain L_{FF} using the following way

$$\begin{aligned} \mathcal{L}_{FF} &= \frac{d\mathcal{L}_F}{dF} = \frac{d\mathcal{L}_F}{dr}\frac{dr}{dF}, \\ &= \left(\frac{56\pi(q^2+r^2)^{\frac{5}{2}}}{15mr^6} - \frac{16\pi(q^2+r^2)^{\frac{7}{2}}}{5mr^7}\right)\left(\frac{1575m^2q^2r^9}{64\pi^2(q^2+r^2)^8} - \frac{225m^2q^2r^7}{16\pi^2(q62+r^2)^7}\right)^{-1}. \end{aligned} \tag{18}$$

The, $h(r)$ can be obtained

$$h(r)_{Rodrigues-deSilva} = \left(1 - \frac{3375m^3q^2r^{14}\left(\frac{56\pi(q^2+r^2)^{\frac{5}{2}}}{15mr^5}\frac{16\pi(q^2+r^2)^{\frac{7}{2}}}{5mr^7}\right)}{128\pi^3(q^2+r^2)^{\frac{21}{2}}\left(\frac{1575m^2q^2r^9}{64\pi^2(q^2+r^2)^8} - \frac{225m^2q^2r^7}{16\pi^2(q^2+r^2)^7}\right)}\right)^{-1}. \tag{19}$$

3.3. Modified Hayward Spacetime with Bronnikov Source

$$h(r)_{Bronnikov} = \left(1 + \frac{4q^2\left(\frac{5(F\beta)^{1/4}\sinh\left((F\beta)^{1/4}\right)}{8\cosh\left((F\beta)^{1/4}\right)} - \frac{3(F\beta)^{1/2}\sinh^2\left((F\beta)^{1/4}\right)}{8\cosh^2\left((F\beta)^{1/4}\right)} + \frac{1}{8}(F\beta)^{1/4}\right)}{Fr^4\left(\frac{(F\beta)^{1/4}\sinh\left((F\beta)^{1/4}\right)}{2\cosh^3\left((F\beta)^{1/4}\right)} - \frac{1}{\cosh^2\left((F\beta)^{1/4}\right)}\right)\cosh^2\left((F\beta)^{1/4}\right)}\right)^{-1}, \tag{20}$$

where $F = \frac{q^2}{2r^4}$.
In this study, we evaluate the curvature of the spacetime by calculating the Ricci scalar of the spacetime figure 1. Where Ricci Scalar $R = g^{\mu\nu}R_{\mu\nu}$ where $R_{\mu\nu}$ is the Ricci tensor for the correspoding length elements for each cases.

4. Equation of Motion and Effective Potentials

Particle trajectory can be obtained from geodesic equation as shown below.

$$\frac{d^2x^\mu}{d\lambda^2} + \Gamma^\mu_{\alpha\beta}\frac{dx^\alpha}{d\lambda}\frac{dx^\beta}{d\lambda} = -\frac{\epsilon}{\mu}F^\nu_\sigma\frac{dx^\sigma}{d\lambda}. \tag{21}$$

$\Gamma^\mu_{\alpha\beta}$ is the Christoffel symbols calculated from the effective metric we use (9) and (10), x^μ is the 4-vector components (t, r, ϕ, θ respectively), and λ is the affine parameter, we can choose λ as proper time τ. For photon, the right hand side will be equal to zero. By calculating the terms of the equation for both magnetic and electric we will obtain the constants of motion for each case.

4.1. Magnetic Case

The terms that will lead us to the constant of motions are $\mu = 0$ and $\mu = 3$ for $\mu = 0$

$$\begin{aligned} \ddot{t} + \frac{f'(r)\dot{t}\dot{r}}{f(r)} &= 0 \\ \frac{d\left(f(r)\dot{t}\right)}{d\tau} &= 0 \\ f(r)\dot{t} = constant &= E \end{aligned} \tag{22}$$

and for $\mu = 3$

$$\begin{aligned} r^2h(r)\ddot{\phi} + (2rh(r) + r^2h'(r))\dot{r}\dot{\phi} &= 0 \\ \frac{d(r^2h(r)\dot{\phi})}{d\tau} &= 0 \\ r^2h(r)\dot{\phi} = constant &= L \\ \dot{\phi} &= \frac{L}{h(r)r^2} \end{aligned} \tag{23}$$

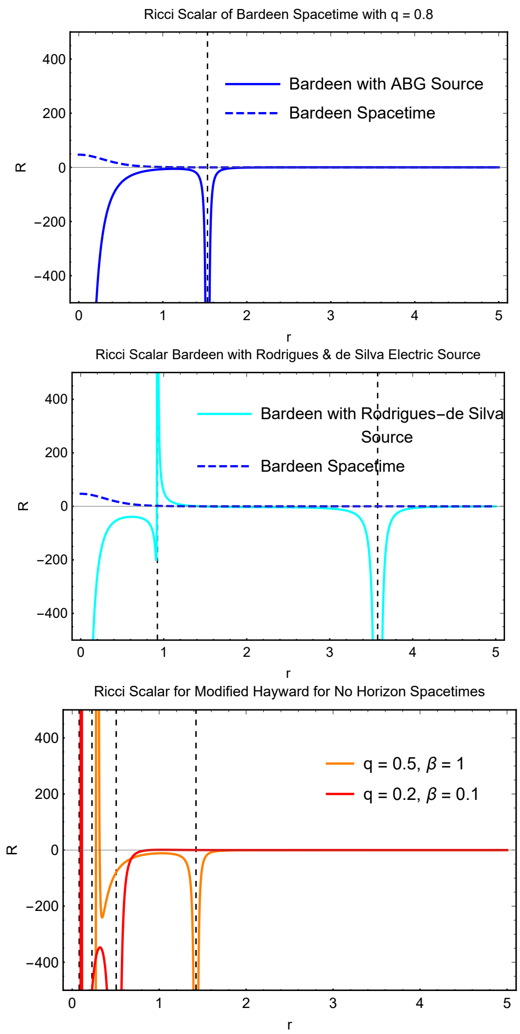

Figure 1. Ricci Scalars for Bardeen with ABG, Bardeen with Rodrigues de Silva, and Modified Hayward

From $\mu = 0$ we obtain energy term $E = f(r)\dot{t}$, and from $\mu = 3$ we obtain the angular momentum term $L = \dot{\phi}h(r)r^2$. Substituting E and L to the effective length element $ds_{eff(m)}$

$$\begin{aligned} ds^2_{eff(m)} &= -f(r)dt^2 + \frac{1}{f(r)}dr^2 + h(r)r^2(d\Omega^2) \times \frac{1}{d\tau^2} \\ \frac{ds^2_{eff(m)}}{d\tau^2} &= -f(r)\dot{t}^2 + \frac{1}{f(r)}\dot{r}^2 + h(r)r^2(\dot{\theta}^2 + Sin^2\dot{\phi}^2) \\ 0 &= -f(r)\dot{t}^2 + \frac{1}{f(r)}\dot{r}^2 + h(r)r^2(\dot{\theta}^2 + Sin^2\theta\dot{\phi}^2) \\ &= -E^2 + \dot{r}^2 + \frac{f(r)}{h(r)}\frac{L^2}{r^2} \\ \dot{r}^2 &= E^2 - \frac{f(r)}{h(r)}\frac{L^2}{r^2} \end{aligned} \tag{24}$$

From the equation above we obtain the $\dot{r}$ for the magnetic case and we can define the effective potential $Veff$ as the second term in the $\dot{r}^2$

$$Veff = \frac{f(r)L^2}{h(r)r^2}. \tag{25}$$

4.2. Electric Case
Similar to the magnetic case, we evaluate the geodesic equation where the value of $\mu = 0$ and $\mu = 3$. For $\mu = 0$

$$\begin{aligned} \frac{d^2t}{d\tau^2} + \left[\frac{f'(r)}{f(r)} + \frac{h'(r)}{h(r)}\right]\frac{dt}{d\tau}\frac{dr}{d\tau} &= 0 \\ \ddot{t} + \left[\frac{f'(r)}{f(r)} + \frac{h'(r)}{h(r)}\right]\dot{t}\dot{r} &= 0 \\ f(r)h(r)\ddot{t} + [f'(r) + h'(r)]\dot{t}\dot{r} &= 0 \\ \frac{d}{d\tau}[f(r)h(r)\dot{t}] &= 0 \\ f(r)h(r)\dot{t} &= constant = E \end{aligned} \tag{26}$$

for $\mu = 3$

$$\begin{aligned} \ddot{\phi} + \left(\frac{1}{r}\dot{r} + \frac{1}{r}\dot{r}\dot{\phi}\right) &= 0 \\ \ddot{\phi} + \frac{2}{r}\dot{r}\dot{\phi} &= 0 \\ r\ddot{\phi} + 2\dot{r}\dot{\phi} &= 0 \\ \frac{1}{r}\frac{d}{d\tau}[r^2 \dot{phi}] &= 0 \\ r^2\dot{\phi} &= constant = L \end{aligned} \tag{27}$$

We obtain $E = f(r)h(r)\dot{t}$, and $L = \dot{\phi}r^2$. We can see the difference between the magnetic and the electric case, the $h(r)$ modified the angular momentum definition whereas it modifies the energy definition for the electric case. Implementing similar method from equation (24) we can obtain the $\dot{r}$ for the electric case

$$\dot{r}^2 = \frac{E^2}{h(r)^2} - \frac{f(r)L^2}{h(r)r^2}, \tag{28}$$

Since the $h(r)$ factor appears on both terms, we cannot define the effective potential only with the second term. Thus, we define the whole $\dot{r}$ as $Ueff$

$$\dot{r}^2 = Ueff = \frac{E^2}{h(r)^2} - \frac{f(r)L^2}{h(r)r^2}. \tag{29}$$

5. Result

The singularity from $h(r)$ is a physical singularity that can be seen on the Ricci Scalar as shown in Figure 1. The additional $h(r)$ factor appears only on the second term of $\dot{r}^2$ for the magnetic case, on the other hand, it appears on both terms in electric case. The existence of singularity from $h(r)$ gives additional zeros to the effective potential for magnetic case. Since it gives

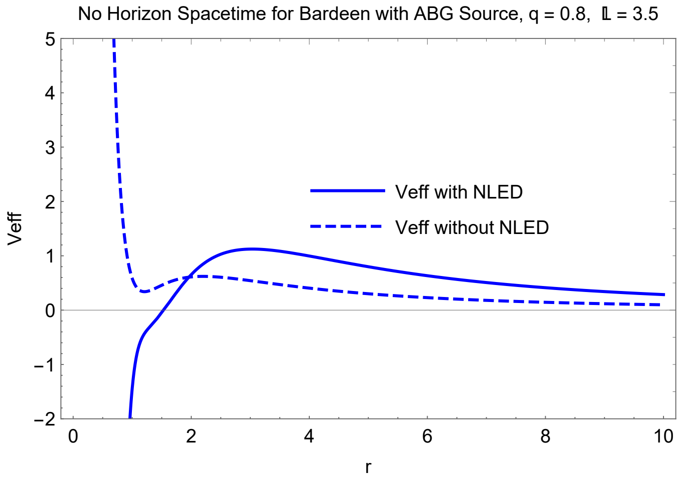

Figure 2. Effective Potential $Veff$ for Bardeen with ABG Source

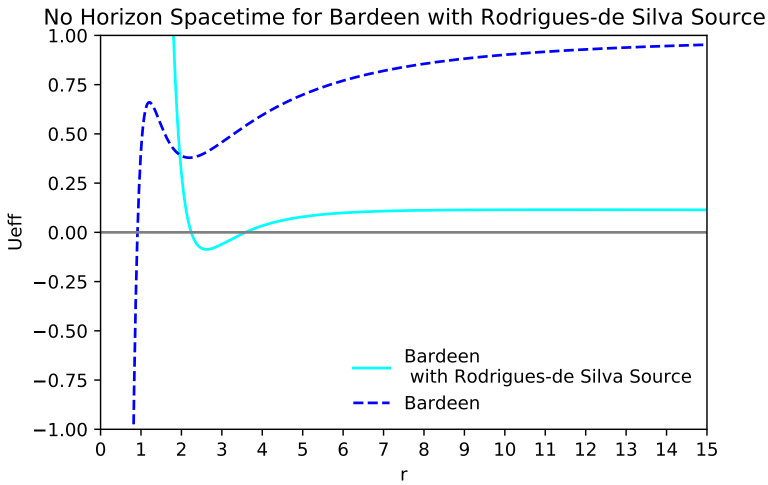

Figure 3. Effective Potential $Ueff$ for Bardeen with Rodrigues-de Silva Source

additional zeros for the $Veff$ a new maxima is generated that represents an unstable orbit. Figure 2 refers to the equation (25) with Bardeen metric in equation (1) for the $f(r)$, we see that the $Veff$ of Bardeen spacetime without NLED (by setting $h(r) = 1$) will have potential barrier near its center, while the one with NLED does not have the potential barrier. In figure 3, the $Ueff$ (29) for Bardeen without NLED, the $Uueff$ only has one root while the one with NLED from Rodrigues-de Silva has two roots. For the modified Hayward case in figure 3, both cases have potential barrier near its center while for the $\beta = 0.1$, the $Veff$ shows a local maxima that indicates an unstable circular orbit.

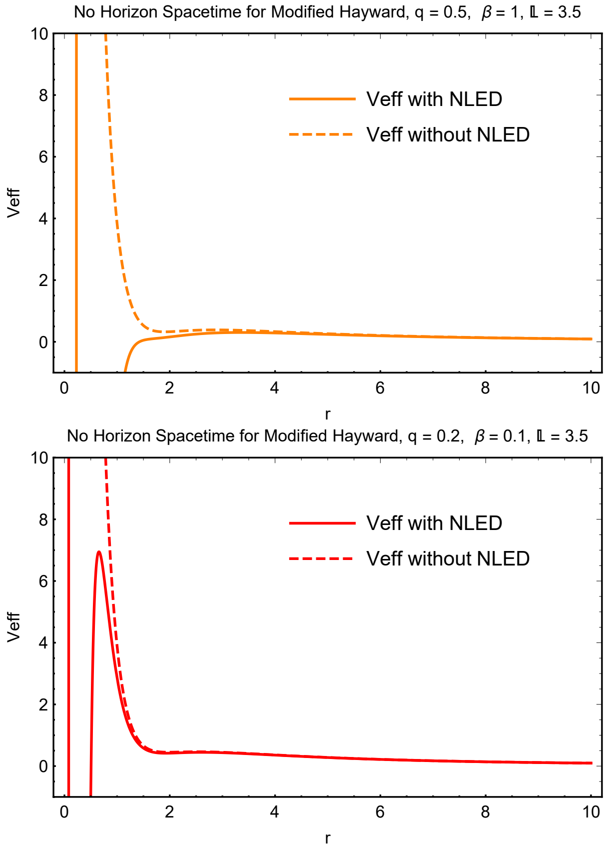

Figure 4. Effective Potentials

6. Conclusion

In this paper, we show that even though the horizonless regular spacetime does not possess naked singularity at its center, nonsingular spacetimes such as the Bardeen and Hayward metrics with NLED could have physical singularities due to the $h(r)$ factor shown by the Ricci scalar. The singularity appears as an additional root for the effective potential thus the types of photon orbits could differ from the one without NLED.

Analyzing the photon geodesic of no horizon spacetimes could enlighten us about the nature of compact objects and singularities. This analysis will require us to explore further about optical features of such objects.

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