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Dynamical Symmetry Breaking in Four–Fermi Interaction Models*

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ABSTRACT

Dynamical symmetry breaking in 4–fermion models in 2+1 dimensions is *quantitatively* studied. We use the $1/N$ expansion and show that these models are renormalizable in this framework, in spite of their non–renormalizability in weak coupling expansion. The particular topics discussed include interactions of composite Goldstone bosons with other particles and among themselves, low–energy theorems, chiral lagrangians, the effects of explicit symmetry breaking, and the thermodynamics of the dynamical symmetry restoring phase transition.

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1. Introduction

The breaking of continuous symmetries in Quantum Field Theory is one of the most important basic phenomena in modern physics. This idea originated in the theory of condensed matter [1] but is widely used in the realm of elementary particles. Here symmetry breaking provides the basis for our understanding of two concrete areas of phenomenology: the unification of weak and electromagnetic forces in the Standard Model [2–4], and the low–energy scattering of pions and nucleons [5,6]. Symmetry breaking also plays a crucial role in attempts to go beyond the Standard Model, such as Grand Unified Theories [7] and "Technicolor" [8].

The symmetry breaking phenomenon is *exhibited* by a non–zero expectation value for some non–singlet field operator known as the "order parameter." In the cases where this order parameter is *linear* in the elementary fields the "classical approximation" to the action is an appropriate one, and the weak coupling (small $\hbar$) expansion can yield a reliable calculational scheme. This is what obtains in the electro–weak theory: the classical potential for the scalar Higgs fields has a maximum at the origin, and (in a gauge-fixed formalism) one *shifts* the fields to one of the minima. Crucially, the Higgs–Kibble mechanism [9] can be seen already at this level of analysis, providing masses for the W and Z^0 bosons and the quarks. Ordinary perturbation theory then yields systematically improvable and *quantitative* predictions for the physics of the electro–weak interactions.

However, there are important cases of symmetry breaking where the order parameter is *composite* in the fields [10,11], such as $\left\langle \overline{\psi}^i_L \psi^j_R \right\rangle$ in QCD, and for these the classical approximation fails completely. Thus, there can be no weak coupling expansion, and the study of "dynamical" symmetry breaking forces us to look for non-perturbative methods. This problem is extremely difficult to overcome in QCD and "realistic" technicolor theories. In the latter case the absence, as yet, of a quantitative analysis is one of the major obstacles to the approach, and one has to rely on analogies and extrapolations from hadron phenomenology[#1]. It therefore seems reasonable to try to study the problem of composite operators in the context of simpler, and better understood models, and this is the main purpose of the present review.

Nambu and Jona-Lasinio were the first to introduce the idea of dynamical symmetry breaking, in the context of the interactions of nucleons and pions [12]. Their model is defined by a 4-fermion interaction lagrangian in $d = 3 + 1$

$$\mathcal{L} = i\overline{\psi}\partial\!\!\!/\psi + \frac{g^2}{2}\ \left[(\overline{\psi}\gamma_\mu \overrightarrow{\tau}\psi)^2 - (\overline{\psi}\gamma_5\gamma_\mu \overrightarrow{\tau}\psi)^2\right] \tag{1.1}$$

where $\overrightarrow{\tau}$ are "isospin" matrices. Classically, the lagrangian has an exact chiral $SU(2)_L \times SU(2)_R$ symmetry, which forbids a mass for the nucleons ψ to all orders in g^2. Nambu and Jona-Lasinio showed that if the coupling g^2 is stronger than a certain critical value then a mass can be generated dynamically in Hartree-Fock approximation. When the mass is non-zero the chiral symmetry is dynamically broken to $SU(2)_V$, and this produces a massless triplet of pseudo-scalars as bound states. (The occurrence, in general, of massless particles corresponding to the broken generators of a rigid symmetry group was proved by J. Goldstone [13].)

#1 The dynamical assumptions are that the QCD lagrangian does indeed reproduce hadronic physics, and that "similar" lagrangians behave "in the same way" up to color factors.

With remarkable insight Nambu and Jona–Lasinio identified these pseudo–scalars as "idealized" pions. Subsequent work then tried to interpret the ***photon*** as a dynamical Goldstone boson as well [14], with the aim of reducing the number of coupling constants needed to specify QED.

This non–perturbative reasoning was motivated from the Bardeen Cooper Scrieffer theory of superconductivity [15]. There is, however, an important physical difference between the models of NJL and BCS, namely that the former does *not* possess a natural candidate for the cutoff Λ which is required to define the Hartree–Fock "gap equation." (In BCS the cutoff corresponds to the Debye frequency, and only in this setting does the gap equation provide a relation between physically measurable quantities.) The cutoff cannot be removed from the scattering amplitudes in the NJL model, but at least the dependence turns out to be only logarithmic [16]. The model can be considered as a low energy *effective* theory for the strong interactions, which might in principle be derivable from QCD [17–19].

The severe problems caused by ultra–violet divergences in $d = 3 + 1$ has made it natural to study field theory in lower dimensions in order to test basic ideas. A great variety of interesting features have been conveniently studied in the framework of exactly solvable models in $d = 1 + 1$ [20–25]. The Gross–Neveu model [26] is a well–known example. The model is $O(2N)$ symmetric with a scalar–scalar 4–fermion interaction

$$\mathcal{L} = i\overline{\psi}_j \partial\!\!\!/ \psi_j + \frac{g^2}{2N} \, (\overline{\psi}_j \psi_j)^2 \, , \tag{1.2}$$

where $j = 1, 2, \ldots, N$. The model was studied by Gross and Neveu in the large N limit, who found that dynamical mass generation occurs just as in (1.1) breaking down the *discrete* $\psi \to \gamma_5\psi$ chiral symmetry. It is of particular interest that the model combines both dynamical symmetry breaking and asymptotic freedom [27], and so resembles QCD in these two key respects.

Though the studies in 1+1 dimensions have been greatly illuminating, the models generally possess some properties which are "unrealistic" from a phenomenological point of view. The first unphysical property is the complete elasticity of the S–matrix, which makes the solvable 1+1–dimensional models more reminiscent of the non–relativistic quantum field theories used in condensed matter physics [15,28]. In the GN model, for example, the *exact* S–matrix of elementary fermions and bound states was found by Zamolodchikov and Zamolodchikov [20],and Shankar and Witten [24]. The solvability of the model followed from the assumption of purely elastic particle scattering, a property which was seen explicitly in the $1/N$ expansion.

The second is the impossibility of the spontaneous breakdown of *continuous* rigid symmetries. A famous theorem, due to Coleman [29] and Mermin and Wagner [30], states that all local operators which are non–singlet under a continuous symmetry group must have *zero* expectation value. This precludes the existence of Nambu–Goldstone bosons[#2], thus to obtain a good analogy to low–energy pion–nucleon physics we must look to a higher dimension. We are constrained to be below $d = 3 + 1$ by the divergence problem, and above $d = 1 + 1$ by Coleman's theorem, so the natural place to try is $d = 2 + 1$!

This idea might appear to be bound to fail since 4–fermion theories are well known to be non–renormalizable in the weak coupling expansion in $d = 2 + 1$.

#2 Long–range order is nevertheless still possible in $2D$ statistical mechanics [31].

However, it was shown in refs.[32–35] that these theories are, in fact, renormalizable in the $1/N$ expansion. Not only is this expansion renormalizable, and hence calculable, the $1/N$ technique reveals a lot of non–perturbative information. This opens up the possibility of *quantitatively* studying dynamically generated (composite) Goldstone bosons. Apart from the pedagogical value of this study, the $2+1$ dimensional models may also have physical application to $2D$ condensed matter systems. Non–relativistic 4–fermion models, such as the Hubbard model [36], are extensively used to describe anti–ferromagnets, including those become high T_c superconductors when properly doped. The continuum (or long wavelength) limit can be approximated by the "relativistic" models described here, where essentially the velocity of light is replaced by the velocity of sound.

This review is organized as follows. Chapter 2 contains a derivation of the $1/N$ expansion for the *discrete* Gross–Neveu lagrangian (1.2). This explains the method that will be used throughout the later chapters. At first we consider a general spacetime dimension d, and describe the phase structure. At *leading* order in $1/N$ the demand of non–trivial interactions in the continuum limit then restricts us to $d = 1+1$ and $2+1$, and we compute the scattering amplitudes. In both dimensions we find a bound state σ–meson at twice the induced mass of the fermions. An interesting difference between the two dimensions is seen in the high energy behavior. In $d = 1+1$ the "invariant charge" $\alpha(E)$ is asymptotically free, as emphasized by Gross and Neveu [26], but in $d = 2+1$ the charge approaches a finite constant. This is similar to the behavior apparently found recently in $3+1$–dimensional QED [37] and we hope that 4–fermion theories in $2+1$ dimensions will provide a good laboratory for studying this phenomenon. Finally, in section

2.6, we present the proof that the Gross–Neveu model is indeed renormalizable *to all orders* in $1/N$ in $d = 2+1$. The reader not interested in the intricacies of regularization and renormalization procedures may skip this subsection, but should note that the proof holds similarly for a wide class of $2+1$–dimensional models.

To conclude, we discuss the validity of the $1/N$ expansion itself. In $1+1$ dimensions the method is by now firmly established by comparison with exact S–matrices [20–25], lattice simulations [38] and semi–classical calculations [39]. However in $d = 2+1$ non–perturbative information is rather fragmental. An exception is the $O(N)$ symmetric σ–model, which has been extensively studied recently [40,41] in connection with 2D antiferromagnets. In particular, Manousakis and Salvador [42] have found a continuum limit of the model using the Monte Carlo method, and their data compares well with the $1/N$ analysis.

Chapter 3 is devoted to the dynamical breaking of *continuous* chiral symmetries in $d = 2+1$. Composite Goldstone bosons are explicitly located as poles in $2 \to 2$ fermion scattering amplitudes, and Goldstone's theorem is proved to all orders in the $1/N$ expansion. We calculate the low–energy interactions of fermions and Goldstone bosons, and verify the Goldberger–Treiman relation. The self–interactions of Goldstone bosons are calculated and summarized in an effective chiral lagrangian.

From a "phenomenological" point of view it is important to understand the physics of an *approximate* chiral symmetry, as was realized in the original work of Nambu and Jona–Lasinio [12]. In the 4–fermion models in $d = 2+1$ we can explicitly break the chiral symmetry, but maintain the renormalizability, by adding mass terms for the fermions. In Chapter 4 we study the interplay between the two mechanisms of symmetry breaking. We calculate the masses of the fermions and pseudo–Goldstone bosons in terms of the explicit breaking parameters, and derive the Gell–Mann–Okubo formula [43]. The bound "σ–meson" present in the chiral limit becomes a resonance.

Chapter 5 contains a discussion of the thermodynamics of chiral symmetry breaking. The *discrete* Gross–Neveu model in 2+1 dimensions undergoes a second–order phase transition at a finite temperature T_c. The mass gap in the high temperature phase is reduced to zero, and the transition is precisely analogous to that in the BCS theory of superconductivity. For the case of *continuous* symmetry one can invoke Coleman's theorem to prove that the critical temperature must be exactly at $T_c = 0$. This infra–red effect is non–analytic in $1/N$, and so cannot be studied directly in the $1/N$ expansion. However, we shall argue that many of the properties of the "would–be" Goldstone bosons do, in fact, survive at low enough temperatures, and for these properties the $1/N$ expansion is reliable.

In Chapter 6 we briefly describe other $2+1$–dimensional renormalizable theories. These include (1) 4–fermion models with vector–vector type interactions; (2) $O(N)$ symmetric non–linear σ–model; (3) supersymmetric σ–model; and (4) $CP(N-1)$ model. These theories have interesting phase structures and probably a rich spectrum of bound states. Moreover, their analogues in $d = 3+1$ are phenomenologically valuable despite the presence of the cutoff [44].

2. The 1/N Expansion and its Renormalizability

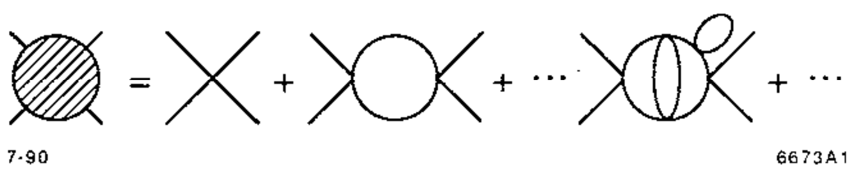
In this chapter we show in detail that 4–fermion theories in $d = 2 + 1$ have a renormalizable $1/N$ expansion. In subsection 2.1 we give a heuristic argument how this can be possible, despite the non–renormalizability in the weak coupling expansion. This is supplemented in 2.6 by a rigorous proof (in the realm of physics) and in 2.7 by a discussion of RG improvement. In order to make the review self–contained the $1/N$ expansion is derived in 2.2, first in a diagrammatic way, and then briefly in a path integral formalism. Subsections 2.3–2.5 contain discussions of the bound states, the phase structure and ultra–violet behavior.

2.1. How a theory nonrenormalizable in weak coupling perturbation theory may nevertheless be renormalizable?

It is widely believed that in quantum field theory the requirement of renormalizability is very strict for spacetime dimensions $d > 1 + 1$, so that the possible list of continuum models is very short. The argument is based on a simple "power–counting" analysis, which we shall review here, to see how it can be transcended.

First one notes that the lagrangians for *free* fields are either second order or first order in derivatives, according to whether the particles are bosons or fermions. The propagators then vary as p^{-2} or p^{-1} for large momenta, and one defines the "ultraviolet dimensions" D_{uv} of the free fields to be $\frac{1}{2}(d - 2)$ and $\frac{1}{2}(d - 1)$, respectively. With this definition the kinetic energy terms have $D_{uv} = d$ and the free action is dimensionless. To obtain a (perturbatively) interacting theory, one may add to the lagrangian operators of total ultra–violet dimension $\leq d$. These are the so–called “relevant” and “marginal” operators [2.1] and comprise just a finite number of polynomials in the fields and their derivatives, for $d > 1+1$. A key point in the power counting argument is that at any *finite* order in perturbation theory the interactions only change the ultra–violet scaling behavior by logarithms. Thus the naïve power counting does indeed work, if it is understood order by order in $\hbar$ [2.2].

Operators with $D_{uv} > d$ can, actually, be included in the action provided there is an ultraviolet cutoff Λ, but then their coupling constants must be tuned to vanish as powers of $1/\Lambda$. As a result these operators have *no* effect at all on the form of Green's functions at a physical energy scale $E \ll \Lambda$, and in this sense they are “irrelevant.” To exemplify this reasoning, let us consider a 4–fermion interaction, as in the Nambu–Jona–Lasinio or Gross–Neveu models. In perturbation theory the connected 4–point function has the diagrammatic expansion



(2.1)

Heuristically, each loop provides a factor $\int \frac{d^dp}{p^2} \sim \Lambda^{d-2}$, which is controllable for $d = 1+1$. However, for $d > 1+1$ the only way to prevent the series from diverging is to give each vertex a weight $g^2 \sim 1/\Lambda^p$ with $p \geq d-2$. Each term in the sum is then suppressed by a power of $1/\Lambda$, and the connected amplitude vanishes in the continuum limit. Thus, we say that in perturbation theory the 4–fermion interaction is “power–wise trivial.”

There have been many attempts over the years to get round the restrictions of perturbative renormalizability. The analytic methods[#3] involve at some point a change in the high-momentum dependence of the propagators, so that the power counting is radically altered. One such approach is to truncate the Dyson–Schwinger equations in some way and solve the resultant system of integral equations "exactly." In this fashion it was found that in QED in $d = 3+1$ the 4-fermion operators have large "anomalous dimensions" and can become relevant [2.3]. Unfortunately, this method is not systematically improvable, and the justification for the truncations remains unclear. Another approach involves the *ad hoc* addition of higher-derivative kinetic energy terms to the action. This method certainly improves the renormalizability but in general suffers from a breakdown of unitarity or causality. The propagators either violate the Källen–Lehmann representation if the number is infinite [2.4], or develop ghost-like poles if the number of derivatives is finite [2.5]. A conjecture by Lee and Wick [2.6] was that by making the ghost-like poles occur as complex conjugate pairs unitarity may be restored, but this idea has not been substantiated to our knowledge[#4].

The basic mechanism of the $1/N$ expansion [2.8,2.9] is the summation of an *infinite* number of Feynman diagrams, and this can go beyond the perturbative power counting. Physically what is happening is that a given cutoff theory has more structure than the naïve continuum model. One then tries to keep some of the new physics as the cutoff is removed, and this leads to unusual renormalisations. A simple example in the context of quantum mechanics is an attractive delta-function

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#3 We shall not attempt to discuss the computational methods on the lattice.
#4 For a review see Ref. [2.7]

potential in a box. The boundary allows the potential to bind, and it is possible to preserve this bound state by adjusting the strength of the delta–function as the walls go off to infinity. Returning to the 4–fermion interaction, consider the subset of diagrams formed by the chain of "bubbles" in geometric series in Fig. 1. If $1/g^2$ is *tuned* to cancel the leading divergence $\sim \Lambda^{d-2}$ in the bubble diagram then the geometric sum can become finite as $\Lambda \to \infty$, and indeed this is what happens in $d = 2 + 1$. Remarkably, the fine–tuning required here is exactly the same as in the Hartree–Fock approximation mentioned in the introduction, and the two sums can be combined. In a diagrammatic notation Hartree–Fock corresponds to the non–perturbative sum of "cactus graphs" [2.10], see Eq. (2.2).

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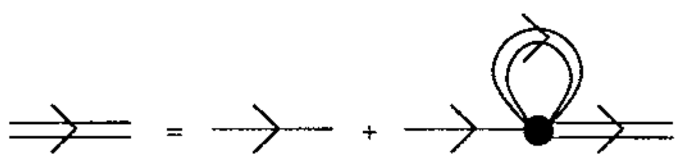
Figure 1. Bubble Chain



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(2.2)

Equation (2.2) yields a "self–consistent" equation for the fermion mass, which can be solved in terms of g^2 in the presence of a ultra–violet cutoff Λ. In this way, the leading order of $1/N$ expansion incorporates both dynamical mass generation *and* non–trivial interactions. As we shall see in the next subsections the $1/N$ expansion can be described systematically and is renormalizable to all orders. An important check of the method is that it is consistent with unitarity, and this is discussed in detail in Appendices C and D.

2.2. THE $1/N$ EXPANSION

Let us consider the simplest possible 4–fermion interaction, namely the Gross–Neveu model (1.2) in an as yet arbitrary spacetime dimension d. Once this case is understood, the method will be easily extendable to a host of more exotic theories. In (1.2) ψ is a Dirac spinor[#5] and the model has a *discrete* "chiral" symmetry $\psi \to \gamma_5 \psi$, which is preserved to all orders in ordinary perturbation theory. This ensures the fermions to be perturbatively massless.[#6]

In the $1/N$ resummation method the *leading–order* contribution for any connected Green's function is defined by the (infinite) set diagrams for which the power of N coming from flavor contractions, call it P, is equal to the number of loops L. (Note that we always have $L \geq P$.) For the 2–point function this set is just the "cactus graphs" of Eq. (2.2) and the sum can be performed provided we take the coupling g^2 to be $O(1)$ in units of N. The full 2–point function is then $O(1)$. For the connected 4–point function the set is the bubble chain of Fig. 1, "dressed" with all possible cacti, giving an overall weight $1/N$. The connected 6–point function has *two* bubble chains dressed with cacti and has weight $1/N^2$, etc.

#5 See Appendix A for a summary of notations for the case $d = 2+1$. For even dimensions the notation is standard [2.11].

#6 In $d = 2+1$ this is not really a chiral symmetry since there are no chiral fermions. Rather it is a Z_2 symmetry which switches the upper and lower components of a Dirac spinor (Appendix A.)

The *next–to–leading–order* contribution for any Green's function is then the set of diagrams for which $P = L - 1$. In general, the n^{th} order of the resummation has $P = L - n$, and as $n \to \infty$ we "diagonally" recover the totality of the diagrams. The summations here can be codified in a convenient way using an auxiliary field formalism [14, 2.12]. The lagrangian (1.2) can be rewritten using a scalar field $\sigma(x)$ as

$$\mathcal{L}(\psi,\overline{\psi},\sigma) = i\overline{\psi}_j \partial\!\!\!/ \psi_j - \sigma\overline{\psi}_j\psi_j - \frac{N\sigma^2}{2g^2} \ , \tag{2.3}$$

where the equivalence is by the *exact* equation of motion for $\sigma(x)$. The discrete chiral symmetry is now written as

$$\psi \to \gamma_5\psi \qquad \overline{\psi} \to -\overline{\psi}\gamma_5 \qquad \sigma \to -\sigma \ . \tag{2.4}$$

We define a functional integral $Z(\xi,\overline{\xi})$ by coupling in sources for the fermionic fields and regularizing with a Lorentz invariant momentum cutoff Λ.#7

$$Z(\xi,\overline{\xi}) \equiv \int D\psi\, D\overline{\psi}\, D\sigma \exp\frac{i}{\hbar}\int d^dx\, \mathcal{L}(\psi,\overline{\psi},\sigma) - \overline{\xi}\psi - \overline{\psi}\xi \ . \tag{2.5}$$

The exponent is purely quadratic in the fermion fields, and we can formally integrate over them. This is the essential trick of introducing $\sigma(x)$. We obtain an "effective action" for the auxiliary field

$$\begin{aligned} Z(\xi,\overline{\xi}) &= \int D\sigma \exp\frac{i}{\hbar}\left[N S_{\text{eff}}[\sigma] + \int d^dx d^dy \overline{\xi}_x (i\partial\!\!\!/ - \sigma)^{-1}_{xy}\sigma_y\right] \\ S_{\text{eff}}[\sigma] &= -\int d^dx\, \frac{\sigma^2}{2g^2} - i\hbar\, \mathrm{tr}\, \ell n(i\partial\!\!\!/ - \sigma) \ . \end{aligned} \tag{2.6}$$

In Eq. (2.6) the N dependence is now fully explicit and $Z(\xi,\overline{\xi})$ can be estimated

#7 The kinetic energy term $\overline{\psi}i\partial\!\!\!/\psi$ is replaced by $\overline{\psi}i\partial\!\!\!/ f(\sigma^2/\Lambda^2)\psi$ for some appropriate cutoff function $f(\cdot)$. Other regularization schemes are discussed in Appendix B.

by the method of stationary phase. The fluctuations around the stationary phase solution then compactly express the generation of the $1/N$ expansion.

To investigate the vacuum structure, let us consider the leading order energy density V_{eff} evaluated for constant fields $\sigma(x) = \sigma$. We have

$$\begin{aligned}\frac{\partial V_{\text{eff}}}{\partial \sigma} &= \frac{\sigma}{g^2} - i\hbar \int\limits_{\Lambda} \frac{d^d p}{(2\pi)^d} \operatorname{tr} \frac{1}{\not{p} - \sigma} \\ &= \frac{\sigma}{g^2} - 4\hbar\sigma \int\limits_{\Lambda} \frac{d^d p_E}{(2\pi)^d} \frac{1}{p_E^2 + \sigma^2} \ .\end{aligned} \tag{2.7}$$

In Eq. (2.7) the contour of integration has been Wick rotated in accordance with the Feynman (causal) boundary condition [2.11], i.e. $p^0 = ip_E^0$. From the effective potential we can see a fundamental distinction between $d = 1+1$ and higher dimensions. Fixing g^2 and Λ there is a logarithmic singularity as $\sigma \to 0$ in $d = 1+1$ which makes $V''_{\text{eff}}(\sigma)$ large and negative. Thus the minimizing value of $\langle\sigma\rangle$ is bounded away from zero no matter how we choose the coupling, and the discrete chiral symmetry *must* be broken.

In $d > 1+1$ there is a two phase structure. $V'_{\text{eff}}(0)$ is positive for *weak* enough coupling, $g^2 < g^2_{\text{crit}}$, and the chiral symmetry is manifested. The critical value of the coupling is

$$g_{\text{crit}}^{-2} = 4\hbar \int\limits_{\Lambda} \frac{d^d p_E}{(2\pi)^d} \frac{1}{p_E^2} \qquad d > 1+1 \ . \tag{2.8}$$

If the coupling is larger than g^2_{crit} then $V'_{\text{eff}}(0) < 0$ and the auxiliary field σ gets a non–zero vacuum expectation value.

Let us consider first the phase of dynamically broken chiral symmetry. In the large N limit $\langle\sigma\rangle = M > 0$ is a physical quantity, namely the induced pole mass of the fermions. In order to have $M \ll \Lambda$ we must *fine-tune* the coupling g^2 in the following way:

$$g^{-2} = 4\hbar \int_{\Lambda} \frac{d^d p_E}{(2\pi)^d} \, \frac{1}{p_E^2 + M^2} \, . \tag{2.9}$$

If this is the case then the effective potentials in $d = 1+1$ and $d = 2+1$ are given by

$$V_{1+1} = \frac{\hbar}{2\pi} \, \sigma^2 \ell n \, \frac{\sigma^2}{eM^2} \tag{2.10a}$$

$$V_{2+1} = \frac{\hbar}{\pi} \left(\frac{1}{3} \, |\sigma|^3 - \frac{1}{2} \, M\sigma^2 \right) \, . \tag{2.10b}$$

The fluctuations about the minimum are now described by the effective Feynman rules in Fig. 2. A key feature of the method is that the auxiliary field $\sigma(x)$ has

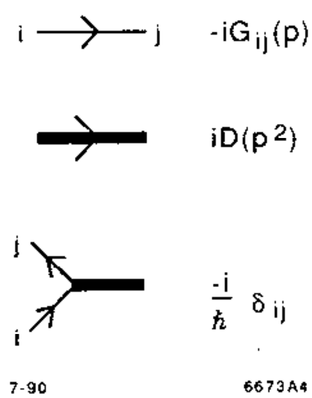


Figure 2. Feynman rules in asymmetric phase.

an *induced* propagator from the double functional derivative of S_{eff}. In Fig. 2 the fermion and auxiliary field propagators are given respectively by

$$G_{ij}(p) = \hbar(\not{p} - M)^{-1} \delta_{ij} \tag{2.11a}$$

$$D(p^2) = \frac{\hbar}{N}\left[\frac{1}{g^2} - i\hbar\int\frac{d^dq}{(2\pi)^d}\,\mathrm{tr}\,\frac{1}{(\not{q}-M)(\not{q}-\not{p}-M)}\right]^{-1} \quad . \tag{2.11b}$$

In this algorithm $Z(\xi,\bar{\xi})$ is given by the set of all effective Feynman diagrams with external fermion "legs," *except* those containing "bubbles" and "tadpoles" as subgraphs, see Fig. 3. The point is that these have already been summed over in Eq. (2.6). Thus each order in $1/N$ is described by a *finite* number of effective graphs.

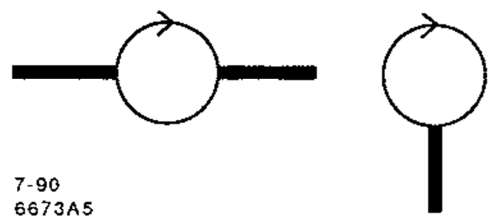


Figure 3. Illegal Subgraphs

From Eq. (2.11b) the reciprocal of $D(p^2)$ contains two divergent integrals of order Λ^{d-2}, and the leading divergence actually cancels for all d.[#8] Thus $D(p^2)$ is finite and *non–vanishing* as $\Lambda \to \infty$ in $d = 1+1$ and $2+1$, and order Λ^{4-d} for $d > 3+1$. The analytic expressions for $d = 2,3$ in the continuum limit are

$$D(p^2)_{1+1} = \frac{\pi}{N}\,\frac{\sqrt{-p^2}}{\sqrt{-p^2+4M^2}}\,\left[\ell n\left(\frac{\sqrt{-p^2+4M^2}+\sqrt{-p^2}}{\sqrt{-p^2+4M^2}-\sqrt{-p^2}}\right)\right]^{-1} \tag{2.12a}$$

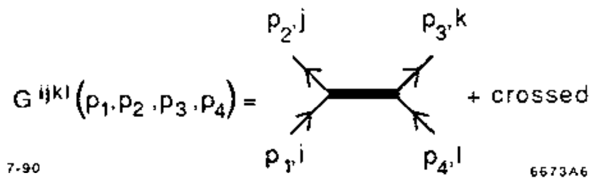
$$D(p^2)_{2+1} = \frac{2\pi}{N}\,\frac{\sqrt{-p^2}}{(-p^2+4M^2)\tan^{-1}(\sqrt{-p^2}/2M)} \tag{2.12b}$$

which are real functions for p^2 negative.

#8 This crucial cancellation is not an artifact of the momentum cutoff, it also holds for a lattice or Pauli–Villars regulator.

2.3. Mesonic Bound State

Let us now consider the physics of the leading-order amplitudes, which are given by the "tree diagrams" of Fig. 2. The fermions have a dynamically–generated mass M, and have interactions beginning in order $1/N$. The 4–point function carries the bulk of the information and is given by



(2.13)

and so essentially reduces to knowing the σ–propagator evaluated as a function of the Mandelstam variables s, t or u. Thus the fine–tuning of the coupling Eq. (2.9) which produced a mass scale small compared to the cutoff, $M \ll \Lambda$, has also produced non–trivial interactions in the cases $d = 1 + 1$ and $2 + 1$.[#9]

If we consider the kinematical region appropriate to the scattering $F_i F_j \to F_i F_j$ then the momentum transfer is *spacelike* and the T–matrix element is real. To obtain the differential cross section we need the $2 \to 2$ phase space factor and the sum over spinor polarizations. In $d = 2 + 1$ we obtain

$$\left(\frac{d\sigma}{d\theta}\right)_{CM} = \frac{1}{16\pi s\sqrt{s - 4M^2}}\left\{(t - 4M^2)^2 D^2(t) \right.$$
$$\left. - \delta_{ij}\left[\frac{1}{4}(u^2 + t^2 - s^2) + 4M^2(s - M^2)\right] D(t)\, D(u)\right\} + t \leftrightarrow u \; . \tag{2.14}$$

In both dimensions there is an exchange channel *pole* at mass $2M$ coincident with

---

#9 For $d = 3 + 1$ there is a subleading logarithmic divergence in Eq. (2.11b). and the 4–point function vanishes as $1/\ell n\Lambda$. The significance of this "logarithmic triviality" will be commented on in the conclusion.

the two fermion threshold. Thus the auxiliary field $\sigma(x)$ not only has become "dynamical" but, in fact, interpolates for a lightly bound meson. (Note that the position of the pole is a pure number in units of M, and it does not need a separate normalisation. This is because the theory has truly only one parameter.)

The meson can also be detected as a resonant contribution to $F_i\overline{F}_j \to F_i\overline{F}_j$, where it appears in the s–channel. Since s is *timelike* the leading order amplitude becomes complex, and is found by analytic continuation of Eq. (2.12). The analytic structure of the amplitude is discussed in Appendix C, and we have in $d = 2+1$

$$\left(\frac{d\sigma}{d\theta}\right)_{CM} = \frac{1}{16\pi\, s\sqrt{s-4M^2}} \Bigg\{ N\delta_{ij}(s-4M^2)^2|D(s)|^2 + (t-4M^2)^2 D^2(t) \\ - \delta_{ij}\left[\frac{1}{2}(t^2-u^2+s^2) + 4M^2(-t+u-s) + 8M^4\right] D(t)\,\mathrm{Re}\, D(s) \Bigg\} .$$

where

$$D(s+i\epsilon) = \frac{-4\pi}{N}\,\frac{\sqrt{s}}{s-4M^2}\left[\ell n\left|\frac{\sqrt{s}+2M}{\sqrt{s}-2M}\right| + i\pi\,\Theta(s-4M^2)\right]^{-1} . \tag{2.15}$$

The "fate" of this bound state beyond leading order is an interesting question. In $d = 1+1$ an extraordinary theoretical breakthrough was made by Zamolodchikov and Zamolodchikov [20], who solved for the S–matrix *exactly*. In the exact $F\overline{F} \to F\overline{F}$ amplitude, for which Eq. (2.12a) indeed gives the large N limit [2.13], the pole has moved away from the 2–body threshold into the bound region, with $\Delta M/M \sim -1/N^2$. For $N \geq 3$ the true σ–meson mass is given by

$$m_2 = M \sin\left(\frac{2\pi}{N-2}\right) \Big/ \sin\left(\frac{\pi}{N-2}\right) . \tag{2.16}$$

In principle the mass formula could be checked from a direct calculation of the

4–point function to higher orders in $1/N$, but this is a very difficult task[#10].

In $d = 2+1$ the exact S–matrix is not known, and we hope that future lattice simulations will shed light on the bound state spectrum. However, one can make a reasonable "analytic" guess using a non–relativistic potential approach, which proved to be very accurate in the $1+1$ case [39]. In Born approximation the scattering amplitude Eq. (2.13) is the Fourier transform of some potential $V(r)$ in an effective non–relativistic Schrödinger equation. Taking the inverse transform of the amplitude we get an *attractive* square–well–like potential with a range $\sim 1/M$ and a depth $\sim M/N$. In $1+1$ dimensions $V(r)$ has at least one bound state, even as $N \to \infty$, with a binding energy $\sim M/N^2$ just as in Eq. (2.16). In $2+1$ the potential is too weak to bind as $N \to \infty$ and the energy shift will be positive. For N small $V(r)$ becomes sufficiently attractive, so heuristically it appears that there is a critical $N = N_c$ below which the σ–meson becomes stable.

2.4. High Energy Behavior

Let us now consider the "deep–euclidean" region of the 4–point function. We can define a dimensionless "charge" $\alpha(E)$, which expresses the strength of the interactions between the fermions, by $\alpha(E) \equiv NE^{\#}G^{(4)}(E)$. Here the 4–point function is taken at the symmetric point, $(p_1+p_2)^2 = (p_1+p_3)^2 = (p_1+p_4)^2 = -E^2$, and $i = j = k = l$. The power # is zero in the case $d = 1+1$ and unity in $d = 2+1$.

#10 Refs. [2.14] claim that the $1/N$ expansion actually gives $\Delta M/M \sim -1/N^4$, in disagreement with Eq. (2.16). It is not clear to us, however, that the renormalization procedures used are consistent with that described in subsection 2.6.

From Eq. (2.12) we obtain

$$\alpha(E)_{1+1} = \frac{\pi E}{\sqrt{E^2+4M^2}} \left[\ell n \left(\frac{\sqrt{E^2+4M^2}+E}{\sqrt{E^2+4M^2}-E} \right) \right]^{-1} \tag{2.17a}$$

$$\alpha(E)_{2+1} = \frac{2\pi E^2}{(E^2+4M^2)\tan^{-1}(E/2M)} \, . \tag{2.17b}$$

In $d = 1+1$ the charge is asymptotically free, decaying as $\ell n^{-1}(E/M)$. This is analogous to the behavior in QCD, and provided the motivation for Gross and Neveu [26] to give a detailed study of the model. One says that the model has undergone "dimensional transmutation," for the classically dimensionless coupling g^2 has been traded for the mass scale M. It is interesting that the non–perturbative result Eq. (2.17a) agrees at high energy with the 1–loop RG–improved weak coupling expansion. There we have

$$\begin{aligned} \alpha(E) &= \hbar g^2(\mu) - \frac{2\hbar^2}{\pi} g^4(\mu)\,\ell n\,(E/\mu) + \ldots \\ &\simeq \frac{\hbar g^2(\mu)}{1+(2\hbar/\pi)g^2(\mu)\ell n(E/\mu)} \to \frac{\pi}{2\ell n(E/\mu)} \, . \end{aligned} \tag{2.18}$$

This gives us some confidence in the use of RG–improvement in QCD. The behavior of $\alpha(E)$ in $d = 1+1$ is expected to be qualitatively the same through all orders in $1/N$ since at high enough energy the charge is arbitrarily small and the order–by–order corrections should be calculable "perturbatively."

In $d = 2+1$ we get a radically different picture. At leading order the charge approaches a finite constant $\alpha(\infty) = 4$, providing a rare example of a non–zero ultra–violet fixed point.#11 But now we no longer have a perturbative argument as

#11 The dependence of $\alpha(\infty)$ on the dimension d is analogous to that in scalar theories [2.15].

to why $1/N$ corrections should be automatically small, and it is a subtle question as to what is the true high-energy behavior of the theory. What is required is a kind of RG-improvement of the $1/N$ expansion itself [2.16], and an investigation of this problem is presented in subsection 2.7.

2.5. The Phase of Unbroken Chiral Symmetry

In the 4-fermion model for $d > 1+1$ the discrete chiral symmetry Eq. (2.4) is dynamically broken provided the coupling g^2 is sufficiently *strong*, as in Eq. (2.9). For completeness, let us consider the symmetric phase. The coupling is now tuned to be slightly *weaker* than the critical value, i.e. we set

$$g^{-2} = \hbar \left(4 \int\limits_{\Lambda} \frac{d^d p_E}{(2\pi)^d} \, \frac{1}{p_E^2} + \frac{v}{4} \right) \; , \qquad d > 1+1 \tag{2.19}$$

where $v > 0$. From the "gap equation" (2.7) we must have $\langle \sigma(x) \rangle = 0$, so the fermions are massless and the chiral symmetry is indeed manifest. In $d = 2+1$ the fermions still have non-trivial (continuum limit) interactions through scalar exchange and the σ-propagator is given by

$$D_\sigma(p^2) = \frac{1}{4N} \left(v + \sqrt{-p^2} \right)^{-1} \; . \tag{2.20}$$

The 4-point function in the symmetric phase thus has a square-root cut at zero momentum, which is a characteristic for massless particles in $d = 2+1$. Note that apparently there is a tachyon-like pole at $p^2 = -v^2$ in Eq. (2.20), but this only occurs on the *second* sheet, and so does not signify a real instability. The σ-field no longer interpolates for a stable physical particle, and it seems reasonable to conjecture that there are no bound states at all in this phase.

2.6. Renormalizability Proof for $d = 2+1$

At non–leading orders in the $1/N$ expansion ultra–violet divergences arise, and we would like to see if these can be systematically controlled. It turns out that the proof of renormalizability is actually simpler for $d = 2+1$ than for $d = 1+1$, which was completed in Ref. [2.17].#12 We consider first the renormalization problem in the symmetric phase. From a "Wilsonian" point of view the result would then carry over automatically to the broken theory [2.1], since the specific realization of a given phase is purely an "infra–red effect."

The Feynman rules in Fig. 2 indicate that the propagators of the ψ and σ fields both behave at large momentum as p^{-1}, so their ultra–violet dimensions are both 1. Note that for the scalar field this is a truly non–perturbative dimension, since an elementary scalar has $D_{\text{uv}} = \frac{1}{2}$, as mentioned in subsection 2.1. Performing the analogue of the power standard counting, the "superficial degree of divergence" D_Λ of a Feynman diagram with E_ψ external fermion legs and E_σ external σ–legs is

$$D_\Lambda = 3 - E_\psi - E_\sigma \ . \tag{2.21}$$

#12 One should not assume that the *perturbative* renormalizability of the $1+1$ model implies its renormalizability in the $1/N$ expansion. After all, the two expansions are completely different. The difficulty, compared with the $2+1$ case, comes from the possibility that a $(\overline{\psi}\psi)^2$ counterterm is required to cancel the divergences, not just the three operators in Eq. (2.3). This would render the $1/N$ expansion inconsistent: suppose the counterterm is induced at rth order in $1/N$ to remove a divergence in the 1PI fermion 4–point function. The counterterm can then generate new graphs for the 2–point function, and then the 4–point function, which are *still* at rth order (due to a factor of N coming from loops.) Thus a *new* rth order counterterm is required, and so it goes on *ad infinitum*. At next–to–leading order, at least, this catastrophe does not happen. The 1PI graph in Fig. 5 is finite in $1+1$ due to the factor of $1/\ell n p^2$ from each of the two σ–propagators. Thus the renormalizability in $1+1$ depends not just on counting *powers* of momenta, but also on powers of *logarithms*!

Thus the only primitively Λ–divergent graphs are those in Fig.4. Notice that,

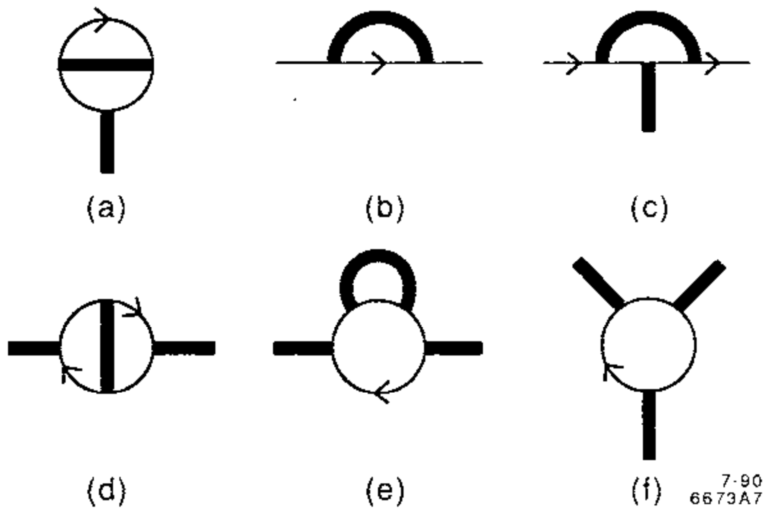


Figure 4. Primiative Divergences

for example, a 4–fermion graph is not superficially Λ–divergent so the "one–loop" graph in Fig. 5 is Λ–finite.

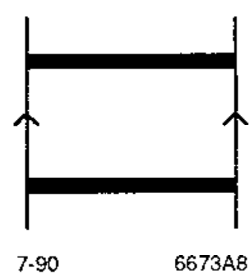


Figure 5. Finite 4–fermion coupling in one–loop

To circumvent a discussion concerning infrared divergences, we imagine carrying out the systematic renormalization using the "method of effective lagrangians," that is by sequentially integrating out high momentum modes [2.1]. A small adaptation of the arguments in [2.2] shows that the "anomalous dimensions" of operators built out of ψ and σ are perturbatively small in $1/N$, so the power counting analysis in Eq. (2.21) is correct. Now from Fig. 4 we may expect to need counterterms for the renormalization of the form $\overline{\psi}\psi$, $\overline{\psi}\partial\!\!\!/\psi$, $\sigma\overline{\psi}\psi$, σ, σ^2, and σ^3. Importantly, this list does *not* contain a kinetic energy term $(\partial_\mu\sigma)^2$, since this operator is dimension 4. This has the consequence that $\sigma(x)$ remains an auxiliary field under all orders renormalization.

Actually, we'll only need counterterms $\overline{\psi}\partial\!\!\!/\psi$, $\sigma\overline{\psi}\psi$ and σ^2, which are precisely the original operators in the lagrangian Eq. (2.3). The others are forbidden by the chiral symmetry: Figs. 4a,f vanish identically in the symmetric phase, as does the momentum–independent divergence in Fig. 4b. Thus the auxiliary field retains the original form of the equation of motion $\sigma(x) \simeq \frac{1}{\Lambda}\overline{\psi}\psi(x)$, and the model can still be represented as a 4–fermion interaction. To discuss the broken case $M \neq 0$ is now easy, since this is generated by a *finite shift* in the σ–field. As is simply proved in this method, this shifting has no effect at all on the ultra–violet structure [2.2,2.5] and this completes the proof.

In summary, the renormalized Green's functions depend on just three subtraction constants, and we define c.f. Eq. (2.3)

$$\mathcal{L}_{\text{bare}}(\psi, \overline{\psi}, \sigma) = iZ_1\overline{\psi}_j\partial\!\!\!/\psi_j - Z_2\sigma\overline{\psi}_j\psi_j - \frac{NZ_3Z_2^2}{2g^2Z_1^2}\,\sigma^2 \ . \tag{2.22}$$

In Eq. (2.22) Z_1 and Z_2 are just overall normalizations for the fields, and will drop out of the S–matrix. Thus the physical amplitudes will depend on just *one* parameter and in this respect the model is like QCD. One can define a "physical" renormalization scheme by stipulating the pole mass of the fermions, or one can specify instead a dimensionless interaction strength at some reference scale μ (see subsection 2.7).

At this point the reader may wonder whether the preceeding formal argument is really true, and that the three constants Z_1, Z_2 and Z_3 are sufficient to renormalize all correlators. In particular, the *language* of the argument was much the same as is used in ordinary perturbation theory, where the bare propagators are much simpler functions of momentum than the bare σ propagator of Eq. (2.12). To allay any fears let us explicitly verify the renormalizability at next–to–leading order.

It is convenient for these purposes to use a euclidean space formulation, so we have the Wick rotated functional integral

$$\begin{aligned} Z(\xi,\overline{\xi}) &= \int D\psi\, D\overline{\psi}\, D\sigma \exp - \int d^3x\, \mathcal{L}_E - \overline{\xi}\psi - \overline{\psi}\xi \\ \mathcal{L}_E &= Z_1 \overline{\psi}_j \not{\partial}_E \psi_j + Z_2 \sigma \overline{\psi}_j \psi_j + \frac{N Z_3 Z_2^2}{2g^2 Z_1^2}\, \sigma^2 \; . \end{aligned} \tag{2.23}$$

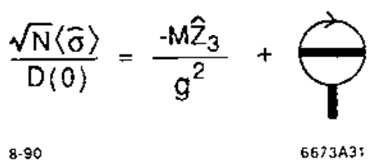
In Eq. (2.23) we have set $\hbar = 1$ and use euclidean γ–matrices, see Appendix A. The Feynman rules from Eq. (2.23) are similar to those in Fig. 2. We keep g^2 as in Eq. (2.9), so that we are in the *broken* phase#13, and set $Z_i = 1 + \widehat{Z}_i/N$. Expanding $\sigma = \frac{Z_1}{Z_2}(M + \frac{1}{\sqrt{N}}\widehat{\sigma})$, the only changes are that the sources have a rescaling, there is a 1–point $\widehat{\sigma}$–vertex of weight $-\widehat{Z}_3 M/g^2\sqrt{N}$, and an extra 2–point $\widehat{\sigma}$–vertex of weight $-\widehat{Z}_3/2g^2\sqrt{N}$.

#13 In an explicit calculation, as opposed to a general all orders argument, it is easiest to think of renormalization in terms of subtractions at zero momentum. This can only be done in the massive case. Moreover, the analysis is directly applicable to $d = 1+1$ by just changing the dimension of the integrals.

For the renormalization it is sufficient to calculate the full vacuum expectation value of $\sigma(x)$, and the connected Green's functions $\langle\psi(x)\overline{\psi}(y)\rangle$, $\langle\sigma(x)\sigma(y)\rangle$ and $\langle\psi(x)\overline{\psi}(y)\sigma(z)\rangle$. We have at next to leading order

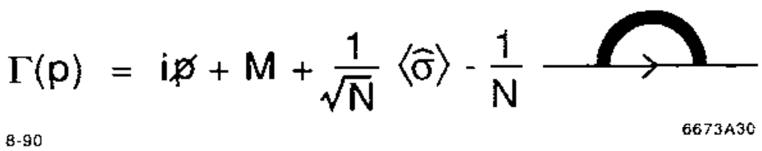
$$\frac{Z_2}{Z_1}\langle\sigma(x)\rangle = M + \frac{1}{\sqrt{N}}\langle\widehat{\sigma}\rangle \tag{2.24a}$$

where



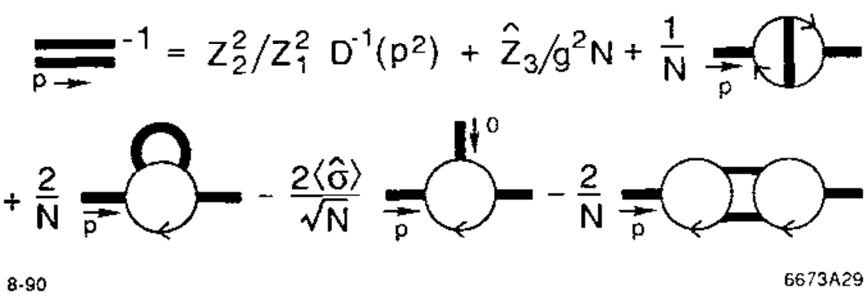
(2.24b)

The inverse fermionic 2–point function is given by $\Gamma^{ij}(p) = Z_1\delta^{ij}\Gamma(p)$, where



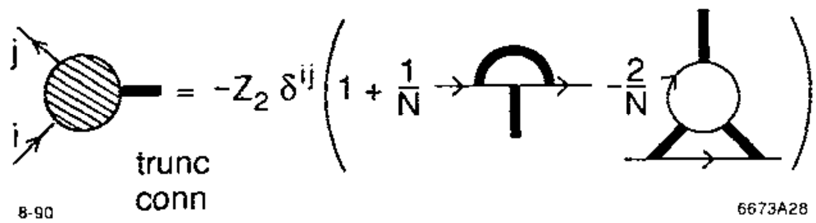
(2.25)

and the inverse of the connected mesonic 2–point function is



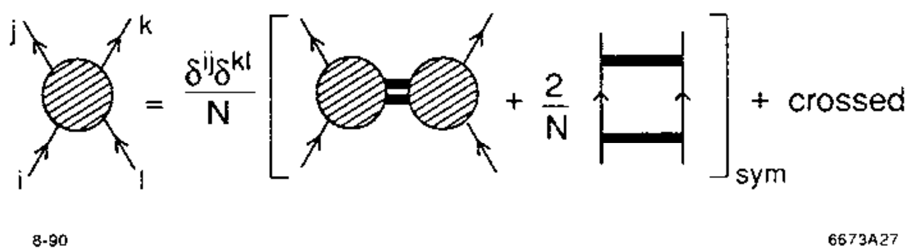
(2.26)

Finally, the connected and truncated vertex function is



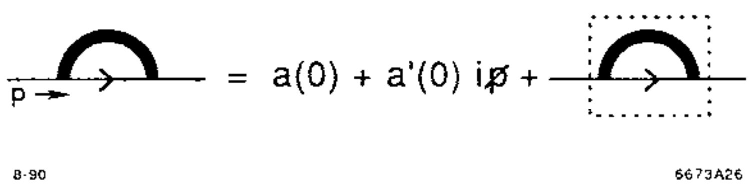
(2.27)

where by truncation we mean that the connected vertex function has been multiplied by the full inverse 2-point functions on all legs. If it is the case that Z_1, Z_2 and Z_3 really do renormalize Eq. (2.24) through Eq. (2.27), then all Green's functions will be renormalized. For example, the connected and truncated fermion 4-point function will indeed be made finite, since

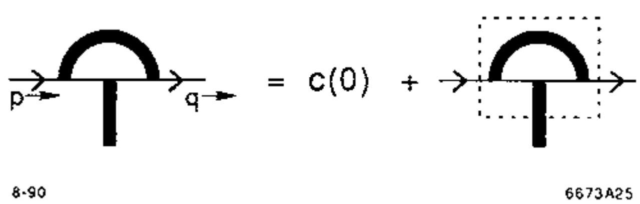


(2.28)

and the "box" diagrams in Eq. (2.28) are explicitly convergent. Now let us define, in the spirit of BPHZ renormalization, Taylor expansions of the divergent diagrams about zero momentum[#14]. First we have



(2.29a)



(2.29b)

The subtracted loop integrals, denoted by the dotted box surrounding them, are explicitly finite. The renormalization conditions from $\langle\sigma(x)\rangle$, $\langle\psi(x)\overline{\psi}(y)\rangle$ and

#14 The argument in this section can be repeated, with some care, using an *arbitrary* subtraction point μ.

$\langle \psi(x)\overline{\psi}(y)\sigma(z) \rangle$ now read respectively

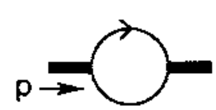
$$
\begin{aligned}
0 &\doteq (\widehat{Z}_1 - \widehat{Z}_2)M + \sqrt{N}\,\langle \widehat{\sigma} \rangle \\
0 &\doteq M\widehat{Z}_1 + \sqrt{N}\,\langle \widehat{\sigma} \rangle - a(0) \\
0 &\doteq \widehat{Z}_1 - a'(0) \\
0 &\doteq \widehat{Z}_2 + c(0) \ ,
\end{aligned}
\tag{2.30}
$$

where the symbol $\doteq$ means equality up to finite parts. In Eq. (2.30) we have used the fact that the last graph in Eq. (2.27) is convergent. These four equations over–determine $\widehat{Z}_1$, $\widehat{Z}_2$ and $\widehat{Z}_3$, and are only consistent provided

$$
0 \doteq a(0) \ + \ M\,c(0) \ . \tag{2.31}
$$

The equality of the divergences in Eq. (2.31) does indeed hold, as can be seen by differentiating Eq. (2.29a) with respect to M, so the consistency is due (as we expect) to the underlying chiral symmetry. What we must prove now is that the $\widehat{Z}_i$ determined from Eq. (2.30) render $\langle \sigma(x)\sigma(y) \rangle$ finite also. This is not at all obvious, since some of the divergent contributions have a complicated momentum dependence. The key to the proof is to use the diagrammatic relation Eq. 2.11b

$$D^{-1}(p^2) = 1/g^2 \ + $$



(2.32)

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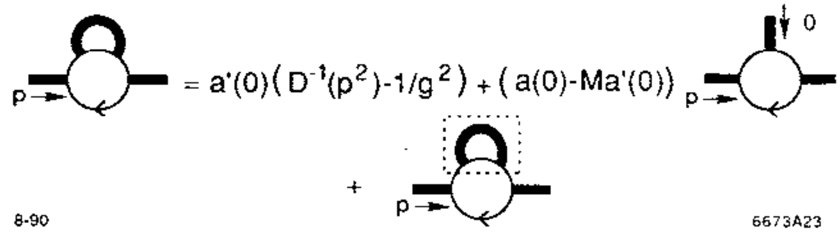
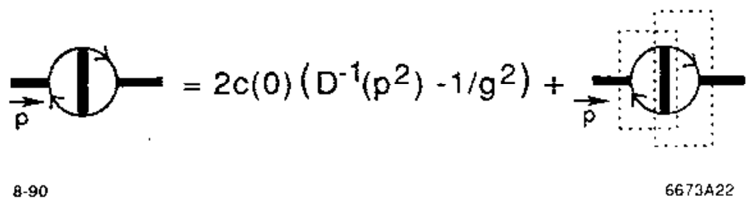
so that



(2.33a)



(2.33b)

Substituting Eq. (2.33) into Eq. (2.26) and using Eq. (2.30) we get an expression for the boson propagator where the possible divergence is, at least, manifestly momentum independent. We check in Appendix E that $\widehat{Z}_3$ as defined by Eqs. (2.24b) and (2.30) does cancel the remaining divergence and once again this is a consequence of chiral symmetry. This completes the somewhat lengthy demonstration that the 4–fermion model is renormalizable at next–to–leading order.

2.7. RG Improvement of $1/N$ Expansion

Equations (2.9) and (2.17) apparently provide a rare example of a finite, non–zero UV fixed point with associated β–function

$$\Lambda \frac{\partial \lambda}{\partial \Lambda} = \frac{\lambda}{\lambda_c} (\lambda - \lambda_c) \; . \tag{2.34}$$

for the coupling $\lambda = g^2 \Lambda$. However, this result does not necessarily establish the UV fixed point beyond leading order in $1/N$, even accepting the all orders renormalizability of the model. This is due to an important difference between the $1/N$

expansion, and the more familiar weak coupling expansion, concerning RG improvement. The point is that in weak coupling perturbation theory the expansion parameter is precisely the quantity whose β–function is calculated. In the case of asymptotic freedom, the expansion parameter is small exactly in the ultra–violet region, so the β–function calculated in the first non–trivial order becomes reliable at a sufficiently high scale. Therefore the RG flow of the coupling near the UV fixed point can be firmly established. In our case however, $1/N$ is not a running coupling. One calculates the β–function of the coupling λ as series in $1/N$ and cannot use the previous reasoning. There is no guarantee a priori that the higher order corrections in the ultra–violet region are less important than the leading order, since there is no connection between large momentum scales and the $N \to \infty$ limit. Thus it could happen, for example, that logarithmic corrections in higher orders force λ_c to infinity analogously to the perturbative running in QED_{3+1}. This would create serious doubts about the reliability of the $1/N$ expansion, and indeed the existence of the phase transition and interacting theory for any finite N.[#15] It is therefore of some interest to see what happens to the finite UV fixed point at next to leading order.

The renormalisation constants Z_i defined in the previous subsection are calculated in Appendix E. In $d = 2 + 1$ we have

$$Z_1 \;=\; 1 - \frac{2}{3\pi^2 N}\ell n\frac{\Lambda}{\mu} \tag{2.35a}$$

$$Z_2 \;=\; 1 + \frac{2}{\pi^2 N}\ell n\frac{\Lambda}{\mu} \tag{2.35b}$$

#15 Alternatively λ_c could be driven to zero, as in QCD, but this seems unlikely since the weak coupling expansion preserves the chiral symmetry to all orders, and does not connect to the broken phase.

$$\frac{Z_3}{g^2} = \frac{2}{\pi^2}(1-\frac{1}{N})\Lambda - \frac{M}{\pi}\left(1+\frac{8}{3\pi^2 N}\ell n\frac{\Lambda}{\mu}\right) \quad . \tag{2.35c}$$

Now the question is what about the appearance of the extra parameter μ in Eq. (2.35). The *exact* S-matrix must depend on just one scale M_{phys} so the bare invariant charge g^2/Z_3 should depend only on Λ and M_{phys}. What has happened is that the leading order parameter M is no longer equal to M_{phys}, and should be thought of as a function of μ.[#16] Differentiating Eq. (2.35c) with respect to μ and setting $\mu = \Lambda$ we get

$$\frac{\mu}{M}\frac{dM}{d\mu} = \frac{8}{3\pi^2 N} \tag{2.36a}$$

i.e. upon integration

$$M(\mu) \sim \mu^{8/3N\pi^2} \quad . \tag{2.36b}$$

Substituting Eq. (2.36b) into Eq. (2.35c) and exponentiating the series the μ dependence drops out and the "improved" invariant charge is now

$$\frac{Z_3}{g^2\Lambda} = \frac{2}{\pi^2}(1-\frac{1}{N}) - \frac{1}{\pi}\left(\frac{M_{\text{phys}}}{\Lambda}\right)^{1-8/3N\pi^2} \quad . \tag{2.37}$$

The finite UV fixed point still exists, albeit shifted downwards by a fraction $1/N$, and the crucial point was the absence of $\Lambda\ell n\Lambda$ divergences in Eq. (2.35c). This is good evidence that the structure of two phases separated by a critical theory survives to finite N. We notice also that the β-function slope has been reduced to $(1-8/3\pi^2 N)$; it must be *positive* for consistency and this holds true even for $N = 1$.

#16 At leading order we had the luxury of choosing a physical normalization condition because the amplitudes were well-behaved for all energy scales E. At subsequent orders such a normalization condition can be defined, but this leads to the problem of large logarithms: the $1/N$ expansion is not *uniform* in E, and this is "cured" by introducing μ.

The deep euclidean behavior of the various Green's functions are just power laws, as in the theory of critical phenomena. The powers can be obtained simply from knowing the renormalization constants in Eq. (2.35), so, for example, the fermion propagator behaves as $1/\not{p}p^{2/3N\pi^2}$. The RG has exponentiated the $\ell n p$ dependence of the "rainbow" diagram in Fig. 4b, and this corresponds to the summation of "nested rainbows." We can codify this power by defining the ultra–violet (critical) dimension of the fermion field

$$[\psi] \;=\; 1 + \frac{1}{3N\pi^2} \quad . \tag{2.38}$$

Similarly, the meson two–point function depends on Z_1 and Z_2, and behaves as $p^{-(1+16/3N\pi^2)}$. Thus

$$[\sigma] \;=\; 1 - \frac{8}{3N\pi^2} \quad . \tag{2.39}$$

The high energy behavior of the connected, truncated Green's function with n fermion legs and m σ–legs is then $\sim E^P$ where $P \;=\; 3 - n[\psi] - m[\sigma]$.

3. Dynamical Symmetry Breaking

In the previous chapter we studied the Gross–Neveu model [26] and the dynamical breaking of its *discrete* chiral symmetry subgroup. The demand of strict renormalizability picked out two special dimensions, $d = 1 + 1$ and $d = 2 + 1$.

Qualitatively new phenomena appear when it is a *continuous* symmetry group which is broken [13], and this is our main subject of interest. In this chapter we consider the simplest 4–fermion theories which have a continuous chiral symmetry, namely the $U(1)_L \times U(1)_R$ model

$$\mathcal{L} = i\overline{\psi}_j \partial\!\!\!/ \psi_j + \frac{g^2}{2N} \left[(\overline{\psi}_j \psi_j)^2 - (\overline{\psi}_j \gamma_5 \psi_j)^2 \right] , \tag{3.1}$$

and its generalizations to $U(n)_L \times U(n)_R$. In the $1/N$ expansion there is a range of values of g^2 for which the global symmetry in (3.1) is dynamically broken to the subgroup $U(n)_V$ in $d = 2+1$. This produces composite Goldstone bosons which are exactly massless to all orders in $1/N$, and we compare and contrast the analysis to the analogous case in $d = 1+1$. Finally, we explore the interactions of the Goldstone bosons with themselves and the elementary fermions, and calculate the low energy effective chiral lagrangian.

3.1. Composite Goldstone Bosons

The derivation of the $1/N$ expansion is similar to that in the discrete Gross–Neveu model. In the case of lagrangian (3.1) we need, however, *two* auxiliary fields:

$$\mathcal{L} = i\overline{\psi}\partial\!\!\!/\psi - \overline{\psi}(\sigma + i\boldsymbol{\pi}\gamma_5)\psi - \frac{N}{2g^2}(\sigma^2 + \boldsymbol{\pi}^2) . \tag{3.2}$$

For notational clarity we drop the "flavour" index j on the fermions. The global $U(1)_V \times U(1)_A$ invariance is given by

$$\begin{aligned} \psi_L &\to e^{i\alpha_L}\psi_L \\ \psi_R &\to e^{i\alpha_R}\psi_R \\ (\boldsymbol{\sigma} + i\boldsymbol{\pi}) &\to e^{i(\alpha_L - \alpha_R)}(\boldsymbol{\sigma} + i\boldsymbol{\pi}) . \end{aligned} \tag{3.3}$$

For all values of the coupling g^2 the effective potential can be obtained from the discrete case version by the simple replacement $\sigma^2 \to \boldsymbol{\sigma}^2 + \boldsymbol{\pi}^2$. Thus for the phase of dynamically broken chiral symmetry we tune g^2 exactly as in (2.9) and obtain as the cutoff is removed

$$V(\boldsymbol{\sigma}, \boldsymbol{\pi}) = \frac{1}{\pi} \left[\frac{1}{3}\left|\sqrt{\boldsymbol{\sigma}^2 + \boldsymbol{\pi}^2}\right|^3 - \frac{M}{2}\left(\boldsymbol{\sigma}^2 + \boldsymbol{\pi}^2\right)\right] . \tag{3.4}$$

The effective potential has a minimum away from the origin, so one linear combination of the composite fields will get a non-zero expectation value. We arbitrarily pick $\langle\boldsymbol{\pi}\rangle = 0$, hence $\langle\boldsymbol{\sigma}\rangle = M$, and the fermions once again have mass M. The Feynman rules are as in Fig. 2, supplemented by a $FF\boldsymbol{\pi}$ vertex carrying a weight γ_5. The pseudo–scalar $\boldsymbol{\pi}$ has also an induced propagator from the double derivative of the effective action, analogously to the scalar field $\boldsymbol{\sigma}$. The γ_5 coupling makes a crucial difference, however, and we have

$$\begin{aligned} D_{\boldsymbol{\pi}}(p^2) &= \frac{\hbar}{N}\left[\frac{1}{g^2} - i\hbar\int \frac{d^3q}{(2\pi)^3}\,\mathrm{tr}\, i\gamma_5(\not{q} - M)^{-1} i\gamma_5(\not{q} - \not{p} - M)^{-1}\right] \\ &= \frac{2\pi}{N}\,\frac{1}{\sqrt{-p^2}\,\tan^{-1}(\sqrt{-p^2}/2M)} . \end{aligned} \tag{3.5}$$

Thus in the large N limit we have not only a bound–state meson of mass $2M$ but also one of mass zero. Here is a strikingly explicit example of Goldstone's theorem in action.

The composite nature of the Goldstone boson can be seen from the *cut* in $D_{\boldsymbol{\pi}}(p^2)$ at the two fermion threshold. This provides a non–trivial "spectral function," which can be calculated from (3.5) through the use of a dispersion relation. We have

$$D_{\boldsymbol{\pi}}(p^2) = \frac{-4\pi M}{N}\left[\frac{1}{p^2} + \int_{4M^2}^{\infty} \frac{\rho(s)\,ds}{p^2 - s + i\epsilon}\right] ,$$

where

$$\rho(s) = \frac{1}{M\sqrt{s}} \left[\ell n^2 \left| \frac{\sqrt{s}+2M}{\sqrt{s}-2M} \right| + \pi^2 \right]^{-1} . \tag{3.6}$$

The spectral function $\rho(s)$ is real and positive and has support only for $s \geq 4M^2$. This is in accordance with the "asymptotic theory," due to Källen and Lehmann, describing quantum fields which interpolate stable physical particles [2.4,2.5].

3.2. Proof of Goldstone's Theorem to all orders in $1/N$

To discuss the properties of the $\boldsymbol{\pi}$–particles to higher orders in $1/N$, we must first renormalize all fermionic and mesonic Green's functions consistently with chiral symmetry. Since the deep euclidean behavior of the $\boldsymbol{\pi}$ and $\boldsymbol{\sigma}$ propagators are the same, the renormalization requires the introduction of three subtraction constants, just as in the discrete model of Section 2. The proof of renormalizability goes over *mutatis mutandis*, and we do not repeat it. The "bare" lagrangian is

$$\mathcal{L}_{\text{bare}} = iZ_1\overline{\psi}\partial\!\!\!/\psi - Z_2\overline{\psi}(\boldsymbol{\sigma} + i\boldsymbol{\pi}\gamma_5)\psi - \frac{NZ_3Z_2^2}{2g^2Z_1^2}\,(\boldsymbol{\sigma}^2 + \boldsymbol{\pi}^2)\ , \tag{3.7}$$

where $Z_1,\ Z_2,\ Z_3$ are written as a power series in $1/N$. In order to choose the $\boldsymbol{\sigma}$–direction for the breaking of the symmetry we couple in a constant, external source h which is later to be removed. Thus we consider the renormalized functional integral

$$\begin{aligned} Z_{\text{ren}}(\zeta,\overline{\zeta},J_\sigma,J_\pi,h) \equiv \int D\psi\, D\overline{\psi}\, D\boldsymbol{\sigma}\, D\boldsymbol{\pi} \exp i \int d^3x \Big[&\mathcal{L}_{\text{bare}} + \\ &\overline{\zeta}\psi + \overline{\psi}\zeta + (h + J_\sigma)(\boldsymbol{\sigma} - \langle\sigma\rangle_h) + J_\pi\boldsymbol{\pi} \Big] . \end{aligned} \tag{3.8}$$

where $\langle\boldsymbol{\sigma}\rangle_h$ is the *exact* expectation value of the mesonic field in the presence of h. Clearly $\langle \boldsymbol{\pi} \rangle_h$ vanishes identically, since it involves a Dirac trace over an odd number[#17] of γ_5's.

We can now generate an infinite series of Ward identities which reflect the dynamical breaking of chiral symmetry. Consider a change of integration variables in (3.8) corresponding to an infinitesimal, *local* $U(1)_A$ rotation, (i.e. let $\alpha_L = -\alpha_R = \alpha(x)$ in (3.3).) It is easily proved [3.1] that such a change of variables leaves $Z_{\rm ren}$ invariant, when understood as a generator of "Feynman-like" diagrams. The jacobian of the transformation is unity, so the response comes purely from $\mathcal{L}_{\rm bare}$ and the source terms. At linear order in $\alpha(x)$ we obtain the functional identity

$$0 = \left\langle Z_1 \partial^\mu (\overline{\psi} \gamma_\mu \gamma_5 \psi)_x + i(\overline{\zeta} \gamma_5 \psi + \overline{\psi} \gamma_5 \zeta)_x - 2[(h + J_\sigma) \boldsymbol{\pi} - J_\pi \sigma]_x \right\rangle_{\rm ren} . \qquad (3.9)$$

Note that (3.9) indicates that Green's functions with an operator insertion of $J_{\mu 5}(x) \equiv Z_1 \overline{\psi} \gamma_\mu \gamma_5 \psi(x)$ are already renormalized, so that the "axial current" requires no further subtractions [2.11].

The identity (3.9) contains a lot of information which is hard to derive directly from the Feynman diagrams. We extract the relevant equations by functionally differentiating with respect to the sources ξ, J_σ and J_π some number of times and then setting the sources to zero. The first such equation is that of current conservation $0 = \partial^\mu \langle J_{\mu 5}(x) \rangle_h$, which is the quantum analogue of Nöther's theorem. Differentiating (3.9) with respect to J_π once we derive

$$0 = i\, \partial_x^\mu \, \langle J_{\mu 5}(x) \boldsymbol{\pi}(y) \rangle_h - 2h \, \langle \boldsymbol{\pi}(x)\, \boldsymbol{\pi}(y) \rangle_y + 2 \langle \sigma \rangle_h \delta^3(x - y) \; , \qquad (3.10)$$

and this is the equation which proves the masslessness of the pions. Multiplying

#17 γ_5 counting shows that all purely mesonic correlation functions with an odd number of $\boldsymbol{\pi}$ legs vanish.

through by the inverse of the $\boldsymbol{\pi}$–propagator, and taking a Fourier transform to momentum space, Eq. (3.10) becomes

$$0 = p^{\mu} \left\langle J_{\mu 5}(p)\boldsymbol{\pi}(-p)\right\rangle^{h}_{\text{trunc}} - 2h + 2\langle\sigma\rangle_h \Gamma^{h}_{\pi\pi}(p^2) \ . \tag{3.11}$$

Provided the limit $h \to 0$ is smooth in all Green's functions the result is at hand. (This infra–red regularity is indeed expected in 2+1 dimensions.) By Lorentz invariance the matrix element $\langle J_{\mu 5}(p)\boldsymbol{\pi}(-p)\rangle_{\text{trunc}}$ is of the form $-ip_{\mu}f(p^2)$, so as the external field is removed we have

$$\Gamma^{h}_{\pi\pi}(p^2) = \frac{ip^2 f(p^2)}{2\,\langle\sigma\rangle} \ . \tag{3.12}$$

Finally we must argue that $f(p^2)$ goes to a finite constant as $p^2 \to 0$, in which case the $\boldsymbol{\pi}$–propagator has a pole at exactly zero momentum. This is certainly the case at leading order in $1/N$, where $f(p^2)$ is calculated from the diagram in Fig. 6a.

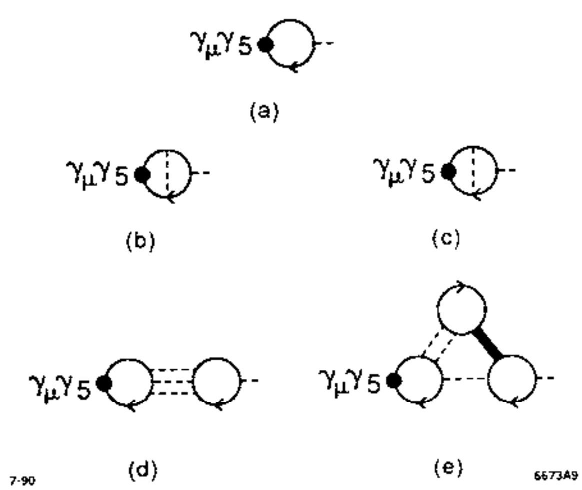


Figure 6. Insertions of Axial Current.

This yields

$$f(p^2) = \frac{NM \tan^{-1}(\sqrt{-p^2}/2M)}{\pi\sqrt{-p^2}} \; p^2 \underset{=}{\rightarrow} 0 \; \frac{N}{2\pi} \,, \tag{3.13}$$

in agreement with (3.5). The residue of the pole is $4\pi M/N$ so the conventionally normalized $\boldsymbol{\pi}$-field is obtained by rescaling by $\sqrt{4\pi M/N}$. The on-shell matrix element $\langle 0|J_{\mu 5}|\boldsymbol{\pi}(p)\rangle$ then reduces to $-ip_\mu f_\pi$, where

$$f_\pi = \sqrt{\frac{MN}{\pi}} \,. \tag{3.14}$$

This quantity is known as the "pion decay constant" and plays a fundamental role in the phenomenology of the pion interactions.

Beyond leading order $f(p^2)$ has potentially dangerous contributions from diagrams with internal pion lines. (Clearly intermediate fermions and $\boldsymbol{\sigma}$-mesons cause no problem since their propagators are massive.) At O(1) we have the graphs of Fig. 6b,c but in fact these are infra-red finite by power-counting. The first interesting cases occur at O(1/N), where the diagrams of Fig. 6d,e separately diverge as $\ell n\ p^2$. At least there are no contributions with a *single* intermediate pion, since these have been removed by the truncation, so we do not have to worry about simple poles. The key point now is that the *on shell* pion amplitudes $\boldsymbol{\pi} \rightarrow 3\boldsymbol{\pi},\ 5\boldsymbol{\pi},\ \ldots$ $(2n+1)\boldsymbol{\pi}$ all *vanish*, since the pions are effectively derivatively coupled. This is verified explicitly at the leading order for each of these processes, i.e. order $1/N^n$, in subsection 3.6. Thus the sum of the right-hand sub-diagrams of Fig. 6d,e actually cancels on shell, and the $O(1/N)$ contribution to $f(p^2)$ is analytic at the origin.

This proves Goldstone's theorem up to *two* sub-leading orders in $1/N$, thankfully without having to calculate $\Gamma_{\pi\pi}(p^2)$ directly. To complete the argument to all orders in $1/N$ is a little tricky. It is crucial that the π-fields remain derivatively coupled, and a discussion of this point is deferred to Section 3.6, which describes the method of effective lagrangians.

3.3. Comparison to 1+1: How Coleman's Theorem Works

The derivation from (3.7) to (3.11) is valid order by order in $1/N$ in *both* $d = 1 + 1$ and $d = 2 + 1$, for it is essentially a combinatoric argument. The crucial difference, however, between the two dimensions comes when we try to take the limit $h \to 0$ in (3.11). In $d = 1 + 1$ the limit $h \to 0$ is potentially infra-red singular. It is true that at leading order in large N the π-particle has mass zero and $\langle\sigma\rangle \neq 0$, but the higher-order terms are divergent and the identities from (3.11) are rendered invalid. As examples, the two next-to-leading order graphs in Fig. 7 are logarithmically divergent in the infra-red, and cause the standard $1/N$ expansion to fail. (In $d = 2 + 1$ the diagrams are infra-red safe.)

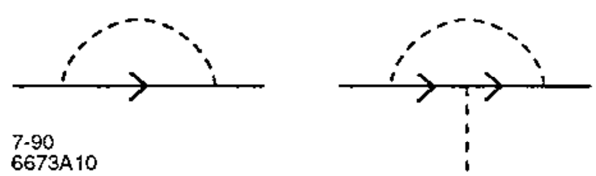


Figure 7. Infra-red divergent diagrams in 1+1.

As was argued by Witten [3.2], the fields σ and π are no longer good interpolating fields as $h \to 0$. In particular the π-field has a spacetime correlation function which grows with distance, and this is unphysical. Witten suggested that the appropriate quantum fields are the modulus and phase of $(\boldsymbol{\sigma} + i\boldsymbol{\pi})$, so we consider the change of variables

$$
\begin{aligned}
\boldsymbol{\sigma} + i\boldsymbol{\pi} &\equiv e^{i\theta}\rho \\
\chi_L &\equiv e^{-i\theta/2}\psi_L \\
\chi_R &\equiv e^{i\theta/2}\psi_R \,.
\end{aligned}
\tag{3.15}
$$

The functional integral for the $U(1) \times U(1)$ model can then be formally rewritten as

$$
\begin{aligned}
Z &= \int D\chi\, D\overline{\chi}\, D\rho^2 D\theta \exp i \int d^3x\, \mathcal{L}(\chi, \rho, \theta) \\
\mathcal{L}(\chi, \rho, \theta) &= \overline{\chi}\left(i\partial\!\!\!/ - \frac{1}{2}\gamma_5 \partial\!\!\!/\theta - \rho\right)\chi - \frac{N}{2g^2}\,\rho^2 \,.
\end{aligned}
\tag{3.16}
$$

In the large N limit, at least, this redefinition makes sense. The modulus field $\rho(x)$ gets an expectation value M, but this is now in accordance with Coleman's theorem [29,30] since ρ is *neutral* under $U(1)_A$. The fermions χ get an induced mass and are also neutral. The θ–field has a propagator proportional to the old $\boldsymbol{\pi}$–propagator,

$$
\begin{aligned}
D_\theta(p^2) &= \frac{i\pi}{M^2 N}\,\frac{\sqrt{-p^2+4M^2}}{\sqrt{-p^2}}\left[\ell n\left(\frac{\sqrt{-p^2+4M^2}+\sqrt{-p^2}}{\sqrt{-p^2+4M^2}-\sqrt{-p^2}}\right)\right]^{-1} \\
&\underset{=}{p^2 \to 0} \quad \frac{-2\pi i}{Np^2}\,,
\end{aligned}
\tag{3.17}
$$

and since its interactions in (3.16) are purely derivative the infra–red divergences have been "absorbed." The θ–field still has a bad spacetime correlator, but the physical quantum field is not $\theta(x)$, rather it is $e^{i\theta(x)}$. Since the interactions are suppressed by powers of $1/N$ we have

$$\left\langle e^{-i\theta(x)} e^{i\theta(0)} \right\rangle = \exp\left\langle \theta(x)\theta(0) \right\rangle \simeq \exp -\frac{1}{2N} \ell n(x^2) \ , \tag{3.18}$$

and this is now an acceptable correlation function since it decays with distance for any finite N. The phase field is closely related to, but is "not quite," a Goldstone boson, and was first conjectured to exist in the context of the 2D XY–model in statistical mechanics [31].

Beyond leading order in $1/N$ the above analysis is problematic, since the correlation functions from (3.16) are *not* manifestly renormalizable. This is due to the local operator products in (3.15), and one imagines that a "smeared out" version of the fields will define sensible amplitudes. (This is a reasonable hope, from the evidence that the S–matrix is factorized [3.3], and the θ–field decouples exactly [3.4].[#18])

3.4. Low–Energy Theorems, Goldberger–Treiman Relation and Other Consequences of DSB

Let us now return to $d = 2 + 1$ to explore the various consequences of the symmetry breaking. For fermion–Goldstone boson interactions the Feynman rules

---

#18 The argument presented in [3.4] uses the serendipitous technique of bosonization [3.5]. The lagrangian (3.1) is apparently equivalent to the bosonic form

$$\mathcal{L} = \sum_{a=1}^{N} \frac{1}{2} (\partial_\mu \phi_a)^2 + \mu_0^2 \sum_{a,b} \cos(2\sqrt{\pi}(\phi_a - \phi_b)) \ .$$

Clearly the potential depends only on the differences of the bosons ϕ_a, so there is a combination $\Phi = \sum \phi_a$ which is decoupled (and massless).

directly give the $\pi F\overline{F}$ coupling to be $i\gamma_5$ when the π field is normalized as in (3.5). Rescaling to make the pole residue unity, we get a physical coupling

$$g_{\pi F\overline{F}} = \sqrt{\frac{4\pi M}{N}} \; . \tag{3.19}$$

For the analogue of PCAC the matrix element of $J_{\mu 5}$ between two fermionic states is given by the diagrams of Fig. 8. The possible tensors are

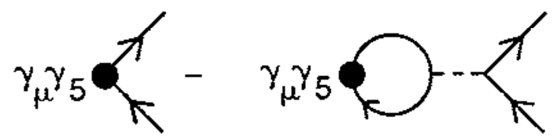


Figure 8. PCAC

$$\langle\psi(P')\,|\,J_{\mu 5}\,|\,\psi(P)\rangle = i\overline{U}(P')\;\left[g_A(q^2)\gamma_\mu\gamma_5 + g_P(q^2)\gamma_5 q_\mu + g_s(q^2)\epsilon_{\mu\nu\rho}q_\nu\gamma_\rho\right]\;U(P) \tag{3.20}$$

and the direct calculation gives

$$g_A = 1\,, \quad g_P = -\frac{2M}{q^2}\,, \quad g_s = 0\;. \tag{3.21}$$

Note that the cut due to the fermion/anti-fermion intermediate state cancels and (3.21) is correct for all q^2. This is in accordance with the Ward identities generated from (3.9).#19 Recalling the "pion decay constant" f_π from (3.14) we obtain the

#19 The situation is a little peculiar in 1 + 1 dimensions where the RHS of (3.20) cancels identically for on-shell spinors [3.3]. This supports the contention in subsection 3.3 that the true physical states are massive, neutral fermions.

analogue of the Goldberger–Treiman relation [3.6]

$$f_\pi = \frac{2Mg_A}{g_{\pi F\overline{F}}} \, . \tag{3.22}$$

We can now compute the on–shell scattering amplitude $\mathcal{A}$ for $F\pi \to F\pi$. This is given by the diagrams in Fig. 9.

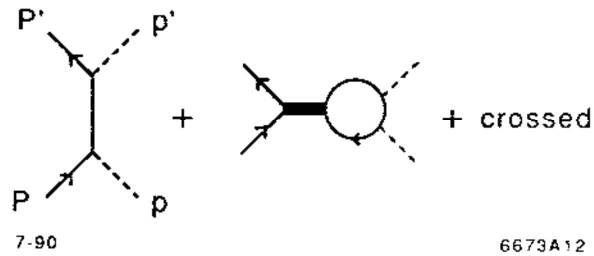


Figure 9. $F\pi \to F\pi$ scattering amplitude

$$\begin{aligned}\mathcal{A} = -\frac{4\pi Mi}{N}\overline{U}(P',s')\Bigg[\frac{\not{p}}{2P\cdot p} + \frac{\not{p}'}{2P\cdot p'} - \frac{4M}{(-q^2+4M^2)} \\ - \frac{\sqrt{-q^2}}{(-q^2+4M^2)} \cdot \frac{\ell n(1-q^2/4M^2)}{\tan^{-1}(\sqrt{-q^2}/2M)}\Bigg] U(P,s) \, .\end{aligned} \tag{3.23}$$

For small pion momenta the amplitude goes to *zero*, in accordance with "low–energy theorems" [3.7,8]. Indeed, the spin–averaged cross–section is purely p–wave in this limit, and is given by

$$\frac{d\sigma}{d\theta}\,(|\overrightarrow{p}| \ll M) = \frac{\pi}{2N^2}\,\frac{|\overrightarrow{p}|}{M^2}\,\sin^2\theta \, . \tag{3.24}$$

In order to investigate the interactions of Goldstone bosons with themselves, we generalize the model to have a $U(n)_L \times U(n)_R$ symmetry. This can be done by promoting $\boldsymbol{\sigma}$ and $\boldsymbol{\pi}$ in (3.2) to $n \times n$ hermitian matrices and letting the fermions be in the fundamental of $U(n)$. So we consider the lagrangian

$$\mathcal{L} = i\overline{\psi}\partial\!\!\!/\psi - \overline{\psi}(\underline{\boldsymbol{\sigma}} + i\underline{\boldsymbol{\pi}}\gamma_5)\psi - \frac{N}{2g^2}\,\mathrm{tr}\,(\underline{\boldsymbol{\sigma}}^2 + \underline{\boldsymbol{\pi}}^2)\ , \tag{3.25}$$

where $\underline{\boldsymbol{\sigma}} = \sigma^\alpha \mathbf{T}^\alpha$, and similarly for $\underline{\boldsymbol{\pi}}$, and $\mathbf{T}^\alpha$ are the generators of $U(n)$[#20]. The rigid invariance is given by

$$\begin{aligned} \psi_L &\to L\,\psi_L \\ \psi_R &\to R\,\psi_R \\ (\underline{\boldsymbol{\sigma}} + i\underline{\boldsymbol{\pi}}) &\to L(\underline{\boldsymbol{\sigma}} + i\underline{\boldsymbol{\pi}})\,R^+\ , \end{aligned} \tag{3.26}$$

which is dynamically broken to $U(n)_V$ when the coupling is tuned as in (2.9). In

#20 $\mathbf{T}^0 = \frac{1}{\sqrt{n}}\mathbf{1}$, $\mathbf{T}^a = \sqrt{2}T^a$ where T^a are the conventionally normalized generators of $SU(n)$ in the fundamental representation [3.9].

this phase the Feynman rules are given in Fig. 10. The propagators are essentially

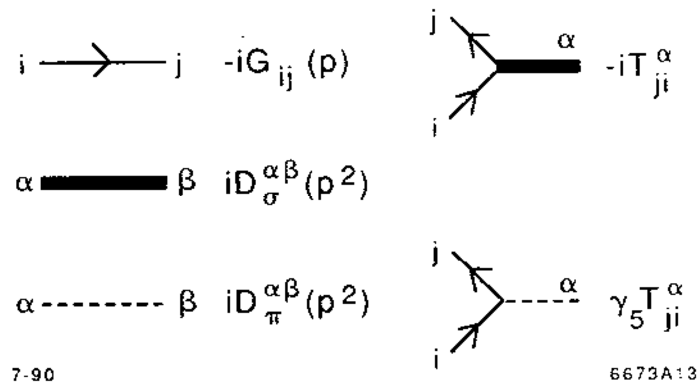


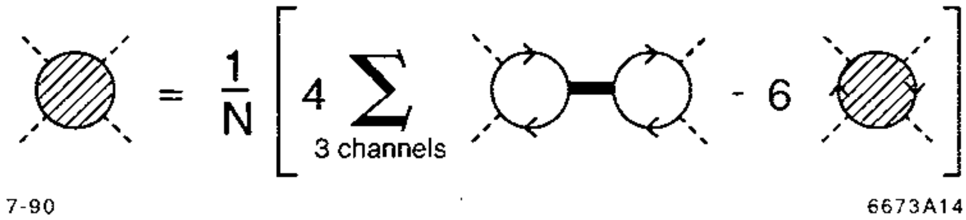
Figure 10. Feynman rules in $U(n)_L \times U(n)_R$ model.

the same as before, but now have color factors,

$$
\begin{aligned}
G_{ij}(p) &= \delta_{ij}(\not p - M)^{-1} \\
D_{\sigma}^{\alpha\beta}(p^2) &= \delta_{\alpha\beta}\,\frac{2\pi}{N}\,\frac{\sqrt{-p^2}}{(-p^2+4M^2)\tan^{-1}(\sqrt{-p^2}/2M)} \\
D_{\pi}^{\alpha\beta}(p^2) &= \delta_{\alpha\beta}\,\frac{2\pi}{N}\,\frac{1}{\sqrt{-p^2}\tan^{-1}(\sqrt{-p^2}/2M)}\,.
\end{aligned}
\tag{3.27}
$$

The $\underline{\pi}$ fields interpolate the n^2 Goldstone bosons corresponding to the n^2 broken generators of $U(n)_A$.

The 4-point function of pions is given at leading order in $1/N$ by the diagrams



(3.28)

The amplitude is expandable in momenta, since at this order there are no intermediate massless particles. At precisely zero momentum the two diagrams in (3.28) exactly cancel, and the matrix element for $\underline{\pi\pi} \to \underline{\pi\pi}$ vanishes. This is dictated by the underlying chiral symmetry, and holds, in fact, for all n–point Green's functions of pions. At second order in momenta the *on–shell* amplitude must be a linear function of the Mandelstam variables s, t, u which satisfies Bose symmetry and crossing. This constrains the S–matrix to vanish unless all four pions belong to the $SU(n)_A$ subgroup of $U(n)_A$, in which case we have

$$A^{(2)}(\pi^a\,\pi^b \to \pi^c\,\pi^d)$$
$$= -\frac{2\pi}{3NM}\ \left\{ f^{ade} f^{bce}(s-t) + f^{abe} f^{cde}(t-u) + f^{ace} f^{dbe}(u-s) \right\}\ . \tag{3.29}$$

The overall magnitude in (3.29) is not, of course, restricted by Bose symmetry and crossing, but *is* determined by chiral symmetry [3.7]. The coefficient of the square brackets must be $2/3f_\pi^2$, where f_π is the pion decay constant. In (3.2) we directly computed $f_\pi = \sqrt{NM/\pi}$, so Eq. (3.29) is in agreement with the general current algebra arguments [3.7].

With regard to continuing the Taylor expansion of the amplitude (3.28), the greatest "phenomenological" interest lies in the next term, i.e. fourth order in derivatives [3.10]. We have

$$
\begin{aligned}
A^{(4)}(\pi^a\pi^b \to \pi^c\pi^d) &= \frac{1}{4}\,\frac{\pi}{NM^3}\,(s^2+t^2+u^2)\,\left(\frac{1}{n}\,S^{abcd}+\frac{1}{2}\,D^{abcd}\right) \\
&\qquad - \frac{5}{12}\,\frac{\pi}{NM^3}\left\{s^2\left(\frac{1}{n}\,\delta^{ab}\delta^{cd}+\frac{1}{2}\,d^{abe}d^{cde}\right)+\text{crossed}\right\} \\
A^{(4)}(\pi^0\pi^0 \to \pi^0\pi^0) &= \frac{1}{3}\,\frac{\pi}{NM^3}\,(s^2+t^2+u^2)\,.
\end{aligned}
\tag{3.30}
$$

In (3.30) the tensor D^{abcd} denotes $(d^{abe}d^{cde}+d^{ace}d^{bde}+d^{ade}d^{bce})$ and S^{abcd} denotes $(\delta^{ab}\delta^{cd}+\delta^{ac}\delta^{bd}+\delta^{ad}\delta^{bc})$. For the cases of two $U(1)_A$ – two $SU(n)_A$ pions, and one $U(1)_A$ – three $SU(n)_A$ pions the amplitudes are given by (3.30), multiplied by the color factors δ^{ab} and $\sqrt{n/2}\;d^{abc}$ respectively.

3.6. THE METHOD OF EFFECTIVE LAGRANGIANS

The low energy theorems are not confined to $\pi\pi \to \pi\pi$ scattering, indeed there is a prediction for the low energy behavior of all n–point pion amplitudes. It would be quite tedious to verify the predictions case by case, and so it would be advantageous to have a formalism which encompasses all the amplitudes at once. Thus we are led to the notion of an "effective lagrangian" [3.11].

A low energy effective lagrangian is a *quasi–local* functional which reproduces the correct low energy amplitudes of a given model. The demand that the lagrangian be expandable in derivatives precludes us from considering $\Gamma_{1\text{PI}}(\underline{\boldsymbol{\pi}})$, the generator of one–particle irreducible diagrams. This is because the presence of multi–pion intermediate states renders $\Gamma_{1\text{PI}}(\underline{\boldsymbol{\pi}})$ to be highly non–analytic. Thus we need to construct an effective Lagrangian whose *full* Feynman diagram expansion (not just "trees") yields the S–matrix. This definition does not specify the lagrangian uniquely, due to the freedom of local field redefinitions [3.12], so in any *calculation* of an effective lagrangian one must first specify the choice of field variables.

One way to define an effective lagrangian for the pions in our model (3.25) is simply to integrate over the massive fields ψ and $\underline{\sigma}$. That is define

$$\exp\, i \int d^3x\, \mathcal{L}_{\text{eff}}(\underline{\pi}) \equiv \int D\underline{\sigma}\, D\psi\, D\overline{\psi} \exp\, i \int d^3x\, \mathcal{L}(\psi, \overline{\psi}, \underline{\sigma}, \underline{\pi}) \,. \tag{3.31}$$

Clearly $\mathcal{L}_{\text{eff}}(\underline{\pi})$ does yield the correct S-matrix for the Goldstone bosons, and it is explicitly expandable in derivatives. Moreover, it also provides renormalizable off-shell amplitudes to all orders in $1/N$. However, $\mathcal{L}_{\text{eff}}(\underline{\pi})$ does not provide a non-linear realization of the chiral symmetry (3.26), and the expected properties of the low energy scattering amplitudes are not manifest.

From a geometric point of view the $\underline{\pi}$ fields live in the "wrong" space, at least with regard to the manifestation of chiral symmetry. $\underline{\pi}$ takes values in the *tangent* to the coset space $U(n)_L \otimes U(n)_R / U(n)_V$ upon which the "canonical" chiral lagrangian is naturally defined [3.3,3.11]. To bring out this structure we need a non-linear change of variables. At least in leading order in $1/N$ this will be a well-defined procedure for the pion S-matrix. However, see below for some comments on the nature of this step.

First, define a unitary matrix ξ, and a hermitian, positive semi-definite matrix ρ as the "polar decomposition" of the fields $\underline{\sigma}$ and $\underline{\pi}$ [3.7,3.13]

$$\underline{\sigma} + i\underline{\pi} = \xi\rho\xi \,. \tag{3.32}$$

On making a *local*, chiral rotation on the fermion fields, $\chi_L \equiv \xi^\dagger \psi_L, \chi_R \equiv \xi\psi_R$, the lagrangian becomes

$$\mathcal{L} = \overline{\chi}(i\not\partial + i\not V(\xi) - \gamma_5 \not A(\xi) - \rho)\,\chi - \frac{N}{2g^2}\,\mathrm{tr}\,\rho^2 \ ,$$

where

$$\begin{aligned} V_\mu(\xi) &\equiv \frac{1}{2}\,(\xi^\dagger \partial_\mu \xi + \xi \partial_\mu \xi^\dagger) \\ A_\mu(\xi) &\equiv -\frac{i}{2}\,(\xi^\dagger \partial_\mu \xi - \xi \partial_\mu \xi^\dagger) \ . \end{aligned} \tag{3.33}$$

The rigid $U(n)_L \times U(n)_R$ invariance (3.26) then transpires to have a hidden *local* form

$$\xi \to L\xi U^\dagger \equiv U\xi R^\dagger \tag{3.34a}$$

$$\begin{aligned} \chi &\to U\chi \\ \rho &\to U\rho U^\dagger \\ A_\mu &\to UA_\mu U^\dagger \\ V_\mu &\to UV_\mu U^\dagger + U\partial_\mu U^\dagger \ , \end{aligned} \tag{3.34b}$$

where the unitary matrix U is a nonlinear function of the field ξ as defined by Eq. (3.34a). Integration over the fermions χ now yields a form suitable for a large N analysis:

$$\begin{aligned} Z = &\int D\rho\, D\xi (\text{Jacobian}) \\ &\times \exp N \left[\frac{-i}{2g^2}\int d^3x\,\mathrm{tr}\,\rho^2 - \mathrm{tr}\,\ell n(i\not\partial - \rho + i\not V - \gamma_5 \not A)\right] \ . \end{aligned} \tag{3.35}$$

The jacobian in (3.35) comes from the non–linear transformation (3.32) and is rather complicated. Fortunately it is *independent* of N, and so when exponentiated it does not affect the leading–order stationary phase approximation. There is also in principle a jacobian from the local chiral rotation $\psi \to \chi$, but this is in fact unity, as can be checked explicitly using a Pauli–Villars regulator. This last corresponds to the statement that there is no "continuous anomaly" in $2+1$ dimensions [3.14].

The canonical chiral lagrangian $\mathcal{L}_{\text{chiral}}\,(\xi)$ is at last obtained by integration over the massive fields ρ in (3.35). Heuristically, Eqs. (3.32)–(3.35) show that $\mathcal{L}_{\text{chiral}}(\xi)$ is lagrangian that (a) reproduces the S–matrix of the Goldstone bosons, (b) is expandable in derivatives, and (c) respects the local "gauge" symmetry (3.35). Thus, the only terms that appear are built out of the gauge covariant quantities A_μ and $D_\mu \equiv \partial_\mu + [V_\mu, \quad]$, and this is the celebrated result of Ref. [3.11].

At leading order in $1/N$ all contributions to $\mathcal{L}_{\text{chiral}}(\xi)$ are explicitly finite, and these conclusions about its properties are valid. (This proves the claim in subsection 3.2 that the pions are derivatively coupled for all processes.) Before presenting the results of the leading–order calculation, however, we comment on the difficulties involved in justifying the analysis at higher orders in $1/N$.

The main problem is that the coefficients in the chiral lagrangian will be ultraviolet divergent, so the notion of calculating the coefficients in $\mathcal{L}_{\text{chiral}}(\xi)$ becomes obscure. One could perhaps avoid this "technicality" by considering a "Wilsonian" version of the chiral lagrangian: the definition would now include an "integration out" of the high momentum modes of ξ, leaving a *finite* cutoff $\Lambda \sim f_\pi^2$. For this purpose we would need a gauge invariant cutoff regularization, such as that constructed in Ref. [3.15]. It is plausible that this modification may allow the calculation of systematic corrections in $1/N$ to the leading–order result. Secondly, the non–linear transformation from $(\underline{\pi}, \underline{\sigma})$ to the (ρ, ξ) involves nonpolynomial operator products.

While the off-shell correlation functions of the π-fields are renormalizable to all orders in $1/N$ and are dependent (up to normalization) on only one parameter f_π, these properties do *not* hold for the correlators of ξ (or $\ell n\,\xi$) beyond leading order. At each successive order in $1/N$ more and more "unphysical" subtraction constants appear in $\mathcal{L}_{\text{chiral}}(\xi)$. These constants will drop out the S-matrix of pions (by Borcher's theorem [3.12]) and so from this point of view do not matter. However, if one wishes to treat $\mathcal{L}_{\text{chiral}}(\xi)$ semi-classically and look for solitonic solutions, as in the Skyrme model of hadrons [3.16], then it is far from clear that the physics is indeed independent of the extra parameters.#21

Returning to Eq. (3.35), we integrate over ρ to obtain $\mathcal{L}_{\text{chiral}}(\xi)$. At second order in derivatives of ξ one can just set ρ to its vacuum expectation value and take A_μ, V_μ to be constants, then

$$\mathcal{L}^{(2)}_{\text{chiral}}(\xi) = \frac{N}{2}$$

$$= \frac{NM}{2\pi}\,\mathrm{tr}\,A_\mu^2$$

$$= \frac{1}{8f_\pi^2}\,\mathrm{tr}\,\partial_\mu\,\xi^2\,\partial_\mu\,\xi^{-2} \qquad (3.36)$$

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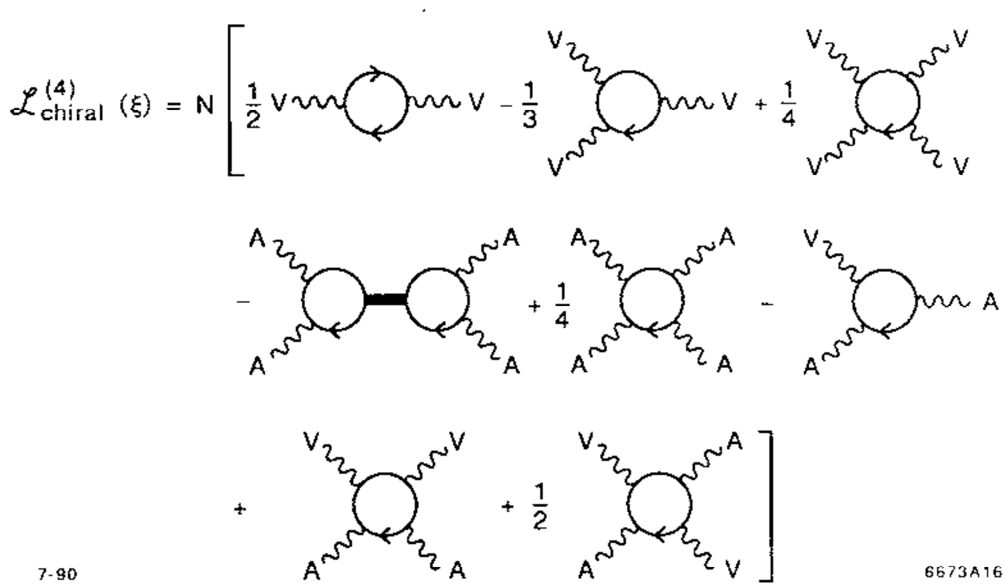
This calculation recovers the familiar action of the non-linear σ-model. The prop-

#21 One may be able to address this question in the context of the 4-fermion model due to Kovner and Eliezer [3.17]. Their symmetry breaking pattern is $SU(N) \to U(N-1)$, so the "low energy manifold" is $CP(N-1)$. This admits of solitons in $2+1$ dimensions.

erly normalized Goldstone boson fields θ^α are given by $\xi = \exp(i\theta^\alpha \mathbf{T}^\alpha / f_\pi)$. At fourth order in derivatives the contributing diagrams are



$$= -\frac{N}{24\pi M}\ \mathrm{tr}\ \left\{ A^2A^2 + 3A_\mu A_\nu A_\mu A_\nu + (D_\mu A_\nu)^2 - 2(D_\mu A_\nu)(D_\nu A_\mu) \right\}\ . \quad (3.37)$$

It is a nice consistency check on the calculation that the $2 \to 2$ scattering amplitude derived from (3.37) does agree with that found previously in Eqs. (3.29)–(3.30).

4. Explicit Versus Dynamical Symmetry Breaking

In Chapter 3 we studied the *dynamical* breaking of chiral $U(n)_L \times U(n)_R$ symmetry down to $U(n)_V$ in the models Eqs. (3.2)–(3.25). It was shown that there exist exactly massless Goldstone bosons $\underline{\boldsymbol{\pi}}$ and massive bound-state mesons $\underline{\boldsymbol{\sigma}}$. In this chapter we explore the consequences of adding bare terms to the 4–fermion lagrangian which break the chiral symmetry *explicitly*. The term which is easiest to interpret physically is a bare mass for the fermions

$$\Delta\mathcal{L} = \mu\overline{\psi}\psi \ . \tag{4.1}$$

We will find that the pions become massive and the $\underline{\boldsymbol{\sigma}}$–mesons move away from the threshold.

The system resembles QCD in the respect that the chiral symmetry is both broken dynamically and explicitly. The relative strength of the dynamical and explicit breaking mechanisms in QCD depends on the flavour of the quarks. For the u and d quarks the dynamical mechanism is dominant, and the physical pions behave as an $SU(2)$ triplet of pseudo–Goldstone bosons [12]. For the heavy quarks, c and b, the dynamical breaking is negligible, and their bound states are non–relativistic [4.1]. For the s quark the two sources are comparable, and this makes the phenomenology very complicated [4.2]. In our toy models one can investigate quantitatively the transition between the two extreme regimes.

4.1. Pseudo–Goldstone Bosons

Let us examine first the effects of adding explicit symmetry breaking terms in the $U(1)_L \times U(1)_R$ model (3.2) in $d = 2+1$. We wish to preserve renormalizability, so we can only add operators of ultra–violet dimension ≤ 3. The allowed symmetry breaking operators are

$$\overline{\psi}\psi,\ \overline{\psi}\gamma_5\psi,\ \overline{\psi}(\boldsymbol{\sigma} - i\boldsymbol{\pi}\gamma_5)\psi,\ \boldsymbol{\sigma},\ \boldsymbol{\pi},\ \boldsymbol{\sigma}\boldsymbol{\pi},\ (\boldsymbol{\sigma}^2 - \boldsymbol{\pi}^2),\ \boldsymbol{\sigma}^3,\ \boldsymbol{\pi}^3,\ \boldsymbol{\sigma}^2\boldsymbol{\pi},\ \boldsymbol{\pi}^2\boldsymbol{\sigma} \ . \tag{4.2}$$

Clearly the most general renormalizable lagrangian contains quite a lot of parameters. We can simplify things a little by using the freedom of *linear* redefinitions of the auxiliary fields, i.e. $\boldsymbol{\sigma} \to \boldsymbol{\sigma}' = A\boldsymbol{\sigma} + B$ and $\boldsymbol{\pi} \to \boldsymbol{\pi}' = C\boldsymbol{\pi} + D$. This freedom can be used to remove the first three operators in the list (4.2), leaving only the "potential energy" operators depending only on $\boldsymbol{\sigma}$ and $\boldsymbol{\pi}$.[#22]

At leading order in $1/N$ the effective potential is given by the symmetric form (3.4) with the breaking terms added as classical contributions. We define the coupling g^2 as in (2.9), so

$$V(\boldsymbol{\sigma}, \boldsymbol{\pi}) = \frac{1}{3\pi} \left| \sqrt{\boldsymbol{\sigma}^2 + \boldsymbol{\pi}^2} \right|^3 - \frac{M}{2\pi} \left(\boldsymbol{\sigma}^2 + \boldsymbol{\pi}^2 \right) - \Delta V$$

where

$$\begin{aligned} \Delta V = a_1 \boldsymbol{\sigma} + a_2 \boldsymbol{\pi} + a_3 \boldsymbol{\sigma}\boldsymbol{\pi} + a_4 (\boldsymbol{\sigma}^2 - \boldsymbol{\pi}^2) + a_5 \boldsymbol{\sigma}^3 \\ + a_6 \boldsymbol{\pi}^3 + a_7 \boldsymbol{\sigma}^2 \boldsymbol{\pi} + a_8 \boldsymbol{\pi}^2 \boldsymbol{\sigma} \ . \end{aligned} \tag{4.3}$$

The normalization conditions on Green's functions now tell us that all the parameters $a_1 \ldots a_8$ are finite quantities. Their physical meaning is elucidated, however, not from the potential directly, but from solving for the S–matrix elements. Note that there is still one degree of freedom left, namely a rigid chiral rotation, so in total there are *seven* breaking parameters that appear in the scattering amplitudes.

A full discussion of the physics from (4.3) would be rather complicated, but we can make some general statements. Firstly, the demand that $V(\boldsymbol{\sigma}, \boldsymbol{\pi})$ is bounded from below implies that the cubic couplings cannot be too large. (This is reminiscent of the $\eta \, \vec{\phi}^{\,6}$ coupling in $O(N)$ scalar theories in $d = 2 + 1$ [4.3].) Given this

#22 In QFT, as opposed to statistical mechanics, the VEVs of the auxiliary fields are not physical quantities, since they do not appear in the S–matrix. The dependence on the extra parameters in (4.2) is therefore irrelevant.

condition the breaking terms cause there to be an isolated global minimum of the potential. The two principle curvatures at the minimum will be both positive, so the former Goldstone bosons $\boldsymbol{\pi}$ are forced to become massive.

A curious feature of the cubic terms is that they cause the auxiliary field equations of motion to be quadratic (and not linear) so the interaction in the model is no longer four–Fermi. In fact it becomes non–polynomial in the fermion bilinears. Let us leave these "exotic" cases aside, for it is in the spirit of this review to consider only the more basic features of explicit symmetry breaking.

We can make a drastic reduction in the number of parameters by setting all the coefficients of the quadratic and cubic terms in (4.3) to zero. This restriction is preserved under all orders renormalization due to a general theorem of Symanzik [4.4]#23, and so is consistent. Using the freedom of chiral rotation to set $a_2 = 0$

#23 In the exact chiral limit *all* terms in (4.3) are zero, and can not be induced under renormalization on grounds of symmetry. Insertions of the *linear* terms $\boldsymbol{\sigma}$ and $\boldsymbol{\pi}$ do break the symmetry, but carry a vertex of mass dimension +2. This causes diagrams with two or three auxiliary field legs to be superficially convergent, so quadratic and cubic counterterms are not required.

Now the ultra–violet divergences can be removed by introducing *four* renormalization constants, when we take into account the freedom of chiral rotations. The full bare lagrangian can be written as

$$\mathcal{L}_{\text{bare}} = Z_1 i\overline{\psi}\partial\!\!\!/\psi - Z_2\overline{\psi}(\boldsymbol{\sigma} + i\boldsymbol{\pi}\gamma_5)\psi - \frac{N Z_3 Z_2^2}{2g^2 Z_1^2}\,(\boldsymbol{\sigma}^2 + \boldsymbol{\pi}^2) - \frac{N}{4\pi}\,m^2\,\frac{Z_4 Z_2}{Z_1}\,\boldsymbol{\sigma}\ .$$

The constants Z_i are formal power series in $1/N$ depending only *logarithmically* on Λ/M. Integration over $\boldsymbol{\sigma}, \boldsymbol{\pi}$ then gives the form of the bare lagrangian in terms of the fermions only,

$$\mathcal{L}_{\text{bare}} = Z_1 i\overline{\psi}\partial\!\!\!/\psi + \frac{Z_1^2 g^2}{2Z_3 N}\,\left((\overline{\psi}\psi)^2 - (\overline{\psi}\gamma_5\psi)^2\right) + m^2 g^2\,\frac{Z_4 Z_1}{4Z_3\pi}\,(\overline{\psi}\psi)\ .$$

Comparing with (4.1) we see the bare mass of the fermions is given by

$$\mu \;=\; m^2\,g^2\,\frac{Z_4}{4Z_3\pi}\ .$$

and a_1 to a positive value, we end up with just *one* breaking parameter: the mass scale m where $a_1 \equiv m^2/4\pi$.

In this simple case the potential (4.3) then has two stationary points at

$$\langle \boldsymbol{\pi} \rangle = 0\,, \qquad \langle \boldsymbol{\sigma} \rangle = \pm\frac{1}{2}\,\left(M + \sqrt{M^2 \pm m^2}\right)\,, \tag{4.4}$$

and $V(\boldsymbol{\sigma}, 0)$ is sketched for various values of m/M in Fig. 11. The $1/N$ perturbation series must be defined around the positive solution in (4.4), for it is the only true minimum. The negative solution is a saddle, and the $\boldsymbol{\pi}$–fluctuations around it are tachyonic.

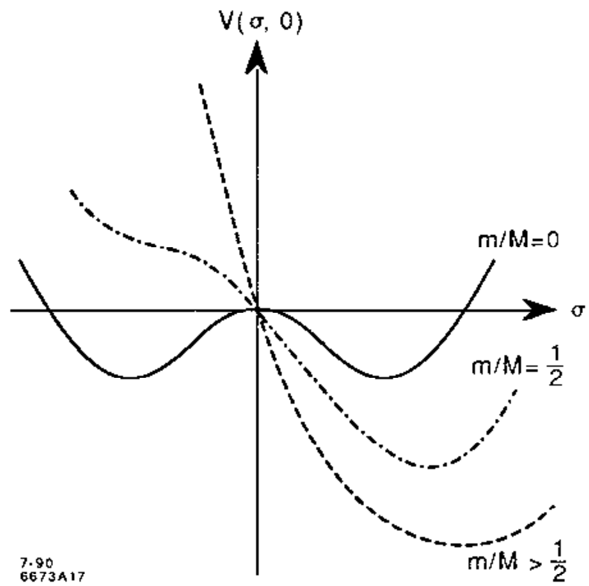


Figure 11. Potential with explicit symmetry breaking.

We note that because $Z_3 g^{-2}$ diverges linearly with the cutoff, see (2.9), the bare mass must be turned to *vanish* as $1/\Lambda$. A bare mass independent of Λ (or only logarithmically dependent) would force the model into the "weak coupling regime" and the correlation functions would become free. This fact is of importance for lattice simulations of the model.

The Feynman rules for the broken theory are the same as for the exactly symmetric version, except that the propagators have been modified. We now have

$$
\begin{aligned}
G(p) &= (\not{p} - M_f)^{-1} \\
D_\sigma(p^2) &= \frac{2\pi}{N}\left[2(M_f - M) + \frac{(-p^2 + 4M_f^2)}{\sqrt{-p^2}}\tan^{-1}\frac{\sqrt{-p^2}}{2M_f}\right]^{-1} \\
D_\pi(p^2) &= \frac{2\pi}{N}\left[2(M_f - M) + \sqrt{-p^2}\tan^{-1}\frac{\sqrt{-p^2}}{2M_f}\right]^{-1}
\end{aligned}
\tag{4.5}
$$

where

$$
M_f = \langle\sigma\rangle_{\min} = \frac{1}{2}\left(M + \sqrt{M^2 + m^2}\right) . \tag{4.6}
$$

Let us now study the Green's functions in detail. The fermion has a Dirac propagator with a mass given by Eq. (4.6). In the chiral limit, $m/M \ll 1$, M_f reduces to M and in the opposite extreme the mass approaches m. The parameter m is thus playing the role of a "current mass," and Eq. (4.6) shows how the "dynamical" and "current" contributions to the fermion mass combine.

The 4-point function of the fermions is given by the diagrams in Fig. 12.

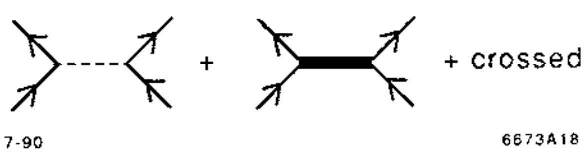


Figure 12. Four-point function of fermions.

The π–exchange diagram in Fig. 12 contains a real pole *below* the two–fermion threshold for all values of M and m. (See Appendix C.) This corresponds to a stable "pseudo–Goldstone boson." The mass m_π is conveniently measured in units of M_f, and becomes a natural function of the combination $\alpha = (M_f - M)/M_f$, where $0 \leq \alpha \leq 1$. The mass ratio $r \equiv m_\pi/2M_f$ is plotted as a function of the explicit breaking parameter α in Fig. 13, and duly vanishes as $\alpha \to 0$. The defining equation is

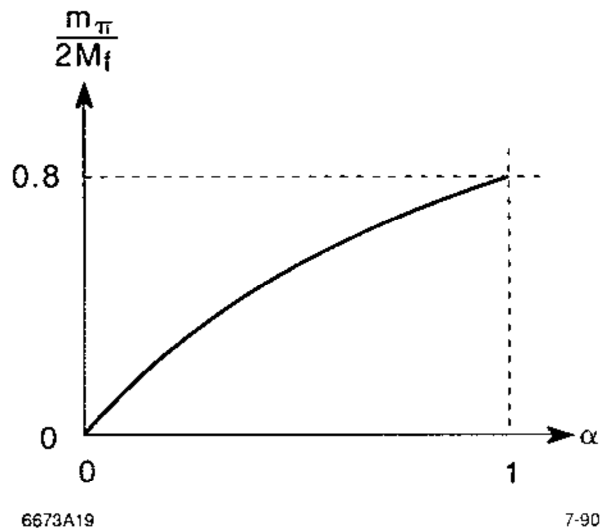


Figure 13. Mass of Pseudo–Goldstone bosons.

$$2\alpha = r\,\ell n\left(\frac{1+r}{1-r}\right) \ . \tag{4.7}$$

In the limit of *dominant* explicit breaking ($M_f \to \infty$, $\alpha \to 1$) the pion remains *tightly* bound, with a mass some 17% lower than $2M_f$. This indicates that the pion cannot accurately be regarded as a purely two–body bound state (a non–relativistic concept) even when its "constituents" are heavy.

The second contribution to the fermionic 4–point function in Fig. 12 is due to the $\boldsymbol{\sigma}$–particle, and it has an illuminating analytic structure. The continuation of

$D_\sigma(p^2)$ to complex values of p^2 has a cut at $p^2 = 4M_f^2$, with an infinite number of Riemann sheets. (So does $D_\pi(p^2)$, of course, and this corresponds to the threshold.) Moreover, as we show in Appendix C, there exists a *unique* sheet on which $D_\sigma(p^2)$ is analytic, with all the other sheets having a pair of conjugate complex poles. This structure is in accordance with S-matrix theory, for the unique analytic sheet is identified as the physical sheet of the $2 \to 2$ amplitude [4.5]. The $\boldsymbol{\sigma}$-particle has turned into a resonance, with a "mass" and "width" given by the position of the poles on the *neighboring* (second) sheet.[#24]

4.2. The Gell–Mann Okubo Formula

Let us turn to the case of $U(n)_L \times U(n)_R$ dynamically broken to $U(n)_V$. The symmetric lagrangian was given in (3.25), and the number of renormalizable symmetry breaking parameters is now even larger than the list in (4.2). This is because we can take different traces over the various monomials. However, it is still true that we can restrict ourselves to a single term linear in $\underline{\boldsymbol{\sigma}}$, since the argument in footnote (F23) is still valid. Thus we consider the lagrangian

$$\mathcal{L} = i\overline{\psi}\not{\partial}\psi - \overline{\psi}(\underline{\boldsymbol{\sigma}} + i\underline{\boldsymbol{\pi}}\gamma_5) - \frac{N}{2g^2}\,\mathrm{tr}\,(\underline{\boldsymbol{\sigma}}^2 + \underline{\boldsymbol{\pi}}^2) + \mathrm{tr}\,\underline{a}\,\underline{\boldsymbol{\sigma}}\ , \tag{4.8}$$

where $\underline{a}$ can be taken to be diagonal and positive semi-definite. There are n breaking parameters which appear in the S-matrix, $a_i \equiv m_i^2/4\pi$. The n-plet of

#24 The shape of the resonance does not, however, fit a Breit–Wigner form, even in the case of large M_f.

fundamental fermions become non-degenerate and have masses, c.f. (4.6),

$$M_{f(i)} = \frac{1}{2} \left(M + \sqrt{M^2 + m_i^2} \right) . \tag{4.9}$$

The model has a "nonet" of n^2 pseudo-Goldstone bosons, and we can derive the analogue of the Gell-Mann Okubo formula [43]. It is convenient to denote the diagonal elements of the $\underline{\pi}(x)$ matrix as $\pi_i(x)$, and the off-diagonal elements as $\pi_{ij}(x)$, where $\pi^*_{ij}(x) = \pi_{ji}(x)$. In this non-canonical basis the real $\pi_i(x)$ fields couple only to the i th fermions, and have propagators just like (4.5):

$$D_{\pi_i}(p^2) = \frac{2\pi}{N} \left[2(M_{f(i)} - M) + \sqrt{-p^2} \tan^{-1} \frac{\sqrt{-p^2}}{2M_{f(i)}} \right]^{-1} . \tag{4.10}$$

The "off-diagonal" pions are a little more complicated. Upon expanding the effective action we see that $\pi_{ij}(x)$ couples both to the i th and j th fermions, so the inverse propagator is given by a "hybrid" fermionic bubble containing two different masses:

$$\begin{aligned} D_{\pi_{ij}}(p^2) &= \frac{1}{N} \left[\frac{1}{g^2} - i \int \frac{d^3 q}{(2\pi)^3} \operatorname{tr} i\gamma_5 (\not{q} - M_i)^{-1} i\gamma_5 (\not{q} - \not{p} - M_j)^{-1} \right] \\ &= \frac{2\pi}{N} \left\{ M_i + M_j - 2M + (-p^2 + (M_i - M_j)^2) I_{ij}(p^2) \right\}^{-1} \end{aligned} \tag{4.11a}$$

where $I_{ij}(p^2)$ is 4π times the bosonic hybrid bubble, and is given by

$$I_{ij}(p^2) = \frac{1}{2\sqrt{-p^2}} \tan^{-1} \left\{ \frac{\sqrt{-p^2}}{2M_i} \cdot \left(1 + \frac{M_j^2 - M_i^2}{-p^2} \right) \right\} + i \leftrightarrow j . \tag{4.11b}$$

The 2-fermion threshold is at $p^2 = (M_i + M_j)^2$ below which the propagator is purely real. For all values of the parameters there is a real pole in the range

$$|M_i - M_j| \; < \; m_{\pi_{ij}} \; < \; M_i + M_j \tag{4.12}$$

corresponding to a stable bound state, (see Appendix C.) In the chiral limit $m_i \ll M$ the poles are easily located, and we have

$$m^2_{\pi(i)} = m_i^2 \,, \qquad m^2_{\pi(ij)} = \frac{1}{2}\,(m_i^2 + m_j^2) \;. \tag{4.13}$$

The mass formulae Eq. (4.13) have computable corrections in powers of m_i^2/M^2. We note that we do not get quite the same pattern familiar from QCD because there the η' is not a pseudo–Goldstone boson [4.6]. Instead, the η' becomes heavy due to the effects of instantons, so the symmetry breaking pattern is rather $SU(n)\times SU(n)\times U(1)_V \to SU(n)_V \times U(1)_V$. This can be reproduced "phenomenologically" in our model by adding a term Re $\det(\underline{\sigma} + i\underline{\pi})$ to the lagrangian. At least for $n \leq 3$ such a term would not destroy the property of renormalizability.

5. Thermodynamics

In this final section we study the thermodynamics of chiral symmetry breaking. In $d = 2+1$ the *discrete* Gross–Neveu model (2.3) undergoes a symmetry restoring phase transition at a finite temperature T_c, the main features of which are similar to the "superconducting" to "normal" transition in metals [5.1].

An extra motivation for this chapter is that it would be very interesting to verify the method of the $1/N$ expansion for the $d = 2+1$ models on the lattice. The calculations at finite temperature can then be used to simulate "finite–size

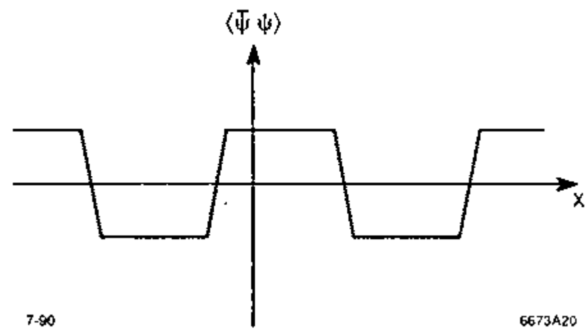


Figure 14. Segments of 1+1 Gross–Neveu.

effects." Similar calculations have been done using various methods in the $1+1$-dimensional case [5.2].

5.1. The Fundamental Difference between the $1+1$ and $2+1$ Cases

The thermodynamics of the discrete GN lagrangian (2.3) was first studied for $d = 1+1$. At zero temperature the chiral symmetry is dynamically broken, generating a mass M for the fermions. At non–zero temperatures a naïve application of the $1/N$ expansion yields a finite critical temperature for symmetry restoration, given in leading order by [5.3]

$$T_c = 0.57\, M \ . \tag{5.1}$$

However, this result is *wrong,* as was illuminated by Ref. [5.4]. In fact, for any finite N the critical temperature is rigorously *zero,* and the situation is analogous to the Ising model in one space dimension. In both 1D models "kink" configurations are unsuppressed, because their cost in energy is *independent* of their length. At low temperatures the system is segmented into regions of *alternating signs* of the order parameter, and this situation is depicted in Fig. 14.

The net average value of the order parameter is then zero, and the symmetry is restored. The $1/N$ expansion in $d = 1 + 1$ misses this effect because the energy per kink goes to infinity as $N \to \infty$. The number density of kinks is proportional to the Boltzman factor $e^{-N\beta M}$, so in leading order the system really is spatially homogeneous. The expansion in $1/N$ just measures the effects of small fluctuations about this state, and cannot reproduce the Boltzman factor since it is non–analytic in $1/N$. To sum up, in one space dimension there is no finite temperature phase transition, but this fact is obscured in the $1/N$ expansion.

The situation changes, however, when we turn to the case of two space dimensions. A simple energy–entropy argument now shows that "domains" *will* be suppressed at low enough temperatures, so that the critical temperature will now be finite. This phenomenon was first calculated in the analogous case of the Ising model by Onsager [5.5]. Thus, we expect a large N calculation in the 4–fermion model to be reliable in $d = 2 + 1$, and our result for the critical temperature is[#25]

$$T_c \;=\; \frac{M}{2\ell n2} \quad . \tag{5.2}$$

In the next subsection we review the finite temperature formalism and compute the Landau Free Energy function. This yields the critical temperature (5.2), and shows the transition to be second order. Next, we introduce a chemical potential μ to probe the effects of a finite fermion density, and construct the phase diagram. Other thermodynamic quantities and the critical exponents are given in Ref. [5.7].

#25 This is in agreement with the calculations of [5.6]

5.2. Finite Temperature Formalism and Phase Diagram

The quantum statistical partition function is defined by

$$Z(\beta) \;=\; Tr\ \exp -(\beta H) \ ,$$

where the hamiltonian of the GN model is

$$H = \int d^2x \left[-i\overline{\psi} \left(\gamma^1 \frac{\partial}{\partial x^1} + \gamma^2 \frac{\partial}{\partial x^2} \right) \psi - \frac{g^2}{2N} (\overline{\psi}\psi)^2 \right] \ . \tag{5.3}$$

To calculate $Z(\beta)$ it is simpler to use the equivalent form [5.8]

$$Z(\beta) = \int D\psi(x)\, D\overline{\psi}(x) \exp - \int_\beta d^3x \left[\overline{\psi} \partial\!\!\!/ \psi + \frac{g^2}{2N} (\overline{\psi}\psi)^2 \right] \tag{5.4}$$

where the fermion fields are anti–periodic functions on $R^2 \times [0, \beta]$. After introducing auxiliary field σ, the functional integral becomes quadratic in the fermion fields, and we can repeat the steps in section 2. The $1/N$ expansion at finite β is guaranteed to be renormalizable by the proof for $\beta = \infty$, when the coupling g^2 is taken equal to its $\beta = \infty$ value (2.9). This is because the temperature is acting simply as an infra–red cutoff, and has no effect on the ultra–violet behavior [2.1]. The effective action for constant configurations is, c.f. (2.7)

$$S_{\text{eff}} = N\beta \ (\text{Area}) \ \mathrm{F}_{\text{Lan}}(\sigma, \beta) \ ,$$

where

$$\frac{\partial}{\partial \sigma} F_{\text{Lan}}(\sigma, \beta) = \sigma \left[-\frac{1}{g^2} - \frac{4}{\beta} \sum_{n=-\infty}^{\infty} \int \frac{d^2\boldsymbol{p}}{(2\pi)^2} \, \frac{1}{(2n+1)^2(\pi^2/\beta^2) + \boldsymbol{p}^2 + \sigma^2} \right] . \tag{5.5}$$

Here F_{Lan} is the Landau's generalized free energy function, whose value at the minimum gives the standard Helmholtz free energy. Using a Poisson summation formula and a contour rotation we obtain (see Ref. [5.4] for details)

$$
\begin{aligned}
\frac{\partial F_{\mathrm{Lan}}}{\partial \sigma} &= 4\sigma \int \frac{d^2 p}{2(2\pi)^2} \left[\frac{1}{\sqrt{p^2+M^2}} - \frac{1}{\sqrt{p^2+\sigma^2}} \left(1 - \frac{2}{\left(e^{\beta\sqrt{p^2+\sigma^2}}\right)+1} \right) \right] \\
&= \frac{\sigma}{\pi} \left[|\sigma| - M + \frac{2}{\beta} \ell n \left(1 + e^{-\beta|\sigma|}\right) \right] \ .
\end{aligned}
\tag{5.6}
$$

For low temperatures the non-zero root of (5.6) gives the thermal average $\langle\sigma\rangle_\beta$, since the origin is a *maximum*. We can solve explicitly to get

$$
\langle\sigma\rangle_\beta = M + \frac{2}{\beta} \ell n \left(\frac{1+\sqrt{1-4\exp(-\beta M)}}{2} \right) \ . \tag{5.7}
$$

The order parameter $\langle\sigma\rangle_\beta$ decreases continuously with temperature, reaching zero at $\beta = \beta_c$, where $\beta_c M = 2\ell n 2$. At higher temperatures $\langle\sigma\rangle_\beta$ is identically zero, and the discrete chiral symmetry is restored.

5.3. Chemical Potential

The effects of a chemical potential μ are given by shifting the energy levels by μ [5.4]. The identical manipulations as before now yield in correspondence to (5.6)

$$
\frac{\partial F_{\mathrm{Lan}}}{\partial \sigma}(\sigma,\beta,\gamma) = \frac{\sigma}{\pi} \left[|\sigma| - M + \frac{1}{\beta} \ell n \left(1 + 2e^{-\beta|\sigma|} \cosh \mu\beta + e^{-2\beta|\sigma|} \right) \right] \ . \tag{5.8}
$$

At zero temperature we see that if $\mu < M$ the absolute minimum still occurs at $|\sigma| = M$. If $\mu > M$ then there is just a unique minimum at $\sigma = 0$, so $\mu = M$ is a critical value. For higher temperatures the critical value of μ drops and reaches zero at $T = T_c$. Thus, we have the phase diagram of Fig. 15 and we have a *line* of

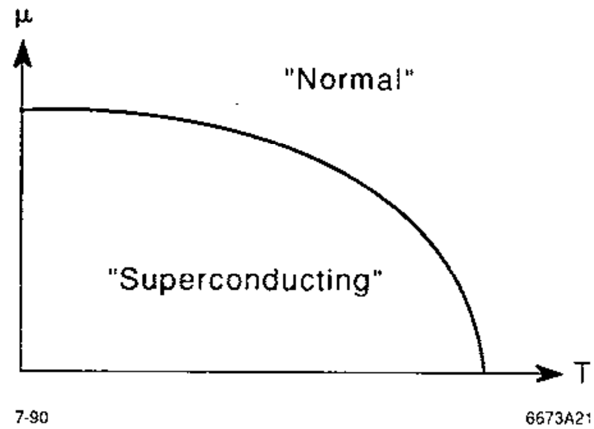


Figure 15. Phase diagram of Plane Superconductor.

second–order phase transitions given analytically by

$$\beta M \;=\; \ell n(2 + 2\cosh \mu\beta) \quad .$$

The phase diagram is analogous to that in superconductivity, where we would plot applied magnetic field versus temperature [5.1].

5.4. Continuous Symmetry

At leading order in $1/N$ the $U(1)_L \times U(1)_R$ model also seems to have a symmetry restoring phase transition at the critical temperature (5.2). But this conclusion runs afoul of Coleman's theorem [29,30], which stipulates that the (continuous) chiral symmetry must be manifested for *all* $T > 0$.

The failure of the $1/N$ expansion can be traced by similar reasoning to that in subsection 3.3. The non–leading diagrams, e.g. Fig. 7, become logarithmically infra–red divergent at non–zero temperature. This is because the Feynman diagrams have had energy *integrals* replaced by energy *sums*, thus making the $E = 0$ terms "effectively two–dimensional."

However, as we saw in subsection 3.3, the $1/N$ expansion is not entirely wrong, since dynamical mass generation *does* occur in the $1+1$ model. The fermions are massive, but "dress" themselves with a coherent state of pions to become *neutral* under $U(1)_A$. It seems reasonable to imagine that this is also what happens in the $2+1$ model at low temperatures.

To understand better the $T_c = 0^+$ phase transition in $d = 2+1$, the framework of the $1/N$ expansion works best in models where the number of Goldstone modes is $O(N)$ rather than $O(1)$ as here. For example we have studied the thermodynamics of the S_{N-1} non–linear σ–model [41], where the non–analyticity in temperature appears already at the leading order in $1/N$.

The critical temperature (5.2) then presumably corresponds to a transition between two *different* chirally symmetric phases: massive and massless, and this is the famous "vortex–condensation" transition of Kosterlitz and Thouless [31]. It would be of particular interest to have these speculations tested in lattice simulations.

6. Conclusions and Discussion

This review describes the phenomenon of dynamical symmetry breaking in the $d = 2+1$ 4–fermion models. These models provide a rare "laboratory" for this interesting phenomenon, in that they admit a non–perturbative calculational scheme (the $1/N$ expansion) which is both systematic and renormalizable.

The low energy excitations are subject to the general theorems of current algebra, which are derived purely from symmetry considerations. While these general theorems are obviously of great value, especially in low energy QCD phenomenology [3.10], a calculable model in which these theorems are explicitly seen in action, has its own advantages. We have verified the following consequences of the breaking of continuous global symmetry:

(1) Goldstone's theorem to all orders in $1/N$.

(2) PCAC and the Goldberger–Treiman relation.

(3) The derivative interactions of the Goldstone bosons in the processes $F\boldsymbol{\pi} \to F\boldsymbol{\pi}$ and $\boldsymbol{\pi}\boldsymbol{\pi} \to \boldsymbol{\pi}\boldsymbol{\pi}$.

(4) The derivative interactions of all multiple $\boldsymbol{\pi}$ amplitudes, as codified by the chiral lagrangian.

(5) The interplay between explicit and dynamical symmetry breaking, in particular the Gell–Mann–Okubo mass formula for pseudo–Goldstone bosons.

(6) The analyticity properties of the S–matrix elements consistent with unitarity.

(7) The thermodynamics of the symmetry restoring phase transition.

There are many other interesting "non-perturbative" problems that may be addressed using the $1/N$ expansion. In this expansion (unlike in weak coupling) one directly sees formation of bound states. The fact that, for example, the $\boldsymbol{\pi}$–particle is composite rather than elementary is clearly demonstrated by its spectral function (3.6), and the high energy dependence of the fermion–meson scattering amplitude, c.f. (3.23). In the CM frame the cross section is identically zero for forward scattering ($\theta = 0$), and for backwards scattering ($\theta = \pi$)

$$\frac{d\sigma}{d\theta}(\theta = \pi) \ = \ \frac{4\pi}{N^2 E} \ ,$$

for large CM total energy E. However at a fixed angle $\theta \neq 0, \pi$ we have

$$\frac{d\sigma}{d\theta} \ = \ \frac{8M^2\pi}{N^2E^3}\left[\frac{2}{\pi^2}\ell n\left(\frac{E^2(1-\cos\theta)}{8M^2}\right) + \frac{(1+\cos\theta)}{2} + \frac{2}{(1+\cos\theta)} - \frac{(1-\cos\theta)^2}{4(1+\cos\theta)}\right] \ .$$

The strong angular dependence and the $1/E^3$ behavior should be contrasted with the expected $1/E$ dependence for "elementary" bosons. It is this "softness" of the composite Goldstone bosons which is the key idea in technicolor models [8].

We have limited ourselves in this review to 4–fermion theories of the simplest kind: scalar–scalar interactions. However, in 2+1 dimensions one can construct another types of interaction terms which become renormalizable by similar methods. For example, the vector–vector interaction was considered briefly by Parisi [32], and the model achieves the old goal [14] of calculating the fine–structure constant.

Outside the class of 4–fermion models there exist many perturbatively non–renormalizable theories which become renormalizable in the $1/N$ expansion, even apparently in the presence of gravity [6.1]. We will mention a few of them:

The S_{N-1} non-linear σ-model

$$\mathcal{L} = \frac{1}{2}(\partial_\mu n^i)^2 + \alpha(n^i n^i - \frac{1}{g^2}) \ .$$

The σ-model was the first non-trivial theory to be shown to be renormalizable in the $1/N$ expansion in $d = 2+1$ [6.2]. At exactly *zero* temperature the σ-model has a two phase structure, distinguished by the order parameter $\langle n^i \rangle$. In the broken phase the Goldstone bosons are "elementary," and have high energy scattering amplitudes quite different from the composite Goldstone bosons of section 3. The theory has recently attracted attention in solid-state physics because it describes two-dimensional anti-ferromagnets like $LaCuO_4$. These substances, when properly doped, become high T_c superconductors [40]. Recent lattice simulations of this model [42] are consistent with the predictions of the $1/N$ expansion for the temperature dependence of the correlation length [41], [2.16]. The evidence for *non-trivial* interactions is, however, still rather weak, and more numerical simulations are required to make it convincing. In the case of $d = 1+1$ the σ-model has been exhaustively studied by a variety of non-perturbative methods [6.3]. Here the correctness of the $1/N$ expansion has been established by comparison with the *exact* S-matrix [20].

SUSY σ-model

This is a coupled combination of the Gross-Neveu and the non-linear σ-models. The supersymmetric action is [6.4]

$$S = \frac{1}{4}\int d^3x \, d^2\theta \left[(D\Phi)^2 + \Psi\left(\Phi^2 - \frac{1}{f}\right)\right] \ ,$$

where the superfields can be expanded in components

$$\Phi^i = \phi^i + \theta\psi^i + \frac{1}{2}\bar{\theta}\theta F$$

$$\Psi = \alpha + \theta u + \sigma \ .$$

The model has two phases: $O(N)$ symmetric and $O(N-1)$ symmetric. In both cases supersymmetry remains unbroken. In the ordered phase quasi–Goldstone fermions appear together with Goldstone bosons.

CP^{N-1} Model.

This model is a non-linear σ–model defined on the projective N–dimensional complex space $SU(N)/SU(N-1) \times U(1)$. In certain coördinates the lagrangian takes the form

$$\mathcal{L} = \partial_\mu \bar{n} \partial_\mu n + \frac{1}{4}\left(\bar{n}\overleftrightarrow{\partial}_\mu n\right)^2 + \alpha\left(\bar{n}n - \frac{1}{g^2}\right) \ .$$

The model has been studied in $2+1$ by Aref'era and Azakov [6.5]. The $1/N$ resummation is made a little more complicated by the existence of a *hidden* gauge symmetry, which must be gauge–fixed in order to define the expansion. Introducing an auxiliary vector field $A_\mu = \frac{1}{2}i(\bar{n} \cdot \overleftrightarrow{\partial}_\mu n)$, the lagrangian has the local $U(1)$ invariance

$$n_j(x) \rightarrow e^{i\Lambda(x)} n_j(x)$$

$$A_\mu(x) \rightarrow A_\mu(x) - \partial_\mu \Lambda(x) \ .$$

In the $O(N)$ symmetric phase, the operator A_μ interpolates a massless particle. The strong logarithmic attraction caused by the exchange of this particle presumably results in the confinement of n and $\bar{n}$. Only bound states will appear in the spectrum, just as in the $1+1$ case [6.6]. There is also an ordered phase which is peculiar to $2+1$. The vector particles disappear from the spectrum, the confinement is lost, and some of the n particles become Goldstone bosons. An interesting variant of the model involves adding a "topological" term $\mathcal{L}_{\rm top} = \theta\varepsilon_{\mu\nu\rho}A_\mu\partial_\nu A_\rho$, which is *marginal* (and not super–renormalizable) operator in $1/N$. There has been great recent interest in the topological CP^{N-1} model for it may describe the fractional quantum Hall effect [6.7].

It is possible to consider many more complicated theories in $2+1$ that are renormalizable in the $1/N$ expansion. The key construction is the geometric sum of "bubble diagrams," which was motivated in Section 2.1. The multiplicity of theories comes about because the particles which "pair up" can be bosons, fermions, or hybrids of both. This great variety of non–trivial theories may be used not only as a testing ground for non–perturbative methods, but possibly as models describing condensed matter systems.

Finally, let us consider the situation in $d = 3+1$. It was remarked in section 2 that the 4–fermion theories are logarithmically trivial, i.e. the connected correlation functions vanish with the cutoff as $1/\ell n\Lambda$. In fact all the examples of $1/N$ resummation have this problem, due to the logarithmic divergence of the (basic) bosonic bubble diagram. It does no good to try to *subtract* this divergence: the connected correlators become non–zero, but break unitarity. A Landau ghost is induced at the subtraction point, so *de facto* there is still a cutoff.

This behavior is suspected also to occur for $\lambda\phi^4$, since it is not asymptotically free [6.8]. However, logarithmic trivialiality doesn't necessarily mean that a theory is useless for phenomenology, as is testified by the Higgs sector of the Weinberg–

Salam model. After all, the cutoff can be taken all the way to the Planck scale, and $\ell n(m_{\rm Pl}/M_W)$ is still only of order ~ 50. In this spirit, attempts have been made to use 4–fermion interactions in $d = 3+1$ for technicolor–type composite models [44], [2.12], [6.9], and this was one of the original motivations for their consideration.

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APPENDIX A

Spinors in 2+1 Dimensions

In this appendix we specify notations used in this paper [A1]. The metric in the 2+1–dimensional Minkowski space is $g_{\mu\nu} = \mathrm{diag}(1, -1, -1)$. The Dirac algebra $\{\gamma_\mu, \gamma_\nu\} = 2g_{\mu\nu}$ has the following two dimensional representation:

$$\gamma^0 = \sigma^2 \ , \quad \gamma^1 = i\sigma^3 \ , \quad \gamma^2 = i\sigma^2 \ , \tag{A.1}$$

where σ_i are the Pauli matrices. Note that there is *no* 2×2 matrix which anti–commutes with all the γ^μ. The momentum operator $i\partial\!\!\!/$ is real in this (Majorana) representation, so the fundamental spinor χ can be taken to be real. The conjugate spinor $\overline{\chi}$ is then defined by $\chi^T\gamma^0$, and Lorentz invariants can be formed in the usual way from χ and $\overline{\chi}$.

The discrete operations of parity and time reversal act on the Majorana spinors as

$$\begin{aligned} P \ &: \ (x^0, x^1, x^2) \to (x^0, -x^1, x^2) \qquad \chi \to \sigma_3\chi \\ T \ &: \ (x^0, x^1, x^2) \to (-x^0, x^1, x^2) \qquad \chi \to i\sigma_2\chi \ . \end{aligned} \tag{A.2}$$

A complex spinor can be formed from two real spinors

$$\psi = \chi_1 + i\chi_2 \ , \tag{A.3}$$

and this allows a definition of charge conjugation

$$C \ : \ \psi \to \psi^* \ . \tag{A.4}$$

The free Dirac action is then a sum of two terms with a manifest $O(2)$ symmetry:

$$\mathcal{L} = \overline{\psi}(i\partial\!\!\!/ - m)\psi = \sum_{j=1,2} \overline{\chi}_j(i\partial\!\!\!/ - m)\chi_j \ . \tag{A.5}$$

There is an alternative (complex) representation for the γ–matrices which obscures the decomposition into Majorana spinors, but is useful for calculations:

$$\gamma^0 = \sigma^3 \ , \quad \gamma^1 = i\sigma^1 \ , \quad \gamma^2 = i\sigma^2 \ . \tag{A.6}$$

This is found by the unitary transformation $\psi \to U\psi$ where $U = \frac{1}{2}(1 + i\sigma_1 + i\sigma_2 + i\sigma_3)$. In either representation the γ–matrices have the property

$$\gamma^\mu\gamma^\nu = g^{\mu\nu} - i\epsilon^{\mu\nu\rho}\gamma_\rho \ , \tag{A.7}$$

The action in (A.5) is invariant under C, but the mass term breaks P and T. To restore these symmetries we must consider a *doublet* of complex fermions

$$\mathcal{L} = \sum_{A=1,2} \overline{\psi}_A i\partial\!\!\!/\psi_A - m(\overline{\psi}_1\psi_1 - \overline{\psi}_2\psi_2) \ , \tag{A.8}$$

where P and T are now defined to include the Z_2 switching $\psi_1 \leftrightarrow \psi_2$. The doublet ψ_A can be assembled into a four component Dirac spinor,

$$\Psi = \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} \tag{A.9}$$

for which we define the 4×4 Dirac matrices

$$\gamma^\mu = \gamma^\mu \otimes \begin{pmatrix} \mathcal{I} & 0 \\ 0 & -\mathcal{I} \end{pmatrix} \ . \tag{A.10}$$

The action (A.8) then takes on the simple form

$$\mathcal{L} = \overline{\Psi}(i\not\partial - m)\Psi \ , \tag{A.11}$$

where $\overline{\Psi} = \Psi^\dagger\gamma^0$. If there are N flavours of Dirac fermions the symmetry group acting on the Majorana spinors is $O(2N)\times O(2N)$, elevated to $O(4N)$ when $m = 0$. It is also possible to consider the mass terms $\overline{\Psi}i\tau_1\Psi$, $\overline{\Psi}i\gamma_5\Psi$ and $\overline{\Psi}\tau_3\Psi$, where

$$\tau_1 = \begin{pmatrix} 0 & \mathcal{I} \\ \mathcal{I} & 0 \end{pmatrix} \ , \quad i\gamma_5 = \begin{pmatrix} 0 & \mathcal{I} \\ -\mathcal{I} & 0 \end{pmatrix} \ , \quad \tau_3 = \begin{pmatrix} \mathcal{I} & 0 \\ 0 & -\mathcal{I} \end{pmatrix} \tag{A.12}$$

of which the first is also P–even and the last two P–odd. However these new bilinears are connected by rotations and there are really just *two* kinds of masses, which we can take to be $\mathbf{1}$ and τ_3.

Finally, in Section 2.6 on renormalizability we use a euclidean space notation. This involves a Wick rotation $t = i\tau$ [2.11], and a redefinition of the metric to $g^E_{\mu\nu} = \mathrm{diag}(1,1,1)$. The euclidean γ–matrices are conveniently defined from the representation (A.6)

$$\gamma^E_0 = \gamma^0 \quad \gamma^E_j = -i\gamma^j \ ; \tag{A.13}$$

this choice satisfies the euclideanized Dirac algebra and is hermitian.

APPENDIX B

Pauli–Villars Regularization

Throughout this review the regularization employed has been a momentum cutoff Λ. This has had the advantages of being easy to interpret physically and being close in spirit to a lattice regularization, and may be of assistance to future Monte Carlo simulations of the 4–fermion models.

The purpose of this appendix is first to show that the use of a quite different regularization scheme, namely Pauli–Villars, does yield the same results for the renormalized correlation functions (in the context of the $1/N$ expansion.) A second point is that some of the leading–order calculations are actually simpler with Pauli–Villars; these include the loop diagrams with vectorial couplings or with particles having different masses.

Consider, once again, the scalar–scalar model of Section 2, this time coupled to a collection of massive Pauli–Villars fields χ. For the case of Dirac spinors in $2+1$ we may choose χ to have τ_3 masses (A.12), in order to preserve the γ_5 "chiral" symmetry. (Recall that γ_5 and τ_3 anti–commute.)

$$\mathcal{L} = \overline{\psi}(i\partial\!\!\!/ - \sigma)\psi - \frac{N\sigma^2}{2g^2} + \sum \overline{\chi}(i\partial\!\!\!/ - \sigma - \Lambda\tau_3)\chi \ . \tag{B.1}$$

In (B.1) and the following we suppress an index on the regulator fields r. Note that these fields may be bosonic or fermionic, since they are not required to satisfy the spin–statistics theorem. The gap equations read, c.f. (2.7),

$$\frac{\partial V}{\partial \sigma} = \frac{\sigma}{g^2} - i\int_p \mathrm{tr}\,\frac{1}{p\!\!\!/ - \sigma} - i\sum\int_p \mathrm{tr}\,\frac{1}{p\!\!\!/ - \sigma - \Lambda\tau_3} \ . \tag{B.2}$$

From (B.2) we see that for $\partial V/\partial\sigma$ to be regulated we require $\sum 1 = -1$, (where this is a *graded* sum [3.15], counting +1 for fermions and −1 for bosons.) The integrals can now be evaluated, and we have

$$\begin{aligned}\frac{\partial V}{\partial \sigma} &= \frac{\sigma}{g^2} + \frac{\sigma|\sigma|}{\pi} + \frac{1}{2\pi}\sum\{(\sigma+\Lambda)|\sigma+\Lambda| + (\sigma-\Lambda)|\sigma-\Lambda|\} \\ &= \frac{\sigma}{g^2} + \frac{\sigma|\sigma|}{\pi} + \frac{2\sigma}{\pi}\sum\Lambda\,,\end{aligned} \tag{B.3}$$

where we take all Λ masses to be positive. Note that in (B.3) the "dangerous" terms σ^2 and Λ^2 cancelled. (This would not have been possible in a theory with an odd number of 2–component spinors.) Thus (B.3) indeed is odd under the reflection symmetry $\sigma \to -\sigma$, and we set

$$\frac{1}{g^2} + \frac{2}{\pi}\sum\Lambda = \text{finite} = -\frac{M}{\pi}\,. \tag{B.4}$$

This recovers Eq. (2.10b) for the potential. We can check the σ–propagator also, c.f. (2.11), by expanding around $\langle\sigma\rangle = M$.

$$\begin{aligned}\frac{D^{-1}_{PV}(p^2)}{N} &= \frac{1}{g^2} - i\int_q \mathrm{tr}\,\frac{1}{(\not q - M)(\not q - \not p - M)} \\ &\quad - i\sum\int_q \mathrm{tr}\,\frac{1}{(\not q - M - \Lambda\tau_3)(\not q - \not p - M - \Lambda\tau_3)}\,.\end{aligned} \tag{B.5}$$

Once again we only need the condition $\sum I = -1$ for the regularisation, as can be seen by adding and subtracting a zero momentum "fish" of mass M to each term. The integrals become

$$\begin{aligned}\frac{D^{-1}_{PV}(p^2)}{N} &= \frac{1}{g^2} + D^{-1}(p^2, M) + \frac{1}{2}\sum\left(\frac{\Lambda}{\pi} + D^{-1}(p^2, \Lambda + M)\right) \\ &\quad + \frac{1}{2}\sum\left(\frac{\Lambda - 2M}{\pi} + D^{-1}(p^2, \Lambda - M)\right)\end{aligned} \tag{B.6}$$

where $D(p^2, M)$ is the function in (2.12). We now can recover the original expression by letting $\Lambda \to \infty$ for fixed p^2, M. The extra term in (B.6) reduces to

$$\frac{1}{g^2} + \frac{2}{\pi} \sum \Lambda - \frac{M}{\pi} \sum 1 \tag{B.7}$$

and by (B.4) this identically vanishes.

In summary the Pauli–Villars scheme unambiguously reproduces the low energy Green's functions, provided that we require that (a) the divergences are indeed regularised, and (b) the chiral symmetry is not broken explicitly.

APPENDIX C

Analytic Structure of Mesonic Propagators

At leading order in $1/N$ the mesonic propagators in all the models have a non-trivial and illuminating analytic structure. The σ-propagator in (2.12b) and (4.5) is, up to a constant of proportionality

$$\begin{aligned} D_\sigma(p^2) &\simeq 1/f(z) \\ f(z) &= a + \frac{z^2+1}{z} \tan^{-1} z \ , \end{aligned} \tag{C.1}$$

where $z = \sqrt{-p^2}/2M_f$ and a is a real number $0 \le a \le 1$. (In the chirally symmetric theory $a = 0$.) Using the identity

$$\tan^{-1} z \ \equiv \ \frac{1}{2i} \ell n \left(\frac{1+iz}{1-iz} \right) \ , \tag{C.2}$$

we see that $D_\sigma(p^2)$ has infinitely many Riemann sheets joined by a cut along the real p^2 axis from $4M_f^2 \to \infty$. It is natural to define $w = \frac{1+iz}{1-iz} \equiv re^{i\theta}$, so that

$$f(z) \;=\; g(w) \;=\; a + \frac{2w}{w^2-1}\,\ell n w \quad . \tag{C.3}$$

The sheets can be labelled by evaluating the logarithm function as $\ell n w = \ell n r + i\theta + 2\pi n i$, where $-\pi \leq \theta \leq \pi$. The *principle* sheet has $n = 0$, for which $D_\sigma(p^2)$ is purely real when p^2 is spacelike (i.e. z real, $|w| = 1$.)

We wish to show that $D_\sigma(p^2)$ is analytic everywhere on the principle sheet away from the cut, i.e. $g(w)$ has no zeroes. Consider first real, spacelike momentum. A zero of $g(w)$ would correspond to the equation

$$0 \;=\; a + \frac{\theta}{\sin\theta} \quad . \tag{C.4}$$

Clearly (C.4) has no solution unless $a < -1$, contrary to fact. This preliminary result means that there are no tachyons. Next we consider the region $0 \leq p^2 < 4M_f^2$, ie $\theta = 0$. For a zero we have

$$0 \;=\; a + \frac{2r}{r^2-1}\,\ell n r \tag{C.5}$$

and this has no solutions unless $a \leq 0$, (where the $a = 0$ solution is at the branch point.) This implies that the σ–propagator (C.1) has no bound state except possibly at the threshold. Finally, for the rest of the complex w–plane, we note that if w is a zero of $g(w)$ then so are $1/w$ and w^*. Thus we can restrict ourselves to $r > 1$ and $0 < \theta \leq \pi$, for which we have the inequalities

$$0 \;<\; \arg \ell n w \;<\; \pi/2$$

and

$$-\pi < \arg\frac{2w}{w^2-1} < 0 \ ,$$

whereupon

$$-\pi < \arg\frac{2w}{w^2-1}\ell n w < \pi/2 \ . \tag{C.6}$$

Thus there can be no zero of $g(w)$ unless $a < 0$, and this completes the proof of analyticity. All other sheets have isolated poles of $D_\sigma(p^2)$ coming in complex conjugate pairs.

Turning now to the π-propagator in (3.5) and (4.5) we have

$$\begin{aligned} D_\pi(p^2) &\simeq 1/h(z) \\ h(z) &= a + z\tan^{-1}z \ , \end{aligned} \tag{C.7}$$

which is again an infinitely sheeted function. On the principle sheet $D_\pi(p^2)$ has a real pole in the region $0 \le p^2 < 4M_f^2$, located at $z \doteq \pm iy$ where

$$2a = y\,\ell n\frac{1+y}{1-y} \ . \tag{C.8}$$

This has a solution for all $a \geq 0$ since the RHS runs from 0 to ∞. There is no zero of $h(z)$ for z real, because in this region the RHS of (C.7) is positive definite, and in fact there are no other zeroes anywhere on the principle sheet.

We can see this by noting that if z is a zero of $h(z)$ then so is z^*, so we can restrict to the lower half plane. In that case we have, if $\mathrm{Re}\, z > 0$,

$$0 < \arg\ell n\left(\frac{1+iz}{1-iz}\right) < \frac{\pi}{2}$$

so that

$$\mathrm{Re}\, z\ell n \left(\frac{1+iz}{1-iz}\right) \; > \; 0 \quad . \tag{C.9}$$

Similarly, if $\mathrm{Re}\, z < 0$ then the logarithm lies in the fourth quadrant and the LHS of (C.9) is negative definite. In both cases, therefore, there can be no zero of $h(z)$, and this proves that on the principle sheet $D_\pi(p^2)$ is analytic everywhere away from the cut except for a simple pole in the bound state region.

Finally, the hybrid propagators of section 4.2 are the most complicated. We can at least prove that the "off–diagonal" pions with propagator (4.11a) remain stable. The function in (4.11b),

$$\frac{1}{\sqrt{-p^2}} \tan^{-1}\left\{\frac{\sqrt{-p^2}}{2M_i}\cdot\left(1+\frac{M_j^2-M_i^2}{-p^2}\right)\right\} \; ,$$

has branch points at $\sqrt{-p^2} = \pm i(M_i - M_j)$ and $\sqrt{-p^2} = \pm i(M_i - M_j)$, since this is where the curly brackets take on the value $\pm i$. However the propagator depends on $I_{ij}(p^2)$, which symmetrizes the above function in i and j, and this *vanishes* on the principle sheet at $\sqrt{-p^2} = \pm i(M_i - M_j)$. Thus we find that $I_{ij}(p^2)$ is real and positive in the range $(M_i - M_j)^2 \leq p^2 \leq (M_i + M_j)^2$, and runs from 0 to ∞. Following the previous reasoning for the pionic propagators, this shows that $D_{\pi_{ij}}(p^2)$ has a simple pole in this range.

APPENDIX D

Unitarity

One of the foundational results in Quantum Field Theory is that the weak coupling expansion is consistent with unitarity and relativistic invariance to all orders. The proof has to deal with the complicated structure of "dressed" propagators and also problems related to renormalization. In the context of spontaneously broken gauge theories the demonstration of perturbative unitarity was a landmark achievement [2.5].

However, the perturbative S-matrix does not contain bound states or resonances. Thus in a fully dynamical setting our understanding of unitarity is incomplete. (For a review of the unitarity problems in a non-perturbative treatment of Bethe-Salpeter equations, see Ref. [D1].)

The 1/N expansion differs markedly from perturbation theory in that bound states and resonances can occur explicitly at leading order. In the appendix we show that the leading order amplitudes are indeed consistent with unitarity. An all orders proof, which is by no means guaranteed by the perturbative theorems, would be very important to establish.

Let us consider first the 4-point fermion function in the simple $d = 2+1$ scalar-scalar model of section 2. The $2 \to 2$ amplitude (2.13) is real in the euclidean kinematic region, but by analytic continuation becomes complex above a threshold in the Minkowski region. (See Eqs. (C.1)–(C.2) with the parameter $a = 0$.) In a given channel we have

$$
\begin{aligned}
D &= \frac{2\pi}{N} \; \frac{\sqrt{-p^2}}{(-p^2+4M^2)\tan^{-1}(\sqrt{-p^2}/2M)} \qquad p^2 < 0 \\
&= \frac{4\pi}{N} \; \frac{\sqrt{p^2}}{-p^2+4M^2-i\epsilon} \; \left[\ell n \left| \frac{\sqrt{p^2}+2M}{\sqrt{p^2}-2M} \right| + i\pi\Theta(p^2-4M^2) \right]^{-1} \qquad p^2 > 0
\end{aligned}
\tag{D.1}
$$

where $p^2 = s,\ t$, or u. The amplitude $\mathcal{A}$ is then given by D multiplied by the appropriate Dirac wavefunctions. Taking the imaginary part,

$$
2\,\mathrm{Im}\,D(s) = N \cdot \frac{(-s+4M^2)}{2\sqrt{s}} \; |D(s)|^2 \quad . \tag{D.2}
$$

In (D.2) the imaginary part from the pole is killed by the logarithmic singularity in the denominator. (This "difficulty" of coincident pole and threshold is removed in the models with explicitly broken chiral symmetry, and we discuss this case presently.) Equation (D.2) is now of exactly the right form for $\mathcal{A}$ to be interpreted as the leading order T–matrix. The pre–factors on the RHS are given by the two–body phase space,

$$
\Omega_{2+1} = \frac{1}{2\pi} \int \frac{d^2k_3 d^2k_4}{4\omega_3\omega_4} \, \delta^3\Big((k_3+k_4) - (\sqrt{s},0)\Big) = \frac{1}{4\sqrt{s}} \quad , \tag{D.3}
$$

and the Dirac sum for emission of a fermion anti–fermion pair [2.5],

$$
\sum |\,\mathrm{wavefunction}\,|^2 = \mathrm{tr}\,(\not{k}_3 + M)(-\not{k}_4 + M) = 2(-s+4M^2) \quad . \tag{D.4}
$$

There is finally a factor N in (D.2) for the number of flavour channels available in the annihilation process.

We can also check the unitarity bounds at high energy [2.11]. As $s \to \infty$ the cross-section (2.15) becomes

$$\left(\frac{d\sigma}{d\Theta}\right)_{CM} \simeq \frac{1}{N\pi\sqrt{s}} \tag{D.5}$$

and the s-wave is not saturated, even for N as low as one.

Turning now to the more complicated models of section 4, the amplitude for σ-exchange gets modified to Eq. (4.5). The analytic structure is described in Appendix C, which shows that the σ-meson is now a true resonance. However, Eq. (D.2) still holds identically in the case, so the S-matrix is unitary just as before. Finally we remark that a similar analysis obtains for the π-exchange amplitudes of sections 3 and 4, so that unitarity is verified for all the leading order processes.

APPENDIX E

Next-to-Leading Order

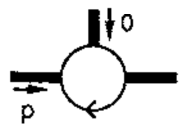
In this appendix we calculate the renormalisation constants Z_i for the scalar-scalar model of section 2. This will verify the consistency of the renormalisation conditions as described in subsection 2.6.

Using a momentum cutoff Λ the divergences in (2.29) can be obtained directly, so that in $d = 2+1$,

$$\begin{aligned} \widehat{Z}_1 &\doteq a'(0) \doteq -\frac{2}{3\pi^2}\,\ell n\,\Lambda \\ \widehat{Z}_2 &\doteq \frac{a(0)}{M} \doteq \frac{2}{\pi^2}\,\ell n\,\Lambda \,. \end{aligned} \tag{E.1}$$

Here the coefficients of $\ell n \Lambda$ are unambiguously determined, and are what appear in the RG equations of subsection 2.7.

The charge renormalisation $\widehat{Z}_3$ involves the somewhat more complicated graphs of (2.24b) or (2.26). We only need to evaluate the divergences of these graphs at zero momentum, and it turn out that it is easier to make the estimate *without* BPHZ subtracting on the sub-graphs. First, consider the identify obtained by differentiating the diagrammatic (2.32) once with respect to M.



$$= -\frac{2M}{\pi\sqrt{p^2}}\tan^{-1}\frac{\sqrt{p^2}}{2M} \tag{E.2}$$

8-90 6673A32

Multiplying by a $\boldsymbol{\sigma}$-propagator and integrating then yields

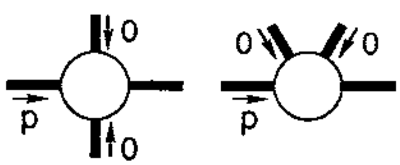
$$= -\frac{2M}{\pi}\left(\frac{\Lambda}{\pi} - M\right) \tag{E.3}$$

9-90 6673A34

whereupon from (2.24b) the charge is computed to be

$$\frac{\widehat{Z}_3}{g^2} \doteq -\frac{2\Lambda}{\pi^2} - \frac{8M}{3\pi^3}\,\ell n\,\Lambda\ . \tag{E.4}$$

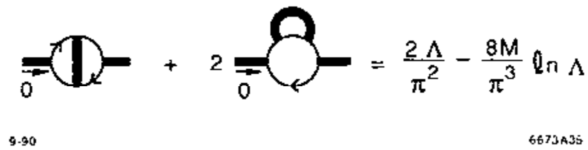
To show consistency with $\langle\sigma(x)\sigma(y)\rangle$ in (2.26) we generate the relevant graphs by differentiating Eq. (E.2) once more:



$$= \frac{2}{\pi\sqrt{p^2}}\left(\tan^{-1}\frac{\sqrt{p^2}}{2M} - \frac{2M\sqrt{p^2}}{p^2+4M^2}\right) \tag{E.5}$$

8-90 6673A33

Looping around with a σ–field then gives



(E.6)

and we find that $\langle \sigma(x)\sigma(y) \rangle$ is indeed made finite by the charge renormalisation (E.4). This analysis goes through identically in the case of $1+1$ dimensions, and for completeness we provide the renormalisation constants in this case

$$
\begin{aligned}
\widehat{Z}_1 &= 0 \\
\widehat{Z}_2 &= \frac{1}{4}\, \ell n\, \ell n\, \frac{\Lambda}{M} \\
\frac{\widehat{Z}_3}{g^2} &= -\frac{1}{\pi}\, \ell n\, \frac{\Lambda}{M} \; - \frac{1}{2\pi}\, \ell n\, \ell n\, \frac{\Lambda}{M}\,.
\end{aligned}
\tag{E.7}
$$

This results are consistency with previous 2–loop calculations in the Gross–Neveu model [E1].

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