Utrecht University

Institute of Theoretical Physics

Carroll Gravity
from the
Polyakov Action

Master's Thesis

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Academic Year 2024 - 2025

Thesis Title	Carroll Gravity from the Polyakov Action
Master's Degree	Theoretical Physics
Institution	Utrecht University - Institute of Theoretical Physics
Master's Degree	MSc in Theoretical Physics - CROHO 60202
Faculty	Science - Graduate School of Natural Science GSNS
Date of submission	July the 4th, 2025

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Abstract

This Master's Thesis project originates from the expectation that the equations of motion of Carroll gravity at leading order could emerge from the context of string theory, by applying the Carroll limit to the Polyakov action and imposing scale invariance. In our primary approach, we employ vielbein formalism to expand the curvature tensor in powers of the speed of light, and isolate the leading-order contribution of the Polyakov action in Riemann normal coordinates. We show that, by demanding the vanishing of the one-loop beta function, one of the three equations of the electric theory is recovered. We further discuss a possible secondary procedure, where the Carrollian analogue of normal coordinates is presented. The work opens with a pedagogical introduction to the basics of Carroll transformations and the Carroll algebra, followed by an overview of Carroll geometry and Carroll gravity.

to my aunt Rosanna

Contents

Introduction

With the term *Carroll Gravity* one refers to the theory of General Relativity (GR) in the limit of vanishing speed of light $c \to 0$, i.e. the *Carroll limit*. Now, given that Special Relativity (SR) is founded and constructed on the fundamental assumption that the speed of light has a constant value for all inertial observers, and that modern physics has been developed with Lorentz invariance as one of the essential and pivotal elements of any reasonable theory of motion, at first glance the idea of treating c as a parameter that can be run, even down to negligible values, could seem questionable. Despite appearances, it has indeed been possible to formalise this limiting procedure with rigour, and that shall be the baseline of our discussion.

In recent years, physics in the limit of vanishing speed of light has been receiving considerable attention, due to the large spectrum of applicability that the Carrollian framework has to offer. The first revival of the *Carroll group* came rather later than its independent introduction by Lévy-Leblond [1] and Sen Gupta [2], when Marc Henneaux formally related it to the study of 3+1 gravity as a zero-signature limit of Lorentzian geometry [3, 4], but since then, the interest of the scientific community has surged only in the last decade.

The study of the geometrical structure designed by the Carroll group and its gravitational implications dates back again to Henneaux [3], while expansions in small c have been first carried out by Dautcourt [5] in analogy with the Post-Newtonian (PN) expansions of GR. Subsequent efforts to address the matter and its dualism to Newton-Cartan theories have been made, leading to the notion of two distinct *electric* and *magnetic* Carroll limits [6], named in analogy with the two non-relativistic contractions of Galilean electromagnetism proposed by Lévy-Leblond himself and Le Bellac [7]. Carroll gravity has strong analogies to the Hamiltonian picture of GR and its 3+1 decomposition [8], it has been useful for studying near-singularities dynamics [9] due to its similarity to the Beliniski-Khalatnikov-Lifshitz (BKL) limit, and finds its place also in the study of black holes geometry, since null hypersurfaces are examples of Carrollian manifolds [10, 11], and because Carroll gravity admits various types of solutions, including black-hole-like ones, even with definable thermal properties [12]. More generally, Carroll gravity and geometry have appeared in a broad variety of recent studies [13–27].

Besides its genealogical branches, the study of systems with Carrollian symmetries has been prominently related to flat space holography. The reason is that it has been shown [6] that the asymptotic Bondi–Metzner–Sachs (BMS) group of symmetries in flat space is isomorphic to a conformal extension of the Carroll group on the boundary at null infinity, which led to numerous developments especially in 2+1 bulk dimensions [28–33], and in 3+1 bulk dimensions [34, 35]. This has brought to the suggestion that Conformal Carroll field theories (CCFTs) could be dual to quantum gravity in asymptotically flat spacetime [36], and they have been studied as an approach to flat space duality parallel to celestial holography [34]. Moreover, Carrollian physics has proven to be relevant in cosmology and inflation [37], as Carroll symmetry constraints on the stress-energy tensor provide equations of state with negative pressure $w = -1$. The implementation of Carroll symmetries in hydrodynamical systems has brought to the definition of Carroll fluids [38, 39] most importantly due to its relevance in flat holography [40]. Finally, Carrollian physics has found applications in the realm of condensed matter, for the study of fractons [41, 42], in models that deal with spacetime subsymmetries [43, 44] and for flat bands [45].

Our work primarily concerns the geometric and gravitational aspects of the Carroll picture. The question we aim to answer is whether the equations of motion of electric Carroll gravity can be recovered from bosonic string theory. Specifically, we expect that imposing scale invariance on the Polyakov action at the quantum level – i.e. killing the Weyl anomaly – for strings that propagates in a curved Carrollian target space, would reproduce the equations that define Carroll gravity at leading order. This expectation takes inspiration from the well-known analogous result [46] where Einstein's vacuum equations emerge by demanding that the beta function of the worldsheet CFT vanishes. If this is proven to be the case for a Carroll background as well, it would mean that this bridge between string theory and gravity holds even in the *zero-signature* regime set up by the Carroll limit.

The present document is structured as follows. We begin Chapter 1 by introducing the Carroll limit in comparison to the Newton one. We then discuss the key features of Carroll transformations, as the $c \to 0$ limit of Poincaré ones, in Section 1.1.1 and 1.1.3, and outline the basics of the Carroll group and its algebra in Section 1.1.4. Chapter 2 serves to introduce the tools to deal with Carroll symmetries on a curved spacetime. We define the vielbein fields and their relations in Section 2.0.1, and write their Carroll expansion in Section 2.0.2. Then, an overview of the relevant geometric characters of Carrollian manifolds is given in Section 2.0.3, followed by the introduction of the Carroll compatible connection (CCC) in Section 2.0.4. Section 2.0.5 provides a general review of Carroll gravity, where we summarise the results of [17] for the theory at leading-order (LO) in the c expansion of GR. These first two chapters build the necessary setup for fitting the Polyakov action in the Carroll framework, and stand in as a pedagogical introduction to the latter. Chapter 3 is dedicated to our main calculation, where we show how one of the LO equations of Carroll gravity is recovered from standard string theory. This is done in Section 3.2 by requiring vanishing beta function of the CFT described by the Polyakov action at LO in its Carroll expansion. In preparation to this purpose, we review in Section 3.1 how this is done for the standard case of Lorentzian geometry. Chapter 4 comprises other possible approaches to the matter. In Section 4.1 we examine a second procedure, that exploits the torsion-full CCC to construct a Carrollian analogue of normal coordinates, without reference to any c expansion. We further mark the pros and cons of our two methodologies, and discuss potential alternatives in Section 4.2. Finally, we present our conclusions and close with some considerations with regard to the overall methodology adopted in this work.

Chapter 1

The Carroll limit

As the Carroll limit is formally the opposite of what is generally referred to in literature as the *Newtonian limit* $c \to \infty$, it is pedagogically instructive to start with a brief historical prelude to address the latter, as the geometric and algebraic structures and the power-expansion formalism used to tackle Carroll physics - and gravity in particular - are mirrored on it.

The Newtonian limit

Since the very birth of SR, it was manifest that Lorentz transformations reduce to Galilei's ones in the approximation of small velocities ($v/c \ll 1$), and that consequently the laws of Newtonian dynamics in inertial (Galilean) frames are recovered from SR. Soon after that, Poisson's equation for the gravitational field could be recovered from Einstein's field equations of GR in the same limit, as it was natural to expect. Einstein himself commented on this explicitly [47], and many others soon after, like Weyl [48] and von Laue [49] to cite a few.

The first strong formal bridge between General Relativity and Newtonian dynamics is to be found in the independent works of Élie Cartan [50] and Kurt Friedrichs [51], fathers of what nowadays is commonly known as Newton-Cartan theory (NC), or Geometrised Newtonian Theory of Gravitation. In this framework, spacetime is described by a quadruple $(\mathscr{N}, \tau, h, \nabla)$, with τ and h being singular (degenerate) timelike and spacelike metrics respectively, on a smooth manifold $\mathscr{N}$ (N as in Newtonian, in contrast with the popular notations $\mathscr{M}$ for (pseudo)Riemannian and $\mathscr{C}$ Carrollian manifolds). This is equipped with a compatible Galilei covariant derivative ∇ that defines geodesics along which test particles move. In Cartan's approach the presence of two metrics is crucially related to the fact that the tensorial laws that hold on $\mathscr{N}$ must respect covariance under Galilean transformations, where distances in space are relative, but time is absolute. If one wants to incorporate this into a four-dimensional geometry, all spacelike vectors must vanish along the τ direction, and the kernel of h must contain all timelike vectors, so h has null signature, with nullity 1. One simple way of obtaining Cartan's geometrical structures from the framework of GR is to suitably take the limit $c \to \infty$ (see [52] or [53] for a modern discussion). Indeed, from a purely analytical point of view, making the speed of light divergent is formally equivalent to studying the regime of small velocities in kinematics and the weak field approximation in gravitation, as Taylor (Laurent) series expansions in powers of $1/c$. Friedrichs [51] already proposed the geometrisation of Newton Gravity as arising from a first order expansion in small $1/c^2$, though later works with more systematic approaches to multiple orders expansions were carried out mainly by Dautcourt [54], and by Chandrasekhar [55], the latter in the context of hydrodynamical approximations of field equations, and independently by Fock [56]. These works, which built on the much earlier contributions by Eddington [57], have been the foundational basis for what now is generically referred to as Post-Newtonian formalism (see [58], [59] or [60] for a comprehensive review).

The first rigorous explanation of what it actually means to perform the Newtonian limit in terms of a system's algebraic properties was given by Wigner and Inönü in their pioneering paper on group contractions of 1953 [61]. Here the speed of light is viewed as a quantity on which elements of the Lorentz group depend parametrically, and the limit $\epsilon = 1/c \to 0$ as the operation that defines a contraction from the Lorentz group to the Galilei one. It was elaborating on this work that in 1965 Jean-Marc Lévy-Leblond [1] first introduced the Carroll Group as a contraction of the Poincaré group in the limit of vanishing c: the Carroll Limit.

1.1 Carroll transformations

The Poincaré group comprises the transformations of space and time rigid translations, rotations in space and Lorentz boosts. Carroll enters physics as a set of coordinates transformations recovered from Lorentz ones in the limit vanishing speed of light $c \to 0$, while the opposite $c \to \infty$ retrieves Galilei's transformations.

Specifics on the limiting procedure

From a simple arithmetical point of view, the speed of light is the dimension-full parameter required to add coordinates of different causal character, and thus, strictly speaking, even varying it to other finite values, while keeping units fixed, is physically and mathematically wrong. For this reason the Carroll limit $c \to 0$ is to be seen as a short hand that refers to approximations where some characteristic velocity u^* is large compared to c, i.e. $u^*/c \gg 1$ (respectively $u^*/c \ll 1$ for the Newtonian limit). For instance, cosmic recessional velocities at super-Hubble scales stand as a valid domain of application. With this in mind, one is further allowed to this limiting procedure jargon in a more formal way, when dealing with the algebras describing the symmetries and the transformation properties holding in the approximation regime of validity. Specifically, the algebraic structure of the Carroll group is blind to the particular dynamics of a system subjected to Carroll symmetries, so in algebraic terms per se, the Carroll limit cannot be related to any external large characteristic speed. This is why authors often proceed first redefining $c \to \epsilon c$, with ϵ adimensional and positive, and then performing the limit as $\epsilon \to 0$ ($\epsilon \to \infty$ for the Newtonian case). Finally, in general one has to decide which quantities to keep fixed when c is sent to zero, and this can lead to different and also inequivalent outcomes.

1.1.1 Carroll from Lorentz

Before sending c anywhere, it is important to see how Carroll and Galilei's transformations arise from Lorentz's *both* in the slow speed regime[1], but as descriptions of events with distinct causal characters. In one dimension of space, the transformations that relate two inertial frames O and O', moving with speed $u = c\beta$ relative to one another are the well known Lorentz transformations

$$\begin{cases} \Delta x' &= \gamma[\Delta x - u\Delta t] \\ \Delta t' &= \gamma[\Delta t - (u/c^2)\Delta x], \end{cases} \tag{1.1}$$

with the boost factor $\gamma = 1/\sqrt{1-\beta^2}$.

Transformations of large time intervals

In the low speeds limit $u \ll c$, i.e. Taylor-expanding γ for $\beta \to 0$, (1.1) becomes

$$\begin{cases} \Delta x' &= \Delta x - u\Delta t + o(\beta) \\ \Delta t' &= \Delta t - (u/c^2)\Delta x + o(\beta), \end{cases} \tag{1.2}$$

that are not the Galilean transformations at all. To recover them, one should require an additional condition such that $u\Delta t = O(\Delta x)$, while $(u/c^2)\Delta x = o(\Delta t)$. This is most generally achieved by considering two highly timelike events, for which the space distance Δx between them is small enough compared to the relative distance $u\Delta t$ travelled by the observers during the events time lapse. In other words, Galilei's transformations hold when relative speeds are low (compared to c) *and* when time intervals are large (compared to the time required to move by Δx at the relative speed)

$$\frac{\Delta x}{\Delta t} \ll u, \qquad\qquad u \ll c. \tag{1.3}$$

Imposing these *Galilean* conditions

$$\begin{cases} \Delta x' &= \Delta x - u\Delta t \\ \Delta t' &= \Delta t, \end{cases} \tag{1.4}$$

the laws of Galilei are recovered. Note that the last term in the second line of (1.2) is usually neglected in virtue of its factor of $1/c^2$, bringing no need of conditions others than the small speed one. However, this line of reasoning implicitly assumes that the spacetime distances at play are the ones where Galilean relativity is applicable, i.e. those for which only the aforementioned last term is negligible. This is why the importance of the large time intervals conditions was stressed with particular emphasis by Lévy-Leblond [1]. As a matter of fact, one can retrieve (1.4) from (1.1) by scaling the relative speed and all lengths by the same amount $\beta \to \epsilon\beta$, $\Delta x \to \epsilon\Delta x$ and $\Delta x' \to \epsilon\Delta x'$, expanding around $\epsilon = 0$ and discarding all $o(\epsilon)$ terms.

[1] Meaning that the relative velocity of the two frames is small compared to the speed of light. We are not referring to any object moving within one frame or the other.

Transformations of large space distances

The situation is reflected if, again for small β one considers the opposite case of two strongly spacelike events, where the spatial separation between two events Δx is large compared to distance coverable by moving at the speed of light during time-lapse within the two. The *Carrollian* conditions are then

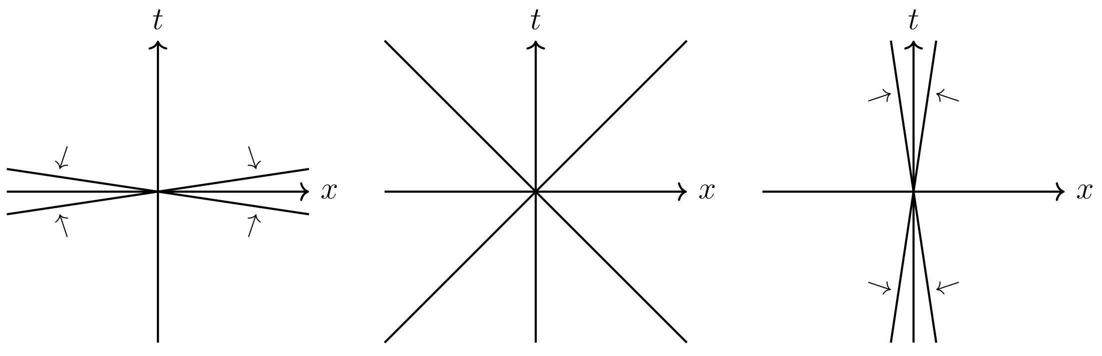
$$\frac{\Delta x}{\Delta t} \gg c \qquad\qquad u \ll c \tag{1.5}$$

that, imposed on (1.1), lead to the Carroll transformations in one dimension of space

$$\begin{cases} \Delta x' &= \Delta x, \\ \Delta t' &= \Delta t - (u/c^2)\Delta x. \end{cases} \tag{1.6}$$

Analytically, this can be achieved by scaling down all time intervals as the relative speed $\Delta t \to \epsilon \Delta t$, $\Delta t' \to \epsilon \Delta t'$, $u \to \epsilon u$ and $\epsilon \to 0$.

It should be clear now that Galilean transformations can describe events that lie close in the same causal sector, while the Carroll ones are suitable for events with a wide causal disconnection. As sketched in Fig.1.1, These opposite causal structures can be thought in terms of the units of measure for time and space that one would choose when dealing with the two situations. For two events satisfying Galilean conditions, a preferable pick could be, for instance, seconds and meters, so that the slope $1/c$ of lightcones in the spacetime diagram would be of order 10^{-9}. By contrast, events meeting Carrollian conditions would require units such as seconds and light years, and lightcones would appear like vertical lines (with a slope of order 10^7).



(a) Galilei lightcones (b) Lorentz lightcones (c) Carroll lightcones

Figure 1.1

Some remarks. First, varying spacetime dimensions to suit the description of strongly spacelike events is interpretable as letting $c \to 0$: lightcones close up onto the time axis, and events are causally correlated only if they happen at the same spatial coordinate. This is why the causal structure of a Carrollian system is often called *ultra-locality*. On the opposite, a system that respect Galilean symmetries is one where all events are causally correlated one another, and lightcones open up as c gets large.

Second, we want to stress that the use the term *ultra-relativistic* in the context of Carroll is highly ambiguous[2], if not wrong: historically, the ultra-relativistic regime has commonly been referred to as the domain of approximation where the velocities of moving objects are close to the speed of light, and that is definitely not the case under discussion. Third, the dichotomy[3] between the Galilean and the Carroll picture is rooted in the different roles played by space and time. In a system invariant under Galilean transformations time is absolute space is relative, and any notion of a universal spatial system of reference is meaningless. Reciprocally, space is an absolute concept in a Carroll universe, while the notion of time between different observers is relative to their particular locations and relative speeds. In fact, note that in (1.6), depending on the direction of u and magnitude of the term $(u/c^2)\Delta x$, time intervals can differ in sign from one frame to another, and no absolute statement can be made on which event happened first. This does not constitute a violation of causality, but it suggests that time is not a good quantity to parametrise evolution of a particle in motion.

1.1.2 Carroll transformations

With our primary references being [1, 62], we now show how the Galilei and Carroll transformations are formally recovered via the Newtonian and Carroll limit respectively, from Poincaré transformations.

The most general Poincaré transformation in 1+3 dimensions, from a coordinate system $(x_0', \boldsymbol{x}')$ to another $(x_0, \boldsymbol{x})$ in its non covariant form is

$$\begin{cases} x_0' &= \gamma(x_0 - \boldsymbol{\beta} \cdot R\boldsymbol{x}) + a_0 \\ \boldsymbol{x}' &= [\gamma^2/(1+\gamma)](\boldsymbol{\beta} \cdot R\boldsymbol{x})\boldsymbol{\beta} - \gamma\boldsymbol{\beta} x_0 + R\boldsymbol{x} + \boldsymbol{a}, \end{cases} \tag{1.7}$$

where $\boldsymbol{\beta}$ is the vector parameter of a Lorentz boost, R is an euclidean rotation matrix, $\boldsymbol{a}$ is a rigid translation in space and a_0 is a time shift. Defining

$$t = \frac{1}{c} x_0, \qquad \boldsymbol{u} = c\boldsymbol{\beta}, \qquad b = \frac{1}{c} a_0, \tag{1.8}$$

and talking the Newtonian limit, keeping the quantities t, $\boldsymbol{u}$ and b fixed as $c \to \infty$, Galilean transformations are retrieved

$$\begin{cases} t' &= t + b \\ \boldsymbol{x}' &= R\boldsymbol{x} - \boldsymbol{u}t + \boldsymbol{a}. \end{cases} \tag{1.9}$$

[2]It is true that the Carroll picture applies in settings where velocities of departure/approach are superluminal, but this by all means does not imply that in a Carrollian system matter can move faster than light. Indeed, as we discuss in the next section, one key feature of Carroll physics is that free particles with non-zero energy cannot move at all, and the only non trivial kinematics is the one of tachyons.

[3]We are reluctant to use the word *duality* here, because Carroll and Galilei are properly dual one another only in $1+1$ dimensions, and $c \to 1/c$ is an actual duality transformation only for equal number of space and time dimensions.

Similarly, if one redefines analogous quantities with the use of a new proportionality variable C as

$$s = Cx_0, \qquad \boldsymbol{v} = C\boldsymbol{\beta}, \qquad f = Ca_0, \tag{1.10}$$

and keep s, $\boldsymbol{v}$ and f fixed when performing the limit $C \to \infty$, the result is a general Carroll transformation in flat spacetime

$$\begin{cases} s' &= s - \boldsymbol{v} \cdot R\boldsymbol{x} + f \\ \boldsymbol{x}' &= R\boldsymbol{x} + \boldsymbol{a}. \end{cases} \tag{1.11}$$

A specification on nomenclature: (1.11) is the most general transformation that preserves all symmetries of the Carroll group, but the term *Carroll boost* refers to the Carroll limit of a Lorentz boost ($f = 0$, $\boldsymbol{a} = \boldsymbol{0}$, $R = I_3$)

$$\begin{cases} s' &= s - \boldsymbol{v} \cdot \boldsymbol{x} \\ \boldsymbol{x}' &= \boldsymbol{x}. \end{cases} \tag{1.12}$$

It is important to note that, if C is taken to have dimensions of a velocity $[C] = \mathrm{LT}^{-1}$, then s (as well as f) acquires dimensions of square-lengths per time $[s] = \mathrm{L}^2\mathrm{T}^{-1}$. This means that in this picture s plays the role of a novel coordinate that has little to share with the physical notion of time provided by the usual Lorentzian interpretation, and thus the word *time* is to be used with caution in Carroll physics. On the other hand, if one replace $C = 1/c$, with $[c] = \mathrm{LT}^{-1}$ and then takes the Carroll limit $c \to 0$, then $[s] = [f] = \mathrm{T}$, but $\boldsymbol{v}$ is not a velocity any more, and so it becomes hard to make physical sense of the parameters that relate different observers. No matter how you look at it, once the Carroll limit is taken, one typical notion of kinematics or the other as to be dropped. However this comes not as a surprise, as one should not expect familiarity in a framework where the concepts of cause and effect become degenerate. It is for this reason that, from now on, we will not address this kind of interpretational issues, unless when relevant or of further instructive.

1.1.3 Energy, velocity and Carroll particles

In this part, we mostly refer to [37]. Although Lorentz boosts take the same form on any four-vector, this is not the case for Carroll boosts. Indeed, Lorentz transformations for energy and momentum

$$\begin{cases} E' &= \gamma(E - c\boldsymbol{\beta} \cdot \boldsymbol{p}) \\ \boldsymbol{p}'_{/\!/} &= \gamma(\boldsymbol{p}_{/\!/} - \boldsymbol{\beta}E/c) \\ \boldsymbol{p}'_\perp &= \boldsymbol{p}_\perp, \end{cases} \tag{1.13}$$

with $\boldsymbol{p} = \boldsymbol{p}_{/\!/} + \boldsymbol{p}_\perp$, become, in the Carroll limit

$$\begin{cases} E' &= E \\ \boldsymbol{p}' &= \boldsymbol{p} - \boldsymbol{v}E, \end{cases} \tag{1.14}$$

where we kept $\boldsymbol{v} = \boldsymbol{\beta}/c$ fixed as $c \to 0$. Energy is then a Carroll invariant quantity, and sets the proportionality of the boosts in momenta. It is also interesting to look at how velocities transform under Carroll boosts. From the differential definition of a velocity over two Carroll frames (cf. (1.12)), we see that

$$\begin{aligned}\boldsymbol{u}' &= \frac{\mathrm{d}\boldsymbol{x}'}{\mathrm{d}s'} = \frac{\mathrm{d}\boldsymbol{x}}{\mathrm{d}(s - \boldsymbol{v}\cdot\boldsymbol{x})} = \frac{\mathrm{d}\boldsymbol{x}}{\mathrm{d}s}\left[\frac{\mathrm{d}}{\mathrm{d}s}\left(s - \boldsymbol{v}\cdot\boldsymbol{x}\right)\right]^{-1} \\ &= \frac{\boldsymbol{u}}{1 - \boldsymbol{v}\cdot\boldsymbol{u}},\end{aligned} \tag{1.15}$$

that is also the result one obtains performing the Carroll limit on the Lorentz composition laws for velocities

$$\begin{aligned}\boldsymbol{u}' &= \boldsymbol{u}'_{/\!/} + \boldsymbol{u}'_{\perp} \\ &= \frac{1}{1 - \beta\cdot\boldsymbol{u}/c}\left[(\boldsymbol{u}_{/\!/} - \boldsymbol{\beta}c) + \gamma^{-1}\boldsymbol{u}_{\perp}\right] \\ &\xrightarrow[c\to 0]{} \frac{\boldsymbol{u}}{1 - \boldsymbol{v}\cdot\boldsymbol{u}}\end{aligned} \tag{1.16}$$

keeping $\boldsymbol{v} = \boldsymbol{\beta}/c$ fixed. Note that the form of these transformations allows for two distinct cases of eternal immobility and perpetual motion: if a particle is motionless in one Carroll frame, then it remains so in any other one; on the other hand, there exists no (finite) Carroll boost that can put a moving particle at rest. This peculiar kinematic feature is closely related to possible violations of causality due to the presence of tachyons, and, as we discuss in the next section, to the distinction of zero and non-zero energy states in the representation of the Carroll algebra.

We shall now bridge to the behaviour of moving objects in the Carroll limit. For this, consider a single point-like free particle with energy E, momentum $\boldsymbol{p}$ and mass m, subjected, in any Lorentz frame, to the relations

$$E = \gamma_u mc^2, \qquad \boldsymbol{p} = \gamma_u m\boldsymbol{u}, \qquad E^2 = m^2c^4 + |\boldsymbol{p}|^2c^2, \tag{1.17}$$

where $\boldsymbol{u}$ is the speed of the particle in the Lorentz frame, $u = |\boldsymbol{u}|$ and $\gamma_u = 1/\sqrt{1 - u^2/c^2}$. Note that now γ_u has not a well-defined limit for $c \to 0$, as this time we are not dealing with the boost factor of a Lorentz transformation between two frames (which always approaches 1 in the Carroll limit). There are two options, depending on whether the particle is at rest or not.

If $u = 0$, then $\gamma_u = 1$ and we are dealing with a particle with zero momentum and with energy $E = mc^2$, that one can keep fixed $E = E_0$ in the Carroll limit by letting the mass scale as ϵ^{-2} while $c \to \epsilon c$, $\epsilon \to 0$. This is the case of eternal immobility.

The other option is $u \neq 0$, for which the factor γ_u becomes, at leading order in $c \to 0$

$$\gamma_u = \pm i\frac{c}{u} + o\left(\frac{c}{u}\right)^2. \tag{1.18}$$

This means that the momentum of the particle becomes imaginary and double-valued $\boldsymbol{p} = \pm imc\hat{\boldsymbol{u}}$, that can be kept fixed by allowing the mass to scale (differently) like ϵ^{-1} as $c \to \epsilon c$, $\epsilon \to 0$. The particle in perpetual motion is therefore a tachyon, and its energy vanishes $E = 0$ due to the exact cancellation in the relativistic dispersion relation (1.17).

The picture fits into the causal structure of Carroll: in the $c \to 0$ limit, lightcones collapse to a line, so the only particles that can respect ultra-locality are the ones in eternal immobility, whose worldlines *are* the lightcone, while particles in perpetual motion are Carroll tachyons, for which ultra-locality is violated. Does this constitute a violation of causality, in the sense of faster-than-light information propagation? Well, yes and no. On the one hand Carroll tachyons, like any other tachyon, do travel faster than c, so they could in principle be used to send faster-than-light information. On the other hand, if c is sent to zero, then everything that moves *is moving* faster than light. The common paradigm of causality already breaks down in the Carroll picture, as any two events happening at different spatial locations can be cause and effect of each other, and vice versa. Moreover, tachyonic particles are zero-energy states with an imaginary mass that diverges in the strict Carroll limit, so it is not even clear how they could propagate information at all. Once again, the bottom line of this discussion on Carroll particles suggests that some questions become ill-posed once the Carroll limit is taken.

1.1.4 Representation of the Carroll Algebra

In 1+3 dimensions, the Carroll group Carr(1,3) is then the Lie group generated by space translations, Euclidean space rotations, Carroll boosts, and time translations, respectively dependent on the ten parameters $\boldsymbol{a}$, $\boldsymbol{v}$, $\boldsymbol{\theta}$ (Euler's angles) and f. If we denote the infinitesimal generators of the restricted Poincaré group $\mathrm{ISO}^+(1,3)$ as S_i, K_i, P_i and P_0, corresponding to rotations, proper orthochronous Lorentz boosts, space and time translations respectively, then Carr(1,3) is obtained via the contraction

$$K_i \to \epsilon K_i, \qquad P_0 \to \epsilon P_0, \qquad \epsilon \to 0, \tag{1.19}$$

while the Galilei group Gal(1,3) reciprocally arises via the contraction

$$K_i \to \epsilon K_i, \qquad P_i \to \epsilon P_i, \qquad \epsilon \to 0. \tag{1.20}$$

A comparison of the algebras of these groups are summarised in Table 1.1.

The main reason we digress on the representation of Carroll Algebra is its relation to the two possible types of Carroll particles, namely the ones with zero and non-zero energy, respectively describing particles at eternal rest and perpetual motion. For a more elaborate discussion, we refer to [37].

Apart from the description of Carr(1,3) as a contraction, the commutation relations

Table 1.1: Algebras of the Carroll, Poincaré and Galilei groups

	Carr(1,3)	$\mathrm{ISO}^+(1,3)$	Gal(1,3)
$[S_i, S_j]$	$\varepsilon_{ijk}S_k$	$\varepsilon_{ijk}S_k$	$\varepsilon_{ijk}S_k$
$[S_i, K_j]$	$\varepsilon_{ijk}K_k$	$\varepsilon_{ijk}K_k$	$\varepsilon_{ijk}K_k$
$[K_i, K_j]$	0	$-\varepsilon_{ijk}S_k$	0
$[S_i, P_j]$	$\varepsilon_{ijk}P_k$	$\varepsilon_{ijk}P_k$	$\varepsilon_{ijk}P_k$
$[K_i, P_j]$	$\delta_{ij}P_0$	$\delta_{ij}P_0$	0
$[S_i, P_0]$	0	0	0
$[K_i, P_0]$	0	P_i	P_i
$[P_i, P_j]$	0	0	0
$[P_i, P_0]$	0	0	0

that define the Carroll algebra can be written as

$$[P_i, C_j] = \delta_{ij}H, \tag{1.21}$$
$$[J_{ij}, P_k] = \delta_{ik}P_j - \delta_{jk}P_i, \tag{1.22}$$
$$[J_{ij}, C_k] = \delta_{ik}C_j - \delta_{jk}C_i, \tag{1.23}$$
$$[J_{ij}, J_{kl}] = \delta_{ik}J_{jl} - \delta_{jk}J_{il} + \delta_{jl}J_{ik} - \delta_{il}J_{jk}, \tag{1.24}$$

and these are equivalent to the rules in Table 1.1 under the redefinitions

$$H = P_0, \qquad C_i = \lim_{c\to 0} K_i, \qquad J_{ij} = \epsilon_{ijk}S_k. \tag{1.25}$$

The Hamiltonian H – the central charge – and the square Casimir element M^2 are the two invariant admitted by the algebra, with the latter defined as

$$M^2 = \frac{1}{2}M_{ij}M_{ij}, \qquad M_{ij} = HJ_{ij} + C_iP_j - C_jP_i, \tag{1.26}$$

while the analogue of the Pauli-Lubanski vector is given by

$$\boldsymbol{W} = H\boldsymbol{S} + \boldsymbol{C}\times\boldsymbol{P}, \qquad M_{ij} = \epsilon_{ijk}W_k. \tag{1.27}$$

The states $|\psi\rangle$ in this representation are simultaneous eigenstates of the two invariants, labelled by their eigenvalues. They split into two families depending on whether the state's energy, as eigenvalue of the Hamiltonian $H|\psi\rangle = E|\psi\rangle$, is zero $E = 0$ or non-zero $E \neq 0$.

Non-zero energy If $E \neq 0$ there always exists a Carroll boost (cf.(1.14)) for which the momentum vanishes ($\boldsymbol{v} = \boldsymbol{p}/E$) in the given frame. The little group – the stabiliser subgroup of Carr(1+3) for which $\boldsymbol{p}$ is invariant – is then SO(3), as a system of a point-particle at rest is invariant under three-dimensional rotations. On such states $|\psi\rangle = |E \neq 0, \boldsymbol{p} = 0\rangle$, the action of the Casimir operator is

$$\hat{M}^2|\psi\rangle = \frac{1}{2}\hat{H}^2\hat{J}_{ij}\hat{J}_{ij}|\psi\rangle = \hat{H}^2\hat{S}_i\hat{S}_i|\psi\rangle = E^2 s(s+1)|\psi\rangle\,, \tag{1.28}$$

where E is the energy of the particle and $s \in \mathbb{N}/2$ its quantised angular momentum, respectively eigenvalues of the Hamiltonian and the total angular momentum operator $\hat{\boldsymbol{S}}^2 = \hat{S}_i\hat{S}_i$. These states can be labelled as $|E, s, m\rangle$, with $m = -s, -s+1, ...s$

and correspond to particles at eternal immobility.

Zero energy When $E = 0$ the momentum $\boldsymbol{p}$ is Carroll-invariant, so one can align it to a preferable direction – say $\hat{\boldsymbol{z}}$ – without loss of generality. Thus, the little group is ISO(2) of Euclidean isometries on the plane orthogonal to the z-axis, spanned by the generators W_x, W_y and by the helicity operator $\hat{L} = \boldsymbol{S} \cdot \hat{\boldsymbol{P}}/\sqrt{\boldsymbol{P}^2}$. In the case $\boldsymbol{W} \cdot \boldsymbol{W} = 0$ the operators $\hat{S}_z$ and $\hat{L}$, and can be used to label zero-energy states, like $|p_z, \lambda\rangle$ where λ is the eigenvalue of $\hat{L}$. These states correspond to particles in perpetual motion, and it would seem that they correspond to Carroll tachyons. However, the generators of spatial translations P_i are not hermitian if $p = |\boldsymbol{p}|$ is an imaginary number, and one requirement [37] under which the representations of zero-energy are constructed is $\boldsymbol{P} \cdot \boldsymbol{P} > 0$. If one perseveres on this line anyway, and interpret these states as tachyonic particles, it would mean that their momentum cannot be a physically measurable quantity, which is not in conflict with common sense. At any rate, we will not focus on these aspects, and leave the issue to future developments.

Chapter 2

Carroll on curved space

In this section we present the essentials to work in the framework of Carroll geometry on curved spacetime, and give outline of Carroll Gravity, as an expansion of GR in powers of vanishing speed of light. The notations and conventions adopted here mimic those of [17], that is the main reference for this part.

A preliminarily observation is in order. In its diagonal form, the metric $g_{\mu\nu}$ depends on a factor of c^2 in the time-like entry, while its inverse $g^{\mu\nu}$ carries a $1/c^2$. Thus, the primary consequence of performing the Carroll limit on Lorentzian geometry is that the former becomes degenerate, and the latter ill-defined, and so it is not possible to use them to lower and raise spacetime indices. For this reason, any two tensor notations that differ in the vertical position of indices must here be treated as different tensorial quantities. As we discuss soon, the only departure from this rule is allowed when one chooses to work in a specific Carroll frame, restricting calculations onto the spatial hypersurface that the frame defines.

2.0.1 Vielbein parametrisation

As we largely discussed in the previous section, the Carroll pictures put apart the roles of space from time in a substantial manner. To formally implement this feature at the geometrical level, the use of *vielbeins*[1] turns out to be of great utility and simplification. The formulation of curved geometry in terms of vielbeins generally goes under the name of *tetrad formalism*, and it was primarily introduced by Weyl, Cartan and Dirac [50, 63, 64]. A vielbein is a set of vector fields (and 1-forms) that are not derived from a particular choice of local charts, but satisfy suitable orthonormality properties on the tangent (cotangent) space at every point on the manifold. This way, one can write the metric tensor and any relevant geometric entity in terms of tetrads, which the burden of coordinate-dependence is delegated to. In the language of fibre bundles, frame fields are the local sections that span the fibre space tangent to the manifold, and dually, coframe fields (the inverse vielbein)

[1]From the German *viel* "many" and *bein* "foot", "leg" or "base", in the geometrical meaning of a basis vector. It refers to the set of generators in the tangent space, but with transliterative abuse authors often take liberty to pluralise it with the suffix -s, indicating the vectors themselves. Spurious variations on the term, like *einbein*, *zweibein*, *dreibein*, *vierbein*... are often used in the literature, alluding to the specific manifold dimension number at play (one, two, tree, four...). In English *tetrad* is the most widely used form in 3+1, while the more general *frame field* applies to any dimension.

are sections living in the union of all cotangent spaces on the manifold.

We want to parametrise the metric tensor in a way that discriminates time-like and space-like entries, and prepares it for its expansion in powers of c. We denote with Θ_A the elements of a general vielbein of the tangent space $T_p\mathscr{M}$, and with E^A their inverses on the cotangent space $T_p^*\mathscr{M}$. They relate to the basis vectors and forms of a coordinate choice x^μ on $\mathscr{M}$ as

$$E^A = E_\mu{}^A \mathrm{d}x^\mu, \qquad \Theta_A = \Theta^\mu{}_A \partial_\mu. \tag{2.1}$$

Capital Latin letters $A = 0, 1, 2, 3$ denote frame components, and lowercase Greek ones are used for spacetime indices as usual. Basis are chosen in such a way that the metric and its inverse are flat on the fiber bundle

$$g_{\mu\nu} = E_\mu{}^A E_\nu{}^B \eta_{AB}, \qquad g^{\mu\nu} = \Theta^\mu{}_A \Theta^\nu{}_B \eta^{AB}, \tag{2.2}$$

with $\eta_{AB} = \eta^{AB} = \mathrm{diag}(-1, 1, 1, 1)$. We want the vielbein structure to give rise to the so called *pre-ultra-local* (PUL) parametrisation of the metric and its inverse

$$g_{\mu\nu} = -\, c^2 T_\mu T_\nu + \Pi_{\mu\nu}, \qquad g^{\mu\nu} = -\,\frac{1}{c^2} V^\mu V^\nu + \Pi^{\mu\nu}, \tag{2.3}$$

that naturally adapts to ultra-local structure arising when performing the Carroll limit. This parametrisation is defined via the introduction of the nowhere-vanishing *temporal vector field* V^μ, the *clock 1-form field* T_μ, the *spatial metrics* $\Pi_{\mu\nu}$ and $\Pi^{\mu\nu}$. To achieve so, we decompose metrics on the bundles as

$$\eta_{AB} = -t_A t_B + S_{AB}, \qquad \eta^{AB} = -t^A t^B + S^{AB}, \tag{2.4}$$

where $t_A t_B$ is a rank 1 matrix that singles out time-like directions, and S_{AB} has rank 3 and projects along space-like directions. On these tensors we require the orthonormality and completeness relations

$$t_A t^A = -1, \qquad t_A S^{AB} = 0, \qquad S_{AB} t^A = 0, \qquad \delta_A^B = -t_A t^B + S_{AC} S^{CB}, \tag{2.5}$$

This choice imposes a split of the tangent bundle structure into a one-dimensional temporal component and a three-dimensional spatial one, orthogonal to each other. We now choose an orthonormal basis on the frame bundle

$$t_A = \delta_A^0, \qquad t^A = -\delta_0^A, \qquad S_{AB} = \delta_A^a \delta_B^b \delta_{ab}, \qquad S^{AB} = \delta_a^A \delta_b^B \delta^{ab}, \tag{2.6}$$

which differentiate indices $(0, a)$ into the temporal one 0 and the spatial ones a. This way one can use the Kronecker deltas δ_{ab} and δ^{ab} to lower and raise spatial indices on the frame bundles. The vielbein relates to the PUL fields through the temporal and spatial projections via this frame basis

$$t_A E_\mu{}^A = c T_\mu \qquad t^A \Theta^\mu{}_A = -\frac{1}{c} V^\mu \tag{2.7}$$

$$\delta_B^a E_\nu{}^B = E_\mu{}^a \qquad \delta_a^B \Theta^\mu{}_B = \Theta^\mu{}_a \tag{2.8}$$

The spatial metric and its inverse are constructed as

$$\Pi_{\mu\nu} = \delta_{ab} E_\mu{}^a E_\nu{}^b, \qquad \Pi^{\mu\nu} = \delta^{ab} \Theta^\mu{}_a \Theta^\nu{}_b. \tag{2.9}$$

This way, the PUL tensors satisfy properties that mirror the ones in (2.5)

$$T_\mu V^\mu = -1, \quad T_\mu \Pi^{\mu\nu} = 0, \quad \Pi_{\mu\nu} V^\nu = 0, \quad \delta^\mu_\nu = -V^\mu T_\nu + \Pi^{\mu\rho}\Pi_{\rho\nu}. \tag{2.10}$$

Note that the spatial metrics contraction does not result into an identity, a further remark on the fact that they are not inverses of each other.

2.0.2 Ultra-local expansion

We now expand the PUL vielbeins in powers of the speed of light as follows

$$T_\mu = \tau_\mu + c^2 N_\mu + o(c^2), \qquad V^\mu = v^\mu + c^2 M^\mu + o(c^2), \tag{2.11}$$
$$E^\mu{}_a = e^\mu{}_a + c^2 \pi^\mu{}_a + o(c^2), \qquad \Theta^\mu{}_a = \theta^\mu{}_a + c^2 \varphi^\mu{}_a + o(c^2), \tag{2.12}$$
$$\Pi_{\mu\nu} = h_{\mu\nu} + c^2 \psi_{\mu\nu} + o(c^2), \qquad \Pi^{\mu\nu} = h^{\mu\nu} + c^2 \psi^{\mu\nu} + o(c^2), \tag{2.13}$$

with spatial metrics at first and second order in c^2 given by

$$h_{\mu\nu} = e_\mu{}^a e_\nu{}^b \delta_{ab}, \qquad h^{\mu\nu} = \theta^\mu{}_a \theta^\nu{}_b \delta^{ab}, \tag{2.14}$$
$$\psi_{\mu\nu} = 2 e_{(\mu}{}^a \pi_{\nu)}{}^b \delta_{ab}, \qquad \psi^{\mu\nu} = 2\theta^{(\mu}{}_a \theta^{\nu)}{}_b \delta^{ab}. \tag{2.15}$$

In doing so, we are implicitly assuming that vielbeins – and basically every quantity we could consider, as the whole fan of geometrical entities at play is expressed in terms of them – are analytic in c^2. Moreover, we restrict to even powers c^{2n} is that we once again want to work in parallel with standard post-Newtonian expansions, where the distinction between expansions in $1/c$ or $1/c^2$ is a matter of notational convention.[2] The metric expansion reads

$$g_{\mu\nu} = h_{\mu\nu} + c^2(-\tau_\mu\tau_\nu + \psi_{\mu\nu}) + o(c^2) \tag{2.16}$$

and its inverse

$$g^{\mu\nu} = -\frac{1}{c^2} v^\mu v^\nu + h_{\mu\nu} - 2v^{(\mu} M^{\nu)} + c^2 M^\mu M^\nu + o(c^2). \tag{2.17}$$

From the relations (2.10) we have subsequent conditions at LO (leading order) and NLO (next leading order)

$$\tau_\mu v^\mu = -1 \qquad v^\mu h_{\mu\nu} = 0 \qquad \tau_\mu h^{\mu\nu} = 0, \tag{2.18}$$
$$\tau_\mu M^\mu = N_\mu v^\nu \qquad M^\mu h_{\mu\nu} = -v^\mu \psi_{\mu\nu} \qquad N_\nu h^{\mu\nu} = -\tau_\mu \psi^{\mu\nu}. \tag{2.19}$$

while from the completeness relation

$$\delta^\nu_\mu = -\tau_\nu v^\mu + h_{\mu\rho} h^{\rho\nu} \qquad h_{\mu\rho}\psi^{\rho\nu} + \psi_{\mu\rho} h^{\rho\nu} = N_\mu v^\nu + \tau_\mu M^\nu \tag{2.20}$$

[2] The main motivation for this is that in the weak field limit, gravitational potentials carry even powers of $1/c$, and no odd-power term can be sourced by these. Furthermore, it has been argued that odd-powers post-Newtonian corrections arise only from sub-sub-lading order onwards [53]. See [65] for a discussion that systematically allow for odd powers expansions.

2.0.3 Carroll manifolds

From a formal point of view, the geometry induced by the Carroll limit can be encoded into the quadruple $(\mathscr{C}, h, v, \tilde{\nabla})$ that defines a Carroll manifold. $\mathscr{C}$ is a smooth variety (a smooth manifold of 1+3 dimensions in our case) endowed with a symmetric tensor $h_{\mu\nu}$, that is degenerate, with order 1 nullity. The kernel $\ker(h)$ is the one-dimensional vector space spanned by the nowhere-vanishing vector field v^μ. Vectors and forms on C are parallel-transported via the covariant derivative $\tilde{\nabla} = \partial + \tilde{C}$, where $\tilde{C}$ is the Carroll compatible connection (CCC). Notation is not chosen without criterion: h is the leading order of the lower-index spatial metric introduced in the PUL parametrisation (2.3). It is the Lorentzian metric g in the Carroll limit, that becomes degenerate once the limit is taken. Similarly v^μ is the PUL temporal vector field at leading order. In this picture, the Carroll group Carr(1+3) comprises all diffeomorphisms that preserve the form of the spatial metric, of the temporal vector field and of the connection, and the Carroll algebra identifies with the Lie algebra of those vectors X for which

$$\mathcal{L}_X V = 0 \qquad \mathcal{L}_X \Pi = 0, \qquad \mathcal{L}_X \tilde{\nabla} = 0. \tag{2.21}$$

Carroll geometry is then the one that can be described with the LO expansion fields. The inclusion of NLO terms and consequent transformation laws expands the geometric structure on the manifold, just like type-II Newton-Cartan theory is an annex of second order terms in the non-relativistic expansion of the Newtonian limit [66]. The geometry arising from first order expansion in vanishing c will suffice, as the present work focuses exclusively on LO theory.

Vielbein transformations

In the frame bundle, vielbeins transform under a general Lorentz transformation Λ just like any other tensor

$$E^A \to \Lambda^A{}_B E^B, \qquad \Theta_A \to -\Lambda^B{}_A \Theta_B. \tag{2.22}$$

With $\Lambda_{AB} = -\Lambda_{BA}$ (indices can be raised and lowered with η in the frame bundle). If we consider infinitesimal transformation that distinguish between Lorentz boosts $\Lambda^0{}_a$ and spatial rotations $\Lambda^a{}_b$, and define a rescaled boost generator $\Lambda_a = c\Lambda^0{}_a$, $\Lambda^a = c\Lambda^a{}_0$ then the vielbeins transform as

$$\begin{aligned} T_\mu &\to T_\mu + \Lambda_a E_\mu{}^a, \qquad & E_\mu{}^a &\to E_\mu{}^a + c^2 \Lambda^a T_\mu + \Lambda^a{}_b E_\mu{}^b, \\ V^\mu &\to V^\mu + c^2 \Lambda^a \Theta^\mu{}_a, \qquad & \Theta^\mu{}_a &\to \Theta^\mu{}_a + \Lambda_a V^\mu - \Lambda^b{}_a \Theta^\mu{}_b. \end{aligned} \tag{2.23}$$

Just like in (1.12), we perform the Carroll limit keeping Λ_a (as well as Λ^a) fixed as c vanishes. The leading order expansion of the infinitesimal transformations

$$\Lambda_a = \lambda_a + O(c^2), \qquad \Lambda^a{}_b = \lambda^a{}_b + O(c^2), \tag{2.24}$$

are now the generators of infinitesimal Carroll boosts and spatial rotations respectively, under which the LO vielbein transform via

$$\begin{aligned} \tau_\mu &\to \tau_\mu + \lambda_a e_\mu{}^a, \qquad & e_\mu{}^a &\to e_\mu{}^a + \lambda^a{}_b e_\mu{}^b, \\ v^\mu &\to v^\mu, \qquad & \theta^\mu{}_a &\to \theta^\mu{}_a + \lambda_a v^\mu - \lambda^b{}_a \theta^\mu{}_b. \end{aligned} \tag{2.25}$$

The temporal vector v^μ and the metric $h_{\mu\nu}$, due to orthogonality of $\lambda^a{}_b$, are Carroll invariant. On the contrary, τ_μ and $h^{\mu\nu}$ do transform under Carroll boosts.

Realisations on a spatial hypersurface

As already discussed, the orthogonality conditions imposed on the vielbeins induce a fibre bundle structure on the manifold: the temporal vector v defines a one-dimensional *vertical* fibration V, orthogonal to the *horizontal* space H spanned by the sections $\theta_\mu{}^a$. Intuitively, this means that one can look at the bundle as the direct sum of the two $T\mathscr{M} = V \oplus H$, and that the manifold can be (locally) foliated into constant Carroll-time hypersurfaces Σ_s, perpendicular to the direction given by v, in a way similar to achronal slices in 1+3 decomposition. Although this sounds natural, it is not possible to do so without choosing an adapted coordinate system on $\mathscr{C}$, i.e. setting in a specific Carroll frame. In fact, spatial (and thus temporal) covectors X can be identified uniquely as $v^\mu X_\mu = 0$, because v is Carroll invariant, but this is not true for spatial vectors Y, as the contraction $\tau_\mu Y^\mu$ is not a Carroll scalar (τ transforms under Carroll boosts). Thus, if one wants to study concrete solutions of GR in the Carroll limit (which is not our case), and perform computation on spatial slices, choosing a boost frame is mandatory.

The most natural choice is to let the Carroll time-like coordinate be parallel to the temporal direction of v

$$v = v^\mu \partial_\mu = \frac{1}{\alpha}\partial_s, \tag{2.26}$$

where α is a non-zero normalisation factor, that can in principle depend on all spacetime coordinates. In this coordinates, the clock 1-form decomposes as

$$\tau = \tau_\mu \mathrm{d}x^\mu = -\alpha \mathrm{d}s + \tau_i \mathrm{d}x^i. \tag{2.27}$$

Now, we have enough gauge freedom to go into a frame where also τ is "purely temporal" $\tau = -\alpha\mathrm{d}s$, performing a Carroll boost $\tau_i \to \tau_i + \lambda_i = 0$. From a geometrical point of view, this correspond to choose an integrable Erhesmann connection on the horizontal sections of the bundle. An Ehresmann connection is a spin connection defined through a 1-form (τ), that parallel transport vectors along the hypersurface which generators of the horizontal space are tangent to. Its integrability is provided if the Frobenius theorem $\tau \wedge \mathrm{d}\tau = 0$ holds. In these adapted coordinates this is true because $\tau \wedge \mathrm{d}\tau = \alpha(\mathrm{d}s \wedge \mathrm{d}\alpha \wedge \mathrm{d}s) = 0$. For a more general choice $\tau_i \neq 0$, one has to require vanishing vorticity ω of v

$$\omega_{ij} = -\alpha\left(\partial_{[i}\tau_{j]} + \tau_{[i|}\partial_s\tau_{|j]}\right) = 0, \tag{2.28}$$

that is equivalent to requiring that the projected connection on the spatial submanifold has no torsion (see [10] for a detailed discussion on the bundle structure of Carroll manifolds). In the simplest case ($\tau_i = 0$), the Carroll metric decomposes as

$$\begin{aligned} v^\mu\partial_\mu &= \alpha^{-1}\partial_s, & h_{\mu\nu}\mathrm{d}x^\mu\mathrm{d}x^\nu &= h_{ij}\mathrm{d}x^i\mathrm{d}x^j, \\ \tau_\mu \mathrm{d}x^\mu &= -\alpha\mathrm{d}s, & h^{\mu\nu}\partial_\mu\partial_\nu &= h^{ij}\partial_i\partial_j, \end{aligned} \tag{2.29}$$

and one can define the perpendicular and parallel projectors onto the horizontal and vertical spaces respectively

$$(P_\perp)^\mu{}_\nu = h^\mu{}_\nu = h^{\mu\rho}h_{\rho\nu}, \qquad\qquad (P_{/\!/})^\mu{}_\nu = -\upsilon^\mu\tau_\nu, \tag{2.30}$$

where all fields are evaluated at a constant value of time s. The covariant derivative $\hat{\nabla}$ on the spatial hypersurface Σ_s is the projection of the Carroll connection $\tilde{\nabla}$ one via $P_\perp$

$$\hat{\nabla}_\rho X^{\mu_1\cdots\mu_n}{}_{\nu_1\cdots\nu_m} = h^\lambda{}_\rho h^{\mu_1}{}_{\alpha_1}\cdots h^{\mu_n}{}_{\alpha_n} h_{\nu_1}{}^{\beta_1}\cdots h_{\nu_m}{}^{\beta_m}\tilde{\nabla}_\lambda X^{\alpha_1\cdots\alpha_n}{}_{\beta_1\cdots\beta_m}, \tag{2.31}$$

and coincides [17] with the Levi-Civita connection on Σ_s constructed from h. The curvature tensor is also constructed through perpendicular projections

$$\hat{R}_{\mu\lambda\nu}{}^\rho = h^\rho{}_\alpha h_\mu{}^\beta h_\lambda{}^\gamma h_\nu{}^\delta \tilde{R}_{\beta\gamma\delta}{}^\alpha. \tag{2.32}$$

Hats denote quantities defined on Σ_s, and tildes refer to Carroll compatible ones. Finally, for further need, we introduce the extrinsic curvature tensor $K_{\mu\nu}$, that quantifies the variation of spatial metric along the temporal direction

$$K_{\mu\nu} = -\frac{1}{2}\mathcal{L}_V\Pi_{\mu\nu} = k_{\mu\nu} + O(c^2), \tag{2.33}$$

where lowercase $k_{\mu\nu}$ stands for the extrinsic curtvature at leading order in c

$$k_{\mu\nu=} -\frac{1}{2}\mathcal{L}_\upsilon h_{\mu\nu}. \tag{2.34}$$

The extrinsic curvature tensor is symmetric and purely spatial

$$V^\mu K_{\mu\nu} = 0. \tag{2.35}$$

This holds by virtue the other equally useful property

$$K_{\mu\nu} = \frac{1}{2}V^\lambda\left[\partial_\mu\Pi_{\nu\lambda} + \partial_\nu\Pi_{\lambda\mu} - \partial_\lambda\Pi_{\mu\nu}\right], \tag{2.36}$$

where here the third equality in (2.10) is exploited. In a hypersurface realisation, the extrinsic curvature is the second fundamental tensor on Σ_s, and reduces to

$$k_{ij} = -\frac{1}{\alpha}\dot{h}_{ij}, \tag{2.37}$$

where the dot denotes a derivative w.r.t. Carroll time s.

2.0.4 Carroll compatible connection

The role of the connection is of large importance in this work, as it will appear in the speed of light expansion of the curvature tensor. We cannot rely on the usual Levi-Civita one, as that defines the correct differentiation under Lorentz-covariance. Instead, we need a connection that is Carroll-covariant, i.e. built to be compatible with the two invariants v^μ and $h_{\mu\nu}$

$$\tilde{\nabla}_\lambda v^\mu = 0, \qquad \tilde{\nabla}_\lambda h_{\mu\nu} = 0. \tag{2.38}$$

One way of achieving this is via *gauging* the Carroll algebra, as done in [14], where one constructs the correct covariant derivative to act on the sections in the Carroll principal bundle. Alternatively, one can require Carroll compatibility on the full (non-expanded) frame fields

$$\tilde{\nabla}_\lambda V^\mu = 0, \qquad \tilde{\nabla}_\lambda \Pi_{\mu\nu} = 0, \tag{2.39}$$

and solve for the connection on the manifold via the vielbein postulate (see below). This has been done in [17], and the CCC obtained is written as

$$\begin{aligned} \tilde{C}^\lambda_{\mu\nu} = &- V^\lambda \left[\partial_{(\mu} T_{\nu)} + T_{(\mu} \mathcal{L}_V T_{\nu)}\right] \\ &+ \frac{1}{2}\Pi^{\lambda\rho}\left(\partial_\mu \Pi_{\nu\rho} + \partial_\nu \Pi_{\rho\mu} - \partial_\rho \Pi_{\mu\nu}\right) \\ &- \Pi^{\lambda\rho} T_\nu K_{\mu\rho}, \end{aligned} \tag{2.40}$$

which we borrow, without reproposing the its full derivation, but highlighting the relevant features of this result. The key ingredient is the so called vielbein postulate

$$\nabla_\lambda E_\mu{}^A = \partial_\lambda E_\mu{}^A - \Gamma^\rho_{\lambda\mu} E_\rho{}^A + \Omega_\lambda{}^A{}_B E_\mu{}^B = 0, \tag{2.41}$$

that is actually not a postulate, but a tautological claim that holds for any connection Γ and spin connection Ω. Here ∇ is the full covariant derivative comprising both connections, taking care of differentiation w.r.t. all indices. (2.41) states that the full covariant derivative on a vielbein has to vanish, [3] and can be used to extract the form of the connection on the manifold given the one on the bundle, and vice versa. To get (2.40) one starts by imposing Carroll compatibility as in (2.39) on the mixed tensor space basis

$$\tilde{\nabla}_\lambda \left(t^A \Theta_A{}^\mu\right) = 0, \qquad \tilde{\nabla}_\lambda \left(S_{AB} E_\mu{}^A E_\nu{}^B\right) = 0, \tag{2.42}$$

and then uses the postulate (2.41) in our chosen frame basis (2.6) solving for $\Gamma = \tilde{C}$. In doing so, multiple constraints on the torsion 2-form $\tilde{T}^A = \mathrm{d}E^A + \tilde{\Omega}^A{}_B \wedge E^B$ are required. Indeed, a fundamental character of any CCC is that it is inevitably torsion-full: its torsion possesses and *intrinsic* part that does not depend on the

[3] Intuitively, the vielbein $E_\mu{}^A$ can be interpreted as nothing but an identity map for mixed-indices tensors in the basis $\mathrm{d}x^\mu \oplus \hat{e}_{(A)}$: it maps vectors from coordinate basis to frame bundle basis. Thus, as ∇ is the correct derivative operator in this mixed tensorial space, it vanishes on the identity element of such space.

spin connection, and thus it cannot be set to zero, unless one specify to a particular Carroll frame. We take the torsion to be given by its intrinsic part only, that in coordinate basis is given by

$$\tilde{T}^{\lambda}{}_{\mu\nu} = 2\tilde{C}^{\lambda}_{[\mu\nu]} = 2\Pi^{\lambda\rho}T_{[\mu}K_{\nu]\rho}, \tag{2.43}$$

that is determined by the extrinsic curvature tensor. CCC are classified in four classes, depending on whether K is zero, traceless, proportional to Π, or neither of these. Finally, we mention the fact that the CCC connection is not even uniquely defined by the strongest torsion constraints, because one cannot use the vielbein postulate to solve for that part of Ω that acts along the t_A direction. This can be manually set to zero only by setting in a particular Carroll frame.

Relation to the Levi-Civita connection

The presence of torsion is the ineludible sign that our CCC cannot be recovered by an expansion in powers of c of the Levi-Civita one Γ. In fact, given the usual form of the Christoffel connection

$$\Gamma^{\lambda}_{\mu\nu} = g^{\lambda\rho}\left[\partial_\mu g_{\nu\rho} + \partial_\nu g_{\rho\mu} - \partial_\rho g_{\mu\nu}\right]. \tag{2.44}$$

Its expression in terms of the PUL fields reads

$$\begin{aligned}\Gamma^{\lambda}_{\mu\nu} = &-\frac{1}{c^2}V^\lambda K_{\mu\nu} \\ &-V^\lambda\left[\partial_{(\mu}T_{\nu)} + T_{(\mu}\mathcal{L}_V T_{\nu)}\right] + \frac{1}{2}\Pi^{\lambda\rho}\left(\partial_\mu\Pi_{\nu\rho} + \partial_\nu\Pi_{\rho\mu} - \partial_\rho\Pi_{\mu\nu}\right) \\ &-c^2 T_{(\mu}\mathrm{d}T_{\nu)\rho}\Pi^{\lambda\rho}.\end{aligned} \tag{2.45}$$

If one is stubbornly tempted to try relate it with $\tilde{C}$:

$$\Gamma^{\lambda}_{\mu\nu} = -\frac{1}{c^2}\overset{(-2)}{\Gamma}{}^{\lambda}_{\mu\nu} + \tilde{C}^{\lambda}_{\mu\nu} + S^{\lambda}{}_{\mu\nu} - c^2\overset{(-2)}{\Gamma}{}^{\lambda}_{\mu\nu} \tag{2.46}$$

the manual inclusion of the last term in (2.40), here denoted as $S^{\lambda}{}_{\mu\nu}$ is forced.

2.0.5 Carroll expansion of General relativity

To study GR in the Carroll limit, one needs to find the equivalent of Einstein's field equations in the small c expansion. This has been done in [17], by expressing the Ricci scalar in terms of the PUL fields inside the Einstein-Hilbert (EH) action, and then expanding in powers of c. We repeat the key passages, and direct the interested reader to the main article for the full picture. To get an expression for R in the PUL parametrisation, one constructs the Ricci tensor from the Levi-Civita connection as

$$R_{\mu\nu} = +\partial_\rho\Gamma^{\rho}_{\mu\nu} - \partial_\mu\Gamma^{\rho}_{\rho\nu} + \Gamma^{\rho}_{\rho\lambda}\Gamma^{\lambda}_{\mu\nu} - \Gamma^{\rho}_{\mu\lambda}\Gamma^{\lambda}_{\rho\nu}, \tag{2.47}$$

and writes Γ as in (2.46), collecting terms that scale with different powers of c

$$R_{\mu\nu} = \frac{1}{c^4}\overset{(-4)}{R}{}_{\mu\nu} + \frac{1}{c^2}\overset{(-2)}{R}{}_{\mu\nu} + \overset{(0)}{R}_{\mu\nu} + c^2\overset{(2)}{R}_{\mu\nu} + c^4\overset{(4)}{R}_{\mu\nu}. \tag{2.48}$$

These terms are given by the following

$$\overset{(-4)}{R}_{\mu\nu} = 0, \tag{2.49a}$$

$$\overset{(-2)}{R}_{\mu\nu} = \tilde{\nabla}_\rho \overset{(-2)}{\Gamma}{}^\rho_{\mu\nu} - 2\tilde{C}^\lambda_{[\mu\rho]} \overset{(-2)}{\Gamma}{}^\rho_{\lambda\nu} - S^\rho_{\mu\lambda} \overset{(-2)}{\Gamma}{}^\lambda_{\rho\nu} + S^\rho_{\rho\lambda} \overset{(-2)}{\Gamma}{}^\lambda_{\mu\nu}, \tag{2.49b}$$

$$\overset{(0)}{R}_{\mu\nu} = \tilde{R}_{\mu\nu} + \tilde{\nabla}_\rho S^\rho_{\mu\nu} - \tilde{\nabla}_\mu S^\rho_{\rho\nu} - 2\tilde{C}^\lambda_{[\mu\rho]} S^\rho_{\lambda\nu} - \overset{(-2)}{\Gamma}{}^\rho_{\mu\lambda} \overset{(2)}{\Gamma}{}^\lambda_{\rho\nu} - \overset{(2)}{\Gamma}{}^\rho_{\mu\lambda} \overset{(-2)}{\Gamma}{}^\lambda_{\rho\nu}, \tag{2.49c}$$

$$\overset{(2)}{R}_{\mu\nu} = \tilde{\nabla}_\rho \overset{(2)}{\Gamma}{}^\rho_{\mu\nu} - 2\tilde{C}^\lambda_{[\mu\rho]} \overset{(2)}{\Gamma}{}^\rho_{\lambda\nu} - \overset{(2)}{\Gamma}{}^\rho_{\mu\lambda} S^\lambda_{\rho\nu}, \tag{2.49d}$$

$$\overset{(4)}{R}_{\mu\nu} = -\overset{(2)}{\Gamma}{}^\rho_{\mu\lambda} \overset{(2)}{\Gamma}{}^\lambda_{\rho\nu}, \tag{2.49e}$$

where $\tilde{R}$ is the Ricci tensor constructed with $\tilde{C}$ only. The Ricci scalar is obtained by contracting the Ricci tensor with the PUL inverse metric, and reads

$$R = \left(-\frac{1}{c^2} V^\mu V^\nu + \Pi^{\mu\nu}\right) R_{\mu\nu} \tag{2.50}$$

$$\begin{aligned} &= -\frac{1}{c^4} V^\mu V^\nu \overset{(-2)}{R}_{\mu\nu} + \frac{1}{c^2}\left(-V^\mu V^\nu \overset{(0)}{R}_{\mu\nu} + \Pi^{\mu\nu} \overset{(-2)}{R}_{\mu\nu}\right) \\ &\quad - V^\mu V^\nu \overset{(2)}{R}_{\mu\nu} + \Pi^{\mu\nu} \overset{(0)}{R}_{\mu\nu} + c^2 \left(-V^\mu V^\nu \overset{(4)}{R}_{\mu\nu} + \Pi^{\mu\nu} \overset{(2)}{R}_{\mu\nu}\right). \end{aligned} \tag{2.51}$$

After some integration by parts, where boundary terms are neglected, the EH action in its PUL parametrisation can be written as

$$S_{\rm EH} = \frac{c^3}{16\pi G} \int_{\mathscr{M}} \mathrm{d}^4x \sqrt{-g}\, R \tag{2.52}$$

$$= \frac{c^2}{16\pi G} \int_{\mathscr{M}} \mathrm{d}^4x\, E \left[\left(K^{\mu\nu} K_{\mu\nu} - K^2\right) + c^2 \Pi^{\mu\nu} \tilde{R}_{\mu\nu} + O(c^4)\right], \tag{2.53}$$

where E is the vielbein determinant $E = \det(T_\mu, E_\mu{}^a)$, $K^{\mu\nu} = K_{\alpha\beta}\Pi^{\mu\alpha}\Pi^{\nu\beta}$ and K being the trace $K = K_{\mu\nu}\Pi^{\mu\nu}$. Note that one can always raise and lower indices of $K_{\mu\nu}$ with the spatial metric $\Pi^{\mu\nu}$, as the extrinsic curvature is purely spatial. The result is an expansion of the action itself,

$$S_{\rm EH} = c^2 S_{\rm LO} + c^4 S_{\rm NLO} + O(c^6), \tag{2.54}$$

and different Carrollian gravity theories correspond to the variation of the action term at one order or the other. The theory arising from varying the LO action is known as the *electric* Carroll limit, while the *magnetic* Carroll limit corresponds to the study of a truncation of the NLO action. In general, the equations of motion obtained by varying the action at order $c^{2(i+1)}$ contain the ones at the precedent order c^{2i}, plus two in addition.

Electric Carroll Gravity

Expanding the vielbein fields and truncating at first order, the LO action is

$$S_{\rm LO} = \frac{1}{16\pi G} \int_{\mathscr{M}} \mathrm{d}^4x\, e \left[k^{\mu\nu} k_{\mu\nu} - k^2\right], \tag{2.55}$$

with $e = \det(\tau_\mu, e_\mu{}^a)$, $k_{\mu\nu}$ is the LO extrinsic curvature of (2.34), $k^{\mu\nu} = k_{\alpha\beta}h^{\mu\alpha}h^{\nu\beta}$, and factors of the speed of light have been absorbed into the gravitational constant G, which can be kept fixed in the Carroll limit. This action was first obtained as a zero signature limit in the Hamiltonian formulation of GR by Henneaux [3]. Its variation w.r.t. the fields v^μ and $h^{\mu\nu}$ respectively produces the two equations

$$\frac{1}{2}\tau_\mu\left(k^{\rho\sigma}k_{\rho\sigma} - k^2\right) - h^{\nu\rho}\tilde{\nabla}_\rho\left(k_{\mu\nu} - kh_{\mu\nu}\right) = 0, \tag{2.56a}$$

$$\frac{1}{2}h_{\mu\nu}\left(k^{\rho\sigma}k_{\rho\sigma} - k^2\right) - k\left(k_{\mu\nu} - kh_{\mu\nu}\right) + v^\rho\tilde{\nabla}_\rho\left(k_{\mu\nu} - kh_{\mu\nu}\right) = 0. \tag{2.56b}$$

Some important remarks are to be made now. First of all, these equations are boost dependent, as they are obtained by varying $h^{\mu\nu}$, which is not a Carroll boost invariant object. This choice is explained by the fact that a variation of S_{LO} w.r.t. the Carroll invariant v^μ and $h_{\mu\nu}$ cannot be used to solve for t_μ and $h^{\mu\nu}$. Secondly, the LO action depends on the extrinsic curvature, and not on the full curvature given by the Ricci tensor $\tilde{R}_{\mu\nu}$ associated with the LO Carroll connection $\tilde{\Gamma}$. Finally, these equations can be projected onto temporal and spatial directions by contractions with v^μ and $h^{\mu\nu}$ respectively. The temporal projection of the second one vanishes, and the other three can be written as

$$k^{\mu\nu}k_{\mu\nu} - k^2 = 0, \tag{2.57a}$$

$$h^{\rho\sigma}\tilde{\nabla}_\rho(k_{\sigma\mu} - kh_{\sigma\mu}) = 0, \tag{2.57b}$$

$$\mathcal{L}_v k_{\mu\nu} = -2k_\mu{}^\rho k_{\rho\nu} + kk_{\mu\nu}. \tag{2.57c}$$

This splitting allows one to interpret the first two as constrain equations that hold separately on every single time slice Σ_s, and that discriminate which initial data of fundamental tensors $h_{\mu\nu}$ and $k_{\mu\nu}$ are allowed. The last one as an evolution equation along the temporal direction, that describe the dynamics of the hypersurface foliations. Note that in (2.57c) only time derivatives are present, and this means that the evolution law of a point on Σ_s along the temporal direction is fully specified by the value of the extrinsic curvature $k(s)_{\mu\nu}$ at that point solely. This is a powerful result because the Cauchy problem, that in GR generally requires to solve highly non trivial PDEs, is here reduced to an ODE in Carroll time only. Moreover, these interpretations are in accordance with the ultra-local causality of the Carroll picture: evolutions at different points on the same time slice are blind one to another, and geodesics can be thought of as a flux normal to the spatial hypersurfaces (thus, parallel to the temporal direction).

The magnetic limit and an apparent contradiction

As already mentioned, the magnetic limit of Carroll gravity arises from the variation of the NLO action. In the present work we are not interested in the study of this limit, and we simply mention few relevant generalities, redirecting the interested reader to the main article [17]. The Carroll magnetic theory can be obtained as a truncated sector of the NLO theory, where one starts with a different action, including an auxiliary tensor field $\chi_{\mu\nu}$ that reproduces the PUL S_{EH} of (2.53) if integrated out. In the limit $c \to 0$ the new action becomes such to produce equations of motions that split into two constraints for the projected Ricci tensor $\tilde{R}_{\mu\nu}$ and for the field $\phi^{\mu\nu}$, and into an evolution equation for $\phi^{\mu\nu}$. The latter can be seen both as the Hamiltonian conjugate momentum of $h^{\mu\nu}$, and as a Lagrange multiplier that fixes $k_{\mu\nu} = 0$. The magnetic limit is a truncation of the NLO Carroll gravity in the sense that it is a simplified case of the more general one, in which one varies v^μ and $h^{\mu\nu}$ after manually setting to zero the fields M^μ and $\psi^{\mu\nu}$ of the subleading c^2 order in the vielbein expansion (2.11). The resulting NLO equations of motion for v^μ and $h_{\mu\nu}$ hold provided that they also obey those of the precedent LO theory.

Before moving to the main part of our work, of which what treated so far serves as preparatory introduction, one important remark has to be made. The machinery outlined in the last sections can be in principle extended to study Carroll Gravity at any order in the speed of light expansion, including new vielbein fields for every order c^{2i}. If these fields are considered as independent of the ones at lower orders, it seems like one has to include new degrees of freedom for every new order that takes into account. This sounds like a foolish inconsistency, as such a philosophy suggests that an infinite number of parameters would be required for an exact study of Carroll gravity. The resolution to this paradox is simple: a Carroll expansion is, above all, an expansion in Taylor series of the relevant geometric objects. For any function, coefficients in its Taylor expansion are recursively calculated as derivatives of the starting function, no additional information required. The story is analogous for Carroll gravity in this picture [17]: in the theory at order $(2i)$ one manually sets the $(2i)$-order fields to zero, and solves for the $(2i-2)$-order ones; the actual number of equations of motion is indeed $2i-2$, because they hold only if the fields involved are solutions to the e.o.m.s at all previous orders. In the end, Carroll gravity at order $(2i)$ is recursively constructed to map out corrections to the equation of motions of the starting LO vielbein fields, the greater i, the finer the corrections to their evolution laws, and the true degrees of freedom are thus the ones of the LO theory only. We will return to this matter in Chapter 4, when discussing the interaction terms to include in the Polyakov action.

Chapter 3

Carroll target space

We now move to the main part of our work, which aims to recover the electric sector of Carroll gravity from the context of string theory. The path followed is parallel to the well-known calculation in Bosonic string theory, in which Einstein's equations in vacuum arise as a consequence of prescribing conformal invariance on the worldsheet. This was first done by Friedan [46], but we point to [67] for a more pedagogical outline. We refer to this calculation, reviewed in the following section, as the *relativistic* case, in opposition to the one in the Carroll limit studied in this chapter.

3.1 The relativistic case

The dynamics of a closed bosonic string propagating in a curved background, in the simplest case where one neglects the couplings of the dilaton and the Kalb-Ramond field, is described by the Polyakov action

$$S_{\mathrm{Pol}} = \frac{T}{2}\int_\Sigma \mathrm{d}\sigma^2 \sqrt{\gamma}\,\gamma^{\alpha\beta}\partial_\alpha X^\mu \partial_\beta X^\nu g_{\mu\nu}(X). \tag{3.1}$$

The integral is performed on the worldsheet Σ, a two-dimensional pseudo-Riemannian variety parametrised by the coordinates $\sigma^\mu = (\sigma^0, \sigma^1) = (\tau, \sigma)$ and endowed with the metric $\gamma_{\alpha\beta}$, γ being its determinant. The embedding X^μ is the spacetime coordinate vector, whose components constitute a collection of $D = 1 + d$ fields that transform like scalars on Σ, and that map from the worldsheet to the target space. This is the spacetime manifold $\mathscr{M}$ with Lorentzian signature, equipped with the metric $g_{\mu\nu}(X)$, function of the embedding fields. Finally, T is the string tension. The Polyakov action is the two-dimensional example of a non-linear sigma model, historically introduced to study the quantisation of spin-zero mesons [68]. Conformal invariance is a gauge symmetry in string theory – thus the basic requirement to work with quantised strings – and it provides the gauge freedom to set the worldsheet metric to a flat one $\gamma_{\alpha\beta} = \eta_{\alpha\beta}$. This is the conformal gauge, adopted for the calculation. When strings move on a flat background $g_{\mu\nu} = \eta_{\mu\nu}$, conformal invariance is ensured if the Polyakov action does not transform under Weyl transformations on the worldsheet – that translates into the requirement that the trace of the stress-energy tensor is zero $T_\alpha{}^\alpha = 0$ – and the field theory that it describes is a non-interacting CFT

$$S_0 = \frac{T}{2}\int_\Sigma \mathrm{d}\sigma^2\, \partial_\alpha X^\mu \partial^\alpha X_\mu. \tag{3.2}$$

However, when the target space possesses non-zero curvature $g_{\mu\nu} \neq \eta_{\mu\nu}$ conformal invariance is broken at the quantum level, because in order to regularise divergences renormalisation procedures either require to introduce some energy-scale cut-off – which physical quantities ultimately depend on – or to analytically continue the number of worldsheet dimensions to values other than two (that is the only dimension where conformal invariance holds classically). At the quantum level, conformal invariance is restored by demanding vanishing beta functions of the effective theory, setting it at a fixed point in the renormalisation group flow.

To achieve this, the issue is addressed perturbatively, by expanding the embedding fields around some vacuum expectation value

$$X^\mu(\sigma) = X_0^\mu + Y^\mu(\sigma), \tag{3.3}$$

for small quantum fluctuations Y^μ. Under this reparametrisation, the metric Taylor-expands[1] as

$$g_{\mu\nu}(X) = g_{\mu\nu}(X_0) + \partial_\lambda g_{\mu\nu}(X_0)Y^\lambda + \frac{1}{2}\partial_\rho\partial_\lambda g_{\mu\nu}(X_0)Y^\lambda Y^\rho + o(Y^2). \tag{3.4}$$

One then exploits the liberty to choose the coordinates of the target space to be the ones of the local inertial frame constructed around the point $p \in \mathscr{M}$, with $X^\mu|_p = X_0^\mu$, i.e. going to Riemann normal coordinates (RNC). Here the metric expansion always takes the form of

$$g_{\mu\nu}(X) = \eta_{\mu\nu} - \frac{1}{3}R_{\mu\lambda\nu\rho}(X_0)Y^\lambda Y^\rho + o(Y^2), \tag{3.5}$$

where $R_{\mu\lambda\nu\rho}$ is the lower-indices curvature tensor of the target space. As we will discuss in the next section, this passage is even more crucial in the Carrollian case, and for this reason it becomes instructive to show how (3.5) is derived, as done in Appendix B. At the level of the CFT, the expansion in RNC is equivalent to considering small fluctuations on a flat background metric

$$g_{\mu\nu}(X) = \eta_{\mu\nu} + f_{\mu\nu}(X), \tag{3.6}$$

and the generating function of the worldsheet path integral becomes a perturbation series

$$\begin{aligned} Z &= Z_0 \int DX^\mu e^{S_0+S_f} && (3.7)\\ &= Z_0 \int DX^\mu e^{S_0}\left[1 + \lambda S_f + \frac{1}{2}\lambda^2 (S_f)^2 + o(\lambda^2)\right], && (3.8)\end{aligned}$$

where the S_0 is the one in (3.2), and the interaction piece S_f is interpreted as the embedding coupling to a graviton

$$S_f = \frac{1}{4\pi\alpha'}\int_\Sigma \mathrm{d}\sigma^2\, \partial_\alpha X^\mu \partial^\alpha X^\nu f_{\mu\nu}(X), \tag{3.9}$$

[1]Notation abuse in the truncation remainder that follows: obviously the expansion may be be seen as $Y^\mu \to \epsilon Y^\mu$ for $\epsilon \to 0$. More specifics on the validity on this expansion soon coming.

i.e. the fields couple to the spacetime disturbance by themselves created. The overall dimensional factor of the action has been changed in terms of the (inverse) Regge slope α' to make sense of the expansion parameter λ in (3.8). As the two expansions (3.5) and (3.6) are equivalent, the regime of large α' – i.e. string fluctuations over large distances – or small energies, or weak coupling, as $[\sqrt{\alpha'}] = \mathrm{L}$ – corresponds to the domain of those displacements Y^μ on $\mathscr{M}$ that are sufficiently small to construct RNC at p. Thus, one is allowed to see λ as an effective dimensionless coupling in the perturbation series, that quantifies how small a fluctuation is, compared to the variation in the manifold curvature:

$$\lambda = \frac{\sqrt{\alpha'}}{r_{\mathscr{M}}}, \tag{3.10}$$

where $r_{\mathscr{M}}$ is the characteristic curvature radius[2] of the the manifold at X_0.

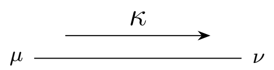
Expanding in RNC, the Polyakov action becomes

$$S_{\text{Pol}} = \frac{T}{2}\int_\Sigma \mathrm{d}\sigma^2 \left[\partial_\alpha Y^\mu \partial^\alpha Y^\nu \eta_{\mu\nu} - \frac{1}{3}R_{\mu\lambda\nu\rho}(X_0)\partial_\alpha Y^\mu \partial^\alpha Y^\nu Y^\lambda Y^\rho\right]. \tag{3.11}$$

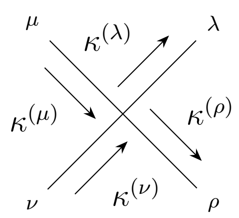
This action describes the CFT of the embedding fields $Y^\mu(\sigma)$ whose interaction is given by the quartic term in (3.11). Note that the target space metric is the actual coupling of the model, so Taylor-expanding it means that one formally deals with an infinite number of interaction terms. Truncating the expansion at order $O(Y^2)$ in RNC corresponds to studying the theory at leading order. Thus, the actual interaction coupling λ is the Riemann tensor of the target space, a tensorial functional of the Y^μ fields evaluated at the fixed solution X_0

$$\lambda = \lambda_{\mu\lambda\nu\rho}(X_0) = -\frac{1}{3}R_{\mu\lambda\nu\rho}(X_0). \tag{3.12}$$

The Feynman rules in momentum space for the propagator and the interaction vertex read



$$\frac{-i\eta^{\mu\nu}}{\kappa^2} \tag{3.13}$$



$$-\frac{i}{3}R_{\mu\lambda\nu\rho}(X_0)\kappa^{(\mu)}_\alpha \kappa^{(\nu)\alpha}\delta^2(\kappa_{\text{in}} - \kappa_{\text{out}}) \tag{3.14}$$

where $\kappa^{(\nu)\alpha}$ stands for α-th component of the momentum of the field Y^μ, and we use the notation $\kappa^\alpha = (\kappa^0, \kappa^1) = (\omega, \kappa)$ for worldsheet momenta. They appear in the

[2]It is inversely proportional to the square root of the Ricci scalar: the smaller $R(X_0)$, the less $\mathscr{M}$ is curved at that point, and the larger its curvature radius.

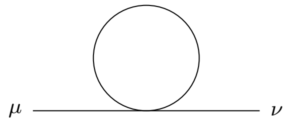


Figure 3.1: Relevant one-loop diagram of the theory

vertex function because we are dealing with a mixed derivative interaction. These rules are used to build the one-loop diagrams and get the correct counterterm that suppresses the divergent part of the action. At one loop the *seagull*[3] diagram in Fig. 3.1 is the only one that contributes. Here we briefly discuss why only logarithmic divergences contribute, and we redirect the reader[4] to [46, 69] for a detailed discussion on renormalisation of the non-linear sigma model. Depending on which vertex legs connect to the propagator in the loop, the integral in momentum-space gives a quadratic, a linear, or a logarithmic divergence. These same divergences arise respectively from the two-point function contractions $\langle Y^\mu(\sigma)Y^\nu(\sigma')\rangle$, $\langle Y^\mu(\sigma)\partial_\alpha Y^\lambda(\sigma')\rangle$ and $\langle \partial_\alpha Y^\mu(\sigma)\partial^\alpha Y^\nu(\sigma')\rangle$ for $(\sigma-\sigma')^\alpha \to 0$. Parametrising in momentum-space polar coordinates $(\omega,\kappa)\to(\theta,p)$, for $|\kappa| = p\to\infty$, the loop integrals blow up in the UV like

$$\int_0^\infty \mathrm{d}p\, \frac{p^{a+1}}{p^2}, \tag{3.15}$$

with $a = 0, 1, 2$ for logarithmic, linear and quadratic divergences respectively.[5] To deal with massless fields in dimensional regularisation, one can introduce an infrared mass cut-off m inside the propagator, that is sent to zero at the end of the computation. Using standard regularisation techniques, for a general number of dimensions $d \in \mathbb{C}$, the three possible values $(a = 0, 1, 2)$ of the full loop integrals evaluate to

$$\int \frac{\mathrm{d}^2\kappa}{(2\pi)^2}\frac{(\kappa^2)^{a/2}}{\kappa^2+m^2}\eta^{\mu_a\nu_a} \underset{2\to d}{\longrightarrow} \mu^{2-d}\int_{S^d}\frac{\mathrm{d}\Omega_d}{(2\pi)^d}\int_0^\infty \mathrm{d}p\,\frac{p^{d-1+a}}{p^2+m^2}\eta^{\mu_a\nu_a} \tag{3.16}$$

$$= \frac{\mu^a}{(4\pi)^{d/2}}\Gamma\left(-\frac{d}{2}\right)\left(\frac{m}{\mu}\right)^{d+a-2}\eta^{\mu_a\nu_a}, \tag{3.17}$$

where $\mathrm{d}\Omega_d$ is the angular measure of the d-dimensional sphere S^d, Γ is the Euler Gamma function, and μ is the energy scale at play, introduced to restore the correct mass dimension of the integral. $\eta^{\mu_a\nu_a}$ comes from the propagator in the loop and, referring to (3.14), its indices $(\mu_a\nu_a)$ correspond to $(\lambda\rho)$ for $a = 0$, to either $(\mu\rho)$, $(\mu\lambda)$, $(\nu\rho)$ or $(\nu\lambda)$ for $a = 1$, and to $(\mu\nu)$ for $a = 2$. Now, for $d\to 2$, the linear $(a = 1)$ and quadratic $(a = 2)$ integrals are zero once the limit $m \to 0$ is taken,

[3]Overstepping the line set by trees and tadpoles, a restrict pool of authors refers to graphs of the shape of the 3.1 as "seagull" diagrams, probably because they are capable to draw one in a way that resembles the feathered mariner.

[4][69] is Polyakov's original article on renormalisation of infrared divergences for interacting Goldstone particles in two dimensions, and [46] is the first paper by Friedan where a general overview of the non-linear sigma model is studied. See Chapter 13 of [70] for a more pedagogical approach, and Chapter 14 of [71] for a general review

[5]Powers of p^a at the numerator depend which legs of the vertex (3.14) are contracted with the propagator in the loop, whether they include none, one or two factors of ∂Y.

recovering the fact that dimensional regularisation sets to zero any UV power-law divergence[6]. For $a = 0$, the m factor in (3.17) carries a power of $d-2$ that sets an ambiguity in its behaviour for $m \to 0$, depending whether one restores the correct dimensions as $d = 2+\epsilon$ or $d = 2-\epsilon$, performing the limit $\epsilon \to 0^+$. However, as explained in [72], the analytic continuation of the integral makes sense only for $\Re(d) \in (2,4)$, so we pick $d = 2+\epsilon$. Expanding (3.17) for ϵ around 0 yields

$$\frac{\eta^{\lambda\rho}}{4\pi}\left[\frac{2}{\epsilon} - \log\left(\frac{\mu^2}{4\pi m^2}\right) + O(\epsilon)\right], \tag{3.18}$$

result that we can use to construct the counterterm for the propagator inside the loop, that cancels the divergence in the action (3.11), that is

$$\frac{1}{12\pi\epsilon}\int \mathrm{d}^2\sigma\ \partial_\alpha Y^\mu \partial^\alpha Y^\nu R_{\mu\nu}(X_0). \tag{3.19}$$

The one-loop β function[7] is proportional to the $1/\epsilon$ pole of the renormalised coupling, namely

$$\beta_{\mu\nu} \propto R_{\mu\nu}(X_0), \tag{3.20}$$

and thus, finally, conformal invariance holds for

$$R_{\mu\nu}(X_0) = 0. \tag{3.21}$$

This concludes the calculation. We must now discuss two critical issues that we had initially swept under the rug: first, renormalisation of theories with massless fields has to take care of IR divergences as well, and the loop integral for $a = 0$ represents a prominent example where they arise severely; second, to be able to give an analytic continuation of the one-loop integral we introduced the mass term m, and this spoils conformal gauge invariance. The resolution of the latter problem can be achieved merely by calculating the integral with other methods that do not involve the introduction of a soft mass term, and in this regard we direct to the work of Leibbrandt [72], where the issue is circumvent by a generalised Gaussian integration redefinition. On the other hand, dimensional regularization itself, as any other renormalisation procedure, breaks conformal symmetry, and this is why the coupling β function does not vanish automatically, but must be set to zero by hand, the whole point of this computation. So, to be fair, introducing the soft mass is a trick that reproduces the right counterterm, but it is not the most correct renormalisation procedure. Regarding IR divergences, we can make sense of why they can be ignored here thanks to the assumptions under which we expanded the action in the first place: as already explained, the perturbative expansion (3.8) – as well as (3.11) – is valid in the domain where fluctuations of the Y^μ fields are small

[6] To be precise, all *even* power-law divergences are null in dimensional regularisation, but one can use similar arguments to claim that this holds for $a = 1$ too in this case, as the procedure is truly sensible to logarithmic divergences only.

[7] We do not show here that the addition of the counterterm can be made absorbed by a wave-function renormalization $\delta_z = (1/6\epsilon)R^\mu{}_\nu Y^\nu$. This is not taken into account, as we are not interested in the exact form of the β function.

compared to the perturbations they create on the target space ($\sqrt{\alpha'} \ll r_{\mathscr{M}}$), and so in the regime of large momenta. This is why we can give cold shoulder to what happens to (3.17) at $p \to 0$.

3.2 The Carroll case

We now repeat what has just been done in the previous section, this time for a target space where the $c \to 0$ takes place: a Carroll manifold. We can allow ourselves to set in a general number of $D = 1 + d$ target space dimensions, as all results of the previous chapters hold (or are quickly generalisable) for any number of spatial directions. First and foremost, we shall preliminarily reveal that our calculation does not recover all three equations of LO Carroll gravity (2.57), but only the first constraint equation. We defer argumentations about the possible reasons for this, and the discussion on the different approaches one could take in the last part of this work. As we there discuss, we see basically two ways to bring the expansion of the Polyakov action in the LO Carroll picture: either working with a Carroll manifold from the beginning – with its defining degenerate metric, temporal vector field and compatible connection – and constructing curvature and normal coordinates without the need to perform any c expansion; or starting from the standard Lorentzian framework of GR and then expanding everything in powers of c, keeping only the first non-vanishing order (LO) of the curvature tensor. Here, we go for the second approach, which follows the trail marked by [17], in the formalism of a Carroll expansion of GR.

3.2.1 Expansion scheme

If we start from a Lorentzian manifold $\mathscr{M}$, we can take for granted that a similar expansion in normal coordinates like (3.11) inside the Polyakov action holds, because first derivatives of the metric tensor are zero at the point p where RNC are defined, as the connection is the nice and symmetric Levi-Civita one Γ (cf. Appendix B). After all, the local vanishing of the metric's first derivatives is the geometric way to phrase the equivalence principle, which must indisputably hold if GR is the starting point. The difference is that now we can express the metric and the Riemann tensor in terms of the PUL vielbein, and expand it in even powers of c. We start with the curvature tensor, which is usually constructed as

$$R_{\mu\lambda\nu}{}^{\sigma} = \partial_\lambda \Gamma^{\sigma}_{\mu\nu} - \partial_\mu \Gamma^{\sigma}_{\lambda\nu} + \Gamma^{\sigma}_{\lambda\tau}\Gamma^{\tau}_{\mu\nu} - \Gamma^{\sigma}_{\mu\tau}\Gamma^{\tau}_{\lambda\nu}. \tag{3.22}$$

Out of an abundance of scrupulousness, we restate (2.46)

$$\Gamma^{\sigma}_{\mu\nu} = \frac{1}{c^2}\overset{(-2)}{\Gamma}{}^{\sigma}_{\mu\nu} + \tilde{C}^{\sigma}_{\mu\nu} + S^{\sigma}{}_{\mu\nu} + c^2\overset{(-2)}{\Gamma}{}^{\sigma}_{\mu\nu}. \tag{3.23}$$

Here, $\tilde{C}$ is the CCC of (2.40), namely

$$\begin{aligned} \tilde{C}^{\sigma}_{\mu\nu} =& - V^{\sigma}\left[\partial_{(\mu}T_{\nu)} + T_{(\mu}\mathcal{L}_V T_{\nu)}\right] \\ &+ \frac{1}{2}\Pi^{\sigma\rho}\left(\partial_\mu \Pi_{\nu\rho} + \partial_\nu \Pi_{\rho\mu} - \partial_\rho \Pi_{\mu\nu}\right) \\ &- \Pi^{\sigma\rho}T_\nu K_{\mu\rho}. \end{aligned} \tag{3.24}$$

The other terms in (3.23) are given by

$$\overset{(-2)}{\Gamma}{}^{\sigma}_{\mu\nu} = -V^{\sigma}K_{\mu\nu}, \tag{3.25}$$

$$S^{\sigma}_{\mu\nu} = \Pi^{\sigma\rho}T_\nu K_{\mu\rho}, \tag{3.26}$$

$$\overset{(2)}{\Gamma}{}^{\sigma}_{\mu\nu} = -\Pi^{\sigma\rho}T_{(\mu}\mathrm{d}T_{\nu)\rho}, \tag{3.27}$$

and again, we stress that these, as all terms denoted with round brackets superscripts, are not yet coefficients of any light speed expansion, but pre-expansion terms that come with different powers of c. Plugging the expression (3.23) inside (3.22) and performing tedious calculations, one obtains an expression for the Riemann tensor that gathers as

$$R_{\mu\lambda\nu}{}^{\sigma} = \frac{1}{c^4}\overset{(-4)}{R}{}_{\mu\lambda\nu}{}^{\sigma} + \frac{1}{c^2}\overset{(-2)}{R}{}_{\mu\lambda\nu}{}^{\sigma} + \overset{(0)}{R}{}_{\mu\lambda\nu}{}^{\sigma} + c^2\overset{(2)}{R}{}_{\mu\lambda\nu}{}^{\sigma} + c^4\overset{(4)}{R}{}_{\mu\lambda\nu}{}^{\sigma}. \tag{3.28}$$

However, for the sake of RNC, we are interested in the all-down-indices version of these tensors. Lowering the upper index with the Lorentzian metric in PUL parametrisation

$$R_{\mu\lambda\nu\rho} = R_{\mu\lambda\nu}{}^{\sigma}g_{\sigma\rho} = R_{\mu\lambda\nu}{}^{\sigma}\left(-c^2T_\sigma T_\rho + \Pi_{\sigma\rho}\right), \tag{3.29}$$

leads to the expression

$$R_{\mu\lambda\nu\rho} = \frac{1}{c^4}\overset{(-4)}{R}{}_{\mu\lambda\nu\rho} + \frac{1}{c^2}\overset{(-2)}{R}{}_{\mu\lambda\nu\rho} + \overset{(0)}{R}{}_{\mu\lambda\nu\rho} + c^2\overset{(2)}{R}{}_{\mu\lambda\nu\rho} + c^4\overset{(4)}{R}{}_{\mu\lambda\nu\rho} + c^6\overset{(6)}{R}{}_{\mu\lambda\nu\rho}. \tag{3.30}$$

The reader could probably find this pitch a bit pedantic, but we want to be sure the expansion scheme is clear, both to ease the process of potential eager cross-checkers, and because one can easily drown in the ocean of terms arising here. A full expression for (3.30) up to order c^2 is given in Appendix C. The c^{-4} term (3.28) is proportional to

$$\overset{(-2)}{\Gamma}{}^{\sigma}_{\lambda\tau}\overset{(-2)}{\Gamma}{}^{\tau}_{\mu\nu} - \overset{(-2)}{\Gamma}{}^{\sigma}_{\mu\tau}\overset{(-2)}{\Gamma}{}^{\tau}_{\lambda\nu}, \tag{3.31}$$

and vanishes due to (2.35). The same goes for its lower-indices sibling in (3.30)

$$\overset{(-4)}{R}{}_{\mu\lambda\nu\rho} = 0 \tag{3.32}$$

and the first non-zero contribution in (3.30) is given by the c^{-2} term. This is schematically given by

$$\overset{(-2)}{R}{}_{\mu\lambda\nu\rho} = \left[\left(\partial_\lambda \overset{(-2)}{\Gamma}{}^{\sigma}_{\mu\nu} + \overset{(-2)}{\Gamma}{}^{\sigma}_{\lambda\tau}\tilde{C}^{\tau}_{\mu\nu} + \tilde{C}^{\sigma}_{\lambda\tau}\overset{(-2)}{\Gamma}{}^{\tau}_{\mu\nu}\right) - (\lambda \rightleftharpoons \mu)\right]\Pi_{\sigma\rho}, \tag{3.33}$$

that, after some manipulation that exploits properties (2.35), (2.36) and all four (2.10), reduces to

$$\overset{(-2)}{R}_{\mu\lambda\nu\rho} = K_{\mu\nu}K_{\lambda\rho} - K_{\mu\rho}K_{\nu\lambda}, \tag{3.34}$$

and it will make up our LO interaction term. Regarding the kinetic part, the term quadratic in Y in the action is the metric tensor at p, and its leading contribution is simply the spatial metric

$$\overset{(0)}{g}_{\mu\nu} = \Pi_{\mu\nu}. \tag{3.35}$$

Now, all pieces together, the Polyakov action reads

$$\begin{aligned} S_{\text{Pol}} = \frac{T}{2}\int_\Sigma \mathrm{d}\sigma^2 \Bigg[&\left(\Pi_{\mu\nu} - c^2 T_\nu T_\nu\right)\partial_\alpha Y^\mu \partial^\alpha Y^\nu \\ &-\frac{1}{3}\left(\frac{1}{c^2}\overset{(-2)}{R}_{\mu\lambda\nu\rho} + \overset{(0)}{R}_{\mu\lambda\nu\rho} + c^2\overset{(2)}{R}_{\mu\lambda\nu\rho} + \cdots\right)\partial_\alpha Y^\mu \partial^\alpha Y^\nu Y^\lambda Y^\rho\Bigg], \end{aligned} \tag{3.36}$$

where the dots stand for the Riemann tensor terms carrying powers higher than c^2, and from here on the X_0 dependence is suppressed, as every functional of Y^μ is to be implicitly evaluated at p, unless differently specified. Looking at (3.36), it seems we are running into a problem, as the first non-vanishing terms – that will constitute the LO contribution in the c expansion – appear with different c powers.

Before solving this, note that we are not working with natural units, because we make $c \neq 1$ appearing everywhere, and some dimensional analysis on the action is necessary. Working in SI units, with worldsheet coordinates taken to be dimensionless, and introducing an auxiliary dimensional parameter ξ

$$[S_{\text{Pol}}] = [\xi]\ [T]\ [Y]^2 = [\xi]\ \text{N m}^2 = [\xi]\ \text{Kg m}^3\ \text{s}^{-2}, \tag{3.37}$$

we see that for the action to have the correct dimensions Kg m^2 s^{-1}, ξ must be an inverse velocity, i.e. $\xi = 1/c$.

Back to (3.36), bringing the first non-vanishing kinetic and vertex terms to the same order can be achieved through the redefinition

$$Y^\mu(\sigma) = cY'^\mu(\sigma). \tag{3.38}$$

Already suppressing the prime on fields, this yields

$$S_{\text{Pol}} = \frac{Tc}{2}\int_\Sigma \mathrm{d}\sigma^2 \left[\quad \Pi_{\mu\nu}\partial_\alpha Y^\mu \partial^\alpha Y^\nu - \frac{1}{3}\overset{(-2)}{R}_{\mu\lambda\nu\rho}\partial_\alpha Y^\mu \partial^\alpha Y^\nu Y^\lambda Y^\rho + \cdots\right], \tag{3.39}$$

where one factor of c gets simplified with the one coming from ξ.

3.2.2 Conformal invariance for Carroll target space

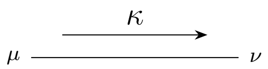
We can now properly expand vielbeins in powers of c, and perform the Carroll limit. In doing so, we define $\tilde{T} = Tc$, and keep it fixed for $c \to 0$. The action in equation (3.39) becomes then

$$\overset{(LO)}{S}_{\text{Pol}} = \frac{\tilde{T}}{2}\int_\Sigma \mathrm{d}\sigma^2 \left[h_{\mu\nu}\partial_\alpha Y^\mu \partial^\alpha Y^\nu - \frac{1}{3}(k_{\mu\nu}k_{\lambda\rho} - k_{\mu\rho}k_{\nu\lambda})\partial_\alpha Y^\mu \partial^\alpha Y^\nu Y^\lambda Y^\rho\right]. \tag{3.40}$$

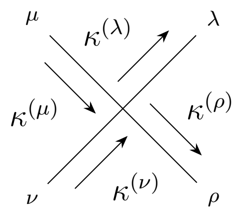
Note that because of the redefinition (3.38), the fields Y' have units of s^{-1}, and restoration of their sensible length dimension shall be done once conformal symmetry is restored. We cannot construct the propagator yet, as the Carroll limit has been taken, and now the matrix in the kinetic term is degenerate, thus not invertible. The only way to circumvent the obstacle is to give up Carroll invariance on the embedding, and perform the calculation in a specific Carroll frame. This compromise is not as bitter as it sounds, because also the equations (2.57) are boost dependent. Moreover, note that both the kinetic and the interaction operators are Carroll-invariant quantities even before choosing a boost frame (because $k_{\mu\nu}$ is constructed with v^λ and $h_{\mu\nu}$ (cf.(2.36)), that do not transform under Carroll boosts). We choose a frame such that the metric data is the one given in (2.29), for which the inverse metric

$$h_{\mu\rho}h^{\rho\nu} = h_{ik}h^{kj} = \delta_i^j \tag{3.41}$$

is properly defined on the foliation of spatial hypersurfaces realised by this choice. In the notation, we keep Greek indices so that the result can be quickly generalised for other Carroll frame choices. The Feynman rules for the theory described by (3.40) are then



$$\frac{-ih^{\mu\nu}}{\kappa^2} \tag{3.42}$$



$$-\frac{i}{3}(k_{\mu\nu}k_{\lambda\rho} - k_{\mu\rho}k_{\nu\lambda})\kappa^{(\mu)}_\alpha \kappa^{(\nu)\alpha}\delta^2(\kappa_{\mathrm{in}} - \kappa_{\mathrm{out}}) \tag{3.43}$$

where all tensors are evaluated at the classical solution $X_0^\mu(\sigma)$, for constant[8] $X_0^0 = s$. The logarithmically divergent counterterm can be calculated like already done in the previous section, just replacing $\eta^{\lambda\rho}$ with $h^{\lambda\rho}$. The result is

$$\delta S = \frac{1}{12\pi\epsilon}\int \mathrm{d}^2\sigma\ \partial_\alpha Y^\mu \partial^\alpha Y^\nu (k_{\mu\nu}k - k_\mu{}^\lambda k_{\nu\lambda}), \tag{3.44}$$

where $h^{\lambda\rho}$ is used to raise and lower indices (and take traces) of tensors proper to the hypersurface realisation (2.29), like the extrinsic curvature. The one-loop β functional is then proportional to

$$\beta_{\mu\nu} \propto (k_{\mu\nu}k - k_\mu{}^\lambda k_{\nu\lambda}), \tag{3.45}$$

and conformal invariance is then restored at the quantum level provided that

$$k_{\mu\nu}k - k_\mu{}^\lambda k_{\nu\lambda} = 0. \tag{3.46}$$

[8]It does not make much of a difference, as we take the expectation value X_0^μ to be a constant classical solution.

Contracting this with $h^{\mu\nu}$ reproduces the first LO Carroll Gravity constraint equation

$$k_{\mu\nu}k^{\mu\nu} - k^2 = 0 \tag{3.47}$$

while any other vielbein contraction leads to a trivial identity. This ends our calculation, and in the following chapter we try to understand why the procedure used here is not able to reproduce all three equations (2.57).

We conclude this chapter with a note on the coordinates expansion validity. Like in the relativistic case, the action (3.40) describes a theory as an approximation that is valid as long as the fluctuations of $Y^\mu(\sigma)$ are small compared to the manifold's curvature at p. However, in this context it is more correct to consider the amplitude of quantum fluctuations relative to the extrinsic curvature of the hypersurface, and specifically to its *mean* curvature. This quantity can be defined [73] as the sum of the principal curvatures of the hypersurface, the eigenvalues of $k_{\mu\nu}$ w.r.t. a orthonormal frame of eigenvectors tangent to the hypersurface (the three $e_\mu{}^a$ in our case), and it is therefore nothing but the trace k. One can then define the characteristic radius of curvature of the spatial hypersurface Σ_s as $r_{\Sigma_s} = 1/\sqrt{k}$, and consider the domain of validity of the approximation as the one for which

$$\lambda = \frac{\sqrt{\alpha'}}{r_{\Sigma_s}} \ll 1 \tag{3.48}$$

holds, being λ the path integral formal expansion parameter.

Chapter 4

Towards other possibilities

At this point, the reader should have realised that the title of this Master's Thesis is susceptible of some reasonable criticism: our calculation has not recovered the full set of LO Carroll Gravity field equations, but solely the first. In this chapter, we first explain the motivation underlying the approach we followed, and then outline the other possible direction that can be – and to some extent, have been – considered to tackle the issue.

4.0.1 Motivation behind the main approach

Multiple factors originally raised expectations for the direction we took to be the most correct one. Above all, the results of [17] suggest that, in the PUL parametrisation, the part of the Ricci tensor that is relevant for generating the LO action of electric Carroll gravity is the one that scales as c^{-2}, that is its first non vanishing component. Our PUL Riemann tensor (3.30) starts being non-zero at the same order, and it is therefore reasonable to expect the c^{-2} term to be the one to single out in the action (3.39). Moreover, this piece contains terms quadratic in the extrinsic curvature, and this is a primary requisite when the aim is to recover (2.57). It is possible that the story does not end with this term alone – meaning that the full set of equations (2.57) requires the inclusion of the (0)-th Riemann term, and/or subsequent ones – but such a modus operandi starts being somewhat arbitrary and ambiguous. Indeed, once the structure of the PUL parametrisation is chosen, the form of the curvature tensor follows, and the fields redefinition (3.38) is the most natural way to obtain an interacting field theory on the worldsheet. Forcing the relevant interaction terms to come from the (0)-th part in (3.30) would require to manually set

$$\overset{(-2)}{R}_{\mu\lambda\nu\rho} = 0, \tag{4.1}$$

without any sensible justification. Alternatively, one could be less restrictive, and impose the above condition to hold only at first order in the c expansion

$$k_{\mu\nu}k_{\lambda\rho} - k_{\mu\lambda}k_{\nu\rho} = 0. \tag{4.2}$$

However, both solutions seem bizarre, as they would imply either that a condition on the same vielbein fields can hold at some orders in c, but not in others, factually discriminating any quantity one the sole basis of its c factors even *before* Carroll-expanding; or, that any term quadratic in $k_{\mu\nu}$ has to vanish at p, but then (2.57) would become a trivial identity carrying no information. To be fair, this is not as strange as it might sound, because the magnetic limit of Carroll gravity requires that the extrinsic curvature vanish, and the equations that define the magnetic sector are seen as corrections to the LO ones [17]. However, we are interested in the electric limit, that correspond to the LO theory; thus, we do not see how including subsequent orders in the action could be admissible for this aim. In addition, if one allows for the inclusion of other orders in c, one way or another, the sub-leading fields N_μ, M^μ, $\psi_{\mu\nu}$ and $\psi^{\mu\nu}$ would inevitably be included in the c expansion of the action at its new relevant order. We know that these do not play any role in the electric theory, and so they should be manually set to zero after taking the Carroll limit. To shed some light on this issue, a sensible cross-check would be to verify the equivalence between three equations (2.57) and

$$\overset{(-2)}{R}_{\mu\nu} = 0, \tag{4.3}$$

(cf. (2.49)) in the $c \to 0$ limit. This should be the case if LO Carroll gravity is indeed equivalent to the theory at order c^{-2}. However, we were not able to show this, as all possible contractions on (4.3) do not seem to provide contents equivalent either to (2.57) or (2.56). Perhaps, this fact could be suggesting us that the issue is not much related to the Polyakov action, but to the expansion scheme itself.

Additionally, the line of action we have chosen in the present work has two advantages: first, following the footsteps of [17] allowed us to make use of an expansion formalism that, if correct, could in principle make any order of Carroll Gravity emerge from the Polyakov action, by requiring conformal invariance to the resulting CFTs at any order; second, constructing the curvature tensor with the Levi-Civita connection substantially simplifies calculations. In fact, the expression for the metric in RNC is relatively simple because Γ is torsionless, and, as we discuss in the next section, things gets more complicated if one include torsion, for the number of terms involved in the coordinate (and consequently in the c) expansion dramatically scales up.

4.1 A parallel direction

As already mentioned in Section 3.2, we have identified two possible ways that should lead to the proper Carroll Polyakov action, and these are schematically represented in Fig. 4.1: the direction followed in the previous chapter has been to first expand in RNC and in powers of c, and then take the Carroll limit; a different line of investigation consists instead in sending c to zero at the beginning, settling into a proper geometric Carroll framework, and expanding the metric in what we call Carroll normal coordinates (CNC).

As LO Carroll gravity is consistently described either as a Carroll expansion of GR [17], or as the electric contraction of Lorentz-invariant field theories [8, 14], we expect that, if this theory can indeed be made arise from the context of strings, the two methods of 4.1 should lead to the same outcome. This said, we emphasise that one cannot a priori assume neither that these two procedures commutes – as the Carroll

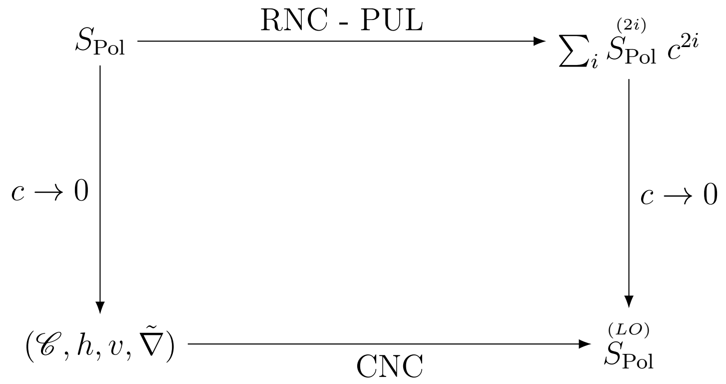


Figure 4.1: Scheme of the two possible paths from the Polyakov action (top-left) to its LO Carroll version (bottom-right). The approach pursued in the third chapter goes clockwise: expanding the metric in RNC, writing it in PUL parametrisation and then performing the Carroll limit to single out LO terms. The second way is anti-clockwise: the Carroll limit is taken at the start, and the expansion in Carroll normal coordinates (CNC) is built using the connection $\tilde{C}$.

geometry is set in a degenerate zero-signature singular manifold – nor that they are in any sense reversible, since the Carroll limit "destroys" the information carried by the fields at sub-leading orders.

4.1.1 Enter Torsion

The other method that has been partially investigated makes no reference to any expansion in the speed of light, but builds on the proper geometric structures of a Carroll manifold. The tetrads employed to described it are denoted as $(v^\mu, {\theta_a}^\mu)$ and $(\tau_\mu, {e_\mu}^a)$, and satisfy conditions identical to (2.18) and (the first of) (2.20)[1]. The first fundamental tensor is the degenerate spatial metric $h_{\mu\nu}$, and its expansion in adapted local coordinates is given by CNC, defined by the connection $\tilde{C}$. This time the CCC is a defining object for the manifold; thus, Carroll covariance and curvature are derived from it. In the relativistic case, RNC serve both to express the metric taylor expansion around p in a covariant form, and to set in a frame where its first derivatives vanish, and – as shown in Appendix B – this is done by working out some combinatorial properties of the Christoffel symbols and their derivatives at the level

[1]The notation adopted here adheres to that used for LO fields of the previous chapters, but we again stress that this is just a matter of convention, as now these quantities are not leading order terms of a Carroll expansion.

of geodesic equations. However, in the case at hand[2] the CCC

$$\begin{aligned}\tilde{C}^{\lambda}_{\mu\nu} =& - v^{\lambda}\left[\partial_{(\mu}\tau_{\nu)} + \tau_{(\mu}\mathcal{L}_{v}\tau_{\nu)}\right] \\ &+ \frac{1}{2}h^{\lambda\rho}\left(\partial_{\mu}h_{\nu\rho} + \partial_{\nu}h_{\rho\mu} - \partial_{\rho}h_{\mu\nu}\right) \\ &- h^{\lambda\rho}\tau_{\nu}k_{\mu\rho}\end{aligned} \tag{4.4}$$

is not symmetric in lower indices because of the intrinsic and irremovable presence of torsion

$$\tilde{T}^{\lambda}{}_{\mu\nu} = 2\tilde{C}^{\lambda}_{[\mu\nu]} = 2h^{\lambda\rho}\tau_{[\mu}k_{\nu]\lambda}. \tag{4.5}$$

To construct CNC we borrow the results of [74], that hold pointwise at p, locus of the normal expansion. For a generic torsion-full connection (already indicated as $\tilde{C}$), and for any covariant (2,0) tensor $A_{\mu\nu}$, with arbitrary signature, the expressions of its first and second derivatives read

$$\partial_{\lambda}A_{\mu\nu} = \tilde{\nabla}_{\lambda}A_{\mu\nu} + \tfrac{1}{2}\tilde{T}^{\sigma}{}_{\lambda\mu}A_{\sigma\nu} + \tfrac{1}{2}\tilde{T}^{\sigma}{}_{\lambda\nu}A_{\mu\sigma}, \tag{4.6}$$

$$\begin{aligned}\partial_{\lambda}\partial_{\rho}A_{\mu\nu} =& \tilde{\nabla}_{(\lambda}\tilde{\nabla}_{\rho)}A_{\mu\nu} + \tilde{T}^{\sigma}{}_{(\lambda\mu}\tilde{\nabla}_{\rho)}A_{\sigma\nu} + \tilde{T}^{\sigma}{}_{(\lambda\nu}\tilde{\nabla}_{\rho)}A_{\mu\sigma} + \tfrac{1}{2}\tilde{T}^{\sigma}{}_{(\lambda\mu}\tilde{T}^{\tau}{}_{\rho)\nu}A_{\sigma\tau} \\ &+ \tfrac{1}{3}\left[-\tilde{R}_{\mu(\lambda\rho)}{}^{\sigma} + 2\tilde{\nabla}_{(\lambda}\tilde{T}^{\sigma}{}_{\rho)\mu} - \tilde{T}^{\tau}{}_{\mu(\lambda}\tilde{T}^{\sigma}{}_{\rho)\tau}\right]A_{\sigma\nu} \\ &+ \tfrac{1}{3}\left[-\tilde{R}_{\nu(\lambda\rho)}{}^{\sigma} + 2\tilde{\nabla}_{(\lambda}\tilde{T}^{\sigma}{}_{\rho)\nu} - \tilde{T}^{\tau}{}_{\nu(\lambda}\tilde{T}^{\sigma}{}_{\rho)\tau}\right]A_{\mu\sigma}.\end{aligned} \tag{4.7}$$

Where the Riemann tensor is defined by the connection as

$$\tilde{R}_{\mu\lambda\nu}{}^{\sigma} = \partial_{\lambda}\tilde{C}^{\sigma}_{\mu\nu} - \partial_{\mu}\tilde{C}^{\sigma}_{\lambda\nu} + \tilde{C}^{\sigma}_{\lambda\tau}\tilde{C}^{\tau}_{\mu\nu} - \tilde{C}^{\sigma}_{\mu\tau}\tilde{C}^{\tau}_{\lambda\nu}. \tag{4.8}$$

If we replace $A_{\mu\nu}$ with $h_{\mu\nu}$ in (4.6), we immediately see that its first derivative is non zero. Plugging (4.5), using (2.38) and some manipulation on the indices symmetries, it reads

$$\partial_{\lambda}h_{\mu\nu} = \tau_{\lambda}k_{\mu\nu} - \tau_{(\mu}k_{\nu)\lambda}. \tag{4.9}$$

Its second derivative, even if (2.38) kills the first three terms in the first line of (4.7) (with h in place of A), still remains a tremendously complicated expression with numerous terms. We report the explicit form of the action in CNC in Appendix C, and give here its condensed quartic interaction term:

$$\Big[\tfrac{1}{12}\left(\tilde{T}^{\sigma}{}_{\lambda\mu}\tilde{T}^{\tau}{}_{\rho\nu} + \tilde{T}^{\sigma}{}_{\mu\rho}\tilde{T}^{\tau}{}_{\lambda\nu}\right)h_{\sigma\tau} + \tfrac{1}{3}\left(-\tilde{R}_{\mu\lambda\rho}{}^{\sigma} + 2\nabla_{\lambda}\tilde{T}^{\sigma}{}_{\rho\mu} - \tilde{T}^{\tau}{}_{\mu\lambda}\tilde{T}^{\sigma}{}_{\rho\tau}\right)h_{\sigma\nu}\Big]\partial_{\alpha}Y^{\mu}\partial^{\alpha}Y^{\nu}Y^{\lambda}Y^{\rho}. \tag{4.10}$$

Here the fields contractions help simplifying the combinatorics of symmetric and antisymmetric parts. When expanding in CNC inside the Polyakov action, the bit

[2] Note that we have not given up on the use of the upper-indices spatial metric $h^{\mu\nu} = e_a{}^{\mu}e_b{}^{\nu}\delta^{ab}$ for convenience. The reader should not attribute any fundamental significance to this quantity, as it is just a tensor defined via tetrads that is not invariant under Carroll boosts, like τ_{μ}.

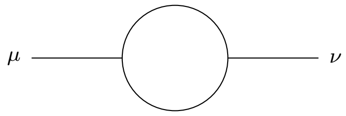
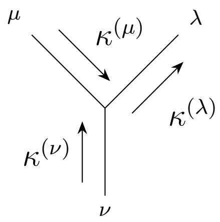


Figure 4.2: First relevant one-loop diagram constructed with cubic vertices

coming from (4.9) brings in cubic interactions for the fields Y^μ, with the corresponding Feynman rule given by



$$-i\left[\tau_\lambda k_{\mu\nu} - \tau_{(\mu}k_{\nu)\lambda}\right]\kappa^{(\mu)}_\alpha \delta^2(\kappa_{\text{in}} - \kappa_{\text{out}}). \tag{4.11}$$

The new one-loop diagram in Fig.4.2 enters the calculation for the beta function, and, up to constant factors and signs, its value is[3]

$$\frac{1}{4}\tau_\mu \tau_\nu k_{\lambda\rho} k^{\lambda\rho} \int \frac{\mathrm{d}^2\kappa}{(2\pi)^2}\, \frac{1}{\kappa^2}. \tag{4.12}$$

As promising as it could seem from the start, this calculation falls short of expectations. After some combinatorics that exploits symmetries of dummy indices, terms quadratic in the extrinsic curvature coming from the quartic interaction bit (cf. Appendix C) sum up to

$$\frac{3}{8}\tau_\lambda \tau_{[\mu} k_{\rho]\sigma} k_\nu{}^\sigma \partial_\alpha Y^\mu \partial^\alpha Y^\nu Y^\lambda Y^\rho, \tag{4.13}$$

and two major problem arise: first, this bit is killed if contracted with the $h^{\lambda\rho}$ of the propagator that gives the logarithmic divergence; second, even if this was not the case, there is no contraction that can produce any term proportional to the squared trace k^2. This latter issue also affects the value (4.12) of the diagram 4.2. Since the goal is to recover (2.57), this term must appear somewhere in the value of the diagrams. Its absence suggests either a mistake in our calculations, or an overlooked contribution, or that this approach is, for some reason, unsuitable for the purpose we intended to pursue.

4.2 Miscellanea

[3]Here again we consider only logarithmic divergences, and rectify that the actual factor of proportionality is a sum of the combinations given by the different ways of contracting legs with internal propagators. This is not relevant for the sake of the calculation at hand, as the diagram factors one gets are equivalent in both cases.

In this final section, we present some further considerations on the potential fallacies of our methods, and suggest alternative approaches that could be inspected to address the issue.

4.2.1 In between the two approaches

Comparing the results of the main procedure and of the secondary one of the last section, we noticed that in (4.7) if one:

- uses the covariant derivative ∇ defined by the Levi-Civita connection Γ, in its PUL decomposition (3.23) in place of the CCC one $\tilde{\nabla}$,
- consequently, uses the Riemann tensor $R_{\mu\lambda\nu}{}^{\sigma}$ of (3.22) in place of $\tilde{R}_{\mu\lambda\nu}{}^{\sigma}$,
- and keeps torsion,

once expanded in c, the c^{-2} order of the bit coming from the second term in parentheses in the second line of (4.10) – the only one that possesses parts scaling as c^{-2} – exactly cancels with the terms from (3.34) at the same order in the expansion

$$\frac{1}{c^2}\left[\tfrac{2}{3}(k_{\mu\nu}k_{\lambda\rho}-k_{\mu\rho}k_{\nu\lambda})-\tfrac{2}{3}(k_{\mu\nu}k_{\lambda\rho}-k_{\mu\rho}k_{\nu\lambda})\right]=0. \tag{4.14}$$

This would mean that, if these replacecments are justifiable, the first non-trivial order to inspect should be the one coming from the c^0 gatherings, an eventuality already considered in 4.0.1. However, we know that such substitutions are not allowed, as if curvature tensors are built with Γ all torsion-dependent terms vanish, while keeping non-zero torsion would imply that the connection at hand cannot be the Levi-Civita one. It might be that we faced a mere coincidence here, but if that is not the case, two consequences would follow. Above all, this evidently wrong calculation would suggest that our approaches are either both wrong, or both correct, although incomplete. Secondly, it would corroborate the fact that: even if the connection is torsionless, the Carroll expansion breaks geodesic symmetry, and the curvature resulting from it describes dynamics of vector fields that are subjected to the presence of a torsion. To be fair, this has been well established already, as it is the reason one can safely decide to describe Carrollian geometry either as the Carroll limit of Lorentzian one, or as the differential topology that arises from the Carroll group as a contraction from the Poincaré one. Nevertheless, it is interesting to consider that this duality can manifest itself in such a practical framework as the one of normal coordinates.

4.2.2 An infinite amount of terms

There is a subtlety involved in the c expansion inside the Polyakov action that so far we partially glossed over, and it comes from propagators. Schematically, the Polyakov action in RNC Carroll-expands as

$$S_{\text{Pol}}\propto T\int \mathrm{d}\sigma^2\sum_{i=0}^{\infty}\left[c^{2i}\overset{(2i)}{\mathcal{O}}_{\text{kin}}(\partial Y)^2+c^{2i+l}\overset{(2i+l)}{\mathcal{O}}_{\text{int}}(\partial Y)^2Y^2\right], \tag{4.15}$$

with $\mathcal{O}_{\text{kin}}$ and $\mathcal{O}_{\text{int}}$ being the kinetic and quartic interaction operators respectively, evaluated at X_0^μ, and l corresponds to the c power of the lowest non-zero contribution in the pre-expansion gathering of the interaction term (cf. (3.39), where $l = -2$). The number of terms in the action is infinite until the Carroll limit is taken, and so far nothing forces us to send $c \to 0$ before singling out the action's order to inspect[4]. Because propagators carry c factors that are inversely proportional to the kinetic term ones, there is an infinite amount of one-loop diagrams that one can construct for *each* order in c. In fact, once the choice of the order to isolate is made – say, the c^{2m} one – the counterterm for the propagator at that order, should be extracted from the $1/\epsilon$ poles of log-divergences in the infinite sum of diagrams as

$$\overset{(2m)}{\delta S} \propto \sum_{j=0}^{\infty} \overset{(2m)}{T} \frac{1}{2\pi\epsilon} \int \mathrm{d}^2\sigma \, \partial_\alpha Y^\mu \partial^\alpha Y^\nu \left(\overset{(2m+2j)}{\mathcal{O}_{\text{kin}}^{-1}} \right)^{\lambda\rho} \left(\overset{(2m+2j-l)}{\mathcal{O}_{\text{int}}} \right)_{\mu\lambda\nu\rho} + o\left(\tfrac{1}{\epsilon}\right), \tag{4.16}$$

The new tension $\overset{(2m)}{T}$ is required to absorb the overall c powers

$$\overset{(2m)}{T} = T c^{2m-1}, \tag{4.17}$$

because of the dimensional factor ξ, (cf. (3.37)) and because of the fields redefinition (3.38). In section 3.2, the counterterm 3.44 corresponds to the summand $j = 0$ in the series (4.16) with $m = 0$, $l = -2$. To avoid getting lost in the notation, a more down-to-earth phrasing might be more pleasing to the reader: we are showing that if one waits to take the Carroll limit until *after* renormalisation, the number of diagrams involved at one-loop is infinite, because one can build them by contracting propagators scaling as c^0, c^{-2}, c^{-4}... with vertices scaling like c^0, c^2, c^4... respectively. What does this mean? Should one face eternal calculation just because the speed of light is sent to zero at one moment instead of a previous one? Of course, the answer is negative. If the task is to calculate one-loop corrections, the scheme laid out here is nothing but a limitless complication that actually takes back into account all terms in the Carroll expansion of either the spatial metric, or the Riemann tensor. This procedure does the opposite of isolating terms of a c order: it mixes them all at every order. Thus, demanding the Carroll limit to be performed after renormalisation is morally equivalent to not having a Carroll expansion at all, and the final result should be the same as the one in the relativistic case. Not only this: even if the number of contributions of the order c^{2m} becomes infinite, it is also incomplete. In fact, to recover the β-function (3.20) – the sole sensible goal that this procedure can aim to, if used correctly – one should not repeat the same calculation for every m, but do that only for $m = 0$, as this would be the only way to properly include all the two infinite towers of kinetic and interaction terms expansions, without missing the nth orders with $n < m$. From this, we can conclude that to make sense of LO, NLO, NNLO... theories as CFTs on different Carrollian embeddings, the corresponding LO, NLO, NNLO... Polyakov actions must be defined as the ones that get singled out by $c \to 0$. Hence, one should first perform the Carroll limit on the action, and *then* study the resulting CFT.

[4] This can be done as in 3.2 by a fields redefinition like (3.38), and reabsorbing the overall common c powers in the tension.

4.2.3 Spatial-temporal splitting

Another possible approach would be along the lines of [75], where the vielbein discriminates between *parallel* and *perpendicular* coordinates (i.e. the scalar fields) using the projectors (2.30) as

$$X^\mu = \left[(P_\perp)^\mu{}_\nu + (P_{/\!/})^\mu{}_\nu\right] X^\nu = \Omega v^\mu X_{/\!/} + X^\mu_\perp, \tag{4.18}$$

where the italic terminology above is to be intended w.r.t. the temporal vector v. The new coordinates are defined as

$$X_{/\!/} = -\frac{1}{\Omega}\tau_\mu X^\mu, \qquad\qquad X^\mu_\perp = h_{\lambda\nu}h^{\lambda\mu}X^\nu. \tag{4.19}$$

This splitting is rigorously defined only on a particular hypersurface realisation, and the normalisation Ω plays the same role as the factor α in (2.29). Thus, perpendicular fields span the three(d)-dimensional hypersurface, the temporal one lives in its vertical one-dimensional orthogonal fibration, and one can always switch to Latin indices for spatial quantities. From here, one can study two distinct geodesic equations, that arise as EOMs under variation of the most general "split" one-particle action

$$S = \int \mathrm{d}\lambda \left[e(mc)^2 + \frac{1}{e}h_{\mu\nu}\dot{X}^\mu_\perp \dot{X}^\nu_\perp + \frac{1}{e}(\dot{X}_{/\!/})^2\right], \tag{4.20}$$

where dots indicate derivatives w.r.t. the affine parameter λ, and e is the auxiliary einbein of the worldline that extends validity for massless particles (cf. Appendix B). The geodesic equations for the fields read

$$\ddot{X}_{/\!/} + \Gamma^{/\!/}_{\mu\nu}\dot{X}^\mu_\perp \dot{X}^\nu_\perp = 0, \tag{4.21a}$$

$$h_{\lambda\mu}\ddot{X}^\mu_\perp + \Gamma^\perp_{\lambda\mu\nu}\dot{X}^\mu_\perp \dot{X}^\nu_\perp = 0, \tag{4.21b}$$

where the Christoffel-like symbols are given by

$$\Gamma^{/\!/}_{\mu\nu} = \frac{1}{2}\partial^{/\!/}h_{\mu\nu}, \qquad\qquad \Gamma^\perp_{\lambda\mu\nu} = \frac{1}{2}\left[\partial^\perp_\mu h_{\nu\lambda} + \partial^\perp_\nu h_{\lambda\mu} - \partial^\perp_\lambda h_{\mu\nu}\right],$$

and the derivatives notation is to be intended w.r.t. parallel and perpendicular fields. From here, one can study the local property of the geodesics, and extract identities such as

$$\partial^{/\!/}h_{\mu\nu} = 0, \qquad\qquad \partial^\perp_\lambda h_{\mu\nu} = 0, \qquad\qquad \partial^\perp_{(\rho|}\Gamma^\perp_{\lambda|\mu\nu)} = 0, \tag{4.22}$$

that can be used to find a manageable expression for a normal coordinates description that is hybrid in the two types of fields. Inside the Polyakov action, the kinetic and interaction terms split consequentially to (4.18), and explicitly adhering to the metric data of the realisation (2.29), it reads

$$\begin{aligned} S_{\mathrm{Pol}} = \frac{\tilde{T}}{2}\int \mathrm{d}^2\sigma\Big[\; & h_{ij}\partial_\alpha Y^i \partial^\alpha Y^j + \partial_\alpha Y_{/\!/}\partial^\alpha Y_{/\!/} \\ & + \frac{1}{2}\partial_k\partial_l h_{ij}Y^l Y^k \partial_\alpha Y^i \partial^\alpha Y^j \\ & - 2\partial_l k_{ij} Y_{/\!/} Y^l \partial_\alpha Y^i \partial^\alpha Y^j \\ & - \frac{1}{\alpha}\dot{k}_{ij}(Y_{/\!/})^2 \partial_\alpha Y^i \partial^\alpha Y^j \quad \Big], \end{aligned} \tag{4.23}$$

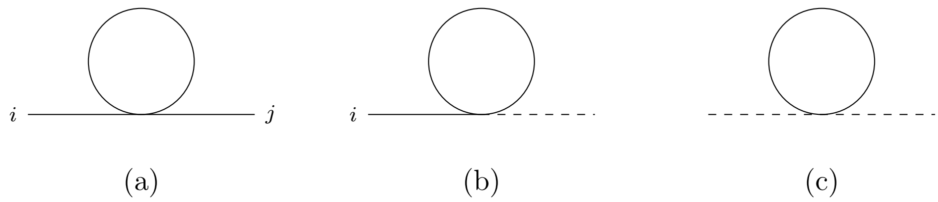


Figure 4.3: Three relevant diagrams that carry log-divergences, obtained contracting the internal spatial propagator with (w.r.t lines of equation (4.23)): (a) the fully spatial vertex of the second line; (b) the odd-mixed vertex of the third line; (c) the even-mixed vertex of the last line.

where kinetic and interaction operators are evaluated at X_0. This action describes the CFT of one temporal scalar and three spatial scalars with three types of mixed interactions. The relevant one-loop diagrams to inspect for extracting log-divergences are given in Fig. 4.3, where full lines propagate spatial fields, and dashed ones are used for the temporal one. This line of work is promising, because we obtained three diagrams that carry the same index structure of the three LO Carroll gravity equations (2.57). However, it has not been explored further because of two reasons. First, we could not find any useful expression for $\partial_l \partial_k h_{ij}$ in terms of the relevant curvature quantities one could build from the split geodesic equations. In fact, describing how the curvature tensor can be rewritten to fit the parallel-perpendicular splitting has not brought to sensible conclusions. Secondly, even if structure and values of loop diagrams seem to point in the correct direction, it is not clear how and how much one is allowed to manipulate it so that (2.57) could be recovered.

4.2.4 Carroll limit on the worldsheet

At some point, when trying to understand which order terms in c would have done the right job, we contemplated the possibility that the proper way to fit the Polyakov action into the Carroll framework could be to give units to worldsheet coordinates. This is typically not the case in the vast majority of literature on the matter, as the parametrisation

$$\sigma \in [0, 2\pi), \qquad\qquad \tau \in \mathbb{R}, \tag{4.24}$$

is straightforwardly provided by natural units, in which $[\sigma] = [\tau] = \varnothing$. However, for some particular gauge choices (static gauge), one indeed identifies two of the embedding fields with σ and τ, by introducing a dimensional constant of proportionality in between. In the end the worldsheet is a manifold with one time-like and one space-like coordinate, so one could feel justified in defining

$$\sigma^\alpha = (\sigma^0, \sigma^1) = (c\tau, \sigma), \qquad\qquad \partial_\alpha = (\partial_0, \partial_1) = \left(-\tfrac{1}{c}\partial_\tau, \partial_\sigma\right), \tag{4.25}$$

with $[\sigma] = \mathrm{m}$, and $[\tau] = \mathrm{s}$. This way, field derivatives inside the Polyakov action split into

$$\partial_\alpha Y^\mu \partial^\alpha Y^\nu = -\frac{1}{c^2}\partial_\tau Y^\mu \partial_\tau Y^\nu + \partial_\sigma Y^\mu \partial_\sigma Y^\nu, \tag{4.26}$$

and contribute to single out different orders in the Carroll expansion of the target space. Even if, from one point of view, this modification might appear as a plausible choice, the attempts to go further on this line have been quickly abandoned for a variety of motivations. First and foremost, just like the static gauge in the relativistic picture, this breaks the conformal invariance of the action at the classical level. Even Lorentz invariance ceases to hold on the worldsheet, but that is far expected, as it is what the Carroll limit does. The absence of these two fundamental symmetries is not an anomaly any more, and it raises doubts about what one actually achieves when demanding scale invariance by killing beta functions. Even if one doggedly takes this direction anyway, the resulting one-loop diagrams are not Lorentz invariant, and require to be calculated with different techniques, and more careful attention. Moreover, it remains unclear what it actually means for a string to propagate on a Carrollian worldsheet, and what the implications would be. For these reasons we leave these questions for future investigations.

Conclusions

The goal of this thesis project was to determine if the LO equations of motion of Carroll gravity can be obtained by requiring, at the quantum level, conformal invariance of the Polyakov action on a Carrollian target space. The calculation we set up mirrored the one of the relativistic case, for standard GR and standard bosonic string theory. We chose the small light speed expansion formalism of GR developed in [17] as our main approach, and after dedicating a comprehensive review to it, we used it to Carroll-expand the Polyakov action in RNC and single out its LO contribution. We showed that for the resulting CFT to retain scale invariance, only the first equation (2.57) of LO Carroll gravity must hold. This procedure seems to be the most natural one to undertake and, if unflawed, could in principle be generalised to recover Carroll Gravity at any order. We then presented an alternative natural approach, that does not involve Carroll expansions, but builds on the geometric entities that characterise a Carroll manifold. We illustrated how to expand the action in CNC, and showed that with this method the relevant counterterm falls short of reproducing the k^2 key term, that has necessarily to appear for the desired set of equations to be recovered. We further discussed the pros and cons of the two approaches, and speculated on the reason of their respective failures. Our conclusion is that, if one is convinced that our procedures are well-founded, the standard methodology to connect gravity with the framework of bosonic strings is either incorrect or in conflict with the zero-signature limit. Of course, on the other hand, it might also be that we are overlooking something in both approaches. The takeaway is that we could not fulfil our initial expectation, and the reader can make use of the present work as a primer on what does not work in the Carroll expansion of the Polyakov action. If the Carroll picture is incompatible with strings, it would be instructive to further investigate the causes of this, and specifically trying to understand in a more rigorous way why the imposition of conformal invariance cannot lead to the fields equations of a singular manifold. If on the opposite, we are just mistaken, it would be beneficial if further research could demonstrate where our line of work proves to be deficient or erroneous. Additionally, it would also be interesting to expand the research upon some points remarked in the last Section. For instance, it is possible that a more scrupulous analysis of the parallel-perpendicular discrimination of fields would turn out to be the proper way to address the issue. Moreover, studying the implications of a Carroll limit on the worldsheet could prove to be related to this splitting, as a reflection in different transformation laws for the fields under Carroll boosts on worldsheet coordinates. Tempting as these prospects may be, they fall outside the scope of this work, and are thus left for future developments.

Excusatio non petita

We conclude by giving a final remark that concerns the overall line of thought and methodology pursued in this project. In our calculations, especially in the main approach, there are major difficulties that concur to make the job hard. The combination of PUL parametrisation, normal coordinates expansion and Carroll expansion is such that the number of terms involved in the Polyakov action is large. And it gets larger the more sub-leading orders are included in the c expansion, although one can often use vielbein relations, and symmetry arguments to scale down their amount. Moreover, these same properties transit from being aids to obstacles when the task is to recognise the bits that, once isolated and suitably contracted, could produce LO Carroll gravity EOMs. Phrasing it like this, it would seem that we tried to force the way to a pre-assumed reachable goal, when it can simply be that *Carroll Gravity from the Polyakov Action* is, despite initial expectations, an unachievable objective, and the sceptical reader could now feel justified to frown upon the philosophy that underlines our lines of investigation. However, in our defense, it must be said that the procedures we developed are indeed subject to a considerable degree of arbitrariness. String theory derives the equations of gravity from a quantum action, a framework that does not clearly indicate whether the Carrollian picture fits more naturally in one way rather than another. Hence, in our opinion, a trial-and-error approach guided by intuition seems to be the most honest way to proceed.

Acknowledgments

Finally, I can speak in first person. I must warn – if the reader has not noticed it already – that verbosity is a well-known trait of mine, and for sure here I will live up to expectations. As this thesis marks the endpoint of my Master's experience in Utrecht University, the words that follow pay tribute all the ones that have been on my side throughout these two years. First of all, I want to thank Prof. Vandoren, who proposed the idea for this project, and guided me throughout the work with punctual help and a good amount of patience. I could not have asked for a better supervisor. Above all, my gratitude goes to Josee Haghedooren, in whom I found the value of a precious friend, the affection of a beloved, and for a significant while, a place to call home. I thank Francesca Sfondrini and the van Beek crew Michiel, Mauro and Livia, for their support, and for being my short-range family in the Netherlands. I thank Esq. Giulio Bonetti, my ontological counterpart, bosom friend and life shareholder. I thank Lucia Righetti, my two-years most stunning discovery, source of pyroclastic dramatic passion and true friendship. I thank my father Antonio, my mother Raffaella, and my sister Chiara for existing, and my grandmother Mariagrazia for always knowing the right thing to say to me at the right time (and for existing as well, that is to say). I thank my aunt Rosanna, for her love. I thank my countless relatives among the Uggés, Gioias, Ajellos, Torrianis, Salas, Goys... and beg your pardon for not naming each of you here, you are too many. I thank Biancamaria Gotti from Bergamo, the coolest, the Greek community of Het Kwadrant, and my friends and colleagues Andri Jauhari, Marco Vecchioni and Raluca Dragonici, for the time spent together and the mutual support in academic suffering. I thank Prof. Spitoni, both as student and collaborator. I thank Carlo Santinelli, most promising among my anagrammer pupils, and our fortuitous encounter in O'Panuozzo. I thank the eminent friends M° Simone Anelli, Dr. Emma Allegra Albertini, Dr. Marco Re and Dr. Gabriele Calcedonio Affuso, for their constant presence in my life. I thank my dear friends Kem, Alice, Isotta, Susanna, Gabriele and Tommaso, the most ancient and weird gathering of people I had the pleasure to be part of. I thank Sara and Fabio Rossi, Vittoria Senzalari, Valentina Procopio, Marco Picione, Margherita Morlacchi a.k.a Mørgiu, Clara Patrini, Giulia Cirigliano and Francesco Andreose; in me there is room for you all. I thank my lovers, whether actual, unreachable or imaginary, that had my heart palpitating in these two years. I thank the students and teachers I met in my teaching assistantships, my colleagues at Spice Monkey and at AH: you made my jobs worth the effort. I thank Radio24, La7, the books, the movies and the animes, for the company provided in solitude. I thank Italy and the Netherlands, Lodi and Utrecht, Milan and Amsterdam; the business with you all seems quite far to be concluded. To close on a bitter note, I have no thanks to offer to those that, either inside or outside University, have managed to make things harder, being that due to incompetence, laziness, self-centrism, ruthlessness, discourtesy, vulgarity and, the hardest to tolerate, lack of elegance.

Appendix A
Glossary

Abbreviations

- GR: General Relativity
- SR: Special Relativity
- CFT: Conformal field theory
- PUL: Pre-ultra-local
- LO: Leading order
- NLO: Next-to-leading order
- NNLO: Next-to-next-to-leading order
- CCC: Carroll compatible connection

Notations

- We use Landau's *big O* and *little o* symbols. In a regime set by a given limit, say $\epsilon \to 0$: a statement like $f = O(g)$ means that f and g scale with the same order; a statement like $f = o(g)$ means that f is negligible w.r.t. g in this limit. Throughout the work, expressions like

  $$F = f + c^2 \overset{(2)}{f} + O(c^4), \qquad F = f + c^2 \overset{(2)}{f} + o(c^2),$$

  are to be intended as equivalent.

- Spatial vectors w, or in general, vectors that live in an equal signature space (Euclidean), are indicated with a bold symbol $\boldsymbol{w}$.

- Γ is the Levi-Civita connection, and ∇ its covariant derivative; $\tilde{C}$ is the CCC, and $\tilde{\nabla}$ its covariant derivative; any tensor denoted with a tilde is relative to the CCC, thus proper to a Carrollian geometry.

- If not differently specified: Greek indices $\{\mu, \nu, \lambda, \rho, \sigma, \delta, \gamma, \tau, \epsilon\}$ denote spacetime components; lowercase Latin $\{i, j, k, l\}$ stand for spatial indices of coordinates; capital Latin indices denote tetrads components on the tangent space; lowercase Latin $\{a, b, c, d\}$ are used for "spatial" components of the latter; Greek indices $\{\alpha, \beta\}$ are used for worldsheet components.

Appendix B

Riemann Normal Coordinates

In this appendix we prove (3.5). As this is an easy exercise, it is often taken for granted in the literature, but we think it could be instructive to show it, in comparison with the Carroll case. The geodesic equation for a free particle, and thus the explicit form of the connection in terms of the metric, can be obtained by minimising the relative one-particle action for free propagation on curved background

$$S = -\frac{1}{2}\int \mathrm{d}\lambda \left[\frac{1}{e} g(x)_{\mu\nu}\frac{\mathrm{d}x^\mu}{\mathrm{d}\lambda}\frac{\mathrm{d}x^\nu}{\mathrm{d}\lambda} - (mc)^2 e\right], \tag{B.1}$$

where e is the auxiliary worldline einbein that extends validity to massless particles[5]. Varying the action with respect to e and x^μ leads to

$$g_{\mu\nu}\frac{\mathrm{d}x^\mu}{\mathrm{d}\lambda}\frac{\mathrm{d}x^\nu}{\mathrm{d}\lambda} + (mc)^2 e = 0, \tag{B.2}$$

$$\frac{1}{e}\frac{\mathrm{d}e}{\mathrm{d}\lambda} g_{\mu\nu}\frac{\mathrm{d}x^\mu}{\mathrm{d}\lambda} + g_{\mu\nu}\frac{\mathrm{d}^2x^\mu}{\mathrm{d}\lambda^2} + \Gamma_{\rho\mu\nu}\frac{\mathrm{d}x^\mu}{\mathrm{d}\lambda}\frac{\mathrm{d}x^\rho}{\mathrm{d}\lambda} = 0. \tag{B.3}$$

(B.2) is the constraint for e, while (B.3) is the evolution equation, and the Christoffel symbols are

$$\Gamma_{\rho\mu\nu} = \frac{1}{2}\left(\partial_\mu g_{\nu\rho} + \partial_\nu g_{\rho\mu} - \partial_\rho g_{\mu\nu}\right), \qquad \Gamma^\rho_{\mu\nu} = g^{\rho\lambda}\Gamma_{\lambda\mu\nu}. \tag{B.4}$$

If now one chooses a constant gauge $\dot{e} = 0$ on the worldline, the result is

$$g_{\rho\mu}\frac{\mathrm{d}^2x^\mu}{\mathrm{d}\lambda^2} + \Gamma_{\rho\mu\nu}\frac{\mathrm{d}x^\mu}{\mathrm{d}\lambda}\frac{\mathrm{d}x^\nu}{\mathrm{d}\lambda} = 0. \tag{B.5}$$

The properties we need come from solving (B.5) locally, around a point $p \in \mathscr{M}$: if one chooses the coordinate chart to identify p at $x^\mu(\lambda = 0)$ then, given initial position[6] and velocity

$$x^\mu\big|_p = x^\mu_0, \qquad \left.\frac{\mathrm{d}x^\mu}{\mathrm{d}\lambda}\right|_p = \xi^\mu, \tag{B.6}$$

the unique solution is simply given by $x^\mu(\lambda) = \lambda\xi^\mu + o(\lambda)$, for which obviously,

$$\left.\frac{\mathrm{d}^2x^\mu}{\mathrm{d}\lambda^2}\right|_p = 0 + o(\lambda). \tag{B.7}$$

[5]To match dimensional units in SI, one has to take $[e] = \mathrm{kg}^{-1}$

[6]We set the coordinate system with origin at p so that $x^\mu_0 = 0$

Plugging this into (B.5) leads to

$$\Gamma_{\rho\mu\nu}\xi^{\mu}\xi^{\nu} = 0. \tag{B.8}$$

Because this must hold locally for any choice of ξ^{μ}, and as Christoffel symbols are symmetric in the last two indices, then

$$\Gamma_{\rho\nu\nu}\big|_p = 0. \tag{B.9}$$

Usually, the next step is raising the free index with the inverse metric, and exploit metric compatibility $\nabla_{\lambda}g_{\mu\nu} = 0$ to get $\partial_{\lambda}g_{\mu\nu} = 0$. In doing this one needs to assume that the Christoffel symbols – i.e. the inverse metric – are well-defined, and this is not the case in the Carroll picture. Hence, we work with lower-indices Christoffel symbols only. The very same conclusion can be obtained by cyclically permuting the three indices of (B.9), to get

$$\partial_{\rho}g_{\mu\nu} = \Gamma_{\mu\nu\rho} + \Gamma_{\nu\rho\mu}. \tag{B.10}$$

Thus, the first derivative of the metric vanishes at the point $\lambda = 0$

$$\partial_{\rho}g_{\mu\nu}\big|_p = 0. \tag{B.11}$$

Now we deal with the quadratic term. Here, we need the actual upper-index Christoffel symbol, for which (B.9) holds too

$$\Gamma^{\rho}_{\mu\nu}\xi^{\mu}\xi^{\nu} = 0. \tag{B.12}$$

We won't use make use of the following results in the Carroll case, as raising indices with he inverse metric is not allowed there. Taking the derivative w.r.t λ of (B.12) leads to

$$\partial_{\lambda}\Gamma^{\rho}{}_{\mu\nu}\xi^{\mu}\xi^{\nu}\xi^{\lambda} + \Gamma^{\rho}{}_{\mu\nu}\xi^{\mu}\xi^{\nu} + \Gamma^{\rho}{}_{\mu\nu}\xi^{\mu}\xi^{\nu} = 0. \tag{B.13}$$

The last two term vanish at p in force of (B.12), leaving

$$\partial_{\lambda}\Gamma^{\rho}{}_{\mu\nu}\xi^{\mu}\xi^{\nu}\xi^{\lambda} = 0. \tag{B.14}$$

This is symmetric in $(\lambda\mu\nu)$, and must hold for generic ξ^{μ}, thus

$$\partial_{\lambda}\Gamma^{\rho}{}_{\mu\nu} + \partial_{\mu}\Gamma^{\rho}{}_{\nu\lambda} + \partial_{\nu}\Gamma^{\rho}{}_{\lambda\mu} = 0. \tag{B.15}$$

Now, at p Christoffel symbols vanish, so the Riemann tensor reads

$$R^{\rho}{}_{\lambda\mu\nu}\big|_p = \partial_{\mu}\Gamma^{\rho}{}_{\nu\lambda} - \partial_{\nu}\Gamma^{\rho}{}_{\mu\lambda}. \tag{B.16}$$

Using (B.15), lowering all indices and using the connection lower indices symmetry, it can be written as

$$R_{\rho\lambda\mu\nu} = g_{\rho\sigma}\left(\partial_{\lambda}\Gamma^{\sigma}{}_{\mu\nu} + 2\partial_{\mu}\Gamma^{\sigma}{}_{\nu\lambda}\right). \tag{B.17}$$

Some further manipulations follo:

$$\begin{aligned}
R_{\rho\lambda\mu\nu} &= -\frac{1}{2}\left(R_{\lambda\rho\mu\nu} + R_{\nu\mu\rho\lambda}\right) && \text{(B.18)}\\
&= -\frac{1}{2}g_{\lambda\sigma}\left(\partial_{\rho}\Gamma^{\sigma}{}_{\mu\nu} + 2\partial_{\mu}\Gamma^{\sigma}{}_{\nu\rho}\right) - \frac{1}{2}g_{\nu\sigma}\left(\partial_{\mu}\Gamma^{\sigma}{}_{\rho\lambda} + 2\partial_{\rho}\Gamma^{\sigma}{}_{\lambda\mu}\right) && \text{(B.19)}\\
&= -\partial_{\rho}\partial_{\mu}g_{\nu\lambda} - \frac{1}{2}\left(g_{\lambda\sigma}\partial_{\mu}\Gamma^{\sigma}{}_{\nu\rho} + g_{\nu\sigma}\partial_{\rho}\Gamma^{\sigma}{}_{\lambda\mu}\right), && \text{(B.20)}
\end{aligned}$$

where in the first line we used the Riemann tensor properties $R_{\rho\lambda\mu\nu} = -R_{\lambda\rho\mu\nu} = -R_{\rho\lambda\nu\mu} = R_{\mu\nu\rho\lambda}$, in the second line used $g\partial\Gamma = \partial(g\Gamma)$ as $\partial g = 0$ at p, and in the third line we used the upper-index equivalent of (B.10)

$$\partial_\lambda g_{\mu\nu} = g_{\mu\sigma}\Gamma^{\sigma}{}_{\lambda\nu} + g_{\nu\sigma}\Gamma^{\sigma}{}_{\lambda\mu}. \tag{B.21}$$

Multiplying by the second order increment, then

$$R_{\rho\lambda\mu\nu}y^\rho y^\mu = -\partial_\rho\partial_\mu g_{\nu\lambda}y^\rho y^\mu - \frac{1}{2}\partial_\mu\left(g_{\lambda\sigma}\Gamma^{\sigma}{}_{\rho\nu} + g_{\nu\sigma}\Gamma^{\sigma}{}_{\rho\lambda}\right)y^\rho y^\mu \tag{B.22}$$

$$= -\frac{3}{2}\partial_\rho\partial_\mu g_{\nu\lambda}y^\rho y^\mu, \tag{B.23}$$

where in the first line we exchanged the dummy indices ($\mu \rightleftharpoons \rho$), used $g\partial\Gamma = \partial(g\Gamma)$, and used (B.21) again in the second line, leading to the proof for the quadratic term of RNC. Finally, the metric at 0th order can always be put into Minkowskian form, simply choosing the coordinates to be the ones adapted to the frame basis of tangent bundle at p.

Appendix C
Explicit expressions

The full expressions for the Riemann tensor in PUL parametrisation(3.30) up to c^2 read

$$
\begin{aligned}
R_{\mu\lambda\nu\rho} = \frac{1}{c^2}\Bigg[& -\Pi_{\mu\sigma}\partial_\nu V^\sigma K_{\rho\lambda} - \frac{1}{2}\left(\partial_\mu\Pi_{\nu\lambda} + \partial_\nu\Pi_{\lambda\mu} - \partial_\lambda\Pi_{\mu\nu}\right) V^\lambda K_{\rho\lambda} - (\nu \rightleftharpoons \rho)\Bigg] \\
+c^0\Bigg[& + T_\mu K_{\rho\lambda}\left(\partial_{(\nu}T_{\gamma)}V^\gamma + T_{(\nu}\mathcal{L}_V T_{\gamma)}V^\gamma - \partial_\nu T_\sigma V^\sigma\right) - T_\mu\partial_\nu K_{\rho\lambda} \\
& - \frac{1}{2}T_\mu K_{\nu\gamma}\Pi^{\gamma\sigma}\left(\partial_\sigma\Pi_{\rho\lambda} + \partial_\rho\Pi_{\lambda\sigma} - \partial_\lambda\Pi_{\sigma\rho}\right) - \Pi_{\mu\sigma}\partial_\nu V^\sigma \partial_{(\rho}T_{\lambda)} \\
& - \Pi_{\mu\sigma}\partial_\nu V^\sigma T_{(\rho}\mathcal{L}_V T_{\lambda)} + \frac{1}{2}\Pi_{\mu\sigma}\partial_\nu\Pi^{\sigma\delta}\left(\partial_\delta\Pi_{\rho\lambda} + \partial_\rho\Pi_{\lambda\delta} - \partial_\lambda\Pi_{\delta\rho}\right) \\
& + \frac{1}{2}\left(\partial_\nu\partial_\mu\Pi_{\rho\lambda} + \partial_\nu\partial_\rho\Pi_{\lambda\mu} - \partial_\nu\partial_\lambda\Pi_{\mu\rho}\right) + \frac{1}{2}T_\mu V^\sigma\left(\partial_\nu\partial_\sigma\Pi_{\rho\lambda} + \partial_\nu\partial_\rho\Pi_{\lambda\sigma} - \partial_\nu\partial_\lambda\Pi_{\sigma\rho}\right) \\
& + \left(\partial_\mu\Pi_{\nu\gamma} + \partial_\nu\Pi_{\gamma\mu} - \partial_\gamma\Pi_{\mu\nu}\right)\Pi^{\gamma\tau}\left(\partial_\tau\Pi_{\rho\lambda} + \partial_\rho\Pi_{\lambda\tau} - \partial_\lambda\Pi_{\tau\rho}\right) \\
& + T_\mu V^\delta\left(\partial_\delta\Pi_{\nu\gamma} + \partial_\nu\Pi_{\gamma\delta} - \partial_\gamma\Pi_{\delta\nu}\right)\Pi^{\gamma\tau}\left(\partial_\tau\Pi_{\rho\lambda} + \partial_\rho\Pi_{\lambda\tau} - \partial_\lambda\Pi_{\tau\rho}\right) \\
& + T_{(\nu}\partial_{\gamma)}T_\mu\,\Pi^{\gamma\tau}T_{(\rho}\partial_{\lambda)}T_\tau + T_\mu V^\sigma T_{(\nu}\partial_{\gamma)}T_\sigma\,\Pi^{\gamma\tau}T_{(\rho}\partial_{\lambda)}T_\tau - (\nu \rightleftharpoons \rho)\Bigg] \\
+c^2\Bigg[& - T_\mu\partial_\nu\left(\partial_{(\rho}T_{\lambda)}\right) - \partial_\nu\left(T_\mu T_{(\rho}\mathcal{L}_V T_{\lambda)}\right) + T_\mu K_{\nu\gamma}\Pi^{\gamma\delta}T_{(\rho}\partial_{\lambda)}T_\delta \\
& + T_\mu\left(\partial_{(\rho}T_{\lambda)} + T_{(\rho}\mathcal{L}_V T_{\lambda)} + T_{(\nu}\mathcal{L}_V T_{\gamma)}\right)\left(T_\sigma\partial_\nu V^\sigma + \partial_{(\nu}T_{\gamma)}V^\gamma\right) \\
& - \frac{1}{2}T_\mu\Pi^{\gamma\sigma}\left(\partial_\sigma\Pi_{\rho\lambda} + \partial_\rho\Pi_{\lambda\sigma} - \partial_\lambda\Pi_{\sigma\rho}\right)\left(\partial_{(\nu}T_{\gamma)} + T_{(\nu}\mathcal{L}_V T_{\gamma)}\right) \\
& + T_{(\rho}\partial_{\gamma)}T_\mu\left(V^\gamma\partial_{(\rho}T_{\lambda)} + V^\lambda T_{(\rho}\mathcal{L}_V T_{\lambda)} + \frac{1}{2}\Pi^{\gamma\sigma}\left(\partial_\sigma\Pi_{\rho\lambda} + \partial_\rho\Pi_{\lambda\sigma} - \partial_\lambda\Pi_{\sigma\rho}\right)\right) \\
& + T_\mu T_{(\rho}\partial_{\gamma)}T_\delta V^\delta\left(V^\gamma\partial_{(\rho}T_{\lambda)} + V^\lambda T_{(\rho}\mathcal{L}_V T_{\lambda)} + \frac{1}{2}\Pi^{\gamma\sigma}\left(\partial_\sigma\Pi_{\rho\lambda} + \partial_\rho\Pi_{\lambda\sigma} - \partial_\lambda\Pi_{\sigma\rho}\right)\right) \\
& - \partial_\nu\left(T_{(\rho}\partial_{\lambda)}T_\delta\,\Pi^{\sigma\delta}\right) - \frac{1}{2}T_{(\rho}\partial_{\lambda)}T_\delta\,\Pi^{\gamma\delta}\left(\partial_\mu\Pi_{\nu\lambda} + \partial_\nu\Pi_{\lambda\mu} - \partial_\lambda\Pi_{\mu\nu}\right) \\
& - \frac{1}{2}T_\mu T_{(\rho}\partial_{\lambda)}T_\delta V^\sigma\left(\partial_\sigma\Pi_{\nu\lambda} + \partial_\nu\Pi_{\lambda\sigma} - \partial_\lambda\Pi_{\sigma\nu}\right) - (\nu \rightleftharpoons \rho)\Bigg] + o(c^2).
\end{aligned}
$$

The full form of the action in CNC in Section 4.1.1, where all quantities but Y^μ fields are intended to be evaluated at the constant X_0^μ, is given by

$$
\begin{aligned}
\overset{(CNC)}{S}_{\text{Pol}} = \frac{\tilde{T}}{2}\int_\Sigma \mathrm{d}^2\sigma \ \Bigg\{ \quad & h_{\mu\nu}\partial_\epsilon Y^\mu \partial^\epsilon Y^\nu + 2\tau_{[\lambda}k_{\mu]\nu}Y^\lambda \partial_\epsilon Y^\mu \partial^\epsilon Y^\nu \\
+ \Bigg[\quad & \frac{1}{3}\left(\tau_{[\lambda}k_{\mu]\epsilon}h^{\epsilon\gamma}\tau_{[\rho}k_{\nu]\gamma}\right) + \frac{1}{3}\left(\tau_{[\mu}k_{\rho]\epsilon}h^{\epsilon\gamma}\tau_{[\lambda}k_{\nu]\gamma}\right) \\
& - \frac{2}{3}\Bigg(- 2v^\delta \tau_{\delta\lambda}\tau_\rho k_{\mu\nu} + 2\tau\rho\partial_\lambda k_{\mu\nu} + 2\tau_\rho\tau_\mu v^\epsilon \partial_\lambda k_{\mu\epsilon} \\
& \qquad + 2\tau_\rho h^{\delta\sigma}\tau_\mu k_{\lambda\sigma}k_{\delta\nu} - \tau_\rho h^{\delta\sigma}k_{\delta\nu}\left(\partial_\lambda h_{\mu\sigma} + \partial_\mu h_{\sigma\lambda} - \partial_\sigma h_{\lambda\mu}\right) \\
& \qquad - \frac{1}{2}\tau_\rho\tau\mu v^\epsilon h^{\delta\sigma}k_{\delta\mu}\left(\partial_\lambda h_{\epsilon\sigma} + \partial_\epsilon h_{\sigma\lambda} - \partial_\sigma h_{\lambda\epsilon}\right) - (\mu \rightleftharpoons \rho)\Bigg) \\
& - \frac{8}{3}\left(\tau_{[\mu}k_{\lambda]\delta}h^{\epsilon\delta}\tau_{[\rho}k_{\epsilon]\nu}\right) \\
& - \frac{2}{3}\Bigg(+ v^\tau \partial_\lambda h_{\tau\nu}\left(\partial_{(\mu}\tau_{\rho)} + \tau_{(\mu}\mathcal{L}_v\tau_{\rho)}\right) \\
& \qquad + \partial_\mu\left(\partial_\mu h_{\rho\nu} + \partial_\rho h_{\nu\mu} - \partial_\nu h_{\mu\rho}\right) \\
& \qquad + h_{\tau\nu}\partial_\lambda h^{\tau\epsilon}\left(\partial_\mu h_{\rho\epsilon} + \partial_\rho h_{\epsilon\mu} - \partial_\epsilon h_{\mu\rho}\right) \\
& \qquad + \tau_\nu v^\epsilon \partial_\lambda\left(\partial_\mu h_{\rho\epsilon} + \partial_\rho h_{\epsilon\mu} - \partial_\epsilon h_{\mu\rho}\right) \\
& \qquad - h_{\tau\nu}\partial_\lambda h^{\tau\epsilon}\tau_\rho k_{\mu\epsilon} + \tau_{[\lambda}k_{\delta]\nu}h^{\delta\gamma}\tau_{[\mu}k_{\rho]\gamma} \\
& \qquad - \partial_\lambda\tau_\rho k_{\mu\nu} - \tau_\rho\partial_\lambda k_{\mu\nu} \\
& \qquad - \tau_\nu v^\lambda \tau_\rho \partial_\lambda k_{\mu\epsilon} - (\mu \rightleftharpoons \lambda)\Bigg) \quad \Bigg]\partial_\epsilon Y^\mu \partial^\epsilon Y^\nu Y^\rho \quad \Bigg\}.
\end{aligned}
$$

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