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THE ROLE OF A MASSIVE STRANGE QUARK IN THE LARGE-N SKYRME MODEL*

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ABSTRACT

We analyze the rigid rotator treatment of the three-flavor Skyrme model in the limit of a large number of colors. This is an approximation to the bound state approach to strangeness. The dynamical picture of baryons that emerges is similar to that of the non-relativistic quark model. In particular, we find a collective excitation of the $\bar{q}q$ background with all the properties of a constituent strange quark. We calculate the nucleon mass and the matrix element $\langle N|\bar{s}s|N\rangle$ as functions of the mass of the strange quark.

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1. Introduction

The strange quark plays a peculiar role in hadronic physics, being neither heavy nor light. Its mass is comparable to Λ_{QCD}, the scale of the strong interactions, and so one might expect virtual strange quarks to be important in the structure of nucleons. The non-relativistic quark model (NRQM) tells us differently though: it suggests 1) that the strange quark is relatively light, since flavor SU(3) seems to be a pretty good symmetry; and 2) that nonetheless the strange quark plays no role in the nucleon since it is not a valence quark. Furthermore, deep inelastic scattering experiments seem to show that the strange quark content of the nucleon is small in that regime.

There are, however, several recent experimental results which indicate that at least two different matrix elements in the proton involving strange quarks are not small at all. One is $\langle p|\bar{s}\gamma^\mu\gamma_5 s|p\rangle$, measured in both νp elastic scattering [1] [2] and in deep inelastic μp scattering [3]. The other is $\langle N|\bar{s}s|N\rangle$, determined indirectly from the measurement in πN scattering of

$$\Sigma_{\pi N} \equiv \bar{m}\langle N|\bar{u}u + \bar{d}d|N\rangle \simeq 45 - 60\text{MeV}, \tag{1.1}$$

where $\bar{m}$ is the average u and d quark mass [1]. (For a review see [5]; the value of $\Sigma_{\pi N}$ is somewhat controversial, and the reader should also see [6].)

If one assumes that the baryon octet masses are well described in first order perturbation with respect to current quark masses, then it is simple to extract from (1.1) the result

$$m_s\langle N|\bar{s}s|N\rangle \simeq 334 \pm 135\text{MeV} \tag{1.2}$$

—implying that if the s quark were massless, the proton mass would be only about 600 MeV! This surprising result is not only of theoretical interest, but has a variety of phenomenological implications: a large value for the above matrix element implies a low critical density for kaon condensation in neutron stars [7] and in heavy ion collisions [8], and affects the couplings to matter of higgs bosons [9], axions [10], and various other dark matter candidates [11].

One must conclude that if the experimental determination (1.1) is correct, then the NRQM is not. For then either 1) the strange quark does have large matrix elements in the

[1] There have recently been proposals to measure the vector strange moments of the proton, $\langle p|\bar{s}\gamma^\mu s|p\rangle$, as well. See [4],[1].

nucleon, or else 2) SU(3) is too badly broken in the baryon sector by the strange quark mass to warrant the use of first order perturbation theory in SU(3) breaking.

The former possibility seems to be the more reasonable, since SU(3) predictions seem to be good for the baryons. The success of the Gell-Mann–Okubo (GMO) baryon mass relations, which follow from the assumption that second-order perturbation in SU(3) symmetry breaking is negligible, is usually considered an indication that m_s can be treated as small. Furthermore, SU(3) apparently works well for mesons. Finally, there seems to be no reason why the underlying QCD theory would suppress the appearence of virtual s quarks. The objection that deep inelastic scattering experiments tell us that strange matrix elements in the nucleon are small is questionable, since these experiments only measure moments of currents, and not $\langle N|\bar{s}s|N\rangle$[2].

An apparently bizarre alternative has been offered by Jaffe [13]. He has performed a chiral bag calculation that suggests that indeed, SU(3) *is* badly broken by the strange quark mass; but that the results from the Gell-Mann–Oakes–Renner (GMOR) analysis of baryon masses (e.g., linear perturbation in the quark masses) are still valid because the symmetry breaking —though nonlinear in the quark masses— remains primarily in the octet channel. This sounds unlikely since one would expect that if SU(3) symmetry breaking depended nonlinearly on the quark masses, then it would be strong in the 10, $\overline{10}$, 27, etc. channels as well. One would therefore expect to see deviations from the Gell-Mann–Okubo formula (which works to about 10 MeV), presumably on the scale of the $N - \Sigma$ splitting. We will argue, however, that large-N_c reasoning alone suggests that the deviations from the GMO mass relations should be on the scale of the $\Lambda - \Sigma$ (hyperfine) splitting, and not on the scale of the $N - \Sigma$ (hypercharge) splitting. While this does not explain why the GMO relations hold to 10 MeV, the numerical serendipity required to achieve the GMO predictions may be less than expected.

Abstracting from the specific model considered by Jaffe, his work suggests that nature may work in the following way. The strange quark mass is too large for the first order perturbation theory in m_s to be a good approximation. The GMO mass relations nevertheless hold for a reason different from the perturbative argument, or simply due to a numerical coincidence. In order to accumulate evidence in favor or against this possibility, we need another model where various baryon masses can be calculated as functions of m_s.

[2] For a recent discussion of nucleon structure functions and strangeness content, see [12]

In this paper we use the rigid rotator approach to Skyrmions as a toy model in which the necessary questions can be posed. We manipulate it into a form where its similarity with the NRQM is quite apparent. Furthermore, we calculate various quark model parameters in terms of the parameters of the chiral lagrangian. Our semiclassical "derivation" of the valence quark model from the chiral lagrangian may shed some new light on the relation between the Skyrme and quark models and, possibly, on the success of the NRQM.

We will demonstrate that first-order perturbative inclusion of m_s is likely to break down badly in the Skyrme model. This does not quite constitute an argument in favor of Jaffe's scenario. The problem is that the $SU(3)$ rigid rotator model is too crude to provide a satisfactory fit to baryon masses. We hope to convince the reader that a chiral model which yields a better agreement with phenomenology, such as the bound state approach to strangeness, may provide a good testing ground for Jaffe's hypothesis as well as for other interesting questions related to strangeness. Since the calculations there are significantly more laborious, we limit ourselves here to a cruder but simpler rigid rotator model.

Specifically, we consider the general chiral Lagrangian for the pseudoscalar mesons, which follows from QCD with N_c colors and three flavors of quarks. The symmetry breaking effects due to the current masses of the quarks are parametrized by m_K^2/Λ^2, m_K being the kaon mass and Λ some mass scale $\simeq 1$ GeV. Thus $SU(3)$ violating effects in the pseudoscalar meson sector are $\lesssim 25\%$—so that SU(3) is a pretty good symmetry in the meson sector.

We then consider the quantized solitons of this theory with the conventional interpretation as the baryons of the underlying QCD theory for a large number of colors, N_c. We find that $SU(3)$ breaking effects in the baryon sector are *not* simply parametrized by m_K^2/Λ^2, as in the meson sector. In addition to the effects parametrized by m_K^2/Λ^2, there are also deviations from $SU(3)$ for the baryons that are analytic functions of m_K^2/M_0^2, where M_0 is a parameter that can be determined in any particular soliton model. M_0 may be much smaller than the symmetry breaking scale Λ in the meson sector, in which case $SU(3)$ should be a much worse symmetry for baryons than for mesons. In fact, the usual $SU(3)$ Skyrme model truncated at four derivatives yields $M_0 \simeq 245$ MeV! Nevertheless, we find that even for $M_0 < m_K$, $SU(3)$ may still *look* like a pretty good symmetry in the baryon sector. In particular, the mass formula one derives is similar to that in the NRQM with the $\vec{j} \cdot \vec{j}$ interactions between quarks[14].

The treatment of $SU(3)$ skyrmions presented in this paper is a simplified version of the bound state approach to strangeness [15], [16]. Our simplification allows us to study the dependence of $\langle N|\bar{s}s|N\rangle$ on m_s (for m_s comparable with M_0) without having to deal with the infinite number of kaon modes. One advantage of both our analysis and the bound state approach is that one can see explicitly how the Skyrme model is able to replicate the results of the quark model. We find a collective excitation that carries all of the quantum numbers of the strange quark and possesses a mass identifiable with the *constituent* mass of the "naive" strange quark. The dependence of the constituent strange quark mass on the bare strange quark mass is calculable in closed form. We believe that the pictures presented here and in the bound state approach to strangeness [15], help to shed light on the success of the NRQM, and go beyond the motivating problem of calculating the $\langle N|\bar{s}s|N\rangle$ matrix element.

The organization of the paper is as follows. In sec. 2 we discuss the calculation of $\langle N|\bar{s}s|N\rangle$ in the SU(3) rigid rotator model with $m_s = 0$, and demonstrate why it is desirable to consider the large N_c limit. In sec. 3 we carry out a systematic expansion in powers of $1/N_c$ and show that a non-zero strange quark mass can be easily included non-perturbatively. The discussion of quantum numbers of the resultant states clarifies the connection with the NRQM picture. In sec. 4 we show that our mass formula resembles the NRQM mass formula. Agreement with the data places tight constraints on the values of all the parameters and seems to rule out the rigid rotator model as a quantitative model for baryons. This exercise also shows why the bound state model works better. In sec. 5 we calculate $\langle N|\bar{s}s|N\rangle$ as a function of m_s and show that it exhibits significant non-linearities. In the Discussion we further compare the rigid rotator and the bound state treatments of Skyrmions. Two appendices discuss the calculation and application of various Clebsch-Gordon coefficients for the baryon representations in the large-N_c QCD.

2. $\langle \mathbf{N}|\bar{\mathbf{s}}\mathbf{s}|\mathbf{N}\rangle$ in the Rigid Rotator at $\mathbf{m_s = 0}$ and Large $\mathbf{N_c}$

To calculate $\langle N|\bar{s}s|N\rangle$ one needs a dynamical model of nucleon wavefunctions. In this paper we investigate the Skyrme model, which allows virtual QCD degrees of freedom to play a large role in the proton, while at the same time sharing many of the group theoretical successes of the quark model.

In this section we address the N_c dependence of $\langle N|\bar{s}s|N\rangle$ while keeping $m_s = 0$. We do this by revisiting the calculation of $\langle N|\bar{s}s|N\rangle$ performed by Donoghue and Nappi [17]

(DN). They used the standard $SU(3)_{flavor}$ Skyrme model wave function for the nucleons [18], [19] with the number of colors $N_c = 3$ and with $m_s = 0$. For the operator $\bar{s}s$ they took

$$\bar{s}s \simeq \mathrm{Tr}\, X\,(U-1) \;+\; h.c. \tag{2.1}$$

where X=diag(0,0,1) and

$$U = e^{2i\pi_a T_a/f} \;, \qquad f \equiv f_\pi = 93 \text{ MeV}, \tag{2.2}$$

where π_a denotes the pseudoscalar octet field, and the T^a are $SU(3)$ generators with Tr $T_a T_b = \frac{1}{2}\delta_{ab}$. In the rigid rotator model one includes field configurations

$$U(\vec{x},t) = \mathcal{A}(t) U_0(\vec{x}) \mathcal{A}^\dagger(t) \;, \qquad \mathcal{A} \in SU(3) \;, \tag{2.3}$$

where U_0 is the hedgehog

$$U_0(\vec{x}) = e^{iF(r)\hat{r}\cdot\vec{\tau}}, \tag{2.4}$$

with $F(r)$ determined by the equations of motion. Quantization of this model reduces to the quantum mechanics of the collective coordinate $\mathcal{A}$. In (2.1) U enters as $(U-1)$ in order to subtract out the vacuum contribution to $\langle \bar{s}s \rangle$. The result that Donoghue and Nappi found was:

$$\mathcal{R} \equiv \frac{\langle N|\bar{s}s|N\rangle}{\langle N|\bar{u}u + \bar{d}d + \bar{s}s|N\rangle} = \frac{7}{30} \tag{2.5}$$

This answer is independent of the shape function $F(r)$. However, it is not "model independent" in the sense that is sometimes used in the Skyrme model literature, which is usually reserved for the results whose validity depends on the rapid convergence of the $1/N_c$ expansion. What we will do now is repeat the DN calculation at arbitrary N_c and show that, as far as the calculation of $\mathcal{R}$ is concerned, the large N_c approximation (on which the semiclassical quantization of the Skyrme model relies) is invalid at $N_c = 3$.

To calculate $\mathcal{R}$ for arbitrary N_c we make use of the results of Manohar [19]. He shows that the matrix element of an operator $\mathcal{O}^{RJ}_{am}$ (belonging to $SU(3)_{flavor} \times SU(2)_{spin}$ representations $\{R, a\}$ and $\{J, m\}$ respectively) in a baryon state $|R'a'; J'm'\rangle$ may be expressed in terms of $SU(3)$ Clebsch-Gordon coefficients as:

$$\langle R'a'; J'm' | \mathcal{O}^{RJ}_{am} | R'a' J'm' \rangle = \mu_{RJ} \begin{pmatrix} R & R' & | & R' \\ a & a' & | & a' \end{pmatrix} \begin{pmatrix} R & R' & | & R' \\ b & b' & | & b' \end{pmatrix} . \tag{2.6}$$

The charges b, b' satisfy

$$\begin{aligned} b &= \{I = J,\ I_3 = -m,\ Y = 0\} \\ b' &= \{I = J',\ I_3 = -m',\ Y = N_c/3\}\ , \end{aligned} \tag{2.7}$$

and μ_{RJ} is the reduced matrix element

$$\mu_{RJ} = \langle U_0 | \mathcal{O}^{RJ} | U_0 \rangle\ . \tag{2.8}$$

The product of C-G coefficients in eq. (2.6) is summed over all possible contractions (e.g., the D and F contractions of $8 \times 8 \to 8$).

Our first observation is that for any quark flavor q, the operator $\bar{q}q$ has $J = 0$ and belongs to the $SU(3)$ representation $R = 1 \oplus 8$. Since R is reducible, one would expect there to be two independent reduced matrix elements μ, rendering the ratio (2.5) dependent not only on the shape function $F(r)$, but also on the exact realization of $\bar{q}q$ in the chiral Lagrangian. Interestingly enough, this is not so. The operator must be realized in the chiral Lagrangian as

$$\bar{q}Xq \to \mathcal{O}_{(1,1)} \times \mathrm{Tr}\ X\mathcal{O}_{(3,\bar{3})}\ +\ h.c. \tag{2.9}$$

where the $\mathcal{O}$'s are operators built out of $(U-1)$ and spatial derivatives of U, transforming as a singlet and a $(3,\bar{3})$ under chiral $SU(3) \times SU(3)$. Time derivatives bring in inverse powers of N_c and are ignored at this point. Since $(U_0 - 1)$ is nonzero only in the $SU(2) \times SU(2)$ block, $\mathbf{1}(U_0 - 1)$ and $\mathbf{T_8}(U_0 - 1)$ are simply related by a factor of $\sqrt{12}$; thus the reduced matrix elements for singlet and octet operators are likewise related:

$$\mu_1 = \sqrt{12}\ \mu_8\ . \tag{2.10}$$

Since diag(0,0,1)=$(\mathbf{1} - \sqrt{12}\ \mathbf{T_8})/3$, we may now write the ratio $\mathcal{R}$ of eq. (2.5) as

$$\mathcal{R} = \frac{1}{3}\left[1 - \sum_{D,F} \begin{pmatrix} 8 & R & | & R \\ \eta & N & | & N \end{pmatrix}^2\right] \tag{2.11}$$

where the nucleons are in the representation R of $SU(3)$. For $N_c = (2n+1)$, R is an $(n+1)(n+3)$ dimensional representation with one upper index and n symmetrized lower indices. In the appendix we show how to work out the C-G coefficient for this representation. Our result is:

$$\mathcal{R} = \frac{2(N_c+4)}{(N_c+3)(N_c+7)} \to \frac{2}{N_c} + \mathcal{O}(\frac{1}{N_c^2})\ . \tag{2.12}$$

This agrees with the DN result (2.5) for $N_c = 3$ but also shows that their result is not "model independent". It does not follow from keeping only the leading term in the semi-classical expansion. In fact, eq. (2.5) relies on the precise definition of $\langle N|\bar{s}s|N\rangle$ in eq. (2.1) and on summing all powers of $1/N_c$. One may check that another definition, such as

$$\bar{q}Xq \simeq \mathrm{Tr}\, X\,(U-1)\left(1+\Lambda^2\partial_\mu U\partial^\mu U^\dagger\right) \;+\; h.c. \tag{2.13}$$

changes the coefficient of $1/N_c^2$ in the expansion of $\mathcal{R}$. This is due to the presence of time derivatives in the new definition of $\bar{s}s$. The only term in eq. (2.12) that seems to be reliable at this stage is the coefficient of $1/N_c$. Keeping only this leading term, we would find $\mathcal{R} = 2/3$ for $N_c = 3$, *i.e.*, that s quarks have larger matrix elements in the nucleon than u or d quarks. In fact, though, simple large-N_c QCD considerations indicate that the coefficient of the leading term in $\mathcal{R}$ found in the Skyrme model cannot be trusted either. In agreement with eq. (2.12), QCD predicts suppression of the strange sea and a value of $\mathcal{R} \sim 1/N_c$. However, to determine the coefficient, one needs to sum an infinite class of diagrams. This is not expected to yield anything as simple as the coefficient 2 in eq. (2.12). ~~In the~~ Skyrme model, this leading coefficient is shifted once the meson loop diagrams are included in the calculation of $\mathcal{R}$. We proceed with the understanding that by ignoring the effects of virtual meson loops, we are working with a toy model. Nevertheless, we shall see that an interesting qualitative picture of the strange quark emerges, if not a quantitative one.

3. Quantization of the Rigid Rotator at $\mathbf{m_s \neq 0}$ and Large $\mathbf{N_c}$

In the absence of first-principles QCD calculations of baryon masses as functions of m_s, one is typically forced to make some dynamical assumptions. A traditional assumption is that these masses do not develop significant non-linearities as m_s is hypothetically turned on from zero to its physical value. Then the effect of m_s on the baryon masses can be calculated in first-order perturbation theory. This is the justification for the DN calculation of $\mathcal{R}$, using the unperturbed $SU(3)$ rotator wavefunctions, described in the previous section. Unfortunately, it has become clear that, at least as far as baryon masses go, the first-order perturbative treatment of $SU(3)$ breaking fails badly in the Skyrme model [19][20] [21]. This suggests that non-linearities may be important and one should try to do better than first-order perturbation theory.

Yabu [22] has presented a calculation of $\mathcal{R}$ in the context of the Yabu-Ando $SU(3)$ rigid rotator solution [23]. This calculation demonstrates the presence of some non-linearity in the evolution of $\mathcal{R}$ with m_s at $N_c = 3$. Yabu's calculation provides a way to extrapolate the DN result to non-zero m_s. However, as explained in the previous section, such calculations are not truly "model independent" because they regard all terms in the $1/N_c$ series on an equal footing. Actually, the calculations of $\mathcal{R}$ are not exceptions in this respect: they highlight a general problem encountered in the quantization of the 3-flavor Skyrme model. It is well known that, in the 2-flavor case, the same quantum numbers $I = J = n+1/2$ arise for all odd N_c, i. e., we can pass to a large N_c without altering the quantum numbers of the low-lying baryons. The situation is more complicated in the 3-flavor case. There the size of the low-lying $SU(3)$ representations grows with increasing N_c. Thus, if one wishes to work explicitly with the octet and the decuplet of baryons, one is forced to consider $N_c = 3$ only. This procedure, which has been used by many authors, has an obvious flaw: one cannot pass to a large N_c where the semiclassical approximation is manifestly justified. In fact, as shown in the example of sec. 2, the calculations relying on this procedure implicitly sum up all powers of $1/N_c$. The answers may come out significantly different from those obtained by retaining only the leading power. Clearly, a lot more has to be done if we wish to study, among other things, the validity of the semiclassical expansion at $N_c = 3$.

With this motivation, we approach the problem from a different angle. We work with a variable N_c and identify the coefficients of the leading terms in the $1/N_c$ expansion. We believe that this is the method most closely analogous to the one adopted in the 2-flavor Skyrme model. Furthermore, our procedure sheds some new light on the connection between the Skyrme model and the NRQM. In fact, to $\mathcal{O}(1/N_c)$ we obtain a mass formula identical in form to the mass formula found in the large-N_c NRQM. It is satisfying that the dependence of the baryon masses on m_s is explicitly calculable to this order. We are not disturbed by the fact that the size of the SU(3) representation depends on the number of colors. We may simply imagine fixing $N_c = 3$ at the end of the calculation to recover the conventional baryon quantum numbers. Also, as shown in the appendices, even at large N_c it may be convenient to identify a subset of the lowest representation, which is a natural large-N_c analogue of the octet.

Let us now set up a formalism suitable for a systematic $1/N_c$ expansion in the 3-flavor case. The basic observation is the following. If we focus on baryons of fixed strangeness and increase N_c, then their relative strangeness content is $\propto 1/N_c$, i. e., they only deviate a little into the strange directions of the collective coordinate space. Perturbation theory in these deviations allows us to construct $1/N_c$ expansions for various baryon observables, with the dependence on m_s calculable analytically. In this section we discuss this quantization procedure, and address the calculation of $\langle N|\bar{s}s|N\rangle$ in sec. 5.

We begin by reviewing the power counting of chiral perturbation theory, following references [24] and [25]. The chiral Lagrangian has two dimensionful scales in it: f, defined in (2.2); and Λ, the scale of the pion momentum expansion. Λ should not be confused with Λ_{QCD}. For example, the chiral Lagrangian derived as a low energy limit of a linear σ-model will be an expansion in ∂/m_σ: in this case $\Lambda = m_\sigma$. Georgi and Manohar argue that $\Lambda \lesssim 4\pi f$, and in fact $\Lambda \simeq 1$ GeV seems to work well in the real world [26]. Symmetry breaking due to the quark mass matrix M is taken into account by insertions of $\mu M/\Lambda^2$, where μ has dimensions of mass; consistent with M's transformation property as a $(3,\bar{3})$ under $SU(3)\times SU(3)$. The Lagrangian then takes the form

$$
\begin{aligned}
\mathcal{L} &= \Lambda^2 f^2 \times \hat{\mathcal{L}}\left(U, U^\dagger, \frac{\partial}{\Lambda}, \frac{\mu M}{\Lambda^2}\right) \\
&= \frac{f^2}{4}\,\mathrm{Tr}\ \partial U \partial U^\dagger \ + \ \frac{f^2}{2}(\mathrm{Tr}\ \mu M U \ + \ h.c.) \ + \ \dots
\end{aligned}
\tag{3.1}
$$

where $\hat{\mathcal{L}}$ consists of all operators consistent with $SU(3)\times SU(3)$, with coefficients of $\mathcal{O}(1)$. The matrix U is defined in (2.2). Note that $\mu m_s \simeq m_K^2$, m_K being the kaon mass, so one expects second order $SU(3)$ symmetry breaking effects to be down from first order by $m_K^2/\Lambda^2 \simeq 25\%$. Including the effects of anomalies is accomplished by adding the usual Wess-Zumino term to the action [27].

We now suppose that the full equations of motion have a soliton solution U_0 of the form (2.4). Derrick's theorem implies that higher derivative terms in $\mathcal{L}$ must be responsible for stabilizing U_0, which means that $\partial/\Lambda = \mathcal{O}(1)$ when acting on U_0, and one cannot *apriori* justify truncating $\mathcal{L}$ at any finite number of derivatives. To quantize the configurations of eq. (2.3), one expands $\mathcal{L}$ to second order in $\mathcal{A}^\dagger\dot{\mathcal{A}}$, and integrates over space to arrive at an effective lagrangian for $\mathcal{A}$. In the absence of quark masses one finds [28]

$$
\begin{aligned}
L(m_s = 0) = &-2\Omega\sum_{i=1}^{3}\left(\mathrm{Tr}\ T_i\mathcal{A}^\dagger\dot{\mathcal{A}}\right)^2 - 2\Phi\sum_{a=4}^{7}\left(\mathrm{Tr}\ T_a\mathcal{A}^\dagger\dot{\mathcal{A}}\right)^2 \\
&-\sqrt{3}N_c\left(\mathrm{Tr}\ T_8\mathcal{A}^\dagger\dot{\mathcal{A}}\right)\ .
\end{aligned}
\tag{3.2}
$$

Both Ω and Φ are spatial integrals of some function of derivatives of U_0, derivable fom the full Lagrangian, and are both $\propto f^2/\Lambda^3 = \mathcal{O}(N_c)$. The $N-\Delta$ splitting depends only on Ω.

When we add a nonzero strange quark mass, we will see that the mass splittings within the octet depend only on Φ. The term with the explicit N_c dependence arises from the Wess-Zumino term.

The $SU(3)$ multiplet containing the particle we will identify with the proton contains baryons with up to $(N_c+1)/2$ strange quarks. This means that the wave function of the proton will extend only $\mathcal{O}(1/N_c)$ into the strange directions of $SU(3)$, in a sense which we will make precise below. This motivates us to write

$$\begin{aligned} &\mathcal{A}(t) = A(t)S(t)\ , \qquad \text{with} \\ &A(t) \in SU(2)\ , \\ &S(t) \equiv \exp\left[2ik_a(t)T_a/f\right]\ ,\ a = 4\ldots 7\ . \end{aligned} \tag{3.3}$$

We then expand perturbatively in the $SU(3)/SU(2)\times U(1)$ coordinates, since the proton wavefunction resides almost entirely in $SU(2)$:

$$S(t) \simeq 1 + 2i(k\cdot T/f) - 2(k\cdot T/f)^2\ldots \tag{3.4}$$

and arrive at a simple quadratic Lagrangian for a "kaon" doublet:

$$\begin{aligned} &L_K(m_s=0) = 4\Phi\dot{K}^\dagger\dot{K} + i\frac{N_c}{2}\left(K^\dagger\dot{K} - \dot{K}^\dagger K\right), \\ &K \equiv \frac{1}{f\sqrt{2}}\begin{pmatrix} k_4 - ik_5 \\ k_6 - ik_7 \end{pmatrix}\ . \end{aligned} \tag{3.5}$$

The reason for doing this expansion is that now we may easily take into account the effects of nonzero m_s. We set $\mu M=$ diag$(\,0, 0, \mu m_s)$ in (3.1). Note that $MU_0 = U_0M = M$. It is simple then to extend (3.5) to nonzero m_s. We find

$$\begin{aligned} &L_K = 4\Phi\dot{K}^\dagger\dot{K} + i\frac{N_c}{2}\left(K^\dagger\dot{K} - \dot{K}^\dagger K\right) - \Gamma m_K^2 K^\dagger K \ +\ \mathcal{O}(\mu m_s/\Lambda^2) \\ &\Gamma \propto \frac{f^2}{\Lambda^3} = \mathcal{O}(N_c) \end{aligned} \tag{3.6}$$

The reader will recognize that this Lagrangian is the same as for a charged particle in two dimensions, with a perpendicular magnetic field and a central harmonic restoring force. The magnetic field strength is $\mathcal{O}(N_c)$. Thus for either large N_c or large m_s, we will find $\langle N|K^\dagger K|N\rangle \ll 1$ and our expansion in powers of K will be validated [3].

The Hamiltonian derived from (3.6) is

$$H = \frac{1}{4\Phi}\Pi^\dagger\Pi + i\frac{N_c}{8\Phi}(\Pi^\dagger K - K^\dagger\Pi) + (\frac{N_c^2}{16\Phi} + \Gamma m_K^2)K^\dagger K \tag{3.7}$$

where $\Pi^\dagger$ is the canonical conjugate of K. In order to diagonalize the Hamiltonian, it is convenient to transform to the basis of creation and annihilation operators:

$$\begin{aligned} K &= \frac{1}{\sqrt{N_c}}\left(1 + \left(\frac{m_K}{M_0}\right)^2\right)^{-1/4}(a + b^\dagger) \\ \Pi^\dagger &= -\frac{i}{2}\sqrt{N_c}\left(1 + \left(\frac{m_K}{M_0}\right)^2\right)^{1/4}(a^\dagger - b)\ , \end{aligned} \tag{3.8}$$

where a and b are doublets, and the parameter M_0 is defined as

$$M_0^2 \equiv N_c^2/16\Phi\Gamma\ . \tag{3.9}$$

Note that M_0 has the dimensions of mass, and is $\mathcal{O}(1)$ as far as N_c counting goes. This transformation brings the Hamiltonian to the diagonal form

$$H = \frac{1}{2}\omega_-\{a, a^\dagger\} + \frac{1}{2}\omega_+\{b, b^\dagger\} \tag{3.10}$$

where

$$\omega_\pm = \frac{N_c}{8\Phi}\left(\sqrt{1 + \left(\frac{m_K}{M_0}\right)^2} \pm 1\right) \tag{3.11}$$

The eigenstates form a Fock space

$$|n_s,\ n_{\bar{s}}\rangle = (a^\dagger)^{n_s}(b^\dagger)^{n_{\bar{s}}}|0\rangle \tag{3.12}$$

[3] Much has been said about how quantizing the Skyrmion with a Wess-Zumino term is analogous to quantizing the motion of a particle around a magnetic (hypercharge) monopole [28], [29]. By expanding in K to quadratic order, we are approximating $SU(3)/SU(2) \times U(1)$ by two flat surfaces, parameterized by K^+ and K^0, with a normal magnetic field. Therefore we find an infinite number of Landau levels in the planes rather than a finite set of quantized states on a compact surface. So long as we stick to Landau levels of small radius ($\langle K^\dagger K\rangle \ll 1$)— or equivalently, to baryons with negative strangeness $\ll N_c$—this is a good approximation.

with $n_s, n_{\bar{s}}$ integers. As is easily surmised, the state $|n_s,\ n_{\bar{s}}\rangle$ looks as if it contains n_s s-quarks and $n_{\bar{s}}$ $\bar{s}$-quarks. This identification will be made more precise in the next section. The states with $n_{\bar{s}} \neq 0$ correspond to the exotic states found in the Skyrme model, which are usually presumed unstable. In the bound state approach to strangeness [15], which has more degrees of freedom than the rigid rotator, these states are indeed unstable. Note that even in the $SU(3)$ limit $m_s \to 0$, it costs energy to produce a state with an $\bar{s}$, but not to replace a u or a d with an s-quark.

So far we have constructed baryon states which are eigenstates of strange-ness but carry no definite isospin or angular momentum. In order to identify the eigenstates of I and J, it is necessary to excite the $SU(2)$ collective coordinate $A(t)$. For each value of strangeness, we will be able to construct a tower of narrowly split states of increasing spin and isospin.

The lagrangian depends on $A(t)$ only through the angular velocity $\dot{\alpha}_i$ defined by

$$A^\dagger \dot{A} = \frac{i}{2}\dot{\alpha}_i \tau_i, \tag{3.13}$$

where τ_i are the Pauli matrices. Since the soliton moment of inertia Ω is $\mathcal{O}(N_c)$, the angular velocities of the low-lying rotational excitations are $\mathcal{O}(1/N_c)$. Thus, in addition to rotation in $K^\dagger K$ space with angular velocity of order 1, the soliton undergoes a slow rigid rotation in isospace. The $\mathcal{O}(1/N_c)$ correction to the lagrangian L_K of eq. (3.6) is

$$\delta L_K = \frac{1}{2}\Omega(\dot{\alpha}_i)^2 + i\dot{\alpha}_i(2\Phi - \Omega)(\dot{K}^\dagger \tau_i K - K^\dagger \tau_i \dot{K}) - \frac{1}{2}N_c \dot{\alpha}_i K^\dagger \tau_i K \tag{3.14}$$

The Hamiltonian obtained from $L_K + \delta L_K$ through the standard canonical procedure can be expanded in powers of $1/N_c$ as

$$H = H_0 + H_1 + \mathcal{O}(1/N_c^2) \tag{3.15}$$

H_0 is the $\mathcal{O}(1)$ hamiltonian of eq. (3.7). As shown in the previous section, this Hamiltonian is diagonal in the basis of Fock states of eq. (3.12). Since the full Hamiltonian H cannot be solved exactly, our goal is to determine the energy levels correct to $\mathcal{O}(1/N_c)$. To this end, we will use first order perturbation theory in

$$H_1 = \frac{1}{2\Omega}\left\{\vec{J}_{ud} - \frac{i}{2}(\Pi^\dagger \vec{\tau} K - K^\dagger \vec{\tau} \Pi)\left(1 - \frac{\Omega}{2\Phi}\right) + \frac{N_c \Omega}{4\Phi}\ K^\dagger \vec{\tau} K\right\}^2, \tag{3.16}$$

where $\vec{J}_{ud}$ is the momentum conjugate to $\vec{\alpha}$. Our notation is intended to remind the reader that, in the quark language $\vec{J}_{ud}$ is the net angular momentum of the u and d quarks. In the soliton model $\vec{J}_{ud}$ is the generator of the right rotations on the collective coordinate A. It measures the angular momentum of $\chi_I(A)$, the collective coordinate component of the wave function. The wave functions can be written as sums of products of the form

$$|n_s,\ n_{\bar{s}}\rangle \chi_I(A). \tag{3.17}$$

$\chi_I(A)$ are the wave functions defined on the $SU(2)$ group manifold which were extensively discussed in ref. [30]. The index I labels the spin-isospin representation. In order to identify the wave functions which correspond to various excitations of the total spin and isospin, we need to construct the $\vec{I}$ and $\vec{J}$ operators.

First, let us consider the transformation properties of K and A. Under spatial rotations $U_0 \to RU_0R^{-1}$, where R is an $SU(2)$ matrix. This transformation acts on K and A as $K \to R^{-1}K$ and $A \to AR$. Using Noether's theorem, we find the total angular momentum to be

$$\vec{J} = \vec{J}_{ud} + \vec{J}_s, \tag{3.18}$$

where

$$\vec{J}_s = \frac{1}{2}(a^\dagger \vec{\tau} a - b\vec{\tau} b^\dagger) \tag{3.19}$$

Thus, $\vec{J}_s$ is the contribution of the strange excitations to the angular momentum. Eq. (3.19) shows that each unit of strangeness carries half a unit of angular momentum. Remembering that the creation operators are two-component objects, we can decompose the Fock states $|n_s,\ n_{\bar{s}}\rangle$ into irreducible representations of $\vec{J}_s$. Then the irreducible representations of $\vec{J}$ can be constructed as sums of products of the form (3.17) using the familiar addition of angular momenta.

Further restriction on the baryon wave functions comes from the fact that they must carry definite total isospin. This restriction is particularly easy to implement. Since K does not transform under isorotations, isospin only acts on $\chi_I(A)$: $\vec{I} = \vec{I}_{ud}$. In other words, the subscript I of the collective coordinate component of the wave function is the total isospin. It follows that each unit of strangeness behaves as an object with no isospin and half a unit of spin. These are the familiar spin-isospin quantum numbers of a strange quark.

Further constraint on quantum numbers follows from the fact that all $SU(2)$ rotator wave functions $\chi_I(A)$ satisfy the $I = J_{ud}$ rule, with I taking on either integral or half-integral values. Witten has shown [27] that, if N_c is odd, the baryon must have half-integral spin. In our model, this leads to the following rule: states with even strangeness carry half-integral isospin, while states with odd strangeness carry integral isospin. This rule leads to the observed baryon quantum numbers.

For example, labeling the states as $|n_s, n_{\bar{s}}\rangle |I, J\rangle$, we see that

$$\begin{aligned}
|N\rangle &= |0,0\rangle|\tfrac{1}{2},\tfrac{1}{2}\rangle \ , \\
|\Delta\rangle &= |0,0\rangle|\tfrac{3}{2},\tfrac{3}{2}\rangle \ , \\
|\Lambda\rangle &= |1,0\rangle|0,0\rangle \ , \\
|\Sigma\rangle &= \left[|1,0\rangle|1,1\rangle\right]_{J=\frac{1}{2}} \ , \\
|\Sigma^*\rangle &= \left[|1,0\rangle|1,1\rangle\right]_{J=\frac{3}{2}} \ ,
\end{aligned} \tag{3.20}$$

and so forth.

4. Mass Formulæ and the Gell-Mann–Okubo Relation

A nonzero strange quark mass breaks $SU(3)$ down to $SU(2) \times U(1)$. Thus the most general form for an effective hamiltonian describing the masses of the N, Λ, Σ and Ξ fields may be expressed as

$$H = a_1 + a_2 Y + a_3 \left[I(I+1) - \tfrac{1}{4}Y^2\right] + a_4 Y^2. \tag{4.1}$$

Empirically, $a_1 = 1116 MeV$, $a_2 = -190 MeV$, $a_3 = 38 MeV$, $a_4 = -11 MeV$ As shown in Appendix B, the leading N_c-dependences of the coefficients are $a_1 \sim N_c$, $a_2 \sim 1$, $a_3 \sim a_4 \sim 1/N_c$. Thus large-N_c reasoning can explain why $|a_3| \ll |a_2|$ and $|a_4| \ll |a_2|$, but it does not by itself explain why $|a_4| \ll |a_3|$. The usual extra ingredient is that, in first order perturbation theory in m_s, one finds that a_4 is zero, and the GMO formula results. If first order perturbation theory is a poor approximation to nature, then the GMO relations may simply be due to a numerical coincidence. Our large-N_c argument shows that this coincidence may not be as drastic as one might have expected.

A mass formula similar to (4.1) follows in the NRQM with magnetic moment interactions between quarks. There the Hamiltonian is taken to be

$$H = m_0 + m_1|S| + m_2 \sum_{i<k} \vec{j}_i \cdot \vec{j}_k + m_2 c \sum_{i,I} \vec{j}_i \cdot \vec{j}_I + m_2 \bar{c} \sum_{I<K} \vec{j}_I \cdot \vec{j}_K \ . \tag{4.2}$$

where the small indices refer to the light quarks and the capital indices – to the strange quarks: m_0 is $\mathcal{O}(N_c)$ because a baryon contains N_c valence quarks. m_1 is the difference between the constituent masses of the strange and the light quarks, which is $\mathcal{O}(1)$. m_2, the strength of the phenomenological magnetic moment interaction, is $\mathcal{O}(1/N_c)$. In the NRQM such interactions are suggested by one gluon exchange calculations [14]. In eq. (4.2), the $SU(3)$ symmetric values are $m_1 = 0$ and $c = \bar{c} = 1$. As we will see, there are significant empirical deviations from the $SU(3)$ symmetry. Eq. (4.2) can be manipulated into

$$H = m_0' + m_1' Y + \frac{m_2}{2}\left\{cJ(J+1) + (1-c)\left[I(I+1) - \tfrac{1}{4}Y^2\right] + \frac{1+\bar{c}-2c}{4}\,Y^2\right\} \quad (4.3)$$

From the point of view of eq. (4.3), the GMO mass relations are successful if

$$1 + \bar{c} - 2c \approx 0 \ . \quad (4.4)$$

Of course, there are other tighter constraints on the values of these parameters which follow from further empirical data. In fact, eq. (4.3) provides a good fit to the masses of all the octet and decuplet baryons with $m_0' \approx 1062 MeV$, $m_1' \approx -192 MeV$, $m_2 \approx 213$ MeV, $c \approx .67$, $\bar{c} \approx .27$ [31].

This fit shows how far c and $\bar{c}$ deviate from their $SU(3)$ symmetric values. To some extent, the fit makes the conventional explanation of the GMO relations suspect: if first order perturbation in m_s changes $\bar{c}$ by 70 %, then why are we allowed to ignore higher order perturbations? Perhaps, there exists a class of models with appreciable non-linearities which manage to yield output parameters close to the empirical. How well does our rigid rotator treatment do in this respect?

First of all, it is not hard to derive a mass formula of the form (4.3) in our approach. In order to calculate the values of m_2 and c, it is sufficient to evaluate the expectation values of H_1 from eq. (3.16) in the baryon states constructed in sec. 3. This simple calculation yields $m_2 = 1/\Omega$ and

$$c = 1 - \frac{4\Omega\omega_-}{8\Phi\omega_- + N_c} \ . \quad (4.5)$$

H_1 does not include the full $1/N_c$ correction to the masses: the terms quartic in K, which we have so far omitted, are also $\mathcal{O}(1/N_c)$. They incorporate the strange quark – strange quark magnetic moment interactions. Thus, the calculation of $\bar{c}$ is a somewhat painful exercise involving the expansion to fourth order in K. Its result will be reported elsewhere. We can, however, compare the results for other parameters with their empirical values. In the model stabilized by the 4-derivative Skyrme term, $\Omega \approx \Gamma \approx 53.5$ and $\Phi \approx 19.5$ in units of $f_\pi^{-1} e^{-3}$. With the values $f_\pi = 64.5 MeV$ and $e = 5.45$ from the 2-flavor fit of ref. [30], we find $\omega_- \approx 248 MeV$, $m_2 \approx 196 MeV$ and $c \approx .25$. Although our input parameters may be far from the best fit in the 3-flavor case, the drastic discrepancy in the calculated value of c is indicative of the breakdown of the rigid rotator treatment reported in other papers [19], [20], [21]. The c being far from its empirical value leads to the incorrect relative splittings between Λ, Σ, and $\Sigma^\star$. Perhaps, our results shed some new light on why the rigid rotator approximation does not work well, and which modifications may lead to a better agreement with the data. For instance, the bound state approach to strangeness also leads to a baryon mass formula of type (4.3), but with a value of c much closer to the empirical. This is due to a treatment of $SU(3)$ breaking which goes beyond the rigid rotator approximation.

5. $\langle \mathrm{N}|\bar{\mathrm{s}}\mathrm{s}|\mathrm{N}\rangle$ in the Rigid Rotator at $m_s \neq 0$

As discussed in the introduction, the measurement of $\Sigma_{\pi N}$ can be related to $\partial M_N / \partial m_s$, which in turn can be related to $\langle N|\bar{s}s|N\rangle|_{m_s=0}$. The question of interest is whether the nucleon mass depends linearly on m_s all the way up to the actual value of m_s. In this section we show that, in our toy model, the mass of the nucleons—and hence $\langle N|\bar{s}s|N\rangle$— may naturally depend non-linearly on m_s.

We have seen in sec. 3 how to construct the quantum states of low strangeness at non-zero strange quark mass. The fluctuating K modes contribute zero-point energy to all of the states. Thus the mass of the nucleon will depend on m_s, even though the nucleon state has $n_s = n_{\bar{s}} = 0$. According to eq. (3.10), this strange quark contribution is simply

$$\delta M_N = \frac{N_c}{4\Phi} \sqrt{1 + (m_K/M_0)^2} \ , \tag{5.1}$$

The ratio $\mathcal{R}$ of eq. (2.5) is also easily extracted from this model. To the leading order in $1/N_c$, it is found to be

$$\mathcal{R} = \langle 0|K^\dagger K|0\rangle = \frac{2}{N_c} \frac{1}{\sqrt{1 + (m_K/M_0)^2}} \tag{5.2}$$

Note that this result agrees with the large N_c value (2.12) for $m_s = 0$, as it should.

We must take care interpreting this result. First of all, there is the problem of the finite renormalization of the Skyrmion mass due to meson loops, which we ignore. Secondly, we must remember that Φ and M_0 are parameters that may be expanded in a power series in $\mu m_s/\Lambda^2 \simeq m_K^2/\Lambda^2$. Thus the form of (5.2) is only meaningful if M_0 evaluated at $m_s = 0$ is significantly smaller than $\Lambda \simeq 1$ Gev. In fact, this condition is met in the usual treatment of the Skyrme model, where the only higher derivative interaction included is the so-called Skyrme term. In this case, using the input parameters displayed in sec. 4, we find

$$M_0 \approx 245 \text{ MeV.} , \tag{5.3}$$

implying

$$\frac{\mathcal{R}(m_K = 495MeV)}{\mathcal{R}(m_K = 0)} \approx 0.45 \ . \tag{5.4}$$

If N_c is large, then $\langle N|\bar{s}s|N\rangle \ll \langle N|\bar{u}u|N\rangle$, in which case (5.4) tells us that

$$\frac{\langle N|\bar{s}s|N\rangle(m_K = 495MeV)}{\langle N|\bar{s}s|N\rangle(m_K = 0)} \approx 0.45 \ . \tag{5.5}$$

This ratio is a measure of deviation of the nucleon wave function from its $SU(3)$ symmetric limit. Can we relate it to any existing data?

If one believes in the parton model, then $\langle N|\bar{s}s|N\rangle$ is a measure of the number of $\bar{s}s$ pairs in the sea. $SU(3)$ symmetry dictates that with $m_K = 0$, this number is equal to the number of $\bar{u}u$ pairs in the sea. Assuming that the number of $\bar{u}u$ pairs does not depend on m_K, eq. (5.5)indicates that, at the actual value $m_K = 495MeV$, the number of the $\bar{s}s$ pairs in the sea is about 0.45 times the number of the $\bar{u}u$ pairs in the sea. According to some experiments, this number is indeed found to be close to $\frac{1}{2}$ [32]. However, the result should be taken with a grain of salt since it is not a direct experimental measurement, but is interpreted in the context of the parton model. Also, it isn't clear that the measurements may be sensibly extrapolated to zero momentum transfer. The experimental result should simply be taken as suggestive that the quark sea in the nucleon may significantly break $SU(3)$ at zero momentum transfer, and that the non-linearity exhibited in (5.2) could be real.

6. Discussion

We would now like to compare our treatment of the $SU(3)$ symmetry breaking with other approaches found in the literature. The simple model considered above is a variant of the $SU(3)$ rigid rotator quantization. The novel feature is that the $SU(3)$ rotations are factored into two parts: the ones that rotate the $SU(2)$ soliton into the "strange directions" ($S(t)$ in eq. (3.3)), and the ones that do not ($A(t)$ in eq. (3.3)). As explained above, the expansion about $S(t) = 1$ provides us with a simple way of performing the $1/N_c$ expansion both in the case of unbroken $SU(3)$ and when arbitrary strange quark mass is included. Thus, we have developed an approximation to the large-N_c analogue of Yabu and Ando's solution [23] of the $SU(3)$ rigid rotator at $N_c = 3$. Our method may prove useful both because of its simplicity and also because the semiclassical quantization is only strictly correct in the large-N_c limit. Let us point out that all approaches based on the rigid $SU(3)$ rotator have a common defficiency once the kaon mass term is included: the rotations in the "strange directions" become only approximate collective coordinates. Strictly speaking, as the soliton rotates into strange directions, it also has a tendency to deform. For a sufficiently large m_s this effect is significant. Further, we might want to study strange deformations which are not smoothly connected to the rigid rotations. With these ideas in mind, ref. [15] developed a treatment of general strange fluctuations about the basic $SU(2)$ skyrmion based on the parametrization

$$U = \sqrt{U_\pi} U_K \sqrt{U_\pi} \tag{6.1}$$

If N_c or m_s is large, the expansion of the lagrangian to second order in K is a good approximation. Then the problem reduces to motion of kaons in the background of the $SU(2)$ skyrmion. In ref. [15] it was shown that the interactions between the kaons and solitons are such that there exist bound states. These bound states carry both baryon number and strangeness and can be naturally identified with hyperons.

The simpler model presented in the previous sections is roughly the bound state model where all the modes except for one kaon and one anti-kaon mode are ignored. Even the modes that are taken into account are treated approximately. This leads to a drastic simplification but also creates some distortions. Let us explain this in more detail. First, our toy model makes it easy to understand why the bound states exist. Consider the $S = -1$ mode energy as a function of m_K. Eq. (3.11) states that $m_K - \omega_-$ is greater than zero and is an increasing function of m_K. Thus the baryon with one $S = -1$ quantum cannot decay into a kaon and a non-strange baryon. It is easy to show that ω_- provides an upper bound on the lowest $S = -1$ mode energy in the bound state approach: in the language of the kaon bound states, the rigid rotator approximation restricts the kaon profile function to be that of the zero mode

$$k(r) \sim \sin(F(r)/2) \tag{6.2}$$

If we allow the kaon profile function to adjust itself with increasing m_K, as we do in the bound state approach, the mode energy can only decrease. The radial function of this lowest $T = 1/2,\ L = 1$ mode is, therefore, smoothly connected to the zero mode of the $m_s = 0$ problem.

The previous paragraph makes it clear that the $S = -1$ excitation in the rigid rotator model of the previous sections is at least a reasonable qualitative approximation to a similar excitation in the bound state approach, where it is created by populating the lowest bound mode. Unfortunately, this connection does not exist for the exotic $S = 1$ mode. Even in the approximation of eq. (3.11), $\omega_+ > m_K$. This means that the $S = 1$ states are unstable and, in a treatment more complete than the rigid rotator model, will decay into mesons and nucleons. Therefore, in the bound state approach their profile functions oscillate at large r and do not resemble the profile of eq. (6.2). Thus, the rigid rotator approximation does not treat these modes correctly. All of this suggests that the bound state approach is a better tool for studying such questions as the strange matrix elements in the proton. However, one has to pay the price of doing much more complicated calculations taking into account the infinity of kaon normal modes. Some work in this direction has already been performed [33].

In spite of the differences listed above, the basic assumption of the bound state approach is the same as in our rigid rotator treatment. Namely, the presence of significant non-linearities in the dependence of observables on m_s is not necessarily in conflict with the successes of the $SU(3)$ phenomenology. Although much more work is needed to substantiate this, the limited phenomenological success of the bound state approach [16][34] raises the speculation that this assumption may be realized in nature.

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Appendix A. Clebsch-Gordon Coefficients for Large N_c

In this Appendix we briefly illustrate the calculation of the D and F Clebsch-Gordon coefficients for the couplings of the spin $\frac{1}{2}$ baryons to an octet operator, for

$$N_c \equiv (2\nu + 1) \tag{A.1}$$

We follow the treatment by A. Manohar [35]; see also [36].

For arbitrary ν, the $SU(3)$ representation R containing the nucleons may be represented by the tensor $\mathcal{B}^a{}_{bcd\ldots}$ with ν symmetric lower indices. We normalize $\mathcal{B}$ so that $\overline{\mathcal{B}}\mathcal{B}$ with contracted indices weights each baryon equally. It is easy to see that this representation contains states with the same hypercharge and isospin quantum numbers as the N, Σ, Λ, and Ξ we have for $N_c = 3$, provided we normalize hypercharge so that the up quark carries $Y = 1/N_c$. One finds:

$$\begin{aligned}
p &= \mathcal{B}^1{}_{333\ldots} \\
n &= \mathcal{B}^2{}_{333\ldots} \\
\Sigma^+ &= \sqrt{\nu}\mathcal{B}^1{}_{233\ldots} \\
\Sigma^- &= \sqrt{\nu}\mathcal{B}^2{}_{133\ldots} \\
\Sigma^0 &= \tfrac{1}{2}\sqrt{2\nu}(\mathcal{B}^1{}_{133\ldots} - \mathcal{B}^2{}_{233\ldots}) \\
\Lambda &= \tfrac{1}{6}\sqrt{(4+2\nu)}(\mathcal{B}^1{}_{133\ldots} + \mathcal{B}^2{}_{233\ldots} - 2\mathcal{B}^3{}_{333\ldots}) \\
\Xi_0 &= \sqrt{\nu(\nu+2)/3}(\mathcal{B}^3{}_{323\ldots} - 2/3\mathcal{B}^2{}_{223\ldots} - 1/3\mathcal{B}^1_{123\ldots})\ ,
\end{aligned} \tag{A.2}$$

where in each case "..." stands for $\nu - 3$ indices equal to "3".

Following Manohar, we now define

$$\begin{aligned}
M_\alpha &\equiv \bar{\mathcal{B}}_a^{b_1\ldots b_\nu}(T_\alpha)^a{}_c\mathcal{B}^c_{b_1\ldots b_\nu} \\
N_\alpha &\equiv \bar{\mathcal{B}}_a^{cb_2\ldots b_\nu}\mathcal{B}^a_{db_2\ldots b_\nu}(T_\alpha)^d{}_c\ ,
\end{aligned} \tag{A.3}$$

where the T_α are SU(3) generators. The F and D clebsch's may then be calculated from

$$\begin{aligned}
\begin{pmatrix} 8 & R & | & R \\ \alpha & B & | & B \end{pmatrix}_F &= g_F\ (M_\alpha - \nu N_\alpha)\ , \\
\begin{pmatrix} 8 & R & | & R \\ \alpha & B & | & B \end{pmatrix}_D &= g_D\ (xM_\alpha + N_\alpha)\ .
\end{aligned} \tag{A.4}$$

"B"is a particular baryon given in (A.2). The parameters g_F and g_D are most easily fixed by normalizing the branching ratios of the proton in both the D and F channels; x is then chosen to make the two channels orthogonal. One finds

$$\begin{aligned} g_F^2 &= \frac{3}{(\nu+2)^2} \\ g_D^2 &= \frac{\nu(\nu+8)^2}{3(\nu+2)^2(\nu+4)} \\ x &= \frac{(2\nu+7)}{(\nu+8)} \ . \end{aligned} \tag{A.5}$$

We are only interested in the baryon couplings to T_8, which can now be simply calculated from the above relations. Defining the following shorthand for the clebsch's:

$$\begin{aligned} \mathcal{F}(B;N_c) &\equiv \begin{pmatrix} 8 & R & \mid & R \\ \eta & B & \mid & B \end{pmatrix}_F , \\ \mathcal{D}(B;N_c) &\equiv \begin{pmatrix} 8 & R & \mid & R \\ \eta & B & \mid & B \end{pmatrix}_D , \end{aligned} \tag{A.6}$$

we find

$$\begin{aligned} \mathcal{F}(N;N_c) &= \frac{N_c}{N_c+3} \\ \mathcal{F}(\Sigma;N_c) &= \frac{N_c-3}{N_c+3} \\ \mathcal{F}(\Lambda;N_c) &= \frac{N_c-3}{N_c+3} \\ \mathcal{F}(\Xi;N_c) &= \frac{N_c-6}{N_c+3} , \end{aligned} \tag{A.7}$$

and

$$\begin{aligned} \mathcal{D}(N;N_c) &= -\frac{3}{N_c+3}\sqrt{\frac{N_c-1}{N_c+7}} \\ \mathcal{D}(\Sigma;N_c) &= -\frac{2(N_c-9)}{(N_c-1)(N_c+3)}\sqrt{\frac{N_c-1}{N_c+7}} \\ \mathcal{D}(\Lambda;N_c) &= -\frac{6}{N_c+3}\sqrt{\frac{N_c-1}{N_c+7}} \\ \mathcal{D}(\Xi;N_c) &= -\frac{(7N_c-15)}{(N_c-1)(N_c+3)}\sqrt{\frac{N_c-1}{N_c+7}} \ . \end{aligned} \tag{A.8}$$

Eq. (2.12) follows directly.

For our discussion of the Gell-Mann–Okubo formula, it is useful to express (A.7) and (A.8) in terms of the 4×4 matrices $\mathbf{1}$, $\mathbf{Y}$, $\vec{\mathbf{I}}^2$, $\mathbf{Y^2}$ acting on the N, Σ, Λ, and Ξ fields:

$$\begin{aligned}\mathcal{F}(N_c) =& \frac{1}{N_c+3}\left[(N_c-3)\mathbf{1}+3\mathbf{Y}\right]\\ \mathcal{D}(N_c) =& \frac{2}{(N_c-1)(N_c+3)}\sqrt{\frac{N_c-1}{N_c+7}}\\ &\times\left[3(N_c-1)\mathbf{1}-(N_c+3)(\vec{\mathbf{I}}^2-\tfrac{1}{4}\mathbf{Y^2})-(N_c-3)\mathbf{Y}\right]\end{aligned} \tag{A.9}$$

where $\mathbf{Y}$ is normalized to unity for the nucleon.

Appendix B. GMOR Analysis and the GMO formula for Large $\mathbf{N_c}$

If SU(3) is only slightly broken by the strange quark mass, then we can follow Gell-Mann–Oakes–Renner and do first order perturbation in SU(3) breaking. This lets one write the contribution to the mass of baryon B from the strange quark as

$$\Delta M_B/m_s = C + F\begin{pmatrix} 8 & R & \mid & R \\ \eta & B & \mid & B\end{pmatrix}_F + D\begin{pmatrix} 8 & R & \mid & R \\ \eta & B & \mid & B\end{pmatrix}_D \tag{B.1}$$

with C, F, and D being undetermined numbers. Restricting ourselves to the eight particles in the representation R with the same quantum numbers as the $N_c = 3$ octet, eq. (A.9) lets us rewrite this as

$$\Delta M_B/m_s = a_1\mathbf{1} + a_2\mathbf{Y} + a_3(\vec{\mathbf{I}}^2 - \tfrac{1}{4}\mathbf{Y^2}) \tag{B.2}$$

which leads to the usual Gell-Mann–Okubo mass relation for the baryons for any value of N_c. Expressing the a_i in terms of D, F, and C is made straight forward by eq. (A.9).

The coefficients D, F, and C are readily calculated in the Skyrme model , in terms of the single reduced matrix element μ_1 and the C-G coefficients from the spin part of the wavefunction, by means of equations (2.6) and (2.10). One finds

$$\begin{aligned}C &= \frac{\mu_1}{3}\\ F &= -\mu_1\begin{pmatrix} 8 & R & \mid & R \\ \eta & N & \mid & N\end{pmatrix}_F = -\frac{\mu_1 N_c}{N_c+3}\\ D &= -\mu_1\begin{pmatrix} 8 & R & \mid & R \\ \eta & N & \mid & N\end{pmatrix}_D = \frac{3\mu_1}{N_c+3}\sqrt{\frac{N_c-1}{N_c+7}}\,.\end{aligned} \tag{B.3}$$

We do not bother to write down here the values for the a_i, but simply note that one finds the generic large-N_c prediction

$$\begin{aligned} a_1 &= \mathcal{O}(N_c) \\ a_2 &= \mathcal{O}(1) \\ a_3 &= \mathcal{O}(1/N_c) \ . \end{aligned} \tag{B.4}$$

Perhaps, the large-N_c reasoning can be invoked to explain why the hyperfine splitting (a_3) is a few times smaller than the hypercharge splitting (a_2).

References

[1] D.B. Kaplan and A. Manohar, Nucl. Phys. B310 (1988) 527.

[2] L.A. Ahrens *et al.* Phys. Rev. D35 (1987) 785.

[3] J. Ashman *et al.* Phys. Lett. 206B (1988) 364.

[4] D. H. Beck, Cal Tech preprint OAP 696 (1988), sumitted to Phys. Rev. D. R. McKeown, Phys. Lett. 219B (1989) 140.

[5] G. Höhler, in Landolt-Börnstein, New Series, Vol. I/9b, part 2, ed. H. Schopper (Springer, Berlin, 1983)

[6] V. P. Efrosini and D. A. Zaikin, Sov. J. Part. Nucl. 16 (6) (1985) 593; T. E. O. Ericson, Phys. Lett. 195B (1987) 116.

[7] D.B. Kaplan and A.E. Nelson, Phys. Lett. 175B (1986) 57; 179B (1986) 409(E); Nucl. Phys. A479 (1988) 273c.

[8] A.E. Nelson, D.B. Kaplan, Phys. Lett. 192B (1987) 193; Nucl. Phys. A479 (1988) 285c

[9] T. P. Cheng, Phys. Rev. D38 (1988) 2869

[10] D.B. Kaplan, Nucl. Phys. B260 (1985) 215; H. Georgi, D.B. Kaplan and L. Randall, Phys. Lett. 169B (1986) 73.

[11] J. Ellis, R. A. Flores, and J. D. Lewin, Phys. Lett. 212B (1988) 375; J. Ellis, R. A. Flores, and S. Ritz, Phys. Lett. 198B (1987) 393.

[12] R. L. Jaffe and A. Manohar, *The g_1 Problem: Fact and Fantasy on the Spin of the Proton*, MIT preprint CTP#1706.

[13] R. L. Jaffe, Phys. Rev. D21 (1980) 3215

[14] A. De Rujula, H. Georgi and S. L. Glashow, Phys. Rev. D12 (1975) 147.

[15] C. Callan and I. Klebanov, Nucl. Phys. B 262 (1985) 365

[16] C. Callan, K. Hornbostel and I. Klebanov, Phys. Lett. 202 B (1988) 269

[17] J. F. Donoghue and C. R. Nappi, Phys. Lett. 168 B (1986) 105.

[18] E. Guadagnini, Nucl. Phys. B 236 (1984) 35

[19] A. Manohar, Nucl. Phys. B 248 (1984) 19

[20] M. Praszalowicz, Phys. Lett. B 158 (1985) 264

[21] M. Chemtob, Nucl. Phys. B 256 (1985) 600

[22] H. Yabu, Phys. Lett. 218 B (1989) 124.

[23] H. Yabu and K. Ando, Nucl. Phys. B 301 (1988) 601

[24] S. Weinberg, Physica 96A (1979) 327.

[25] A. Manohar and H. Georgi, Nucl. Phys. B234 (1984) 189.

[26] J. F. Donoghue, E. Golowich and B. R. Holstein, Phys. Rev. D30 (1984) 587.

[27] E. Witten, Nucl. Phys. B 223 (1983) 422, 433

[28] A. P. Balachandran, Lectures at 1985 TASI, M. Bowick and F. Gursey, eds., World Scientific, 1986

[29] See, for example, I. J. R. Aitchison, Acta Phys. Polon. B18 (1987) 207.

[30] G. Adkins, C. Nappi and E. Witten, Nucl. Phys. B 228 (1983) 552

[31] A. Th. M. Aerts, P. J. G. Mulders and J. J. de Swart, Phys. Rev. D17 (1978) 260.

[32] H. Abramowicz et al., Z. Phys. C15 (1982) 19

[33] J. Blaizot, M. Rho, N. Scoccola, Phys. Lett. 209 B (1988) 27

[34] E. Nyman and D. Riska, Helsinki preprint HU-TFT-88-50; E. Nyman, Helsinki preprint HU-TFT-89-9

[35] A. Manohar, unpublished.

[36] J. Bijnens, H. Sonoda and M. Wise, Can. J. Phys. 64 (1986) 1.