



Coclosed G_2 -structures on $SU(2)^2$ -invariant cohomogeneity one manifolds

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Abstract

We consider two different $SU(2)^2$ -invariant cohomogeneity one manifolds, one non-compact $M = \mathbb{R}^4 \times S^3$ and one compact $M = S^4 \times S^3$, and study the existence of coclosed $SU(2)^2$ -invariant G_2 -structures constructed from half-flat $SU(3)$ -structures. For $\mathbb{R}^4 \times S^3$, we prove the existence of a family of coclosed (but not necessarily torsion-free) G_2 -structures which is given by three smooth functions satisfying certain boundary conditions around the singular orbit and a non-zero parameter. Moreover, any coclosed G_2 -structure constructed from a half-flat $SU(3)$ -structure is in this family. For $S^4 \times S^3$, we prove that there are no $SU(2)^2$ -invariant coclosed G_2 -structures constructed from half-flat $SU(3)$ -structures.

Keywords G_2 -structures · Cohomogeneity one actions · Differential geometry

Mathematics Subject Classification 53C10

1 Introduction

A G_2 -structure in a smooth seven-manifold M is a smooth positive 3-form φ , and so defines a unique Riemannian metric g_φ and an orientation μ_φ on M . A G_2 -structure is called *closed* if $d\varphi = 0$, and *coclosed* if $d*\varphi = 0$, where $*_\varphi$ is the Hodge star operator associated to (g_φ, μ_φ) . Seven-dimensional manifolds with a G_2 -structure have been of great interest in differential geometry, since if a G_2 -structure is both closed and coclosed, the metric g_φ has holonomy contained in G_2 . They have also attracted substantial interest in String Theory and M-theory. In heterotic or type II string theory they give rise to three-dimensional vacua, while in M-theory to four-dimensional $N = 1$ Minkowski vacua. Coclosed G_2 -structures exist on any oriented spin seven-manifold [9]. They appear in the context of the heterotic G_2 system, which is studied both by mathematicians and physicists. When considering compactifications of heterotic string theory down to 3 dimensions, the *heterotic G_2 system* is a system for both geometric fields and gauge fields over a manifold with a G_2 -structure. The two gauge fields are G_2 -instantons and have their curvatures related by the so-called *heterotic Bianchi identity*.

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The heterotic G_2 system also requires that, after a conformal transformation, the G_2 -structure is coclosed. It appears in the broader context of the heterotic systems on manifolds with special geometry, such as the Hull–Strominger system [27, 38]. The heterotic G_2 system was first studied in [16–22, 28, 31]. For a full description of the system, we refer the reader for example to [11].

The main aim of this paper is to find coclosed G_2 -structures in the cohomogeneity one setting. A *cohomogeneity one manifold* is a Riemannian manifold with an action by isometries of a compact Lie group G having a generic orbit of codimension one. We are interested in cohomogeneity one manifolds as this is the next step of complexity after Reidegeld studied homogeneous coclosed G_2 -structures in [36]. In [10], Cleyton and Swann studied coclosed G_2 -structures on cohomogeneity one manifolds where the acting Lie group G was simple. For that reason, we are interested in the situation when G is not simple. Assuming that the G -action is almost effective and that the manifold is simply connected leads us to consider $G = \mathrm{SU}(2)^2$, up to finite quotients. As torsion-free G_2 manifolds have been studied in this case ([3] and see below for references on torsion-free G_2 -structures on $\mathbb{R}^4 \times S^3$), we will focus on the case when the G_2 -structure is coclosed but not necessarily torsion-free.

In this paper, we consider two different $\mathrm{SU}(2)^2$ -invariant cohomogeneity one manifolds, one non-compact $M = \mathbb{R}^4 \times S^3$ and one compact $M = S^4 \times S^3$, and look for coclosed $\mathrm{SU}(2)^2$ -invariant G_2 -structures constructed from half-flat $\mathrm{SU}(3)$ -structures, i.e. structures are the induced structures on hypersurfaces of a G_2 -manifold. By considering this type of G_2 -structure, we are able to write them in a simplified way [34, 37]. This will allow us to write conditions for the structure to be coclosed as a system of linear ordinary differential equations. We first find a large family of structures with these conditions on $\mathbb{R}^4 \times S^3$. The following Theorem summarizes the existence result (see Sect. 4 for a more detailed statement).

Theorem 1.1 *On the cohomogeneity one manifold $M = \mathbb{R}^4 \times S^3$ with group diagram $\mathrm{SU}(2)^2 \supset \Delta\mathrm{SU}(2) \supset \{1\}$, there is a family of $\mathrm{SU}(2)$ -invariant coclosed G_2 -structures which is given by three positive smooth functions satisfying certain boundary conditions around the singular orbit, and a non-zero parameter. Moreover, any $\mathrm{SU}(2)$ -invariant coclosed G_2 -structure constructed from a half flat $\mathrm{SU}(3)$ -structure is in this family.*

On $S^4 \times S^3$, we prove that there are no $\mathrm{SU}(2)^2$ -invariant coclosed G_2 -structures constructed from half-flat $\mathrm{SU}(3)$ -structures.

On the manifold $\mathbb{R}^4 \times S^3$, there are two explicit complete G_2 -holonomy metrics: the Bryant–Salamon (BS) metric [7] and the Brandhuber et al. (BGGG) metric [4]. The first one is one of the three asymptotically conical examples constructed by Bryant and Salamon in 1989, while the second one is a member of a family of complete $(\mathrm{SU}(2)^2 \times \mathrm{U}(1))$ -invariant G_2 -holonomy metrics found by Bogoyavlenskaya in [5], known as the \mathbb{B}_7 family. More recently, Foscolo, Haskins and Nordström constructed infinitely many new 1-parameter families of simply connected complete noncompact G_2 -manifolds [15], which are also $(\mathrm{SU}(2)^2 \times \mathrm{U}(1))$ -invariant. In [35], Podestá proved the existence of a one-parameter family of locally defined nearly parallel G_2 -structures, which are mutually non-isomorphic and invariant under the cohomogeneity one action of the group $\mathrm{SU}(2)^3$.

The family of G_2 -structures from Theorem 1.1 includes a $\mathrm{SU}(2)^3$ -invariant subfamily of explicit structures. It also includes some known families of torsion-free G_2 -structures as particular cases: the Bryant–Salamon G_2 -holonomy metric [7] and the 1-parameter family of complete $(\mathrm{SU}(2)^2 \times \mathrm{U}(1))$ -invariant G_2 -holonomy metrics of Brandhuber et al. [4] and Bogoyavlenskaya [5].

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In Sect. 2, we will start by giving a review of the geometry of G_2 -structures, the associated splitting of differential forms into irreducible G_2 representations and the torsion classes of the structure in 2.1. We also explain how to construct a G_2 -structure from a half-flat $SU(3)$ -structure in 2.2. In Sect. 2.3, we will develop the mathematical background on cohomogeneity one manifolds with two, one or zero ends.

We will start Sect. 3 by describing which seven-dimensional simply connected manifolds can be of cohomogeneity one for the almost effective action of a compact Lie group. Then, in Sect. 3.1, we give a description of a G_2 -structure on a manifold which is invariant under the cohomogeneity one action of $SU(2)^2$. In Sect. 3.2 we discuss the conditions for the metric to be extended smoothly to a singular orbit for the two manifolds of our interest. In Sect. 3.3 we obtain the systems of equations for the G_2 -structure to be coclosed and write them as a system of linear ordinary differential equations. Then we consider the special case of $SU(2)^3$ -invariant coclosed G_2 -structures and give some examples in Sect. 3.4.

In Sect. 4 we find a class of coclosed G_2 -structures on a given seven-dimensional simply connected manifold under the cohomogeneity one action of $SU(2)^2$. In Sect. 4.1 we prove some existence results for the coclosed equations. We present and prove our results for the non-compact manifold $\mathbb{R}^4 \times S^3$ in Sect. 4.2 and for the compact manifold $S^4 \times S^3$ in 4.3. Finally, in Sect. 4.4, we make a few concluding remarks.

2 Preliminaries

2.1 The geometry of G_2 -structures

In this section, we will give a summary of the geometry of G_2 -structures, from a Riemannian geometric point of view, including a discussion of the torsion. Through the past years, G_2 -structures have been extensively studied. For more details, we refer the reader to [30, Section 4] and [29, Chapter 11].

Consider \mathbb{R}^7 with the standard euclidean metric g_0 , for which the standard basis e_1, \dots, e_7 is orthonormal. Let $\mu_0 = e^1 \wedge \dots \wedge e^7$ be the standard volume form associated to g_0 and the standard orientation. Define the ‘‘associative’’ 3-form φ_0 by

$$\varphi_0 = e^{123} - e^{167} - e^{527} - e^{563} - e^{415} - e^{426} - e^{437}, \quad (2.1)$$

where e^1, \dots, e^7 is the standard dual basis of $(\mathbb{R}^7)^*$ and we write $e^{ijk} = e^i \wedge e^j \wedge e^k$, giving a different expression for the associative form. Note that the order of e^1, \dots, e^7 might change between references.

Definition 2.1 Let M be a smooth seven-manifold. A G_2 -structure on M is a smooth 3-form φ on M such that, at every $p \in M$, there exists a linear isomorphism $T_p M \cong \mathbb{R}^7$ with respect to which $\varphi_p \in \Lambda^3(T_p^* M)$ corresponds to $\varphi_0 \in \Lambda^3(\mathbb{R}^7)^*$.

A G_2 -structure φ on M induces a Riemannian metric g_φ and associated Riemannian volume form μ_φ . Let $*_\varphi$ be the Hodge star operator induced from (g_φ, μ_φ) . Then the *coassociative form* is given by

$$\psi = *_\varphi \varphi.$$

A smooth seven-manifold admits a G_2 -structure if and only if it is both orientable and spin [23].

Let $\Omega^k = \Gamma(\Lambda^k(T^*M))$ be the space of smooth k -forms on M , and let Ω_l^k be the irreducible representation of G_2 of (pointwise) dimension l . We can write a decomposition of Ω^2 and Ω^3 into irreducible G_2 representations:

$$\begin{aligned} \Omega^2 &= \Omega_7^2 \oplus \Omega_{14}^2, \\ \Omega^3 &= \Omega_1^3 \oplus \Omega_7^3 \oplus \Omega_{27}^3, \end{aligned}$$

where

$$\begin{aligned} \Omega_7^2 &= \{\beta \in \Omega^2 \mid *(\varphi \wedge \beta) = -2\beta\}, \\ \Omega_{14}^2 &= \{\beta \in \Omega^2 \mid *(\varphi \wedge \beta) = \beta\} = \{\beta \in \Omega^2 \mid \beta \wedge \psi = 0\} \cong \mathfrak{g}_2, \\ \Omega_1^3 &= \{f\varphi \mid f \in \Omega^0\}, \\ \Omega_7^3 &= \{X \lrcorner \psi \mid X \in \Gamma(TM)\}, \\ \Omega_{27}^3 &= \{\gamma \in \Omega^3 \mid \gamma \wedge \varphi = 0, \gamma \wedge \psi = 0\}. \end{aligned} \tag{2.2}$$

Let M be a manifold with G_2 -structure φ and let ∇ be the Levi-Civita covariant derivative associated to the Riemannian metric g_φ .

Definition 2.2 We can write

$$\nabla_X \varphi = T(X) \lrcorner \psi,$$

for some vector field $T(X)$ on M . We call $T \in \Gamma(T^*M \otimes T^*M)$ the *full torsion tensor* of φ .

The following Proposition ([6, Proposition 1], consequence of a classical result of Fernández and Gray [13]) gives a full description of the torsion of φ is in terms of four quantities, known as *intrinsic torsion forms*.

Proposition 2.3 For any G_2 -structure φ , there exist unique differential forms $\tau_0 \in \Omega^0(M)$, $\tau_1 \in \Omega^1$, $\tau_2 \in \Omega_{14}^2$ and $\tau_3 \in \Omega_{27}^3$ so that the following equations hold:

$$d\varphi = \tau_0\psi + 3\tau_1 \wedge \varphi + *\tau_3, \tag{2.3}$$

$$d\psi = 4\tau_1 \wedge \psi + \tau_2 \wedge \varphi. \tag{2.4}$$

Definition 2.4 The G_2 -structure φ is called *torsion-free* if $\nabla\varphi = 0$. We say (M, φ) is a G_2 -manifold if φ is a torsion-free G_2 -structure on M .

Hence, φ is torsion-free if and only if $T = 0$. Vanishing of the torsion is equivalent to the intrinsic torsion forms being all zero, so from (2.3) and (2.4) we also have that φ is torsion-free if and only if both $d\varphi = 0$ and $d\psi = 0$. When φ is torsion-free, g_φ is Ricci-flat and has holonomy contained in G_2 . Moreover, if (M, φ, g_φ) is compact and φ is torsion-free, then the holonomy $\text{Hol}(g_\varphi) = G_2$ if and only if $\pi_1(M)$ is finite (see [29, Chapter 11]).

We can classify G_2 -structures in different types, depending on which components of the torsion vanish:

- (i) Torsion-free ($\nabla\varphi = 0$): $\tau_0 = \tau_1 = \tau_2 = \tau_3 = 0$;
- (ii) Closed ($d\varphi = 0$): $\tau_0 = \tau_1 = \tau_3 = 0$;
- (iii) Coclosed ($d\psi = 0$): $\tau_1 = \tau_2 = 0$;
- (iv) Nearly parallel ($d\varphi = \lambda\psi, \lambda \neq 0$): $\tau_1 = \tau_2 = \tau_3 = 0$.

In this paper, we will be interested in coclosed G_2 -structures. The principal reason for this is that for a manifold to admit a solution to the heterotic G_2 system, it needs to admit a G_2 -structure with intrinsic form $\tau_2 = 0$. After a conformal transformation, we can assume that $\tau_1 = 0$. As proved by Crowley and Nordström in [9], coclosed G_2 -structures exist on any oriented spin seven-manifold. Coclosed structures have also been in the past referred to as cosymplectic.

2.2 Half-flat SU(3)-structures and G_2 -structures

An SU(3)-structure (g, J, Ω) on a six-dimensional smooth manifold M is a Riemannian metric g and a g -orthogonal almost complex structure J together with a $(3, 0)$ -form of non-zero constant norm $\Omega = \Omega_1 + i\Omega_2$ satisfying the *normalization condition*:

$$\Omega_1 \wedge \Omega_2 = \frac{2}{3}\omega^3. \quad (2.5)$$

We say that $\alpha \in \Omega^k(M)$ is *stable* if for every $p \in M$, the orbit of α_p under the action of $\text{GL}(T_p M)$ is open in $\Omega^k(T_p M^*)$. An SU(3)-structure on a six-dimensional smooth manifold can be described in terms of a pair of a 2-form ω and a 3-form Ω_2 that are both stable (with $\lambda(\Omega_2) < 0$) and such that they satisfy the *compatibility condition*:

$$\omega \wedge \Omega_2 = 0,$$

provided that the corresponding metric is positive definite and the normalization condition is satisfied. Given such forms, using Hitchin's construction [24, Section 2] we may recover the SU(3)-structure as $J = J_{\Omega_2}$ (where J_{Ω_2} is an almost complex structure depending only on Ω_2), $g(\cdot, \cdot) := \omega(\cdot, J\cdot)$ and $\Omega := \Omega_1 + i\Omega_2$, where $\Omega_1 := J\Omega_2$. For more details about this process, see for example [1, Section 2.2]. We say that an SU(3)-structure given by the pair (ω, Ω_2) is *half-flat* if

$$d\Omega_1 = 0, \quad d\omega^2 = 0.$$

We consider the 7-dimensional manifold $M = I \times N$, where N is a smooth manifold of dimension 6 and I is an interval with coordinate $t \in \mathbb{R}$. Let $(\omega(t), \Omega_2(t))$ be a 1-parameter family of SU(3)-structures on N parameterized by $t \in I$. Then, the following forms give a G_2 -structure on M :

$$\begin{aligned} \varphi &= dt \wedge \omega(t) + \Omega_1(t), \\ \psi &= \frac{\omega^2(t)}{2} - dt \wedge \Omega_2(t). \end{aligned} \quad (2.6)$$

Every G_2 -structure in the principal part of a cohomogeneity one manifold can be constructed from an SU(3)-structure in the principal orbits. Assume that $(\omega(t), \Omega_2(t))$ is half-flat. This G_2 -structure is closed if the 1-parameter family $(\omega(t), \Omega_2(t))$ is a solution of:

$$\dot{\Omega}_1 = d\omega.$$

The G_2 -structure is coclosed if $(\omega(t), \Omega_2(t))$ is a solution of:

$$\omega \wedge \dot{\omega} = -d\Omega_2.$$

Hence, the G_2 -structure is torsion-free if $(\omega(t), \Omega_2(t))$ is a solution of both equations.

Remark 2.5 Note that a coclosed G_2 -structure whose restriction to each principal orbit is half-flat only requires $d\Omega_1 = 0$ as an extra condition, as $d\omega^2 = 0$ automatically holds.

2.3 Cohomogeneity one manifolds

We will give a description of the structure of cohomogeneity one manifolds and their metrics. For further details, see for instance [2, 25, 26, 40].

Definition 2.6 A Riemannian manifold M is of *cohomogeneity one* for the action of the compact Lie group G if G is a closed subgroup of the isometry group of M and has an orbit of codimension one.

From now on, we assume that M is a simply connected cohomogeneity one manifold, and G is a compact, connected Lie group. By the compactness of G , the action is proper and there exists a G -invariant Riemannian metric g on M ; this is equivalent to saying that G acts on the Riemannian manifold (M, g) by isometries. Moreover, we assume that the cohomogeneity one action is *almost effective*, i.e. that the set of points of G whose action is trivial is discrete. Let $\pi : M \rightarrow M/G$ be the canonical projection onto the orbit space and equip M/G with the quotient topology relative to π . The quotient space M/G is then homeomorphic to a circle or an interval [2]. As we are assuming that M is simply connected, we have that M/G is homeomorphic to an interval I . The inverse images of the interior points of the orbit space M/G are known as *principal orbits*, while the inverse images of the boundary points are called *singular orbits*. We denote by M^{princ} the union of all principal orbits, which is an open dense subset of M , and by G_p the isotropy group at $p \in M$.

First, we will suppose M is compact. It follows that M/G is homeomorphic to the closed interval $I = [0, 1]$. Denote by \mathcal{O}_- and \mathcal{O}_+ the two singular orbits $\pi^{-1}(0)$ and $\pi^{-1}(1)$, respectively. Let $\gamma : [0, 1] \rightarrow M$ be a normal geodesic (i.e. geodesic orthogonal to every principal orbit) between $\pi^{-1}(0)$ and $\pi^{-1}(1)$, which always exists. Up to rescaling, we can always suppose that the orbit space M/G is such that $\pi \circ \gamma = \text{Id}_{[0,1]}$. Then, by Kleiner’s Lemma, there exists a subgroup K of G such that $G_{\gamma(t)} = K$ for all $t \in (0, 1)$ and K is subgroup of $G_{\gamma(0)}$ and $G_{\gamma(1)}$.

For M non-compact, M/G is homeomorphic either to an open interval or to an interval with one closed end. In the former case, M is a product manifold $M \cong I \times G/K$. In the latter case, there exists exactly one singular orbit, and $M/G \cong I$ where $I = [0, \infty)$. Analogously to the compact case, there exists a normal geodesic $\gamma : [0, \infty) \rightarrow M$ such that $\gamma(0) \in \pi^{-1}(0)$ and we can suppose $\pi \circ \gamma = \text{Id}_{[0,\infty)}$. In addition, there exists a subgroup K of G such that $G_{\gamma(t)} = K$ for all $t \in (0, \infty)$ and K is a subgroup of $G_{\gamma(0)}$.

Up to conjugation along the orbits, when M is compact we have three possible isotropy groups $H_- := G_{\gamma(0)}$, $H_+ := G_{\gamma(1)}$ and $K := G_{\gamma(t)}$, $t \in (0, 1)$. When M is non-compact and has one singular orbit, instead, we have two possible isotropy groups $H := G_{\gamma(0)}$ and $K := G_{\gamma(t)}$, $t \in (0, \infty)$. From all of the above, we have that

$$M^{\text{princ}} \cong \overset{\circ}{I} \times G/K,$$

and so, by fixing a suitable global coordinate system, we can decompose the G -invariant metric g as

$$g_{\gamma(t)} = dt^2 + g_t, \tag{2.7}$$

where dt^2 is the $(0, 2)$ -tensor corresponding to the vector field $\xi := \gamma'(t)$ evaluated at the point $\gamma(t)$, and g_t is a G -invariant metric on the homogeneous orbit $G \cdot \gamma(t)$ through the point $\gamma(t) \in M$.

Now, we will assume M is compact. By the density of M^{princ} in M and the Tube Theorem, M is homotopically equivalent to

$$(G \times_{H_-} D_-) \cup_{G/K} (G \times_{H_+} D_+), \tag{2.8}$$

where the geodesic balls $D_{\pm} := \exp(B_{\varepsilon^{\pm}}(0))$, $B_{\varepsilon^-}(0) \subset T_{\gamma(0)}(G \cdot \gamma(0))^{\perp}$, $B_{\varepsilon^+}(0) \subset T_{\gamma(1)}(G \cdot \gamma(1))^{\perp}$, are normal slices to the singular orbits in $\gamma(0)$, $\gamma(1)$. Here, $G \times_{H_{\pm}} D_{\pm}$

is the associated fiber bundle to the principal bundle $G \rightarrow G/H_i$ with type fiber D_{\pm} . By Bochner's linearization theorem, M is also homotopically equivalent to

$$(G \times_{H_-} B_{\varepsilon^-}(0)) \cup_{G/K} (G \times_{H_+} B_{\varepsilon^+}(0)). \quad (2.9)$$

The isotropy groups H_{\pm} act on $B_{\varepsilon^{\pm}}(0)$ via the slice representation and, since the boundary of the tubular neighborhood $\text{Tub}(\mathcal{O}_{\pm}) := G \times_{H_{\pm}} B_{\varepsilon^{\pm}}(0)$, is identified with the principal orbit G/K and the G -action on $\text{Tub}(\mathcal{O}_{\pm})$ is identified with the H_{\pm} -action on $B_{\varepsilon^{\pm}}(0)$, then H_{\pm} acts transitively on the sphere $S^{l_{\pm}} := \partial B_{\varepsilon^{\pm}}(0)$, $l_{\pm} > 0$ still having isotropy K . The normal spheres $S^{l_{\pm}}$ are thus the homogeneous spaces H_{\pm}/K .

The collection of G with its isotropy groups $G \supset H_-, H_+ \supset K$ is called the *group diagram* of the cohomogeneity one manifold M . Viceversa, let $G \supset H_-, H_+ \supset K$ be compact groups with $H_{\pm}/K \cong S^{l_{\pm}}$. By the classification of transitive actions on spheres one has that the H_{\pm} -action on $S^{l_{\pm}}$ is linear and hence it can be extended to an action on $B_{\varepsilon^{\pm}}$ bounded by $S^{l_{\pm}}$. Therefore, (2.9) defines a cohomogeneity one manifold M . Analogously, if M is a non-compact cohomogeneity one manifold with one singular orbit, we define the *group diagram* of M to be the collection of G and the isotropy groups $G \supset H \supset K$, where the homogeneous space H/K will be a sphere. The converse is also true: the group diagram defines a non-compact cohomogeneity one manifold M . In these cases, M is homotopically equivalent to $G \times_H B_{\varepsilon}(0)$, where $B_{\varepsilon}(0) \subseteq T_{\gamma(0)}(G \cdot \gamma(0))^{\perp}$ as before.

3 G_2 -structures on cohomogeneity one manifolds

Let M be a seven-dimensional simply connected manifold of cohomogeneity one for the almost effective action of a compact connected Lie group G . Let φ be a G_2 -structure on M which is preserved by the action, and denote $\psi = *_\varphi\varphi$. Let K be the principal isotropy group.

We require that the action of G preserves the G_2 -structure, so the principal isotropy group K acts on T_pM with $K \subset G_2$, for any principal point p in M . If we denote $\mathfrak{k} = \text{Lie}(K)$, then $\mathfrak{k} \subset \mathfrak{g}_2$. Let $\xi = \partial/\partial t$, where t is the cohomogeneity one parameter. The subgroup of G_2 that fixes the subspace $\langle \xi|_p \rangle$ of T_pM is $\text{SU}(3)$. Hence, the requirement that G acts on M with cohomogeneity one preserving φ implies that the representation of the isotropy group $K = K_p$ on the tangent space of the principal orbit is a subgroup of $\text{SU}(3)$. Therefore at the Lie algebra level $\mathfrak{k} := \text{Lie}(K)$ is $\{0\}$, \mathbb{R} , $\mathfrak{su}(2)$, $2\mathbb{R}$, $\mathfrak{u} = \mathfrak{su}(2) \oplus \mathbb{R}$ or $\mathfrak{su}(3)$. Note that $\mathfrak{su}(2)$ has two different embeddings in $\mathfrak{su}(3)$.

As $\dim \mathfrak{g} - \dim \mathfrak{k} = 6$, we can write an initial list of possible possible decompositions of the Lie algebra \mathfrak{g} into simple summands. This list can be reduced if we assume that the manifold M is simply connected, using [25, Proposition 1.8] for the compact case and [1, Proposition 3.1] for the non-compact case. If $\mathfrak{k} \subset \mathfrak{g}$ is an ideal, then the principal isotropy would act trivially on M and the action would not be almost effective. Then, up to finite quotients, the principal orbits are one of the following types:

- (1) $S^3 \times S^3 = \text{SU}(2)^2 = \frac{\text{SU}(2)^2 \times \text{U}(1)}{\text{U}(1)} = \frac{\text{SU}(2)^3}{\text{SU}(2)}$;
- (2) $S^5 \times S^1 = \frac{\text{SU}(3) \times \text{U}(1)}{\text{SU}(2)}$;
- (3) $F_{1,2} = \frac{\text{SU}(3)}{T^2}$ ($\mathfrak{k} = 2\mathbb{R}$, $\mathfrak{g} = \mathfrak{su}(3)$);
- (4) $\mathbb{C}P(3) = \frac{\text{Sp}(2)}{\text{SU}(2)\text{U}(1)}$ ($\mathfrak{k} = \mathfrak{su}(2) \oplus \mathbb{R}$, $\mathfrak{g} = \mathfrak{sp}(2)$);

$$(5) S^6 = \frac{G_2}{\text{SU}(3)} \ (\mathfrak{k} = \mathfrak{su}(3), \mathfrak{g} = \mathfrak{g}_2).$$

For the last three cases, and up to a finite quotient, the group G acting is $\text{SU}(3)$, $\text{Sp}(2)$ and G_2 , respectively. In [10], Cleyton and Swann studied G_2 -structures with a cohomogeneity one action of a compact Lie group G . They only consider the case when the group G acting is simple, so they consider acting groups $\text{SU}(3)$, $\text{Sp}(2)$ and G_2 . Later, they studied the coclosed (or cosymplectic, as they call them) structures and determined the topological types of manifolds admitting such structures. They also found new examples of compact manifolds with coclosed G_2 -structures. On account of this, our focus will be on the first case, where there is an invariant action of $\text{SU}(2)^2$. We will also consider situations where there are extra symmetries, such as extra $\text{SU}(2)$.

3.1 G_2 -structures on $\text{SU}(2)^2$ -invariant cohomogeneity one manifolds

In this section we construct an $\text{SU}(2)^2$ -invariant half-flat $\text{SU}(3)$ -structure on the principal part $I_t \times N$ of a cohomogeneity one manifold M , where I_t is either $[0, \infty)$ or $[0, 1]$, and $N = G/K$. We will describe the $\text{SU}(3)$ -structure, the associated G_2 -structure given by equation (2.6) and the corresponding metric on M in terms of six real valued functions, by using an orthonormal basis given by a Milnor frame. This follows from the work by Schulte-Hegensbach in his PhD thesis [37] and Madsen and Salamon in [34]. Here we will follow the reformulation by Lotay and Oliveira from [32, Section 2]. We note that the induced $\text{SU}(3)$ -structures $(\omega(t), \Omega_2(t))$ on the hypersurfaces of the manifold N do not need to be half-flat (this is guaranteed when M is a G_2 -manifold, but when the G_2 -structure is coclosed only the condition $d\omega^2 = 0$ is guaranteed). We make this assumption in order to use the description below to facilitate the search for new examples coclosed G_2 -structures.

We can construct a basis of $\mathfrak{su}(2) \oplus \mathfrak{su}(2)$ written as $\mathfrak{su}(2) \oplus \mathfrak{su}(2) = \mathfrak{su}^+(2) \oplus \mathfrak{su}^-(2)$. Let $\{T_i\}_{i=1}^3$ be a basis for $\mathfrak{su}(2)$ such that $[T_i, T_j] = 2\epsilon_{ijk}T_k$. Then

$$T_i^+ = (T_i, T_i), \quad T_i^- = (T_i, -T_i),$$

define a basis for $\mathfrak{su}^+(2)$ and $\mathfrak{su}^-(2)$ respectively. Let $\{\eta_i^+\}_{i=1}^3$ and $\{\eta_i^-\}_{i=1}^3$ be dual basis to $\{T_i^+\}_{i=1}^3$ and $\{T_i^-\}_{i=1}^3$ respectively. Then the structure equations are

$$\begin{aligned} d\eta_i^+ &= -\epsilon_{ijk}(\eta_j^+ \wedge \eta_k^+ + \eta_j^- \wedge \eta_k^-), \\ d\eta_i^- &= -2\epsilon_{ijk}\eta_j^- \wedge \eta_k^+. \end{aligned}$$

We will denote $\eta_{ij}^\pm = \eta_i^\pm \wedge \eta_j^\pm$, and $\eta_{123}^\pm = \eta_1^\pm \wedge \eta_2^\pm \wedge \eta_3^\pm$. A generic $\text{SU}(2)^2$ -invariant half-flat $\text{SU}(3)$ -structure on the principal bundle is given by

$$\begin{aligned} \omega &= 4 \sum_{i=1}^3 A_i B_i \eta_i^- \wedge \eta_i^+, \\ \Omega_1 &= 8B_1 B_2 B_3 \eta_{123}^- - 4 \sum_{i=1}^3 \epsilon_{ijk} A_i A_j B_k \eta_i^+ \wedge \eta_j^+ \wedge \eta_k^-, \\ \Omega_2 &= -8A_1 A_2 A_3 \eta_{123}^+ + 4 \sum_{i=1}^3 \epsilon_{ijk} B_i B_j A_k \eta_i^- \wedge \eta_j^- \wedge \eta_k^+, \end{aligned} \tag{3.1}$$

for six real-valued functions $A_i, B_i : I_t \rightarrow \mathbb{R}, i = 1, 2, 3, A_i(t), B_i(t) \neq 0$ for t in the interior of I_t . The compatible metric determined by this $\text{SU}(3)$ -structure on $\{t\} \times N$ is:

$$g_t = \sum_{i=1}^3 (2A_i)^2 \eta_i^+ \otimes \eta_i^+ + \sum_{i=1}^3 (2B_i)^2 \eta_i^- \otimes \eta_i^-,$$

and the resulting metric on $I_t \times N$, compatible with the G_2 -structure $\varphi = dt \wedge \omega + \Omega_1$, is given by

$$g = dt^2 + g_t. \quad (3.2)$$

Hence, we can see the functions $A_i(t)$ and $B_i(t)$ as describing deformations of the standard cone metric.

Remark 3.1 For a G_2 -structure given by a half-flat $SU(3)$ -structure as before, using the previous expressions and equation (2.6), we have that $\psi \lrcorner d\varphi = 0$, so

$$\tau_0 = 0.$$

This means that for all of the G_2 -structures considered, the scalar curvature is zero.

Remark 3.2 In seven dimensional heterotic string theory, we say that the *flux* is a 3-form given by

$$H = \frac{1}{6} \tau_0 \varphi - \tau_1 \lrcorner \psi - \tau_3.$$

One of the equations of the heterotic G_2 system, known as the *heterotic Bianchi identity* or *anomaly free condition*, relates the exterior differential of the flux to the curvatures of two gauge fields. If the G_2 -structure is coclosed as before, and using that $\tau_0 = 0$ from the previous remark, we get

$$\begin{aligned} dH = \frac{1}{6} & \prod_{i=1}^3 |A_i B_i| \\ & [16(-4A_1 B_1 + (B_1 B_2 B_3)' - (A_2 A_3 B_1)' + (A_3 A_1 B_2)' + (A_1 A_2 B_3)') \eta_{23}^- \wedge \eta_{23}^+ \\ & + 16(-4A_2 B_2 + (B_1 B_2 B_3)' + (A_2 A_3 B_1)' - (A_3 A_1 B_2)' + (A_1 A_2 B_3)') \eta_{13}^- \wedge \eta_{13}^+ \\ & + 16(-4A_3 B_3 + (B_1 B_2 B_3)' + (A_2 A_3 B_1)' + (A_3 A_1 B_2)' - (A_1 A_2 B_3)') \eta_{12}^- \wedge \eta_{12}^+]. \end{aligned}$$

This expression will be relevant when looking for solutions of the heterotic G_2 system for the $SU(2)^2$ -invariant cohomogeneity one manifolds and the described coclosed G_2 -structures. This will be subject of future work.

3.2 Extension to the singular orbits

The union of the principal orbits M^{princ} of the manifold M is a dense subset of M . Hence, it is possible to extend the metric on the principal part to singular orbit(s) to give a metric on the manifold M . However, there are some extra conditions that we need to impose in order to ensure that this extension is smooth. These conditions follow from a method developed by Eschenburg and Wang [12] to find when a metric (or more generally, a tensor) extends smoothly to a singular orbit. In [39], Verdiani and Ziller gave an efficient way of checking the conditions for a smooth extension of the metric. To use this method, we need to fix our cohomogeneity one manifold. We will consider two situations, one of a compact and one of a non-compact manifold.

As explained in Sect. 2.3, a non-compact cohomogeneity one manifold is given by a homogeneous vector bundle, while a compact one by the union of two homogeneous disc bundles. We are interested in the smoothness conditions near a singular orbit, so we restrict ourselves to only one such bundle.

The non-compact manifold that we are going to consider is $M = \mathbb{R}^4 \times S^3$, seen as a cohomogeneity one manifold with group diagram $SU(2) \times SU(2) \supset \Delta SU(2) \supset \{1\}$. In particular, we consider the embedding $\mathbb{R}^4 \times S^3 \hookrightarrow \mathbb{H} \times \mathbb{H}$ and let $SU(2) \times SU(2)$ act via

$$(a_1, a_2) \cdot (p, q) = (a_1 p, a_1 q \bar{a}_2).$$

$\mathbb{R}^4 \times S^3$ is diffeomorphic to the total space of the spinor bundle $S \rightarrow S^3$ over the 3-sphere, which can be described as the quotient

$$\frac{SU(2) \times SU(2) \times \mathbb{R}^4}{\Delta SU(2)},$$

where $SU(2)$ is acting on the right diagonally. One of the reasons why we are interested in this manifold is that it admits torsion-free G_2 -structures [4, 5, 7] and G_2 -instantons [8, 32].

The next Lemma tells us the conditions on the functions from the previous section $A_i, B_i, i = 1, 2, 3$, for the metric to extend to the singular orbit $Q = SU(2)^2/\Delta SU(2) \cong S^3$.

Lemma 3.3 [32, Lemma 8] *The metric g in (3.2) extends smoothly (as a metric) over the singular orbit $Q = SU(2)^2/\Delta SU(2)$ if and only if A_i, B_i are non-zero for $t > 0$ and:*

- (i) the A_i 's are odd with $\dot{A}_i(0) = 1/2$;
- (ii) the B_i 's are even with $B_1(0) = B_2(0) = B_3(0) \neq 0$ and $\ddot{B}_1(0) = \ddot{B}_2(0) = \ddot{B}_3(0)$.

Note that condition (i) says that the metrics have to be almost round near to the singular orbit, while condition (ii) guarantees that the singular orbit is totally geodesic.

Remark 3.4 There is another inequivalent action of $SU(2)^2$ on $\mathbb{R}^4 \times S^3$, with group diagram $SU(2) \times SU(2) \supset SU(2) \times \{1\} \supset \{1\}$. Moreover, there are infinitely many distinct seven-dimensional cohomogeneity one manifold with one singular orbit and an inequivalent action of $SU(2)^2$. They have singular isotropy group $K_{m,n} = \{(e^{i\theta_1}, e^{i\theta_2}) \in T^2 | e^{i(m\theta_1+n\theta_2)} = 1\}$, where m, n are co-prime (this guarantees that the manifold is simply connected), and principal isotropy group isomorphic to $\mathbb{Z}_{2|m+n}$. The manifolds constructed from that action are denoted by $M_{m,n}$. In [15], Foscolo, Haskins and Nordström constructed 1-parameter families of torsion-free G_2 -structures on these manifolds, known respectively as the \mathbb{D}_7 family and the infinite extension of the \mathbb{C}_7 family. It would be interesting to study coclosed G_2 -structures on those manifolds as well, for which the (different) conditions for a smooth extension to the singular orbit can be found in [15, Proposition 4.1].

For a compact manifold, we consider the cohomogeneity one manifold $M = S^4 \times S^3$ with group diagram $SU(2) \times SU(2) \supset \Delta SU(2), \Delta SU(2) \supset \{1\}$. We consider the embedding $S^3 \times S^4 \hookrightarrow \mathbb{H} \times (\mathbb{H} \times \mathbb{R})$, then $S^3 \times S^3$ acts on M via

$$(a_1, a_2) \cdot (p, q, t) = (a_1 p a_2^{-1}, a_2 q, t).$$

The metric g in (3.2) extends smoothly (as a metric) over the singular orbits $Q_{1,2} = SU(2)^2/\Delta SU(2)$ if the conditions for A_i, B_i from the previous lemma, which are now defined on the closed interval $[0, 1]$, are satisfied around both singular orbits. Note that in Lemma 3.3, t is the arclength parameter along a geodesic intersecting the principal orbits orthogonally, but we assume that the length of the interval I_t is 1 instead of an arbitrary positive number, as it does not affect the discussion, and in order to be consistent with the notation of Sect. 2.3.

Remark 3.5 For the two previous situations, the corresponding normal sphere(s) H/K have dimension 3. It would be interesting to explore another example where there is a normal sphere with dimension 1, such as the Bazaikin-Bogoyavlenskaya manifolds [3], which have group diagram $SU(2)^2 \supset U(1) \supset \mathbb{Z}/4$.

3.3 Coclosed equations

We derive equations for the functions A_i, B_i such that the G_2 -structure obtained via (3.1) is coclosed. Recall from Sect. 2.2 that they will be derived from

$$\omega \wedge \dot{\omega} = -d\Omega_2. \tag{3.3}$$

Using the expression for ω and Ω_2 from equation (3.1), we obtain the system of ODEs that will give us the coclosed conditions. We present it in the following Proposition.

Proposition 3.6 *Let M be a seven-dimensional simply connected cohomogeneity one manifold under the action of $SU(2)^2$, with a G_2 -structure constructed from a half-flat $SU(3)$ -structure. Let the $SU(3)$ -structure $(\omega, \Omega_1, \Omega_2)$ be written as in (3.1). Then, the equations for the G_2 -structure to be coclosed are:*

$$\begin{cases} A_2 B_2 (\dot{A}_3 B_3 + A_3 \dot{B}_3) + A_3 B_3 (\dot{A}_2 B_2 + A_2 \dot{B}_2) = \\ \qquad A_1 A_2 A_3 - B_2 B_3 A_1 + B_1 B_3 A_2 + B_1 B_2 A_3, \\ A_1 B_1 (\dot{A}_3 B_3 + A_3 \dot{B}_3) + A_3 B_3 (\dot{A}_1 B_1 + A_1 \dot{B}_1) = \\ \qquad A_1 A_2 A_3 + B_2 B_3 A_1 - B_1 B_3 A_2 + B_1 B_2 A_3, \\ A_1 B_1 (\dot{A}_2 B_2 + A_2 \dot{B}_2) + A_2 B_2 (\dot{A}_1 B_1 + A_1 \dot{B}_1) = \\ \qquad A_1 A_2 A_3 + B_2 B_3 A_1 + B_1 B_3 A_2 - B_1 B_2 A_3. \end{cases} \tag{3.4}$$

If we define

$$\begin{aligned} D_1 &= A_2 B_2 A_3 B_3, \\ D_2 &= A_1 B_1 A_3 B_3, \\ D_3 &= A_1 B_1 A_2 B_2, \end{aligned}$$

then we can write (3.4) as a system of ODEs for the functions D_1, D_2, D_3 , where the coefficients depend on free functions A_1, A_2, A_3 . To ensure that the induced metrics are smooth, Lemma 3.3 leads us to consider A_i 's which are non-zero (positive, more precisely) for t in the interior of I_t and satisfy certain boundary conditions. Hence, this system will be well defined in the interior of I_t and we will study the behaviour for the boundary points, where the A_i 's are zero. Note that the initial conditions from Lemma 3.3 correspond to the following conditions for the functions D_i

$$D_i(t) = \frac{b_0^2}{4} t^2 + O(t^4), \tag{3.5}$$

and D_i even, $i = 1, 2, 3$. The system is

$$\begin{cases} \dot{D}_1 = A_1 A_2 A_3 - \frac{A_1^2 D_1}{A_1 A_2 A_3} + \frac{A_2^2 D_2}{A_1 A_2 A_3} + \frac{A_3^2 D_3}{A_1 A_2 A_3}, \\ \dot{D}_2 = A_1 A_2 A_3 + \frac{A_1^2 D_1}{A_1 A_2 A_3} - \frac{A_2^2 D_2}{A_1 A_2 A_3} + \frac{A_3^2 D_3}{A_1 A_2 A_3}, \\ \dot{D}_3 = A_1 A_2 A_3 + \frac{A_1^2 D_1}{A_1 A_2 A_3} + \frac{A_2^2 D_2}{A_1 A_2 A_3} - \frac{A_3^2 D_3}{A_1 A_2 A_3}. \end{cases} \tag{3.6}$$

Writing $D = (D_1, D_2, D_3)^T$ the system of equations now becomes

$$\dot{D} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} A_1 A_2 A_3 + \frac{1}{A_1 A_2 A_3} \begin{pmatrix} -A_1^2 & A_2^2 & A_3^2 \\ A_1^2 & -A_2^2 & A_3^2 \\ A_1^2 & A_2^2 & -A_3^2 \end{pmatrix} D. \tag{3.7}$$

We will write it with matrix notation. Let

$$M = \frac{1}{A_1 A_2 A_3} \begin{pmatrix} -A_1^2 & A_2^2 & A_3^2 \\ A_1^2 & -A_2^2 & A_3^2 \\ A_1^2 & A_2^2 & -A_3^2 \end{pmatrix}, \quad N = A_1 A_2 A_3 \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}. \tag{3.8}$$

Then our system is

$$\dot{D} = M(t)D + N(t). \tag{3.9}$$

Remark 3.7 We would like to recover the B_i 's from the D_i 's with:

$$B_i(t) = \text{sign}(b_0) \sqrt{\frac{D_j D_k}{D_i A_i^2}}, \quad t > 0, \quad B_i(0) = b_0, \tag{3.10}$$

where $\ddot{D}_i(0) = b_0^2/2$ and $\{i, j, k\}$ is a cyclic permutation of $\{1, 2, 3\}$. We can only do that if $D_i D_j / D_k > 0$ for $t > 0$.

3.4 $SU(2)^3$ -invariant coclosed G_2 -structures

We now study some special cases of coclosed G_2 -structures, more concretely by considering the case where the G_2 -structure enjoys an extra $SU(2)$ -symmetry, i.e. $A_1 = A_2 = A_3$ and $B_1 = B_2 = B_3$. We will also study explicitly the case $A_1 = A_2 = A_3$ and show that the extra $SU(2)$ -symmetry follows.

3.4.1 Case $A_1 = A_2 = A_3$ and $B_1 = B_2 = B_3$

We only have one equation:

$$(A_1 B_1)' = \frac{1}{2} \frac{A_1^2 + B_1^2}{B_1}.$$

If we define $D = A_1^2 B_1^2$, we get

$$\dot{D} = A_1^3 + \frac{D}{A_1}. \tag{3.11}$$

Example 3.8 If we take $A_1 = t/2$, our equation is

$$\dot{D} = \frac{t^3}{8} + \frac{2D}{t}.$$

This has solutions depending on one constant, which we denote c :

$$D = ct^2 + \frac{t^4}{16}.$$

Hence renaming $4c$ by c :

$$B_1 = \sqrt{\frac{t^2}{4} + c}.$$

This is an even function, and $B_1(0) \neq 0$ if and only if $c \neq 0$. For $c > 0$, since it is well defined for $t \in [0, \infty)$, we can smoothly extend the metric to the singular orbit at $t = 0$. The corresponding metric is

$$g = dt^2 + t^2 \sum_{i=1}^3 \eta_i^+ \otimes \eta_i^+ + (t^2 + c) \sum_{i=1}^3 \eta_i^- \otimes \eta_i^-.$$

In the limit $c \rightarrow 0$, the metric is conical.

Example 3.9 One of the first examples of a torsion-free G_2 -structure with a complete metric is the one that gives the Bryant–Salamon metric on $\mathbb{R}^4 \times S^3$ from [7]. The G_2 -structure is then coclosed. The Bryant–Salamon metric is actually $SU(2)^3$ -invariant, and it can be realised as a cohomogeneity one manifold with group diagram $SU(2)^3 \supset SU(2)^2 \supset SU(2)$ (where $SU(2)$ is embedded in $SU(2)^3$ as $1 \times 1 \times SU(2)$ and $SU(2)^2$ as $\Delta_{1,2}SU(2) \times SU(2)$), as well as with an action of $SU(2)^2$ in multiple inequivalent ways. The extra symmetry means that $A_1 = A_2 = A_3$ and $B_1 = B_2 = B_3$, and the torsion-free equations reduce to

$$\dot{A}_1 = \frac{1}{2} \left(1 - \frac{A_1^2}{B_1^2} \right), \quad \dot{B}_1 = \frac{A_1}{B_1}.$$

There is a solution

$$A_1 = \frac{r}{3} \sqrt{1 - r^{-3}}, \quad B_1 = \frac{r}{\sqrt{3}},$$

where $r \in [1, +\infty)$ is a coordinate defined implicitly by

$$t(r) = \int_1^r \frac{ds}{\sqrt{1 - s^{-3}}},$$

and t denotes the arc length parameter. When $t \rightarrow \infty$, $A_1(t) \sim t/3$ and $B_1(t) \sim t/\sqrt{3}$. The metric will then be

$$g = dt^2 + \sum_{i=1}^3 \left(\frac{4r^2}{9} (1 - r^{-3}) \eta_i^+ \otimes \eta_i^+ + \frac{4r^2}{3} \eta_i^- \otimes \eta_i^- \right).$$

The asymptotic cone over of this metric is the standard homogeneous nearly Kähler structure on $S^3 \times S^3$.

The general solution to equation (3.11) is

$$D = ce^{\int_{1/2}^t \frac{1}{A_1(\xi)} d\xi} + e^{\int_{1/2}^t \frac{1}{A_1(\xi)} d\xi} \int_0^t A_1^3(\eta) e^{-\int_{1/2}^\eta \frac{1}{A_1(\xi)} d\xi} d\eta. \tag{3.12}$$

As $\lim_{t \rightarrow 0} D(t)t^2 = 4c$, and we are only interested in positive solutions, we rename c by $b_0^2/16$ so that the behaviour of D for small t agrees with (3.5). Let f_{A_1} be the function defined by

$$f_{A_1} : t \mapsto e^{\int_{1/2}^t \frac{1}{A_1(\xi)} d\xi},$$

we conclude that $D = \frac{b_0^2}{16} f_{A_1}(t) + f_{A_1}(t) \int_0^t A_1^3(\eta) f_{A_1}^{-1}(\eta) d\eta$ and

$$B_1 = \sqrt{A_1^{-2}(t) \left(\frac{b_0^2}{16} f_{A_1}(t) + f_{A_1}(t) \int_0^t A_1^3(\eta) f_{A_1}^{-1}(\eta) d\eta \right)}.$$

For these solutions, there is an extra $SU(2)$ symmetry, and hence they are not only $SU(2)^2$ -invariant but actually $SU(2)^3$ -invariant.

If we consider $\mathbb{R}^4 \times S^3$ and assume a linear asymptotic behaviour of A_1 , i.e.

$$A_1(t) \sim at,$$

$a > 0$, when $t \rightarrow \infty$, we have linear asymptotic behaviour of B_1 (as in the Bryant–Salamon metric, for which $a = 1/3$) where $a > 1/4$ (but only in this case). Specifically,

$$B_1(t) \sim \sqrt{\frac{a^2}{4a - 1}}t.$$

3.4.2 Case $A_1 = A_2 = A_3$

We will now see what happens when $A_1 = A_2 = A_3$. Writing $D = (D_1, D_2, D_3)^T$ the system of equations (3.3) is

$$\dot{D} = \frac{1}{A_1} \begin{pmatrix} -1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{pmatrix} D + \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} A_1^3.$$

We will solve the system for a generic smooth $A_1(t)$, with $A_1(t) \neq 0$ if $t \neq 0$. By first solving the homogeneous system and then using variation of parameters for the the general solution to the in-homogeneous system, we find that the general solution is

$$D = \begin{pmatrix} -f_{A_1}^{-2}(t) & -f_{A_1}^{-2}(t) & f_{A_1}(t) \\ 0 & -f_{A_1}^{-2}(t) & f_{A_1}(t) \\ -f_{A_1}^{-2}(t) & 0 & f_{A_1}(t) \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} + \begin{pmatrix} f_{A_1}(t) \int_0^t A_1^3(\eta) f_{A_1}^{-1}(\eta) d\eta \\ f_{A_1}(t) \int_0^t A_1^3(\eta) f_{A_1}^{-1}(\eta) d\eta \\ f_{A_1}(t) \int_0^t A_1^3(\eta) f_{A_1}^{-1}(\eta) d\eta \end{pmatrix},$$

for some real constants c_1, c_2, c_3 . We observe that although $1/A_1(t)$ might not be locally integrable in a neighborhood of 0, the exponential of the integral $\pm \int_{1/2}^t A_1^{-1}(\xi) d\xi$ need not present a singularity at 0 (see examples below). Similarly for

$$f_{A_1}(t) \int_0^t A_1^3(\eta) f_{A_1}^{-1}(\eta) d\eta.$$

As motivated from the conditions from Lemma 3.3, we now suppose a Taylor expansion of $A_1(t)$ of the following form

$$A_1(t) = \frac{t}{2} + a_{1,3}t^3 + a_{1,5}t^5 + \dots$$

We have then

$$\int_{1/2}^t \frac{1}{A_1(\xi)} d\xi = \int_{1/2}^t \frac{2}{\xi} d\xi + \int_{1/2}^t \frac{a_{1,3}\xi + a_{1,5}\xi^3 + \dots}{\frac{1}{4} + \frac{a_{1,3}}{2}\xi^2 + \frac{a_{1,5}}{2}\xi^4 + \dots} d\xi = 2 \ln t + f(t).$$

The second term is the integral of a smooth function, and we have denoted it by $f(t)$. Hence

$$f_{A_1}(t) = e^{f(t)}t^2, \quad f_{A_1}^{-2}(t) = e^{-2f(t)}t^{-4}.$$

Imposing $D_i(0) = 0$ so that conditions from Lemma 3.3 can be satisfied, we see that we must have $c_1 = c_2 = 0$, so $B_1 = B_2 = B_3$ and we are in special case of 3.4.1. Renaming c_3 as $b_0^2/16$ as before, the general solution becomes

$$D = \begin{pmatrix} \frac{b_0^2}{16} f_{A_1}(t) + f_{A_1}(t) \int_0^t A_1^3(\eta) f_{A_1}^{-1}(\eta) d\eta \\ \frac{b_0^2}{16} f_{A_1}(t) + f_{A_1}(t) \int_0^t A_1^3(\eta) f_{A_1}^{-1}(\eta) d\eta \\ \frac{b_0^2}{16} f_{A_1}(t) + f_{A_1}(t) \int_0^t A_1^3(\eta) f_{A_1}^{-1}(\eta) d\eta \end{pmatrix}.$$

4 Class of coclosed G_2 -structures over a $SU(2)^2$ -invariant manifold

4.1 Existence and uniqueness results

The main tool that we are going to use to get existence and uniqueness results for coclosed G_2 -structures is the following theorem ([33, Theorem 7.1] and [14] for this statement).

Theorem 4.1 [14, Theorem 4.7] *Consider the singular initial value problem*

$$\dot{y} = \frac{1}{t}M_{-1}(y) + M(t, y), \quad y(0) = y_0, \tag{4.1}$$

where y takes values in \mathbb{R}^k , $M_{-1} : \mathbb{R}^k \rightarrow \mathbb{R}^k$ is a smooth function of y in a neighbourhood of y_0 and $M : \mathbb{R} \times \mathbb{R}^k \rightarrow \mathbb{R}^k$ is smooth in t, y in a neighbourhood of $(0, y_0)$. Assume that

- (i) $M_{-1}(y_0) = 0$;
- (ii) $hId - d_{y_0}M_{-1}$ is invertible for all $h \in \mathbb{N}, h \geq 1$.

Then there exists a unique solution $y(t)$ of (4.1). Furthermore y depends continuously on y_0 satisfying (i) and (ii).

Note that this result only gives a short-time solution. However, we will be able to further extend the solution (see Remark 4.2).

We will write our system as in this theorem. Let $A_1, A_2, A_3 : [0, L) \rightarrow \mathbb{R}$ be smooth functions, where L is either infinity or 1, that satisfy the corresponding conditions from Lemma 3.3 (i), i.e. with $A_i(t) > 0$ for $t \in (0, L)$, and such that

- (i) A_i 's are odd;
- (ii) $\dot{A}_i(0) = 1/2$.

Let $D_0 = (0, 0, 0)^T$, as $D_i(0)$ must be 0 if we want an extension to a singular orbit in $t = 0$. Then we can write (3.7) as

$$\dot{D} = \frac{1}{t}M_{-1}(D) + M(t, D),$$

where

$$M_{-1} = 2 \begin{pmatrix} -1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{pmatrix} : \mathbb{R}^3 \rightarrow \mathbb{R}^3,$$

and

$$M(t, D) = A_1A_2A_3 \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + \frac{1}{A_1A_2A_3} \begin{pmatrix} -A_1^2 & A_2^2 & A_3^2 \\ A_1^2 & -A_2^2 & A_3^2 \\ A_1^2 & A_2^2 & -A_3^2 \end{pmatrix} D - \frac{1}{t}M_{-1}(D)$$

is smooth in t, D . The condition (i) from Theorem 4.1 holds as $M_{-1}(D_0) = 0$. However, (ii) does not hold as $d_{y_0}M_{-1}$ has one positive eigenvalue: $d_{y_0}M_{-1} - hId$ is not invertible for $h = 2$:

$$d_{y_0}M_{-1} = 2 \begin{pmatrix} -1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{pmatrix}.$$

Dividing (3.7) by t^2 , we get

$$\frac{\dot{D}}{t^2} = \frac{1}{t}M_{-1} \left(\frac{D}{t^2} \right) + M' \left(t, \frac{D}{t^2} \right),$$

where M_{-1} is as before and

$$M'(t, y) = \frac{1}{t^2}A_1A_2A_3 \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + \frac{1}{A_1A_2A_3} \begin{pmatrix} -A_1^2 & A_2^2 & A_3^2 \\ A_1^2 & -A_2^2 & A_3^2 \\ A_1^2 & A_2^2 & -A_3^2 \end{pmatrix} y - \frac{1}{t}2 \begin{pmatrix} -1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{pmatrix} y$$

The function $M'(t, y)$ is smooth in (t, y) in a neighborhood of $(0, D_0)$. Then

$$\begin{aligned} \frac{d}{dt} \left(\frac{D}{t^2} \right) &= \frac{1}{t}(M_{-1} - 2Id) \left(\frac{D}{t^2} \right) + M' \left(t, \frac{D}{t^2} \right) \\ &= \frac{1}{t}M'_{-1} \left(\frac{D}{t^2} \right) + M' \left(t, \frac{D}{t^2} \right), \end{aligned}$$

where

$$M'_{-1} = 2 \begin{pmatrix} -2 & 1 & 1 \\ 1 & -2 & 1 \\ 1 & 1 & -2 \end{pmatrix} : \mathbb{R}^3 \rightarrow \mathbb{R}^3.$$

Now $d_{y_0}M'_{-1}$ only has non-negative eigenvalues. Fix

$E_0 = (b_0^2/4, b_0^2/4, b_0^2/4)^T$ for some $b_0 \neq 0$. Note that $M'_{-1}(E_0) = 0$. The new singular initial value problem for $E(t) = D(t)/t^2$ is:

$$\dot{E} = \frac{1}{t}M'_{-1}(E) + M'(t, E), \quad E(0) = E_0. \tag{4.2}$$

It satisfies the conditions from Theorem 4.1, meaning there exists a unique solution $E(t)$ in a neighbourhood of 0. Furthermore it depends continuously on E_0 . Then we can define D_1, D_2, D_3 by $D(t) = t^2E(t)$.

Remark 4.2 As for every $i = 1, 2, 3$ we have that A_i is smooth and positive on $(0, L)$, the function $(D, t) \mapsto M(t)D + N(t)$ (with M and N are as in (3.8)) is Lipschitz in D on any closed interval contained in $(0, L)$. Hence, using Picard–Lindelöf Theorem we know that we can extend the solution in a neighbourhood of 0 to $[0, L)$.

We have proved the next Proposition.

Proposition 4.3 *Let $A_1, A_2, A_3 : [0, L) \rightarrow \mathbb{R}$ be smooth functions with $A_i(t) > 0$ for $t \in (0, L)$, where L is either infinity or 1, such that*

- (i) A_i 's are odd;
- (ii) $\dot{A}_i(0) = 1/2$.

Let $b_0 \neq 0$. Consider the singular initial value problem

$$\begin{aligned} \dot{D}_1 &= A_1A_2A_3 - \frac{A_1^2D_1}{A_1A_2A_3} + \frac{A_2^2D_2}{A_1A_2A_3} + \frac{A_3^2D_3}{A_1A_2A_3}, \\ \dot{D}_2 &= A_1A_2A_3 + \frac{A_1^2D_1}{A_1A_2A_3} - \frac{A_2^2D_2}{A_1A_2A_3} + \frac{A_3^2D_3}{A_1A_2A_3}, \\ \dot{D}_3 &= A_1A_2A_3 + \frac{A_1^2D_1}{A_1A_2A_3} + \frac{A_2^2D_2}{A_1A_2A_3} - \frac{A_3^2D_3}{A_1A_2A_3}, \end{aligned}$$

with

$$D_i(0) = \dot{D}_i(0) = 0, \quad \ddot{D}_i(0) = b_0^2/4.$$

Then, there exists a unique solution on $[0, L)$ for this system of ODEs, that depends continuously on b_0 .

We need to check whether we can recover B_1, B_2, B_3 using equation (3.10) to obtain a coclosed G_2 -structure in the principal part. We can only do that if $D_1 D_2 D_3 > 0$. The following Lemma guarantees that this is true.

Lemma 4.4 *Let $D_1, D_2, D_3 : [0, L) \rightarrow \mathbb{R}$ be the unique solutions from Proposition 4.3, for some L which is either a positive number or infinity. Then $D_i(t) > 0$ for $t \in (0, L)$.*

Proof By construction $D_i(t) = \frac{b_0^2}{4}t^2 + O(t^3)$, so in a neighborhood around 0 they are all positive. Suppose the statement is not true, and let $t' > 0$ be the smallest positive real number where one of the D_i 's is 0. Then

$$\dot{D}_i(t') = A_1(t')A_2(t')A_3(t') + \frac{A_j^2(t')D_j(t')}{A_1(t')A_2(t')A_3(t')} + \frac{A_k^2(t')D_k(t')}{A_1(t')A_2(t')A_3(t')} > 0$$

where $\{i, j, k\}$ is a cyclic permutation of $\{1, 2, 3\}$. However, D_i cannot be increasing at t' , so this is a contradiction. \square

Corollary 4.5 *Let $A_1, A_2, A_3 : [0, L) \rightarrow \mathbb{R}$ be smooth functions with $A_i(t) > 0$ for $t \in (0, L)$, where L is either infinity or 1, such that*

- (i) A_i 's are odd;
- (ii) $\dot{A}_i(0) = 1/2$.

Let $b_0 \neq 0$. Then there exist unique functions $B_1, B_2, B_3 : [0, L) \rightarrow \mathbb{R}$ that, together with A_1, A_2, A_3 , will give a solution for the system of ODEs (3.4) with $B_i(0) = b_0$, and the B_i 's depend continuously on b_0 .

We could also study the system (3.6) without appealing to the theory of singular initial value problems. We may do so using the same procedure of Sect. 3.4.2, by first solving it using the general theory of linear systems of ordinary differential equations and then introducing the boundary conditions of the functions A_1, A_2, A_3 . We describe the main points but omit the details, as it follows the same steps as in Sect. 3.4.2. If $\lambda_1(t), \lambda_2(t), \lambda_3(t)$ are the eigenvalues of $M(t)$, then the general solution will be equal to a particular solution plus a linear combination of $c_1 \exp \int_{1/2}^t \lambda_1(\xi)$, $c_2 \exp \int_{1/2}^t \lambda_2(\xi)$, and $c_3 \exp \int_{1/2}^t \lambda_3(\xi)$, where c_1, c_2, c_3 are arbitrary constants. Near the singular orbit and without loss of generality, we have $\lambda_1 = -4/t + O(t)$, $\lambda_2 = -4/t + O(t)$, and $\lambda_3 = 2/t + O(t)$, so imposing $D_i(0) = 0$ we get $c_1 = c_2 = 0$. Finally, c_3 will be uniquely determined by b_0^2 and the desired boundary conditions for the solution (3.5) also follow. While this alternative method requires long computations that the solution above avoided, we note that this can be used to get an explicit solution to (3.6), which will therefore lead to explicit expressions of the G_2 -structure and the metric.

4.2 Extension on $\mathbb{R}^4 \times S^3$

In this section we consider the seven-dimensional non-compact simply connected cohomogeneity one manifold $M = \mathbb{R}^4 \times S^3$ with group diagram $SU(2)^2 \supset \Delta SU(2) \supset \{1\}$. From the

last section, we know that on the principal part of that manifold there is a family of coclosed G_2 -structures. They are constructed from a half-flat $SU(3)$ -structure which is invariant under the cohomogeneity one action. The next step is checking whether it is possible to extend this structure to the singular orbit.

The functions A_i that, together with the constant $b_0 \neq 0$, give us our family of G_2 -structures, satisfy the conditions (i) from Lemma 3.3 by construction. Recall the expression of the B_i 's:

$$B_i(t) = \text{sign}(b_0) \sqrt{\frac{D_j D_k}{D_i A_i^2}}, t \in (0, L), \quad B_i(0) = b_0,$$

where $\{i, j, k\}$ is a cyclic permutation of $\{1, 2, 3\}$. Note that $B_i(t)$ is continuous at $t = 0$. We will check whether they satisfy the conditions (ii) from Lemma 3.3 for the metric to be extended to a singular orbit $Q = SU(2)^2/\Delta SU(2) \cong S^3$. First, a direct consequence from Lemma 4.4 is that B_i 's are sign definite for $t > 0$. The next two Lemmas guarantee that, under the same boundary hypothesis for A_1, A_2, A_3 , the remaining conditions from Lemma 3.3 hold.

Lemma 4.6 *The functions B_1, B_2, B_3 from Corollary 4.5 are even.*

Proof Recall that we can write (3.6) in matrix form as $dD/dt = M(t)D + N(t)$, where $D = (D_1, D_2, D_3)^T$ and both M, N are matrices of odd functions. Let $D(t)$ be the unique solution of this equation in a neighbourhood of 0, say $[0, \varepsilon)$. We extend M and N to $(-\varepsilon, 0)$ by setting $M(-t) := -M(t)$ and $N(-t) := -N(t)$, $t \in (0, \varepsilon)$. Then, since $D(t)$ is a solution of the equation $dy(t)/d(t) = M(t)y(t) + N(t)$, which is equivalent to $dy(t)/d(-t) = M(-t)y(t) + N(-t)$, we can therefore set $D(-t) := D(t)$, which makes D into an even function. Finally, as the D_i 's are even, the B_i 's are even too. □

Lemma 4.7 *The functions B_1, B_2, B_3 from Corollary 4.5 satisfy $\ddot{B}_1(0) = \ddot{B}_2(0) = \ddot{B}_3(0)$.*

Proof We denote by $a_{i,3}$ the coefficient accompanying t^3 in the Taylor expansion of $A_i(t)$, $b_{i,2}$ the coefficient accompanying t^2 in the Taylor expansion of $B_i(t)$ and $d_{i,4}$ the parameter accompanying t^4 in the Taylor expansion of $D_i(t)$. We can then write:

$$\begin{aligned} A_i(t) &= \frac{t}{2} + a_{i,3}t^3 + O(t^5), \\ B_i(t) &= b_0 + b_{i,2}t^2 + O(t^4), \\ D_i(t) &= \frac{b_0^2}{4}t^2 + d_{i,4}t^4 + O(t^6). \end{aligned}$$

The parameters $d_{i,4}$ depend on $a_{i,3}$, and we can compute this dependence from the ODEs for $D(t)$ by considering the Taylor expansion of both sides of

$$A_1 A_2 A_3 \dot{D}_i = (A_1 A_2 A_3)^2 - A_i^2 D_i + A_j^2 D_j + A_k^2 D_k,$$

where $\{i, j, k\}$ is a cyclic permutation of $\{1, 2, 3\}$. After a direct computation we get that

$$d_{i,4} = \frac{1}{16} - \frac{1}{2} b_0^2 a_{i,3}.$$

Now, as

$$D_i(t) = A_j(t)B_j(t)A_k(t)B_k(t),$$

we can also write

$$d_{i,4} = \frac{b_0^2}{2}(a_{j,3} + a_{k,3}) + \frac{b_0}{4}(b_{j,2} + b_{k,2}).$$

Putting these two together, we get

$$b_{1,2} = b_{2,2} = b_{3,2} = \frac{1}{8b_0} - b_0(a_{1,3} + a_{2,3} + a_{3,3}).$$

Hence, the condition $\ddot{B}_1(0) = \ddot{B}_2(0) = \ddot{B}_3(0)$ is always true. □

We can now write the following Proposition.

Proposition 4.8 *Let $M = \mathbb{R}^4 \times S^3$ be the seven-dimensional non-compact simply connected cohomogeneity one manifold with group diagram $SU(2)^2 \supset \Delta SU(2) \supset \{1\}$, with a G_2 -structure constructed from a half-flat $SU(3)$ -structure which is invariant under the cohomogeneity one action. Let the $SU(3)$ -structure $(\omega, \Omega_1, \Omega_2)$ be written as in (3.1). Let $A_1, A_2, A_3 : [0, \infty) \rightarrow \mathbb{R}$ be smooth functions with $A_i(t) > 0$ for $t \in (0, \infty)$ such that*

- (i) A_i 's are odd;
- (ii) $\dot{A}_i(0) = 1/2$.

Let $b_0 \neq 0$. Then there exist unique functions $B_1, B_2, B_3 : [0, \infty) \rightarrow \mathbb{R}$ that, together with A_1, A_2, A_3 , will give a solution to (3.4) with $B_i(0) = b_0$, and the B_i 's depend continuously on $b_0 \neq 0$. The metric g given by (3.2) extends smoothly over the singular orbit.

Proof By Corollary 4.5, there exists unique functions $B_1, B_2, B_3 : [0, \infty) \rightarrow \mathbb{R}$ that, together with A_1, A_2, A_3 , will give a solution to (3.4) with $B_i(0) = b_0$, and the B_i 's depend continuously on $b_0 \neq 0$. Note that the B_i 's are smooth functions in $[0, \infty)$, and $B_1(0) = B_2(0) = B_3(0) \neq 0$. By Lemma 4.6, the functions B_i are even, and by Lemma 4.7, $\ddot{B}_1(0) = \ddot{B}_2(0) = \ddot{B}_3(0)$. They are also sign definite. Hence, by Lemma 3.3, as the conditions for the functions A_i are satisfied by construction, the metric g extends smoothly over the singular orbit. □

The next Theorem summarizes the results from the previous Proposition.

Theorem 4.9 *On the cohomogeneity one manifold $M = \mathbb{R}^4 \times S^3$ with group diagram $SU(2)^2 \supset \Delta SU(2) \supset \{1\}$, there is a family of $SU(2)^2$ -invariant coclosed G_2 -structures which is given by three positive smooth functions $A_1, A_2, A_3 : [0, \infty) \rightarrow \mathbb{R}$ satisfying the boundary conditions at $t = 0$*

$$A_i(t) = \frac{t}{2} + O(t^3),$$

and a non-zero parameter. Moreover, any $SU(2)^2$ -invariant coclosed G_2 -structure constructed from a half flat $SU(3)$ -structure is in this family.

Remark 4.10 The volume of the singular orbit at $t = 0$ is proportional to b_0^3 .

4.3 Extension on $S^4 \times S^3$

In this section we consider the seven-dimensional compact simply connected cohomogeneity one manifold $M = S^4 \times S^3$ with group diagram $SU(2)^2 \supset \Delta SU(2), \Delta SU(2) \supset \{1\}$. From

Sect. 4.1, we know that on the principal part of that manifold there is a family of coclosed G_2 -structures constructed from a half-flat $SU(3)$ -structure. In the last section, we saw that it is possible to extend the structure to one singular orbit $SU(2)^2/\Delta SU(2)$.

The next Proposition shows that the previous structure cannot be smoothly extended to two singular orbits of type $SU(2)^2/\Delta SU(2)$.

Proposition 4.11 *On the cohomogeneity one manifold $M = S^4 \times S^3$ with group diagram $SU(2)^2 \supset \Delta SU(2)$, $\Delta SU(2) \supset \{1\}$, there are no $SU(2)^2$ -invariant coclosed G_2 -structures constructed from half-flat $SU(3)$ -structures.*

Proof Suppose that M has a G_2 -structure constructed from a half-flat $SU(3)$ -structure on its principal part, which is invariant under the cohomogeneity one action. Then the $SU(3)$ -structure $(\omega, \Omega_1, \Omega_2)$ can be written as in (3.1), and there are $A_1, A_2, A_3 : [0, 1] \rightarrow \mathbb{R}$ smooth functions with $A_i(t) > 0$ for $t \in (0, 1)$ such that

- (i) A_i 's are odd around $t = 0$;
- (ii) $\dot{A}_i(0) = 1/2$.

Let $b_0 \neq 0$. By Proposition 4.3 we know that there exists a unique solution of the system (3.6), $D_i : [0, 1] \rightarrow \mathbb{R}$ with initial conditions $D_i(0) = \dot{D}_i(0) = 0, \ddot{D}_i(0) = b_0^2/4$, for $i = 1, 2, 3$. Then, there exist unique functions $B_1, B_2, B_3 : [0, 1] \rightarrow \mathbb{R}$ that, together with A_1, A_2, A_3 , will solve (3.4). Also, by Proposition 4.8 the metric (3.2) can be extended to the singular orbit at $t = 0$. Hence, this solution can be extended to the interval $[0, 1)$. It remains to check whether we can extend the solutions to $t = 1$. In equation (3.6), we add

$$\frac{d}{dt}(D_1 + D_2 + D_3) = 3A_1A_2A_3 + \frac{A_1^2D_1 + A_2^2D_2 + A_3^2D_3}{A_1A_2A_3},$$

In Lemma 4.4 we proved that for $t \in (0, 1)$, $D_i(t) > 0, i = 1, 2, 3$ so $d(D_1 + D_2 + D_3)/dt > 0$. In particular, as $D_1 + D_2 + D_3$ vanishes at $t = 0$, it cannot be 0 at $t = 1$. Therefore, there cannot be an extension to the singular orbit at $t = 1$. This is because the B_i functions obtained from the D_i 's blow up at this orbit.

Alternatively, this result also follows from the smoothness conditions for $A_i, i = 1, 2, 3$ around both singular orbits. From the symmetry of equations (3.4), we deduce that the smoothness conditions at $t = 1$ require that $A_i(1) = 0, \dot{A}_i(1) = 1/2$. This, together with $A_i(0) = 0, \dot{A}_i(0) = 1/2$, implies that there is a point in $(0, 1)$ where the A_i vanishes. This gives a contradiction with the fact that A_i is sign definite in $(0, 1)$. □

4.4 Conclusions and future directions

The first thing that we observe is that on $M = \mathbb{R}^4 \times S^3$ there are many more G_2 -structures constructed from half-flat $SU(3)$ -structures that are coclosed than that are torsion-free (in general, one can perturb a coclosed G_2 -structure by adding a small exact 4-form to construct another coclosed G_2 -structure). In particular, Lotay and Oliveira give a two-parameter family of torsion-free G_2 -structures constructed from half-flat $SU(3)$ -structures [32, Remark 21]. The coclosed family described in this chapter depends on three smooth functions satisfying certain boundary conditions and a non-zero parameter. More generally, known examples of torsion-free G_2 -structures constructed from half-flat $SU(3)$ -structures have an extra $U(1)$ -symmetry. With our previous notation, this means that $A_2 = A_3$ and $B_2 = B_3$. We showed that if we relax the torsion-free condition to coclosed, the A_i 's only need to be equal at orders lower or equal than 1 in the Taylor expansion of the cohomogeneity one parameter around

the singular orbit, and the B_i 's at orders lower or equal than 2. Our families of structures contain the Bryant–Salamon G_2 -holonomy metric [7] and the 1-parameter family of complete $(\mathrm{SU}(2)^2 \times \mathrm{U}(1))$ -invariant G_2 -metrics of Brandhuber et al. [4] and Bogoyavlenskaya [5], also known as the \mathbb{B}_7 family. In particular, we deduce from [15, Theorem 6.16] that if we have an extra $\mathrm{U}(1)$ -symmetry and impose that the metric is G_2 and complete, we get precisely this family.

While the objective of this paper is the construction of new examples of coclosed G_2 -structures which are invariant under the cohomogeneity one action of $\mathrm{SU}(2)^2$, we outline future directions for a systematic study of these structures which could be made by relaxing some of the assumptions made in this paper. We first recall that coclosed G_2 -structures do not need to be half-flat. While this condition is guaranteed to hold for torsion-free G_2 -structures, when our structures are coclosed we only need a weaker condition: $d\omega^2 = 0$. Therefore, we may consider families of $\mathrm{SU}(3)$ -structures $(\omega(t), \Omega_2(t))$ such that $d\omega^2 = 0$ instead of being half-flat, but this would require a different analysis as the description of the structure from equation (3.1) would no longer hold. It would also be interesting to consider seven-dimensional manifolds with a G_2 -structure invariant under a cohomogeneity one action of $\mathrm{SU}(2)^2$ which is different than the one considered in this paper and described in Sect. 3.2 (see Remark 3.4 for other inequivalent actions). As the coclosed equations (3.6) do not depend on the action, these will remain valid for a different action of $\mathrm{SU}(2)^2$, while the conditions for a smooth extension to the singular orbit will be different. A similar analysis of the ODEs with the new boundary conditions could be done for these cases, potentially providing new families of metrics containing \mathbb{D}_7 family and the infinite extension of the \mathbb{C}_7 family as particular cases.

In [1], A. and Salvatore showed that if M is a six-dimensional simply connected cohomogeneity one manifold under the almost effective action of a connected Lie group G , then M admits no G -invariant balanced non-Kähler $\mathrm{SU}(3)$ -structures. As in this paper, the search for balanced $\mathrm{SU}(3)$ -structures was also motivated by heterotic string theory; in particular, the Hull–Strominger system. This means that on most cases (and possibly always), the existence of a balanced structure (which can be seen as the six-dimensional analogue to a coclosed G_2 -structure) in the cohomogeneity one setting forces the structure to be Kähler. We observe that the existence of a class of coclosed G_2 -structures, not necessarily torsion-free, in the cohomogeneity one setting contrasts with this result, as in the 7 dimensional analogue, the cohomogeneity one hypothesis does not force coclosed G_2 -structures to be torsion-free.

Given the structures found in this paper, the question that arises is what are the G_2 -instantons over $\mathbb{R}^4 \times S^3$ with these structures. Another question is whether it is possible to find solutions to the heterotic G_2 system over them. At the present time, the author is working on these problems (see Remark 3.2).

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Author Contributions I am the sole author of the paper.

Data availability No datasets were generated or analysed during the current study.

Declarations

Conflict of interest The author declares that there are no Conflict of interest.

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