

CONFORMAL FIELD THEORY APPROACH TO THE KONDO EFFECT*

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Recently, a new approach, based on boundary conformal field theory, has been applied to a variety of quantum impurity problems in condensed matter and particle physics. A particularly enlightening example is the multi-channel Kondo problem. In this review some earlier approaches to the Kondo problem are discussed, the needed material on boundary conformal field theory is developed and then this new method is applied to the multi-channel Kondo problem.

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1. Renormalization group and Fermi liquid approaches to the Kondo effect

1.1. Introduction to the Kondo effect

Most mechanisms contributing to the resistivity of metals, $\rho(T)$, give either $\rho(T) \rightarrow 0$, as $T \rightarrow 0$ (phonons or electron-electron interactions), or $\rho(T) \rightarrow \text{constant}$, as $T \rightarrow 0$ (non-magnetic impurities). However, metals containing magnetic impurities show a $\rho(T)$ which increases as $T \rightarrow 0$. This was explained by Kondo [1] in 1964 using a simple Hamiltonian:

$$H = \sum_{\vec{k}\alpha} \psi_{\vec{k}}^{\dagger\alpha} \psi_{\vec{k}\alpha} \varepsilon(k) + \lambda \vec{S} \cdot \sum_{\vec{k}\vec{k}'} \psi_{\vec{k}}^{\dagger} \frac{\vec{\sigma}}{2} \psi_{\vec{k}'}, \quad (1.1)$$

where $\psi_{\vec{k}\alpha}$'s are conduction electron annihilation operators, (of momentum $\vec{k}$, spin α) and $\vec{S}$ represents the spin of the magnetic impurity with

$$[S^a, S^b] = i\varepsilon^{abc} S^c.$$

The interaction term represents an impurity spin interacting with the electron spin at $\vec{x} = 0$.

With the above Hamiltonian, Born approximation gives: $\rho(T) \sim \lambda^2$, independent of T . The next order term has a divergent coefficient at $T = 0$:

$$\rho(T) \sim [\lambda + \nu \lambda^2 \ln \frac{D}{T} + \dots]^2. \quad (1.2)$$

Here D is the band-width, ν the density of states. This result stimulated an enormous amount of theoretical work. As Nozières put it, “Theorists ‘diverged’ on their own, leaving the experiment realities way behind”. [2] What happens at low T , i.e. $T \sim T_K = D \exp(-1/\nu\lambda)$? In that case the $O(\lambda^2)$ term will be as big as the term of $O(\lambda)$. What about the $O(\lambda^3)$ term? Such questions helped lead to the development of the renormalization group needed to understand the problem.

In particle physics such a growth of a coupling constant at low energies explains quark confinement (1973) and “asymptotic freedom” at $E \rightarrow \infty$. To solve these problems, Wilson [3] developed a very powerful numerical renormalization group approach. The Kondo model was also “solved” by the Bethe ansatz [4, 5] which gives the specific heat and magnetization. Nozières, [2, 7] following ideas of Anderson [6] and Wilson [3], developed a very simple, and in a sense exact, picture of the low T behaviour. With Ludwig, I have generalized and reformulated Nozières approach [9–16] using recent result in conformal field theory. The latter approach is very general and can be applied to a number of other problems: multi-channel and higher spin Kondo effect [9–16], two (or more)-impurity Kondo effects [17, 18], impurity assisted tunneling [19], impurities in one-dimensional conductors (“quantum wires”) [20] or 1D antiferromagnets [21, 22], baryon-monopole interaction [23]. Some of these problems, including the multi-channel Kondo effect, exhibit non-Fermi liquid behaviour. These are among the very few exactly solved problems that do this (the others are 1D Luttinger liquids). It has been suggested that this may be connected with exotic behaviour of certain compounds, including high- T_c superconductors [24, 25].

1.2. Renormalization group

We could integrate out $\psi(k)$ for k far from k_F , the Fermi wave-vector, and successively reduce the band-width D to obtain a new effective interaction. [See figure 1] This is hard to do exactly. At weak coupling one can do it perturbatively in λ . With the simplest approach, real-time, time-ordered perturbation theory, we expand

$$T \exp \left[-i\lambda \int \vec{S}(t) \cdot \psi^\dagger \frac{\vec{\sigma}}{2} \psi(\vec{0}, t) \right],$$

where the fields are in the interaction picture.

As $\vec{S}(t)$ is independent of t , we simply multiply powers of $\vec{S}$ using

$$[S^a, S^b] = i\epsilon^{abc} S^c, \quad \vec{S}^2 = s(s+1).$$

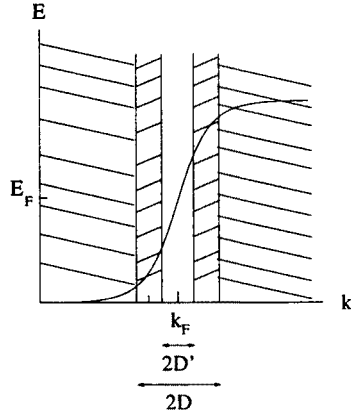


Fig. 1. Reduction of the cut-off from D to D'

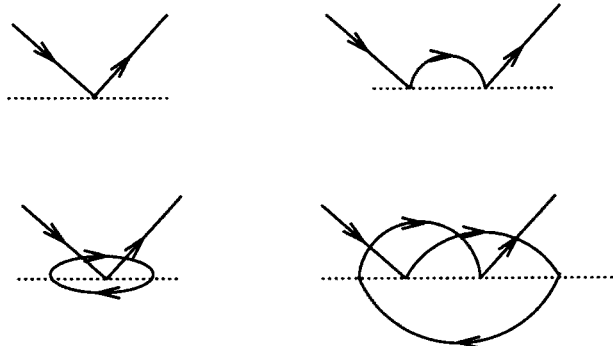


Fig. 2. Feynman diagrams contributing to renormalization of the Kondo coupling constant to third order.

We must time-order $\vec{S}$'s which don't commute. The first few diagrams are shown in figure 2. In 2nd order in λ , we have:

$$-\frac{\lambda^2}{2} \int dt dt' T(S^a(t) S^b(t')) \cdot T[\psi^\dagger(t) \frac{\sigma^a}{2} \psi(t) \psi^\dagger(t') \frac{\sigma^b}{2} \psi(t')],$$

which can be reduced, using Wick's theorem, to:

$$\begin{aligned} -\frac{1}{2} \lambda^2 \int dt dt' \psi^\dagger \left[\frac{\sigma^a}{2}, \frac{\sigma^b}{2} \right] \psi T \langle \psi(t) \psi^\dagger(t') \rangle (\theta(t-t') S^a S^b + \theta(t'-t) S^b S^a) \\ = \frac{\lambda^2}{2} \int dt dt' \psi^\dagger \frac{\vec{\sigma}}{2} \psi \cdot \vec{S} \text{sn}(t-t') \langle \psi(t) \psi^\dagger(t') \rangle, \end{aligned} \tag{1.3}$$

where $\text{sn}(t-t')$ is the sign-function which arises from T -ordering spins.

We see that the integral

$$\int dt \varepsilon(t) G(t) = -i \int \frac{dt}{|t|} \quad (1.4)$$

is divergent in the infrared limit: $t \rightarrow \infty$, where $G(t) = \langle \psi(t) \psi^\dagger(0) \rangle$. But we only integrate out electrons with $D' < k < D$ which gives $\ln D/D'$. To do it explicitly we use the Fourier transformed form:

$$\begin{aligned} \int \frac{d^3 k}{(2\pi)^3} \int \frac{d\omega}{2\pi} \left[\frac{1}{i\omega + \delta} + \frac{1}{i\omega - \delta} \right] \frac{i}{\omega - \varepsilon_k + i\delta \text{sn}(\varepsilon_k)} \\ = \int \frac{d^3 \vec{k}}{(2\pi)^3} \frac{1}{|\varepsilon_k|} \approx 2\nu \int_{D'}^D \frac{d\varepsilon}{\varepsilon} = 2\nu \ln \frac{D}{D'}. \end{aligned} \quad (1.5)$$

Thus

$$\delta\lambda = \nu\lambda^2 \ln \frac{D}{D'}, \quad (1.6)$$

and

$$\frac{d\lambda}{d \ln D} = -\nu\lambda^2. \quad (1.7)$$

We see that lowering the band cut-off increases λ , or defining a length-dependent cut-off, $l \sim v_F/D$,

$$\frac{d\lambda}{d \ln l} = \nu\lambda^2. \quad (1.8)$$

Integrating the equation (equivalent to performing an infinite sum of diagrams), gives:

$$\lambda_{\text{eff}}(D) = \frac{\lambda_0}{1 - \nu\lambda_0 \ln \frac{D_0}{D}}. \quad (1.9)$$

If $\lambda_0 > 0$ (antiferromagnetic), then $\lambda_{\text{eff}}(D)$ diverges at $D \sim T_k \sim D_0 e^{-\frac{1}{\nu\lambda_0}}$, If $\lambda_0 < 0$ (ferromagnetic), $\lambda_{\text{eff}}(D) \rightarrow 0$. See figure 3.

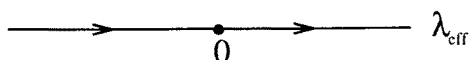


Fig. 3. RG flow of the Kondo coupling

The behaviour at temperature T is determined by $\lambda_{\text{eff}}(T)$: $\rho(T) \rightarrow 0$ as $T \rightarrow 0$ for the ferromagnetic case. What happens for the antiferromagnetic case?

1.3. Mapping to a one-dimensional model

The above discussion can be simplified if we map the model into a one dimensional one. We assume a spherically symmetric $\varepsilon(\vec{k})$,

$$\varepsilon(k) = \frac{k^2}{2m} - \varepsilon_F \approx v_F(k - k_F), \quad (1.10)$$

and a δ -function Kondo interaction. There is only s-wave scattering, *i.e.*

$$\begin{aligned} \psi(\vec{k}) &= \frac{1}{\sqrt{4\pi k}} \psi_0(k) + \text{higher harmonics}, \\ H_0 &= \int dk \psi_0^\dagger \psi_0 \varepsilon(k) + \text{higher harmonics}, \\ H_{\text{int}} &= \lambda v_F \nu \int dk dk' \psi_{0,k}^\dagger \frac{\vec{\sigma}}{2} \psi_{0,k'} \cdot \vec{S}, \end{aligned} \quad (1.11)$$

where $\nu = k_F^2/2\pi^2 v_F$ is the density of states per spin. This can also be written in terms of radial co-ordinate. We eliminate all modes except for a band width $2D$: $|k - k_F| < D$ Defining left and right movers (incoming and out going waves),

$$\Psi_{L,R}(r) \equiv \int_{-\infty}^{\infty} dk e^{\pm ikr} \psi_0(k + k_F), \quad \Rightarrow \quad \psi_L(0) = \psi_R(0), \quad (1.12)$$

we have

$$\begin{aligned} H_0 &= \frac{v_F}{2\pi} \int_0^\infty dx (\psi_L^\dagger i \frac{d}{dr} \psi_L - \psi_R^\dagger i \frac{d}{dr} \psi_R) \text{ (note the unconventional normalization),} \\ H_{\text{int}} &= v_F \lambda \psi_L(0)^\dagger \frac{\vec{\sigma}}{2} \psi_L(0) \cdot \vec{S}. \end{aligned} \quad (1.13)$$

Here we have redefined a dimensionless Kondo coupling, $\lambda \rightarrow \lambda\nu$. Using the notation

$$\psi_L = \psi_L(x, \tau) = \psi_L(z = \tau + ix), \quad \psi_R(x, \tau) = \psi_R(z^* = \tau - ix), \quad (1.14)$$

where τ is imaginary time and $x = r$, (and we set $v_F = 1$) we have

$$\langle \psi_L(z) \psi_L^\dagger(0) \rangle = \frac{1}{z}, \quad \langle \psi_R(z^*) \psi_R^\dagger(0) \rangle = \frac{1}{z^*}. \quad (1.15)$$

Alternatively, since

$$\psi_L(0, \tau) = \psi_R(0, \tau) \quad \psi_L = \psi_L(z), \quad \psi_R = \psi_R(z^*), \quad (1.16)$$

we may consider ψ_R to be the continuation of ψ_L to the negative r -axis:

$$\psi_R(x, \tau) \equiv \psi_L(-x, \tau). \quad (1.7)$$

Now we obtain a relativistic $(1 + 1)$ dimensional field theory (“chiral” one, containing left-movers only) interacting with the impurity at $x = 0$ with

$$H_0 = \frac{v_F}{2\pi} \int_{-\infty}^{\infty} dx \psi_L^\dagger i \frac{d}{dx} \psi_L \quad (1.18)$$

and H_{int} as in Eq. (1.13). See figure 4.

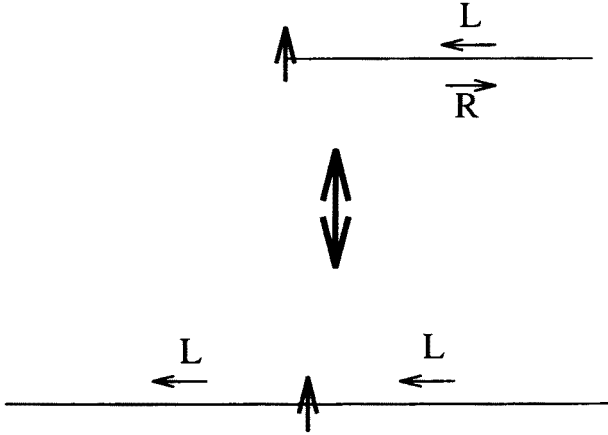


Fig. 4. Reflecting the left-movers to the negative axis

1.4. Fermi liquid approach at low T

What is the $T \rightarrow 0$ behavior of the antiferromagnetic Kondo model? The simplest assumption is $\lambda_{\text{eff}} \rightarrow \infty$. But what does that really mean? Consider the strong coupling limit of a lattice model [2], for convenience,

in spatial dimension $D = 1$. (D does not really matter since we can always reduce the model to $D = 1$.)

$$H = t \sum_i (\psi_i^\dagger \psi_{i+1} + \psi_{i+1}^\dagger \psi_i) + \lambda \vec{S} \cdot \psi_0^\dagger \frac{\vec{\sigma}^2}{2} \psi_0. \quad (1.19)$$

Consider the limit $\lambda \gg |t|$. The groundstate of the interaction term will be the following configuration: one electron at the site 0 forms a singlet with the impurity: $|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle$ (We assume $S_{\text{imp}} = 1/2$). Now we do perturbation theory in t . We have the following low energy states: an arbitrary electron configuration occurs on all other sites-but other electrons or holes are forbidden to enter the site-0, since that would destroy the singlet state, costing an energy, $\Delta E \sim \lambda \gg t$. Thus we simply form free electron Bloch states with the boundary condition $\phi(0) = 0$, where $\phi(i)$ is the single-electron wave-function. Note that at zero Kondo coupling, the parity even single particle wave-functions are of the form $\phi(i) = \cos ki$ and the parity odd ones are of the form $\phi(i) = \sin ki$. On the other hand, at $\lambda \rightarrow \infty$ the parity even wave-functions become $\phi(i) = |\sin ki|$, while the parity odd ones are unaffected.

The behaviour of the parity even channel corresponds to a $\pi/2$ phase shift in the s-wave channel.

$$\phi_j \sim e^{-ik|j|} + e^{+2i\delta} e^{ik|j|}, \quad \delta = \pi/2. \quad (1.20)$$

In terms of left and right movers on $r > 0$ we have changed boundary condition,

$$\begin{aligned} \psi_L(0) &= \psi_R(0), & \lambda &= 0, \\ \psi_L(0) &= -\psi_R(0), & \lambda &= \infty. \end{aligned} \quad (1.21)$$

The strong coupling fixed point is the same as the weak coupling fixed point except for a change in boundary conditions (and the removal of the impurity). In terms of the left-moving description of the P -even sector, the phase of the left-mover is shifted by π as it passes the origin. Imposing another boundary condition a distance l away quantizes k :

$$\begin{aligned} \psi(l) &= \psi_L(l) + \psi_R(l) = \psi_L(l) + \psi_L(-l) = 0, \\ \lambda = 0 : \quad k &= \frac{\pi}{l}(n + \frac{1}{2}) \\ \lambda = \infty : \quad k &= \frac{\pi n}{l}. \end{aligned} \quad (1.22)$$

Near the Fermi surface the energies are linearly spaced. Assuming particle-hole symmetry, the Fermi energy lies midway between levels or on a

level. [See figures 5 and 6.] The two situations switch with the phase shift. Wilson's numerical RG scheme [3] involves calculating the low-lying spectrum numerically and looking for this shift. This indicates that λ renormalizes to ∞ even if it is initially small. However, now we expect the screening to take place over a longer length scale

$$\xi \sim \frac{v_F}{T_K} \sim \frac{v_F}{D} e^{1/\nu\lambda}. \quad (1.23)$$

In other words, the wave function of the screening electron has this scale. We get low energy Bloch states of free electrons only for $|k - k_F| \ll 1/\xi$ (so we must take $l \gg \xi$). [See figure 7.] The free electron theory with a phase shift corresponds to a universal stable low energy fixed point for the Kondo problem. This observation determines the $T = 0$ resistivity for an array of Kondo impurities at random locations of low density n_i . It is the same as for non-magnetic s-wave scatterers with a $\pi/2$ phase shift at the Fermi energy. $\delta = \pi/2$ gives the so-called unitary limit resistivity:

$$\rho_u = \frac{3n_i}{\pi\nu^2 v_F^2 e^2}. \quad (1.24)$$

The low- T behaviour, so far, seems trivial. Much of the interesting behaviour comes from the leading irrelevant operator. The impurity spin has disappeared (screened) from the description of the low- T physics. However certain interactions between electrons are generated (at the impurity site only) in the process of eliminating the impurity spin. We can determine these by simply writing the lowest dimension operators allowed by symmetry.

It is simplest to work in the 1D formulation, with left-movers only. We write the interaction in terms of ψ_L , obeying the new boundary condition (but not the impurity spin). The dimension of the operator is determined as in 1D field theory

$$H = \int dx \psi_L^\dagger i \frac{d}{dx} \psi_L + \dots \quad (1.25)$$

The length and time dimensions are equivalent (we convert with v_F),

$$[H] = E \Rightarrow [\psi] = E^{1/2}.$$

The interactions are local

$$\delta H = \sum_i \lambda_i O_i(x=0), \quad [\lambda_i] + [O_i] = 1.$$

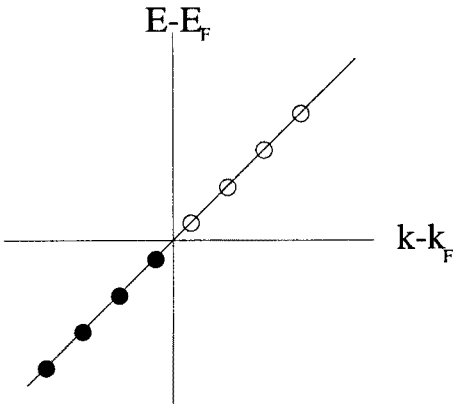


Fig. 5. Free fermion energy levels with antiperiodic boundary conditions.

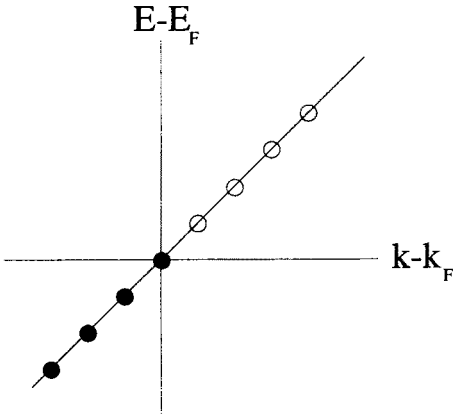


Fig. 6. Free fermion energy levels with periodic boundary conditions.

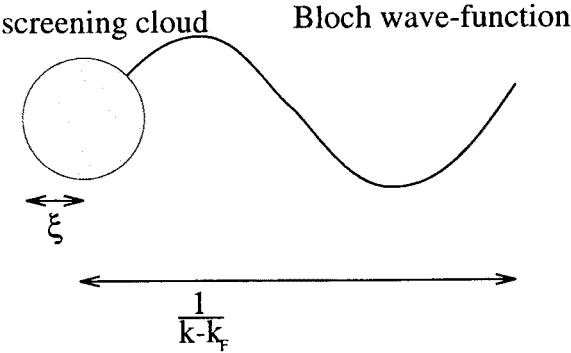


Fig. 7. Non-interacting Bloch states with a vanishing boundary condition occur for $|k - k_F| \ll v_F/T_K$.

So λ_i has negative energy dimension if $[O_i] > 1$, implying that it is irrelevant. In RG theory one usually defines a dimensionless coupling constant by multiplying powers of the cut-off D , if

$$[\lambda_i] = E^{-a}, \quad \tilde{\lambda}_i \equiv \lambda_i D^a,$$

$\tilde{\lambda}_i$ decreases as we lower D :

$$\frac{d\tilde{\lambda}_i}{d \ln D} = a\tilde{\lambda}_i. \quad (1.26)$$

Such a coupling, with $a > 0$, produces no infrared divergences in perturbation theory. The ultraviolet ones are cancelled by the explicit factors of the ultraviolet cut-off, D , appearing in the Lagrangian, $\tilde{\lambda} O/D^a$. What are the lowest dimension operators allowed by symmetry? Consider $\psi^{+\alpha}(0)\psi_{\alpha}(0)$. This has $d = 1$. However, it is not allowed because it breaks particle-hole symmetry. If particle-hole symmetry is broken then we do get this, a potential scattering term; it adds a term to the phase shift. Consider another term,

$$i\psi^{\dagger\alpha} \frac{d}{dx} \psi_{\alpha}(0) - i \frac{d}{dx} \psi^{\dagger\alpha} \psi_{\alpha}(0). \quad (1.27)$$

This has $d = 2$. This term produces a k -dependent phase shift. The only other term with $d \leq 2$ is $\psi^{\dagger\uparrow}\psi_{\uparrow}\psi^{\dagger\downarrow}\psi_{\downarrow}$. This term represents the electron-electron interaction induced by an impurity spin-flip. The first electron flips the impurity spin. This makes it possible for the second electron to flip it back if the electron spin is correct. These are the only $d \leq 2$ operators. There are no relevant ($d \leq 1$) operators, implying the stability of the low energy fixed point. Note that, by contrast, the high energy, zero Kondo coupling, fixed point is unstable. The dimension 1 operator (the Kondo interaction) can occur there because of the presence of the impurity spin.

We can't calculate these two coupling constants exactly except by using complicated methods: Wilson's numerical method or the Bethe ansatz. They both have dimension E^{-1} . We expect them to be $O[1/T_K]$ by a standard scaling argument. *I.e.* functions of the cut-off D and coupling constant λ can be replaced by functions of the reduced cut-off, D' and the renormalized coupling constant, $\lambda_{\text{eff}}(D')$: $f[D, \lambda] = f[D', \lambda_{\text{eff}}(D')]$. We can lower the cut-off down to T_K where λ_{eff} is $O(1)$ so $f = f(T_K, 1) = f(T_K)$. This is a characteristic scale introduced by the infrared divergences of perturbation theory. For $T_K \ll D$ ($\lambda \ll 1$), Nozières argued that the two irrelevant coupling constants have a universal ratio. So there is only one unknown parameter ("the Wilson number"). Essentially all low-temperature information is given by this irrelevant coupling constant, if it was not already determined by the $\pi/2$ phase shift at k_F . I will give a different derivation

of the ratio of the two coupling constants later using conformal field theory. We now simply do perturbation theory in the irrelevant coupling constant $\sim 1/T_K$. We can determine powers of T by dimensional analysis. For the specific heat we find:

$$C \sim \frac{\pi}{3v_F} lT + a \frac{T}{T_K}. \quad (1.28)$$

This is the specific heat for the one-dimensional system with a single impurity at the origin. Note that the first term is simply the specific heat of the free system, proportional to system length. The second term is independent of length and is the impurity specific heat. It is the result of first order perturbation theory in the irrelevant coupling constant, of $O(1/T_K)$. The (linear) power of T can be fixed by dimensional analysis; a is a pure number. Note that while this is formally an "irrelevant" contribution, it in fact gives the leading impurity specific heat at low T . To obtain the specific heat for the three dimensional system we simply multiply the first term by the ratio $\nu V/(l/2\pi v_F)$ i.e. the ratio of densities of states per unit energy. For a dilute random array the last term gets multiplied by the number of impurities. At high T we get, approximately, the entropy for a decoupled $s = 1/2$ impurity:

$$S(T) = \frac{\pi l}{3v_F} T + \ln 2. \quad (1.29)$$

At low T , the impurity entropy decreases to 0:

$$S(T) = \frac{\pi l}{3v_F} T + \frac{aT}{T_K}. \quad (1.30)$$

In general we may write:

$$S(T) - \frac{\pi l}{3v_F} T \equiv S_{\text{imp}} = g\left(\frac{T}{T_K}\right), \quad (1.31)$$

where g is a scaling function which is universal for weak bare coupling. See figure 8. The behaviour of $g(x)$ for small arguments is determined by RG-improved weak coupling perturbation theory. Its behaviour at low T is determined from the theory of the low energy fixed point. Its behaviour at arbitrary T/T_K is a property of the universal crossover between fixed points. It has been found from the Bethe ansatz.

Similarly, the susceptibility, at $T = 0$, is given by:

$$\chi \sim \frac{l}{2\pi v_F} + \frac{b}{T_K}. \quad (1.32)$$

The ratio b/a is universal since the coupling constant ($1/T_K$) drops out. This is known as the Wilson Ratio.

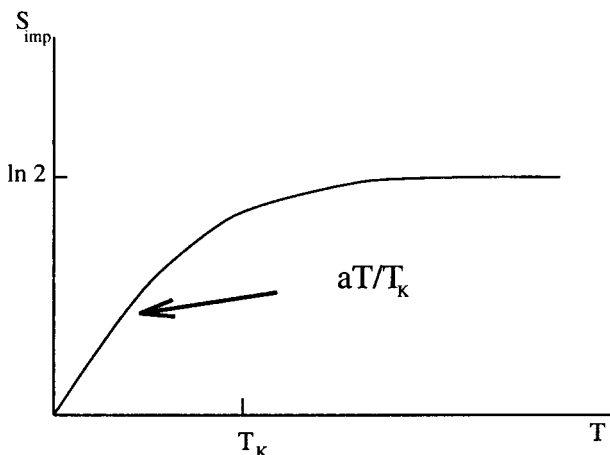


Fig. 8. Qualitative behaviour of the impurity entropy.

At high T , we must (for weak bare coupling) obtain approximately the results for a free spin:

$$\chi \sim \frac{l}{2\pi v_F} + \frac{1}{4T}. \quad (1.33)$$

At lower T , using RG improved perturbation theory this becomes:

$$\chi \sim \frac{l}{2\pi v_F} + \frac{1}{4T} \left[1 - \frac{1}{\ln\left(\frac{T}{T_K}\right)} + \dots \right].$$

In general, we may write:

$$\chi - \frac{l}{2\pi v_F} \equiv \chi_{\text{imp}} = \frac{1}{T} f\left(\frac{T}{T_K}\right),$$

where $f(T/T_K)$ is another universal scaling function. See figure 9.

The temperature dependent part of the low T resistivity for the dilute random array is 2nd order in perturbation theory,

$$\rho = \rho_u \left[1 - d \left(\frac{T}{T_K} \right)^2 \right], \quad (1.36)$$

where d is another dimensionless constant. The second term comes from second order perturbation theory in the irrelevant coupling constant. Another universal ratio can be formed. I will discuss this in Sec. 3, in some detail, using the CFT approach.

We now expect a scaling behaviour:

$$\rho(T) = n_i f\left(\frac{T}{T_K}\right),$$

sketched in figure 10.

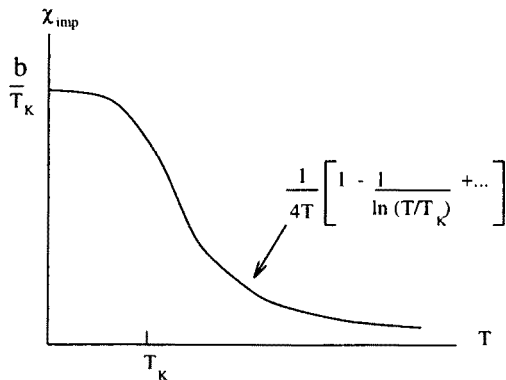


Fig. 9. Qualitative behaviour of the impurity susceptibility

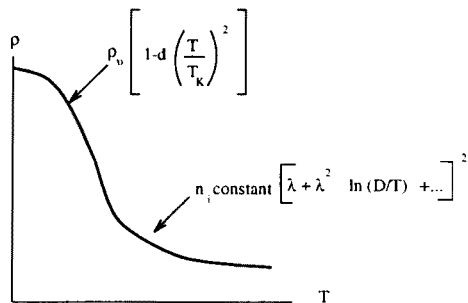


Fig. 10. Qualitative behaviour of the resistivity

2. Conformal field theory techniques

It is very useful to bosonize free fermions to understand the Kondo effect. This allows separation of spin and charge degrees of freedom which greatly simplifies the problem.

We start by considering a left-moving spinless fermion field with Hamiltonian density:

$$\mathcal{H} = \frac{1}{2\pi} \psi_L^\dagger i \frac{d}{dx} \psi_L. \tag{2.1}$$

Define the current (=density) operator,

$$\begin{aligned} J_L(x-t) &= \psi_L^\dagger \psi_L : (x,t) \\ &= \lim_{\varepsilon \rightarrow 0} [\psi_L(x) \psi_L(x+\varepsilon) - \langle 0 | \psi_L(x) \psi_L(x+\varepsilon) | 0 \rangle] \end{aligned} \tag{2.2}$$

(Henceforth we generally drop the subscripts “L”.) We will reformulate the theory in terms of currents (key to bosonization). Consider:

$$\begin{aligned}
 J(x)J(x+\varepsilon) \quad \text{as} \quad \varepsilon \rightarrow 0 \\
 &= : \psi^\dagger(x)\psi(x)\psi^\dagger(x+\varepsilon)\psi(x+\varepsilon) : \\
 &+ [: \psi^\dagger(x)\psi(x+\varepsilon) : + : \psi(x)\psi^\dagger(x+\varepsilon) :] G(\varepsilon) + G(\varepsilon)^2 \\
 G(\varepsilon) &= \langle 0 | \psi(x)\psi^\dagger(x+\varepsilon) | 0 \rangle = \frac{1}{-i\varepsilon}.
 \end{aligned} \tag{2.3}$$

By Fermi statistics the 4-Fermi term vanishes as $\varepsilon \rightarrow 0$

$$: \psi^\dagger(x)\psi(x)\psi^\dagger(x)\psi(x) : = - : \psi^\dagger(x)\psi^\dagger(x)\psi(x)\psi(x) : = 0. \tag{2.4}$$

The second term becomes a derivative,

$$\begin{aligned}
 \lim_{\varepsilon \rightarrow 0} [J(x)J(x+\varepsilon) + \frac{1}{\varepsilon^2}] &= \lim_{\varepsilon \rightarrow 0} \frac{1}{-i\varepsilon} [: \psi^\dagger(x)\psi(x+\varepsilon) : - : \psi^\dagger(x+\varepsilon)\psi(x) :] \\
 &= 2i : \psi^\dagger \frac{d}{dx} \psi : \\
 \mathcal{H} &= \frac{1}{4\pi} J(x)^2 + \text{constant}.
 \end{aligned} \tag{2.5}$$

Now consider the commutator, $[J(x), J(y)]$. The quartic and quadratic terms cancel. We must be careful about the divergent c -number part,

$$\begin{aligned}
 [J(x), J(y)] &= -\frac{1}{(x-y-i\delta)^2} + \frac{1}{(x-y+i\delta)^2} \quad (\delta \rightarrow 0^+) \\
 &= \frac{d}{dx} \left[\frac{1}{x-y-i\delta} - \frac{1}{x-y+i\delta} \right] \\
 &= 2\pi i \frac{d}{dx} \delta(x-y).
 \end{aligned} \tag{2.6}$$

Now consider the free massless boson theory with Hamiltonian density (setting $v_F = 1$):

$$\mathcal{H} = \frac{1}{2} \left(\frac{\partial \phi}{\partial t} \right)^2 + \frac{1}{2} \left(\frac{\partial \phi}{\partial x} \right)^2, \quad \left[\phi(x), \frac{\partial}{\partial t} \phi(y) \right] = i\delta(x-y). \tag{2.7}$$

We can again decompose it into the left- and right-moving parts,

$$\begin{aligned}
 (\partial_t^2 - \partial_x^2)\phi &= (\partial_t + \partial_x)(\partial_t - \partial_x)\phi, \\
 \phi(x, t) &= \phi_L(x+t) + \phi_R(x-t), \\
 (\partial_t - \partial_x)\phi_L &\equiv \partial_- \phi_L = 0, \quad \partial_+ \phi_R = 0, \\
 H &= \frac{1}{4}(\partial_- \phi)^2 + \frac{1}{4}(\partial_+ \phi)^2 = \frac{1}{4}(\partial_- \phi_R)^2 + \frac{1}{4}(\partial_+ \phi_L)^2.
 \end{aligned} \tag{2.8}$$

Consider the Hamiltonian density for a left-moving boson field:

$$\mathcal{H} = \frac{1}{4}(\partial_+ \phi_L)^2,$$

$$[\partial_+ \phi_L(x), \partial_+ \phi_L(y)] = [\dot{\phi} + \phi', \dot{\phi} + \phi'] = 2i \frac{d}{dx} \delta(x - y). \quad (2.9)$$

Comparing to the Fermionic case, we see that:

$$J_L = \sqrt{\pi} \partial_+ \phi_L = \sqrt{\pi} \partial_+ \phi \quad (2.10)$$

since the commutation relations and Hamiltonian are the same. That means the operators are the same with appropriate boundary conditions.

Let us compare the spectra. For the Fermionic case, choose boundary condition:

$$\psi(l) = -\psi(-l) \quad (\text{i.e. } \psi_L(l) + \psi_R(l) = 0), \quad k = \frac{\pi}{l}(n + \frac{1}{2}), \quad n = 0, \pm 1, \pm 2, \dots \quad (2.11)$$

[See figure 11. Note that we have shifted k by k_F .] Consider the minimum energy state of charge Q (relative to the ground state). See figure 11. We have the single Fermion energy:

$$E = v_F k, \quad (2.12)$$

so:

$$E(Q) = v_F \frac{\pi}{l} \sum_{n=0}^{Q-1} (n + \frac{1}{2}) = \frac{v_F \pi}{2l} Q^2. \quad (2.13)$$

Now consider particle hole excitations relative to the Q -ground state: The most general particle-hole excitation is obtained by raising n_m electron by m level, then n_{m-1} electrons by $m-1$ levels, etc. [See figure 12.]

$$E = \frac{\pi v_F}{l} \left(\frac{1}{2} Q^2 + \sum_{m=1}^{\infty} n_m \cdot m \right). \quad (2.14)$$

Now consider the bosonic spectrum. What are the boundary conditions? Try the periodic one,

$$\phi(l) = \phi(-l) \Rightarrow k = \frac{\pi m}{l}. \quad (2.15)$$

The m^{th} single particle level has $E_m = v_F k_m$. The total energy is

$$E = \frac{\pi v_F}{l} \left(\sum_1^{\infty} n_m \cdot m \right), \quad n_m = \text{occupation number : } 0, 1, 2, \dots \quad (2.16)$$

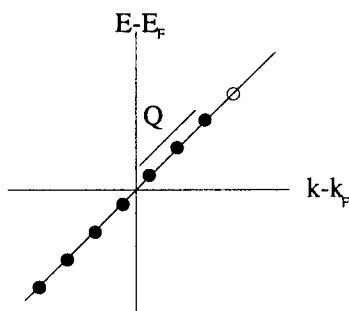


Fig. 11. A state with Q extra electrons added to the groundstate.

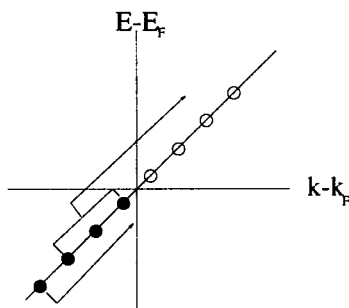


Fig. 12. A particle-hole excitation in which three electrons are raised four levels and then one electron is raised three levels.

Where does the Q^2 term in Eq. (2.14) come from? We need more general boundary condition on the boson field. Let ϕ be an angular variable:

$$\begin{aligned} \phi_L(-l) &= \phi_L(l) + \sqrt{\pi}Q, \quad Q = 0, \pm 1, \pm 2, \dots \\ \Rightarrow \phi_L(x+t) &= \frac{\sqrt{\pi}}{2} \frac{Q}{l} \cdot (x+t) + \sum_{m=1}^{\infty} \frac{1}{\sqrt{4\pi m}} \left(\exp(-i\frac{\pi m}{l}(x+t))a_m + \text{h.c.} \right), \end{aligned} \quad (2.17)$$

where a_n 's are the annihilation operators and Q is the winding number,

$$E = \int_{-l}^l dx \left[\frac{1}{2} \left(\frac{\partial \phi}{\partial t} \right)^2 + \frac{1}{2} \left(\frac{\partial \phi}{\partial x} \right)^2 \right] = \frac{\pi}{l} \left[\frac{1}{2} Q^2 + \dots \right]. \quad (2.18)$$

Here we have set $v_F = 1$. We have the following correspondence:

soliton $\leftrightarrow$ electron, oscillator $\leftrightarrow$ particle-hole pair.

It is also possible to represent fermion operators in terms of the boson,

$$\psi_L \sim e^{i\sqrt{4\pi}\phi_L}, \quad (2.19)$$

which gives the correct Green's function and implies the same angular definition of ϕ_L .

For the Kondo effect we are also interested in the phase-shifted boundary condition: [See figure 6.]

$$\begin{aligned}\psi_L(l) &= +\psi_L(-l), & k &= \frac{\pi}{l}n, \quad (\text{for fermions}) \\ E &= \frac{\pi v_F}{l} \left[\frac{Q(Q-1)}{2} + \sum_1^\infty n_m m \right].\end{aligned}\quad (2.20)$$

We have the degenerate ground state, $Q = 0$ or 1 , which correspond to anti-periodic boundary condition on ϕ ,

$$\begin{aligned}\phi(l) &= \phi(-l) + \sqrt{\pi}(Q - \frac{1}{2}) \\ E &= \frac{\pi}{l} \frac{1}{2} (Q - \frac{1}{2})^2 + \dots = \frac{\pi}{l} (\frac{1}{2}Q(Q-1) + \text{const.} + \dots).\end{aligned}\quad (2.21)$$

Now we include spin, *i.e.* we have 2-component electrons,

$$H_0 = iv_F \psi^{\alpha\dagger} \frac{d}{dx} \psi_\alpha, \quad (\alpha = 1, 2, \text{ summed}). \quad (2.22)$$

Now we have charge and spin currents (or densities). We can write H in a manifestly $SU(2)$ invariant way, quadratic in charge and spin currents:

$$J =: \psi^{\alpha\dagger} \psi_\alpha :, \quad \vec{J} = \psi^{\dagger\alpha} \frac{\vec{\sigma}_\alpha^\beta}{2} \psi_\beta. \quad (2.23)$$

Using:

$$\vec{\sigma}_\alpha^\beta \cdot \vec{\sigma}_\gamma^\delta = 2\delta_\gamma^\beta \delta_\alpha^\delta - \delta_\beta^\alpha \delta_\delta^\gamma \quad (2.24)$$

$$\vec{J}^2 = -\frac{3}{4} : \psi^{\dagger\alpha} \psi_\alpha \psi^{\dagger\beta} \psi_\beta : + \frac{3i}{2} \psi^{\alpha+} \frac{d}{dx} \psi_\alpha + \text{c-number},$$

$$J^2 =: \psi^{\dagger\alpha} \psi_\alpha \psi^{\dagger\beta} \psi_\beta : + 2i \psi^{\alpha+} \frac{d}{dx} \psi_\alpha + \text{c-number},$$

$$\mathcal{H} = \frac{1}{8\pi} J^2 + \frac{1}{6\pi} \vec{J}^2, \quad (2.25)$$

we have the following commutation relations,

$$\begin{aligned}[J(x), J(y)] &= 4\pi i \delta'(x-y), \quad (\text{twice the result for the spinless case}) \\ [J(x), J^z(y)] &= \frac{1}{2} [J_\uparrow + J_\downarrow, J_\uparrow - J_\downarrow] = 0.\end{aligned}\quad (2.26)$$

From $[J, \vec{J}] = 0$, we see that H is sum of commuting charge and spin parts.

$$\begin{aligned} [J^a(x), J^b(y)] &= 2\pi\psi^\dagger \left[\frac{\sigma^a}{2}, \frac{\sigma^b}{2} \right] \psi \cdot \delta(x-y) + \text{Tr} \left[\frac{\sigma^a}{2}, \frac{\sigma^b}{2} \right] 2\pi i \frac{d}{dx} \delta(x-y) \\ &= 2\pi i \varepsilon^{abc} J^c(x) \cdot \delta(x-y) + \pi i \delta^{ab} \frac{d}{dx} \delta(x-y). \end{aligned} \quad (2.27)$$

We obtain the Kac-Moody algebra of central charge $k = 1$. More generally the coefficient of the second term is multiplied by an integer k . Fourier transforming,

$$\begin{aligned} \vec{J}_n &\equiv \frac{1}{2\pi} \int_{-l}^l dx \exp\left(in \frac{\pi}{l} x\right) \vec{J}(x), \\ [J_n^a, J_m^b] &= i \varepsilon^{abc} J_{n+m}^c + \frac{1}{2} n \delta^{ab} \delta_{n,-m}, \end{aligned} \quad (2.28)$$

we have an ∞ -dimensional generalization of the ordinary $\text{SU}(2)$ Lie algebra. The spin part of the Hamiltonian is

$$H_s = \frac{\pi}{l} \frac{1}{3} \sum_{n=-\infty}^{\infty} : \vec{J}_{-n} \cdot \vec{J}_n : .$$

The spectrum of H_s is again determined by the algebra obeyed by the $\vec{J}_n$'s together with boundary conditions. The construction is similar to building representations of $\text{SU}(2)$ from commutation relations, *i.e.* constructing raising operator, *etc.*

In the $k = 1$ case we are considering here it is simplest to use:

$$\begin{aligned} \vec{J}(x)^2 &= 3(J^z(x))^2 \\ \mathcal{H} &= \frac{1}{8\pi} J^2 + \frac{1}{2\pi} (J^z)^2 \\ &= \frac{1}{4\pi} (J_\uparrow^2 + J_\downarrow^2) \\ &= \frac{1}{4} ((\partial_+ \phi_\uparrow)^2 + (\partial_+ \phi_\downarrow)^2) \\ &= \frac{1}{4} [(\partial_+ (\frac{\phi_\uparrow + \phi_\downarrow}{\sqrt{2}}))^2 + (\partial_+ (\frac{\phi_\uparrow - \phi_\downarrow}{\sqrt{2}}))^2] \\ &= \frac{1}{4} ((\partial_+ \phi_c)^2 + (\partial_+ \phi_s)^2). \end{aligned} \quad (2.30)$$

Now we have introduced two commuting charge and spin free massless bosons. $\text{SU}(2)$ symmetry is now concealed but boundary condition on ϕ_s

must respect it. Consider the spectrum of fermion theory with boundary condition: $\psi(l) = -\psi(-l)$,

$$E = \frac{\pi V}{l} \left[\frac{Q_{\uparrow}^2}{2} + \frac{Q_{\downarrow}^2}{2} + \sum_{m=-\infty}^{\infty} m(n_m^{\uparrow} + n_m^{\downarrow}) \right]. \quad (2.31)$$

Change over to ϕ_c and ϕ_s . We can relabel occupation numbers,

$$\begin{aligned} n_m^{\uparrow}, n_m^{\downarrow} &\longrightarrow n_m^c, n_m^s \\ Q &= Q_{\uparrow} + Q_{\downarrow} \\ S^z &= \frac{1}{2}(Q_{\uparrow} - Q_{\downarrow}) \\ E &= \frac{\pi v_F}{l} \left[\frac{1}{4}Q^2 + (S^z)^2 + \sum_1^{\infty} mn_m^c + \sum_1^{\infty} mn_m^s \right] \\ &= E_c + E_s \\ \phi_c &= \frac{\sqrt{\pi}}{2\sqrt{2}} \frac{Q}{l} (x+t) + \dots \\ \phi_s &= \frac{\pi}{\sqrt{2}} \frac{S^z}{l} (x+t) + \dots \end{aligned} \quad (2.32)$$

Actually charge and spin bosons are not completely decoupled; we must require $Q = 2S^z \pmod{2}$, to correctly reproduce the free fermion spectrum. We see that the boundary conditions on ϕ_c and ϕ_s are coupled. Now consider the phase-shifted case.

$$E = \frac{\pi v_F}{l} \left[\frac{1}{4}(Q-1)^2 + (S^z)^2 + \dots \right]. \quad (2.34)$$

Redefine $Q-1 \rightarrow Q$ so

$$E = \frac{\pi v_F}{l} \left[\frac{1}{4}Q^2 + (S^z)^2 + \dots \right], \quad (2.35)$$

the same as before the phase shift, Eq. (2.32). One of the 0-energy single-particle states is filled, for $Q = 0$ and there are 4 groundstates,

$$(Q, S^z) = (0, \pm \frac{1}{2}), (\pm 1, 0). \quad (2.36)$$

Now $Q = 2S^z + 1 \pmod{2}$; *i.e.* we “glue” together charge and spin excitations in two different ways, either

$$\begin{aligned} &(\text{even, integer}) \oplus (\text{odd, half-integer}) \\ \text{or } &(\text{even, half-integer}) \oplus (\text{odd, integer}), \end{aligned} \quad (2.37)$$

depending on the boundary conditions. The $\pi/2$ phase shift simply reverses these “gluing conditions”.

The set of all integer spin states form a “conformal tower”. They can be constructed from the Kac-Moody algebra by applying the raising operators $\vec{J}_{-n}$ to the lowest (singlet) state, with all spacings $(\pi v_F)/l \cdot (\text{integer})$. Likewise for all half-integer spin states, $(s^z)^2 = (1/4) + \text{integer}$. Likewise for even and odd charge states. The KM algebra determines uniquely conformal towers but boundary conditions determine which conformal towers occur in the spectrum and in which spin-charge combinations.

3. Conformal field theory approach to the Kondo effect

The chiral one-dimensional Hamiltonian density of Eq. (1.18) and (1.13) is:

$$\mathcal{H} = \frac{d}{dx} \psi_{L\alpha} + \lambda \psi_L^\dagger \frac{\vec{\sigma}_\alpha^\beta}{2} \psi_{L\beta} \cdot \vec{S} \delta(x) \quad (\text{left-movers only}). \quad (3.1)$$

We rewrite it in terms of spin and charge currents only,

$$\mathcal{H} = \frac{1}{8\pi} J^2 + \frac{1}{6\pi} (\vec{J})^2 + \lambda \vec{J} \cdot \vec{S} \delta(x). \quad (3.2)$$

The Kondo interaction only involves spin fields only, not charge fields: $H = H_s + H_c$. Henceforth we only consider the spin part. In Fourier transformed form,

$$\begin{aligned} H_s &= \frac{\pi}{l} \left(\frac{1}{3} \sum_{n=-\infty}^{\infty} \vec{J}_{-n} \cdot \vec{J}_n + \lambda \sum_{n=-\infty}^{\infty} \vec{J}_n \cdot \vec{S} \right) \\ [J_n^a, J_m^b] &= i\epsilon^{abc} J_{n+m}^c + \frac{n}{2} \delta^{ab} \delta_{n,-m} \\ [S^a, S^b] &= i\epsilon^{abc} S^c \\ [S^a, J_n^b] &= 0. \end{aligned} \quad (3.3)$$

From calculating Green's functions for $\vec{J}(x)$ we could again reproduce perturbation theory $(d\lambda)/(d\ln D) = -\lambda^2 + \dots$, i.e. a small $\lambda > 0$ grows. What is the infrared stable fixed point? Consider $\lambda = 2/3$, where we may “complete the square”.

$$\begin{aligned} H &= \frac{\pi V}{3l} \sum_{n=-\infty}^{\infty} \left[(\vec{J}_{-n} + \vec{S}) \cdot (\vec{J}_n + \vec{S}) - \frac{3}{4} \right] \\ [J_n^a + S^a, J_m^b + S^b] &= i\epsilon^{abc} (J_{n+m}^c + S^c) + \frac{n}{2} \delta^{ab} \delta_{n,-m}. \end{aligned} \quad (3.4)$$

H is quadratic in the new currents, $\vec{\mathcal{J}}_n \equiv \vec{J}_n + \vec{S}$, which obey the same Kac-Moody algebra! What is the spectrum of $H(\lambda = 2/3)$? We must get back to Kac-Moody conformal towers for integer and half-integer spin. This follows from the KM algebra and the form of H (i.e. starting from the lowest state we produce the entire tower by applying the raising operators, $\vec{J}_{-n}$).

Thus we find that *the strong-coupling fixed point is the same as the weak-coupling fixed point*. However, the total spin operator is now $\vec{\mathcal{J}}_0 = \vec{J}_0 + \vec{S}$. We consider impurity spin magnitude, $s = 1/2$. Any integer-spin state becomes a 1/2-integer spin state and vice versa.

$$\text{integer} \leftrightarrow 1/2\text{-integer}.$$

Presumably $\lambda = 2/3$ is the strong coupling fixed point in this formulation of the problem. ∞ -coupling can become finite coupling under a redefinition, e.g.

$$\lambda_{\text{Lattice}} = \frac{\lambda_{\text{KM}}}{1 - \frac{3}{2}\lambda_{\text{KM}}}. \quad (3.6)$$

We expect the low-energy, large l spectrum to be KM conformal towers for any λ . The effect of the Kondo interaction is to interchange the two conformal towers, integer $\leftrightarrow$ 1/2-integer. See figure 13. This is equivalent to a $\pi/2$ phase-shift,

$$\begin{aligned} &(\text{even, integer}) \oplus (\text{odd, } \tfrac{1}{2}\text{-integer}) \\ &\quad \Updownarrow \\ &(\text{even, } \tfrac{1}{2}\text{-integer}) \oplus (\text{odd, integer}) \end{aligned} \quad (3.7)$$

3.1. Leading irrelevant operator: specific heat, susceptibility, Wilson ratio, resistivity at $T > 0$.

At the stable fixed point $\vec{S}$ has disappeared; i.e. it is absorbed into $\vec{\mathcal{J}}$,

$$\vec{\mathcal{J}}(x) = \vec{J}(x) + 2\pi\vec{S} \delta(x). \quad (3.8)$$

What interactions could be generated in H_{eff} ? These only involve $\vec{\mathcal{J}}$, not $\vec{S}$.

$$H_s = \frac{1}{6\pi} \vec{\mathcal{J}}(x)^2 + \lambda_1 \vec{\mathcal{J}}(0)^2 \delta(x). \quad (3.9)$$

This is the only dimension-2 rotationally invariant operator in the spin sector. We have succeeded in reducing two dimension-2 operators to one. The other one is the charge-operator $\lambda_2 J(0)^2 \delta(x)$, $\lambda_2 = 0$ because there

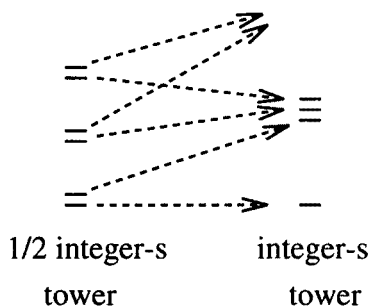


Fig. 13. At $\lambda = 2/3$ the $1/2$ -integer-spin conformal tower is mapped into the integer-spin conformal tower.

is no interaction in the charge sector (with other regularization we expect $\lambda_1 \sim (1/T_K)$, $\lambda_2 \sim (1/D) \ll \lambda_1$).

Now we calculate the specific heat and susceptibility to 1st order in λ_1 . Susceptibility of left-moving free fermions:

$$\begin{aligned}
 \text{0-th order} \quad M &= \frac{1}{2}(n_{\uparrow} - n_{\downarrow}) = l \int d\varepsilon \nu(\varepsilon) \left[n\left(\varepsilon + \frac{h}{2}\right) - n\left(\varepsilon - \frac{h}{2}\right) \right] \\
 \chi &= \frac{l}{2\pi} \quad (\text{for } T \ll D) \\
 \text{1-st order} \quad \chi &= \frac{1}{3T} \langle [\int dx \tilde{\mathcal{J}}(x)]^2 \rangle_{\lambda_1} \\
 &= \chi_0 - \frac{\lambda_1}{3T^2} \left\langle \left[\int dx \tilde{\mathcal{J}}(x) \right]^2 \tilde{\mathcal{J}}(0)^2 \right\rangle + \dots
 \end{aligned} \tag{3.10}$$

A simplifying trick is to replace:

$$\delta\mathcal{H} = \lambda_1 \tilde{\mathcal{J}}^2(0) \delta(x) \longrightarrow \frac{\lambda_1}{2l} \tilde{\mathcal{J}}^2(x), \tag{3.12}$$

which gives the same result to first order in λ (only) by translational invariance of H at $\lambda = 0$. Now the Hamiltonian density changes into

$$\mathcal{H} \rightarrow \left(\frac{1}{6\pi} + \frac{\lambda_1}{2l} \right) \tilde{\mathcal{J}}^2(x). \tag{3.12}$$

We simply rescale H by a factor

$$H \rightarrow \left(1 + \frac{3\pi\lambda_1}{l} \right) H. \tag{3.13}$$

Equivalently in a thermal average,

$$T \rightarrow \frac{T}{1 + \frac{3\pi\lambda_1}{l}} \equiv T(\lambda_1), \tag{3.14}$$

$$\begin{aligned}
 \chi(\lambda_1, T) &= \frac{1}{3T} \left\langle \left(\int \vec{\mathcal{J}} \right)^2 \right\rangle_{T(\lambda_1)} \\
 &= \frac{1}{1 + \frac{3\pi\lambda_1}{l}} \chi(0, T(\lambda_1)) \\
 &\approx \left[1 - \frac{3\pi\lambda_1}{l} \right] \chi_0 \\
 &= \frac{l}{2\pi} - \frac{3\lambda_1}{2}, \tag{3.15}
 \end{aligned}$$

where in the last equality the first term represents the bulk part and the second one, of order $\sim 1/T_K$, comes from the impurity part. Specific Heat:

$$\text{0-th order} \quad C = C_c + C_s, \quad C_c = C_s = \frac{\pi l T}{3}. \tag{3.16}$$

Each free left-moving boson makes an identical contribution.

$$\begin{aligned}
 \text{1st order in } \lambda_1 \quad C_s(\lambda_1, T) &= \frac{\partial}{\partial T} \langle H(\lambda_1) \rangle_{\lambda_1} \\
 &= C_s(0, T(\lambda_1)) \\
 &= \frac{\pi l}{3} \frac{T}{1 + \frac{3\pi\lambda_1}{l}} \\
 &\approx \frac{\pi l T}{3} - \pi^2 \lambda_1 T. \tag{3.17}
 \end{aligned}$$

$$\frac{\delta C_s}{C_s} = -\frac{3\pi\lambda_1}{l} = 2 \frac{\delta C_s}{C}. \tag{3.18}$$

The Wilson Ratio:

$$R_w \equiv \frac{\delta\chi}{\chi} \left(\frac{\delta C}{C} \right)^{-1} = 2 = \frac{C}{C_s} \tag{3.19}$$

measures the fraction of C coming from the spin degrees of freedom.

Doing more work, we can calculate the resistivity to $O(\lambda^2)$ [7, 15]. First we get the electron lifetime from the self-energy. The change in the 3D Green's function comes only from the 1D s-wave part:

$$\begin{aligned}
 &G_3(\vec{r}_1, \vec{r}_2) - G_3^0(|\vec{r}_1 - \vec{r}_2|) \\
 &= \frac{1}{8\pi^2 r_1 r_2} [e^{-ik_F(r_1+r_2)} (G_{\text{LR}}(r_1, r_2) - G_{\text{LR},0}(r_1, r_2)) + \text{h.c.}] \\
 &= G_3^0(r_1) \Sigma G_3^0(r_2). \tag{3.20}
 \end{aligned}$$

The self-energy Σ depends only on the frequency. It gets multiplied by the impurity concentration for a finite density (in the dilute limit). We must calculate the 1D Green's function $G_{\text{LR}}(r_1, r_2, \omega)$ perturbatively in λ

$$\begin{aligned} O(\lambda_1^0) : \quad G_{\text{LR}}(r_1, r_2) &= -G_{\text{LL}}^0(r_1, -r_2) \\ &= -G_{\text{LL}}^0(r_1 + r_2) \\ &= -G_{\text{LR}}^0(r_1, r_2), \end{aligned} \quad (3.21)$$

where the $(-)$ sign comes from the change in boundary conditions,

$$G_{\text{LR}} - G_{\text{LR}}^0 = -2G_{\text{LR}}^0 + O(\lambda_1). \quad (3.22)$$

To calculate to higher orders it is convenient to write the interaction as:

$$\vec{J}^2 = -\frac{3}{4} : \psi^\dagger \alpha \psi_\alpha \psi^\dagger \beta \psi_\beta : + \frac{3i}{4} \left(\psi^\dagger \alpha \frac{d}{dx} \psi_\alpha - \frac{d\psi^\dagger \alpha}{dx} \psi_\alpha \right). \quad (3.23)$$

To second order in λ_1 , we have the Feynman diagrams shown in figure 14, giving:

$$\Sigma^R(\omega) = \frac{-in}{2\pi\nu} \left[2 + 3\pi i \lambda_1 \omega - \frac{1}{2}(3\pi \lambda_1)^2 \omega^2 - \frac{1}{4}(3\pi \lambda_1)^2 (\omega^2 + \pi^2 T^2) \right]. \quad (3.24)$$

The first three terms give a phase-shift and the last term represents inelastic scattering.

$$\begin{aligned} \Sigma^R(\omega) &= \frac{-in_i}{2\pi\nu} [1 - e^{2i\delta(\omega)}] + \Sigma_{\text{inel}}^R(\omega) \\ \delta &= \frac{\pi}{2} + \frac{3\pi \lambda_1 \omega}{2} + \dots \\ \frac{1}{\tau} &= -2I_m \Sigma_R(\omega) \\ \frac{1}{\tau(\omega)} &= \frac{n_i}{\pi\nu} \left[2 - \frac{1}{2}(3\pi \lambda_1)^2 \omega^2 - \frac{1}{4}(3\pi \lambda_1)^2 (\omega^2 + \pi^2 T^2) \right]. \end{aligned} \quad (3.25)$$

The leading λ_1 dependence is $O(\lambda_1^2)$ in this case. The $O(\lambda_1)$ term in Σ^R is real. We calculate the conductivity from the Kubo formula. (There is no contribution from the scattering vertex for pure s-wave scattering.)

$$\begin{aligned} \sigma(T) &= \frac{2e^2}{3m^3} \int \frac{d^3 \vec{k}}{(2\pi)^3} \left[-\frac{\partial n}{\partial \varepsilon_k} \right] \vec{k}^2 \tau(\varepsilon_k), \\ \tau(\varepsilon_k) &\approx \frac{\pi\nu}{2n_i} \left[1 + \frac{1}{4}(3\pi \lambda_1^2) \varepsilon_k^2 + \frac{1}{8}(3\pi \lambda_1)^2 (\varepsilon_k^2 + (\pi^2 T^2)) \right], \\ \rho(T) &= \frac{1}{\sigma(T)} = \frac{3n_i}{\pi(e\nu_F\nu)^2} \left[1 - \frac{9}{4}\pi^4 \lambda_1^2 T^2 \right]. \end{aligned} \quad (3.26)$$

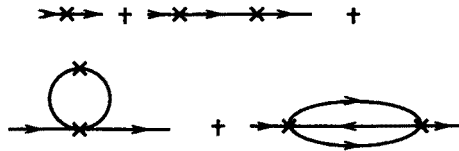


Fig. 14. Feynman diagrams contributing to the electron self-energy up to second order in the leading irrelevant coupling constant of Eq. (3.8).

All low-temperature properties are determined in terms of one unknown coupling constant $\lambda_1 \sim 1/T_K$. Numerical or Bethe ansatz methods are needed to find the precise value of $\lambda_1(D, \lambda) \propto (1/D) \exp(1/\lambda)$.

4. Multi-channel Kondo effect

Normally there are several “channels” of electrons — *e.g.* different *d*-shell orbitals. A very simple and symmetric model is:

$$H = \sum_{\vec{k}, \alpha, i=1,2,\dots,k} \varepsilon_{\vec{k}} \psi_{\vec{k}}^{\dagger \alpha i} \psi_{\vec{k} \alpha i} + \lambda \vec{S} \cdot \sum_{\vec{k}, \vec{k}', \alpha, \beta i} \psi_{\vec{k}}^{\dagger \alpha i} \vec{\sigma}_{\alpha}^{\beta} \psi_{\vec{k}' \beta i}. \quad (4.1)$$

This model has $SU(2) \times SU(k) \times U(1)$ symmetry. Realistic systems do not have this full symmetry. To understand the potential applicability of this model we need to analyse the relevance of various types of symmetry breaking [14]. An interesting possible experimental application of the model was proposed by Ralph, Ludwig, von Delft and Buhrman [19]. In general, we let the impurity have an arbitrary spin, *s*, as well.

Perturbation theory in λ is similar to the result mentioned before:

$$\frac{d\lambda}{d \ln D} = -\nu \lambda^2 + \frac{k}{2} \nu^2 \lambda^3 + O[k s(s+1) \lambda^4] \\ \vec{S}^2 = s(s+1). \quad (4.2)$$

Does $\lambda \rightarrow \infty$ as $T \rightarrow 0$? Let's suppose it does and check consistency. What is the groundstate for the lattice model of Eq. (1.19), generalized to arbitrary *k* and *s*, at $\lambda/t \rightarrow \infty$? In the limit we just consider the single-site model:

$$H = \lambda \vec{S} \cdot \psi_0^{\dagger} \frac{\vec{\sigma}}{2} \psi_0, \quad (4.3)$$

for $\lambda > 0$ (antiferromagnetic case) the minimum energy state has maximum spin for electrons at $\vec{0}$ *i.e.* spin = $k/2$. Coupling this spin- $k/2$ to a spin-*s*, we do not get a singlet if $s \neq k/2$, but rather an effective spin of size $|s - k/2|$.

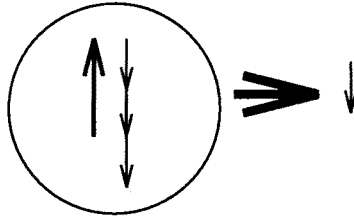


Fig. 15. Formation of an effective spin at strong Kondo coupling. $k = 3$, $s = 1$ and $s_{\text{eff}} = 1/2$.

[See figure 15.] The impurity is underscreened ($k/2 < s$) or overscreened ($k/2 > s$).

Now let $(t/\lambda) \ll 1$ be finite. Electrons on site ± 1 can exchange an electron with 0. This gives an effective Kondo interaction:

$$\lambda_{\text{eff}} \sim \frac{t^2}{\lambda} \ll 1.$$

See figure 16. What is the sign of λ_{eff} ? The coupling of the electron spins is antiferromagnetic: $\lambda_{\text{eff}} \vec{S}_{e1,0} \cdot \vec{S}_{e1,1}$, with $\lambda_{\text{eff}} > 0$ (as in the Hubbard model). But we must combine spins

$$\vec{S}_{\text{eff}} = \vec{S} + \vec{S}_{e1,0}. \quad (4.4)$$

For $(k/2) < s$, $\vec{S}_{\text{eff}} \parallel -\vec{S}_{e1,0}$ but, for $(k/2) > s$, $\vec{S}_{\text{eff}} \parallel +\vec{S}_{e1,0}$. So, ultimately, $\lambda_{\text{eff}} < 0$ in the underscreened case and $\lambda_{\text{eff}} > 0$ in the overscreened case. In the first (underscreened) case, the assumption $\lambda \rightarrow \infty$ was consistent since a ferromagnetic $\lambda_{\text{eff}} \rightarrow 0$ under renormalization and this implies $\lambda \rightarrow \infty$, since $\lambda_{\text{eff}} \sim -\frac{t^2}{\lambda}$. In this case we expect the infrared fixed point to correspond to a decoupled spin of size $s_{\text{eff}} = s - k/2$ and free electrons with a $\pi/2$ phase shift. In the second (overscreened) case the ∞ -coupling fixed point is not consistent. Hence the fixed point occurs at intermediate coupling: This fixed point does not correspond to a simple boundary condition on electrons, instead it is a Non-Fermi-Liquid Fixed Point. See figure 17.

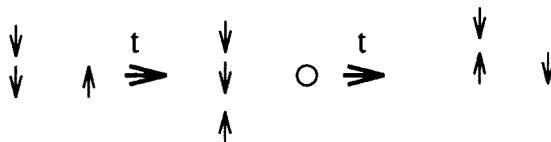


Fig. 16. Effective Kondo interaction of $O(t^2)$.

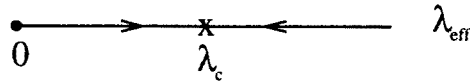


Fig. 17. RG flow of the Kondo coupling in the overscreened case.

For the $k = 2$, $s = 1/2$ case we may think of the electrons (one from each channel) in the first layer around the impurity as aligning antiferromagnetically with the impurity. This overscreens it, leaving an effective $s = 1/2$ impurity. The electrons in the next layer then overscreen this effective impurity, *etc.* At each stage we have an effective $s = 1/2$ impurity. [See figure 18.] Note in this special case that there is a duality between the weak and strong coupling unstable fixed points: they both contain an $s = 1/2$ impurity.

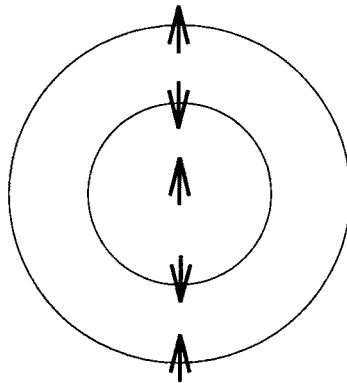


Fig. 18. The overscreened case with $s = 1/2$, $k = 2$.

4.1. Large k limit

The β -function is:

$$\beta = \lambda^2 - \frac{k}{2}\lambda^3 + O(\lambda^4). \tag{4.5}$$

If we only consider the first two terms, there is a fixed point at:

$$\lambda_c \approx \frac{2}{k}. \tag{4.6}$$

At this (small) value of λ the quartic term, and all higher terms are $O(1/k^4)$, whereas the quadratic and cubic terms are $O(1/k^2)$. Thus we may ignore

all terms but the quadratic and cubic ones, for large k . The slope of the β -function at the critical point is:

$$\left. \frac{d\beta}{dk} \right|_{\lambda_c} = 2\lambda_c - \frac{3}{2}\lambda_c^2 = -\frac{2}{k}. \quad (4.7)$$

This implies that the leading irrelevant coupling constant at the non-trivial (infrared) fixed point has dimension $2/k$ at large k , so that $(\lambda - \lambda_c)$ scales as $\Lambda^{2/k}$. Thus the leading irrelevant operator has dimension $(1 + 2/k)$. This is not an integer! This implies that this critical point is not a Fermi liquid.

4.2. Current algebra approach

We can gain some insight into the nature of the non-trivial critical point using the current algebra approach discussed in the previous section for the $k = 1$ case. It is now convenient to use a form of bosonization which separates spin, charge and *flavour* (*i.e.* channel) degrees of freedom. This representation is known as a conformal embedding. We introduce charge (J), spin ($\vec{J}$) and flavour (J^A) currents. A runs over the $k^2 - 1$ generators of $SU(k)$. The corresponding elements of the algebra are written T^A . These are traceless Hermitean matrices normalized so that:

$$\text{Tr } T^A T^B = \frac{1}{2} \delta^{AB}, \quad (4.8)$$

and obeying the completeness relation:

$$\sum_A (T^A)^b_a (T^A)^d_c = \frac{1}{2} \left[\delta^b_c \delta^d_a - \frac{1}{k} \delta^b_a \delta^d_c \right], \quad (4.9)$$

and the commutation relations:

$$[T^A, T^B] = i f^{ABC} T^C, \quad (4.10)$$

where the f^{ABC} are the $SU(k)$ structure constants. Thus the currents are:

$$J \equiv: \psi^{\dagger i\alpha} \psi_{i\alpha} :, \quad \vec{J} \equiv \psi^{\dagger i\alpha} \frac{\vec{\sigma}^{\beta}_{\alpha}}{2} \psi_{i\beta}, \quad J^A \equiv \psi^{\dagger i\alpha} (T^A)^j_i \psi_{j\alpha}. \quad (4.11)$$

(All repeated indices are summed.) It can be seen using Eq. (4.9) that the free fermion Hamiltonian can be written in terms of these currents as:

$$\mathcal{H} = \frac{1}{8\pi k} J^2 + \frac{1}{2\pi(k+2)} \vec{J}^2 + \frac{1}{2\pi(k+2)} J^A J^A. \quad (4.12)$$

The currents $\vec{J}$ obey the $SU(2)$ Kac-Moody algebra with central charge k and the currents J^A obey the $SU(k)$ Kac-Moody algebra with central charge 2:

$$[J_n^A, J_m^B] = if^{ABC} J_{n+m}^C + n\delta^{AB} \delta_{n,-m}. \tag{4.13}$$

The three types of currents commute with each of the other two types, as do the three parts of the Hamiltonian. Thus we have succeeded in expressing the Hamiltonian in terms of these three types of excitations: charge, spin and flavour. The Virasoro central charge c (proportional to the specific heat) for a Hamiltonian quadratic in currents of a general group G at level k is [26]:

$$c_{G,k} = \frac{\text{Dim}(G) \cdot k}{C_V(G) + k}, \tag{4.14}$$

where $\text{Dim}(G)$ is the dimension of the group and $C_V(G)$ is the quadratic Casimir in the fundamental representation. For $SU(k)$ this has the value:

$$C_V(SU(k)) = k. \tag{4.15}$$

Thus the total value of the central charge, c , is:

$$c_{\text{tot}} = 1 + \frac{3k}{k+2} + \frac{(k^2-1)2}{k+2} = 2k, \tag{4.16}$$

the correct value for $2k$ species of free fermions. Complicated “gluing conditions” must be imposed to correctly reproduce the free fermion spectra, with various boundary conditions. These were worked out in general by Altshuler, Bauer and Itzykson [27]. The $SU(2)_k$ sector consists of $k+1$ conformal towers, labelled by the spin of the lowest energy (“highest weight”) state: $s = 0, 1/2, 1, \dots, k/2$ [32, 33].

We may now treat the Kondo interaction much as in the single channel case. It only involves the spin sector which now becomes:

$$\mathcal{H}_s = \frac{1}{2\pi(k+2)} \vec{J}^2 + \lambda \vec{J} \cdot \vec{S} \delta(x). \tag{4.17}$$

We see that we can always “complete the square” at a special value of λ :

$$\lambda_c = \frac{2}{2+k}, \tag{4.18}$$

where the Hamiltonian reduces to its free form after a shift of the current operators by $\vec{S}$ which preserves the KM algebra. We note that at large k this special value of λ reduces to the one corresponding to the critical point: $\lambda_c \rightarrow 2/k$.

While this observation is tantalizing, it leaves many open questions. We might expect that some rearranging of the $(k+1)$ $SU(2)_k$ conformal towers takes place at the critical point but precisely what is it? Does it correspond to some sort of boundary condition? If so what? How can we calculate thermodynamic quantities and Green's functions? To answer these questions we need to understand some more technical aspects of CFT in the presence of boundaries.

5. Boundary conformal field theory

We will assume that the critical point corresponds to a conformally invariant boundary condition on the free theory. Using the general theory of conformally invariant boundary conditions developed by Cardy [28] we can completely solve for the critical properties of the model. Why assume that the critical point corresponds to such a boundary condition? It is convenient to work in the space-(imaginary) time picture. The impurity then sits at the boundary, $r = 0$ of the half-plane $r > 0$ on which the Kondo effect is defined. If we consider calculating a two-point Green's function when both points are taken very far from the boundary (with their separation held fixed) then we expect to obtain bulk behaviour, unaffected by the boundary. [See figure 19.] This, at long distances and times is the conformally invariant behaviour of the free fermion system. Very close to the boundary, we certainly do not expect the behaviour to be scale invariant (let alone conformally invariant) because various microscopic scales become important. The longest of these scales is presumably the Kondo scale, $\xi_K \approx v_F/T_L \approx a \exp(1/\nu\lambda)$. Beyond this distance, it is reasonable to expect scale-invariant behaviour. However, if the two points are far from each other compared to their distance from the boundary [figure 20] then the behaviour is still influenced by the boundary even when both points are far from it. We have a sort of boundary-dependent termination of the bulk conformally invariant behaviour. The dependence on the details of the boundary (such as the value of ξ_K) drops out. We may think of various types of boundaries as falling into universality classes, each corresponding to a type of conformally invariant behaviour. Rather remarkably, the above statements hold true whether we are dealing with a 2-dimensional classical statistical system with some boundary condition imposed, or dealing with a $(1+1)$ -dimensional quantum system with some dynamical degrees of freedom living on the boundary. In fact, we already saw an example of this in the single-channel Kondo problem. The dynamical impurity drops out of the description of the low-energy physics and is replaced by a simple, scale-invariant boundary condition, $\psi_L = -\psi_R$.

Precisely what is meant by a conformally invariant boundary condition? Without boundaries, conformal transformations are analytic mappings of

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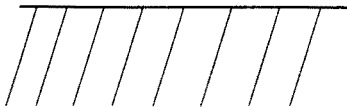


Fig. 19. The bulk limit.

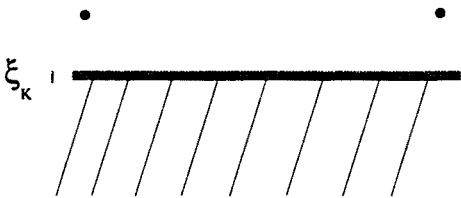


Fig. 20. The boundary limit.

the complex plane:

$$z \equiv \tau + i x , \tag{5.1}$$

into itself:

$$z \rightarrow w(z) . \tag{5.2}$$

(Henceforth, we set the Fermi velocity, $v_F = 1$.) We may Taylor expand an arbitrary conformal transformation around the origin:

$$w(z) = \sum_0^\infty a_n z^n , \tag{5.3}$$

where the a_n 's are arbitrary complex coefficients. They label the various generators of the conformal group. It is the fact that there is an infinite number of generators (*i.e.* coefficients) which makes conformal invariance so powerful in $(1 + 1)$ dimensions. Now suppose that we have a boundary at $x = 0$, the real axis. At best, we might hope to have invariance under all transformations which leave the boundary fixed. This implies the condition:

$$w(\tau)^* = w(\tau) . \tag{5.4}$$

We see that there is still an infinite number of generators, corresponding to the a_n 's of Eq. (5.3) except that now we must impose the conditions:

$$a_n^* = a_n . \tag{5.5}$$

We have reduced the (still ∞) number of generators by a factor of $1/2$. The fact that there is still an ∞ number of generators, even in the presence of a boundary, means that this boundary conformal symmetry remains extremely powerful.

To exploit this symmetry, following Cardy, it is very convenient to consider a conformally invariant system defined on a cylinder of circumference β in the τ -direction and length l in the x direction, with conformally invariant boundary conditions A and B at the two ends. [See figure 21.] From the quantum mechanical point of view, this corresponds to a finite temperature, $T = 1/\beta$. The partition function for this system is:

$$Z_{AB} = \text{Tr} \exp(-\beta H_{AB}^l), \quad (5.6)$$

where we are careful to label the Hamiltonian by the boundary conditions as well as the length of the spatial interval, both of which help to determine the spectrum. Alternatively, we may make a modular transformation, $\tau \leftrightarrow x$. Now the spatial interval, of length, β , is periodic. We write the corresponding Hamiltonian as H_P^β . The system propagates for a time interval l between initial and final states A and B . Thus we may equally well write:

$$Z_{AB} = \langle A | \exp(-l H_P^\beta) | B \rangle. \quad (5.7)$$

Equating these two expressions, Eqs (5.6) and (5.7) gives powerful constraints which allow us to determine the conformally invariant boundary conditions.

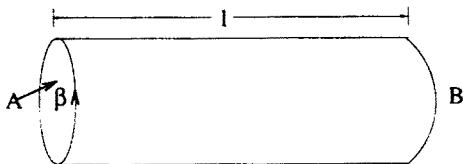


Fig. 21. Cylinder of length l , circumference β with boundary conditions A and B at the two ends.

To proceed, we make a further weak assumption about the boundary conditions of interest. We assume that the momentum density operator, $T - \bar{T}$ vanishes at the boundary. This amounts to a type of unitarity condition. In the free fermion theory this becomes:

$$\psi_L^{\dagger\alpha i} \psi_{L\alpha i}(t, 0) - \psi_R^{\dagger\alpha i} \psi_{R\alpha i}(t, 0) = 0. \quad (5.8)$$

Note that this is consistent with both boundary conditions that occurred in the one-channel Kondo problem: $\psi_L = \pm \psi_R$.

Since $T(t, x) = T(t + x)$ and $\bar{T}(t, x) = \bar{T}(t - x)$, it follows that

$$\bar{T}(t, x) = T(t, -x). \quad (5.9)$$

i.e. we may regard $\bar{T}$ as the analytic continuation of T to the negative axis. Thus, as in our previous discussion, instead of working with left and right movers on the half-line we may work with left-movers only on the entire line. Basically, the energy momentum density, T is unaware of the boundary condition. Hence, in calculating the spectrum of the system with boundary conditions A and B introduced above, we may regard the system as being defined periodically on a torus of length $2l$ with left-movers only. The conformal towers of T are unaffected by the boundary conditions, A , B . However, which conformal towers occur *does* depend on these boundary conditions. We introduce the characters of the Virasoro algebra, for the various conformal towers:

$$\chi_a \left(\exp \frac{-\pi\beta}{l} \right) \equiv \sum_i \exp(-\beta E_i^a(2l)), \quad (5.10)$$

where $E_i^a(2l)$ are the energies in the a^{th} conformal tower for length $2l$. *i.e.:*

$$E_i^a(2l) = \frac{\pi}{l} x_i^a - \frac{\pi c}{24l}, \quad (5.11)$$

where the x_i^a 's correspond to the (left) scaling dimensions of the operators in the theory and c is the conformal anomaly. The spectrum of H_{AB}^l can only consist of some combination of these conformal towers. *i.e.:*

$$Z_{AB} = \sum_a n_{AB}^a \chi_a \left(\exp \frac{-\pi\beta}{l} \right), \quad (5.12)$$

where the n_{AB}^a are some non-negative integers giving the multiplicity with which the various conformal towers occur. Importantly, only these multiplicities depend on the boundary conditions, not the characters, which are a property of the bulk left-moving system. Thus, a specification of all possible multiplicities, n_{AB}^a amounts to a specification of all possible boundary conditions A . The problem of specifying conformally invariant boundary conditions has been reduced to determining sets of integers, n_{AB}^a . For rational conformal field theories, where the number of conformal towers is finite, only a finite number of integers needs to be specified.

Now let us focus on the boundary states, $|A\rangle$. These must obey the operator condition:

$$[T(x) - \bar{T}(x)]|A\rangle = 0 \quad (\forall x). \quad (5.13)$$

Fourier transforming with respect to x , this becomes:

$$[L_n - \bar{L}_n]|A\rangle = 0. \quad (5.14)$$

This implies that all boundary states, $|A\rangle$ must be linear combinations of the “Ishibashi states” [29]:

$$|a\rangle \equiv \sum_m |a; m\rangle \otimes \overline{|a; m\rangle}. \quad (5.15)$$

Here m labels all states in the a^{th} conformal tower. The first and second factors in Eq. (5.15) refer to the left and right-moving sectors of the Hilbert Space. Thus we may write:

$$|A\rangle = \sum_a |a\rangle \langle a0|A\rangle. \quad (5.16)$$

Here,

$$|a0\rangle \equiv |a; 0\rangle \otimes \overline{|a; 0\rangle}. \quad (5.17)$$

(Note that while the states, $|a; m\rangle \otimes \overline{|b; n\rangle}$ form a complete orthonormal set, the Ishibashi states, $|a\rangle$ do not have finite norm.) Thus, specification of boundary states is reduced to determining the matrix elements, $\langle a0|A\rangle$. (For rational conformal field theories, there is a finite number of such matrix elements.) Thus the partition function becomes:

$$Z_{AB} = \sum_a \langle A|a0\rangle \langle a0|B\rangle \langle a| \exp(-lH_P^\beta) |a\rangle. \quad (5.18)$$

From the definition of the Ishibashi state, $|a\rangle$ we see that:

$$\langle a| \exp(-lH_P^\beta) |a\rangle = \sum_m \exp(-2lE_m^a(\beta)), \quad (5.19)$$

the factor of 2 in the exponent arising from the equal contribution to the energy from T and $\bar{T}$. This can be written in terms of the characters:

$$\langle a| \exp(-lH_P^\beta) |a\rangle = \chi_a \exp\left(\frac{-4\pi l}{\beta}\right). \quad (20)$$

We are now in a position to equate these two expressions for Z_{AB} :

$$Z_{AB} = \sum_a \langle A|a0\rangle \langle a0|B\rangle \chi_a \exp\left(\frac{-4\pi l}{\beta}\right) = \sum_a n_{AB}^a \chi_a \exp\left(\frac{-\pi\beta}{l}\right). \quad (5.21)$$

This equation must be true for all values of l/β . It is very convenient to use the modular transformation of the characters [30, 31]:

$$\chi_a \exp\left(\frac{-\pi\beta}{l}\right) = \sum_b S_a^b \chi_b \exp\left(\frac{-4\pi l}{\beta}\right). \quad (5.22)$$

Here S_b^a is known as the “modular S-matrix”. (This name is rather unfortunate since this matrix has no connection with the scattering-matrix.) We thus obtain a set of equations relating the multiplicities, n_{AB}^a which determine the spectrum for a pair of boundary conditions and the matrix elements $\langle a0|A\rangle$ determining the boundary states:

$$\sum_b S_b^a n_{AB}^b = \langle A|a0\rangle \langle a0|B\rangle. \quad (5.23)$$

We refer to these as Cardy’s equations. They basically allow a determination of the boundary states and spectrum.

How do we go about constructing boundary states and multiplicities which satisfy these equations? Generally, boundary states corresponding to trivial boundary conditions can be found by inspection. *I.e.*, given n_{AA}^b we can find $\langle a|A\rangle$. We can then generate new (sometimes non-trivial) boundary states by *fusion*. *I.e.* given any conformal tower, c , we can obtain a new boundary state $|B\rangle$ and new spectrum n_{AB}^a from the “fusion rule coefficients”, N_{ab}^c . These non-negative integers are defined by the operator product expansion (OPE) for (chiral) primary operators, ϕ_a . In general the (OPE) of ϕ_a with ϕ_b contains the operator ϕ_c N_{ab}^c times. In simple cases, such as occur in the Kondo problem, the N_{ab}^c ’s are all 0 or 1. In the case of $SU(2)_k$, which will be relevant for the Kondo problem, the OPE is [32, 33]:

$$j \otimes j' = |j - j'|, |j - j'| + 1, |j - j'| + 2, \dots \min\{j + j', k - j - j'\}. \quad (5.24)$$

Note that this generalizes the ordinary angular momentum addition rules in a way which is consistent with the conformal tower structure of the theories (*i.e.* the fact that primaries only exist with $j \leq k/2$). Thus,

$$N_{jj'}^{j''} = \begin{cases} 1 & (|j - j'| \leq j'' \leq \min\{j + j', k - j - j'\}) \\ 0 & \text{otherwise} \end{cases}. \quad (5.25)$$

The new boundary state, $|B\rangle$, and multiplicities obtained by fusion with the conformal tower c are given by:

$$\begin{aligned} \langle a0|B\rangle &= \langle a0|A\rangle \frac{S_c^a}{S_0^a} \\ n_{AB}^a &= \sum_b N_{bc}^a n_{AA}^b. \end{aligned} \quad (5.26)$$

Here 0 labels the conformal tower of the identity operator. Importantly, the new boundary state and multiplicites so obtained, obey Cardy's equation. The right-hand side of Eq. (5.23) becomes:

$$\langle A|a0\rangle\langle a0|B\rangle = \langle A|a0\rangle\langle a0|A\rangle\frac{S_c^a}{S_0^a}. \quad (5.27)$$

The left-hand side becomes:

$$\sum_b S_b^a n_{AB}^b = \sum_{b,d} S_b^a N_{dc}^b n_{AA}^d. \quad (5.28)$$

We now use a remarkable identity relating the modular S-matrix to the fusion rule coefficients, known as the Verlinde formula [34]:

$$\sum_b S_b^a N_{dc}^b = \frac{S_d^a S_c^a}{S_0^a}. \quad (5.29)$$

This gives:

$$\sum_b S_b^a n_{AB}^b = \frac{S_c^a}{S_0^a} \sum_d S_d^a n_{AA}^d = \frac{S_c^a}{S_0^a} \langle A|a0\rangle\langle a0|A\rangle = \langle A|a0\rangle\langle a0|B\rangle, \quad (5.30)$$

proving that fusion does indeed give a new solution of Cardy's equations. The multiplicites, n_{BB}^a are given by double fusion:

$$n_{BB}^a = \sum_{b,d} N_{bc}^a N_{dc}^b n_{AA}^d. \quad (5.31)$$

[Recall that $|B\rangle$ is obtained from $|A\rangle$ by fusion with the primary operator c .] It can be checked that the Cardy equation with $A = B$ is then obeyed. It is expected that, in general, we can generate a complete set of boundary states from an appropriate reference state by fusion with all possible conformal towers.

6. Boundary conformal field theory results on the multi-channel Kondo effect

6.1. Fusion and the finite-size spectrum

We are now in a position to bring to bear the full power of boundary conformal field theory on the Kondo problem. By the arguments at the beginning of Sec. 5, we expect that the infrared fixed points describing

the low- T properties of the Kondo Hamiltonian correspond to conformally invariant boundary conditions on free fermions. We might also expect that we could determine these boundary conditions and corresponding boundary states by fusion with appropriate operators beginning from some convenient, trivial, reference state.

We actually already saw a simple example of this in Sec. 3 in the single channel, $s = 1/2$, Kondo problem. There we observed that the free fermion spectrum, with convenient boundary conditions could be written:

$$(0, \text{even}) \oplus (\tfrac{1}{2}, \text{odd}). \quad (6.1)$$

Here 0 and $1/2$ label the $SU(2)_1$ KM conformal towers in the spin sector, while "even" and "odd" label the conformal towers in the charge sector. We argued that, after screening of the impurity spin, the infrared fixed point was described by free fermions with a $\pi/2$ phase shift, corresponding to a spectrum:

$$(\tfrac{1}{2}, \text{even}) \oplus (0, \text{odd}). \quad (6.2)$$

The change in the spectrum corresponds to the interchange of $SU(2)_1$ conformal towers:

$$0 \leftrightarrow \tfrac{1}{2}. \quad (6.3)$$

This indeed corresponds to fusion, with the spin- $1/2$ primary field of the WZW model. To see this note that the fusion rules for $SU(2)_1$ are simply [from Eq. (5.25)]:

$$0 \otimes \tfrac{1}{2} = \tfrac{1}{2} \quad \tfrac{1}{2} \otimes \tfrac{1}{2} = 0. \quad (6.4)$$

Thus for an $s = 1/2$ impurity, the infrared fixed point is given by fusion with the $j = 1/2$ primary. This is related to our completing the square argument. The new currents at the infrared fixed point, $\vec{J}$, are related to the old ones, $\vec{J}$, by:

$$\vec{J}_n = \vec{J}_n + \vec{S}. \quad (6.5)$$

If $\vec{J}$ and $\vec{J}$ were ordinary spin operators, then the new spectrum would be given by the ordinary angular momentum addition rules. In the case at hand, where $\vec{J}$ and $\vec{J}$ are KM current operators, it is plausible that the spectrum is given by fusion with the spin- s representation, generalizing the ordinary angular momentum addition rules in a way which is consistent with the structure of the KM CFT. In particular, Eq. (6.5) implies, for half-integer s , that states of integer total spin are mapped into states of half-integer total spin, and vice versa, a property which follows from fusion with $j = 1/2$.

This immediately suggests a way of determining the boundary condition for arbitrary number of channels, k and impurity spin magnitude, s :

fusion with spin- s . Actually, while this is possible for $s \leq k/2$, corresponding to exact or overscreening, it is not possible in the underscreened case since there is no spin- s primary with which to fuse for $s > k/2$. Instead, in the underscreened case, we assume fusion with the maximal possible spin, namely $k/2$. This seems to correspond to the (in this case stable) strong coupling fixed point described in Sec. 3. $k/2$ electrons partially screen the impurity. The fact that further screening is not possible is related to Fermi statistics. The maximal possible conduction electron spin state at the origin, for k channels is $k/2$. This is also essentially the reason why there are no primaries with larger spin, as can be seen from the corresponding bosonization of free fermions. We reiterate this essential point: *The infrared fixed point in the k -channel spin- s Kondo problem is given by fusion with the spin- s primary for $s \leq k/2$ or with the spin $k/2$ primary for $s > k/2$.* We have referred to this as the “fusion rules hypothesis”. If the general assumption that the infrared fixed point should be described by a conformally invariant boundary condition is accepted, then this hypothesis starts to seem very plausible. The general method for generating new boundary conditions is by fusion. Since the Kondo interaction appears entirely in the spin sector of the theory we should expect that the fusion occurs in that sector. The current redefinition $\vec{J} \rightarrow \vec{\tilde{J}}$ and various self-consistency checks all point towards this particular set of fusions.

An immediate way of checking the fusion rule hypothesis, and more generally the applicability of the boundary CFT framework to this problem, is to work out in detail the finite size spectrum for a few values of k and s and compare with spectra obtained by numerical methods.

Let us first consider the exactly screened and underscreened cases, $s \geq k/2$, where fusion occurs with the spin $k/2$ primary. In this case the fusion rules are simply:

$$j \otimes \frac{k}{2} = \frac{k}{2} - j. \quad (6.6)$$

Each conformal tower is mapped into a unique conformal tower. It can be shown that this gives the free fermion spectrum with a $\pi/2$ phase shift [10].

We demonstrate the case $k = 2$, $s = 1$ in Tables I and II. Let us start with antiperiodic boundary conditions in the left-moving formalism:

$$\psi_L(l) = -\psi_L(-l). \quad (6.7)$$

Let us express this free fermion spectrum, for 2 spin components and 2 channels, in terms of products of conformal towers in the charge, spin and flavour sectors. In this case, the flavour sector corresponds to $SU(2)_2$ as does the spin sector. We will refer to the corresponding quantum numbers as j for ordinary spin and j_f for flavour (or “pseudo-spin”). We need the energy of the “highest weight state” (i.e. groundstate) of each conformal tower. For

the Kac-Moody conformal towers, the highest weight state transforming under the representation R of the group G at level k has energy [26]:

$$E_R = \frac{\pi}{l} \frac{C_R}{k + C_A}, \quad (6.8)$$

where C_R is the quadratic Casimir in the R representation and A refers to the fundamental representation. For the case of $SU(2)$ the representations are labelled by their spin, j and the Casimirs are:

$$C_j = j(j+1). \quad (6.9)$$

We also need the energy for the charge sector. These can be worked out by generalizing the method used in the $k = 1$ case in Sec. 2. The energy for the lowest charge Q excitation, for each species of fermion is:

$$E = \frac{\pi}{l} \frac{Q^2}{2}, \quad (6.10)$$

as shown in Eq. (2.14). Altogether we obtain 4 terms like this for the 4 species of fermions (2 spin $\times$ 2 flavours). We can express the total energy in terms of the total charge:

$$Q \equiv Q_{11} + Q_{12} + Q_{21} + Q_{22}, \quad (6.11)$$

and various difference variables. This gives:

$$E = \frac{\pi}{l} \frac{Q^2}{8} + \dots \quad (6.12)$$

This gives the energy of the charge Q primary. In addition to the energy of the primary state we obtain additional terms in the energy corresponding to the excitation level in the charge, spin and flavour conformal towers: n_Q , n_s and n_f . These are non-negative integers. Altogether, we may write the energy of any state as:

$$E = \frac{\pi}{l} \left[\frac{Q^2}{8} + \frac{j(j+1)}{4} + \frac{j_f(j_f+1)}{4} + n_Q + n_s + n_f \right]. \quad (6.13)$$

For primary states, $n_Q = n_s = n_f = 0$. Q must be integer and j and j_f must be integer or half-integer. The allowed combinations of Q , j and j_f are what we refer to as "gluing conditions". They depend on the boundary conditions. For antiperiodic boundary conditions, the allowed fermion momenta are

$$k = \pi \frac{(n + \frac{1}{2})}{l}. \quad (6.14)$$

The corresponding energy levels are drawn in figure 5. Note that the ground-state is unique. It has $j = j_f = Q = 0$. Thus we must include the corresponding product of conformal towers in the spectrum. The single particle or single hole excitation has $j = j_f = 1/2$ and $Q = \pm 1$. The energy is:

$$E = \frac{\pi}{l} \left[\frac{1}{8} + \frac{3/4}{4} + \frac{3/4}{4} \right] = \frac{\pi}{l} \frac{1}{2}, \quad (6.15)$$

the right value. 2-particle excitations have $lE/\pi = 1$, $Q = 2$ and either have $j = 1, j_f = 0$ or $j = 0, j_f = 1$, due to Fermi statistics. Again the energy is given correctly by Eq. (6.13). The lowest energy particle-hole excitations, with $lE/\pi = 1$, have $Q = 0$ and various values of j and j_f . In particular, they can have $j = j_f = 1$. It turns out that these excitations are Kac-Moody primaries. Again the energy is given correctly by Eq. (6.13). It can be shown [27] that these are all conformal towers that occur, except that we must allow arbitrary values of Q , mod 4. These conformal towers are summarized in Table I.

TABLE I

Conformal towers appearing in the $k = 2$ free fermion spectrum with anti-periodic boundary conditions.

$Q \pmod{4}$	j	j_f	$(El/\pi)_{\min}$
0	0	0	0
0	1	1	1
± 1	$1/2$	$1/2$	$1/2$
2	0	1	1
2	1	0	1

Now consider fusion with the $j = 1$ primary. This has the effect of shuffling the spin conformal towers in the following way:

$$\begin{aligned} 0 &\rightarrow 1 \\ 1/2 &\rightarrow 1/2 \\ 1 &\rightarrow 0. \end{aligned} \quad (6.16)$$

The spectrum of Table I goes into that of Table II under this shuffling. It can be checked that this corresponds to free fermions with a $\pi/2$ phase shift, i.e. periodic boundary conditions, or a shift of the Fermi energy by $1/2$ a level spacing, drawn in figure 6. Now note that the groundstate is $2^4 = 16$ -fold degenerate, since the zero-energy level may be filled or empty for each species of fermion. The charge Q , in Table II is now measured relative to the symmetric case where 2 of these levels are filled and 2 are empty. Also note that if make the replacement:

$$Q \rightarrow Q - 2, \quad (6.17)$$

we get back the previous spectrum of Table I. Making this replacement in Eq. (6.13), we obtain:

$$\frac{El}{\pi} \rightarrow \frac{El}{\pi} + \frac{Q}{2} \tag{6.18}$$

(ignoring a constant). This corresponds to shifting the Fermi energy by 1/2-spacing; i.e. a $\pi/2$ phase shift.

TABLE II

Conformal towers appearing in the $k = 2$ free fermion spectrum with periodic boundary conditions, obtained from the Table I by fusion with $j = 1$.

$Q \pmod{4}$	j	j_f	$(El/\pi)_{\min}$
0	1	0	1/2
0	0	1	1/2
± 1	1/2	1/2	1/2
2	1	1	3/2
2	0	0	1/2

In the overscreened case the fusion rules are more interesting. They lead to spectra which cannot be obtained by applying any simple linear boundary conditions to the free fermions. Thus we may refer to these as non-Fermi liquid fixed points. It might be possible to find some kind of non-linear description of the boundary conditions in this case. But note that a non-linear boundary condition effectively introduces an interaction into the theory at the boundary. A boundary condition quadratic in the fermion fields might induce an additional condition quartic in fields, etc. Thus specification of non-linear boundary conditions could be very difficult. Cardy's formalism cleverly sidesteps this problem by the device of focussing on the boundary *states* instead of boundary conditions, and providing a method (fusion) for producing these boundary states. As was stated above, and we will continue to see in what follows, knowledge of the boundary states will determine all physical properties of the theory so nothing is lost by using this abstract description of the boundary condition.

For the $k = 2, s = 1/2$ example, the fusion rules give:

$$\begin{aligned} 0 &\rightarrow \frac{1}{2}, \\ \frac{1}{2} &\rightarrow 0 \oplus 1, \\ 1 &\rightarrow \frac{1}{2}. \end{aligned} \tag{6.19}$$

Now we get a larger number of conformal towers in the spectrum with this boundary condition, shown in Table III. We have shifted the groundstate energy to 0, in this Table. Note that energies $El/\pi = 1/8$ and $5/8$ now

occur. These do not correspond to any possible linear boundary conditions on free fermions. Note in particular that, due to particle-hole symmetry, only phase shifts of 0 or $\pi/2$ are allowed. These give half-integer energies, as we saw above and in Tables I and II.

TABLE III

Conformal towers appearing in the $k = 2$ free fermion spectrum after fusion with $j = 1/2$.

$Q \pmod{4}$	j	j_f	$(El/\pi)_{\min}$
0	1/2	0	0
0	1/2	1	1/2
± 1	0	1/2	1/8
± 1	1	1/2	5/8
2	1/2	0	1/2
2	1/2	1	1

This spectrum was compared with numerical work on the $k = 2$, $s = 1/2$ Kondo effect and the agreement was excellent (to within 5% for several of the lowest energy states) [14]. This provides evidence that the fusion rule hypothesis is correct in the overscreened case.

6.2. Impurity entropy

We define the impurity entropy as:

$$S_{\text{imp}}(T) \equiv \lim_{l \rightarrow \infty} [S(l, T) - S_0(l, T)], \quad (6.20)$$

where $S_0(l, T)$ is the free fermion entropy, proportional to l , in the absence of the impurity. We will find an interesting, non-zero value for $S_{\text{imp}}(0)$. Note that, for zero Kondo coupling, $S_{\text{imp}} = \ln[s(s+1)]$, simply reflecting the groundstate degeneracy of the free spin. In the case of exact screening, ($k = 2s$), $S_{\text{imp}}(0) = 0$. For underscreening,

$$S_{\text{imp}}(0) = \ln[s'(s'+1)], \quad (6.21)$$

where $s' \equiv s - k/2$. What happens for overscreening? Surprisingly, we will obtain, in general, the log of a non-integer, implying a sort of “non-integer groundstate degeneracy”.

To proceed, we show how to calculate $S_{\text{imp}}(0)$ from the boundary state. All calculations are done in the scaling limit, ignoring irrelevant operators, so that $S_{\text{imp}}(T)$ is a constant, independent of T , and characterizing the particular boundary condition. It is important, however, that we take the

limit $l \rightarrow \infty$ first, as specified in Eq. (6.20), at fixed, non-zero T . *I.e.* we are interested in the limit, $l/\beta \rightarrow \infty$. Thus it is convenient to use the first expression for the partition function, Z_{AB} in Eq. (5.7)

$$Z_{AB} = \sum_a \langle A|a0\rangle \langle a0|B\rangle \chi_a \exp\left(\frac{-4\pi l}{\beta}\right) \rightarrow \exp\left(\frac{\pi l c}{6\beta}\right) \langle A|00\rangle \langle 00|B\rangle. \quad (6.22)$$

Here $|00\rangle$ labels the groundstate in the conformal tower of the identity operator. c is the conformal anomaly. Thus the free energy is:

$$F_{AB} = -\frac{\pi c T^2 l}{6} - T \ln \langle A|00\rangle \langle 00|B\rangle. \quad (6.23)$$

The first term gives the specific heat:

$$C = \frac{\pi c T l}{3} \quad (6.24)$$

and the second gives the impurity entropy:

$$S_{\text{imp}} = \ln \langle A|00\rangle \langle 00|B\rangle. \quad (6.25)$$

This is a sum of contributions from the two boundaries,

$$S_{\text{imp}} = S_A + S_B. \quad (6.26)$$

Thus we see that the "groundstate degeneracy" g_A , associated with boundary condition A is:

$$\exp[S_{\text{imp}A}] = \langle A|00\rangle \equiv g_A. \quad (6.27)$$

Here we have used our freedom to choose the phase of the boundary state so that $g_A \geq 0$. For our original, anti-periodic, boundary condition, $g = 0$. For the Kondo problem we expect the low T impurity entropy to be given by the value at the infrared fixed point. Since this is obtained by fusion with the spin- s (or $k/2$) operator, we obtain from Eq. (5.26),

$$g = \frac{S_s^0}{S_0^0}. \quad (6.28)$$

The modular S-matrix for $SU(2)_k$ is [27, 30]:

$$S_{j',j}^j(k) = \sqrt{\frac{2}{2+k}} \sin \left[\frac{\pi(2j+1)(2j'+1)}{2+k} \right], \quad (6.29)$$

so

$$g(s, k) = \frac{\sin \frac{\pi(2s+1)}{(2+k)}}{\sin \frac{\pi}{(2+k)}}. \quad (6.30)$$

This formula agrees exactly with the Bethe ansatz result [36]. This formula has various interesting properties. Recall that in the case of exact or underscreening ($s \geq k/2$) we must replace s by $k/2$ in this formula, in which case it reduces to 1. Thus the groundstate degeneracy is 1 for exact screening. For underscreening we must multiply g by $(2s' + 1)$ to account for the decoupled, partially screened impurity. Note that, in the overscreened case, where $s < k/2$, we have:

$$\frac{1}{2+k} < \frac{2s+1}{2+k} < 1 - \frac{1}{2+k}, \quad (6.31)$$

so $g > 1$. In the case $k \rightarrow \infty$ with s held fixed, $g \rightarrow 2s + 1$, i.e. the entropy of the impurity spin is hardly reduced at all by the Kondo interaction, corresponding to the fact that the critical point occurs at weak coupling. In general, for underscreening:

$$1 < g < 2s + 1. \quad (6.32)$$

i.e. the free spin entropy is somewhat reduced, but not completely eliminated. Furthermore, g is not, in general, an integer. For instance, for $k = 2$ and $s = 1/2$, $g = \sqrt{2}$. Thus we may say that there is a non-integer "groundstate degeneracy". Note that in all cases the groundstate degeneracy is reduced under renormalization from the zero Kondo coupling fixed point to the infrared stable fixed point. This is a special case of what we believe to be a general result: *the groundstate degeneracy always decreases under renormalization*. This appears to be related to Zamolodchikov's c -theorem [37] which states that the conformal anomaly parameter, c , always decreases under renormalization. The intuitive explanation of the c -theorem is that, as we probe lower energy scales, degrees of freedom which appeared approximately massless start to exhibit a mass. This freezes out their contribution to the specific heat, the slope of which can be taken as the definition of c . In the case of the " g -theorem" the intuitive explanation is that, as we probe lower energy scales, approximately degenerate levels of impurities exhibit small splittings, reducing the degeneracy.

So far, only a perturbative proof of the g -theorem has been given [15]. It is completely analogous to a perturbative proof of the c -theorem given by Cardy and Ludwig [38], independently of Zamolodchikov's more general proof. For the g -theorem proof, we consider perturbing around a boundary

CFT fixed point with a barely relevant boundary operator. *I.e.* the action is:

$$S = S_0 - \lambda \int_0^\beta d\tau \phi(0, \tau), \quad (6.33)$$

where ϕ has dimension $1 - y$ with $0 < y \ll 1$. The β -function has the form:

$$\beta = y\lambda - b\lambda^2, \quad (6.34)$$

for some constant, b . There is a nearby fixed point at:

$$\lambda_c = \frac{y}{b}. \quad (6.35)$$

It is possible to calculate the small change in g using renormalization group improved perturbation theory. This gives:

$$\frac{\delta g}{g} = -\frac{\pi^2 y^3}{3b^2} < 0. \quad (6.36)$$

6.3. Boundary Green's functions: two-point functions, $T = 0$ resistivity

In this sub-section we explain the basic concepts for calculation of Green's functions in the presence of a conformally invariant boundary condition [35]. We then work out the case of two-point functions in detail. Finally we show how this gives information about the Kondo problem.

The most important point is the consequence of the identification of left- and right-moving sectors, discussed in Sec. 5. In general, in the bulk theory, a typical local operator is a product of left- and right-moving factors:

$$\phi(x) = \phi_L(x) \bar{\phi}_R(x). \quad (6.37)$$

Here x is the spatial co-ordinate; we suppress the time-dependence. However, in the presence of a boundary, we use:

$$\bar{\phi}_R(x) = \bar{\phi}_L(-x). \quad (6.38)$$

Thus a local operator with left and right-moving factors becomes a bilocal operator with only left-moving factors:

$$\phi(x) \rightarrow \phi_L(x) \bar{\phi}_L(-x). \quad (6.39)$$

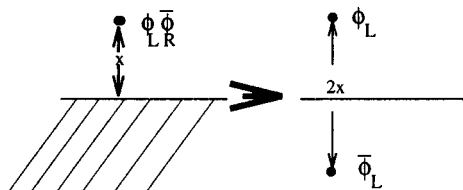


Fig. 22. A local operator becomes effectively bilocal in the presence of a boundary.

[See figure 22.] Thus a one-point function becomes a two-point function, two-point becomes four-point *etc.*

Henceforth, the number of points in the Green's function will refer to the larger number after this doubling due to the identification of left with right. In the remainder of this sub-section we show how to calculate boundary two-point functions. Four point functions are discussed in the next sub-section.

Our bulk operators are normally defined so that $\langle \phi(x) \rangle_{\text{bulk}} = 0$. For a semi-infinite plane with a boundary, this one-point function essentially becomes a two-point function which may have a non-zero value:

$$\langle \phi(x) \rangle_A = \langle \phi_L(x) \bar{\phi}_L(-x) \rangle_A = \frac{C_A}{(2x)^{2d}}. \quad (6.40)$$

Here d is the scaling dimension of ϕ_L , which does not depend on the boundary condition, A . On the other hand, the coefficient, C_A does depend on the boundary condition. Following Cardy and Lewellen [31], we may calculate C_A in terms of the boundary state $|A\rangle$. We assume that ϕ is a primary field.

This is done by making a conformal mapping from the semi-infinite cylinder to the semi-infinite plane:

$$z = i \tanh \frac{\pi w}{\beta}. \quad (6.41)$$

Writing:

$$z = \tau + ix, \quad w = \tau' + ix', \quad (6.42)$$

we see that, as x' goes from $-\beta/2$ to $\beta/2$, τ goes from $-\infty$ to ∞ . [See figure 23.] The semi-infinite plane is $x > 0$ and the semi-infinite cylinder is $\tau' > 0$. Note that we are regarding x' (space) as the periodic variable on the cylinder. *I.e.* we are calculating imaginary-time propagation from the boundary state $|A\rangle$. We will calculate the one-point function on the cylinder, first directly, then by obtaining it from the half-plane by conformal mapping. To obtain the correlation function on the *infinite* half-cylinder, it is convenient to take the limit of a finite cylinder, with boundary state $|00\rangle$

(the highest weight state of the identity conformal tower) at the other end. Thus, on the infinite cylinder,

$$\langle \phi(\tau', 0) \rangle_A = \lim_{T \rightarrow \infty} \frac{\langle 00 | \exp(-(T - \tau')H_P^\beta) \phi(0, 0) \exp(-\tau' H_P^\beta) | A \rangle}{\langle 00 | \exp(-TH_P^\beta) | A \rangle}. \quad (6.43)$$

Now we insert a complete set of states between ϕ and $|A\rangle$. Since ϕ is a primary field, $\phi|00\rangle$ gives only a sum of states in the conformal tower of ϕ . For convenience, we also consider the limit, $\tau' \rightarrow \infty$, so that only the highest weight state, $|\phi 0\rangle$ survives. Thus,

$$\langle \phi(\tau') \rangle_A \rightarrow_{\tau' \rightarrow \infty} \frac{\langle 00 | \phi | \phi 0 \rangle \langle \phi 0 | A \rangle}{\langle 00 | A \rangle} \exp\left(-\left(\frac{2\pi}{\beta}\right) 2d\tau'\right). \quad (6.44)$$

Note that

$$E_\phi = \left(\frac{2\pi}{\beta}\right) 2d. \quad (6.45)$$

We need the matrix element $\langle 00 | \phi | \phi 0 \rangle$, for the periodic Hamiltonian H_P^β . This can be obtained from the Green's function $\langle 00 | \phi(\tau'_1) \phi(\tau'_2) | 00 \rangle$, arising from a calculation on the infinite cylinder of radius β [31]. This can be obtained by a conformal mapping from the infinite plane, giving:

$$\begin{aligned} \langle 00 | \phi(\tau'_1) \phi(\tau'_2) | 00 \rangle &= \left[\frac{\beta}{\pi} \sinh \frac{\pi}{\beta} (\tau'_1 - \tau'_2) \right]^{-4d} \\ &\rightarrow_{\tau'_1 - \tau'_2 \rightarrow \infty} \left(\frac{2\pi}{\beta} \right)^{4d} \exp\left(-4\pi d \frac{(\tau'_1 - \tau'_2)}{\beta}\right) \\ &= |\langle 00 | \phi | \phi 0 \rangle|^2 \exp(-E_\phi(\tau'_1 - \tau'_2)). \end{aligned} \quad (6.46)$$

Thus we obtain:

$$\langle 00 | \phi | \phi \rangle = \left(\frac{2\pi}{\beta} \right)^{2d}.$$

Thus the desired one-point function, on the half-cylinder is:

$$\langle \phi(\tau') \rangle_A \rightarrow_{\tau' \rightarrow \infty} \left(\frac{2\pi}{\beta} \right)^{2d} \frac{\langle \phi 0 | A \rangle}{\langle 00 | A \rangle} e^{-(2\pi/\beta) 2d\tau'}. \quad (6.48)$$

Now let us repeat this calculation, by conformal transformation from the semi-infinite plane. As argued above, the result on the half-plane takes the form:

$$\langle \phi(x) \rangle_A = \frac{C_A}{(2x)^{2d}}. \quad (6.49)$$

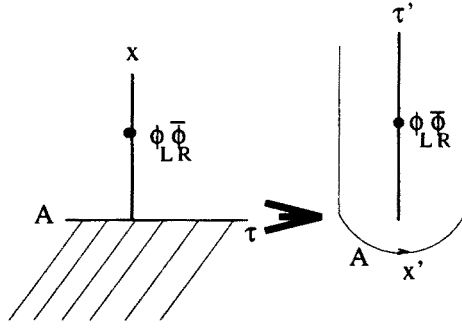


Fig. 23. Conformal mapping of a two-point function from the semi-infinite plane to the semi-infinite cylinder.

We now obtain the result on the half-cylinder by the conformal transformation of Eq. (6.41). This gives:

$$\begin{aligned} \langle \phi(\tau) \rangle_A^{1/2\text{-cylinder}} &= \left| \frac{dz}{dw} \right|^{2d} \langle \phi(x) \rangle_A^{\text{plane}} = \left(\frac{\pi}{\beta} \operatorname{sech}^2 \frac{\pi \tau'}{\beta} \right)^{2d} \frac{C_A}{(2x)^{2d}} \\ &= C_A \left(\frac{\pi}{\beta} \frac{\operatorname{sech}^2 \frac{\pi \tau'}{\beta}}{2 \tanh \frac{\pi \tau'}{\beta}} \right)^{2d} \rightarrow C_A \left(\frac{2\pi}{\beta} \exp \left(\frac{-2\pi \tau'}{\beta} \right) \right)^{2d}. \end{aligned} \quad (6.50)$$

Thus, equating Eqs (6.48) and (6.50), we finally obtain the desired formula for the coefficient C_A , defined in Eq. (6.40):

$$C_A = \frac{\langle \phi 0 | A \rangle}{\langle 00 | A \rangle}. \quad (6.51)$$

If $|A\rangle$ is obtained by fusion with primary operator c from some reference state $|F\rangle$ (standing for free), then:

$$C_A = C_A^{\text{free}} \frac{S_c^\phi}{S_0^\phi} \left(\frac{S_c^0}{S_0^0} \right)^{-1}. \quad (6.52)$$

Eq. (6.40) can be immediately generalized to the case where the left and right factors occur at different points in the upper half-plane:

$$\langle \phi_L(z) \bar{\phi}_R(z') \rangle_A = \frac{C_A}{[(\tau - \tau') + i(x + x')]^{2d}}. \quad (6.53)$$

An important application of this formula in the Kondo problem is to the single fermion Green's function, $\langle \psi_L(z)^\dagger \psi_R(z') \rangle$. In the case of periodic

boundary conditions:

$$\langle \psi_L^{\dagger i\alpha}(x) \psi_{Rj\beta}(x) \rangle_{\text{free}} = \langle \psi_L^{\dagger i\alpha}(x) \psi_{Lj\beta}(-x) \rangle = \frac{\delta_j^i \delta_\beta^\alpha}{2x}. \quad (6.54)$$

We may obtain the one-point function at the Kondo fixed point by fusion. The fermion operator can be written as a product of spin flavour and charge operators:

$$\psi_{Lj\alpha} \propto g_\alpha h_j \exp \left(i \sqrt{\frac{2\pi}{k}} \phi \right). \quad (6.55)$$

Here g and h are the left moving factors of the primary fields of the WZW models, transforming under the fundamental representation, *i.e.* g has $s = 1/2$. We may use Eq. (6.52). The Kondo boundary condition is obtained by fusion with the spin- s conformal tower, where s is the spin of the impurity. Thus:

$$\begin{aligned} \langle \psi_L^{\dagger i\alpha}(x) \psi_{Rj\beta}(x) \rangle_s^{\text{Kondo}} &= \frac{\delta_j^i \delta_\beta^\alpha S_s^{1/2}}{2x S_0^{1/2}} \left(\frac{S_s^0}{S_0^0} \right)^{-1} \\ &= \frac{\cos \frac{\pi(2s+1)}{(2+k)}}{\cos \frac{\pi}{(2+k)}} \frac{\delta_j^i \delta_\beta^\alpha}{2x}. \end{aligned} \quad (6.56)$$

Here we have used the $SU(2)$ modular S-matrix given in Eq. (6.30). There are several interesting points to notice about this formula. First of all, we see that for the case of exact or underscreening, where $s = k/2$, the cosine in the numerator in Eq. (6.56) becomes:

$$\cos \frac{\pi(k+1)}{(k+2)} = -\cos \frac{\pi}{(k+2)}. \quad (6.57)$$

Thus:

$$\langle \psi_L^{\dagger} \psi_R \rangle_{k/2}^{\text{Kondo}} = -\langle \psi_L^{\dagger} \psi_R \rangle_{\text{free}}. \quad (6.58)$$

As expected, this corresponds to a $\pi/2$ phase shift:

$$\psi_R(0) = e^{2i\delta} \psi_L(0) = -\psi_L(0). \quad (6.59)$$

In general, we may define a one-particle into one-particle S-matrix element, $S^{(1)}$ by:

$$\langle \psi_L^{\dagger} \psi_R \rangle = S^{(1)} \langle \psi_L^{\dagger} \psi_R \rangle_{\text{free}}, \quad (6.60)$$

with:

$$S^{(1)} = \frac{\cos \frac{\pi(2s+1)}{(2+k)}}{\cos \frac{\pi}{(2+k)}}. \quad (6.61)$$

We see that in the overscreened case, $|S^{(1)}| < 1$. $S^{(1)}$ is the matrix element, at the Fermi energy, for a single electron to scatter off the impurity into a single electron. In the case where $|S^{(1)}| = 1$, we see, by unitarity, that there is zero probability for a single electron to scatter into anything but a single electron at the Fermi energy, at zero temperature. This is precisely the starting point for Landau's Fermi liquid theory. Thus we refer to such cases as Fermi liquid boundary conditions. In the overscreened case, where $|S^{(1)}| < 1$, this inelastic scattering probability is non-zero so we have non-Fermi liquid boundary conditions. It is interesting to note that in the large k limit (with s held fixed), $S^{(1)} \rightarrow 1$, corresponding to the fixed point occurring at weak coupling. We also note that, for $k = 2$, $s = 1/2$, $S^{(1)} = 0$. This is related to the symmetry between the zero coupling ($S^{(1)} = 1$) and infinite coupling ($S^{(1)} = -1$) fixed point, mentioned above. In this case the one particle to one particle scattering rate vanishes!

From the electron self-energy we can obtain the lifetime and hence the resistivity for a dilute array of impurities by the Kubo formula [15] giving a $T = 0$ resistivity:

$$\rho(0) = \frac{3n_i}{k\pi(e\nu v_F)^2} \left[\frac{1 - S^{(1)}}{2} \right]. \quad (6.62)$$

Here n_i is the impurity density. The first factor is the "unitary limit"; *i.e.* this is the largest possible resistivity that can occur for a dilute array of non-magnetic impurities. It is only realized in the Kondo problem in the Fermi liquid case (exact or overscreened). Otherwise the resistivity is reduced by the factor $(1 - S^{(1)})/2$, which goes to zero for large k .

6.4. Four point boundary Green's functions, spin-density Green's function

In this sub-section we sketch how four-point functions of chiral operators (or two-point functions of non-chiral ones) are calculated in the presence of a boundary, using the particular example of the fermion four-point function with the Kondo boundary condition. For more details, see Ref. [16]. Similarly to the case of the two-point function, we will find that the four-point function is determined by bulk properties up to a constant which can be expressed in terms of matrix elements involving the boundary state.

We consider the Green's function:

$$G \equiv \langle \psi_{L\alpha i}(z_1) \psi_L^{\dagger\bar{\beta}\bar{j}}(z_2) \psi_{R\beta j}(z_3) \psi_R^{\dagger\bar{\alpha}\bar{i}}(z_4) \rangle. \quad (6.63)$$

We suppress spin and flavour indices in what follows, but it must be understood that we are dealing with tensors throughout. The first step is to regard the right-moving fields as reflected left-moving ones:

$$\psi_R(z) \rightarrow \psi_L(z^*). \quad (6.64)$$

Then we use Eq. (6.55) to express G as a product of charge, spin and flavour Green's functions:

$$G = G_c G_s G_f.$$

The form of G_c is unique up to a multiplicative constant, and is unaffected by the boundary. G_s and G_f are partially determined by the bulk conformal field theories. *I.e.* they must be solutions of the linear Knizhnik-Zamolodchikov (KZ) equations [26]. These equations have two solutions in both cases. We label them $G_s(p)$ and $G_f(q)$ where p and q take values 0 and 1. They are tensors in spin and flavour space; we suppress these indices. Thus we may schematically write the solution as:

$$G = \sum_{p,q=0}^1 a_{p,q} G_c G_s(p) G_f(q). \quad (6.66)$$

The four constants, $a_{p,q}$ remain to be determined. It turns out that three of them can be determined from general considerations, with only one ($a_{1,1}$) depending on the boundary conditions.

To see this, and to understand better how the solutions $G_s(p)$ and $G_f(q)$ are defined, it is convenient to consider the limit $z_1 \rightarrow z_2$ and $z_3 \rightarrow z_4$, illustrated in figure 24. In the limit we may use the operator product expansion of $\psi_L(z_1)$ with $\psi_L^\dagger(z_2)$. Since these points are at a fixed distance from the boundary as they approach each other, the bulk OPE applies. This is simply the trivial OPE of free fermions:

$$\psi(z_1)\psi^\dagger(z_2) \sim \frac{1}{z_1 - z_2} + J + \vec{J} + J^A + \mathcal{O}_s^{\text{ad}} \mathcal{O}_f^{\text{ad}} + \dots \quad (6.67)$$

Here we have used the fact that the spin and flavour currents are bilinear in the fermion fields. $\mathcal{O}_s^{\text{ad}}$ and $\mathcal{O}_f^{\text{ad}}$ are the primary fields in the adjointed representation of the spin and flavour groups. These have scaling dimension $2/(2+k)$ and $k/(2+k)$ respectively, so their product has scaling dimension 1 and corresponds to the product of fermion fields with spin and flavour traces subtracted. Note, importantly, that neither adjoint primary can appear by itself, since it has a fractional scaling dimension that does not occur in the free fermion OPE.

Now let us consider the bosonized expression for the fermion fields, of Eq. (6.55). Consider the OPE of each factor separately:

$$\begin{aligned} e^{i\sqrt{2\pi/k}\phi}(z_1)e^{-i\sqrt{2\pi/k}\phi}(z_2) &\sim (z_1 - z_2)^{-1/2k} + \dots \\ g(z_1)g^\dagger(z_2) &\sim (z_1 - z_2)^{-3/[2(2+k)]} + (z_1 - z_2)^{1/[2(2+k)]}\mathcal{O}_s^{\text{ad}} + \dots \\ h(z_1)h^\dagger(z_2) &\sim (z_1 - z_2)^{-(k^2-1)/[k(2+k)]} + (z_1 - z_2)^{1/[k(2+k)]}\mathcal{O}_f^{\text{ad}} + \dots \end{aligned} \quad (6.68)$$

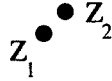


Fig. 24. The bulk limit, $z_1 \rightarrow z_2$, $z_3 \rightarrow z_4$.

A general solution of the KZ equations for the four-point spin and flavour Green's functions, G_s and G_f will have singularities corresponding to both the singlet and adjoint terms in the OPE shown above. The two independent solutions, $G_s(p)$ and $G_f(q)$ can be chosen so that only the singlet appears for $p = 0$ and $q = 0$ and only the adjoint appears for $p = 1$ and $q = 1$. It is now clear that, in order for G to reproduce free fermion behaviour in the limit $z_1 \rightarrow z_2$, we must demand that the coefficients $a_{1,0}$ and $a_{0,1}$ vanish in Eq. (6.66) so that the spin or flavour adjoint field doesn't occur without being multiplied by the other adjoint field. Similarly, requiring the correct normalization for the singlet-singlet singularity in this limit:

$$G \rightarrow \frac{1}{(z_1 - z_2)} \frac{1}{(z_3^* - z_4^*)}, \quad (6.69)$$

determines the non-zero value of the coefficient $a_{0,0}$. I.e. it has the same value as in the free fermion case, independent of the boundary conditions. Thus only $a_{1,1}$ remains to be determined by detailed consideration of the boundary condition.

From considering the limit $z_1 \rightarrow z_2$, $z_3 \rightarrow z_4$, we see that:

$$a_{1,1} \propto \langle [\mathcal{O}_s^{\text{ad}} \mathcal{O}_f^{\text{ad}}](z_1) [\mathcal{O}_s^{\text{ad}} \mathcal{O}_f^{\text{ad}}](z_3^*) \rangle_A. \quad (6.70)$$

This is a two-point Green's function with the two points straddling the boundary. Thus its normalization *does* depend on the particular boundary condition, A . In fact this normalization is precisely the coefficient C_A which was calculated in the previous sub-section, Eq. (6.52). Thus we obtain, upon identifying ϕ in Eq. (6.52) with the adjoint (spin 1) primary in the spin sector, for the Kondo boundary condition:

$$a_{1,1} = \frac{S_s^1}{S_0^1} \left(\frac{S_s^0}{S_0^0} \right)^{-1} a_{1,1}^{\text{free}}. \quad (6.71)$$

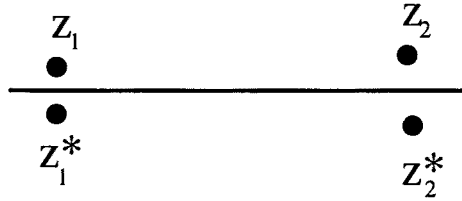


Fig. 25. The boundary limit $z_1 \rightarrow z_1^*$, $z_2 \rightarrow z_2^*$.

This result completes the determination of the four point function. The general form of G is rather complicated, involving hypergeometric functions. We only give explicit results here for the simplest non-trivial case, $k = 2$, $s = 1/2$. We consider only the most singular ($2k_F$ part) of the spin density Green's function. Writing:

$$S^a(\vec{r}, \tau) \equiv \exp(2ik_F r) \psi_L^\dagger \frac{\sigma^a}{2} \psi_R + \text{h.c.}, \quad (6.72)$$

this becomes:

$$\begin{aligned} \langle S^a(\vec{r}_1, \tau_1) S^b(\vec{r}_2, \tau_2) \rangle &= \frac{\delta^{ab}}{8\pi^4 r_1^2 r_2^2} \frac{\eta^{-1/2}}{|z_1 - z_2^*|^2} \\ &\times \left[2 \cos 2k_F(r_1 + r_2) + \left(2 + \frac{\eta}{1 - \eta} \right) \cos 2k_F(r_1 - r_2) \right] + \dots \end{aligned} \quad (6.73)$$

Here the ... represents less singular terms and η is the cross-ratio:

$$\eta \equiv -\frac{(z_1 - z_1^*)(z_2 - z_2^*)}{(z_1 - z_2^*)(z_2 - z_1^*)} = \frac{4r_1 r_2}{(\tau_1 - \tau_2)^2 + (r_1 + r_2)^2}. \quad (6.74)$$

Note that in the bulk limit, $|z_1 - z_2| \ll r_1, r_2$, [figure 24 with $z_1 = z_3$ and $z_2 = z_4$] $\eta \rightarrow 1$ and the Green's function reduces to its bulk form:

$$\langle S^a(\vec{r}_1, \tau_1) S^b(\vec{r}_2, \tau_2) \rangle \rightarrow \frac{\delta^{ab}}{8\pi^4 r_1^2 r_2^2} \frac{\cos 2k_F(r_1 - r_2)}{(\tau_1 - \tau_2)^2 + (r_1 - r_2)^2}, \quad (6.75)$$

the free fermion result. On the other hand, in the boundary limit, [figure 25] $r_1, r_2 \ll |\tau_1 - \tau_2|$, $\eta \rightarrow 0$ and

$$\langle S^a(\vec{r}_1, \tau_1) S^b(\vec{r}_2, \tau_2) \rangle \rightarrow \frac{\delta^{ab} \cos 2k_F r_1 \cos 2k_F r_2}{4\pi^4 (r_1 r_2)^{3/2} |\tau_1 - \tau_2|}. \quad (6.76)$$

The τ -dependence implies that S^a has scaling dimension $1/2$ at the boundary, although it has dimension 1 in the bulk, corresponding to the spin current operator.

This change at the boundary can be understood as follows. As explained in the previous sub-section, in the presence of a boundary, the spin density becomes a bilocal operator, at the points z and z^* . As $r \rightarrow 0$ we use the O.P.E. to express it as a sum of local operators. In general, any operator in the O.P.E. of the two left-moving operators $\mathcal{O}_L$ and $\bar{\mathcal{O}}_L$ may appear. The coefficients of the various terms in the O.P.E. (some of which may be zero) depend on the particular boundary condition. This is very natural since we are taking an O.P.E. for two points which straddle the boundary. [See figure 22.] In this case we are obtaining the spin adjoint primary field in the O.P.E., unaccompanied by the flavour adjoint, as it is in the bulk O.P.E. We give a general prescription for determining the boundary operator content in the next sub-section.

*6.5. Boundary operator content and leading irrelevant operator:
specific heat, susceptibility, Wilson ratio, resistivity at $T > 0$*

As we saw in Secs 1 and 3, the leading irrelevant operator plays a very important role in the Kondo problem, determining the low temperature behaviour of the specific heat, susceptibility and resistivity. One of the novel features of a non-Fermi liquid boundary condition is that boundary operators may appear which do not occur in the bulk theory. As explained in the previous sub-sections, this is a consequence of the bilocal nature of operators in the presence of a boundary and the fact that the O.P.E. coefficients depend on the particular boundary condition. In this sub-section we derive a general formula which gives all boundary operators that occur with a particular boundary condition. Then we analyse the effect of the leading irrelevant operator in the overscreened Kondo problem.

We can identify the boundary operator content from a general relationship between the finite-size spectrum and operator content. This is established by a conformal mapping from the semi-infinite plane to the infinite strip. We consider a correlation function for some primary operator, $\mathcal{O}$, on the semi-infinite plane:

$$\langle \mathcal{O}(\tau_1) \mathcal{O}^\dagger(\tau_2) \rangle_A. \quad (6.77)$$

[See figure 26.] Now we make the conformal mapping:

$$z = l e^{\pi w/l}. \quad (6.78)$$

Here w is on the strip:

$$-\frac{l}{2} < \text{Im } w < \frac{l}{2}. \quad (6.79)$$

We define:

$$z = \tau + ix, \quad w = u + iv. \quad (6.80)$$

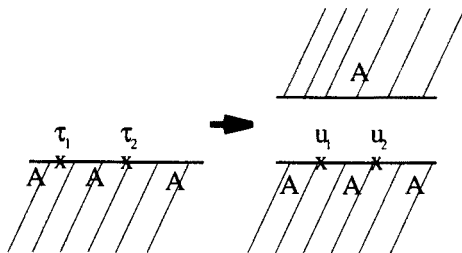


Fig. 26. Conformal mapping of a boundary two-point function from the semi-infinite plane to the finite strip.

It is convenient to assume $\tau_1, \tau_2 > 0$ so that $v_1 = v_2 = 0$, as shown in figure 26. The correlation function on the infinite plane has the form:

$$\langle \mathcal{O}(\tau_1) \mathcal{O}^\dagger(\tau_2) \rangle_A = \frac{1}{(\tau_1 - \tau_2)^{2x}}. \quad (6.81)$$

From this we obtain the correlation function on the strip:

$$\begin{aligned} \langle \mathcal{O}(u_1) \mathcal{O}^\dagger(u_2) \rangle_{AA} &= \left\{ \frac{\frac{\partial z}{\partial w}(u_1) \frac{\partial z}{\partial w}(u_2)}{[z(u_1) - z(u_2)]^2} \right\}^x \\ &= \left[\frac{2l}{\pi} \sinh \frac{\pi}{2l}(u_1 - u_2) \right]^{2x}. \end{aligned} \quad (6.82)$$

Here AA denotes the correlation function on the strip with boundary condition A on both sides. As $u_2 - u_1 \rightarrow \infty$ this approaches

$$\langle \mathcal{O}(\tau_1) \mathcal{O}^\dagger(\tau_2) \rangle_A \rightarrow \left(\frac{\pi}{l} \right)^{2x} \exp \left(-\pi x \frac{(u_2 - u_1)}{l} \right). \quad (6.83)$$

Alternatively, we may evaluate the correlation function on the strip by inserting a complete set of states:

$$\langle \mathcal{O}(u_1) \mathcal{O}^\dagger(u_2) \rangle_{AA} = \sum_n |\langle 0 | \mathcal{O} | n \rangle_{AA}|^2 e^{-E_n(u_2 - u_1)}. \quad (6.84)$$

Here we get the eigenstates on the strip with boundary conditions of type ' A ' on both sides. As $u_2 - u_1 \rightarrow \infty$, the lowest energy state created from the groundstate by $\mathcal{O}$ dominates. This is simply the highest weight state corresponding to the primary field $\mathcal{O}$. Clearly, the corresponding energy is:

$$E_n = \pi \frac{x}{l}. \quad (6.85)$$

Thus we see that the primary boundary operators with boundary condition A , are in one-to correspondence with the conformal towers appearing in the spectrum with boundary conditions AA .

Actually we have made an assumption here, that the operator $\mathcal{O}$ does not change the boundary condition. This appears to be reasonable for the Kondo problem, but in some situations, such as the X-ray edge singularity, it is necessary to consider more general boundary condition changing operators [39].

Thus to obtain the boundary operator content with a boundary condition A , we simply need to know the finite size spectrum with the boundary condition A at both sides. In the case of the Kondo boundary condition, this is obtained by “double fusion”. *I.e.* beginning with the free fermion spectrum, fusing once with the spin- s primary gives the spectrum with free fermion boundary conditions at one end and Kondo boundary conditions at the other. Fusing a second time with the spin- s primary gives the spectrum with Kondo boundary conditions at both ends, as mentioned in Sec. 3. The fact that double fusion gives the spectrum with Kondo boundary conditions at both ends can also be seen by a “completing the square” argument. *I.e.*, if we have Kondo impurities at $x = 0$ and $x = l$, then at a special value of both Kondo couplings, $\lambda_1 = \lambda_2 = 2/3$, the Hamiltonian returns to its free form if we redefine the $SU(2)$ currents by:

$$\vec{\mathcal{J}}_n \equiv \vec{J}_n + \vec{S}_1 + (-1)^n \vec{S}_2. \quad (6.86)$$

Double fusion represents a generalization of ordinary angular momentum addition rules, for the case of two impurity spins, which is consistent with the conformal tower structure of the WZW model. With free fermion boundary conditions, the boundary operators are simply the bulk free fermion operators. After double fusion, new boundary operators, not present in the free fermion theory, are generally present. However, an exception occurs in the case $s = k/2$ corresponding to exact or underscreening. In this case under double fusion the j th conformal tower is mapped into itself:

$$j \rightarrow k/2 - j \rightarrow j.$$

Thus the free fermion operator content is recovered, corresponding to a Fermi liquid boundary condition.

On the other hand, in the overscreened case, $k > 2s$, non-trivial operators are always obtained. In particular, the spin $j = 1$, charge $Q = 0$ flavour singlet operator, $\vec{\phi}$, always occurs. This follows from double fusion with the identity operator:

$$\begin{aligned} (Q = 0, j = 0, \text{ flavour singlet}) &\rightarrow (Q = 0, j = s, \text{ flavour singlet}) \\ &\rightarrow (Q = 0, j = 0, 1, \dots, \min(2s, k - 2s), \text{ flavour singlet}). \end{aligned} \quad (6.88)$$

For overscreening, $\min(2s, k - 2s) \geq 1$, so $\vec{\phi}$ (with $j = 1$) always occurs. $\vec{\phi}$ itself, cannot appear in the effective Hamiltonian, since it is not a spin singlet. However, its first descendent, $\vec{J}_{-1} \cdot \vec{\phi}$ can appear since it is invariant under all the symmetries of the underlying Kondo Hamiltonian. It has dimension,

$$x = 1 + \frac{2}{2+k}. \quad (6.89)$$

This dimension is > 1 meaning that the operator is irrelevant. On the other hand, the dimension is < 2 so it is more relevant than $\vec{J}^2$, the leading irrelevant operator in the Fermi liquid case. It can be seen to be the leading irrelevant operator in all overscreened cases [11].

We now discuss the calculation of various quantities to lowest non-vanishing order in the leading irrelevant term in the effective Hamiltonian:

$$\delta H = \lambda_1 \vec{J}_{-1} \cdot \vec{\phi}. \quad (6.90)$$

We begin with the specific heat. The simple change of variables trick that we used in Sec. 3 for the Fermi liquid case doesn't work here. In fact, since $\vec{J}_{-1} \cdot \vec{\phi}$ is a Virasoro primary operator (although a Kac-Moody descendant) its one-point function continues to vanish even at finite temperature. Consequently, the leading contribution to the specific heat is second order in λ_1 . The contribution to the impurity free energy of $O(\lambda_1^2)$ is:

$$-\beta f_{\text{imp}} = \frac{\lambda_1^2}{2} \int_{-\beta/2}^{\beta/2} d\tau_1 \int_{-\beta/2}^{\beta/2} d\tau_2 T \langle \vec{J}_{-1} \cdot \vec{\phi}(\tau_1) \vec{J}_{-1} \cdot \vec{\phi}(\tau_2) \rangle. \quad (6.91)$$

Since $\vec{J}_{-1} \cdot \vec{\phi}$ is a Virasoro primary, this two-point function is given by:

$$\langle \vec{J}_{-1} \cdot \vec{\phi}(\tau_1) \vec{J}_{-1} \cdot \vec{\phi}(\tau_2) \rangle = \frac{3(2+k/2)}{|\tau_1 - \tau_2|^{2(1+\Delta)}} \rightarrow \frac{3(2+k/2)}{|\frac{\beta}{\pi} \sin \frac{\pi}{\beta}(\tau_1 - \tau_2)|^{2(1+\Delta)}}. \quad (6.92)$$

Here

$$\Delta \equiv \frac{2}{2+k}, \quad (6.93)$$

and we have assumed that $\vec{\phi}$ has unit normalization:

$$\langle \phi^a(\tau_1) \phi^b(\tau_2) \rangle = \frac{\delta^{ab}}{|\tau_1 - \tau_2|^{2\Delta}}. \quad (6.94)$$

The normalization of $\vec{J}_{-1} \cdot \vec{\phi}$ can then be fixed using the Kac-Moody algebra. A naive rescaling argument implies that:

$$f_{\text{imp}} \propto T^{1+2\Delta}. \quad (6.95)$$

Actually the integral in Eq. (6.91) requires an ultraviolet cut-off. However, this only leads to additional terms in f_{imp} that vanish more rapidly as $T \rightarrow 0$. Evaluating the universal term in Eq. (6.91) explicitly, we obtain:

$$\begin{aligned} C_{\text{imp}} &= -T \frac{\partial^2 f_{\text{imp}}}{\partial T^2} \\ &= \lambda_1^2 \left[T^{2\Delta} \pi^{1+2\Delta} (2\Delta)^2 3 \left(\frac{k}{2} + 2 \right) \frac{1}{2} \frac{\Gamma(1/2 - \Delta) \Gamma(1/2)}{\Gamma(1 - \Delta)} + O(T) \right] \\ &\quad + O(\lambda_1^3 T^{3\Delta}) + \dots \\ &\quad (\text{for } k > 2). \end{aligned} \quad (6.96)$$

Here $\Gamma(x)$ is Euler's Gamma function. Note that this result is independent of the magnitude of the impurity spin, s . Also note that since, for $k > 2$, $\Delta = 2/(2 + k) < 1/2$, so the impurity specific heat is more singular than in the Fermi liquid case. In the particular case $k = 2$, where $\Delta = 1/2$, we obtain from Eq. (6.91):

$$C_{\text{imp}} = \lambda_1^2 \pi^2 9T \ln \left(\frac{T_K}{T} \right) + \dots \quad (6.97)$$

Note that this is only more singular by a log than in the Fermi liquid case. The impurity susceptibility is also second order in λ_1 ,

$$\chi_{\text{imp}} \propto \beta \lambda_1^2 \int d\tau_1 d\tau_2 dx_1 dx_2 \langle \vec{J}_{-1} \cdot \vec{\phi}(\tau_1) \vec{J}_{-1} \cdot \vec{\phi}(\tau_2) J^3(0, x_1) J^3(0, x_2) \rangle. \quad (6.98)$$

Again, by a scaling argument we see that:

$$\chi_{\text{imp}} \propto \lambda_1^2 T^{2\Delta-1}. \quad (6.99)$$

By evaluating the integral in Eq. (6.98) explicitly, we can calculate the Wilson ratio, R_W for the overscreened Kondo problem:

$$\frac{\chi_{\text{imp}} C_{\text{bulk}}}{C_{\text{imp}} \chi_{\text{bulk}}} \equiv R_W = \frac{(2 + k/2)(2 + k)^2}{18}. \quad (6.100)$$

We can calculate the Wilson ratio in the exactly screened, Fermi liquid, case, much more simply. In this case, the simple rescaling argument of Sec. 3 always works giving:

$$\frac{C_{\text{imp}}}{C_s} \left(\frac{\chi_{\text{imp}}}{\chi} \right)^{-1} = 1. \quad (6.101)$$

Here C_s is the part of the bulk specific heat coming from the spin degrees of freedom. Hence the Wilson ratio is given by

$$R_W = \frac{C}{C_s}. \quad (6.102)$$

It simply measures the ratio of the total bulk specific heat to that coming from the spin degrees of freedom. Using:

$$C = \frac{\pi}{3} c T, \quad (6.103)$$

where c is the conformal anomaly, and the fact that $c = 2k$ for $2k$ species of free fermions and $c = 3k/(2+k)$ for the level k $SU(2)$ WZW model, we obtain:

$$R_W = \frac{2(2+k)}{3}, \quad (6.104)$$

for the exactly screened Kondo problem.

Finally we consider the T -dependence of the impurity resistivity. For more details, see Ref. [15]. We may write the electron Green's function in the one-dimensional theory with the impurity at the origin in the form:

$$G(r_1, -r_2, \omega_n) - G_0(r_1 + r_2, \omega_n) = G_0(r_1, \omega_n) \Sigma_1(\omega_n) G_0(r_2, \omega_n). \quad (6.105)$$

In the three dimensional theory with a dilute random array of Kondo impurities, of density n_i the self-energy becomes:

$$\Sigma_3 = \frac{n_i}{\nu} \Sigma_1. \quad (6.106)$$

The electron life-time is given by:

$$\frac{1}{\tau} = -2 \operatorname{Im} \Sigma_3^R, \quad (6.107)$$

where the superscript R denotes the retarded Green's function, obtained by analytic continuation from the Matsubara Green's function. The conductivity is then expressed in terms of the lifetime in the standard way:

$$\sigma = \frac{2e^2}{3m^2} \int \frac{d^3 p}{(2\pi)^3} \left[\frac{-dn_F}{d\varepsilon_p} \right] \vec{p}^2 \tau(\varepsilon_p). \quad (6.108)$$

Thus, we just need to calculate the one-dimensional self-energy, Σ_1 . This was already done, at $T = 0$, in Subsection (VIC). There we showed that:

$$G = G_0 S^{(1)}, \quad (6.109)$$

where $S^{(1)}$ is the scattering matrix element for 1-electron to go into 1-electron, given explicitly in Eq. (6.61). This gave the resistivity:

$$\rho = \frac{1 - S^{(1)}}{2} \rho_u, \quad (6.110)$$

where ρ_u is the unitary limit resistivity. To obtain the leading T -dependence, we do perturbation theory in the leading irrelevant coupling constant, λ_1 . In this case, there is a non-zero first order contribution:

$$\delta \Sigma_1 \propto \lambda_1 \langle \psi \vec{J}_{-1} \cdot \vec{\phi} \psi^\dagger \rangle. \quad (6.111)$$

Since λ_1 has scaling dimension Δ , it follows that:

$$\rho = \rho_u \frac{1 - S^{(1)}}{2} \left[1 + \alpha \lambda_1 T^\Delta + \dots \right], \quad (6.112)$$

where α is a dimensionless constant which can be obtained by explicit evaluation of first order perturbation theory. After a rather long calculation, α can be expressed in terms of an integral over hypergeometric functions. We evaluated this integral numerically for the case $k = 2$, where $\Delta = 1/2$, obtaining:

$$\alpha = 4\sqrt{\pi}. \quad (6.113)$$

Thus we can form another universal ratio involving the square of the temperature-dependent term in the resistivity, and the specific heat coefficient.

Note that in the Fermi liquid case the resistivity had the form:

$$\rho = \rho_u [1 - \alpha \lambda_1^2 T^2]. \quad (6.114)$$

The leading T -dependence is second order in λ_1 and T . In the non-Fermi liquid case the sign of the leading T -dependence depends on the sign of the irrelevant coupling constant, λ_1 . As we increase the Kondo coupling constant, so as to pass through the non-trivial critical point, at Kondo couplings of $O(1)$, the leading irrelevant coupling constant, λ_1 , should change sign. Thus the sign of the leading T -dependent term in the resistivity actually switches at the critical point. We cannot determine the value of λ_1 from our methods. However, if we make the plausible assumption that $\lambda_1 < 0$ for weak Kondo coupling and hence $\lambda_1 > 0$ for strong Kondo coupling, then the resistivity behaves, at low T as shown in figure 27 (since the constant, $\alpha > 0$). For weak Kondo coupling the resistivity decreases with T , exhibiting a power law singularity at $T = 0$. On the other hand, for strong Kondo coupling, the resistivity initially *increases* with T [figure 27]. This is quite reasonable for very strong Kondo coupling where, at high T the

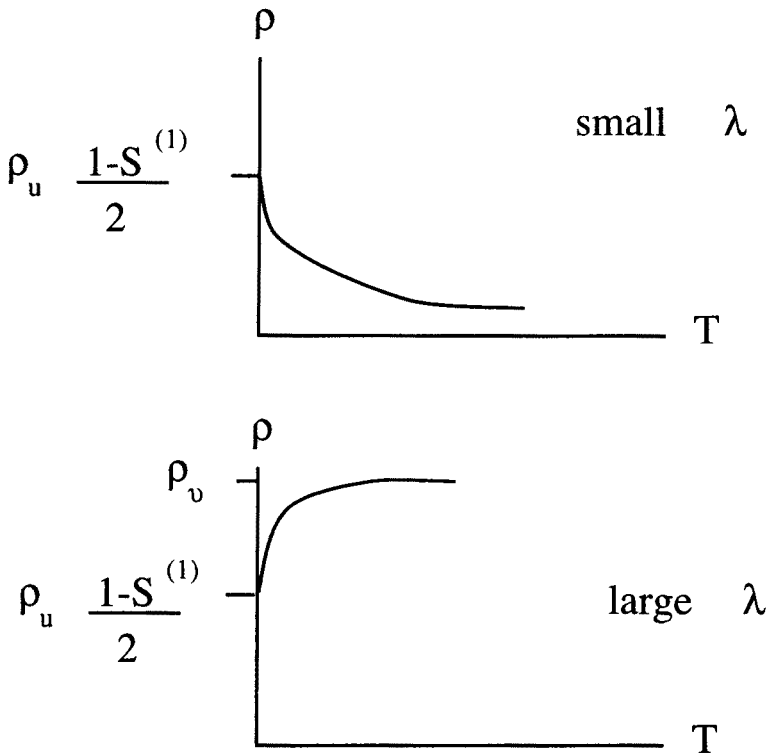


Fig. 27. Qualitative behaviour of the resistivity in the overscreened case.

effective renormalized Kondo coupling is close to the unstable fixed point at ∞ . Hence, we expect to obtain the unitary limit resistivity at high T in this case. This requires ρ to increase with T .

This $\sqrt{T}$ behaviour of the resistivity, and other scaling behaviour predicted by the present approach for the case $k = 2$, $s = 1/2$, were observed in recent experiments [19]. However, the interpretation of these experiments in terms of the 2-channel Kondo problem is controversial at present [40–42].

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