THE COMPOSITION FACTORS OF KAC MODULES OF $sl(M/N)$

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Throughout this article we adopt the notation and conventions of the article [1] entitled *Atypical modules of the Lie superalgebra* $gl(m/n)$ based on the Colloquium talk by Dr J. Van der Jeugt and published elsewhere in these Proceedings. The complex Lie superalgebra $sl(m/n)$ is the subalgebra of $gl(m/n)$ consisting of matrices $x = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$, with A, B, C and D matrices of size $m \times m, m \times n, n \times m$ and $n \times n$, respectively, with $\mathrm{str}\, x = \mathrm{tr}\, A - \mathrm{tr}\, D = 0$. $G = sl(m/n)$ admits a grading $G = G_{-1} \oplus G_0 \oplus G_1$ with $G_0 = sl(m) \oplus \mathrm{C} \oplus sl(n)$, the even subalgebra of $sl(m/n)$. The universal enveloping algebras of G, G_0 and G_{-1} are denoted by $U(G), U(G_0)$ and $U(G_{-1})$, respectively.

The weight space H^* is the dual of the Cartan subalgebra H of $sl(m/n)$. It is spanned by the forms ϵ_i $(i = 1, \dots, m)$ and δ_j $(j = 1, \dots, n)$, with $\sum_{i=1}^{m} \epsilon_i - \sum_{j=1}^{n} \delta_j = 0$, and is equipped with an inner product such that $\langle \epsilon_i | \epsilon_j \rangle = \delta_{ij}$, $\langle \epsilon_i | \delta_j \rangle = 0$, $\langle \delta_i | \delta_j \rangle = -\delta_{ij}$, where δ_{ij} is the usual Kronecker symbol. The set of simple roots is taken to be the distinguished set $\{\epsilon_i - \epsilon_{i+1}, i = 1, 2, \dots, m-1; \epsilon_m - \delta_1; \delta_j - \delta_{j+1}, j = 1, 2, \dots, n-1\}$, so that the sets of positive even and odd roots are given by $\Delta_0^+ = \{\epsilon_i - \epsilon_j, 1 \le i < j \le m; \delta_i - \delta_j, 1 \le i < j \le n\}$ and $\Delta_1^+ = \{\beta_{ij} = \epsilon_i - \delta_j, 1 \le i \le m, 1 \le j \le n\}$, respectively. It is convenient to define $\rho = \rho_0 - \rho_1$ with $\rho_0 = \frac{1}{2} \sum_{\alpha \in \Delta_0^+} \alpha$, and $\rho_1 = \frac{1}{2} \sum_{\beta \in \Delta_1^+} \beta$.

In the $\epsilon\delta$-basis a weight $\Lambda \in H^*$ takes the form $\Lambda = \sum_{i=1}^{m} \mu_i \epsilon_i + \sum_{j=1}^{n} \nu_j \delta_j$. The corresponding Kac-Dynkin labels are defined by $a_i = \mu_i - \mu_{i+1}$ for $i = 1, 2, \dots, m-1$; $a_m = \mu_m + \nu_1$ and $a_{m+j} = \nu_j - \nu_{j+1}$ for $j = 1, 2, \dots, n-1$. With these two conventions we write $\Lambda = (\mu_1 \mu_2 \dots \mu_m | \nu_1 \nu_2 \dots \nu_n) = [a_1 a_2 \dots a_{m-1}; a_m; a_{m+1} \dots a_{m+n-1}]$. The $\epsilon\delta$-notation has a built-in redundancy thanks to the identity $\sum_{i=1}^{m} \epsilon_i - \sum_{j=1}^{n} \delta_j = 0$. This may be exploited to ensure that $\mu_m \ge 0$ and $\nu_1 \le 0$. In what follows it is convenient to denote all negative integers $-k$ by $\bar{k}$.

A weight $\Lambda \in H^*$ is said to be integral dominant if and only if $a_i \in \mathrm{N}$ for $i \ne m$ and $a_m \in \mathrm{C}$. Corresponding to each integral dominant weight Λ there exists an irreducible finite-dimensional highest weight module $V_0(\Lambda) = U(G_0) v_\Lambda$ of $sl(m) \oplus \mathrm{C} \oplus sl(n)$. Extending this to a $G_0 \oplus G_{+1}$ module by setting $G_{+1} V_0(\Lambda) = 0$ and inducing to G then gives the Kac-module [2] $\overline{V}(\Lambda)$ of $sl(m/n)$. This is isomorphic to $U(G_{-1}) \otimes V_0(\Lambda)$ and is generated through the action on $V_0(\Lambda)$ of the exterior algebra over $e(-\beta)$ with $\beta \in \Delta_1^+$. Thus $\overline{V}(\Lambda)$ and $V_0(\Lambda)$ share the same highest weight vector v_Λ, and $\dim \overline{V}(\Lambda) = 2^{mn}\, dim V_0(\Lambda)$. In general the Kac-module $\overline{V}(\Lambda)$ of $sl(m/n)$ is indecomposable but reducible, with composition factors isomorphic to various irreducible modules of $sl(m/n)$. Our aim is to present a prescription for determining all these composition factors. First we have two theorems:

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Theorem 1. (Kac [2]) Let $M(\Lambda)$ be the unique maximal submodule of $\overline{V}(\Lambda)$, then $V(\Lambda) = \overline{V}(\Lambda)/M(\Lambda)$ is irreducible.

Theorem 2. (Gould [3]) Let $v_\Omega = \Pi_{\beta \in \Delta_1^+} e(-\beta) v_\Lambda$, then $X(\Lambda) = U(G)v_\Omega$ is irreducible and $X(\Lambda) = V(\Gamma)$ for some Γ.

As pointed out elsewhere [4] a key construct in discussing the structure of $\overline{V}(\Lambda)$ is the atypicality matrix $A(\Lambda)$. Its matrix elements are given by $A(\Lambda)_{ij} = \langle \Lambda + \rho | \beta_{ij} \rangle$ with $\beta_{ij} = \epsilon_i - \delta_j \in \Delta_1^+$ for $i = 1, 2, \ldots, m$ and $j = 1, 2, \ldots, n$. The integral dominant weight Λ is said to be typical if $A(\Lambda)$ contains no zeros, and atypical of degree r if $A(\Lambda)$ contains r zeros. With this definition we have two more theorems:

Theorem 3. (Kac [2]) If Λ is typical then $\overline{V}(\Lambda) = V(\Lambda)$ is irreducible, and thus consists of a single composition factor.

Theorem 4. (Van der Jeugt et al. [5]) If Λ is singly atypical of type β then $\overline{V}(\Lambda)$ is the semi-direct sum of $V(\Lambda) = \overline{V}(\Lambda)/M(\Lambda)$ and $V(\Phi) = X(\Lambda) = M(\Lambda)$. The algorithm for determining Φ is as follows: Construct the sequence $S_\beta = (\beta_1 \beta_2 \ldots \beta_k)$ of positive odd roots with $\beta_1 = \beta$, such that $\langle \Lambda + \rho | \beta_1 \rangle = 0$ with $\Lambda - \beta_1$ non-dominant; $\langle \Lambda + \rho - \beta_1 | \beta_2 \rangle = 0$ with $\Lambda - \beta_1 - \beta_2$ non-dominant; and so on, until the sequence terminates with $\langle \Lambda + \rho - \beta_1 - \beta_2 - \cdots - \beta_{k-1} | \beta_k \rangle = 0$ with $\Lambda - \beta_1 - \beta_2 - \cdots - \beta_k$ dominant. Then $\Phi = \Lambda - S(\beta)$, where $S(\beta) = \beta_1 + \beta_2 + \cdots + \beta_k$.

This procedure can be implemented diagramatically [1,4,5], see for example the singly atypical case $\Lambda = (7663|\bar{1}\bar{1}\bar{3}\bar{3}\bar{5}\bar{5}) = [103; 2; 02020]$ of $sl(4/6)$ illustrated in [1] for which $\Phi = (5533|\bar{1}\bar{1}\bar{2}\bar{2}\bar{2}\bar{4}) = [020; 2; 01002]$. $S(\beta)$ defines, and is defined by, the removal of a continuous boundary strip of boxes from each portion of the composite Young diagram $F^{\bar{\kappa}';\mu}$ with the row lengths of F^μ and the column lengths of $F^{\bar{\kappa}'}$ determined by the parts of the partitions $\mu = (\mu_1 \mu_2 \ldots \mu_m)$ and $\kappa = (-\nu_n \ldots - \nu_2 - \nu_1)$.

The problem is to find the generalisation of these results appropriate to the multiply atypical case. Consider the case for which Λ is doubly atypical of type $\beta_1 = \epsilon_i - \delta_j$ and $\beta_2 = \epsilon_k - \delta_l$, with $k < i$ and $j < l$, so that $\langle \Lambda + \rho | \beta \rangle = 0$ for $\beta \in \Delta_1^+$ if and only if $\beta = \beta_1$ or β_2. Let $A(\Lambda)_{kj} = x = -A(\Lambda)_{il}$ and $h = i - k + l - j - 1$, the hook length between the two zeros of the atypicality matrix, then Λ is said to be normal if $x \geq h + 2$, quasi-critical if $x = h + 1$ and critical if $x = h$. On the basis of extensive investigations we conjecture the following:

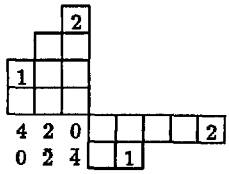
Conjecture 5. (i) If Λ is doubly atypical and normal then $\overline{V}(\Lambda)$ contains four composition factors isomorphic to irreducible modules with highest weights Λ, $\Lambda - S(\beta_1)$, $\Lambda - S(\beta_2)$ and $\Lambda - S(\beta_1) - S(\beta_2)$.

(ii) If Λ is doubly atypical and quasi-critical then $\overline{V}(\Lambda)$ contains five composition factors isomorphic to irreducible modules with highest weights Λ, $\Lambda - S(\beta_1)$, $\Lambda - S(\beta_2)$, $\Lambda - S(\beta_1) - S(\beta_2)$ and $\Lambda - S(\beta_1 L \beta_2)$, where $S(\beta_1 L \beta_2)$ is defined by the removal of continuous boundary strips starting from the position specified by β_2 and continuing until they link with and include the strips associated with $S(\beta_1)$.

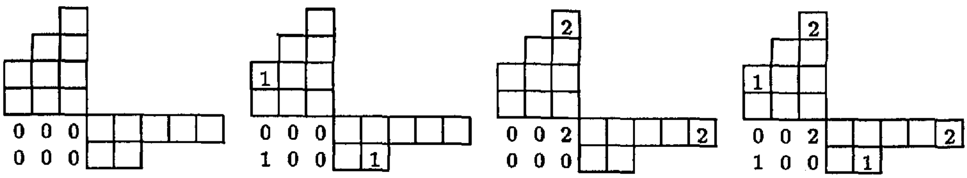
(iii) If Λ is doubly atypical and critical then $\overline{V}(\Lambda)$ contains three composition factors isomorphic to irreducible modules with highest weights Λ, $\Lambda - S(\beta_1)$ and $\Lambda - S(\beta_1 W \beta_2)$, where $S(\beta_1 W \beta_2)$ is obtained by first removing the strips defined by $S(\beta 1)$ and then removing further strips starting this time from the positions specified by β_2 which wrap around the first strips and continue until the resulting diagram is once more regular.

These three possibilities are illustrated in the following examples:

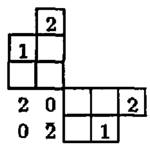
(i) The $sl(2/3)$ case $\Lambda = (52|\bar{2}\bar{3}\bar{4}) = [3;0;11]$ is doubly atypical of type $\beta_1 = \beta_{21}$ and $\beta_2 = \beta_{14}$. The atypicality matrix and the starting points of the strips to be removed are indicated in the following diagram:



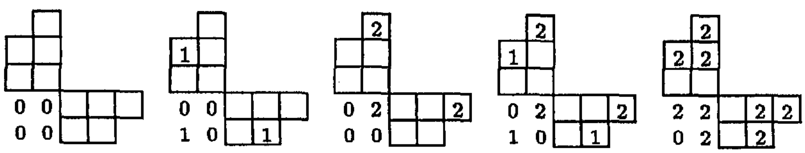
Λ is normal since $x = 4$ and $h = 2$. In this case $S(\beta_1) = \beta_{21}$ and $S(\beta_2) = \beta_{14}$. The four composition factors have highest weights: $(52|\bar{2}\bar{3}\bar{4}) = [3;0;11]$, $(51|\bar{1}\bar{3}\bar{4}) = [4;0;21]$, $(42|\bar{2}\bar{3}\bar{3}) = [2;0;10]$ and $(41|\bar{1}\bar{3}\bar{3}) = [3;0;20]$ corresponding to the strip removals:



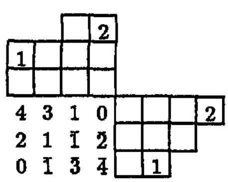
(ii) The $sl(2/2)$ case $\Lambda = (32|\bar{2}\bar{3}) = [1;0;1]$ is doubly atypical of type $\beta = \beta_{21}$ and $\beta = \beta_{12}$:



In this case Λ is quasi-critical since $x = 2$ and $h = 1$. Now $S(\beta_1) = \beta_{21}$, $S(\beta_2) = \beta_{12}$ and $S(\beta_1 L \beta_2) = \beta_{11} + \beta_{12} + \beta_{22}$. The five composition factors have highest weights $(32|\bar{2}\bar{3}) = [1;0;1]$, $(31|\bar{1}\bar{3}) = [2;0;2]$, $(22|\bar{2}\bar{2}) = [0;0;0]$, $(21|\bar{1}\bar{2}) = [1;0;1]$ and $(11|\bar{1}\bar{1}) = [0;0;0]$ corresponding to the strip removals:

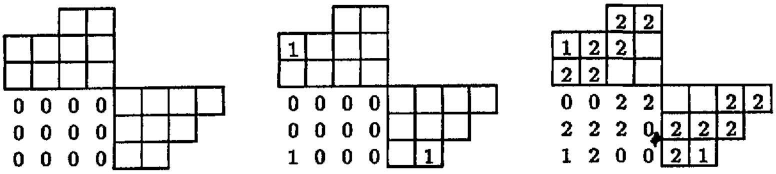


(iii) Finally, the $sl(3/4)$ case $\Lambda = (432|\bar{2}\bar{2}\bar{3}\bar{3}) = [11;0;010]$ is doubly atypical of type $\beta_1 = \beta_{31}$ and $\beta_2 = \beta_{14}$:



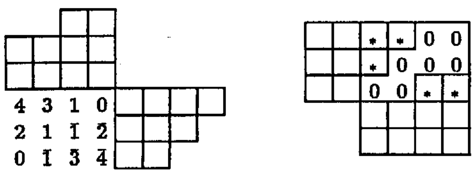
Λ is critical since $x = 4$ and $h = 4$. This time $S(\beta_1) = \beta_{31}$ and the usual method of determining $S(\beta_2)$ leads to a strip which reaches and wraps around that associated with β_1. There are now only three composition factors with highest weights $(432|\bar{2}\bar{2}\bar{3}\bar{3}) =$

[11; 0; 010], $(431|\bar{1}\bar{2}\bar{3}\bar{3}) = [12;0;110]$ and $(432|\bar{2}\bar{2}\bar{3}\bar{3}) = [11;0;010]$ obtained by means of the strip removals:



To deal with cases for which the degree of atypicality is greater than two an algorithm has been developed, based on the notion of a strip removal scheme in which the question of linking and wrapping is determined from a criticality matrix. The whole process is codified, and an algorithm has been constructed and implemented on a computer. Many checks have been carried out covering cases of atypicality degree as large as five, for which the number of composition factors rises as high as 132.

The converse problem of determing all those Kac-modules $\overline{V}(\Lambda)$ which contain a specific irreducible module $V(\Sigma)$ as a composition factor turns out to have a simpler solution. The algorithm for its solution starts from the atypicality matrix $A(\Sigma)$. The first step is to determine those β's which belong to a set $\Delta_S(\Sigma) \subseteq \Delta_1^+$. This may be done in several ways [4] but for our purposes here a diagramatic way is preferable. It is illustrated in the following diagram in which the entries $*$ specify the β's belonging to $\Delta_S(\Sigma)$ for $sl(3/4)$ with $\Sigma = (432|\bar{2}\bar{2}\bar{3}\bar{3}) = [11;010]$.



The entries $*$ are those covered by boxes of F^{μ} and $F^{\bar{\kappa}'}$ positioned so that the ith row of F^{μ} and the jth column of $F^{\bar{\kappa}'}$ terminate just to the left and just below the position of the leftmost 0 in $A(\Sigma)$. If overlapping had occurred it would have been necessary to truncate the diagram and reposition a new portion around the position of the next zero in the atypicality matrix. In the case of no overlap, as above, the top boundary of F^{μ} and the right hand boundary of $F^{\bar{\kappa}'}$ are extended until they meet.

The algorithm is then as follows. Each connected set of zeros in the matrix of $*$'s and 0's constructed as above is renumbered consecutively step by step in a shifting process whereby at every stage F^{μ} and $F^{\bar{\kappa}'}$ either slide one step south and one step west, respectively, or one step east and one step north, respectively. In these two cases the zeros covered in this way are all to be renumbered either 1 or 1′, appropriately. The process is then repeated until all zeros are covered. At each stage there is a choice of a south-west (SW) or north-east (NE) slide leading to new unprimed and primed entries. The process terminates after precisely r steps, leading to a total of 2^r distinctly labelled matrices $CF(\Sigma)$. The significance of this labelling lies in the following:

Conjecture 6. Let Σ be multiply atypical of degree r. Each of the 2^r matrices $CF(\Sigma)$ defines Λ such that $V(\Sigma)$ is a composition factor of $\overline{V}(\Lambda)$. Λ is found by adding to Σ those β's associated with the positions of the unprimed numbers.

The procedure is exemplified as follows in the doubly atypical $sl(3/4)$ case $\Sigma = (432|\bar{2}\bar{2}\bar{3}\bar{3}) = [11;0;010]$.

$$\begin{array}{cccc} * & * & 0 & 0 \\ * & 0 & 0 & 0 \\ 0 & 0 & * & * \end{array} \;\nearrow\; NE \;\begin{array}{cccc} * & * & 1' & 0 \\ * & 1' & 1 & 1 \\ 1' & 1' & * & * \end{array} \;\begin{array}{l} \nearrow\; NE \;\begin{array}{cccc} * & * & 1' & 2' \\ * & 1' & 1' & 1' \\ 1' & 1' & * & * \end{array} \\ \searrow\; SW \;\begin{array}{cccc} * & * & 1' & 2 \\ * & 1' & 1' & 1' \\ 1' & 1' & * & * \end{array} \end{array}$$

$$\begin{array}{cccc} * & * & 0 & 0 \\ * & 0 & 0 & 0 \\ 0 & 0 & * & * \end{array} \;\searrow\; SW \;\begin{array}{cccc} * & * & 1 & 1 \\ * & 1 & 0 & 1 \\ 1 & 1 & * & * \end{array} \;\begin{array}{l} \nearrow\; NE \;\begin{array}{cccc} * & * & 1 & 1 \\ * & 1 & 2' & 1 \\ 1 & 1 & * & * \end{array} \\ \searrow\; SW \;\begin{array}{cccc} * & * & 1 & 1 \\ * & 1 & 2 & 1 \\ 1 & 1 & * & * \end{array} \end{array}$$

It can be inferred from the final four diagrams that $V(\Sigma)$, with $\Sigma = (432|\bar{2}\bar{2}\bar{3}\bar{3}) = [11;0;010]$, is a composition factor of four Kac-modules $\overline{V}(\Lambda)$ having highest weights:

$$\begin{aligned} (432|\bar{2}\bar{2}\bar{3}\bar{3}) + (000|0000) &= (432|\bar{2}\bar{2}\bar{3}\bar{3}) = [11;0;010]; \\ (432|\bar{2}\bar{2}\bar{3}\bar{3}) + (100|000\bar{1}) &= (532|\bar{2}\bar{2}\bar{3}\bar{4}) = [21;0;011]; \\ (432|\bar{2}\bar{2}\bar{3}\bar{3}) + (222|\bar{1}\bar{2}\bar{1}\bar{2}) &= (654|\bar{3}\bar{4}\bar{4}\bar{5}) = [11;1;101]; \\ (432|\bar{2}\bar{2}\bar{3}\bar{3}) + (232|\bar{1}\bar{2}\bar{2}\bar{2}) &= (664|\bar{3}\bar{4}\bar{5}\bar{5}) = [02;1;110]. \end{aligned}$$

This procedure is very easy to program and all our checks to date indicate that the results are entirely consistent with the determination of composition factors of Kac-modules by means of the algorithm based on criticality and strip removals. Indeed it was the nice combinatorial features of this algorithm in its identification of β's to be subtracted from Λ to give Σ that led to the discovery of the very simple converse procedure just described for obtaining all 2^r possible Λ from a knowledge of the r-fold atypical Σ.

References

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