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Modave lecture notes: Field exploration in phases of matter

Antoine Pasternak*

INFN, Sezione di Milano
Via Celoria 16, I-20133 Milano, Italy

E-mail: antoinepasternak14@gmail.com

A new look has been given to symmetries in quantum field theory in the last ten years. Generalized symmetries broadened our vision of their role in constraining the renormalization group flow. Together with the knowledge of their anomalies, they can be used to classify phases of matter that fall outside the scope of Landau's paradigm with ordinary symmetries. Examples include superfluids on thin layers, deconfined phases of gauge theories, and field theories with topological order. These lectures will introduce these notions and work out examples of quantum field theories that exhibit such phases.

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1. Introduction

Field theory is a successful framework and a powerful tool in theoretical physics for investigating the fundamental properties of matter. It is used to study physical processes in various contexts, as it finds applications across different energy and density scales. Below are some notable examples:

- Electroweak and strong interactions in particle physics.
- Phase transitions in condensed matter.
- Inflation in cosmology.
- Quantum gravity via holography.

Advances in these areas have notably relied on simplifications arising from symmetries, and Landau's theory of phase transitions is no exception. In this case, symmetries serve as an organizing principle for describing phases of matter. They offer insights into vacuum degeneracy and systematically construct the lowest-energy excitations upon it in a bottom-up manner. According to Landau, the low-energy versions of different theories displaying the same symmetry pattern belong to the same phase and can be described universally. However, over the past century, examples that do not fit Landau's paradigm have been found. Famous examples include superfluids on thin materials, deconfined phases of gauge theories, topological field theories, and the list continues to grow today.

More recently, Gaiotto, Kapustin, Seiberg, and Willett brought the concept of higher-form symmetries to the forefront [1]. Along with 't Hooft anomalies, these symmetries have provided physicists with new tools to uncover the low-energy mysteries of field theories. In particular, McGreevy emphasized in a review [2] that they can revitalize Landau's idea. Higher-form symmetries can be identified in the aforementioned outliers and many more, reigniting the quest for a classification of phases of matter, provided we generalize our approach to symmetries.

In these lecture notes, we follow the spirit of [2] and use phases of matter as motivation to introduce higher-form symmetries and related tools, such as 't Hooft anomalies. We aim to illustrate this introduction with simple examples from scalar field theories and electromagnetism to cover both older and more recent results on symmetries, their breaking, and their anomalies. Unfortunately, these notes will end prematurely, as many other examples could have been included, given that they were designed for only six hours of lectures. For instance, non-invertible and subsystem symmetries will not be introduced here.

Our main references are [1–3]. Additionally, these notes have drawn upon preexisting reviews [4–11] and teaching material [12–17].

2. Review on fields, symmetries, and phases

2.1 Field theories

Quantum field theory casts knowledge about physical phenomena into correlation functions for its different observables O. For local observables, these are evaluated at several points x in a d-dimensional spacetime manifold $\mathscr{M}_d$. When the field theory is formulated in terms of a Lagrangian, correlation functions can be computed via a path integral:

$$\langle O_1(x_1)\cdots O_N(x_N)\rangle = \frac{1}{Z}\int D\Phi\, O_1(\Phi(x_1))\cdots O_N(\Phi(x_N))e^{iS[\Phi]/\hbar}\,, \tag{1}$$

where Z is the partition function of the field theory:

$$Z = \int D\Phi\, e^{iS[\Phi]/\hbar}\,. \tag{2}$$

In the path integral formulation, we sum over field configurations $\Phi(x)$ with complex weights provided by the action $S[\Phi]$. Each O on the LHS of Equation (1) is an operator acting on the Hilbert space of the field theory. On the RHS, it is a function of the fields Φ.

To ensure that the path integral converges, it may be convenient to convert the complex weight into a real one by performing a Wick rotation:

$$t \to -i\tau \,, \tag{3}$$

together with introducing an Euclidean action S_e:

$$iS[\Phi]/\hbar \to -S_e[\Phi]/\hbar \,. \tag{4}$$

This defines the path integral for the Euclidean theory.

The Wick rotation transforms the path integral into a familiar form for statistical physicists. Indeed, the RHS of Equation (4) is reminiscent of the free energy for a static field configuration, as shown by the following identification:

$$-\, S_e[\Phi]/\hbar \leftrightarrow -\beta F[\Phi] \,. \tag{5}$$

The reduced Planck constant $\hbar$ is replaced with the inverse temperature $\beta = 1/k_B T$ of the statistical system. Instead of summing over quantum fluctuations, statistical physicists sum over thermal fluctuations, but the computational device is similar. An important caveat is that the Euclidean action S_e is evaluated on fields in a d-dimensional Euclidean spacetime, while the free energy F is evaluated on *static* field configurations in a d-dimensional space without a time component. The key takeaway is that some results from Euclidean QFT in a d-dimensional spacetime, like the falloff of certain correlation functions, can be exported to classical statistical systems in $d + 1$ dimensions, and vice-versa. This already illustrates the universal power of field theory in a broad sense.

In both quantum and statistical field theories, an important question is whether we can use the microscopic details of the field theory, or a lattice version, to predict its behavior at low energies and momenta. Ultimately, this would allow us to understand the phase of matter it describes. The renormalization group flow answers this question in many cases. It has been put forward by Wilson in 1975 to solve the Kondo problem [18]. This field theory tool is instrumental in our quest to classify the phases of matter, as illustrated in the next sections.

2.2 Renormalization group flow

One way to introduce the renormalization group flow (RG flow) is to separate low and high-frequency modes in the partition function. This is the *frequency separation scheme*, which is pedagogically presented in [12]. The field Φ being a sum of Fourier modes, we may formally distinguish its *infrared* (IR) and *ultraviolet* (UV) contributions with respect to an arbitrary energy scale Λ:

$$\Phi = \Phi_{\rm IR} + \Phi_{\rm UV} \,, \quad \mathrm{IR} \leqslant \Lambda < \mathrm{UV} \,. \tag{6}$$

We can then think about integrating the UV modes in the path integral to produce an effective action $S_{\rm eff}$ for the IR modes only:

$$Z = \int D\Phi_{\rm IR} D\Phi_{\rm UV} \, e^{iS[\Phi_{\rm IR},\Phi_{\rm UV}]} = \int D\Phi_{\rm IR} \, e^{iS_{\rm eff}[\Phi_{\rm IR};\Lambda]} \,. \tag{7}$$

The effective action is useful for studying processes with energy transfers that are not larger than Λ.

In more detail, the RG flow maps the Lagrangian theory of S to an effective one, equipped with the action S_{eff}, through a continuous deformation that coarse-grains the theory. Focusing on low frequencies is achieved with a scaling:

$$(E, \vec{k}) \to \lambda^{-1}(E, \vec{k}) , \quad (t, \vec{x}) \to \lambda(t, \vec{x}) , \quad \lambda > 1 . \tag{8}$$

Relevant interactions will increase in strength in the infrared, while *irrelevant* ones will lose importance. There are also *marginal* couplings unaffected by the RG flow. This qualification is dictated by the dimension of the coupling.

For example, in the free theory of a massive field with mass m, such that $[m] = 1$, the adimensional mass $\nu = m/E$ gains importance in the IR:

$$\nu \to \lambda \nu , \quad \lambda > 1 . \tag{9}$$

As we go down in the IR, the mass m is increasingly felt. Ultimately, if $\Lambda < m$, the energy transfers will not be sufficient to create any excitation related to the massive field, and the latter can be removed, or *integrated out*, from the effective field theory. This demonstrates that, in parallel to changes in the coupling, the RG flow also adapts the degrees of freedom of theory. Contrary to what is formally suggested in Equation (6), Φ_{UV} and Φ_{IR} may have different natures. In the following, we will use these symbols to represent the various degrees of freedom in the UV and IR theories, not merely as different modes of a single field Φ.

Note that this presentation refers to the Wilsonian scheme of renormalization, which should be contrasted with the 1PI effective action where the path integral is performed over all frequencies. However, these schemes may coincide in the absence of interacting massless particles.

These notions could be reviewed in greater detail, but we will instead focus on a limited set of hopefully enlightening physical examples.

Weak interaction

In our modern understanding of particle physics, the weak interaction is responsible for β decay:

$$n(udd) \longrightarrow p^+(uud) + e^- + \bar{\nu}_e . \tag{10}$$

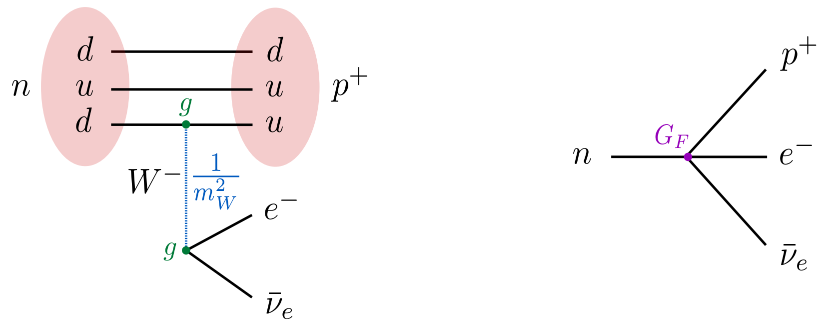
More precisely, the Salam-Weinberg model of 1968 tells us that the decay of the neutron is mediated by the boson W^-, as shown in Figure 1 (a). In the corresponding Lagrangian, one finds the following terms involving W^- and fermionic matter ψ:

$$\mathcal{L}_{\text{SW}} \supset g\bar{\psi}\gamma^\mu W_\mu \psi + m_W W^\mu W_\mu . \tag{11}$$

These terms explain how one quark d in the neutron can be converted into a quark u by emitting a W^-, turning the neutron into a proton. The W^- then converts into a pair of fermions: an electron e^- and an antineutrino $\bar{\nu}_e$.

The mass of the W^- is given by $m_W = 80\,\text{GeV}$. Below this scale, the W^- is integrated out, leaving only fermions. The interaction is effectively seen as a quartic interaction between fermions, as shown in Figure 1 (b) and prescribed by Fermi in 1933:

$$\mathcal{L}_{\text{F}} \supset G_{\text{F}}\bar{\psi}\psi\bar{\psi}\psi . \tag{12}$$



(a) Salam-Weinberg model of the weak interaction. **(b)** Fermi interaction.

Figure 1: Two views on the β decay.

The couplings in both models are matched by the RG flow at the tree level:

$$G_{\mathrm{F}} \propto \frac{g^2}{m_W^2} \,. \tag{13}$$

In Fermi's interaction, the coupling G_{F} is found to be irrelevant and, as a result, tends effectively to zero as we flow down to the IR. This explains the name "weak interaction".

Strong interaction

In quantum chromodynamics (QCD), the gauge coupling g is classically marginal but becomes relevant once loop corrections are added. Its running with energy is prescribed by the so-called *beta function equation*:

$$\beta(E) = E \frac{\partial}{\partial E} g(E) = b\, g(E)^2 + \cdots \,, \tag{14}$$

where $b < 0$ is a numerical factor which characterizes one-loop contributions. To integrate this equation, we need to introduce the dynamical scale Λ_{QCD} such that

$$g(E) = \frac{g(\Lambda_{\mathrm{QCD}})}{1 + b\, g(\Lambda_{\mathrm{QCD}}) \ln(\Lambda_{\mathrm{QCD}}/E)} \,. \tag{15}$$

The coupling grows strong near $E \sim \Lambda_{\mathrm{QCD}}\, e^{-1/bg(\Lambda_{\mathrm{QCD}})}$.

In fact, the gauge coupling in QCD grows stronger and stronger as we go down in energy. Bound states form, and the correct degrees of freedom are not simply the quarks and gluons obtained after frequency separation, but rather composite fields like the mesons and baryons. One-loop contributions in the beta function are not enough to capture these strong-coupling features and solving the low-energy regime of QCD remains an outstanding problem nowadays.

Possible ends of the renormalization group flow

The RG flow ends on a *fixed point* of the beta function and the later determines the deep IR of the theory. Since the RG flow may remove some fields and adapt the couplings, it is not surprising that a single fixed point may connect different UV theories. For instance, consider two theories, A and B, where B is just theory A plus some extra decoupled massive fields. It is then expected that A and B flow to the same IR fixed point. This illustrates the fact that the RG is not a group and that it erases some information (see a, c, F-theorems).

In general, many different UV theories share the same *universal* IR, and fall in the same phase of matter. Sometimes, theory A may be a bona fide quantum field theory, while B could be a quantum theory on a lattice. This may seem trivial if the lattice is just a regularization of the quantum field theory spacetime, but in principle, there might be some more interesting scenarios.

It is tempting to take these observations to motivate the classification of different endpoints of the RG flow. The first important criterion is whether the IR theory is *gapped* or *gapless*.

- Gapped theory: all excitations remain massive at infinite volume.
  - Trivially gapped theory: the vacuum is unique (example: a theory of massive scalars).
  - Non-trivially gapped theory: displaying degeneracy of the vacuum, long-range order, etc. (example: the topological theory of Chern-Simons).
- Gapless theory: there are massless excitations at infinite volume. We can further distinguish two cases:
  - Free theory (example: a theory of free massless scalars).
  - Interacting theory (example: Wilson-Fisher fixed point).

  Since all scales are washed out by the RG flow, the last two cases are scale-invariant theories, which generally fall under the class of conformal field theories (CFT).

How do we know in which kind of vacuum we end up at the end of an RG flow?

Even though the renormalization group had not yet been studied at the time, Landau hypothesized that the properties of the infrared should be characterized by the *symmetries* of the model [19]. We will now explore reasons why this idea is far from anachronistic.

2.3 Symmetries and Landau's paradigm

The symmetries of a quantum field theory interestingly characterize its RG flow:

(a) If G is the symmetry group of a UV theory, then it cannot be explicitly broken by the RG flow. In other words, the effective action will never contain a term that is not invariant under G. Note however that G may be replaced with a larger structure, such as an emergent symmetry group.

(b) Since the field content changes from Φ_{UV} to Φ_{IR}, representations of G in the IR may differ from the ones in the UV. In general, only a subgroup of G will act non-trivially, or faithfully, on the degrees of freedom in the IR, and so actually counts as a symmetry. The emergent symmetry group is faithful by definition.

(c) If Φ_{IR} transforms non-trivially under G, or one of its subgroups, then its vacuum expectation value (VEV) is an *order parameter*. More precisely, if $\langle \Phi_{\text{IR}} \rangle \neq 0$ is preserved by $H \subset G$, we say that G is spontaneously broken down to H.

The idea of Landau is to use that knowledge to characterize different *phases* of matter. The notion of phase is more general than the notion of vacua discussed before. For example, we say that different gapped vacua, possibly from different UV theories, belong to the same phase if one can go from one to the other by performing an adiabatic change without closing the gap. This may involve modifying the couplings of the theory and adding or removing degrees of freedom. In this case, phase transitions thus signal the apparition of massless degrees of freedom.

Landau's paradigm

(1) Different phases correspond to different combinations (G, H).

(2) Massless degrees of freedom at the critical point are given by the fluctuations of the order parameter $\langle \Phi_{\text{IR}} \rangle$.

In these lectures, we will mainly be concerned with (1) and will illustrate it right away with a famous example from statistical physics.

Ising model

The Ising model is a lattice theory with N spins $\sigma_i = \pm 1$ located at sites i. Consider the classical version of the model in $d \geqslant 2+1$ spacetime dimensions and subject to thermal fluctuations. The Hamiltonian exhibits a nearest-neighbor interaction with interaction strength $J > 0$:

$$H = -J \sum_{\langle ij \rangle}^{N} \sigma_i \sigma_j \,. \tag{16}$$

It is minimized when all spins either point downward $\sigma_i = -1$, or upward $\sigma_i = +1$. Those are called *ordered* or *ferromagnetic* configurations. Thermal fluctuations will introduce some randomness in the spin configuration. In particular, the model has a critical temperature T_c over which the spins average to zero.[1] This is a *disordered* or *paramagnetic* configuration.

This discussion can be rephrased in terms of symmetries G and H. The Hamiltonian is invariant under a symmetry group $G = \mathbb{Z}_2$ which acts as follows:

$$\sigma_i \to -\sigma_i \,. \tag{17}$$

In this case, an order parameter is given by the average magnetization of the spin-lattice:

$$m = \frac{1}{N} \sum_{i=1}^{N} \sigma_i \,. \tag{18}$$

Indeed, it transforms non-trivially under $G = \mathbb{Z}_2$: $m \to -m$.

- Disordered phase at $T > T_c$: $\langle m \rangle = 0$ and $G = H = \mathbb{Z}_2$. The symmetry group is preserved and the vacuum is unique.
- Ordered phase at $T < T_c$: $\langle m \rangle = \pm 1$ and $G = \mathbb{Z}_2$, $H = \varnothing$. The symmetry group is spontaneously broken and there are two vacua related to each other by $\mathbb{Z}_2$.

[1] In 1944, Onsager solved analytically the Ising model for $d = 2+1$ and found $T_c = 2J/k_B \ln(1+\sqrt{2})$.

This example illustrates well how the VEV of the order parameter, and hence the couple (G, H), determines the phase in which our vacua belongs. Moreover, they illustrate a general lesson related to the groundstate degeneracy (GSD) of theories with a discrete symmetry (in the higher-form context, call it a *zero-form* discrete symmetry).

Breaking of a discrete symmetry

The breaking of discrete zero-form symmetry G into H leads to a groundstate degeneracy:

$$\text{GSD} = |G/H| = \frac{|G|}{|H|} \,. \tag{19}$$

Beyond Landau's paradigm

After Landau formulated his proposal for classifying phases of matter, much water flowed under the bridges, and physicists went on to discover many phases of field theories that fall outside the aforementioned paradigm. A non-exhaustive list includes the following examples:

- Superfluids on thin layers.
- Gapless phases of gauge theories.
- Quantum Hall effect and topological order.
- Symmetry-protected topological order.
- Fractons.
- Landau Fermi liquids.

The object of these lectures is to slightly deviate from Landau's original paradigm, while keeping symmetries as the central aspect for classifying phases of matter, following [2]. We will find that introducing anomalies and higher-form symmetries allows us to reincorporate some of the examples in this list into phases that can be classified in terms of symmetries. But first, we review some basic aspects of symmetries in quantum field theory.

2.4 From currents and charges to topological operators

Continuous symmetries are traditionally introduced with a Lie group G.[2] Let $g \in G$. The element g can be obtained as an expansion around the identity 1:

$$g = 1 + i\alpha^a t^a + \cdots \,, \quad a = 1, \cdots, \dim\mathfrak{g} \,. \tag{20}$$

where α^a is a set of parameters and t^a are generators of the Lie algebra $\mathfrak{g}$. In the following, we will refer to a Lie group element by g or α interchangeably. Let Φ^g and Φ^{t^a} be the fields obtained after transformation under g and t^a respectively. We have

$$\Phi^g = \Phi + i\alpha^a \Phi^{t^a} + \cdots \,. \tag{21}$$

[2]Non-invertible symmetries are not be covered in these lecture notes. See [7, 11] for introductions to this topic.

For example, $G = U(1)$ is defined together with a periodic parameter $\alpha \sim \alpha + 2\pi$ such that

$$\Phi^g = e^{i\alpha q}\Phi\,, \quad \Phi^t = q\Phi\,. \tag{22}$$

Importantly, in this example, $q \in \mathbb{Z}$, otherwise $\alpha = 2\pi$ cannot consistently reproduce the identity of the group $U(1)$. The charge q may vary from one field to another.

Symmetry currents and charges

G is a symmetry of the classical field theory if all of its elements g leave the action $S[\Phi]$ invariant up to boundary terms:

$$S\left[\Phi^g\right] = S\left[\Phi\right] + \text{boundary terms}\,. \tag{23}$$

Then, Noether's theorem proves the existence of conserved currents:

$$j^a_\mu = \frac{\partial L}{\partial \partial^\mu \Phi}\Phi^{t^a} - K_\mu\,, \tag{24}$$

in the sense that j^a_μ satisfies the continuity equation on-shell:

$$\partial_\mu j^{a\mu} = 0\,. \tag{25}$$

These are often called *Noether currents*, in contrast to *topological currents* which satisfy the continuity equation even off-shell.

Let $\mathscr{M}_d = \mathbb{R} \times \mathscr{N}_{d-1}$, where $\mathbb{R}$ and $\mathscr{N}_{d-1}$ are the time and spatial components respectively. Noether charges are defined as follows:

$$Q^a(t) \equiv \int_{\mathscr{N}_{d-1}} \mathrm{d}^{d-1}x \;\; j^a_t(t,\vec{x})\,. \tag{26}$$

Each charge is conserved on-shell as a result of Noether's theorem:

$$\frac{\mathrm{d}Q^a}{\mathrm{d}t} = \int_{\mathscr{N}_{d-1}} \mathrm{d}^{d-1}x \;\; \partial_t j^a_t = \int_{\mathscr{N}_{d-1}} \mathrm{d}^{d-1}x \;\; \vec{\nabla}\cdot\vec{j}^a = \int_{\partial\mathscr{N}_{d-1}} \mathrm{d}^{d-2}x \;\; \vec{n}\cdot\vec{j}^a = 0\,. \tag{27}$$

The last equality is obtained trivially when $\mathscr{N}_{d-1}$ is a closed manifold, i.e. without boundaries: $\partial\mathscr{N}_{d-1} = 0$. Otherwise, we must assume good fall-off conditions for the field Φ near the boundary.

At the quantum level, the conservation of the current is generalized by Ward's identities:

$$\begin{aligned} \langle \partial_\mu j^{a\mu}(x) \rangle &= 0\,, \\ \langle \partial_\mu j^{a\mu}(x)\Phi(y) \rangle &= i\langle \Phi^{t^a}(y)\rangle \delta^d(x-y)\,, \\ \langle \partial_\mu j^{a\mu}(x)\Phi(y)\Phi(z) \rangle &= i\langle \Phi^{t^a}(y)\Phi(z)\rangle \delta^d(x-y) + i\langle \Phi(y)\Phi^{t^a}(z)\rangle \delta^d(x-z)\,, \\ &\text{etc.} \end{aligned} \tag{28}$$

Symmetry operator

In the quantum theory, Q and Φ are promoted to operators acting on the Hilbert space. Note that Q was defined at some time t, but could equivalently be defined by $\mathscr{N}_{d-1}$, the hypersurface on which it is integrated. We drop the index a of the Lie generator from now on. The operators obey the following commutation relation:

$$[Q(\mathcal{N}_{d-1}), \Phi(x)] = i\Phi^t(x) I(x, \mathcal{N}_{d-1}) , \tag{29}$$

where $I(x, \mathcal{N}_{d-1})$ is the intersection number between x and $\mathcal{N}_{d-1}$:

$$I(x, \mathcal{N}_{d-1}) = \begin{cases} 1 & \text{if } x \in \mathcal{N}_{d-1} , \\ 0 & \text{otherwise} . \end{cases} \tag{30}$$

We can make our life easy by picking up $G = U(1)$:

$$[Q(\mathcal{N}_{d-1}), \Phi(x)] = iq\Phi(x) I(x, \mathcal{N}_{d-1}) , \quad q \in \mathbb{Z} . \tag{31}$$

In general, Q defines a unitary action for $g \in G$:

$$U_\alpha(\mathcal{N}_{d-1}) \equiv e^{i\alpha Q(\mathcal{N}_{d-1})} . \tag{32}$$

We call U_α the symmetry operator. It obeys the following set of useful identities:

$$U_\alpha(\mathcal{N}_{d-1})^{-1} = U_\alpha(\mathcal{N}_{d-1})^\dagger = U_{-\alpha}(\mathcal{N}_{d-1}) = U_\alpha(-\mathcal{N}_{d-1}) , \tag{33}$$

where $-\mathcal{N}_{d-1}$ is $\mathcal{N}_{d-1}$ with reversed orientation. It acts on operators by conjugation:[3]

$$U_\alpha(\mathcal{N}_{d-1})\Phi(x)U_\alpha(\mathcal{N}_{d-1})^{-1} = e^{i\alpha q I(x,\mathcal{N}_{d-1})}\Phi(x) . \tag{34}$$

See Figure 2 (a) for an illustration of this equality. The symmetry operator satisfies the following fusion rule inherited from group law:

$$U_\alpha(\mathcal{N}_{d-1})U_\beta(\mathcal{N}_{d-1}) = U_{\alpha\cdot\beta}(\mathcal{N}_{d-1}) . \tag{35}$$

Note that U_α exists for discrete symmetries too.

Symmetry defect

While we defined $U_\alpha(\mathcal{N}_{d-1})$ as an operator acting on an Hilbert space $\mathcal{H}(\mathcal{N}_{d-1})$, one can also introduce a timelike hypersurface $\mathcal{T}_{d-1}$ such that $U_\alpha(\mathcal{T}_{d-1})$ appears in correlation functions as a *defect*. Its effect is to modify the quantum theory by imposing a boundary condition. If the location of the hypersurface $\mathcal{T}_{d-1}$ matches with some location y, quantification should impose the condition:

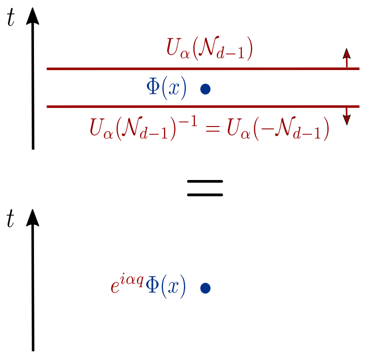
$$\lim_{\epsilon\to 0} \Phi(y-\epsilon) = e^{i\alpha q} \lim_{\epsilon\to 0} \Phi(y+\epsilon) , \tag{36}$$

and hence modifying the Hilbert space $\mathcal{H}(\mathcal{N}_{d-1})$ without affecting the set of operators.

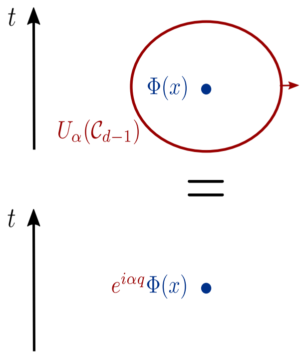
In the Euclidean theory, the defect is naturally obtained from the operator with a spacetime rotation. In fact, any topological deformation of $\mathcal{N}_{d-1}$ can be conceived in the Euclidean spacetime. Following this insight, consider a closed hypersurface $\mathcal{C}_{d-1}$ in $\mathcal{M}_d$, i.e. $\partial\mathcal{C}_{d-1} = 0$. First of all, note that it generalizes the definition of charge accordingly:

$$Q(\mathcal{C}_{d-1}) \equiv \oint_{\mathcal{C}_{d-1}} \mathrm{d}^{d-1}x \;\; n^\mu j_\mu , \tag{37}$$

[3]The action by conjugation is a general feature. Note that it allows for projective representations as well.



(a) U_α is defined on a spacelike surface $\mathscr{N}_{d-1}$ that intersects with x (distances are exagerated in the picture for clarity).



(b) U_α is defined on a closed hypersurface $\mathscr{C}_{d-1}$ in spacetime that links with x.

Figure 2: Action of the symmetry operator U_α on the local operator $\Phi(x)$.

where n^μ is the normal vector to $\mathscr{C}_{d-1}$. Now, smoothly deform $\mathscr{C}_{d-1}$ into $\mathscr{C}'_{d-1}$.

$$Q(\mathscr{C}'_{d-1}) - Q(\mathscr{C}_{d-1}) = \oint_{\mathscr{C}'_{d-1}} \mathrm{d}^{d-1}x \; n^\mu j_\mu - \oint_{\mathscr{C}_{d-1}} \mathrm{d}^{d-1}x \; n^\mu j_\mu = \int_{\mathscr{X}_d} \mathrm{d}^d x \; \partial^\mu j_\mu \,, \tag{38}$$

where $\mathscr{X}_d$ is chosen such that $\partial\mathscr{X}_d = \mathscr{C}'_{d-1} - \mathscr{C}_{d-1}$. The last integral vanishes if no charge is present in $\mathscr{X}_d$. So, the insertion of $Q(\mathscr{C}'_{d-1})$ in a correlation function will not differ from choosing to insert $Q(\mathscr{C}_{d-1})$ instead if it does not include a charged operator $\Phi(x)$ with $x \in \mathscr{X}_d$. Put differently, we should be able to deform $Q(\mathscr{C}'_{d-1})$ into $Q(\mathscr{C}_{d-1})$ without crossing $\Phi(x)$. This shows that the dependence of the charge operator in $\mathscr{C}_{d-1}$ is only homological.

The same is true for $U_\alpha(Q(\mathscr{C}_{d-1}))$. Moreover, one can show that its action can be generalized as follows:

$$U_\alpha(\mathscr{C}_{d-1})\Phi(x) = e^{i\alpha q L(x,\mathscr{C}_{d-1})}\Phi(x) \,, \tag{39}$$

where the linking number $L(x, \mathscr{C}_{d-1})$ is more suited to this definition than the intersection number. See Figure 2 (b) for an illustration This confirms the homological nature of the symmetry operator, hence it is a *topological operator*.

2.5 Anomalies

The current j_μ associated with a continuous symmetry is sourced in the path integral by a *classical background* A_μ. It is often implemented with a minimal coupling:

$$Z[A] = \int D\Phi \, e^{iS[\Phi;A]} \,, \quad S[\Phi;A] \supset \int \mathrm{d}^d x \, A_\mu j^\mu \,, \tag{40}$$

such that

$$\langle j^\mu(x) \cdots \rangle_A = \frac{1}{Z[A]}(-i)\frac{\delta}{\delta A_\mu(x)} \cdots Z[A] \,. \tag{41}$$

The notation $\langle\cdot\rangle_A$ means that the background field is not set to zero when evaluating the correlation function.

A general feature is that the symmetry transformation is promoted to a gauge transformation in the presence of the background:

$$g = g(x) \quad : \quad \Phi \to \Phi^g \,, \quad A_\mu \to A_\mu^g = gA^\mu g^{-1} + ig\partial_\mu g^{-1} \,. \tag{42}$$

If $G = U(1)$, then we find

$$g = e^{i\alpha(x)} \quad : \quad \Phi \to e^{i\alpha}\Phi \,, \quad A_\mu \to A^\mu + \partial_\mu \alpha \,. \tag{43}$$

At this point, a few remarks are in order.

- The classical background A_μ should not be confused with the dynamical gauge field a_μ of a gauge theory. In the latter case, the path integral sums over different field configurations for a_μ:

$$Z_{\text{gauge}} = \int DaD\Phi\, e^{iS[a,\Phi]} \,. \tag{44}$$

  Moreover, terms involving the field strength $f_{\mu\nu}$ in the action rule its dynamics:

$$S[a,\Phi] \supset -\frac{1}{4}\int \mathrm{d}^d x\, f_{\mu\nu} f^{\mu\nu} \quad \text{with} \quad f_{\mu\nu} = \partial_\mu a_\nu - \partial_\nu a_\mu \,. \tag{45}$$

- In contrast with the previous point, we appreciate that $Z[A]$ is evaluated on a single configuration for the gauge field A_μ. Moreover, if we set A_μ to the rather singular choice:

$$A_\mu = \begin{cases} \alpha n_\mu & x \in \mathscr{C}_{d-1} \,, \\ 0 & \text{elsewhere,} \end{cases} \tag{46}$$

  where n_μ the orthonormal vector to the hypersurface $\mathscr{C}_{d-1}$ and α is constant, then it may be observed that

$$Z[A] = \int D\Phi\, e^{i\alpha \oint_{\mathscr{C}_{d-1}} \mathrm{d}^{d-1}x\, n^\mu j_\mu} e^{iS[\Phi]} = \int D\Phi\, U_\alpha(Q(\mathscr{C}_{d-1})) e^{iS[\Phi]} \,. \tag{47}$$

  So, evaluating the correlation function with such a classical background is equivalent to introducing a symmetry operator:

$$\langle \cdots \rangle_A = \langle U_\alpha(Q(\mathscr{C}_{d-1})) \cdots \rangle \,. \tag{48}$$

  In general, a fixed classical background introduces a mesh of symmetry operators. A gauge transformation on A introduces a change in the mesh.

  This justifies a posteriori why the global symmetry becomes a gauge symmetry in the gauge version of the theory. When we sum over all configurations a in the path integral of Equation (44), we sum over all possible insertions of symmetry operators and keep only the invariant pieces, the same way a projection would operate.

’t Hooft anomalies

One could ask whether the path integral is invariant under the symmetry G when a classical background is added:

$$Z[A^g] \stackrel{?}{=} Z[A] \,. \tag{49}$$

This is not always the case. When it is not, the non-invariance is encapsulated in a term $f(A, g)$:

$$D\Phi^g \, e^{iS[\Phi^g;A^g]} = D\Phi \, e^{iS[\Phi;A]} e^{i\int \mathrm{d}^d x \, f(A,g)} \tag{50}$$

There is still hope to cancel this variation with the introduction of local counterterms $c[\Phi; A]$:

$$S[\Phi; A] \longrightarrow S[\Phi; A] + c[\Phi; A] \,. \tag{51}$$

For instance, these would modify the minimal coupling of Equation (40).

The addition of counterterms is not always enough to get rid of the non-invariance. In this case, this signals the presence of a *’t Hooft anomaly* in the field theory. It should not be seen as a flaw. The field theory is perfectly sane as it is and this anomaly can be used to infer some of its properties. A first observation is that we cannot simply sum over gauge field configurations to gauge the symmetry G. There is an ambiguity in the definition of $Z[A]$.

Anomaly inflow

There is one way of getting rid of the ’t Hooft anomaly. It can be done by claiming that $\mathscr{M}_d$ is the boundary of a higher-dimensional bulk $\mathscr{X}_{d+1}$ where the classical background A lives too. We say that the bulk supports the *anomaly theory*, which is an action for the classical field:

$$\mathcal{A}_{d+1}[A] = 2\pi \int_{\mathscr{X}_{d+1}} \mathrm{d}^{d+1}x \, \omega(A) \,. \tag{52}$$

Then, we ask that

$$\omega(A^g) = \omega(A) - \partial_\mu(n^\mu f(A, g)) \,. \tag{53}$$

We are then prepared to introduce a partition function for the whole bulk-boundary system:

$$Z'[A] \equiv Z[A] e^{i\mathcal{A}_{d+1}[A]} \,. \tag{54}$$

This one is invariant under the symmetry transformation:

$$Z'[A^g] \equiv Z'[A] \,. \tag{55}$$

The anomaly theory fully characterizes the anomaly and allows us to gauge the symmetry at the condition to keep the bulk and boundary together.

Anomalies and the infrared

The anomaly is an RG flow invariant. If G bears an anomaly in the UV, it does in the IR too. Importantly then, the IR cannot be trivially gapped! In line with what we said about the possible ends of the RG flow in section 2.2, we can infer some general remarks on the presence of ’t Hooft anomalies:

- $d = 0 + 1$: 't Hooft anomalies correspond to projective representations of symmetries.
- $d = 1 + 1$: Topological order is not possible in this dimension. So, anomalies signal either gapless or degenerate gapped vacua.
- $d \geqslant 2 + 1$: Anomalies signal topological order, gapless, or degenerate gapped vacua.

Note that discrete symmetries may feature anomalies too.

3. Superfluids

At temperatures below 2.2 K, liquid helium-4 (^{4}He) is in a *superfluid phase* [20]. It deserves this qualification since it satisfies the following characteristics:

- The flow of the fluid preserves the number of ^{4}He atoms because these are stable isotopes.
- All ^{4}He atoms may occupy the same lowest energy state, forming a Bose-Einstein condensate.
- The flow of the fluid is non-viscous.

Drawing parallels to solids in frictionless contact, it costs arbitrarily small amounts of energy to generate motion in a non-viscous fluid. Hence, a massless pole should be found in the correlation functions of the field theory of a superfluid. We say that superfluids are gapless phases of matter, as opposed to normal fluids with their gapped excitations.

The stability of ^{4}He isotopes signals that their number should be a conserved charge, naturally fitting in a representation of $U(1)$, meaning it belongs to $\mathbb{Z}$. This is the crucial hint that the corresponding field theory should be invariant under a $U(1)$ symmetry acting on a certain field Φ:

$$U(1) \quad : \quad \Phi \to e^{i\alpha}\Phi \,. \tag{56}$$

Then, Goldstone's theorem nicely connects the Bose-Einstein condensate, here the finite VEV of the field,

$$\langle \Phi \rangle = v \,, \tag{57}$$

to the presence of a massless mode, the Goldstone boson [21]. The classical version of the theorem will reviewed in the following section.

This brief account on superfluids is another illustration of how symmetries, and in this case Goldstone's theorem, characterize effectively some gapless phases of matter. However, the theorem has known limitations in field theories with low dimensions due to infrared fluctuations, as shown by the Coleman-Hohenberg-Mermin-Wagner theorem [22–24]. This is our motivation to introduce an alternative and more recent argument based solely on symmetries and their anomalies to exhibit massless poles [3]. This can be done in any dimension by introducing higher-form symmetries.

3.1 Spontaneous symmetry breaking

For a quantum field theory in d spacetime dimensions to be symmetric under a $U(1)$ group, we need some operator charged under it. So, we introduce a single complex scalar Φ that transforms as follows:

$$U(1) : \quad \Phi \to e^{i\alpha}\Phi\,, \quad \alpha \sim \alpha + 2\pi\,. \tag{58}$$

It is natural to fix its charge to +1 so that the charges of the excited states will match the number of helium isotopes.

To generate a condensate in the vacuum, we ask that the scalar feels a Ginzburg-Landau potential:

$$V(\Phi^*\Phi) = -\frac{m^2}{2}\Phi^*\Phi + \frac{\lambda}{4}(\Phi^*\Phi)^2\,, \tag{59}$$

This potential qualifies because it is invariant under the $U(1)$ symmetry. The energy is bounded below for $\lambda > 0$. The sign of the quadratic term is chosen together with $m^2 > 0$ such that, classically, the energy is minimized whenever

$$|\langle\Phi\rangle| = v = \frac{m}{\sqrt{\lambda}}\,. \tag{60}$$

We say that the scalar develops a non-zero vacuum expectation value v, which is equivalent to a vacuum condensate.

The vacua solutions to Equation (60) form a circle in field space, and the $U(1)$ symmetry moves you continuously from one vacuum to another. In other words, the vacua are not symmetric under $U(1)$ because they fix the phase of $\langle\Phi\rangle$ to a certain value, so $U(1)$ is spontaneously broken. We find the symmetry groups of the theory and vacua to be

$$G = U(1)\,, \quad H = 1\,. \tag{61}$$

Goldstone's theorem

Goldstone's theorem famously states that a spontaneously broken continuous symmetry signals the presence of a massless particle in the spectrum: the *Goldstone boson*. In consequence, the infrared phase of the model must be gapless.

The Goldstone boson can be found classically by expanding the Ginzburg-Landau potential around a spontaneously selected vacuum. Taking $\langle\Phi\rangle = v$ to be purely real suggests the following expansion:

$$\Phi(x) = (v + \rho(x))e^{i\phi(x)}\,. \tag{62}$$

Now, we have two real scalars at our disposal: the non-compact ρ and compact $\phi \sim \phi + 2\pi$. With this expansion, the complete action of the model reads

$$S[\rho,\phi] = \int_{\mathscr{M}_d} \mathrm{d}^d x \left[-\frac{1}{2}\partial_\mu\rho\partial^\mu\rho - \frac{1}{2}(v+\rho)^2\partial_\mu\phi\partial^\mu\phi + m^2\rho^2 + \frac{m^2}{v}\rho^3 + \frac{m^2}{4v^2}\rho^4 - \frac{m^2v^2}{4}\right]. \tag{63}$$

This makes it clear that ρ has a mass m. The compact ϕ is massless and happens to be the Goldstone boson for the broken $U(1)$.

Goldstone, Salam, and Weinberg proved in a seminal paper that this idea generalizes at the quantum level [21]. The result is usually quoted as *Goldstone's theorem*. It may be formulated as follows:

Goldstone's theorem

Consider a quantum field theory with a continuous and internal symmetry G. If the theory develops a vacuum expectation value $\langle \Phi \rangle = v$ preserved by $H \subset G$, and such that $H \neq G$, it contains a massless particle in its spectrum.

The theorem is sometimes refined to include the counting rules for the number of Goldstone bosons, which may be subtle in the context of non-relativistic field theories or for the breaking of spacetime symmetries. See [25] for a recent review on this matter.

Effective field theory for the Goldstone boson

For energies below m, the field ρ can be integrated out from the expanded action in Equation (63). The effective action for the Goldstone boson ϕ then takes into account the interactions between ρ and $\partial\phi$ in the form of quartic and higher couplings:

$$S[\phi] = \int_{\mathscr{M}_d} \mathrm{d}^d x \left[-\frac{v^2}{2} \partial_\mu \phi \partial^\mu \phi + \frac{a_4}{m^2} (\partial_\mu \phi \partial^\mu \phi)^2 + \frac{a_6}{m^4} (\partial_\mu \phi \partial^\mu \phi)^3 + \cdots \right], \tag{64}$$

where a_4 and a_6 are numerical factors. The expansion goes on with higher derivative terms. At the lowest energies, one retains only the term with fewer derivatives:

$$S[\phi] = -\frac{v^2}{2} \int_{\mathscr{M}_d} \mathrm{d}^d x \, \partial_\mu \phi \partial^\mu \phi \, . \tag{65}$$

In both actions, the spontaneously broken $U(1)$ symmetry is realized as a shift, or momentum, symmetry:

$$U(1)_{\mathrm{m}} : \quad \phi \to \phi + \alpha \, , \quad \alpha \sim \alpha + 2\pi \, . \tag{66}$$

Normal fluids

Notice that if the sign of the quadratic term in Equation (59) was positive, then we would be dealing with a field theory of (anti)particles with a mass proportional to m. In this gapped phase, the potential is minimized with

$$\langle \Phi \rangle = 0 \, , \tag{67}$$

and $G = H = U(1)$. This is show that the VEV $\langle \Phi \rangle$ is the order parameter for the phase transition.

3.2 Infrared fluctuations in low dimensions

A key assumption in the proof of Goldstone's theorem is that the complex scalar, that serves as an order parameter in this context, develops *long-range order*:

$$\langle \Phi \rangle = v \, , \tag{68}$$

where we arbitrarily set the phase to zero. We already saw that this is realized at the level of classical field theory by minimizing a Ginzburg-Landau potential, but the vacuum expectation value may actually suffer from quantum or thermal fluctuations that dominate the infrared.

This problem of fluctuations does not concern the massive field ρ whose infrared behavior is regulated by its mass m. We can forget about it and directly assume

$$\langle\Phi\rangle = v\langle e^{i\phi}\rangle \tag{69}$$

as starting point and investigate on its compatibility with Equation (68). In the quantum theory of Equation (65), this is done by separating the positive and negative energy modes of the compact field:

$$\phi = \phi^+ + \phi^- \,, \quad \text{such that} \quad \phi^- \left|0\right\rangle = 0\,, \quad \left\langle 0\right| \phi^+ = 0\,. \tag{70}$$

Using Baker–Campbell–Hausdorff formula together with $[\phi^\pm, [\phi^+, \phi^-]] = 0$, one finds

$$\langle e^{i\phi}\rangle = \langle e^{i\phi^+} e^{[\phi^+,\phi^-]/2} e^{i\phi^-}\rangle = e^{-\langle[\phi^-,\phi^+]\rangle/2}\,. \tag{71}$$

Note that

$$\langle\phi(x)\phi(y)\rangle = \langle\phi^-(x)\phi^+(y)\rangle = \langle[\phi^-(x), \phi^+(y)]\rangle\,, \tag{72}$$

so we can write

$$\langle e^{i\phi}\rangle = e^{-\langle\phi(x)\phi(x)\rangle/2}\,. \tag{73}$$

The vacuum expectation value is expressed in terms of a two-point function evaluated at coincident points, so it will carry UV divergences. It is customary to take care of these in quantum field theory with renormalization. The real trouble will come from potential IR divergences also present in the two-point function.

We can evaluate the two-point function in an Euclidean d-dimensional spacetime using an IR cut-off Λ_{IR}:

$$\langle\phi(x)\phi(y)\rangle = \frac{1}{v^2}\int_{\Lambda_{\text{IR}}} \frac{\mathrm{d}^d k}{(2\pi)^d}\frac{e^{-ik\cdot(x-y)}}{k^2} \propto \begin{cases} -\log(\Lambda_{\text{IR}}|x-y|) & d = 1+1 \\ -|x-y|^{2-d} + \Lambda_{\text{IR}}^{d-2} & d > 1+1 \end{cases}\,. \tag{74}$$

The harmless UV divergence can be seen in both expressions when $x \to y$. An IR divergence is also present for $d = 1+1$ in the limit of $\Lambda_{\text{IR}} \to 0$. The latter cannot be treated with counterterms in the action and reveals a true physical feature of the theory, that is

$$\langle e^{i\phi}\rangle \overset{\Lambda_{\text{IR}}\to 0}{\longrightarrow} 0 \quad \Rightarrow \quad \langle\Phi\rangle = 0 \quad \text{for} \quad d = 1+1\,. \tag{75}$$

The outcome is that there is no long range order for a relativistic quantum field theory in $d = 1+1$. On the other side, spontaneous symmetry breaking remains possible in higher dimensions.

This result generalizes with thermal fluctuations in statistical field theories in one dimension higher. It also has a counterpart for discrete symmetries in lower dimensions. We summarize all of theses results below:

Coleman-Mermin-Wagner theorems

There are spacetime dimensions d for which long-range order, $\langle\Phi\rangle = v$, and so spontaneous symmetry breaking, are spoiled by infrared fluctuations:

$\langle\Phi\rangle = 0$ is imposed by	quantum fluctuations	thermal fluctuations
Continuous symmetry	$d \leqslant 1+1$	$d \leqslant 2+1$
Discrete symmetry	$d = 0+1$	$d = 1+1$

The conclusion of this section is that the gaplessness of the superfluid phase in $d > 1+1$ in a quantum theory follows from a symmetry principle thanks to Goldstone's theorem. But if we were to find a gapless mode in a $d = 1+1$ scalar theory, some other principle should be invoked. The same can be said for a statistical theory in $d = 2+1$.

3.3 The compact scalar in $d = 1+1$

Can superfluids be accommodated in quantum field theory when $d = 1+1$? In statistical physics, this is equivalent to asking whether a superfluid phase exists in two-dimensional materials, or thin layers. The answer is that such phases do exist. However, we have already seen that they cannot coexist with long-range order, so there must be something special about these dimensions. Indeed, the phase transition to these states is known to be of infinite-order and is called the Berezinskii-Kosterlitz-Thouless transition, sometimes referred to as a topological phase transition.

As we will see, the model of Equation (65) with $d = 1+1$ is a CFT (with central charge $c = 1$) and is known to exhibit a massless pole. A priori, this action qualifies to study a bosonic superfluid and it is natural to start here with respect to higher dimensions. So, what does protect the masslessness in this case?

We will explore in more details the theory of a free compact scalar in $1+1$-dimensions.

$$S[\phi] = -\frac{R^2}{4\pi}\int_{\mathscr{M}_2} \mathrm{d}^2x\ \partial_\mu\phi\partial^\mu\phi\,. \tag{76}$$

Note the change of notations with respect to the previous section ($R^2 = 2\pi v^2$). The dimensions are $[\phi] = [R] = 0$. For what follows, the compactness of ϕ is crucial, and we recall that the field is normalized such that

$$\phi \sim \phi + 2\pi\,. \tag{77}$$

The theory displays a *momentum symmetry* that acts a shift on the compact field:

$$U(1)_{\mathrm{m}}\ :\quad \phi \longrightarrow \phi + \alpha\,,\quad \alpha \sim \alpha + 2\pi\,. \tag{78}$$

This $U(1)_{\mathrm{m}}$ is the analogue of the one that was spontaneously broken in higher dimensions. The corresponding conserved current is given by

$$j_\mu = \frac{R^2}{2\pi}\partial_\mu\phi\,. \tag{79}$$

It is easy to check that it is a Noether current and its conservation follows simply from the equation of motion:

$$\frac{\delta S}{\delta\phi} = \frac{R^2}{2\pi}\partial_\mu\partial^\mu\phi = 0\,. \tag{80}$$

If our spacetime decomposes as $\mathscr{M}_2 = \mathbb{R}\times\mathscr{N}_1$ with $\mathscr{N}_1$ the purely spatial component, then the following charge is conserved over time:

$$Q(\mathscr{N}_1) = \frac{R^2}{2\pi}\int_{\mathscr{N}_1}\mathrm{d}x\ \partial_t\phi\,. \tag{81}$$

In an Euclidean spacetime $\mathscr{M}_2$, the charge density can be integrated over any closed contour $\mathscr{C}_1$:

$$Q(\mathscr{C}_1) = \frac{R^2}{2\pi} \oint_{\mathscr{C}_1} \mathrm{d}x^\mu \; \epsilon_{\mu\nu} \partial^\nu \phi \,. \tag{82}$$

Whenever a current is a gradient, we can use the epsilon symbol to construct another conserved current in the theory:

$$\widetilde{j}_\mu = \frac{\epsilon_{\mu\nu}}{R^2} j^\nu = \frac{\epsilon_{\mu\nu}}{2\pi} \partial^\nu \phi \,. \tag{83}$$

Its conservation is guaranteed by topology (as long as ϕ is not singular) and does not rely on the equation of motion. This is an example of *topological current*. The corresponding charge that is conserved over time writes as follows:

$$\widetilde{Q}(\mathscr{N}_1) = \frac{1}{2\pi} \int_{\mathscr{N}_1} \mathrm{d}x \; \partial_x \phi \,. \tag{84}$$

In an Euclidean spacetime, we will find

$$\widetilde{Q}(\mathscr{C}_1) = \frac{1}{2\pi} \oint_{\mathscr{C}_1} \mathrm{d}x^\mu \; \partial_\mu \phi \,. \tag{85}$$

But what does it measure?

Winding

If the field $\mathscr{C}_1$ is a contour on a plane where ϕ is smooth and well-defined, then Equation (85) vanishes trivially and the charge does not measure anything. To get a non-trivial answer, we have to resort to non-trivial topological configurations.

Once again, the compact scalar is subject to the following identification:

$$\phi \sim \phi + 2\pi \,. \tag{86}$$

This redundancy in ϕ can be used on a generic spacetime manifold $\mathscr{M}_2$ while gluing different patches into an atlas. In the following, we will treat $\mathscr{M}_2$ as an Euclidean manifold.

For instance, consider the Euclidean plane with an hole in the middle. The pierced plane can be covered with three patches $\mathscr{U}_2^{(i)} \subset \mathscr{M}_2$, $i = 1, 2, 3$, where ϕ is represented by $\phi^{(i)}$. The patching is illustrated on Figure 3. We chose the patches such that every intersection $\mathscr{U}_2^{(i)} \cap \mathscr{U}_2^{(j)}$ is connected. Since the scalar is compact, this implies that $\phi^{(i)}$ and $\phi^{(j)}$ may differ by an unique multiple of 2π:

$$\phi^{(i)} - \phi^{(j)} = 2\pi m^{ij} \,, \quad m^{ij} \in \mathbb{Z} \,, \quad \text{on } \mathscr{U}_2^{(i)} \cap \mathscr{U}_2^{(j)} \,. \tag{87}$$

In our example, we will choose

$$\begin{aligned} \phi^{(2)} - \phi^{(1)} &= 0 && \text{on } \mathscr{U}_2^{(2)} \cap \mathscr{U}_2^{(1)} \,, \\ \phi^{(3)} - \phi^{(2)} &= 0 && \text{on } \mathscr{U}_2^{(3)} \cap \mathscr{U}_2^{(2)} \,, \\ \phi^{(1)} - \phi^{(3)} &= 2\pi m && \text{on } \mathscr{U}_2^{(1)} \cap \mathscr{U}_2^{(3)} \,. \end{aligned} \tag{88}$$

In general, consistency imposes a cocycle condition on triple intersections, if any:

$$m^{ij} + m^{jk} + m^{ki} = 0 \quad \text{on } \mathscr{U}_2^{(i)} \cap \mathscr{U}_2^{(j)} \cap \mathscr{U}_2^{(k)} \,. \tag{89}$$

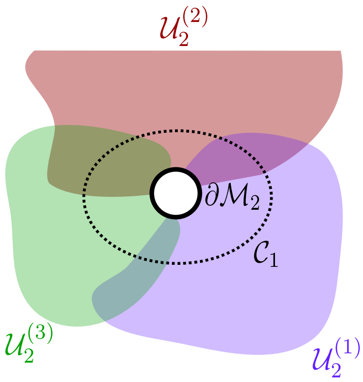


Figure 3: Three patches $\mathscr{U}_2^{(i)}$, with $i = 1, 2, 3$, are introduced in the holed spacetime $\mathscr{M}_2$. The closed contour $\mathscr{C}_1$ can be used to integrate the winding charge $\widetilde{Q}$.

Draw a closed contour $\mathscr{C}_1$ around the hole in our example such that it goes through every of the three patches, as shown on Figure 3. The topological charge along this contour is given by

$$\widetilde{Q}(\mathscr{C}_1) = \frac{1}{2\pi} \oint_{\mathscr{C}_1} \mathrm{d}x^\mu \; \partial_\mu \phi \, . \tag{90}$$

Find three points $x_{ij} \in \mathscr{U}^{(i)} \cap \mathscr{U}^{(j)}$ and such that they belong to the contour, $x_{ij} \in \mathscr{C}_1$. We can finally compute the topological charge:

$$\begin{aligned} \widetilde{Q}(\mathscr{C}_1) &= \tfrac{1}{2\pi} \left[\left(\phi^{(1)}(x_{13}) - \phi^{(1)}(x_{21})\right) + \left(\phi^{(2)}(x_{21}) - \phi^{(2)}(x_{32})\right) + \left(\phi^{(3)}(x_{32}) - \phi^{(3)}(x_{13})\right) \right] \\ &= m \, . \end{aligned} \tag{91}$$

The topological charge measures the *winding* of the field around the hole. It counts how much the field increases or decreases around a non-trivial cycle in the spacetime geometry. All of this topological information is contained in the integers used to glue the patches.

When the field is in such a topologically non-trivial configuration, say with $m \neq 0$, we call it a *vortex* or *instanton*. If we introduce polar coordinates $(r > 0, \varphi \sim \varphi + 2\pi)$ on the pierced plane such that the hole is located at $r = 0$, then a minimal example of vortex solution with winding m can be written as

$$\phi(r, \varphi) = m\varphi \, . \tag{92}$$

More generally, we can introduce a m-vortex by removing a disk $\mathscr{D}_2$ (or tube in higher d) from the spacetime geometry together with a condition on the boundary $\partial\mathscr{D}_2$ for the compact field:

$$\frac{1}{2\pi} \oint_{\partial\mathscr{D}_2} \mathrm{d}\phi = m \tag{93}$$

imposed as a boundary condition.

The winding charges are always integers, and so suggests that the topological symmetry is actually a $U(1)_\mathrm{w}$. Since it is a topological symmetry, it is not seen as acting on ϕ. We will see next how to define operators charged under $U(1)_\mathrm{w}$.

Dual field

Vortices are intrinsically non-perturbative objects. To be seen as usual perturbative excitations, we need a dual description of the theory in terms of a *dual field*.

Start with the partition function (with loose notation)

$$Z = \int D\phi \, e^{-i\int \frac{R^2}{4\pi}(\partial\phi)^2} . \tag{94}$$

Notice that the action does only depend on $\partial\phi$. Thus we can treat $\partial\phi$ as a field on its own in the partition function at the price of specifying explicitly the topological condition,

$$\frac{\epsilon^{\mu\nu}}{2\pi}\partial_\mu\partial_\nu\phi = 0 , \tag{95}$$

with the help of a Lagrange multiplier $\widetilde{\phi}$. Neglecting meaningless factors in front of the path integral and integrating by parts on the Lagrange multiplier term, we find

$$Z = \int D\partial\phi D\widetilde{\phi} \, e^{-i\int \frac{R^2}{4\pi}(\partial\phi)^2 - \widetilde{\phi}\frac{\epsilon^{\mu\nu}}{2\pi}\partial_\mu\partial_\nu\phi} = \int D\partial\phi D\widetilde{\phi} \, e^{-i\int \frac{R^2}{4\pi}(\partial\phi)^2 + \frac{\epsilon^{\mu\nu}}{2\pi}\partial_\mu\widetilde{\phi}\partial_\nu\phi} . \tag{96}$$

The equations of motion are

$$2\pi\frac{\delta S}{\delta\partial^\mu\phi} = R^2\partial_\mu\phi - \epsilon_{\mu\nu}\partial^\nu\widetilde{\phi} = 0 , \quad 2\pi\frac{\delta S}{\delta\widetilde{\phi}} = \epsilon^{\mu\nu}\partial_\mu\partial_\nu\phi = 0 . \tag{97}$$

The first equation shows how to relate $\widetilde{\phi}$ to ϕ, and the second just imposes the topological condition, as expected.

Because the theory is free, it is easy now to integrate out $\partial\phi$, using its equation of motion. We find

$$Z = \int D\widetilde{\phi} \, e^{-i\int \frac{1}{4\pi R^2}(\partial\widetilde{\phi})^2} . \tag{98}$$

This exactly the same theory as the original with the coupling R replaced by $1/R$. This is T-duality in action, and $\widetilde{\phi} \sim \widetilde{\phi} + 2\pi$ is the compact scalar field T-dual to ϕ.

From the first equation of motion in Equation (97), we find that the winding charge for ϕ is a momentum charge for $\widetilde{\phi}$:

$$\widetilde{Q}(\mathscr{C}_1) = \frac{1}{2\pi}\oint_{\mathscr{C}_1} \mathrm{d}x^\mu \; \partial_\mu\phi = \frac{1}{2\pi R^2}\oint_{\mathscr{C}_1} \mathrm{d}x^\mu \; \epsilon_{\mu\nu}\partial^\nu\widetilde{\phi} . \tag{99}$$

So, the *winding symmetry* $U(1)_\mathrm{w}$ acts as a momentum symmetry on the dual field. The continuous symmetries we found are summarized below:[4]

$$\begin{aligned} U(1)_\mathrm{m} \quad &: \quad \phi \to \phi + \alpha , \quad \alpha \sim \alpha + 2\pi . \\ U(1)_\mathrm{w} \quad &: \quad \widetilde{\phi} \to \widetilde{\phi} + \beta , \quad \beta \sim \beta + 2\pi . \end{aligned} \tag{101}$$

[4]Note in passing that the theory of the compact scalar also enjoys a discrete symmetry, the conjugation operation:

$$\mathbb{Z}_2 : \quad (\phi, \widetilde{\phi}) \to -(\phi, \widetilde{\phi}) . \tag{100}$$

The operators respectively charged under $U(1)_\mathrm{m}$ and $U(1)_\mathrm{w}$ are

$$e^{in\phi} \quad \text{and} \quad e^{im\widetilde{\phi}}\,, \quad \text{for} \quad n,m\in\mathbb{Z}\,. \tag{102}$$

The operator $e^{im\widetilde{\phi}}$ gives an alternative way to introduce a vortex in a correlation function.

Returning to the statistical physics perspective, in $d=2+1$, the dual framework is particularly convenient for studying the phase at high temperatures, i.e. above the temperature at which the Berezinskii-Kosterlitz-Thouless transition occurs. In this phase, vortex configurations are favored and proliferate.

Mixed 't Hooft anomaly

There is something special about the momentum and winding symmetries that we will be able to use in order to show the existence of a massless mode in $d=1+1$, a *mixed 't Hooft anomaly*.

We introduce two background sources A_μ and $\widetilde{A}_\mu$ for the currents j_μ and $\widetilde{j}_\mu$ respectively. They transform under an improved $U(1)_\mathrm{m}\times U(1)_\mathrm{w}$:

$$\begin{aligned} U(1)_\mathrm{m} \quad &: \quad \begin{cases} \phi\to\phi+\alpha\,, \\ A_\mu\to A_\mu+\partial_\mu\alpha\,, \end{cases} \quad \alpha\sim\alpha+2\pi\,. \\ U(1)_\mathrm{w} \quad &: \quad \begin{cases} \widetilde{\phi}\to\widetilde{\phi}+\beta\,, \\ \widetilde{A}_\mu\to\widetilde{A}_\mu+\partial_\mu\beta\,, \end{cases} \quad \beta\sim\beta+2\pi\,. \end{aligned} \tag{103}$$

where α and β may vary in spacetime. We must then check if the partition function with sources is still invariant under the symmetries:

$$Z[A_\mu+\partial_\mu\alpha,\widetilde{A}_\mu+\partial_\mu\beta] \stackrel{?}{=} Z[A_\mu,\widetilde{A}_\mu]\,. \tag{104}$$

If not, this signals the presence of a 't Hooft anomaly.

In our free theory, this question can already be asked at the classical level, i.e. with the action alone. Introduce first the background A_μ for $U(1)_\mathrm{m}$:

$$S[\phi;A_\mu] = \int_{\mathcal{M}_2}\mathrm{d}^2x\,\left[-\frac{R^2}{4\pi}\partial_\mu\phi\partial^\mu\phi + j_\mu A^\mu - \frac{R^2}{4\pi}A_\mu A^\mu\right] = -\frac{R^2}{4\pi}\int_{\mathcal{M}_2}\mathrm{d}^2x\,(\partial_\mu\phi-A_\mu)(\partial^\mu\phi-A^\mu)\,. \tag{105}$$

Note the necessity to introduce a quadratic term in A_μ, also called *Meissner term*, in order to maintain invariance under the improved $U(1)_\mathrm{m}$. This is typical from situations where A_μ couples to a field that shifts under a symmetry. In fact, the Meissner term shows that $U(1)_\mathrm{m}$ alone is not anomalous because both $S[\phi;A_\mu]$ and the measure in the path integral are invariant under the momentum symmetry.

This action can be used to extract the *improved current* of $U(1)_\mathrm{m}$:

$$J_\mu = \frac{\delta S}{\delta A^\mu} = \frac{R^2}{2\pi}(\partial_\mu\phi - A_\mu)\,. \tag{106}$$

The latter is simply the gauge-invariant version of j_μ, also called supercurrent in the context of superfluidity. The equation of motion indicates that this current is conserved even in the presence of a background:

$$\frac{\delta S}{\delta\phi} = \frac{R^2}{2\pi}\partial_\mu(\partial^\mu\phi - A^\mu) = \partial_\mu J^\mu = 0\,. \tag{107}$$

This is another way to see that $U(1)_\mathrm{m}$ alone is not anomalous.

Try now to add a background $\widetilde{A}_\mu$ for $U(1)_\mathrm{w}$ in the usual manner:

$$S'[\phi; A_\mu, \widetilde{A}_\mu] = \int_{\mathscr{M}_2} \mathrm{d}^2x \left[-\frac{R^2}{4\pi}(\partial_\mu\phi - A_\mu)(\partial^\mu\phi - A^\mu) + \widetilde{A}_\mu \frac{\epsilon^{\mu\nu}}{2\pi}\partial_\nu\phi \right] . \tag{108}$$

This action is invariant up to a boundary term under the improved $U(1)_\mathrm{w}$:

$$\delta_\beta S'[\phi; A_\mu, \widetilde{A}_\mu] = \int_{\mathscr{M}_2} \mathrm{d}^2x \; \partial_\mu \left(\beta \frac{\epsilon^{\mu\nu}}{2\pi}\partial_\nu\phi \right) , \tag{109}$$

which is acceptable. But it is not invariant under $U(1)_\mathrm{m}$:

$$\delta_\alpha S'[\phi; A_\mu, \widetilde{A}_\mu] = \int_{\mathscr{M}_2} \mathrm{d}^2x \; \widetilde{A}_\mu \frac{\epsilon^{\mu\nu}}{2\pi}\partial_\nu\alpha . \tag{110}$$

You can try to fix the situation by adding a counterterm:

$$S''[\phi; A_\mu, \widetilde{A}_\mu] = S'[\phi; A_\mu, \widetilde{A}_\mu] - \int_{\mathscr{M}_2} \mathrm{d}^2x \; \widetilde{A}_\mu \frac{\epsilon^{\mu\nu}}{2\pi} A_\nu , \tag{111}$$

but now we find a lack of invariance under $U(1)_\mathrm{w}$:

$$\delta_\beta S''[\phi; A_\mu, \widetilde{A}_\mu] = -\int_{\mathscr{M}_2} \mathrm{d}^2x \; \partial_\mu\beta \frac{\epsilon^{\mu\nu}}{2\pi} A_\nu . \tag{112}$$

There is in fact no way to get rid of this non-invariance. The impossibility to gauge both $U(1)_\mathrm{m}$ and $U(1)_\mathrm{w}$ at the same time signals the presence of a mixed 't Hooft anomaly. The anomaly theory in $d = 2 + 1$ is

$$\mathcal{A}[A_\mu, \widetilde{A}_\mu] = -\int_{\mathscr{X}_3} \mathrm{d}^3x \; \frac{\epsilon^{\mu\nu\rho}}{2\pi} \widetilde{A}_\mu \partial_\nu A_\rho . \tag{113}$$

In fact, because we have an intimate relation between the two currents in absence of sources:

$$\widetilde{j}_\mu = \frac{\epsilon_{\mu\nu}}{R^2} j^\nu , \tag{114}$$

we can actually use a single background to extract both improved currents from the action in Equation (105). Consider

$$\widetilde{A}_\mu = R^2 \epsilon_{\mu\nu} A^\nu , \qquad \frac{\delta}{\delta\widetilde{A}^\mu} = \frac{\epsilon_{\mu\nu}}{R^2}\frac{\delta}{\delta A_\nu} . \tag{115}$$

We can use $S[\phi, A_\mu]$ from Equation (105) to extract the improved topological current:

$$\widetilde{J}_\mu = \frac{\delta S}{\delta\widetilde{A}^\mu} = \frac{\epsilon_{\mu\nu}}{2\pi}(\partial^\nu\phi - A^\nu) . \tag{116}$$

The latter is gauge invariant, but is no longer conserved when the $U(1)_\mathrm{m}$ classical background is curved:

$$\partial_\mu \widetilde{J}^\mu = -\frac{\epsilon^{\mu\nu}}{2\pi}\partial_\mu A_\nu . \tag{117}$$

This modified Ward identity represents the well-known fact that applying an external electric field generates winding planes in a superfluid. In the dual picture, one finds the opposite situation: $\widetilde{J}_\mu$ is conserved, but the conservation of J_μ is spoiled by $\epsilon^{\mu\nu}\partial_\mu\widetilde{A}_\nu$.

The massless pole

The (non)-conservation laws derived above lead to the following Ward identities:

$$\partial_\mu \langle J^\mu \rangle_A = 0\,, \quad \partial_\mu \langle \widetilde{J}^\mu \rangle_A = -\frac{\epsilon_{\mu\nu}}{2\pi} \partial_\mu A_\nu\,. \tag{118}$$

Consider the following two-point function involving the improved currents:

$$\langle J_\mu(x) J_\nu(y) \rangle = \int \frac{\mathrm{d}^2 x}{(2\pi)^2} e^{-ik\cdot(x-y)} \Pi_{\mu\nu}(k)\,. \tag{119}$$

We find that $\Pi_{\mu\nu}$ also determines mixed two-point functions:

$$\langle J_\mu(x) \widetilde{J}_\nu(y) \rangle_A = \frac{\epsilon_{\nu\sigma}}{R^2} \langle J_\mu(x) J^\sigma(y) \rangle_A\,. \tag{120}$$

From its definition, it is clear that $\Pi_{\mu\nu}$ has an obvious constraint from the permutation of indices:

$$\Pi_{\mu\nu}(k) = \Pi_{\nu\mu}(-k)\,. \tag{121}$$

Using Lorentz invariance, this leads to

$$\Pi_{\mu\nu}(k) = f(k^2) k^2 g_{\mu\nu} + g(k^2) k_\mu k_\nu\,, \tag{122}$$

with f and g still unknown.[5]

We can also use

$$\langle J_\mu(x) J_\nu(y) \rangle_A = \frac{1}{Z[A]} \frac{\delta}{\delta A^\mu(x)} \frac{\delta}{\delta A^\nu(y)} Z[A] = \frac{1}{Z[A]} \frac{\delta}{\delta A^\mu(x)} \left(Z[A] \langle J_\nu(y) \rangle_A \right) \tag{123}$$

to show with Equation (118) that

$$\langle J_\mu(x) \partial^\nu J_\nu(y) \rangle_A = \frac{1}{Z[A]} \frac{\delta}{\delta A^\mu(x)} \left(Z[A] \langle \partial^\nu J_\nu(y) \rangle_A \right) = 0\,. \tag{124}$$

This implies the following relation:

$$ik^\nu \Gamma_{\mu\nu} = 0 \quad \Leftrightarrow \quad f(k^2) k^2 + g(k^2) k^2 = 0\,. \tag{125}$$

Similarly, one can derive that

$$\langle J_\mu(x) \partial^\nu \widetilde{J}_\nu(y) \rangle_A = \frac{1}{Z[A]} \frac{\delta}{\delta A^\mu(x)} \left(Z[A] \langle \partial^\nu \widetilde{J}_\nu(y) \rangle_A \right) = -\frac{\epsilon_{\nu\sigma}}{2\pi} \langle J_\mu(x) \rangle_A \partial^\nu A^\sigma + \frac{\epsilon_{\mu\nu}}{2\pi} \partial^\nu \delta^2(x-y)\,. \tag{126}$$

Putting the source at zero, $A = 0$, this gives

$$\langle J_\mu(x) \partial^\nu \widetilde{J}_\nu(y) \rangle = \frac{\epsilon_{\mu\nu}}{2\pi} \partial^\nu \delta^2(x-y)\,, \tag{127}$$

and so

$$ik_\nu \frac{\epsilon^{\nu\sigma}}{R^2} \Pi_{\mu\sigma}(k) = i\frac{\epsilon_{\mu\nu}}{2\pi} k^\nu \quad \Leftrightarrow \quad f(k^2) = -\frac{R^2}{2\pi} \frac{1}{k^2}\,. \tag{128}$$

[5] Because the theory is conformal, additional symmetry arguments could be employed to constrain the functions f and g. However, we choose to rely on the anomaly instead, as this approach is the only one that generalizes to higher dimensions.

Eventually, we find

$$\Pi_{\mu\nu}(k) = \frac{R^2}{2\pi}\frac{k_\mu k_\nu - k^2 g_{\mu\nu}}{k^2}\,. \tag{129}$$

The propagator exhibits a pole at $k^2 = 0$ associated to a massless mode.

This entire discussion applies to a broad class of models in any dimension d, provided they exhibit the appropriate symmetry structure:

$$S[\phi; A_\mu] = \int \mathrm{d}^d x\, \mathcal{F}(\partial_\mu\phi - A_\mu)\,, \tag{130}$$

where the function $\mathcal{F}$ may be non-analytic. However, to arrive at this conclusion, it is necessary to introduce higher-form symmetries.

4. Higher-form symmetries

4.1 More winding symmetries

Let us rewrite the free action for a compact scalar $\phi \sim \phi + 2\pi$ in any spacetime $\mathscr{M}_d$ as follows:

$$S[\phi] = -\frac{R^2}{4\pi}\int_{\mathscr{M}_d} \mathrm{d}^d x\; \partial_\mu\phi\partial^\mu\phi\,. \tag{131}$$

We argued that it was always symmetric under a $U(1)_{\mathrm{m}}$ momentum symmetry, whose associated current is

$$j_\mu = \frac{R^2}{2\pi}\partial_\mu\phi\,, \quad \partial_\mu j^\mu = 0\,. \tag{132}$$

This current always all has a topological counterpart, irrespectively of the dimension d:

$$\widetilde{j}_{[\mu_1\cdots\mu_{d-1}]} = \frac{1}{R^2}\epsilon_{\mu_1\cdots\mu_d} j^{\mu_d} = \frac{1}{2\pi}\epsilon_{\mu_1\cdots\mu_d}\partial^{\mu_d}\phi\,, \quad \partial_{\mu_1}\widetilde{j}^{[\mu_1\cdots\mu_{d-1}]} = 0\,, \tag{133}$$

which is conserved even off-shell. The corresponding charge counts the winding of $(d-2)$-dimensional vortices in a $(d-1)$-dimensional space. So the integral is always performed over $(d-1)-(d-2) = 1$ dimension:

$$\widetilde{Q}(\mathscr{C}_1) = \frac{1}{(d-1)!}\oint_{\mathscr{C}_1} \mathrm{d}x^\mu \epsilon_{\mu\nu_1\cdots\nu_{d-1}}\widetilde{j}^{\nu_1\cdots\nu_{d-1}} = \frac{1}{2\pi}\oint_{\mathscr{C}_1}\mathrm{d}x^\mu\partial_\mu\phi\,. \tag{134}$$

In contrast, the momentum charge is still integrated over a $(d-1)$-dimensional manifold:

$$Q(\mathscr{C}_{d-1}) = \frac{1}{(d-1)!}\oint_{\mathscr{C}_{d-1}} \mathrm{d}x^{\mu_1}\cdots\mathrm{d}x^{\mu_{d-1}}\epsilon_{\mu\cdots\mu_{d-1}\nu} j^\nu = \frac{R^2}{2\pi}\oint_{\mathscr{C}_{d-1}}\mathrm{d}^{d-1}x\, n^\mu\partial_\mu\phi\,. \tag{135}$$

The story of the dual field in Section 3.3 can be repeated, with the difference that $\widetilde{\phi}$ now appears as an antisymmetric object with $d-2$ indices:

$$S[\partial\phi, \widetilde{\phi}] \supset \frac{1}{2\pi}\int \mathrm{d}^d x\, \epsilon^{\mu\nu_1\cdots\nu_{d-2}\rho}\partial_\mu\widetilde{\phi}_{\nu_1\cdots\nu_{d-2}}\partial_\rho\phi\,. \tag{136}$$

This modified action is still invariant under a shift of the dual field associated with the winding charge:

$$U(1)^{(d-2)}_{\mathrm{w}} \quad : \quad \widetilde{\phi}_{\nu_1\cdots\nu_{d-2}} \to \widetilde{\phi}_{\nu_1\cdots\nu_{d-2}} + \beta_{\nu_1\cdots\nu_{d-2}}\,, \quad \partial_{[\mu}\beta_{\nu_1\cdots\nu_{d-2}]} = 0\,, \tag{137}$$

with a parameter that satisfies the Bianchi identity. It generalizes that the shift parameter α is constant for the momentum symmetry in any d and that the winding shift parameter β was constant for $d = 1 + 1$.

The correct mathematical language to talk about antisymmetric objects like $\widetilde{\phi}$ and β is the one of differential forms. Because they have $d-2$ indices, they are examples of $(d-2)$-forms. Hence, we say that the winding symmetry is a $(d-2)$-form symmetry, noted $U(1)_{\mathrm{w}}^{(d-2)}$ with charges still being integers. We shall introduce these notions in the next section.

Note in passing that, just like in Section 3.3, the scalar in a d-dimensional spacetime enjoys a $U(1)_{\mathrm{m}}^{(0)} \times U(1)_{\mathrm{w}}^{(d-2)}$ symmetry with a mixed 't Hooft anomaly. So, the later constrains the correlation functions to have a massless pole that now coincides with the Goldstone boson of Section 3.1.

Superfluids

All scalar field theories with the action

$$S[\phi; A_\mu] = \int \mathrm{d}^d x\, \mathcal{F}(\partial_\mu \phi - A_\mu)\,, \tag{138}$$

where $\mathcal{F}$ may be non-analytic, exhibit higher-form symmetries $U(1)_{\mathrm{m}}^{(0)} \times U(1)_{\mathrm{w}}^{(d-2)}$, subject to a mixed 't Hooft anomaly. This anomaly requires the presence of a gapless mode, which corresponds to the Goldstone boson for $d > 1 + 1$.

4.2 Differential forms

Consider an antisymmetric tensor:

$$f_{\mu_1 \cdots \mu_p} = f_{[\mu_1 \cdots \mu_p]}\,, \tag{139}$$

that is smooth and uniquely defined on $\mathscr{M}_d$. It defines a p-form:

$$f = \frac{1}{p!} f_{\mu_1 \cdots \mu_p} \mathrm{d}x^{\mu_1} \wedge \cdots \wedge \mathrm{d}x^{\mu_p}\,, \tag{140}$$

where the definition of the wedge product involves the permutation group S_p:

$$\mathrm{d}x^{\mu_1} \wedge \cdots \wedge \mathrm{d}x^{\mu_p} = \sum_{s \in S_p} \mathrm{sign}(s)\, \mathrm{d}x^{s(\mu_1)} \otimes \cdots \otimes \mathrm{d}x^{s(\mu_p)}\,. \tag{141}$$

The p-form is in fact a mathematical device that just wants to be integrated over a p-dimensional manifold $\mathscr{L}_p$:

$$\int_{\mathscr{L}_p} f = \sum_{\mu_1 < \cdots < \mu_p} \int_{\mathscr{L}_p} \mathrm{d}^p y\, \frac{\partial(x^{\mu_1}, \cdots, x^{\mu_p})}{\partial(y^1, \cdots, y^p)} f_{\mu_1 \cdots \mu_p}\,. \tag{142}$$

For example, take f to represent the electromagnetic tensor with $f_{0i} = \mathcal{E}_i$ being the electric field and $f_{ij} = \epsilon_{ijk}\mathcal{B}_k$ involving the magnetic field. Take the closed 2-dimensional manifold $\mathscr{C}_2$ in space. We find the magnetic flux through $\mathscr{C}_2$ as follows:

$$\oint_{\mathscr{C}_2} f = \frac{1}{2} \oint_{\mathscr{C}_2} \mathrm{d}^2 y\, n_i \epsilon^{ijk} f_{jk} = \oint_{\mathscr{C}_2} \mathrm{d}^2 y\, \vec{n} \cdot \vec{\mathcal{B}}\,, \tag{143}$$

where $\vec{n}$ is the orthonormal normal to $\mathscr{C}_2$.

The exterior derivative

We introduce the *exterior derivative*:

$$\mathrm{d}f = \frac{1}{p!}\partial_{\mu_{p+1}} f_{\mu_1\cdots\mu_p} \mathrm{d}x^{\mu_{p+1}} \wedge \mathrm{d}x^{\mu_1} \wedge \cdots \wedge \mathrm{d}x^{\mu_p} \,. \tag{144}$$

Notably, antisymmetry implies that

$$\mathrm{d}^2 = 0 \,. \tag{145}$$

A p-form f is said to be *closed* if

$$\mathrm{d}f = 0 \quad \Leftrightarrow \quad \partial_{[\mu_{p+1}} f_{\mu_1\cdots\mu_p]} = 0 \,, \tag{146}$$

and it is *exact* if there exists a $(p-1)$-form h such that

$$f = \mathrm{d}h \quad \Leftrightarrow \quad f_{\mu_1\cdots\mu_p} = p\partial_{[\mu_p} h_{\mu_1\cdots\mu_{p-1}]} \,. \tag{147}$$

Due to Equation (145), an exact form is always closed, but the opposite is not true.

Using this formalism, Stoke's theorem takes the following form:

$$\int_{\mathscr{L}_{p+1}} \mathrm{d}f = \oint_{\partial\mathscr{L}_{p+1}} f \,, \tag{148}$$

where $\mathscr{L}_{p+1}$ is a smooth, oriented $(p+1)$-dimensional manifold. Note that, in the context of vortices, we have

$$\mathrm{d}\phi = \partial_\mu \phi \, \mathrm{d}x^\mu \quad \text{with} \quad \frac{1}{2\pi}\oint_{\partial\mathscr{U}_2} \mathrm{d}\phi = \frac{1}{2\pi}\int_{\mathscr{U}_2} \mathrm{d}^2\phi = m \in \mathbb{Z} \,. \tag{149}$$

This does not contradict Equation (145) because ϕ is not uniquely defined on the open manifold $\mathscr{U}_2$ when $m \neq 0$, and so $\mathrm{d}\phi$ is not an exact form, despite what is suggested by the notation.

Another important result is Poincaré's lemma. It states that if a p-form f is closed on an open ball $\mathscr{U}_d \subset \mathscr{M}_d$, then one can always find a $(p-1)$-form h such that the following relation is true locally on $\mathscr{U}_d$:

$$f = \mathrm{d}h \,. \tag{150}$$

Hodge dual

For every p-form f, we may introduce a $(d-p)$-form as its *Hodge dual*:

$$\star f = \frac{\sqrt{|g|}}{p!(d-p)!} f^{\mu_1\cdots\mu_p} \epsilon_{\mu_1\cdots\mu_p\nu_{p+1}\cdots\nu_d} \mathrm{d}x^{\nu^{p+1}} \wedge \cdots \wedge \mathrm{d}x^{\nu^d} \,. \tag{151}$$

Coming back to our example with the electromagnetic tensor, we write the electric flux through $\mathscr{C}_2$ with the Hodge dual form:

$$\oint_{\mathscr{C}_2} \star f = \frac{1}{2}\oint_{\mathscr{C}_2} \mathrm{d}x^\mu \mathrm{d}x^\nu \, \epsilon_{\mu\nu\rho\sigma} f^{\rho\sigma} = \oint_{\mathscr{C}_2} \mathrm{d}^2x \, \vec{n} \cdot \vec{\mathcal{E}} \,. \tag{152}$$

The Hodge dual can be used to construct a volume form, or d-form, that will appear in integrals over all spacetime $\mathscr{M}_d$:

$$f \wedge \star f = \frac{\sqrt{|g|}}{p!} f^{\mu_1 \cdots \mu_p} f_{\mu_1 \cdots \mu_p} \mathrm{d}x^1 \wedge \cdots \wedge \mathrm{d}x^d \,. \tag{153}$$

These will naturally appear in actions of physical theories. For example, we have

$$S[\phi] = -\frac{v^2}{2} \int_{\mathscr{M}_d} \mathrm{d}\phi \wedge \star \mathrm{d}\phi = -\frac{v^2}{2} \int_{\mathscr{M}_d} \mathrm{d}^d x \sqrt{|g|} \partial_\mu \phi \partial^\mu \phi \,, \tag{154}$$

and

$$S[a] = -\frac{1}{2e^2} \int_{\mathscr{M}_d} f \wedge \star f = -\frac{1}{4e^2} \int_{\mathscr{M}_d} \mathrm{d}^d x \sqrt{|g|} f_{\mu\nu} f^{\mu\nu} \,. \tag{155}$$

Finally, note that the exterior derivative and Hodge duality can be combined to reproduce conservation equations. For instance, Maxwell's equations in the vacuum become

$$\mathrm{d} \star f = 0 \quad \Leftrightarrow \quad \partial_\mu f^{\mu\nu} = 0 \quad \Leftrightarrow \quad \begin{cases} \vec{\nabla} \cdot \vec{\mathcal{E}} = 0 \,, \\ \vec{\nabla} \times \vec{\mathcal{B}} - \frac{1}{c^2} \frac{\partial \vec{\mathcal{E}}}{\partial t} = \vec{0} \,, \end{cases} \tag{156}$$

and

$$\mathrm{d} f = 0 \quad \Leftrightarrow \quad \partial_{[\mu} f_{\nu\rho]} = 0 \quad \Leftrightarrow \quad \begin{cases} \vec{\nabla} \cdot \vec{\mathcal{B}} = 0 \,, \\ \vec{\nabla} \times \vec{\mathcal{E}} + \frac{\partial \vec{\mathcal{B}}}{\partial t} = \vec{0} \,. \end{cases} \tag{157}$$

4.3 Properties of the higher-form symmetries

A physical theory is constrained by a continuous higher-form symmetry, say a continuous p-form symmetry, if it admits a conserved $(p+1)$-form current j:

$$\mathrm{d} \star j = 0 \,. \tag{158}$$

This current is integrated over a closed manifold $\mathscr{C}_{d-p-1}$ of codimension $p+1$ to define the conserved charge:

$$Q(\mathscr{C}_{d-p-1}) = \oint_{\mathscr{C}_{d-p-1}} \star j \,. \tag{159}$$

The corresponding symmetry operator takes the following form:

$$U_\alpha(\mathscr{C}_{d-p-1}) = e^{i\alpha Q(\mathscr{C}_{d-p-1})} \,. \tag{160}$$

Charged operators W carrying charge q now extend over p directions in spacetime. We find

$$U_\alpha(\mathscr{C}_{d-p-1}) W(\mathscr{L}_p) = e^{i\alpha q L(\mathscr{C}_{d-p-1}, \mathscr{L}_p)} W(\mathscr{L}_p) \,. \tag{161}$$

See Figure 4 for an illustration of the linking of the operators above.

We list some key properties of p-form symmetries:

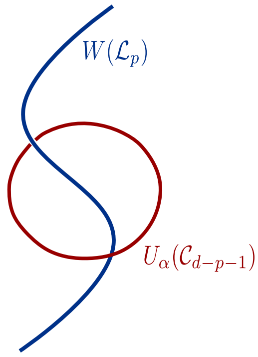


Figure 4: Action of the symmetry operator $U_\alpha(\mathscr{C}_{d-p-1})$ on the line operator $W(\mathscr{L}_p)$. On the picture, $d = 3$ and $p = 1$.

- The symmetry operator U_α is again a topological operator. Let $\mathscr{C}'_{d-p-1}$ be a smoothly deformed version of $\mathscr{C}_{d-p-1}$ along $\mathscr{X}_{d-p}$, see Figure 5. Then,

$$Q(\mathscr{C}_{d-p-1}) - Q(\mathscr{C}'_{d-p-1}) = \oint_{\mathscr{C}_{d-p-1}} \star j - \oint_{\mathscr{C}'_{d-p-1}} \star j = \int_{\mathscr{X}_{d-p}} \mathrm{d} \star j = 0\,. \tag{162}$$

  Note that this result relies on Lorentz invariance.[6]

- For $p > 0$, and assuming spacetime has no torsion, there is no preferred order of symmetry operations, allowing operators to be rearranged freely. In consequence, the fusion rule may only be abelian:

$$U_\alpha(\mathscr{C}_{d-p-1})U_\beta(\mathscr{C}_{d-p-1}) = U_\beta(\mathscr{C}_{d-p-1})U_\alpha(\mathscr{C}_{d-p-1}) = U_{\alpha+\beta}(\mathscr{C}_{d-p-1})\,. \tag{164}$$

- A classical background source can be introduced in the form of a classical $p+1$-gauge field B:

$$S[\Phi; B] \supset \int B \wedge \star j\,. \tag{165}$$

  In particular, (mixed) 't Hooft anomalies may still be present.

- A p-form symmetry may be also discrete, in which case the above properties remain valid, despite the absence of current.

Higher-form symmetry breaking

While anomalies are a powerful tool for elucidating vacuum properties, symmetry breaking remains a valuable concept that we aim to generalize to higher-form symmetries.

[6] Consider the current of a 1-form symmetry, $j^{[\mu\nu]}$. In a relativistic theory, we write its conservation in components:

$$\partial_t j^{[tk]} + \partial_i j^{[ik]} = 0\,, \quad \partial_i j^{[it]} = 0\,. \tag{163}$$

The second equation is a constraint imposed by Lorentz invariance, and is not required for a 1-form symmetry current to be conserved in non-relativistic field theories. However, it is instrumental in Equation (162) [26].

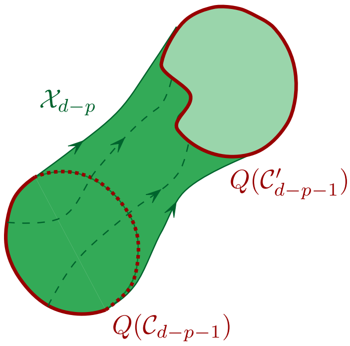


Figure 5: The deformation of $\mathscr{C}_{d-p-1}$ into $\mathscr{C}'_{d-p-1}$ generates a hypersurface $\mathscr{X}_{d-p}$ with two boundaries.

For traditional 0-form symmetries, as discussed in Section 3.1, symmetry breaking is signaled by the value of the order parameter $\langle \Phi \rangle$. Spontaneous symmetry breaking is also reflected in the infrared fluctuations of the two-point functions $\langle \Phi(x)\Phi(0) \rangle$.

$$\begin{array}{llcl} & \text{Order parameter} & & \text{Fluctuations} \\ \text{Spontaneously broken symmetry:} & \langle \Phi \rangle = v \neq 0 & \longrightarrow & \langle \Phi(x)\Phi(0) \rangle \overset{x\to\infty}{=} v^2 \\ \text{Preserved symmetry:} & \langle \Phi \rangle = 0 & \longrightarrow & \langle \Phi(x)\Phi(0) \rangle \overset{x\to\infty}{\propto} e^{-x/L} \end{array} \tag{166}$$

The parameter L is the correlation length of the system.

These properties of two-point functions generalize to p-form symmetries through the *perimeter* and *area* laws for the VEV of the charged operator $W(\mathscr{C}_p)$, defined on a closed $\mathscr{C}_p$.

$$\begin{array}{ll} \text{Spontaneously broken symmetry:} & \langle W(\mathscr{C}_p) \rangle \sim e^{-\text{Perimeter}(\mathscr{L}_p)} \neq 0 \\ \text{Preserved symmetry:} & \langle W(\mathscr{C}_p) \rangle \sim e^{-\text{Area}(\mathscr{L}_p)} \longrightarrow 0 \end{array} \tag{167}$$

Here, the perimeter corresponds to the volume of $\mathscr{C}_p$, while the area refers to the volume enclosed by $\mathscr{C}_p$ in one higher dimension. The perimeter law can be absorbed into a renormalization term of the action, effectively corresponding to a non-zero value for the VEV, reminiscent of spontaneous symmetry breaking. The area law, however, cannot be absorbed through renormalization and will vanish in the limit of a large loop $\mathscr{C}_p$, indicating preserved symmetry.

Generalizations of Goldstone and Coleman's theorems exist for higher-form symmetries [1, 27]. For example, long-range order and spontaneous breaking of a continuous p-form symmetry are prohibited for quantum field theories in a spacetime with dimension $d \leqslant p+2$. In higher dimensions, non-vanishing $\langle W(\mathscr{C}_p) \rangle$ signals the presence of a Goldstone boson in a p-form field correlation function.

5. Electrodynamics

5.1 Maxwell's theory

Let a_μ be a dynamical gauge field. In $d = 1 + 3$, Maxwell's action is written as follows:

$$S = -\frac{1}{2e^2} \int_{\mathscr{M}_4} f \wedge \star f\,, \quad f = \mathrm{d}a\,. \tag{168}$$

This leads to the following well-known equation of motion:

$$\frac{1}{e^2}\mathrm{d}\star f = 0\,. \tag{169}$$

It is often accompanied by a topological constraint of f, known as the Bianchi identity:

$$\frac{1}{2\pi}\mathrm{d}f = 0\,, \tag{170}$$

which follows from the definition of f in term of a. These two equations can be expressed as conservation laws for the electric and magnetic 2-form currents, j and $\widetilde{j}$, respectively:

$$j = \frac{1}{e^2}f\,, \quad \widetilde{j} = \frac{1}{2\pi}\star f\,. \tag{171}$$

The constant factors differ from those in Equations (156) and (157), ensuring that the corresponding charges are integers.

Electric symmetry

The electric symmetry, associated with the conservation of the 2-form electric current j, manifests in Maxwell's action as a shift transformation of a_μ, making it a 1-form symmetry:

$$U(1)^{(1)}_{\mathrm{e}} \quad : \quad a \to a + \alpha\,, \quad \mathrm{d}\alpha = 0\,. \tag{172}$$

However, a_μ is not a gauge-invariant object and does not directly define an operator in Maxwell's theory. Gauge transformation act as follows:

$$U(1)_{\mathrm{gauge}} \quad : \quad a \to a + \mathrm{d}\lambda\,, \quad \lambda \sim \lambda + 2\pi\,, \tag{173}$$

The integral of a_μ over a closed contour $\mathscr{C}_1$ is invariant under all exact gauge transformations, but it remains sensitive to the ones with a period:

$$\oint a \to \oint a + 2\pi p \quad \text{if} \quad \oint \mathrm{d}\lambda = 2\pi p\,, \quad p \in \mathbb{Z}\,. \tag{174}$$

The true gauge-invariant object are the *Wilson lines*:

$$W(\mathscr{C}_1) = e^{in\oint_{\mathscr{C}_1} a}\,, \quad n \in \mathbb{Z}\,. \tag{175}$$

We find that the 1-form electric symmetry acts on them as follows:

$$U(1)^{(1)}_{\mathrm{e}} \quad : \quad W(\mathscr{C}_1) \to e^{in\bar{\alpha}}W(\mathscr{C}_1)\,, \quad \bar{\alpha} = \oint_{\mathscr{C}_1} \alpha\,. \tag{176}$$

The insertion of a Wilson line operator is reminiscent of introducing a point charge in the path integral:

$$\langle W(\mathscr{L}_1)\rangle = \int Da\, e^{-\frac{i}{2e^2}\int_{\mathscr{M}_4} f\wedge\star f + in\int_{\mathscr{L}_1} a}\,, \tag{177}$$

where $\mathscr{L}_1$ is the wordline of a particle with electric charge n. The line integral in the Wilson line can be expressed in terms of the electric potential a_t and the vector potential $\vec{a}$:

$$\int_{\mathscr{L}_1} a = -\int_{\mathscr{L}_1} \mathrm{d}t\left(a_t + \frac{\mathrm{d}\vec{x}}{\mathrm{d}t}\cdot\vec{a}\right)\,. \tag{178}$$

This term changes the equation of motion by introducing a point source of electric charge n:

$$\frac{1}{e^2}\mathrm{d}\star f = n\,\delta^d(x \in \mathscr{L}_1)\,. \tag{179}$$

Magnetic symmetry

The story of the dual field for the compact scalar of Section 3.3 is mirrored with the dynamical gauge field a_μ by introducing the *magnetic photon* $\widetilde{a}_\mu$. The magnetic theory is then just Maxwell's theory with an inverted coupling:

$$a \to \widetilde{a}\,, \quad e \to 1/e\,. \tag{180}$$

The dual field strength is written as follows:

$$\widetilde{f} = \mathrm{d}\widetilde{a} = \frac{1}{2\pi e^2} \star f\,. \tag{181}$$

On $\widetilde{a}_\mu$, the magnetic symmetry acts as a shift transformation:

$$U(1)^{(1)}_{\mathrm{m}} \quad : \quad \widetilde{a} \to \widetilde{a} + \beta\,, \quad \mathrm{d}\beta = 0\,. \tag{182}$$

't Hooft lines are the magnetic equivalent of Wilson lines:

$$H(\mathscr{C}_1) = e^{im\oint_{\mathscr{C}_1}\widetilde{a}}\,, \quad m \in \mathbb{Z}\,, \tag{183}$$

and the magnetic 1-form symmetry acts on them:

$$U(1)^{(1)}_{\mathrm{m}} \quad : \quad H(\mathscr{C}_1) \to e^{im\bar{\beta}} H(\mathscr{C}_1)\,, \quad \bar{\beta} = \oint_{\mathscr{C}_1} \beta\,. \tag{184}$$

The insertion of a 't Hooft line in the path integral corresponds to introducing a monopole with magnetic charge m in spacetime. In terms of a_μ, the magnetic monopole is a singular solution requiring a patching of space. We introduce two hemispheres, S^2_{N} (North) and S^2_{S} (South), defined with an azimutal angle $\varphi \in [0, 2\pi[$ and a polar angle $\theta \in [0, \pi]$. The patch $\theta \in [0, \pi/2]$ is covered by S^2_{N}, and $\theta = [\pi/2, \pi]$ by S^2_{S}. Consider the following expressions for the gauge field on each patch:

$$a_{\mathrm{N}} = \frac{m}{2}(1 - \cos\theta)\mathrm{d}\varphi\,, \quad a_{\mathrm{S}} = \frac{m}{2}(-1 - \cos\theta)\mathrm{d}\varphi\,, \tag{185}$$

with $m \in \mathbb{Z}$. The difference between these definitions is a $U(1)$ gauge transformation $g(\varphi)$:

$$a_{\mathrm{N}} - a_{\mathrm{S}} = ig\mathrm{d}g^{-1} = m\mathrm{d}\varphi \quad \text{where} \quad g(\varphi) = e^{im\varphi}\,. \tag{186}$$

The magnetic flux through the whole sphere is computed as follows:

$$\oint_{S^2} f = \int_{S^2_{\mathrm{N}}} \mathrm{d}a_{\mathrm{N}} + \int_{S^2_{\mathrm{S}}} \mathrm{d}a_{\mathrm{S}} = \oint_{\text{equator}} (a_{\mathrm{N}} - a_{\mathrm{S}}) = m \oint_{\text{equator}} \mathrm{d}\varphi = m2\pi\,. \tag{187}$$

The photon as a Goldstone boson

Introduce a Wilson line on a rectangular circuit $\mathscr{C}_1$ with sides of lengths T and R. In the axial gauge, $a_t = 0$, we find

$$\langle W(\mathscr{C}_1)\rangle = \langle e^{-i\int_R \mathrm{d}x\, a_x(T)} e^{i\int_R \mathrm{d}x\, a_x(0)}\rangle\,. \tag{188}$$

In the Euclidean space,

$$\begin{aligned} \langle W(\mathscr{C})\rangle &= \langle 0|\, e^{TH} e^{-i\int_R \mathrm{d}x\, a_x(0)} e^{-TH} e^{i\int_R \mathrm{d}x\, a_x(0)}\, |0\rangle \\ &= \textstyle\sum_n \langle 0|\, e^{-i\int_R \mathrm{d}x\, a_x(0)} e^{-TE_n(R)}\, |n\rangle \langle n|\, e^{i\int_R \mathrm{d}x\, a_x(0)}\, |0\rangle \\ &= \textstyle\sum_n e^{-TE_n(R)} |\langle n|\, e^{i\int_R \mathrm{d}x\, a_x(0)}\, |0\rangle|^2 \end{aligned} \tag{189}$$

Now, take $T \to \infty$ to find

$$\lim_{T\to\infty} \langle W(\mathscr{C}_1)\rangle \propto e^{-TE_0(R)} \,. \tag{190}$$

In the conformal phase of electromagnetism, the energy of the static configuration goes like $E_0(R) \propto 1/R$:

$$\lim_{T\to\infty} \langle W(\mathscr{C}_1)\rangle \propto e^{-T/R} \quad \text{with} \quad T \gg R \,. \tag{191}$$

The decrease is slower than the perimeter law, which is reminiscent of spontaneous symmetry breaking. Moreover the one-photon state clearly overlaps with the current $f^{\mu\nu}$:

$$\langle 0|\, f^{\mu\nu}(x)\, |\xi, k\rangle = (\xi^\mu k^\nu - k^\mu \xi^\nu) e^{-ik\cdot x} \,. \tag{192}$$

So, it is natural to identify the photon with a Goldstone boson.

Mixed 't Hooft anomaly

It is straightforward to generalize the anomaly argument for the compact scalar presented in Section 3.3 to the case of electromagnetism by improving the field strength with a background gauge field B:

$$f^{\mu\nu} \to f^{\mu\nu} - B^{\mu\nu} \,. \tag{193}$$

The improved electric and magnetic currents are the following:

$$J^{\mu\nu} = \frac{1}{e^2}(f^{\mu\nu} - B^{\mu\nu}) \,, \quad \widetilde{J}^{\mu\nu} = \frac{1}{2\pi}\epsilon^{\mu\nu\rho\sigma}(f_{\rho\sigma} - B_{\rho\sigma}) \,. \tag{194}$$

The following Ward identities protect the masslessness of the photon:

$$\langle \partial_\mu J^{\mu\nu}\rangle_B = 0 \,, \quad \langle \partial_\mu \widetilde{J}^{\mu\nu}\rangle_B = -\frac{1}{2\pi}\epsilon^{\nu\lambda\rho\sigma}\partial_\lambda B_{\rho\sigma} \,. \tag{195}$$

The anomaly theory for $U(1)_{\mathrm{e}}^{(1)} \times U(1)_{\mathrm{m}}^{(1)}$ is written as follows:

$$\mathscr{A}[B, \widetilde{B}] = -\frac{1}{2\pi}\int_{\mathscr{M}_5} B \wedge \mathrm{d}\widetilde{B} \,. \tag{196}$$

Electromagnetism

In absence of dynamical charges, Maxwell's theory is symmetric under the 1-form symmetries $U(1)_{\mathrm{e}}^{(1)} \times U(1)_{\mathrm{m}}^{(1)}$. The later is subject to a mixed 't Hooft anomaly that requires the presence of a gapless mode, the photon.

5.2 With charged matter

Coulomb phase

Quantum Electrodynamics (QED) is the four-dimensional theory of Maxwell coupled to fermionic matter fields ψ. The action for QED combines the Maxwell term, a coupling to matter, and the matter field's dynamics:

$$S_{\mathrm{QED}}[a, \psi] = -\frac{1}{2e^2}\int f \wedge \star f + \int a \wedge \star j_{\mathrm{mat}}(\psi) + S_{\mathrm{mat}}[\psi] \,. \tag{197}$$

The equation of motion for the gauge field a_μ acquires a source term from matter fields:

$$\mathrm{d} \star f = e^2 \star j_{\text{mat}}(\psi) . \tag{198}$$

This breaks explicitly the electric 1-form symmetry $U(1)_{\text{e}}^{(1)}$. The physical reason is that, in QED, Wilson lines can terminate with insertions of matter fields ψ:

$$W(x,y) = \bar{\psi}(x) e^{i \int_y^x a} \psi(y) . \tag{199}$$

In other words, Wilson lines get *screened* by local terms associated with matter fields.

The electric potential runs with the energy scale, which, in the Wilsonian scheme of renormalization, is interpreted as the running of the coupling e^2:

$$V(R) = \frac{e(R)^2}{4\pi R} , \quad e(R)^2 = \frac{e(\Lambda_*^{-1})^2}{1 + \gamma e(\Lambda_*^{-1}) \ln(\Lambda_* R)} . \tag{200}$$

Here, Λ_* is the energy scale associated with the Landau pole and γ is a numerical factor. If ψ has a mass m_ψ, this expression holds up until $R = m_\psi^{-1}$. Beyond this distance, ψ decouples and is integrated out, leaving:

$$V(R) = \frac{e(m_\psi^{-1})^2}{4\pi R} , \quad R > m_\psi^{-1} . \tag{201}$$

If ψ is massless, the potential continues to decrease:

$$V(R) \overset{R\to\infty}{\sim} \frac{1}{4\pi\gamma R \ln(\Lambda_* R)} . \tag{202}$$

In fact, independently of the value of the mass m_ψ, the electric potential decreases fast enough so that the fields ψ decouple and the electric symmetric 1-form symmetry is restored as an emergent symmetry. The deep IR theory becomes symmetric under $U(1)_{\text{e}}^{(1)} \times U(1)_{\text{m}}^{(1)}$. This symmetry structure is characterized by the presence of a mixed 't Hooft anomaly between the electric and magnetic 1-form symmetries, ensuring that the photon remains massless in the IR and the phase is gapless.

Coulomb phase of electromagnetism

Dynamical charges explicitly break the electric 1-form symmetry of Maxwell's theory. However, these charges decouple in the deep IR, leaving behind an emergent 1-form symmetry group $U(1)_{\text{e}}^{(1)} \times U(1)_{\text{m}}^{(1)}$, together with a mixed 't Hooft anomaly, and the massless photon.

Higgs phase with charge 1

We now explore the scalar version of QED to reach the Higgs phase of electromagnetism. Consider a complex scalar Φ with electric charge 1. The Lagrangian is

$$\mathcal{L}_{\text{Higgs}} = -\frac{1}{2} D_\mu \Phi^* D^\mu \Phi - \frac{1}{4e^2} f_{\mu\nu} f^{\mu\nu} - V(\Phi^*\Phi) . \tag{203}$$

The potential V is chosen to be minimized for $|\langle\Phi\rangle| = v$. The scalar field can then be expanded around v:

$$\Phi = (v + \rho) e^{i\phi} . \tag{204}$$

As usual, ρ is massive and can be integrated out. The remaining part of the Lagrangian is

$$\mathcal{L}_{\text{Higgs}} = -\frac{1}{4e^2} f_{\mu\nu} f^{\mu\nu} - \frac{v^2}{2}(a_\mu - \partial_\mu \phi)(a^\mu - \partial^\mu \phi)\,. \tag{205}$$

The shift $\partial_\mu \phi$ can be interpreted as a gauge transformation, such that it may be absorbed in the definition of a_μ:

$$a_\mu - \partial_\mu \phi \to a_\mu\,. \tag{206}$$

The gauge field is clearly massive, with a squared mass proportional to v^2. Thus, in the Higgs phase, the photon acquires a mass and the system becomes gapped.

The equation of motion for a_μ is

$$\partial_\mu f^{\mu\nu} = e^2 v^2 a^\nu\,. \tag{207}$$

In components, we get

$$\vec{\nabla} \cdot \vec{\mathcal{E}} = -m_\gamma^2 V\,, \quad \vec{\nabla} \times \vec{\mathcal{B}} = -m_\gamma^2 \vec{a} \tag{208}$$

where $m_\gamma^2 = e^2 v^2$ and $a_t = V$. This leads to the following equations for V and $\vec{\mathcal{B}}$:

$$\nabla^2 V = m_\gamma^2 V\,, \quad \nabla^2 \vec{\mathcal{B}} = m_\gamma^2 \vec{\mathcal{B}}\,. \tag{209}$$

Both the electric potential and the magnetic fields decay exponentially. Specifically, the electric potential vanishes at distances larger than m_γ^{-1} and so the electric charges do not feel any force, they move freely. The Higgs phase is a *superconducting phase*. On the other side, the magnetic lines do not spread over distances larger than m_γ^{-1} and so are confined in tubes thinner than this distance, which we call vortices.

The electric 1-form symmetry is explicitly broken by the condensate of electric charges, while the magnetic 1-form symmetry is preserved (and not spontaneously broken). At sufficiently large distances, it can be shown that

$$V_{\text{e}}(R) \overset{R\to\infty}{\sim} 0\,, \quad V_{\text{m}}(R) \overset{R\to\infty}{\sim} \sigma R\,, \tag{210}$$

where σ gives the tension of the magnetic flux vortices. This expression for the magnetic potential translates into an area-law for the 't Hooft lines.

Higgs phase of electromagnetism

The vacuum condensate explicitly breaks the electric 1-form symmetry of Maxwell's theory. Through the Higgs mechanism, the photon acquires a mass, rendering the Higgs phase gapped. Magnetic fluxes are confined into vortices and protected by a preserved 1-form magnetic symmetry $U(1)_{\text{m}}^{(1)}$.

Higgs phase with charge N

If the scalar field Φ has an electric charge $N > 1$, the scenario becomes more interesting. After integrating out the Higgs field ρ, we obtain the following Lagrangian:

$$\mathcal{L}_{\text{Higgs}} = -\frac{1}{4e^2} f_{\mu\nu} f^{\mu\nu} - \frac{v^2}{2}(N a_\mu - \partial_\mu \phi)(N a^\mu - \partial^\mu \phi)\,. \tag{211}$$

The equation of motion for ϕ imposes that

$$a = \frac{1}{N}\mathrm{d}\phi\,, \quad \phi \sim \phi + 2\pi\,. \tag{212}$$

When $N = 1$, this corresponds to a vanishing profile for a, as a result of a gauge transformation. However, when $N > 1$, there is the possibility for a to acquire a fractional period over a closed contour $\mathscr{C}_1$:

$$\frac{N}{2\pi}\oint_{\mathscr{C}_1} a = \frac{1}{2\pi}\oint_{\mathscr{C}_1}\mathrm{d}\phi \in \mathbb{Z}\,. \tag{213}$$

From all these periods, there are N distinct values that the Wilson loop will recognize:

$$\oint_{\mathscr{C}_1} a = k\frac{2\pi}{N}\,, \quad k = 0, \cdots, N-1\,. \tag{214}$$

Each of these values defines a different vacuum, and these vacua are connected by a subgroup $\mathbb{Z}_N^{(1)}$ of the original electric 1-form symmetry group $U(1)_{\mathrm{e}}^{(1)}$. This subgroup remains intact while the rest of the electric group is explicitly broken by the condensate with charge N. The vacuum degeneracy indicates that this discrete subgroup is spontaneously broken. Notably, the magnetic 1-form symmetry $U(1)_{\mathrm{m}}^{(1)}$ is not spontaneously broken, otherwise that would render the photon massless. Furthermore, we observe long-range order from the vacuum expectation value (VEV) of the Wilson loop:

$$\langle W(\mathscr{C}_1)\rangle = e^{ik\frac{2\pi}{N}}\,, \quad k = 0, \cdots, N-1\,. \tag{215}$$

The degeneracy of the gapped vacuum and the presence of long-range order associated with loop operators are key indicators that this Higgs phase exhibits *topological order*. Notably, Equation (215) holds only if a non-contractible cycle $\mathscr{C}_1$ exists in the spacetime geometry. Without such a cycle, the loop operator could be deformed into a trivial configuration, forcing its VEV to vanish. Consequently, the groundstate degeneracy depends on the topology of the spacetime geometry.

For a more detailed derivation of these results, the reader is referred to [28], where the dualization of the theory is used to show that its infrared behavior is governed by the *BF theory*:

$$S_{\mathrm{BF}} = \frac{N}{2\pi}\int b\wedge \mathrm{d}a\,, \tag{216}$$

where b is a 2-form field dual to ϕ. This is a well-known example of *topological field theory* (TQFT), which characterizes the Higgs phase with topological order.

Higgs phase with topological order

When the vacuum condensate has a charge $N > 1$, a subgroup $\mathbb{Z}_N^{(1)}$ of Maxwell's electric symmetry survives the explicit breaking. This discrete 1-form symmetry is spontaneously broken in the vacuum, leading to topological order: the vacuum is non-trivially gapped with a groundstate degeneracy that depends on the spacetime topology, along with long-range order for loop operators.

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