

Introduction to Axion Photon Interaction in Particle Physics and Photon Dispersion in Magnetized Media

Avijit K. Ganguly
*Banaras Hindu University (MMV), Varanasi,
 India*

1. Introduction

Symmetries, global or local, always play an important role in the conceptual aspects of physics be in broken or unbroken phase. Spontaneous breaking of the continuous symmetries always generates various excitations with varying mass spectra. Axion is one of that type, generated via spontaneous breaking of a global Chiral $U(1)$ symmetry named after its discoverers, Peccei and Queen. This symmetry is usually denoted by $U(1)_{PQ}$. To give a brief introduction to this particle and its origin we have to turn our attention to the development of the standard model of particle physics and its associated symmetries. The standard model of particle physics describes the strong, weak and electromagnetic interactions among elementary particles. The symmetry group for this model is, $SU_c(3) \times SU(2) \times U(1)$. The strong interaction (Quantum Chromo Dynamics (QCD)) part of the Lagrangian has $SU(3)$ color symmetry and it is given by,

$$\mathcal{L} = -\frac{1}{2g^2} \text{Tr } F_{\mu\nu}^a F_a^{\mu\nu} + \bar{q}(i\mathcal{D} - m)q. \quad (1)$$

It was realized long ago that, in the limit of vanishingly small quark masses (chiral limit), Strong interaction lagrangian has a global $U(2)_V \times U(2)_A$ symmetry. This symmetry group would further break spontaneously to produce the hadron multiplets. The vector part of the symmetry breaks to iso-spin times baryon number symmetry given by $U(2)_V = SU(2)_I \times U(1)_B$. In nature baryon number is seen to be conserved and the mass spectra of nucleon and pion multiplets indicate that the isospin part is also conserved approximately.

So one is left with the axial vector symmetry. QCD being a nonabelian gauge it is believed that this theory is confining in the infrared region. The confining property of the theory is likely to generate condensates of antiquark quark pairs. Thus u - and d quark condensates would have non-zero vacuum expectation values, i.e.,

$$\langle 0|\bar{u}(0)u(0)|0 \rangle = \langle 0|\bar{d}(0)d(0)|0 \rangle \neq 0. \quad (2)$$

and they would break the $U(2)_A$ symmetry. Now if the axial symmetry is broken, we would expect nearly four degenerate and massless pseudoscalar mesons. Interestingly enough, out of the four we observe three light pseudoscalar Nambu Goldstone (NG) Bosons in nature, i.e., the pions. They are light, $m_\pi \simeq 0$, but the other one (with approximately same mass) is not

to be found. Eta meson though is a pseudoscalar meson, but it has mass much greater than the pion ($m_\eta \gg m_\pi$). So the presence of another light pseudoscalar meson in the hadronic spectrum, seem to be missing. This is usually referred in the literature [(Steven Weinberg , 1975)] as the $U(1)_A$ problem.

1.1 Strong CP problem and neutron dipole moment

Soon after the identification of QCD as the correct theory of strong interaction physics, instanton solutions [(Belavin Polyakov Shvarts and Tyupkin , 1975)] for non-abelian gauge theory was discovered. Subsequently, through his pioneering work, 't Hooft [(t Hooft , 1976a),(t Hooft , 1976b)] established that a θ term must be added to the QCD Lagrangian. The expression of this piece is,

$$\mathcal{L}_\theta = \theta \frac{g^2}{32\pi^2} F_a^{\mu\nu} \tilde{F}_{a\mu\nu}. \quad (3)$$

But in the presence of this term *the axial symmetry* is no more a realizable symmetry for QCD. This term violates Parity and Time reversal invariance, but conserves charge conjugation invariance, so it violates CP. Such a term if present in the lagrangian would predict neutron electric dipole moment. The observed neutron electric dipole moment [(R. J. Crewther, 1978)] is $|d_n| < 3 \times 10^{-26}$ ecm and that requires the angle θ to be extremely small [$d_n \simeq e\theta m_q / M_N^2$ indicating [(V. Baluni , 1979; R. J. Crewther et. al. , 1980)] $\theta < 10^{-9}$]. This came to be known as the strong CP problem. In order to overcome this problem, Pecci and Queen subsequently Weinberg and Wilczek [(R. Peccei and H. Quinn , 1977; S. Weinberg , 1978; F. Wilczek , 1978)] postulated the parameter θ to be a dynamical field with odd parity arising out of some chiral symmetry breaking taking place at some energy scale f_{PQ} . With this identification the θ term of the QCD Lagrangian now changes to,

$$\mathcal{L}_a = \frac{g^2}{32\pi^2} a F_a^{\mu\nu} \tilde{F}_{a\mu\nu}, \quad (4)$$

where a is the axion field. They[(R. Peccei and H. Quinn , 1977; S. Weinberg , 1978; F. Wilczek , 1978)] also provided an estimate of the mass of this light pseudoscalar boson. Although these ultra light objects were envisioned originally to provide an elegant solution to the strong CP problem [(R. Peccei and H. Quinn , 1977),WW,wilczek] (see (R. Peccei , 1996)] for details) but it was realized later on that their presence may also solve some of the outstanding problems in cosmology, like the dark matter or dark energy problem (related to the closure of the universe). Further more their presence if established, may add a new paradigm to our understanding of the stellar evolution. A detailed discussion on the astrophysical and cosmological aspects of axion physics can be found in [(M.S. Turner , 1990; G. G. Raffelt , 1990; G. G. Raffelt , 1997; G .G .Raffelt , 1996; J. Preskill et al , 1983)]. In all the models of axions, the axion photon coupling is realized through the following term in the Lagrangian,

$$\mathcal{L} = \frac{1}{M} a \mathbf{E} \cdot \mathbf{B}. \quad (5)$$

Where $M \propto f_a$ the axion coupling mass scale or the symmetry breaking scale and a stands for the axion field. The original version of Axion model, usually known as Peccei-Queen Weinberg-Wilczek model (PQWW), had a symmetry breaking scale that was close to weak scale, f_w . Very soon after its inception, the original model, associated with the spontaneous breakdown of the global PQ symmetry at the Electro Weak scale (EW) f_w , was experimentally

ruled out. However modified versions of the same with their associated axions are still of interest with the symmetry breaking scale lying between EW scale and 10^{12} GeV. Since the axion photon/matter coupling constant, is inversely proportional to the breaking scale of the PQ symmetry, f_a and is much larger than the electroweak scale $f_a \gg f_w$, the resulting axion turns out to be very weakly interacting. And is also very light ($m_a \sim f_a^{-1}$) therefore it is often called “the invisible axion model” [(M.Dine et al. , 1981; J. E. Kim , 1979)]. For very good introduction to this part one may refer to[(R. Peccei , 1996)].

There are various proposals to detect axions in laboratory. One of them is the solar axion experiment. The idea behind this is the following, if axions are produced at the core of the Sun, they should certainly cross earth on it's out ward journey from the Sun. From equation [5], it can be established that in an external magnetic field an axion can oscillate in to a photon and vice versa. Hence if one sets up an external magnetic field in a cavity, an axion would convert itself into a photon inside the cavity. This experiment has been set up in CERN, and is usually referred as CAST experiment[(K. Zioutas et al., 2005)]. The conversion rate inside the cavity, would depend on the value of the coupling constant ($\frac{1}{M}$), axion mass and the axion flux. Since inside the sun axions are dominantly produced by Primakoff and compton effects. One can compute the axion flux by calculating the axion production rate via primakoff & compton process using the available temp and density informations inside the sun. Therefore by observing the rate of axion photon conversion in a cavity on can estimate the axion parameters. The study of solar axion puts experimental bound on M to be, $M > 1.7 \times 10^{11}$ GeV [(Moriyama et al. , 1985),(Moriyama et al. , 1998b)].

The same can be estimated from astrophysical observations. In this situation, it possible to estimate the rate at which the axions would draw energy away form the steller atmosphere by calculating the axion flux (i.e. is axion luminosity) from the following reactions[7]

$$e^+ + e^- \rightarrow \gamma + a , \quad e^- + \gamma \rightarrow e^- + a \quad (6)$$

&

$$\gamma_{plasmon} \rightarrow \gamma + a , \quad \gamma + \gamma \rightarrow a. \quad (7)$$

Axions being weakly interacting particles, would escape the steller atmosphere and the star would lose energy. Thus it would affect the age vs luminosity relation of the star. Comparison of the same with observations yields bounds on e.g., axion mass m_a and M . A detailed survey of various astrophysical bounds on the parameters of axion models and constraints on them, can be found in ref. [(G .G .Raffelt , 1996)].

In the astrophysical and cosmological studies, mentioned above, medium and a magnetic field are always present. So it becomes important to seek the modification of the axion coupling to photon, in presence of a medium or magnetic field or both. Particularly in some astrophysical situations where the magnetic component, along with medium (usually referred as magnetized medium) dominates. Examples being, the Active Galactic Nuclei (AGN), Quasars, Supernova, the Coalescing Neutron Stars or Nascent Neutron Stars, Magnetars etc.. The magnetic field strength in these situations vary between, $B \sim 10^6 - 10^{17}$ G, where some are significantly above the critical, Schwinger value[(J. Schwinger , 1951)]

$$\mathcal{B}_e = m_e^2 / e \simeq 4.41 \times 10^{13} \text{ G} \quad (8)$$

[(M. Ruderman , 1991; Duncan & Thompson , 1992)]. In view of this observation and the possibilities of applications of axion physics to these astrophysical as well as cosmological scenarios, it is pertinent to find out the effect of medium and magnetic field to axion photon coupling.

As we already have noted, the axion physics is sensitive to presence of medium and magnetic field. In most of the astrophysical or cosmological situations these two effects are dominant. In view of this it becomes reasonable to study how matter and magnetic field effect can affect the axion photon vertex. Modification to axion photon vertex in a magnetized media was studied in [(A. K. Ganguly , 2006)]. In this document we would present that work and discuss new correction to $a - \gamma$ vertex in a magnetized media. In the next section that we would focus on axion photon mixing effect with tree level axion photon vertex and show how this effect can change the polarization angle and ellipticity of a propagating plane polarized light beam passing through a magnetic field. After that we would elaborate on how the same predictions would get modified if the same process takes place in a magnetized media. This particular study involves diagonalisation of a 3×3 matrix, so at the end we have added an appendix showing how to construct the diagonalizing matrix to diagonalize a 3×3 symmetric matrix.

2. The loop induced vertex

The axion-fermion (lepton in this note) interaction¹ — with $g'_{af} = (X_f m_f / f_a)$ the Yukawa coupling constant, X_f , the model-dependent factors for the PQ charges for different generations of quarks and leptons [(G.G.Raffelt , 1996)], and fermion mass m_f is given by, [(M.Dine et al. , 1981)],

$$\mathcal{L}_{af} = \frac{g'_{af}}{m_f} \sum_f (\bar{\Psi}_f \gamma_\mu \gamma_5 \Psi_f) \partial^\mu a, \quad (2.9)$$

The sum over f , in eqn. [2.9], stands for sum over all the fermions, from each family. Although, in some studies, instead of using [2.9], the following Lagrangian has been employed,

$$\mathcal{L}_{af} = -2ig'_{af} \sum_f (\bar{\Psi}_f \gamma_5 \Psi_f) a, \quad (2.10)$$

but, Raffelt and Seckel [(G. Raffelt , 1988)] has pointed out the correctness of using [2.9]. We for our purpose we will make use of [2.9]. We would like to note that the usual axion photon mixing Lagrangian in an external magnetic field turns out to be,

$$\mathcal{L}_{a\gamma} = -g_{a\gamma\gamma} \frac{e^2}{32\pi^2} a \mathbf{F} \tilde{\mathbf{F}}^{\text{Ext}}. \quad (2.11)$$

In equation [2.11] the axion photon coupling constant is described by,

$$g_{a\gamma\gamma} = \frac{1}{f_a} \left[A_{PQ}^{\text{em}} - A_{PQ}^c \frac{2(4+z)}{3(1+z)} \right], \quad (2.12)$$

with $z = \frac{m_u}{m_d}$, where m_u and m_d are the masses of the light quarks. Anomaly factors are given by the following relations, $A_{PQ}^{\text{em}} = \text{Tr}(Q_f^2) X_f$ and $\delta_{ab} A_c^{\text{em}} = \text{Tr}(\lambda_a \lambda_b X_f)$ (and the trace is over

¹ Some of the issues related axion fermion coupling had been reviewed in [(A. K. Ganguly , 2006)], one can see the references there.

the fermion species). We would like to add that, for the sake of brevity at places, we may use g instead of $g_{a\gamma\gamma}$ at some places in the rest of this paper. Therefore the additional contribution to the axion photon effective lagrangian from the new vertex would add to the existing one i.e., eqn. [2.11].

3. Expression for photon axion vertex in presence of uniform background magnetic field and material medium

In order to estimate the loop induced $\gamma - a$ coupling, one can start with the Lagrangian given by Eqn. [2.9]. Defining $p' = p + k$ the effective vertex for the $\gamma - a$ coupling turns out to be,

$$i\Gamma_\nu(k) = g_{af} e Q_f \int \frac{d^4 p}{(2\pi)^4} k^\mu \text{Tr} [\gamma_\mu \gamma_5 iS(p) \gamma_\nu iS(p')] . \quad (3.13)$$

The effective vertex given by [3.13], is computed from the diagram given in [Fig.1]. In eqn. [3.13] $S(p)$ is the in medium fermionic propagator in external magnetic field, computed to all orders in field strength. The structure of the same can be found in [(A. K. Ganguly , 2006)]. One can easily recognize that, eqn. [3.13], has the following structure, $\Gamma_\nu(k) = k^\mu \Pi_{\mu\nu}^A(k)$. Where $\Pi_{\mu\nu}^A$, is the axial polarization tensor, comes from the axial coupling of the axions to the leptons and it's:

$$i\Pi_{\mu\nu}^A(k) = g_{af} e Q_f \int \frac{d^4 p}{(2\pi)^4} \text{Tr} [\gamma_\mu \gamma_5 iS(p) \gamma_\nu iS(p')] . \quad (3.14)$$

In general the axial polarization tensor, $\Pi_{\mu\nu}^A$ (some times called the VA response function), would have contributions from pure magnetic field background, as well as magnetic field plus medium, i.e., magnetized medium. The contribution from only magnetic field and the one with magnetized medium effects, are given in the following expression,

$$\begin{aligned} i\Pi_{\mu\nu}^A(k) = g_{af} e Q_f \int \frac{d^4 p}{(2\pi)^4} \text{Tr} & \left[\gamma_\mu \gamma_5 iS_B^V(p) \gamma_\nu iS_B^V(p') + \gamma_\mu \gamma_5 S_B^\eta(p) \gamma_\nu iS_B^V(p') \right. \\ & \left. + \gamma_\mu \gamma_5 iS_B^V(p) \gamma_\nu S_B^\eta(p') \right] . \quad (3.15) \end{aligned}$$

The pure magnetic field contribution to $\Pi_{\mu\nu}^A(k)$ has been estimated in [(A. K. Ganguly , 2006; D. V. Galtsov , 1972; L. L.DeRaad *et al.* , 1976; A. N. Ioannisian *et al.* , 1997; C. Schubert , 2000)]. The expression of the would be provided in the next section, after that the thermal part contribution to the same would be reported .

3.1 Magnetized vacuum contribution

The VA response function in a magnetic field Π^A has been evaluated in [(A. K. Ganguly , 2006; D. V. Galtsov , 1972; L. L.DeRaad *et al.* , 1976; A. N. Ioannisian *et al.* , 1997; C. Schubert , 2000)], with varying choice of metric; we have reevaluated it according to our metric convention $g_{\mu\nu} \equiv \text{diag} (+1, -1, -1, -1)$. The expression for the same according our convention is:

$$\Pi_{\mu\nu}^A(k) = \frac{i g_{af} (e Q_f)^2}{(4\pi)^2} \int_0^\infty dt \int_{-1}^{+1} dv e^{\phi_0} \left\{ \left(\frac{1-v^2}{2} k_{\parallel}^2 - 2m_e^2 \right) \tilde{F}_{\mu\nu} - (1-v^2) k_{\mu\parallel} (\tilde{F}k)_\nu \right. \\ \left. + R \left[k_{\nu\perp} (k\tilde{F})_\mu + k_{\mu\perp} (k\tilde{F})_\nu \right] \right\}, \quad (3.16)$$

Where, $R = \left[\frac{1-v \sin \mathcal{Z} v \sin \mathcal{Z} - \cos \mathcal{Z} \cos \mathcal{Z} v}{\sin^2 \mathcal{Z}} \right]$ and $\phi_0 = it \left[\frac{1-v^2}{4} k_{\parallel}^2 - m^2 - \frac{\cos v \mathcal{Z} - \cos \mathcal{Z}}{2\mathcal{Z} \sin \mathcal{Z}} k_{\perp}^2 \right]$. In the above expression, $\tilde{F}^{\mu\nu} = \frac{1}{2} \epsilon^{\mu\nu\rho\sigma} F_{\rho\sigma}$, and $\epsilon^{0123} = 1$ is the dual of the field-strength tensor, with $\mathcal{Z} = eQ_f B t$. Therefore, following eqn. [3.13], the photon axion vertex in a purely magnetized vacuum, would be, $\Gamma^\nu(k) = k^\mu \Pi_{\mu\nu}^{A_B}(k)$ i.e.,

$$\Gamma^\nu(k) = \frac{i g_{af} (e Q_f)^2}{(4\pi)^2} \int_0^\infty dt \int_{-1}^{+1} dv e^{\phi_0} k^\mu \left\{ \left(\frac{1-v^2}{2} k_{\parallel}^2 - 2m_e^2 \right) \tilde{F}_{\mu\nu} - (1-v^2) k_{\mu\parallel} (\tilde{F}k)_\nu \right. \\ \left. + R \left[k_{\nu\perp} (k\tilde{F})_\mu + k_{\mu\perp} (k\tilde{F})_\nu \right] \right\}, \quad (3.17)$$

This result is not gauge invariant. However following [(A. K. Ganguly , 2006; A. N. Ioannisian et al. , 1997)], one may integrate the first term under the integral, and arrive at the expression for, the Effective Lagrangian for loop induced axion photon coupling in a magnetized vacuum, to be given by,

$$\mathcal{L}_{a\gamma}^B = a A^\nu \Gamma_\nu(k) \quad (3.18)$$

In eqn.[3.18],we define the axion field by a and $(k\tilde{F})^\nu = k_\mu \tilde{F}^{\mu\nu}$ and $(\tilde{F}k)^\nu = \tilde{F}^{\nu\mu} k_\mu$. Finally the loop induced contribution to the axion photon effective Lagrangian is,

$$\mathcal{L}_{a\gamma}^B = -\frac{1}{32\pi^2} g_{af} (e Q_f)^2 \left[4 + \frac{4}{3} \left(\frac{k_{\parallel}^2}{m^2} \right) \right] a F_{\mu\nu} \tilde{F}^{\mu\nu}. \quad (3.19)$$

Since we are interested in $\omega < m$, so the magnitude of the factor $\left(\frac{k_{\parallel}}{m} \right)^2 \ll 1$, thus the order of magnitude estimate estimate of this contribution is of $O(1)$. However some of the factors there are momentum dependent, so it may affect the dispersion relation for photon and axion.

4. Contribution from the magnetized medium

Having estimated the effective axion photon vertex in a purely magnetic environment, we would focus on the contribution from the magnetized medium. As before, one can evaluate the same by using the expression for a fermion propagator in external magnetic field and medium; the result is:

$$\Pi_{\mu\nu}^{A_B}(k) = (ig_{af} e Q_f) \int \frac{d^4 p}{(2\pi)^4} \int_{-\infty}^{\infty} ds e^{\Phi(p,s)} \int_0^{\infty} ds' e^{\Phi(p',s')} \text{Tr} \left[\right. \\ \left. [\gamma_\mu \gamma_5 G(p,s) \gamma_\nu G(p',s')] \eta_F(p) + [\gamma_\mu \gamma_5 G(-p',s') \gamma_\nu G(-p,s)] \eta_F(-p) \right] \\ = (ig_{af} e Q_f) \int \frac{d^4 p}{(2\pi)^4} \int_{-\infty}^{\infty} ds e^{\Phi(p,s)} \int_0^{\infty} ds' e^{\Phi(p',s')} R_{\mu\nu}(p, p', s, s') \quad (4.20)$$

where $R_{\mu\nu}(p, p', s, s')$ contains the trace part. $R_{\mu\nu}(p, p', s, s')$ is a polynomial in powers of the external magnetic field with even and odd powers of \mathcal{B} , can be presented as,

$$R_{\mu\nu}(p, p', s, s') = R_{\mu\nu}^{(E)}(p, p', s, s') + R_{\mu\nu}^{(O)}(p, p', s, s') \quad (4.21)$$

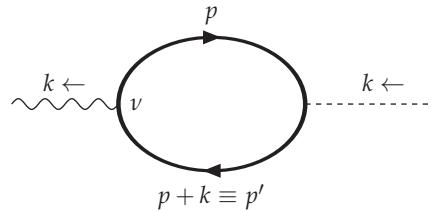


Fig. 1. One-loop diagram for the effective axion electromagnetic vertex .

We have denoted the pieces with even and odd powers in the external magnetic field strength \mathcal{B} in $R_{\mu\nu}$, as $R_{\mu\nu}^{(E)}$ and $R_{\mu\nu}^{(O)}$. In addition to being just even and odd in powers of $eQ_f\mathcal{B}$, they are also odd and even in powers of chemical potential, therefore, under charge conjugation they would transform as, $\mathcal{B} \& \mu \leftrightarrow (-\mu) \& (-\mathcal{B})$, i.e., both behave differently. More over their parity structures are also different. These properties come very useful while analyzing, the structure of axion photon coupling, using discrete symmetry arguments to justify the presence or absence of either of the two; that is the reason, why they should be treated separately. The details of this analysis can be found in [(A. K. Ganguly , 2006)].

4.0.1 Vertex function: even powers in \mathbf{B}

The expression for the $R_{\mu\nu}^E$, (that is the term with even powers of the magnetic field), comes out to be,

$$R_{\mu\nu}^{(E)} \stackrel{\circ}{=} 4i\eta_-(p_0) \left[\varepsilon_{\mu\nu\alpha\beta} p^\alpha k^\beta (1 + \tan(eQ_f\mathcal{B}s) \tan(eQ_f\mathcal{B}s')) + \varepsilon_{\mu\nu\alpha\beta_\perp} k^\alpha k^{\beta_\perp} \times \tan(eQ_f\mathcal{B}s) \tan(eQ_f\mathcal{B}s') \frac{\tan(eQ_f\mathcal{B}s) - \tan(eQ_f\mathcal{B}s')}{\tan(eQ_f\mathcal{B}s) + \tan(eQ_f\mathcal{B}s')} \right]. \quad (4.22)$$

Because of the presence of $\varepsilon_{\mu\nu\alpha\beta} k^\beta$ and $\varepsilon_{\mu\nu\alpha\beta_\perp} k^\alpha$, it vanishes on contraction $R_{\mu\nu}^{(E)}$ with k^ν .

The two point VA response function $\Pi^A(k)$, can be interpreted as a (one particle irreducible) two point vertex; with one point for the external axion line and the other one (Lorentz indexed) for the external photon line. But since the evaluations are done in presence of external magnetic field \mathcal{B} they correspond to soft external photon line insertions. That is their four momenta $k^\alpha \rightarrow 0$. If each soft external photon line contributes either +1 or -1 to the total spin (angular momentum) of the effective vertex, then, for an even order term in external field strength \mathcal{B} the total spin of this piece would be a coherent sum of all the contributions from all the odd number of soft photon lines \mathcal{B} . Now recall that in order arrive at the the expression for the effective interaction Lagrangian for $\gamma - a$ from $\Pi_{\mu\nu}^A(x)$ —we need to multiply the same (with some sort of naivete) by $a(x)F^{\mu\nu}(x)$. Therefore, it is worth noting that, if

we multiply $\Pi_{\mu\nu}^A$ (Even B) with $a(x)F^{\mu\nu}(x)$, the number of photon lines become odd and number of spin zero pseudoscalar is also odd. Since the effective Lagrangian can be related to the generating functional of the vertex for transition of photons to axion, then for this case it would mean, odd number of photons are going to produce a spin zero pseudoscalar. That is odd number of spin one photons would combine to produce a spin zero axion— *which is impossible*, hence such a term better not exist. Interestingly enough, *that is what* we get to see here.

4.0.2 Vertex function: odd powers in B

The nonzero contribution to the vertex function would be coming from $R_{\mu\nu}^O$. More precisely, from the following term,

$$ik^\mu \mathcal{R}_{\mu\nu}^{(O)} = 8m^2 \eta_+(p) \left[k^\mu \epsilon_{\mu\nu 12} (\tan(eQ_f \mathcal{B}s) + \tan(eQ_f \mathcal{B}s')) \right], \quad (4.23)$$

Placing all the factors and integral signs, the vertex function $\Gamma_\nu(k)$ can be written as,

$$\begin{aligned} \Gamma_\nu(k) = (g_{af} e Q_f) \left(8m^2 k^\mu \epsilon_{\mu\nu 12} \right) & \int \frac{d^4 p}{(2\pi)^4} \eta_+(p) \int_{-\infty}^{\infty} ds \int_0^{\infty} ds' e^{\Phi(p,s) + \Phi(p',s')} \\ & \times \left[\tan(eQ_f \mathcal{B}s) + \tan(eQ_f \mathcal{B}s') \right] \end{aligned} \quad (4.24)$$

Upon performing the gaussian integrals for the perpendicular momentum components, there after taking limit $|k| \rightarrow 0$ and assuming photon energy $\omega < m_f$ one arrives at,

$$\Gamma_\nu(k) = -16(g_{af}(eQ_f)^2) \left(\frac{k^\mu \tilde{F}_{\mu\nu}}{16\pi^2} \right) \Lambda(k_{\parallel}^2, k \cdot u, \beta, \mu). \quad (4.25)$$

All the informations about the medium, are contained in $\Lambda(k_{\parallel}^2, k \cdot u, \beta, \mu)$ and it is given by.

$$\Lambda(k_{\parallel}^2, k \cdot u, \beta, \mu) = \int d^2 p_{\parallel} \left[n_F(|p_0|, \mu) + n_F(|p_0|, -\mu) \right] \left(\frac{m^2 \delta(p_{\parallel}^2 - m^2)}{(k_{\parallel}^2 + 2(p \cdot k)_{\parallel})} \right) \quad (4.26)$$

In the expression above the temperature of the medium ($\beta = 1/T$), number density of the fermions (which in turn is related to μ), mass of the particles in the loop (m), energy and longitudinal momentum of the photon (i.e. k_{\parallel}). The statistical factor has already been evaluated in [(A. K. Ganguly , 2006)], in various limits. So instead of providing the same we state the result obtained in the limits $m \ll \mu$, and $\text{limit} T \rightarrow 0$. The value of the same in this limit is

$$\text{Lt}_{T \rightarrow 0} \Lambda \simeq -\frac{1}{2} \frac{\left| \frac{\mu}{m} \right|}{\sqrt{\left(1 + \left[\frac{\mu}{m} \right]^2 \right)}} \quad (4.27)$$

In the limit $\mu \gg m$, the right hand side of Eqn. [4.27] $\sim \frac{1}{2}$ and when $\mu \sim m$, it would turn out to be $\sim \frac{1}{2\sqrt{2}}$

In the light of these estimates, it is possible to write down the axion photon mixing Lagrangian, for low frequency photons in an external magnetic field, in the following way:

$$\mathcal{L}_{a\gamma}^{Total} = \mathcal{L}_{a\gamma}^{vac} + \mathcal{L}_{a\gamma}^{\mathcal{B}} + \mathcal{L}_{a\gamma}^{\mathcal{B},\mu,\beta}. \quad (4.28)$$

Where each of the terms are given by,

$$\begin{aligned} \mathcal{L}_{a\gamma}^{vac} &= -g_{a\gamma\gamma} \frac{e^2}{32\pi^2} a \mathbf{F} \tilde{\mathbf{F}}, \\ \mathcal{L}_{a\gamma}^{\mathcal{B}} &= \frac{-1}{32\pi^2} \left[4 + \frac{4}{3} \left(\frac{k_{\parallel}}{m} \right)^2 \right] \sum_f g_{af} (eQ_f)^2 a \mathbf{F} \tilde{\mathbf{F}}, \\ \mathcal{L}_{a\gamma}^{\mathcal{B},\mu,\beta} &= \frac{32}{32\pi^2} \cdot \left(\frac{k_{\parallel}}{\omega} \right)^2 (\Lambda) \sum_f g_{af} (eQ_f)^2 a \mathbf{F} \tilde{\mathbf{F}}. \end{aligned} \quad (4.29)$$

Therefore, in the limit of $|k_{\perp}| \rightarrow 0$ and $\omega \ll m_f$, one can write the total axion photon effective Lagrangian using eqn. [4.29], in the following form.

$$\mathcal{L}_{a\gamma}^{Total} = - \left[g_{a\gamma\gamma} + \left(4 + \frac{4}{3} \left(\frac{k_{\parallel}}{m} \right)^2 \right) \sum_f g_{af} (Q_f)^2 - 32 \left(\frac{k_{\parallel}}{\omega} \right)^2 \Lambda \sum_f g_{af} (Q_f)^2 \right] \frac{e^2}{32\pi^2} a \mathbf{F} \tilde{\mathbf{F}}. \quad (4.30)$$

We would like to point out that, the in medium corrections doesn't alter the tensorial structure of the same. It remains intact. However the parameter M , doesn't remain so. Apart from numerical factors it also starts depending on the kinematic factors. It is worth noting that, all the terms generated by loop induced corrections do respect **CPT**. Additionally, as we have analyzed already the total spin angular momentum is also conserved. The tree level photon axion interaction term in the Lagrangian as found in the literature is of the following form,

$$\frac{1}{M} a \mathbf{F}^{\mu\nu} \cdot \tilde{\mathbf{F}}_{\mu\nu}^{ext}, \quad (4.31)$$

The bounds on various axion parameters are obtained by using this Lagrangian. As we have seen the medium and other corrections can affect the magnitude of M . Since M is related to the symmetry breaking scale, a change in the estimates of M would have reflection on the symmetry breaking scale and other axion parameters. This is the primary motivation for our dwelling on this part of the problem before moving into aspects of axion electrodynamics, that affects photon polarization.

5. Axion photon mixing

Now that we are equipped with the necessary details of axion interactions with other particles, we can write down the relevant part of the Lagrangian that describes the Axion photon interaction. The tree level Lagrangian that describes the axion photon dynamics is given by,

$$\mathcal{L} = -\frac{1}{4} \mathbf{F}^{\mu\nu} \mathbf{F}_{\mu\nu} + \frac{1}{4M} \mathbf{F}^{\mu\nu} \tilde{\mathbf{F}}_{\mu\nu} + \frac{1}{2} \left(\partial_{\mu} a \partial^{\mu} a - m_a^2 a^2 \right), \quad (5.32)$$

here m_a , is the axion mass and other quantities have their usual meaning. This effective Lagrangian shows the effect of mixing of a spin zero pseudo-scalar with two photons. If one of the dynamical photon field in eqn. [5.32] is replaced by an external magnetic field, one would recover the Lagrangian given by eqn.[4.31]. This mixing part can give rise to various interesting observable effects; however in this section we would consider, the change in the state of polarization of a plane polarized light beam, propagating in an external magnetic field, due to axion photon mixing. In order to perform that analysis, we start with the equation of motion for the photons and the axions, in an external magnetic field \mathcal{B} , that follows from the interaction part of the Lagrangian in eqn. [5.32], as we replace one of the dynamical photon field by external magnetic field field.

This system that we are going to study involve the dynamics of three field Degrees Of Freedom (DOF). As we all know, that the massless spin one gauge fields in vacuum have just two degrees of freedom; so we have those two DOF and the last one is for the spin zero pseudoscalar Boson. In this simple illustrative analysis, we would ignore the transverse component of the momentum k_\perp . With this simplification in mind we have three equations of motion, one each for: $A_\perp(z)$, $A_\parallel(z)$ and $a(z)$ -i.e., the three dynamical fields. Where $A_\perp(z)$, the photon/gauge field with polarization vector directed along the perpendicular direction to the magnetic field, $A_\parallel(z)$ the remaining component of the photon/gauge field having polarization vector lying along the magnetic field and "a(z)" the pseudoscalar Axion field. These three equations can be written in a compact form e.g.,

$$\left[(\omega^2 + \partial_z^2) \mathbf{I} + \mathcal{M} \right] \begin{pmatrix} A_\perp \\ A_\parallel \\ a \end{pmatrix} = 0. \quad (5.33)$$

where \mathbf{I} is a 3×3 identity matrix and \mathcal{M} is the short hand notation for the following matrix.

$$\mathcal{M} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & ig\mathcal{B}\omega \\ 0 & -ig\mathcal{B}\omega & -m_a^2 \end{pmatrix}, \quad (5.34)$$

usually termed as axion photon mixing matrix or simply the mixing matrix. As can be seen from eqn.[5.33], the transverse gauge degree of freedom gets decoupled from the rest, and the other two i.e., the longitudinal gauge degrees of freedom and pseudoscalar degree of freedom are coupled with each other. It is because of this particular way of evolution of the transverse and the parallel components of the gauge field, even magnetized vacuum would show dichoric effect.

In the off diagonal element of the matrix [5.34] given by, $\pm ig\mathcal{B}\omega$, $\mathcal{B} = \mathcal{B}^E \sin(\hat{\alpha})$, is the transverse part of the external magnetic field \mathcal{B}^E and $\hat{\alpha}$ is the angle between the wave vector \vec{k} and the external magnetic field \mathcal{B}^E and lastly in a short hand notation, $g = \frac{1}{M}$. The nondiagonal part of the 3×3 matrix, in eqn. [5.34] can be written as,

$$M_{2 \times 2} = \begin{pmatrix} 0 & ig\mathcal{B}\omega \\ -ig\mathcal{B}\omega & -m_a^2 \end{pmatrix}. \quad (5.35)$$

One can solve for the eigen values of the eqn. [5.35], from the determinantal equation,

$$\begin{vmatrix} M_j & -ig\mathcal{B}\omega \\ ig\mathcal{B}\omega & m_a^2 + M_j \end{vmatrix} = 0. \quad (5.36)$$

In eqn. [5.36] j can take either of the two values $+$ or $-$, and the roots are as follows:

$$M_{\pm} = -\frac{m_a^2}{2} \pm \sqrt{\left[\left(\frac{m_a^2}{2}\right)^2 + (g\mathcal{B}\omega)^2\right]}. \quad (5.37)$$

6. Equation of motion

The equations of motion for the photon field with polarization vector in the perpendicular direction to the external magnetic field is,

$$\left[(\omega^2 + \partial_z^2)\right] (A_{\perp}) = 0. \quad (6.38)$$

The remaining single physical degree freedom for the photon, with polarization along the external magnetic field, gets coupled with the axion; and the equation of motion turns out to be,

$$\left[(\omega^2 + \partial_z^2) \mathbf{I} + M_{2 \times 2}\right] \left(\begin{array}{c} A_{\parallel} \\ a \end{array}\right) = 0. \quad (6.39)$$

It is possible to diagonalize eqn.[6.39] by a similarity transformation. We would denote the diagonalizing matrix by O , given by,

$$O = \begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix} \equiv \begin{pmatrix} c & -s \\ s & c \end{pmatrix}, \quad (6.40)$$

in short. The diagonal matrix can further be written as,

$$M_D = O^T M_{2 \times 2} O = \begin{pmatrix} c & s \\ -s & c \end{pmatrix} \begin{pmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{pmatrix} \begin{pmatrix} c & -s \\ s & c \end{pmatrix}, \quad (6.41)$$

with the following forms for the elements of the matrix $M_{2 \times 2}$, given by: $M_{11} = 0$, $M_{12} = ig\mathcal{B}\omega$, $M_{21} = -ig\mathcal{B}\omega$ and lastly $M_{22} = -m_a^2$. The value of the parameter θ is fixed from the equality,

$$M_D = \begin{pmatrix} c & s \\ -s & c \end{pmatrix} \begin{pmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{pmatrix} \begin{pmatrix} c & -s \\ s & c \end{pmatrix} = \begin{pmatrix} M_+ & 0 \\ 0 & M_- \end{pmatrix}, \quad (6.42)$$

leading to,

$$\begin{pmatrix} c^2 M_{11} + s^2 M_{22} + 2csM_{12} & M_{12}(c^2 - s^2) + cs(M_{22} - M_{11}) \\ M_{12}(c^2 - s^2) + cs(M_{22} - M_{11}) & s^2 M_{11} + c^2 M_{22} - 2csM_{12} \end{pmatrix} = \begin{pmatrix} M_+ & 0 \\ 0 & M_- \end{pmatrix}, \quad (6.43)$$

Now equating the components of the matrix equation [6.43], one arrives at:

$$\tan(2\theta) = \frac{2M_{12}}{M_{11} - M_{22}} = \frac{2ig\mathcal{B}\omega}{m_a^2}. \quad (6.44)$$

Therefore upon using this similarity transformation, the coupled Axion photon differential equation can further be brought to the following form,

$$\left[(\omega^2 + \partial_z^2) \mathbf{I} + M_D \right] \begin{pmatrix} \bar{A}_{||} \\ \bar{a} \end{pmatrix} = 0. \quad (6.45)$$

7. Dispersion relations

Defining the wave vectors in terms of k_i 's, as:

$$\begin{aligned} k_{\perp} &= \omega \\ k_+ &= \sqrt{\omega^2 + M_+} \\ k_- &= -\sqrt{\omega^2 + M_+} \end{aligned} \quad (7.46)$$

and

$$\begin{aligned} k'_+ &= \sqrt{\omega^2 + M_-} \\ k'_- &= -\sqrt{\omega^2 + M_-} \end{aligned} \quad (7.47)$$

8. Solutions

The solutions for the gauge field and the axion field, given by [6.45] as well as the solution for eqn. for A_{\perp} in k space can be written as,

$$\bar{A}_{||}(z) = \bar{A}_{||+}(0)e^{ik_+z} + \bar{A}_{||-}(0)e^{-ik_-z} \quad (8.48)$$

$$\bar{a}(z) = \bar{a}_+(0) e^{ik'_+z} + \bar{a}_-(0) e^{-ik'_-z} \quad (8.49)$$

$$A_{\perp}(z) = A_{\perp+}(0)e^{ik_{\perp}z} + A_{\perp-}(0)e^{-ik_{\perp}z} \quad (8.50)$$

9. Correlation functions

The solutions for propagation along the +ve z axis, is given by,

$$\bar{A}_{||}(z) = \bar{A}_{||+}(0)e^{ik_+z} \quad (9.51)$$

$$\bar{a}(z) = \bar{a}_+(0) e^{ik'_+z} \quad (9.52)$$

that can further be written in the following form,

$$\begin{pmatrix} \bar{A}_{||}(z) \\ \bar{a}(z) \end{pmatrix} = \begin{pmatrix} e^{ik_+z} & 0 \\ 0 & e^{ik'_+z} \end{pmatrix} \begin{pmatrix} \bar{A}_{||}(0) \\ \bar{a}(0) \end{pmatrix}. \quad (9.53)$$

Since,

$$\begin{pmatrix} \bar{A}_{||}(z/0) \\ \bar{a}(z/0) \end{pmatrix} = O^T \begin{pmatrix} A_{||}(z/0) \\ a(z/0) \end{pmatrix}. \quad (9.54)$$

it follows from there that,

$$\begin{pmatrix} A_{||}(z) \\ a(z) \end{pmatrix} = O \begin{pmatrix} e^{ik_+ z} & 0 \\ 0 & e^{ik'_+ z} \end{pmatrix} O^T \begin{pmatrix} A_{||}(0) \\ a(0) \end{pmatrix}. \quad (9.55)$$

Using eqn.[9.55] we arrive at the relation,

$$A_{||}(z) = \left[e^{ik_+ z} \cos^2 \theta + e^{ik'_+ z} \sin^2 \theta \right] A_{||}(0) + \left[e^{ik_+ z} - e^{ik'_+ z} \right] \cos \theta \sin \theta a(0) \quad (9.56)$$

$$a(z) = \left[e^{ik_+ z} - e^{ik'_+ z} \right] \cos \theta \sin \theta A_{||}(0) + \left[e^{ik_+ z} \sin^2 \theta + e^{ik'_+ z} \cos^2 \theta \right] a(0) \quad (9.57)$$

If we assume the axion field to be zero, to begin with, i.e., $a(0) = 0$, then the solution for the gauge fields take the following form,

$$\begin{aligned} A_{||}(z) &= \left[e^{ik_+ z} \cos^2 \theta + e^{ik'_+ z} \sin^2 \theta \right] A_{||}(0) \\ A_{\perp}(z) &= e^{ik_+ z} A_{\perp}(0). \end{aligned} \quad (9.58)$$

Now we can compute various correlation functions with the photon field. The correlation functions of parallel and perpendicular components of the photon field take the following form:

$$\begin{aligned} < A_{||}^*(z) A_{||}(z) > &= \left[\cos^4 \theta + \sin^4 \theta + 2 \sin^2 \theta \cos^2 \theta \cos [(k_+ - k'_+) z] \right] < A_{||}^*(0) A_{||}(0) > \\ < A_{||}^*(z) A_{\perp}(z) > &= \left[\cos^2 \theta e^{i(k_+ - k'_+) z} + \sin^2 \theta e^{i(k_+ - k'_+) z} \right] < A_{||}^*(0) A_{\perp}(0) > \\ < A_{\perp}^*(z) A_{\perp}(z) > &= < A_{\perp}^*(0) A_{\perp}(0) > \end{aligned} \quad (9.59)$$

10. Digression on stokes parameters

Various optical parameters like polarization, ellipticity and degree of polarization of a given light beam can be found from the coherency matrix constructed from various correlation functions given above. The coherency matrix, for a system with two degree of freedom is defined as an ensemble average of direct product of two vectors:

$$\rho(z) = \langle \begin{pmatrix} A_{||}(z) \\ A_{\perp}(z) \end{pmatrix} \otimes \begin{pmatrix} A_{||}(z) & A_{\perp}(z) \end{pmatrix}^* \rangle = \begin{pmatrix} \langle A_{||}(z) A_{||}^*(z) \rangle & \langle A_{||}(z) A_{\perp}^*(z) \rangle \\ \langle A_{\perp}^*(z) A_{||}(z) \rangle & \langle A_{\perp}(z) A_{\perp}^*(z) \rangle \end{pmatrix} \quad (10.60)$$

The important thing to note here is that, under any anticlock-wise rotation α about an axis perpendicular the $||$ and \perp components, would convert:

$$\rho(z) \rightarrow \rho'(z) = \langle \mathcal{R}(\alpha) \begin{pmatrix} A_{||}(z) \\ A_{\perp}(z) \end{pmatrix} \otimes \begin{pmatrix} A_{||}(z) & A_{\perp}(z) \end{pmatrix}^* \mathcal{R}^{-1}(\alpha) \rangle \quad (10.61)$$

where $\mathcal{R}(\alpha)$ is the rotation matrix. Now from the relations between the components of coherency matrix and the stokes parameters:

$$\begin{aligned} I &= < A_{||}^*(z) A_{||}(z) > + < A_{\perp}^*(z) A_{\perp}(z) >, \\ Q &= < A_{||}^*(z) A_{||}(z) > - < A_{\perp}^*(z) A_{\perp}(z) >, \\ U &= 2\text{Re} < A_{||}^*(z) A_{\perp}(z) >, \\ V &= 2\text{Im} < A_{||}^*(z) A_{\perp}(z) >. \end{aligned} \quad (10.62)$$

It is easy to establish that,

$$\rho(z) = \frac{1}{2} \begin{pmatrix} I(z) + Q(z) & U(z) - iV(z) \\ U(z) + iV(z) & I(z) - Q(z) \end{pmatrix} \quad (10.63)$$

Therefore, under an anticlock wise rotation by an angle α , about an axis perpendicular to the plane containing $A_{||}(z)$ and $A_{\perp}(z)$, the density matrix transforms as: $\rho(z) \rightarrow \rho'(z)$; the same in the rotated frame would be given by,

$$\rho'(z) = \frac{1}{2} \mathcal{R}(\alpha) \begin{pmatrix} I(z) + Q(z) & U(z) - iV(z) \\ U(z) + iV(z) & I(z) - Q(z) \end{pmatrix} \mathcal{R}^{-1}(\alpha). \quad (10.64)$$

For a rotation by an angle α –in the anticlock direction– about an axis perpendicular to $A_{||}$ and A_{\perp} plane, the rotation matrix $\mathcal{R}(\alpha)$ is,

$$\mathcal{R}(\alpha) = \begin{pmatrix} \cos \alpha & \sin \alpha \\ -\sin \alpha & \cos \alpha \end{pmatrix}. \quad (10.65)$$

From the relations above, its easy to convince oneself that, in the rotated frame of reference the two stokes parameters, Q and U get related to the same in the unrotated frame, by the following relation.

$$\begin{pmatrix} Q'(z) \\ U'(z) \end{pmatrix} = \begin{pmatrix} \cos 2\alpha & \sin 2\alpha \\ -\sin 2\alpha & \cos 2\alpha \end{pmatrix} \begin{pmatrix} Q(z) \\ U(z) \end{pmatrix} \quad (10.66)$$

The other two parameters, i.e., I and V remain unaltered. It is for this reason that some times I and V are termed invariants under rotation.

For a little digression, we would like to point out that, in a particular frame, the Stokes parameters are expressed in terms of two angular variables χ and ψ usually called the ellipticity parameter and polarization angle, defined as,

$$\begin{aligned} I &= I_p \\ Q &= I_p \cos 2\psi \cos 2\chi \\ U &= I_p \sin 2\psi \cos 2\chi \\ V &= I_p \sin 2\chi. \end{aligned} \quad (10.67)$$

The ellipticity angle, χ , following [10.67], can be shown to be equal to,

$$\tan 2\chi = \frac{V}{\sqrt{Q^2 + U^2}}, \quad (10.68)$$

and the polarization angle can be shown to be equal to.

$$\tan 2\psi = \frac{U}{Q} \quad (10.69)$$

From the relations given above, it is easy to see that, under the frame rotation,

$$\mathcal{R}(\alpha) = \begin{pmatrix} \cos 2\alpha & \sin 2\alpha \\ -\sin 2\alpha & \cos 2\alpha \end{pmatrix} \quad (10.70)$$

the Tangent of χ , i.e., $\tan \chi$ remains invariant, however the tangent of the polarization angle gets additional increment by twice the rotation angle, i.e.,

$$\begin{aligned} \tan(2\chi) &\rightarrow \tan(2\chi) \\ \tan(2\psi) &\rightarrow \tan(2\alpha + 2\psi). \end{aligned} \quad (10.71)$$

It is worth noting that the two angles are not quite independent of each other, in fact they are related to each other. Finally we end the discussion of use of stokes parameters by noting that, the degree of polarization is usually expressed by,

$$p = \frac{\sqrt{Q^2 + U^2 + V^2}}{I_{P_T}} \quad (10.72)$$

where I_{P_T} is the total intensity of the light beam.

11. Evaluation of ellipticity (χ) and polarization (ψ) angles

Now we would proceed further from the formula given in the previous sections, to evaluate the ellipticity and polarization angles for a beam of plane polarized light propagating in the z direction. Since we are interested in finding out the effect of axion photon mixing, we need the expressions for the Stokes parameters with the Axion photon mixing effect and with that we would evaluate the ellipticity angle χ and polarization angle ψ at a distance z from the source. Using the expressions for the correlators (i.e., eqns. [9.59]), one can evaluate the stokes parameters and they turn out to be

$$\begin{aligned} I &= \left[\cos^4 \theta + \sin^4 \theta + 2 \sin^2 \theta \cos^2 \theta \cos [(k_+ - k'_+) z] \right] < A_{||}^*(0) A_{||}(0) > + < A_{\perp}^*(0) A_{\perp}(0) > \\ Q &= \left[\cos^4 \theta + \sin^4 \theta + 2 \sin^2 \theta \cos^2 \theta \cos [(k_+ - k'_+) z] \right] < A_{||}^*(0) A_{||}(0) > - < A_{\perp}^*(0) A_{\perp}(0) > \\ U &= 2 \left(\left[\cos^2 \theta \cos [(k_{\perp} - k_+) z] \right] + \sin^2 \theta \cos [(k_{\perp} - k'_+) z] \right) < A_{||}^*(0) A_{\perp}(0) > \\ V &= 2 \left(\left[\cos^2 \theta \sin [(k_{\perp} - k_+) z] \right] + \sin^2 \theta \sin [(k_{\perp} - k'_+) z] \right) < A_{||}^*(0) A_{\perp}(0) > \end{aligned} \quad (11.73)$$

Till this point, the expressions we obtain are very general i. e., no approximations were made. However for predicting or explaining the experimental outcome one would have to choose some initial conditions and make some approximations to evaluate the physical quantities of interest. In that spirit, in this analysis we would take the initial beam of light to be plane polarized, with the plane of polarization making an angle $\frac{\pi}{4}$ with the external magnetic field. And their amplitude would be assumed to be unity; therefore under this approximation $A_{||}(0) = A_{\perp}(0) = \frac{1}{\sqrt{2}}$.

It is important to note that, for axion detection through polarization measurements or astrophysical observations, the parameter $\theta \ll 1$. Also we can define another dimension full parameter, $\delta = \frac{g}{m_a^2}$. With the current experimental bounds for Axion mass and coupling constant $\delta \ll 1$. So we can safely take $\cos\theta \sim 1$ and $\sin\theta \sim \theta$. Now going back to eqns., (7.46) and (7.47) one can see that the dispersion relations for the wave vectors are given by,

$$\begin{aligned} k_{\perp} &\simeq \omega, \\ k_{+} &\simeq \omega + \frac{(g\mathcal{B}\omega)^2}{2m_a^2\omega}, \\ k'_{+} &\simeq \omega - \frac{m_a^2}{2\omega} - \frac{(g\mathcal{B}\omega)^2}{2m_a^2\omega} \\ \theta &= \frac{g\mathcal{B}\omega}{m_a^2} \end{aligned} \quad (11.74)$$

Since the ratio $\frac{g}{m_a^2} = \delta \ll 1$, we can always neglect their higher order contributions in any expansion involving δ . Therefore making the same, Q can be shown to be close to zero and the Stokes parameter U turns out to be:

$$U = 1 + O(\delta^n) \text{ when } n \geq 1 \dots \quad (11.75)$$

Before proceeding further, we note the following relations,

$$\begin{aligned} k_{+} - k_{\perp} &= \frac{m_a^2\theta^2}{2\omega} \\ k'_{+} - k_{\perp} &= -\frac{m_a^2}{2\omega}, \\ k_{+} - k'_{+} &\simeq \frac{m_a^2}{2\omega}. \end{aligned} \quad (11.76)$$

they would be useful to find out the other Stokes parameter V . In terms of these, V comes out to be,

$$V = \sin\left(-\frac{m_a^2\theta^2 z}{2\omega}\right) + \theta^2 \sin(m_a^2 z / 2\omega) \quad (11.77)$$

If we retain terms of order θ^2 only, in eqn. [11.77], then, we find, $V = \frac{1}{48} \frac{\theta^2 m_a^6 z^3}{\omega^3}$, where an overall sign has been ignored. Finally substituting the values of θ and other quantities, the ellipticity angle χ is turns out to be

$$\chi = \frac{1}{96\omega} \left(\frac{(\mathcal{B}m_a^2)}{M} \right)^2 z^3. \quad (11.78)$$

The expression of the ellipticity angle χ as given by eqn. [11.78], found to be consistent with the same in (R. Cameron et al. , 1993). It should however be noted that, for interferometer based experiments, if the path length between the mirrors is given by l , and there are n reflections that take place between the mirrors then $\chi(nl) = n\chi(l)$, i.e. the coherent addition of ellipticity per-pass. The reason is the following: every time the beam falls on the mirror the photons get reflected, the axions are lost, they don't get reflected from the mirror.

Having evaluated the ellipticity parameter, we would move on to calculate the polarization angle from the expression

$$\tan(2\psi) = \frac{U}{Q}.$$

However there is little subtlety involved in this estimation; recall that the beam is initially polarized at an angle 45° with the external magnetic field. So to find out the final polarization after it has traversed a length z , we need to rotate our coordinate system by the same angle and evaluate the cumulative change in the polarization angle. We have already noted in the previous section, the effect of such a rotation on the stokes parameters and hence on the polarization angle; so following the same procedure, we evaluate the angle Ψ from the following relation,

$$\tan(2\psi + \frac{\pi}{2}) = \frac{U}{Q}. \quad (11.79)$$

We have already noted (eq. [11.75]) that for the magnitudes of the parameters of interest, the stokes parameter $U \sim 1$; and that makes the angle 2ψ inversely proportional to Q , where the proportionality constant turns out to be unity. Therefore we need to evaluate just Q , using the approximations as stated before. Recalling the fact that, the mixing angle θ is much less than one, we can expand all the θ dependent terms in the expression for Q , and retain terms up to order θ^2 . Once this is done, we arrive at:

$$Q = -2\theta^2 \left(\sin^2 \left(\frac{(k_+ - k'_+)z}{2} \right) \right), \quad (11.80)$$

Now one can substitute the necessary relations given in eqns. [11.77] in eqn. [11.80] to arrive at the expression for ψ . Once substituted the polarization angle turns out to be.

$$\psi = \frac{(\mathcal{B}^E z)^2}{16M^2\omega}. \quad (11.81)$$

We would like to point out that, the angle of polarization as given by [11.81] also happens to be consistent with the same given in reference [(R. Cameron et al. , 1993)] where the authors had evaluated the same using a different method. In the light of this, we conclude this section by noting that, all the polarization dependent observables related to optical activity can be obtained independently by various methods, for the parameter ranges of interest or

instrument sensitivity, the results obtained using stokes parameters turns out to be consistent with the alternative ones.

12. Axion electrodynamics in a magnetized media

In the earlier section we have detailed the procedure of getting axion photon modified equation of presence of tree level axion photon interaction Lagrangian. And this equation of motion would be valid in vacuum, but in nature most of the physical processes take place in the presence of a medium, ideal vacuum is hardly available. Therefore to study the axion photon system and their evolution one needs to take the effect of magnetized vacuum into account. This could be done by taking an effective Lagrangian, that incorporates the magnetized matter effects. This Lagrangian is provided in [(A. K. Ganguly P.K. Jain and S. Mandal , 2009)].

In momentum space this effective Lagrangian is given by:,

$$\mathcal{L} = \frac{1}{2} \left[-A_\mu k^2 \tilde{g}^{\mu\nu} A_\nu + A_\mu \tilde{\Pi}^{\mu\nu} A_\nu + i \frac{\tilde{\mathcal{F}}^{\mu\nu} k_\mu A_\nu a}{M_a} - a(k^2 - m_a^2)a \right]. \quad (12.82)$$

The notations in eqn. [12.82] are the following, $\tilde{g}^{\mu\nu} = \left(g^{\mu\nu} - \frac{k^\mu k^\nu}{k^2} \right)$, $\tilde{\mathcal{F}}^{\mu\nu}$ is the field strength tensor of the external field, $\frac{1}{M_a} \simeq \frac{1}{M}$ the axion photon coupling constant, $\tilde{\Pi}^{\mu\nu}$ is polarization tensor including Faraday contribution and is given by,

$$\tilde{\Pi}^{\mu\nu}(k) = \Pi_T(k) R^{\mu\nu} + \Pi_L(k) Q^{\mu\nu}(k) + \Pi_p(k) P^{\mu\nu}. \quad (12.83)$$

Usually in the thermal field theory notations, the cyclotron frequency is given by, $\omega_B = \frac{eB}{m}$ and plasma frequency (in terms of electron density n_e and temperature T) in written as, $\omega_p = \sqrt{\frac{4\pi n_e}{m} \left(1 - \frac{5T}{2m} \right)}$. In terms of these expressions, the longitudinal form factor Π_L , transverse form factor Π_T and Faraday form factor Π_p along with their projection operators $Q^{\mu\nu}$, $R^{\mu\nu}$ and $P^{\mu\nu}$ are given by,

$$\Pi_L(k) = k^2 \omega_p^2 \left(\frac{1}{\omega^2} + 3 \frac{|\vec{k}|^2}{\omega^4} \frac{T}{m} \right), \quad \Pi^p(k) = \frac{\omega \omega_B \omega_p^2}{\omega^2 - \omega_B^2} \quad \text{and} \quad \Pi_T = \omega_p^2 \left(1 + \frac{|\vec{k}|^2}{\omega^2} \frac{T}{m} \right)$$

where
$$\begin{cases} Q_{\mu\nu} = \frac{\tilde{u}_\mu \tilde{u}_\nu}{\tilde{u}^2} \\ R_{\mu\nu} = \tilde{g}_{\mu\nu} - Q_{\mu\nu}, \\ P_{\mu\nu} = i \epsilon_{\mu\perp\nu\alpha\beta} \frac{k^\alpha}{|k|} u^\beta. \end{cases}$$

The equations of motion for Gauge pseudoscalar fields that follows from the Lagrangian (12.82) are the following:

$$\left(-k^2 \tilde{g}_{\alpha\nu} + \tilde{\Pi}_{\alpha\nu}(k) \right) A^\nu(k) = -i \frac{k^\mu \tilde{\mathcal{F}}_{\mu\alpha} a}{2M_a} \quad (12.84)$$

$$\left(k^2 - m^2 \right) a = i \frac{b_\mu^{(2)} A^\mu(k)}{2M_a}. \quad (12.85)$$

For the problem in hand we have two vectors and one tensor at our disposal, frame velocity of the medium u^μ , 4 momentum of the photon k^μ and external magnetic field strength tensor $\mathcal{F}^{\mu\nu}$. To describe the dynamics of the 4 component gauge field, we need to expand them in an orthonormal basis. One can construct the basis in terms of the following 4-vectors:

$$b^{(1)\nu} = k_\mu \mathcal{F}^{\mu\nu}, \quad b^{(2)\nu} = k_\mu \tilde{\mathcal{F}}^{\mu\nu}, \quad I^\nu = \left(b^{(2)\nu} - \frac{(\tilde{u}^\mu b_\mu^{(2)})}{\tilde{u}^\mu} \tilde{u}^\nu \right), \quad \text{and } k^\mu. \quad (12.86)$$

In eqn. [12.86] we have made use of the additional vector, $\tilde{u}^\nu = \tilde{g}^{\nu\mu} u_\mu$ ($u^\mu = (1, 0, 0, 0)$).

$$\begin{aligned} N_1 &= \frac{1}{\sqrt{-b_\mu^{(1)} b^{(1)\mu}}} = \frac{1}{B_z K_\perp} \\ N_2 &= \frac{1}{\sqrt{-I_\mu I^\mu}} = \frac{|\vec{K}|}{\omega K_\perp B_z} \\ N_L &= \frac{1}{\sqrt{-\tilde{u}_\mu \tilde{u}^\mu}} = \frac{K}{|\vec{K}|}, \end{aligned} \quad (12.87)$$

The negative sign under the square roots are taken to make the vectors real. The Gauge field or photon field now can be expanded in this new basis,

$$A_\alpha(k) = A_1(k) N_1 b_\alpha^{(1)} + A_2(k) N_2 I_\alpha + A_L(k) N_L \tilde{u}_\alpha + k_\alpha N_{||} A_{||}(k). \quad (12.88)$$

The form factor $A_{||}(k)$ is associated with the gauge degrees of freedom and would be set to zero. It is easy to see that, this construction satisfies the Lorentz Gauge condition $k^\mu A_\mu = 0$. The equations of motion for the axions and photon form factors are given by,

$$\begin{aligned} (k^2 - \Pi_T(k)) A_2(k) - i \Pi_p N_1 N_2 \left[\epsilon_{\mu \perp \nu \perp 30} b^{(1)\nu} I^\mu \right] N_1 A_1(k) &= - \frac{(i N_2 b_\mu^{(2)} I^\mu) a}{M_a}, \\ (k^2 - \Pi_T(k)) A_1(k) + i \Pi_p N_1 N_2 \left[\epsilon_{\mu \perp \nu \perp 30} b^{(1)\mu} I^\nu \right] A_2(k) &= 0, \\ (k^2 - \Pi_L(k)) A_L(k) &= \frac{i N_L (b_\mu^{(2)} \tilde{u}^\mu) a}{M_a}, \\ \left[\frac{(i b_\mu^{(2)} I^\mu)}{M_a} N_2 A_2(k) + \frac{(i b_\mu^{(2)} \tilde{u}^\mu)}{M_a} N_L A_L(k) \right] &= (k^2 - m^2) a. \end{aligned} \quad (12.89)$$

As in the previous case, in this case too we would assume the wave propagation to be in the z direction. and a generic solution written as $\Phi_i(t, z)$ for all the dynamical degrees of freedom would be assumed to be of the form, $\Phi_i(t, z) = e^{-i\omega t} \Phi_i(0, z)$. As we had done before, now we may express Eqs. (12.89), in real space in the matrix form

$$\left[(\omega^2 + \partial_z^2) \mathbf{I} - M \right] \begin{pmatrix} A_1(k) \\ A_2(k) \\ A_L(k) \\ a(k) \end{pmatrix} = 0. \quad (12.90)$$

where \mathbf{I} is a 4×4 identity matrix and the modified mixing matrix, because of magnetized medium, turns out to be,

$$M = \begin{pmatrix} \Pi_T & -iN_1N_2\Pi_p\epsilon_{\mu\perp\nu\perp}30b^{(1)\mu}I^\nu & 0 & 0 \\ iN_1N_2\Pi_p\epsilon_{\mu\perp\nu\perp}30b^{(1)\nu}I^\mu & +\Pi_T & 0 & -i\frac{N_2b_\mu^{(2)}I^\mu}{M_a} \\ 0 & 0 & \Pi_L & -i\frac{N_1b_\mu^{(2)}\tilde{u}^\mu}{M_a} \\ 0 & i\frac{N_2b_\mu^{(2)}I^\mu}{M_a} & i\frac{N_Lb_\mu^{(2)}\tilde{u}^\mu}{M_a} & m_a^2 \end{pmatrix}. \quad (12.91)$$

Solving this problem exactly is a difficult task, however in the low density limit one can usually ignore the effect of longitudinal field and Π_L . Again if we assume the $\omega \gg \omega_p$, then we can simplify the faraday contribution further. Incorporating these effects, the mixing matrix in this case turns out to be a 3×3 matrix, given by:

$$M = \begin{pmatrix} \omega_p^2 & i\omega_B\omega_p^2 \cos \theta'/\omega & 0 \\ -i\omega_B\omega_p^2 \cos \theta'/\omega & \omega_p^2 & -ig\mathcal{B}\omega \\ 0 & ig\mathcal{B}\omega & m_a^2 \end{pmatrix}, \quad (12.92)$$

The angle θ' is the angle between the magnetic field and the photon momentum \vec{k} . The other symbols are the same as used in the previously. This matrix can be diagonalized and one can obtain the exact result. The method of exact diagonalization of this matrix is relegated to the appendix.

The matrix given by eqn. [12.92] has been diagonalized and its eigen values have been evaluated perturbatively [(A. K. Ganguly P.K. Jain and S. Mandal , 2009)], in the limit $g\mathcal{B}\omega \gg \frac{\omega_B\omega_p^2 \cos \theta'}{\omega} \gg |m_a^2 - \omega_p^2|$. The construction of the density (or coherency) matrix from there is a straight forward exercise as illustrated before. Therefore instead of repeating the same here we would provide the values of the stokes parameters, computed from various components of the density matrix ⁽²⁾. In this analysis we assume plane polarized light, with the following initial conditions $a(0) = 0$ and $A_1(0) = A_2(0) = \frac{1}{\sqrt{2}}$. That is the initial angle the beam makes with the direction of I^μ is $\pi/4$. The resulting stoke parameters are,

$$Q = -\sin(\Delta z), \quad I = 1, \\ V = \frac{(g\mathcal{B})^2 \omega^3 \sin\left(\frac{\Delta z}{2}\right) \cos\left(\frac{\Delta z}{2} - \frac{\pi}{4}\right)}{\sqrt{2}\omega_B \omega_p^2 \cos \theta' (m_a^2 - \omega_p^2)}, \quad U = \cos(\Delta z), \quad (12.93)$$

where in eqn. [12.93], the parameter Δ is given by, $\Delta = -2\frac{\omega_B\omega_p^2 \cos \theta'}{\omega^2}$. Since V is associated with circular/ elliptic polarization, we can see from eqn. [12.93] that, even if one starts with a plane polarized wave, to begin with, it can become circularly or elliptically polarized light because of axion photon interaction and faraday effect. The ellipticity of the propagating wave turns out to be,

$$\chi = \frac{1}{2} \tan^{-1} \left(\frac{(g\mathcal{B})^2 \omega^3 \sin\left(\frac{\Delta z}{2}\right) \cos\left(\frac{\Delta z}{2} - \frac{\pi}{4}\right)}{\sqrt{2}\omega_B \omega_p^2 \cos \theta' (m_a^2 - \omega_p^2)} \right). \quad (12.94)$$

⁽²⁾ See for instance equation. [5.14], in [(A. K. Ganguly P.K. Jain and S. Mandal , 2009)]

and the polarization angle, ψ would be given by,:

$$\tan(\psi + \pi/2) = -\cot(\Delta z). \quad (12.95)$$

when z is the path length traversed by the beam, in the magnetized media. We would like to emphasize here that, even in the limit of weak external magnetic field, it may not be prudent to ignore the contribution of Faraday effect. If we define a new energy scale ω_s , such that

$$\omega_s = \left| \frac{\omega_B \left(\omega_p^4 - \omega_p^2 m_a^2 \right) M^2 \cos \theta}{(\mathcal{B}^E)^2 \sin^2 \theta} \right|, \quad (12.96)$$

then for $\omega_s \gg \omega$, to estimate χ , one should consider the Faraday effect simultaneously.

We conclude here by noting that in this write up, we have tried to provide a comprehensive study of axion photon mixing and the associated observables of a photon beam. We have employed the coherency matrix formulation for studying the polarization properties; Starting with tree level axion photon interaction Lagrangian, we have demonstrated explicitly, how to construct the Stokes parameters from there. From there we have shown how to calculate the ellipticity angle and polarization angle from the Stokes Parameters. The relevant findings or questions pertaining to the current or proposed experiments in this area involve inclusion of matter effects, consideration of very strong magnetic field, dynamics of very high energy photon in such a scenario. Except the last, we have discussed the issues relevant for the first two. We end here by hoping that this elementary write up would help those who would like to take up advanced level investigations in this direction.

13. Acknowledgment

Many of the ideas I have presented here, took its shape during my collaborations with Prof. P. K. Jain and Dr. Subhayan Mandal. I am acknowledge them here in this note. I also would like to thank my wife, Dr. Archana Puri for her patience and understanding.

14. Appendix: Constructing the orthogonal matrix for diagonalization

Here we out line diagonalization of a 3×3 matrix given by eqn. (12.92), i.e., a symmetric matrix of the following type,

$$\mathbf{X}_3 = \begin{bmatrix} a & b & 0 \\ b & c & d \\ 0 & d & g \end{bmatrix}. \quad (14.97)$$

Generalizing it to a hermitian matrix of the kind we have is trivial, so we would concentrate on diagonalizing the type given by eqn. (14.97). As noted already, the Cayley-Hamilton characteristic equation for this matrix looks like, $|\mathbf{X}_3 - \lambda_i| = 0$. for the i 'th eigen value. Or for that matter, for any of the three eigen values, one should have:

$$\begin{vmatrix} a - \lambda_i & b & 0 \\ b & c - \lambda_i & d \\ 0 & d & g - \lambda_i \end{vmatrix} = 0 \quad (14.98)$$

Which when written in algebraic form looks like,

$$\lambda^3 - \lambda^2(a + c + g) + \lambda(gc + ga + ac - d^2 - b^2) + (ad^2 + gb^2 - gac) = 0 \quad (14.99)$$

Recalling that, the three roots of eqn. (14.99) satisfies the following relations

$$\lambda_1 + \lambda_2 + \lambda_3 = (a + c + g) \quad (14.100)$$

$$\lambda_1\lambda_2 + \lambda_2\lambda_3 + \lambda_3\lambda_1 = (gc + ga + ac - d^2 - b^2) \quad (14.101)$$

$$\lambda_1\lambda_2\lambda_3 = -(ad^2 + gb^2 - gac) \quad (14.102)$$

We should have for any value of $i(1, 2 \text{ or } 3)$,

$$\begin{bmatrix} a - \lambda_i & b & 0 \\ b & c - \lambda_i & d \\ 0 & d & g - \lambda_i \end{bmatrix} \begin{pmatrix} u_i \\ v_i \\ w_i \end{pmatrix} = 0, \quad (14.103)$$

with corresponding eigen-vector

$$\mathbf{V}_i = \begin{pmatrix} u_i \\ v_i \\ w_i \end{pmatrix}, \quad (14.104)$$

All that we need to prove is ,

$$\mathbf{V}_i \cdot \mathbf{V}_j = \delta_{ij}. \quad (14.105)$$

when suitably normalized. Next, assuming the eigen vectors to be normalized, we would demonstrate the necessary identities they need to satisfy. The proof should follow by explicit use of the values of λ_i 's in (14.105) (which is laborious) or by some other less laborious method. Here we explore the last option. We write down the generic eqns. satisfied by the components of the eigen vectors

$$\begin{aligned} (a - \lambda)u + bv &= 0 \\ bu + (c - \lambda)v + dw &= 0 \\ dv + (g - \lambda)w &= 0. \end{aligned} \quad (14.106)$$

It's easy to find out the nontrivial solns of (14.106) (for any of the three eigenvalues) by inspection and they are:

$$\begin{aligned} u &= -b(g - \lambda) \\ v &= (a - \lambda)(g - \lambda) \\ w &= -d(a - \lambda). \end{aligned} \quad (14.107)$$

All that is to be shown is $\mathbf{V}_1 \cdot \mathbf{V}_2 = 0$ and other similar relations. We would prove the previous relation, others can be done using similar method. To begin with note that,

$$\begin{aligned} \mathbf{V}_1 \cdot \mathbf{V}_2 &= \left[b^2(g - \lambda_1)(g - \lambda_2) + d^2(a - \lambda_1)(a - \lambda_2) \right. \\ &\quad \left. + (g - \lambda_1)(g - \lambda_2)(a - \lambda_1) \times (a - \lambda_2) \right] \end{aligned} \quad (14.108)$$

which is trivial to check. Next we start from,

$$[(g - \lambda_1)(g - \lambda_2)] = g^2 - g(\lambda_1 + \lambda_2) + \lambda_1 \lambda_2. \quad (14.109)$$

Eqn. (14.109) is a function of λ_1 and λ_2 , and we need to convert it to a function of a single variable λ_3 . To do that we would make use of the following tricks,

$$\begin{aligned} \lambda_1 + \lambda_2 &= [\lambda_1 + \lambda_2 + \lambda_3] - \lambda_3 \\ \lambda_1 \lambda_2 &= [\lambda_1 \lambda_2 + \lambda_2 \lambda_3 + \lambda_3 \lambda_1] - \lambda_3(\lambda_2 + \lambda_1) \end{aligned} \quad (14.110)$$

Now one can use the relations (14.101, 14.102 and 14.102), to replace the expressions inside the square bracket in eqns. (14.110) to get a function of only λ_3 . i.e.

$$\begin{aligned} \lambda_1 + \lambda_2 &= a + c + g - \lambda_3 \\ \lambda_1 \lambda_2 &= gc + ga + ac - d^2 - b^2 - \lambda_3(a + c + g - \lambda_3). \end{aligned} \quad (14.111)$$

As one uses eqns. (14.111) in eqn. (14.109) one arrives at,

$$g^2 - g(\lambda_1 + \lambda_2) + (\lambda_1 \lambda_2) = (\lambda_3 - a)(\lambda_3 - c) - b^2 - d^2. \quad (14.112)$$

so

$$b^2(g - \lambda_1)(g - \lambda_2) = b^2[(\lambda_3 - a)(\lambda_3 - c) - b^2 - d^2]. \quad (14.113)$$

Similarly one can show that,

$$d^2(a - \lambda_1)(a - \lambda_2) = d^2[(\lambda_3 - g)(\lambda_3 - c) - b^2 - d^2]. \quad (14.114)$$

Finally as we substitute in eqn. (14.108), the results of eqns. (14.113) and (14.114), we get after some cancellations,

$$\mathbf{V}_1 \cdot \mathbf{V}_2 = (c - \lambda_3) \left[(a - \lambda_3)(g - \lambda_3)(c - \lambda_3) - b^2(g - \lambda_3) - d^2(a - \lambda_3) \right] = 0, \quad (14.115)$$

because the expression inside the square bracket of eqn. (14.115) after the first = sign, is zero, as can be seen by expanding the determinant, i.e., eqn. (14.98) after taking λ_i to be λ_3 . In a similar fashion it can be shown that,

$$\mathbf{V}_1 \mathbf{V}_2 = \mathbf{V}_2 \mathbf{V}_3 = \mathbf{V}_3 \mathbf{V}_1 = 0. \quad (14.116)$$

15. Proof: \mathbf{V} 's actually diagonalize the mixing matrix

Lets start from:

$$\begin{aligned} \begin{bmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{bmatrix}^T \begin{pmatrix} a & b & 0 \\ b & c & d \\ 0 & d & g \end{pmatrix} \begin{bmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{bmatrix} &= \\ \begin{bmatrix} u_1 a + b v_1 & u_1 b + v_1 c + w_1 d & v_1 d + g w_1 \\ u_2 a + b v_2 & u_2 b + v_2 c + w_2 d & v_2 d + g w_2 \\ u_3 a + b v_3 & u_3 b + v_3 c + w_3 d & v_3 d + g w_3 \end{bmatrix} \begin{pmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{pmatrix} & \end{aligned} \quad (15.117)$$

Now if we recall (14.106), we see that,

$$\begin{aligned} au_1 + bv_1 &= \lambda_1 u_1 \\ bu_1 + cv_1 + dw_1 &= \lambda_1 v_1 \\ dv_1 + gw_1 &= \lambda_1 w_1. \end{aligned} \quad (15.118)$$

Similarly,

$$\begin{aligned} au_2 + bv_2 &= \lambda_2 u_2 \\ bu_2 + cv_2 + dw_2 &= \lambda_2 v_2 \\ dv_2 + gw_2 &= \lambda_2 w_2. \end{aligned} \quad (15.119)$$

And

$$\begin{aligned} au_3 + bv_3 &= \lambda_3 u_3 \\ bu_3 + cv_3 + dw_3 &= \lambda_3 v_3 \\ dv_3 + gw_3 &= \lambda_3 w_3. \end{aligned} \quad (15.120)$$

So we can substitute eqns. (15.118) to (15.120) in eqns. (15.118), to get:

$$\begin{bmatrix} u_1a + bv_1 & u_1b + v_1c + w_1d & v_1d + gw_1 \\ u_2a + bv_2 & u_2b + v_2c + w_2d & v_2d + gw_2 \\ u_3a + bv_3 & u_3b + v_3c + w_3d & v_3d + gw_3 \end{bmatrix} \begin{pmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{pmatrix} = \begin{bmatrix} u_1\lambda_1 & v_1\lambda_1 & w_1\lambda_1 \\ u_2\lambda_2 & v_2\lambda_2 & w_2\lambda_2 \\ u_3\lambda_3 & v_3\lambda_3 & w_3\lambda_3 \end{bmatrix} \quad (15.121)$$

$$\times \begin{pmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{pmatrix} = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix}$$

So we have checked that, the transformation matrix, constructed from the orthogonal vectors, diagonalize the mixing matrix.

16. References

Weinberg S. (1975). The U(1) problem. *Physical Review D* 11, 3583. (1975)

Belavin A.A., Polyakov A.M., Shvarts, A. S. and Tyupkin, yu. S. (1975). Inatanton Solutions In Nonabelian Gauge Theories. *Physics Letters B* 59, 85, (1975)

't Hooft, G. (1976). Symmetry Breaking Through Bell-Jackiw anomalies *Physical Review Letters*, Vol.37, 8, (1976).

't Hooft, G. (1976). Computation of the quantum effects due to a four dimensional pseudoparticle. *Physical Review D* 14, 3432 (1978); (E) *ibid.* 18, 2199, (1978).

R.J. Crewther (1978), "Effects of topological charge in gauge theories. *Acta Phys. Austriaca Suppl.* 19, 47.

V. Baluni(1979) "CP violating effects in QCD", *Physical Review D* 19, 2227.

R. J. Crewther, P. Di. Vecchia, G. Veneziano, E. Witten (1980) , "Chiral estimates of the electric dipole moment of the neutron in quantum chromodynamics ", *Physics Letters B*, 88, 123; (E) *ibid.* B91, 487 (1980).

R. D. Peccei and H. R. Quinn (1977) , "CP Conservation in the Presence of Instantons," *Physical Review Letters*, Vol. 38, 1440. R. D. Peccei and H. R. Quinn(1977) , "Constraints

Imposed by CP Conservation in the Presence of Instantons," *Physical Review D* 16, 1791.

S. Weinberg (1978) , "A New Light Boson?," *Physical Review Letters*, Vol 40, 223.

F. Wilczek (1978), "Problem of Strong p and t Invariance in the Presence of Instantons," *Physical Review Letters*, Vol 40, 279.

R.D. Peccei (1996), (QCD, Strong CP and Axions). *Journal of Korean Physics Society*, 29, S199. [arXiv:hep-ph/9606475]. For a more extensive review of the strong CP problem, see in *CP Violation*, edited by C. Jarlskog (World Scientific, Singapore, 1989).

M. S. Turner (1990) , "Windows on the Axion," *Physics Reports*, Vol 197, 67.

G. G. Raffelt (1990), "Astrophysical methods to constrain axions and other novel particle *Physics Reports* Vol 198, 1. G.G. Raffelt, in *Proceedings of Beyond the Desert*, edited by H.V. Klapdor-Kleingrothaus and H. Paes (Institute of Physics, Bristol, 1998), p. 808.

G.G. Raffelt (1998) , in *Proceedings of 1997 European School of High-Energy Physics*, edited by N. Ellis and M. Neubert (CERN, Geneva, 1998), p. 235. Report No. hep-ph/9712538.

G.G. Raffelt (1996), *Stars as Laboratories for Fundamental Physics* (University of Chicago Press, Chicago, 1996).

J. Preskill M. B. Wise and F. Wilczek (1983) *Physics Letters*, B 120, 127.

M. Dine, W. Fischler and M. Srednicki (1981) , "A Simple Solution to the Strong CP Problem with a Harmless Axion," *Physics Letters*, B104, 199.

J. E. Kim (1979), "Weak Interaction Singlet and Strong CP Invariance," *Physical Review Letters*, Vol 43, 103.

L. M. Krauss, J. Moody and F. Wilczek (1985), *Physical Review Letters*, Vol.55, 1797.

S . Moriyama, M. Minowa, T. Namba, Y. Inoue, Y. Takasu and A. Yamamoto (1998), "Direct search for solar axions by using strong magnetic field and X-ray detectors," *Physics Letters B* 434, 147. [hep-ex/9805026].

K. Zioutas et al.(2005) , [CAST Collaboration] , First results from the CERN axion solar telescope. *Physical Review Letters*, Vol 94, 121301.

M. Ruderman (1991), in *Neutron Stars: Theory and Observation*, edited by J. Ventura and D. Pines (Kluwer Academic, Dordrecht, 1991); G.S. Bisnovatyi-Kogan and S.G. Moiseenko (1992), *Astron. Zh.* Vol. 69, 563 (1992) [*Sov. Astron.* Vol. 36, 285 (1992)];

R.C. Duncan and C. Thompson (1992), *Astrophysical Journal*, 392, L9; C. Thompson and R.C. Duncan (1995) , *Mon. Not. R. Astron. Soc.* Vol. 275, 255 ; M. Bocquet (1995) et al., *Astron. Astrophys.* 301, 757.

A. K, Ganguly (2006),"Axion Photon Mixing in Magnetized media", *Annals Of Physics* (N. Y.), Vol 321, 1457.

J. S. Schwinger (1951), "On gauge invariance and vacuum polarization," *Physical Review*, Vol. 82, 664.

G. Raffelt and D. Seckel (1988), "Bounds on Exotic Particle Interactions from SN 1987a," *Physical Review Letters*, Vol. 60, 1793.

D. V. Galtsov and N. S. Nikitina (1972), "Photoneutrino processes in a strong field," *Zh. Eksp. Teor. Fiz.* Vol. 62, 2008.

L. L. . DeRaad, K. A. Milton and N. D. Hari Dass (1976), "Photon Decay Into Neutrinos in a Strong Magnetic Field," *Physical Review D* 14, 3326.

A. N. Ioannisian and G. G. Raffelt (1997), "Cherenkov radiation by massless neutrinos in a magnetic field," *Physical Review D* 55, 7038. [arXiv:hep-ph/9612285].

C. Schubert (2000), "Vacuum polarization tensors in constant electromagnetic fields. Part 2," *Nuclear Physics B* Vol. 585, 429. [arXiv:hep-ph/0002276].

A. K. Ganguly, S. Konar and P. B. Pal (1999), *Physical Review D* 60 105014. [arXiv:hep-ph/9905206].

K. Bhattacharya and A. K. Ganguly (2003) , "The Axialvector vector amplitude and neutrino effective charge in a magnetized medium," *Physical Review D* 68, 053011. [arXiv:hep-ph/0308063].

See for instance (A. K. Ganguly et al. , 1999).

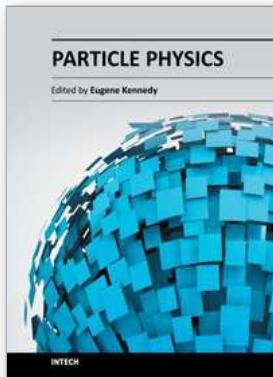
M. Giovannini (2005), "Magnetized birefringence and CMB polarization," *Physical Review D* 71, 021301. [arXiv:hep-ph/0410387].

R. Cameron *et al.* (1993), "Search for nearly massless, weakly coupled particles by optical *Physical Review D* 47, 3707.

K. Bhattacharya and A. K. Ganguly (2003), "The Axialvector vector amplitude and neutrino effective charge in a *Physical Review D* 68, 053011. [arXiv:hep-ph/0308063].

R . Cameron. et al., (1993). Search for nearly massless, weakly coupled particles by optical techniques. *Physical Review D*, 47, 3707.

Avijit K. Ganguly, Pankaj Jain and Subhayan Mandal (2009): Photon and Axion Oscillation in a magnetized medium: A general treatment, *Physical Review D* 79, 115014.



Particle Physics

Edited by Dr. Eugene Kennedy

ISBN 978-953-51-0481-0

Hard cover, 114 pages

Publisher InTech

Published online 20, April, 2012

Published in print edition April, 2012

Stimulated by the Large Hadron Collider and the search for the elusive Higgs Boson, interest in particle physics continues at a high level among scientists and the general public. This book includes theoretical aspects, with chapters outlining the generation model and a charged Higgs boson model as alternative scenarios to the Standard Model. An introduction is provided to postulated axion photon interactions and associated photon dispersion in magnetized media. The complexity of particle physics research requiring the synergistic combination of theory, hardware and computation is described in terms of the e-science paradigm. The book concludes with a chapter tackling potential radiation hazards associated with extremely weakly interacting neutrinos if produced in copious amounts with future high-energy muon-collider facilities.

How to reference

In order to correctly reference this scholarly work, feel free to copy and paste the following:

Avijit K. Ganguly (2012). Introduction to Axion Photon Interaction in Particle Physics and Photon Dispersion in Magnetized Media, Particle Physics, Dr. Eugene Kennedy (Ed.), ISBN: 978-953-51-0481-0, InTech, Available from: <http://www.intechopen.com/books/particle-physics/introduction-to-axion-photon-interaction-in-particle-physics-and-photon-dispersion-in-magnetized-med>



InTech Europe

University Campus STeP Ri
Slavka Krautzeka 83/A
51000 Rijeka, Croatia
Phone: +385 (51) 770 447
Fax: +385 (51) 686 166
www.intechopen.com

InTech China

Unit 405, Office Block, Hotel Equatorial Shanghai
No.65, Yan An Road (West), Shanghai, 200040, China
中国上海市延安西路65号上海国际贵都大饭店办公楼405单元
Phone: +86-21-62489820
Fax: +86-21-62489821