Reconstruction of a star motion in the vicinity of black hole from the redshift of the electromagnetic spectrum

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The problem of calculating the redshift of electromagnetic spectrum of the star, moving in the vicinity of Schwarzschild black hole is solved within the framework of the General Theory of Relativity. The inverse problem — determination the parameters of the motion of a star from observational data of redshift is considered. The approach that gives possibilities to solve the inverse problem is proposed. The approach is tested on the numerical model that gives possibilities to calculate redshift as function of time of observation for a star moving in the vicinity of Schwarzschild black hole. The parameters of the star in numerical model are close to parameters of the S-stars, moving in the vicinity of the Sgr A*.

Keywords: Black hole; gravitational redshift.

1. Introduction

It is well-known from astrophysical observations that supermassive black hole with mass $m_{SBH} \approx 4 \cdot 10^6 m_\odot$,[1,2] where $m_\odot$ is the mass of the Sun, exists in the Galactic Center.[1,3–5] Apart from this, large amount of stars exists in this region. For example, S-cluster includes stars closest to supermassive black hole.[2,6–8]

Astrophysical observations of such stars give possibility to study the structure of the Galactic Center and to test theories of gravity. The main source of information about the motion of these stars is their electromagnetic radiation. In the present work, we have performed theoretical investigation of the redshift of electromagnetic radiation of a star moving in the vicinity of a black hole. This problem includes two parts: the direct problem — calculation the redshift of the electromagnetic radiation of a star moving in external gravitational field and the inverse problem: — determination the motion of the star in external gravitational field if redshift as function of time of observation is known.

The direct problem within the framework of the General Relativity is considered in many papers see, e.g.,[6–12]. The mentioned studies consider different general relativistic effects such as Shapiro delay, gravitational redshift, and Doppler shift. To solve the direct problem, one needs to solve the boundary value problem for the isotropic geodesic that connects the source and the observer. In the cited studies, if the corresponding general relativistic effects are taken into account, this problem is solved using tables of impact parameters that make the accuracy of the solution limited by the step of the data in the tables. In our previous papers,[13–15] we developed a covariant approach that allows one to obtain compact expressions for redshift as a function of observation time. We have solved the boundary value problem by numerically solving a non-linear equation, which allows for much more accurate solutions than table-based methods.

For the solution of inverse problem it is necessary to use statistical methods such us MCMC method.[6,8–10,12] But mentioned studies do not contain any approach for constructing initial guess for the solution of the problem. The method that is presented in this paper gives possibility to obtain such guesses from graphics.

As an example of solving the inverse problem, we consider a mathematical model of a star moving close to a supermassive black hole. For the demonstration purpose, we chose the parameters of motion corresponding to a star on a slightly closer orbit around the black hole than the orbits of known S-stars (see, e.g.,[2,5]). Such orbits allow us to test the approach in the strong-field regime apply to the cases when the sources are on very tight orbits near the Galactic Center. Sources with such orbits may be found with future observations.

In this paper, we use a system of units where the speed of light in the vacuum is equal to unity ($c = 1$), and the metric has signature $-\,+\,++$.

2. Theoretical model

In the present paper consider only the case of spherically symmetric non-charged black hole. In General Relativity such a black hole can be described by the Schwarzschild metric (see, e.g.,[16]):

$$\mathrm{d}s^2 = \frac{\mathrm{d}r^2}{1-2M/r} + r^2\mathrm{d}\theta^2 + r^2\sin^2\theta\mathrm{d}\varphi^2 - \left(1-\frac{2M}{r}\right)\mathrm{d}t^2\,. \tag{1}$$

Here, $x^i = \{t,\, r,\, \theta,\, \varphi\}$ are Schwarzschild coordinates. Furthermore, $M = Gm_{SBH}$, where G is the gravitational constant. In our model, the mass of the black hole m_{SBH} is much larger than the mass of the star $m_s \sim m_\odot$. Because of this, we model source stars as test particles moving in the external gravitational field of the supermassive black hole. One can obtain the 4-velocity components of the star from the geodesic equation. They have the following form (see, e.g.,[16]):

$$\begin{aligned}
u^0 &= \frac{\mathrm{d}t}{\mathrm{d}\tau} = \frac{E}{(1-2M/r)}\,;\\
u^1 &= \frac{\mathrm{d}r}{\mathrm{d}\tau} = e_s\sqrt{E^2-(1-2M/r)(1+L^2/r^2)}\,;\\
u^2 &= \frac{\mathrm{d}\theta}{\mathrm{d}\tau} = 0\,;\\
u^3 &= \frac{\mathrm{d}\varphi}{\mathrm{d}\tau} = \frac{L}{r^2}\,,
\end{aligned} \tag{2}$$

where we chose the orientation of the spatial part of the coordinate system in such a way that the trajectory of the star lies in the plane $\theta = \pi/2$. Here L is the specific angular momentum of the star (in mass), E is its specific energy (in mass), and τ is its proper time. Factor e_s describes whether the considered part of the trajectory is receding or approaching.

From the system of equations (2), one can find the trajectory of the star in analytic form. In the case of finite motion, it has the following form (see, e.g.,[17]):

$$\frac{1}{r} = \frac{1}{r_s(\varphi,\, \delta,\, p_1,\, p_2)} = \frac{1}{p_1} - \frac{p_2 - p_1}{p_1 p_2} \operatorname{sn}^2 \left[\frac{(\varphi - \delta)}{2} \sqrt{1 - \frac{4M}{p_2} - \frac{2M}{p_1}},\, k_s \right] , \tag{3}$$

where

$$k_s = \sqrt{\frac{p_2 - p_1}{p_1 p_2/(2M) - p_2 - 2p_1}} \,,$$

where $\operatorname{sn}[\varphi,\, k]$ is the Jacobi sine of the first kind (see[18] for definition), δ is the longitude of pericenter, and p_1 and p_2 are pericenter and apocenter distances respectively. They are uniquely related to E and L as follows:

$$L = \frac{p_1 p_2}{\sqrt{\left(1 + \dfrac{p_1 + p_2}{2M}\right) p_1 p_2 - (p_1 + p_2)^2}} \,; E = \sqrt{\frac{(p_1 + p_2)\left(2M + \dfrac{p_1 p_2}{2M} - p_1 - p_2\right)}{\left(1 + \dfrac{p_1 + p_2}{2M}\right) p_1 p_2 - (p_1 + p_2)^2}} \,. \tag{4}$$

The proper time of the star τ can be expressed as a function of its angular coordinate φ by using the well-known analytic formula (see, e.g.,[17]):

$$\tau = \tau_s(\varphi,\, p_1,\, p_2) + \tau_0 \,. \tag{5}$$

Here, $\tau_s(\delta,\, p_1,\, p_2) = 0$. We will not write down this expression explicitly due to cumbersomeness.

Astrophysical observations of stars in the vicinity of the Galactic Center use electromagnetic radiation in the wavelength range of $1\mu\text{m} - 10\text{m}$ (including observations of pulsars, see, e.g.,[3]). Such wavelengths are small compared to the typical orbit sizes of S-stars, which allows us to use the geometric optics approximation (see, e.g.,[19]). In this approximation, electromagnetic radiation propagates along a null geodesic with tangent vector k_i that satisfies the following relations: $k_j k^j = 0$ and $k_{i;j} k^j = 0$. Here we chose the following coordinate frame $\tilde{K} : \{t,\, r,\, \tilde{\theta},\, \tilde{\varphi}\}$. Therefore, the observer resides on the axis $\tilde{\theta} = 0$, $\tilde{\varphi} = 0$, we find that the trajectory of the light ray lies in the plane $\tilde{\varphi} = \mathit{const}$ and obtain (see, e.g.,[16]):

$$\begin{aligned} k^0 &= \frac{\mathrm{d}t}{\mathrm{d}\nu} = \frac{1}{(1 - 2M/r)} \,; \\ k^1 &= \frac{\mathrm{d}r}{\mathrm{d}\nu} = e_r \sqrt{1 - (1 - 2M/r) D^2/r^2} \,; \\ k^2 &= \frac{\mathrm{d}\tilde{\theta}}{\mathrm{d}\nu} = -\frac{D}{r^2} \,; \\ k^3 &= \frac{\mathrm{d}\tilde{\varphi}}{\mathrm{d}\nu} = 0 \,, \end{aligned} \tag{6}$$

where ν is an affine parameter along the ray and D is the impact parameter. Factor e_s describes whether the considered light trajectory is receding or approaching.

The sign in the expression for $\tilde{\theta}$ in equation (6) is chosen so that $D > 0$. We only consider zeroth-order trajectories, i.e. trajectories for which the increment of $\tilde{\theta}$ from the source to observer is less then π (see, e.g.,[20, 21]).

From equations (6) and by using the boundary condition $r \to \infty$ for $\tilde{\theta} = 0$, we obtain the following analytic expression for the trajectory of the ray:

$$\frac{1}{r} = \frac{1}{r_r(\tilde{\theta},\, D)} = \frac{1}{P} - \frac{Qk^2}{2PM}\,\mathrm{cn}^2\left[\frac{\tilde{\theta}}{2}\sqrt{\frac{Q}{P}} + \mathrm{F}\left[\arccos\left(\sqrt{\frac{2M}{Qk^2}}\right), k\right], k\right],$$

where $r_r = r$ is satisfied for the points on the world line of the ray,

$$Q = \sqrt{P^2 + 4PM - 12M}\,; \qquad k = \sqrt{\frac{Q - P + 6M}{2Q}}\,, \tag{7}$$

$\mathrm{cn}\,[\varphi,\, k]$ and $\mathrm{F}\,[\varphi,\, k]$ are the Jacobi cosine and the elliptic integral of the first kind, respectively (see[18] for definition). If real, P has a physical meaning of the closest approach distance (see, e.g.,[17]). However, whether P is real or complex, it can be expressed through the impact parameter D as follows:

$$P = -\frac{2}{\sqrt{3}} D \sin\left[\frac{1}{3}\arcsin\left(\frac{3\sqrt{3}M}{D}\right) - \frac{\pi}{3}\right].$$

The angular coordinates in both coordinate systems are connected by the following relation (see Fig. 1):

$$\tilde{\theta} = \arccos[\cos(\varphi)\sin(i_0)]\,, \tag{8}$$

where angle i_0 is the inclination of the orbit of the star.

Redshift of the spectrum of electromagnetic radiation can be calculated from the formula (see, e.g.,[10]):

$$z = \frac{\delta\lambda}{\lambda} = \frac{(u_i)_s(k^i)_s}{(u_j)_o(k^j)_o} - 1\,. \tag{9}$$

Here, λ is the wavelength of emitted light, $\delta\lambda$ is the wavelength difference between received and emitted light. $(k_i)_o$ and $(k_i)_s$ denote the wave vector at the location of the observer and source, respectively. Likewise, $(u_i)_o$ and $(u_i)_s$ denote the 4-velocity vector of the observer and the star, respectively.

Consider the case of a stationary observer located at spatial infinity. Therefore, the spacetime around the observer is described to good accuracy by the Minkowsky metric. Therefore, for the observer we have $(u^i)_o = \{1,\, 0,\, 0,\, 0\}$. To calculate the components of the wave vector $(k_i)_s$, it is necessary to solve the boundary value problem for the system of differential equations (6) (see Fig. 2). The zeroth-order trajectory corresponds to the maximal intensity, and one may find it almost in all cases for S-stars (see, e.g.,[20, 21]). Because of this, we will consider only light rays of zeroth order. Therefore, the solution is unique. For the chosen assumptions, solving the mentioned boundary value problem reduces to solving the following non-linear ordinary equation for the impact parameter D:

$$r_s(\varphi,\, p_1,\, p_2) = r_r(\tilde{\theta},\, D)\,. \tag{10}$$

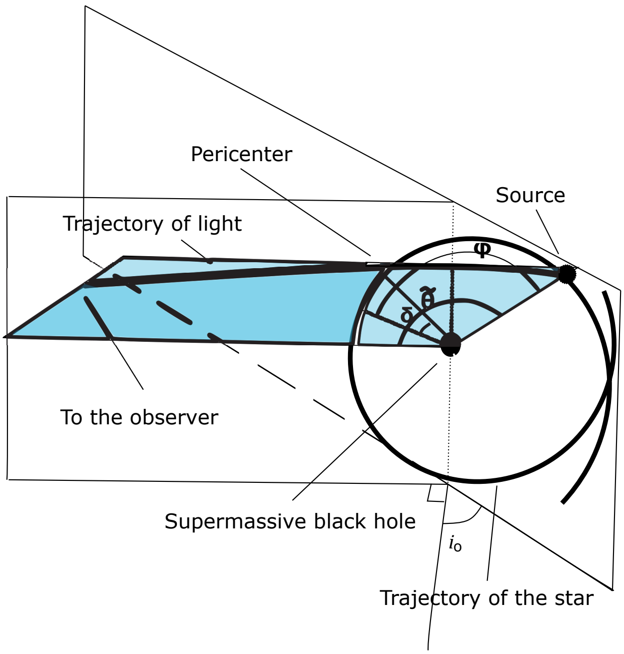

Fig. 1. For the derivation of formula (8)

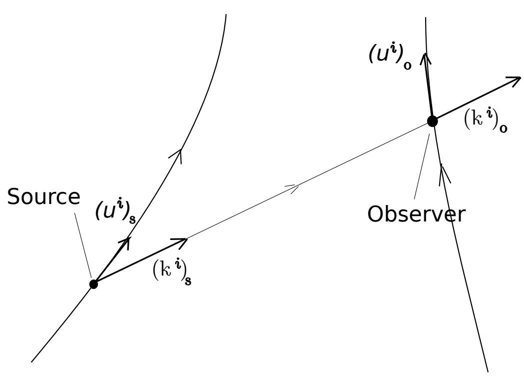

Fig. 2. $(k^i)_o$ and $(k^i)_s$ are tangent vectors to null geodesic that intersect both the worldline of the source and the worldline of the observer

Taking into account the stationarity of the observer, the relation between the angles (8), and substituting expressions (2), (6) into (9), we obtain the redshift in the following form:

$$z = -1 + \frac{E}{q} + \frac{DL}{r^2}\beta - e_s e_r \frac{1}{q}\sqrt{\left(E^2 - q\left(1 + \frac{L^2}{r^2}\right)\right)\left(1 - q\frac{D^2}{r^2}\right)}. \qquad (11)$$

Here, we denote $1 - 2M/r = q$ and $\beta = \sin(i_0)\sin(\varphi)/\sin(\tilde{\theta})$. The presented equations allow one to solve the direct problem: calculating the redshift of a star moving in an external gravitational field of a supermassive black hole as a function of observation time. We illustrate the method by using a numerical model as shown in Fig. 3.

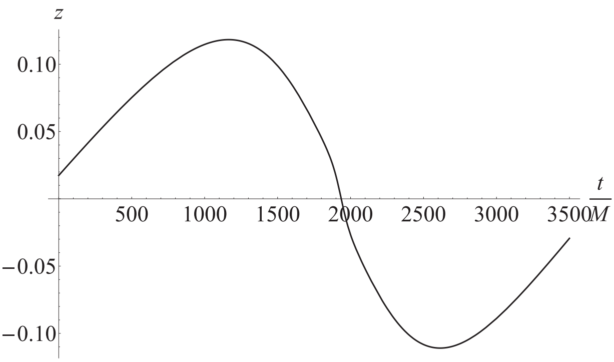

Fig. 3. Redshift of electromagnetic spectrum of a star in an external Schwarzschild gravitational field as a function observation time t. The pericenter distance of the stellar orbit is $p_1 = 60M$, its apocenter distance is $p_2 = 90M$, its inclination is $i_0 = 1.4\,\mathrm{rad}$, the longitude of pericenter is $\delta = 1\,\mathrm{rad}$ and the initial time of pericenter passage is $\tau_0 = 0M$

However, it is more interesting for astrophysical purposes to solve the inverse problem: determining the parameters of motion of a star in the external gravitational field of a supermassive black hole based on its redshift data. In the literature, the inverse problem is solved by minimizing the χ^2 function (see, e.g.,[9, 10]):

$$\chi^2 = \sum_{j=1}^{N} \left[\frac{(z_j - z_{\mathrm{obs},j})^2}{\sigma_Z^2} \right] , \tag{12}$$

where z_j and $z_{\mathrm{obs},j}$ are the theoretical and observed values of the redshift, respectively, for the times of observation t_j ($j \in [1,\, N]$). σ_Z^2 is the dispersion of the redshift observation data. Since function $z_{\mathrm{obs},j}(t)$ has no explicit expression (at least, because D is the solution of non-linear equation 10), minimizing χ^2 can only be performed numerically (for example, using the Metropolis-Hastings algorithm[22]).

In this work, we present another approach based on deriving a system of equations expressed explicitly using elementary or special functions of the parameters of motion of the star. To obtain such equations for the inverse problem, one has to find expressions not only for z but also for $\mathrm{d}z/\mathrm{d}\tau$. We describe this calculation in the following section.

3. Derivative of the redshift function

3.1. *Newman-Penrose null tetrad and optical scalars*

In this section we will use the Newman-Penrose null tetrad (see, e.g.,[23, 24]), determined along the world line of the ray emitted by the star:

$$k^i,\, n^i,\, m^i\, \bar{m}^i\, . \tag{13}$$

Here k^i is the wave vector of the ray. Symbol $^-$ denotes the complex conjugation. All vectors in (13) are null. All scalar products between vectors in (13) are equal to 0 apart from

$$k_i n^i = -1;\; m_i \bar{m}^i = 1\,. \tag{14}$$

Consider a congruence of isotropic geodesics that have tangent vectors k^i and intersect the world line of the observer at time t_o. Also, consider the Newman-Penrose tetrad (13) at all points of this congruence. Then the components of the vectors of the tetrad in the coordinate basis of $\tilde{K}$ have the form (we chose the affine parameter ν such that k_0=-1):

$$\begin{aligned}
&k_j = \left\{-1,\, e_r\frac{\sqrt{1-(1-2M/r)D^2/r^2}}{1-2M/r},\, -D,\, 0\right\}\,;\\
&n_j = \left\{-\frac{1}{2}(1-2M/r),\, -\frac{e_r}{2}\sqrt{1-(1-2M/r)D^2/r^2},\right.\\
&\left.\frac{D}{2}(1-2M/r),\, 0\right\}\,;\\
&m_j =\\
&\frac{1}{\sqrt{2}}\left\{0,\, i\frac{D}{r},\, ie_r r\sqrt{1-(1-2M/r)D^2/r^2},\, r\sin\tilde{\theta}\right\}\,;\\
&\hat{m}_j =\\
&\frac{1}{\sqrt{2}}\left\{0,\, -i\frac{D}{r},\, -ie_r r\sqrt{1-(1-2M/r)D^2/r^2},\, r\sin\tilde{\theta}\right\}\,.
\end{aligned} \tag{15}$$

For the considered congruence, one can obtain the following equations (see, e.g.,[24–26], $\epsilon = k_{[i;j]} = 0$[14])

$$k_{i;j}m^i\bar{m}^j = -\rho\,, k_{i;j}m^i m^j = -\sigma\,, \qquad k_{i:j}k^j = k_{i;j}k^i = 0\,. \tag{16}$$

Here, ρ and σ are optical scalars. They are can be found numerically from well-known equations (see, e.g.,[23–26]): Only the sum $\rho+\sigma$ admits an analytical expression (see, e.g.,[25]):

$$\begin{aligned}
\rho+\sigma = -\frac{\frac{\mathrm{d}}{\mathrm{d}\nu}\left(r\sin\tilde{\theta}\right)}{r\sin\tilde{\theta}} =\\
-\frac{e_r}{r}\sqrt{1-\left(1-\frac{2M}{r}\right)\frac{D^2}{r^2}} + \frac{D}{r^2}\cot\tilde{\theta}\,.
\end{aligned} \tag{17}$$

We now write down components of the vector of 4-velocity of the star in the basis of the null tetrad (15):

$$u^j = \frac{1}{\sqrt{2}}\left(\bar{A}m^j + A\bar{m}^j\right) + Bk^j + Cn^j\,. \tag{18}$$

Here A, B, C — are coefficients of decomposition. We obtain $-k_j u^j = (1+z) = C$. Denoting the components of the Killing vector $\frac{\partial}{\partial t}$ as ξ^j, we have

$$\xi^j = \frac{1}{2}\left(1 - \frac{2M}{r}\right) k^j + n^j \,. \tag{19}$$

Furthermore

$$E = -u^i \xi_i = B + \frac{1}{2}\left(1 - \frac{2M}{r}\right)(1+z)\,. \tag{20}$$

From the relation for the norm of u_i, we obtain

$$u_j u^j = |A|^2 - 2(1+z)B = -1\,. \tag{21}$$

From (20) and (21), it follows that

$$B = \frac{1+|A|^2}{2(1+z)}\,, \qquad |A|^2 = -1 + 2E(1+z) - (1-2M/r)(1+z)^2\,.$$

Now, we express the time derivative of redshift, using the relation $k_{[j;l]} = 0$ satisfied for the considered congruence in Schwarzschild spacetime (see, e.g.,[14]):

$$\begin{aligned}
&\frac{\mathrm{d}z}{\mathrm{d}\tau} = -k_{j;l}u^j u^l = -|A|^2 k_{j;l} m^j \bar{m}^l - A^2 k_{j;l} m^j m^l - \\
&\bar{A}^2 k_{j;l} \bar{m}^j \bar{m}^l - 2\bar{A}(1+z) k_{j;l} n^j m^l - 2A(1+z) k_{j;l} n^j \bar{m}^l = \\
&|A|^2(\rho + \sigma \cos(2P_A)) - 2\sqrt{2}(1+z)\frac{D}{r^3}|A| \sin(P_A) + \\
&\frac{e_r}{r^2}(1+z)^2 \sqrt{1 - \left(1 - \frac{2M}{r}\right)\frac{D^2}{r^2}}\,.
\end{aligned} \tag{22}$$

Here we use $A = |A|e^{iP_A}$, where $|A|$ and P_A are real. Numerical calculations show that optical scalar σ have very small value comparing to other terms in 22. Due to this will neglect the value of σ in calculations.

From equations (22) and (17) for the time derivative of redshift, we obtain

$$\begin{aligned}
&\frac{\mathrm{d}z}{\mathrm{d}\tau} = \left[2E(1+z) - \left(1 - \frac{2M}{r}\right)(1+z)^2 - 1\right] \times \\
&\left[-\frac{e_r}{r}\sqrt{1 - \left(1 - \frac{2M}{r}\right)\frac{D^2}{r^2}} + \frac{D}{r^2}\cot\hat{\theta}\right] - \\
&2\frac{D}{r^3}\sin P_A \sqrt{2E(1+z) - \left(1 - \frac{2M}{r}\right)(1+z)^2 - 1} + \\
&\frac{e_r}{r^2}(1+z)^2\sqrt{1 - \left(1 - \frac{2M}{r}\right)\frac{D^2}{r^2}}\,.
\end{aligned} \tag{23}$$

To obtain an analytical formula, it is convenient to exclude the impact parameter D from equations (11) and (23). We obtain:

$$\frac{D}{r} = \qquad (24)$$

$$\frac{\frac{L}{r}\beta\left((1+z)\left(1-\frac{2M}{r}\right)-E\right) \pm \sqrt{\left(E^2-\left(1-\frac{2M}{r}\right)\left(1+\frac{L^2}{r^2}\right)\right)\left(|A|^2+(\beta^2-1)\frac{L^2}{r^2}\right)}}{E^2-1+\frac{2}{r}+\frac{L^2}{r^2}(\beta^2-1)(1-\frac{2}{r})} =$$

$$\mathcal{F}_1(r,\, z,\, p_1,\, p_2,\, i_0,\, \tilde{\theta})\,;$$

$$\text{and } \frac{D}{r} = \mathcal{F}_2(r,\, z,\, \frac{\mathrm{d}z}{\mathrm{d}\tau},\, p_1,\, p_2,\, i_0,\, \tilde{\theta}) =$$

$$\frac{r\frac{\mathrm{d}z}{\mathrm{d}\tau}\left(|A|^2\cot\tilde{\theta}-\frac{2(1+z)}{r}|A|S\right)}{\left(1-\frac{2M}{r}\right)\left(\frac{(1+z)^2}{r}-|A|^2\right)^2+\left(|A|^2\cot\hat{\theta}-\frac{2(1+z)}{r}|A|S\right)^2} \pm$$

$$\frac{\sqrt{\left(1-\frac{2M}{r}\right)\left[\left(\frac{(1+z)^2}{r}-|A|^2\right)^2-r^2\left(\frac{\mathrm{d}z}{\mathrm{d}\tau}\right)^2\right]+\left(|A|^2\cot\tilde{\theta}-\frac{2(1+z)}{r}|A|S\right)^2}}{\left(1-\frac{2M}{r}\right)\left(\frac{(1+z)^2}{r}-|A|^2\right)^2+\left(|A|^2\cot\hat{\theta}-\frac{2(1+z)}{r}|A|S\right)^2} \times$$

$$\left(\frac{(1+z)^2}{r}-|A|^2\right). \qquad (25)$$

Here, $S = \sin P_A$. To find an exact expression for $\sin P_A$, one may use the law of angular momentum conservation:

$$u_i\Psi^i = L = const\,, \qquad (26)$$

where Ψ^i is the Killing vector field associated with the symmetry of the Schwarzschild metric relative to spatial rotation around an arbitrary axis (we chose it to be orthogonal to the orbit plane). Components of Ψ^i in the coordinate basis of $\hat{K}$ are given by (see, e.g.,[19])

$$\Psi^j = \left\{0,\, 0,\, (\cos i + \sin i\cot\tilde{\theta}\cos\tilde{\varphi}),\, \sin i\sin\tilde{\varphi}\right\}. \qquad (27)$$

Equation of the orbital plane has the following form

$$-\sin i\sin\tilde{\theta}\cos\tilde{\varphi} + \cos i\cos\tilde{\theta} = 0\,. \qquad (28)$$

From equations (26), (27), (28), (18) and (15), it follows

$$e_r\frac{l}{r}\frac{1-\beta^2}{|A|\sqrt{1-\left(1-\frac{2M}{r}\right)\frac{D^2}{r^2}}} + e_P\beta\sqrt{1-\frac{l^2}{r^2}\frac{(1-\beta^2)}{|A|^2}} =$$

$$\beta\sin P_A + e_r\frac{\sqrt{1-\beta^2}}{\sqrt{1-\left(1-\frac{2M}{r}\right)\frac{D^2}{r^2}}}\cos P_A\,, \qquad (29)$$

where e_P is defined as

$$e_P = \text{sign}\left[e_r\beta l\sqrt{1-\left(1-\frac{2M}{r}\right)\frac{D^2}{r^2}} + e_s D\sqrt{E^2-\left(1-\frac{2M}{r}\right)\left(1+\frac{L^2}{r^2}\right)}\right]. \tag{30}$$

The exact solution of (29) has the form

$$\sin P_A = e_P\sqrt{1-\frac{L^2}{r^2}\frac{(1-\beta^2)}{|A|^2}}. \tag{31}$$

4. The inverse problem

4.1. *The surface of parameters of motion*

The main purpose of the present subsection is to obtain the relation between the parameters of motion of the star from one hand and the redshift z and the derivative $\mathrm{d}z/\mathrm{d}\tau$ for certain moments of proper time from another. From (25) and (25), we obtain:

$$\mathcal{F}_1(r,\, z(t_o),\, p_1,\, p_2,\, i_0,\, \tilde{\theta}) = \mathcal{F}_2(r,\, z(t_o),\, \frac{\mathrm{d}z}{\mathrm{d}\tau}(t_o),\, p_1,\, p_2,\, i_0,\, \tilde{\theta}). \tag{32}$$

For the known redshift data $z(t_o)$ and $\frac{\mathrm{d}z}{\mathrm{d}\tau}(t_o) = (z(t_o)+1)\frac{\mathrm{d}z}{\mathrm{d}\tau}(t_o)$ (t_o is a certain observation time), equation (32) allows one to implicitly express the constant parameters of motion p_1, p_2 and i_0 in the case, when the radial location of emission r and the angle $\tilde{\theta}$ are known. Therefore, more equations are needed to solve the problem. For this purpose, one can use equations (25) and (10). We express $\tilde{\theta}$ from (10). The impact parameter D in equation (10) can be expressed using (25). This way, we obtain:

$$\tilde{\theta} = f(r,\, \mathcal{F}_1(r,\, z(t_o),\, p_1,\, p_2,\, i_0,\, \tilde{\theta})). \tag{33}$$

Here f is some known explicit function. This equation can be solved for $\tilde{\theta}$ using the iteration method. Because the right-hand side of (33) depends on $\tilde{\theta}$ only through the small optical scalar ρ, it has little influence on the whole expression, and the solution of (33) converges quickly. For our numerical model, we have used only two iterations to obtain the solution in explicit form. Therefore, we obtain the following equation:

$$r = r_s(\varphi(\tilde{\theta}(r,\, z(t_o),\, p_1,\, p_2,\, i),\, i) - \delta,\, p_1,\, p_2). \tag{34}$$

The system (32), (34 obtained for certain observation time t_o contains 2 equations and 5 unknown variables: p_1, p_2, i_0, δ, r. Because of this, the solution is a 3-surface in a corresponding 5-dimensional space. Since we are only interested in the relations that connect the parameters of motion we must numerically calculate a projection of this space into a 4-dimensional space (with four coordinates: p_1, p_2, i_0, δ). The calculation results for our numerical model of the radiation of the star are presented (see Fig. 4–7) for different points of redshift data. To uniquely visualise the solution, we present it graphically on 2-dimensional sections of the mentioned 4-dimensional space. As seen from Fig. 4–5, the 3-surfaces obtained for each data point do not coincide. Therefore, the intersection point of the surfaces gives an exact value of the parameters of motion of the star. This point can be determined from the obtained figures with high accuracy.

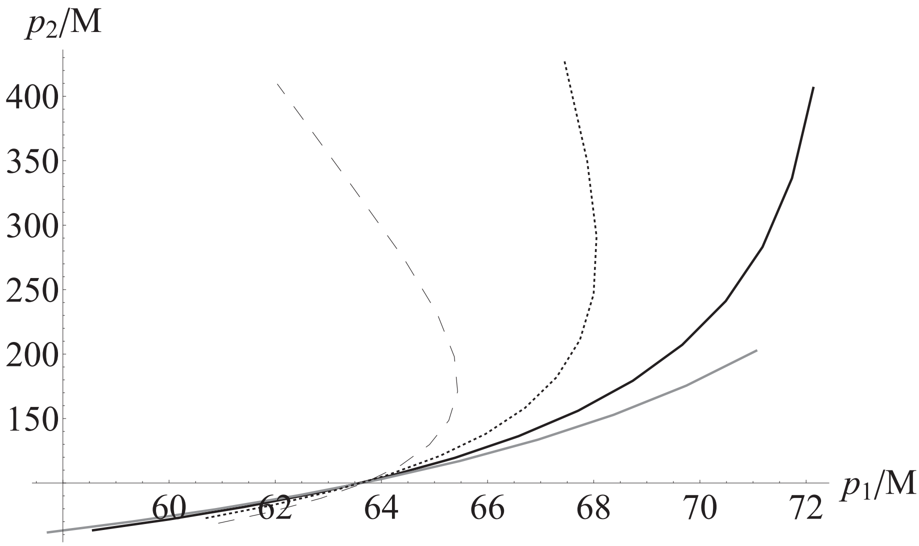

Fig. 4. 2-sections of the solution of (32), (34) for $i_0 = \pi/2$ and $\delta = 1$ by plane p_1, p_2 for different points of data (dashed: $t = 777M$, dotted: $t = 851M$, black: $t = 923M$, gray: $t = 992M$, see also Fig. 3)

Figures 4–7 also illustrate that in the case when the angular parameters are chosen to coincide with the exact solution, sections have a unique point of intersection (Fig. 6) that corresponds to the solution of the inverse problem. At the same time, if the angular parameters are not exact, a unique intersection point does not exist (Fig. 7). This is because the chosen 2-dimensional surface in the last case does not intersect with the solution in the whole 4-dimensional space of motion parameters. By using these figures, one may find if there exists a unique point of intersection in a certain region of the surface. Therefore, an initial approximation to the exact solution of the inverse problem can be found from these figures with a fairly good accuracy.

4.2. *Solution of the inverse problem*

A further improvement of the results can be obtain using the least squares method. For this purpose, we use functions $\tau_s(\varphi, p_1, p_2)$ and $r_s(\varphi, p_1, p_2)$. From (32), obtain the following equation:

$$\begin{aligned}&\mathcal{F}_1(r_s(\varphi+\delta, E, L), z(\tau_s+\tau_0), p_1, p_2, i, \tilde{\theta}(\varphi, i)) = \\ &\mathcal{F}_2(r_s(\varphi+\delta, E, L), z(\tau_s+\tau_0), \frac{\mathrm{d}}{\mathrm{d}\tau} z(\tau_s+\tau_0), p_1, p_2, i, \tilde{\theta}(\varphi, i)) . \qquad (35)\end{aligned}$$

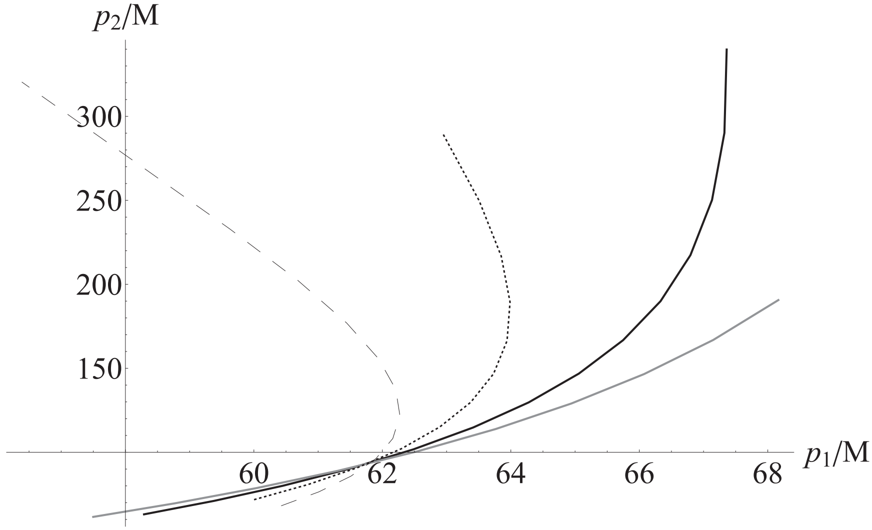

Fig. 5. 2-sections of solution of (32), (34) for $i_0 = \pi/2$ and $\delta = 1, 4$ by plane p_1, p_2 for different points of data (dashed: $t = 777M$, dotted: $t = 851M$, black: $t = 923M$, gray: $t = 992M$, see also Fig. 3)

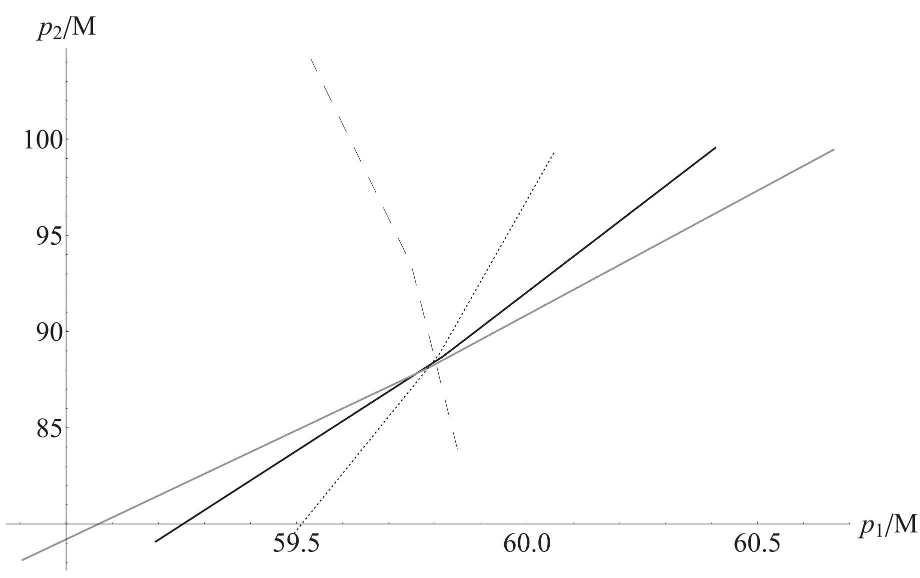

Fig. 6. Magnification of a part of figure (4)

Here, function $z(\tau)$ can be constructed based on the observational data for $z(t)$ and from function $\tau(t)$ given as an implicit function from:

$$\tau(t) = \int_0^t \frac{\mathrm{d}t'}{(1 + z(t'))}\,. \tag{36}$$

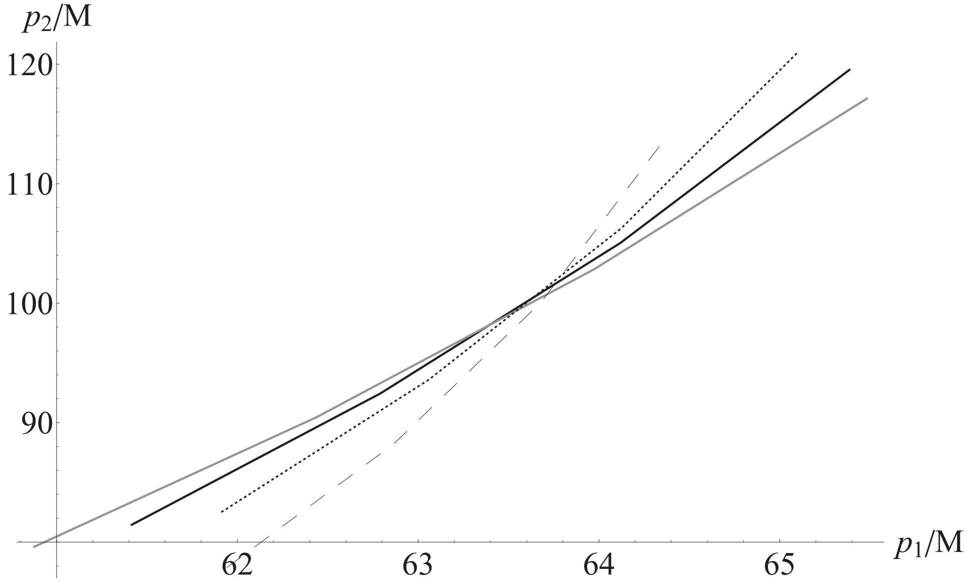

Fig. 7. Magnification of a part of figure (5)

Table 1. Results

Parameter	Initial approximation	Reconstructed value	Exact value
Pericenter distance, p_1/M	62.0	60.1	60.0
Apocenter distance, p_2/M	95.0	89.1	90.0
Orbital inclination, i_0, rad	1.5	1.48	1.4
Initial phase, δ, rad	0.9	1.0	1.0
Initial time of pericenter passage, τ_0/M	–	0.0	0.0

The left-hand side of (35) and the right-hand side of (35) are certain functions of φ. As follows from (35), these functions must be equal for a certain set of unknown parameters E, L, i_0, δ, τ_0. Therefore, one may find these parameters by using the least-squares method. As an example, we choose 10 points for different values of φ in the range [1rad, 1.4rad]. We obtain the initial approximation using the results from the previous subsection. We show the obtained results in Table 1.

5. Conclusion

The presented approach allows one to solve the inverse problem: reconstructing the motion of a star moving in the external gravitational field of a supermassive black hole based on its redshift. The approach uses the properties of congruences of isotropic geodesics to account for the difference between trajectories of light that come to the observer from the different locations of the star during the observation time.

As the main result of the paper, we have provided a method for obtaining good starting values for the parameters of motion of a star. These starting values allow one to solve the inverse problem more accurately by using statistical methods. For this purpose, we used the graphs of the surfaces in the space of parameters of motion. However, the approach may also be formulated in terms of solving a system of equations. If one writes down equations (32) and (34) for four moments of time of observation (t_{o1}, t_{o2}, t_{o3}, t_{o4}), one may obtain a system of 8 equations for 8 unknown variables (p_1, p_2, δ, i_0 and four values of the radius of radiation: r_1, r_2, r_3, r_4). Therefore, one will obtain a complete system of equations. Even numerically, it is not easy to solve a system of 8 non-linear equations. However, in future work, developed mathematical methods will allow one to efficiently solve this system of equations and analyse the conditions for obtaining non-unique solutions. The problem of non-unique solutions can be solved, for instance, by adding more equations to the considered system. The last parameter of motion τ_0 can be determined from the least-squares method. Therefore, in principle, equations (32) and (34) can be used to obtain a unique solution of the inverse problem, rather than the graphic solution presented in our paper.

The obtained equations are exact equations in General Relativity (we only neglect the optical scalar σ). Therefore, one may use the presented approach for all possible sources moving at arbitrary distances from the black hole (if they can be approximated as test particles in an external gravitational field of the black hole).

Furthermore, the approach can be directly applied to pulsar timing data for a pulsar moving in an external gravitational field. A large number of pulsars will likely be detected in the Galactic Center in the near future (see, e.g.,[12]). Pulsars can move closer to the supermassive black hole than S-stars. This way, they may be even more interesting for testing theories of gravity. The arrival times of the radio pulses can be expressed through the redshift by using:

$$t_{TOA}^{(N)} = t_{TOA}^{(N-1)} + T_p(z+1) = t_{TOA}^{(N-1)} + T_p \frac{(k^i u_i)_s}{(k^i u_i)_o} \,.$$

Here T_p — is the pulsar period in the reference frame of the pulsar, z is the redshift, $t_{TOA}^{(j)}$ — is the time of arrival of the j-th the pulse. In the problem of reconstructing the pulsar motion in the neighbourhood of a supermassive black hole, there exists one more unknown parameter — T_p.

Another interesting application of the results of this paper is reconstructing the motion of a binary star in the vicinity of a black hole. Determining the motion of such objects is a very important problem in astrophysics and stellar mechanics (see, e.g.,[27–31]). An approach for solving the problem of determining only the relative motion of the binary components was presented in our previous paper.[32]

The approach can be applied directly to the redshift data for the stars, moving near the supermassive black hole in the Galactic Center (for example, the S62 star[2]) to test General Relativity. To do this, one may obtain the parameters of motion of a star by using the presented algorithm and calculate the redshift as a function of observation time for future observations. Then, comparing the obtained curve with the observational data will allow one to test the theory.

One may use statistical methods (for example, Metropolis-Hastings algorithm, see[22]) to increase the accuracy of reconstructing the motion of the star. Such methods allow one to calculate the likelihood probability distribution and optimise values of parameters of motion of the star according to the distribution. The approach can be generalised to reconstruct the motion of a star in the vicinity of a rotating black hole. We leave this problem for future work.

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