symmetry

IMPACT FACTOR 2.940

CITESCORE 4.3

Article

Self-Consistency Equations for Composite Operators in Models of Quantum Field Theory

Yury Pismak and Artem Pismensky

Special Issue

New Trends in High Energy Physics

Edited by

Prof. Dr. Vladimir V. Vechernin and Prof. Dr. Yury M. Pismak

MDPI

https://doi.org/10.3390/sym15010132

symmetry

MDPI

Article

Self-Consistency Equations for Composite Operators in Models of Quantum Field Theory

Yury Pismak [1,2,*] and Artem Pismensky [2,3]

[1] Department of High Energy and Elementary Particle Physics, Faculty of Physics, St. Petersburg State University, Ulyanovskaya Str. 1-3, Peterhof, St. Petersburg 198504, Russia
[2] Department of Physics, St . Petersburg Electrotechnical University, Professor Popov Str., 5, St. Petersburg 197022, Russia
[3] Theoretical Physics Division NRC "Kurchatov Institute" Petersburg Nuclear Physics Institute, Orlova Roscha, Gatchina, St. Petersburg 188300, Russia
* Correspondence: y.pismak@spbu.ru; Tel.: +7-812-428-4553

Abstract: The technique of functional Legendre transforms is used to develop an effective method for calculating the characteristics of critical phenomena in quantum field theory models in the Euclidean space of dimension d. Based on the diagrammatic representation of the second Legendre transform in the theory with a cubic interaction potential, the construction of self-consistent equations is carried out, the solution of which makes it possible to find the dimensions not only of the main fields, but also of the quadratic on the composite operators within the $1/n$-expansion. Application of the proposed methods in the model F has given the opportunity to calculate in the main approximation by $1/n$ the anomalous dimensions of both scalar and tensor composite operators quadratic on the fields ϕ. For them, as functions of the spatial dimension d, we obtained explicit analytical expressions in the form of relations of two polynomials with integer coefficients.

Keywords: theory of critical phenomena; scale invariance; functional Legendre transforms; skeleton diagrams; critical dimensions; 1/n-expansion; composite operators

check for updates

Citation: Pismak, Y.; Pismensky, A. Self-Consistency Equations for Composite Operators in Models of Quantum Field Theory. *Symmetry* **2023**, *15*, 132. https://doi.org/10.3390/sym15010132

Academic Editor: Ignatios Antoniadis

Received: 16 November 2022
Revised: 27 December 2022
Accepted: 28 December 2022
Published: 2 January 2023

1. Introduction

One of the most popular calculation methods in quantum field theory and statistical physics is perturbation theory. However, there are problems for which such methods are not applicable. An example of this is the problem of calculating the characteristics of critical phenomena. In this field, there emerged methods for obtaining approximate results in the form of initial parts of power series, but the interaction constants are not their expansion parameters. Such approaches can be considered as a modification of the perturbation theory based on the use for its construction of quantities that can be viewed as small in the considered physical situation.

Models of quantum field theory in Euclidean space are often applied to quantitative description of critical phenomena and their characteristics are calculated within ϵ—or $1/n$—expansions. The parameter ϵ is the deviation of the space dimension d from some of its critical value d_{cr} specific to each model, and n is the number of field components [1–5]. Many results of various investigations obtained by such methods are presented in [6].

In order to construct an effective perturbation theory for the description of critical phenomena the symmetry properties of systems are of great importance. In quantum-field models, besides invariance with respect to translations and rotations in Euclidean d-dimensional space, scale and conformal invariance appear also very important.

An element S_λ of the continuous abelian group of scale transformations is given by the number $\lambda > 0$.We call the parameters of the model its constants, which can be set in different ways. They can, in particular, change with scale transformations. For the parameter α, by definition

$$S_\lambda \alpha = \alpha_\lambda = \lambda^{\Delta_\alpha} \alpha$$

where the number Δ_α is called the dimension of α. If $\Delta_\alpha \neq 0$, then α is called a dimensional parameter, and if $\Delta_\alpha = 0$, it is dimensionless. By definition, λ which defines the S_λ transformation is assumed to be dimensionless.

For the function $F(\alpha^1, \dots, \alpha^n)$

$$S_\lambda F(\alpha^1, \dots, \alpha^n) = F_\lambda(\alpha^1, \dots, \alpha^n) = F(S_\lambda \alpha^1, \dots, S_\lambda \alpha^n) = F(\alpha^1_\lambda, \dots, \alpha^n_\lambda).$$

Therefore,

$$S_\lambda \left(\prod_{k=1}^{n} \alpha_k \right) = \prod_{k=1}^{n} \alpha^k_\lambda = \lambda^{\Delta^{(n)}_\alpha} \prod_{k=1}^{n} \alpha^k, \ \Delta^{(n)}_\alpha = \sum_{k=1}^{n} \Delta_{\alpha^k},$$
$$S_{\lambda_2}(S_{\lambda_1} \alpha) = \lambda_1^{\Delta_\alpha} S_{\lambda_2} \alpha = \lambda_1^{\Delta_\alpha} \lambda_2^{\Delta_\alpha} \alpha = (\lambda_1 \lambda_2)^{\Delta_\alpha} \alpha = S_{\lambda_1 \lambda_2} \alpha.$$

Thus, the dimension of the product of the parameters is equal to the sum of their dimensions, and $S_{\lambda_2} S_{\lambda_1} = S_{\lambda_2 \lambda_1} = S_{\lambda_1} S_{\lambda_2}$. For $\lambda = 1$ the transformation $S_1 = 1$ is identical and $(S_\lambda)^{-1} = S_{\lambda^{-1}}$.

Functions $F(\alpha^1, \dots, \alpha^n)$ for which

$$S_\lambda F(\alpha^1, \dots, \alpha^n) = F(\lambda^{\Delta_{\alpha^1}} \alpha^1, \dots, \lambda^{\Delta_{\alpha^n}} \alpha^n) = \lambda^{\Delta_F} F(\alpha^1, \dots, \alpha^n), \tag{1}$$

are called generalized homogeneous. They play an important role in the theory of critical phenomena. If in (1) we put $\lambda = |\alpha^k|^{-1/\Delta_{\alpha^k}}$, then we get

$$F(\alpha^1, \dots, \alpha^n) = |\alpha^k|^{\frac{\Delta_F}{\Delta_{\alpha^k}}} F^{(k)}(a^1, \dots, a^n),$$

where $a^j = \alpha^j |\alpha^k|^{-\frac{\Delta_{\alpha^j}}{\Delta_{\alpha^k}}}$ and $F^{(k)}(a^1, \dots, a^n) = F(a^1, \dots a^{k-1}, 1, a^{k+1}, \dots, a^n)$ is a function of $n-1$ variables.

For description of critical phenomena, we will employ models of quantum field theory in Euclidean space of dimension d . Scale transformation of coordinates x_i, $i = 1, \dots, d$, of its points we will define as follows: $x^i_\lambda = x^i/\lambda$, that is, $\Delta_{x^i} = -1$, and $x_\lambda = x/\lambda$, $\Delta_x = -1$ for vector $x = \{x^1, \dots, x^n\}$.

The generating functional of Green's functions in the quantum field theory model with the action $S(\varphi) = S_0(\varphi) + S_{int}(\varphi)$ is written as

$$G(J) = c \int e^{iS(\varphi) + J\varphi} D\varphi, \ c^{-1} = \int e^{iS_0(\varphi)} D\varphi,$$

where $S_0(\varphi)$ is the free theory action and the functional $S_{int}(\varphi)$ describes the interactions in the model. Differentiating $G(J)$ with respect to J and setting $J = 0$, we get the Green's functions

$$G_n(x_1, \dots, x_n) = c \int e^{iS(\varphi)} \varphi(x_1) \dots \varphi(x_n) D\varphi.$$

If the action functional $S(\varphi)$ is local and $\varrho = \{\varrho \dots \varrho_k\}$ is the set of its parameters, then it is assumed that there is a scaling transformation $S^K_\lambda \varrho$ of parameters $S^K_\lambda \varrho_i = \lambda^{d_i} \varrho_i$, and replacement $\varphi(x) \to S^K_\lambda \varphi(x) = \lambda^{d_\varphi} \varphi(\lambda x)$ is defined by the dimensions d_φ, d_{ϱ_i} for which $S^K_\lambda S(\varphi, \varrho) = S(S^K_\lambda \varphi, S^K_\lambda \varrho) = S(\varphi, \varrho)$. In this sense, the action is said to be scale-invariant and dimensionless. The dimensions d_φ, d_{ϱ_i}, as well as the transformation S^K_λ are called canonical.

The composite operator $\mathbf{V}(\hat{\varphi}(x))$ of the field $\hat{\varphi}$ in the framework of the functional approach corresponds to the local functional $V(\varphi(x))$, from fields φ and its derivatives. The properties of the system associated with the composite operator $\mathbf{V}(\hat{\varphi})$ are characterized by the functions

$$G_n(V, x, x_1, \ldots, x_n) = c \int V(\varphi(x))\, e^{iS(\varphi)} \varphi(x_1) \ldots \varphi(x_n) D\varphi.$$

In this paper, we restrict ourselves simply to the study of related functions of the form

$$G_n(V, x_1, \ldots, x_n) = \int G_n(V, x, x_1, \ldots, x_n) dx.$$

Their generating functional is written as

$$G(V, J) = c \int V(\varphi)\, e^{iS(\varphi)+J\varphi} D\varphi,\ V(\varphi) = \int V(\varphi(x)) dx.$$

If we denote

$$S'(V, \varphi) = S(\varphi) + \rho V(\varphi),\ G'(V, J) = c \int e^{iS'(\varphi)+J\varphi} D\varphi,\ G'_n(V, x_1, \ldots, x_n) = \frac{\delta^k G(V, J)}{\delta J(x_1) \cdots \delta J(x_n)},$$

then

$$G(V, J) = -i \frac{\partial}{\partial \rho} G'(J)\Big|_{\rho=0}.$$

We assume that the action functionals $S(\varphi),\ S'(V, \rho, \varphi)$ are local and dimensionless. If $\rho' = \{\rho, \rho_1, \ldots \rho_p\}$ is the set of all parameters of the functional $S'(\varphi)$, by making the change variables $\varphi(x) \to S^k_\lambda \varphi(x)$ we obtain, due to the dimensionlessness of $S'(\varphi)$,

$$\begin{aligned} G'_n(\lambda x_1, \ldots, \lambda x_n) &= c \int e^{iS(\varphi, \rho_S)} V(\varphi, \rho_V) \varphi(\lambda x_1) \ldots \varphi(\lambda x_n) D\varphi = \\ &= \lambda^{-nd_\varphi - d_V} c \int e^{iS(\varphi, S^K_\lambda \rho_S)} V(\varphi,\ S^K_\lambda \rho_V) \varphi(x_1) \ldots \varphi(x_n) D\varphi. \end{aligned}$$

Here, ρ_S, ρ_V are the parameter sets of the action functional $S(\varphi)$ and $V(\varphi)$, respectively. Thus, the asymptotic of large distances is determined by the action with large values of the interaction constants of $S(\varphi)$, the dimensions of which are non-negative due to the renormalizability of the theory, which is assumed. Therefore, to analyze the asymptotic of the Green functions, it is necessary to sum the contributions of the perturbation theory series. As a result of such summations, it turns out that, in a critical state, the system is described by the functions with generalized homogeneous asymptotic of the form

$$G_n(V, \lambda x_1, \ldots, \lambda x_n) = \lambda^{-n\Delta_\varphi - \Delta_V} G_n(V, x_1, \ldots, x_n)$$

with critical indices Δ_φ, Δ_V not coinciding in canonical dimensions d_φ, d_V. The difference between the critical and canonical dimensions is called the anomalous dimension.

In this paper, we will consider quantum field models in Euclidean space with local action functional and composite operators of polynomial form. We will obtain self-consistency equations for calculating the anomalous dimensions of composite operators and demonstrate their application in the model of the n-component scalar field with $O_n(\varphi^2)^2$ interaction .

Choosing as the action of the most general form,

$$S(A, \varphi) = -\sum_{k=1}^{n} \frac{1}{k!} A_k \varphi^k D\varphi.$$

With functions of k arguments $A_k = A_k(x_1, \ldots, x_k)$, which we will call potentials, we will use the expression

$$\mathcal{G}(A) = \int e^{-\mathcal{S}(A,\varphi)} D\varphi = \int \exp\left\{\sum_{k=1}^{n} \frac{1}{k!} A_k \varphi^k\right\} D\varphi.$$

as a generating functional of Green's functions [6]. Denoting $A = \{A_1, \ldots, A_n\}$ the set of potentials of the theory, we consider them as arguments of $\mathcal{G}(A)$.Here, we do not suppose that the potentials $A_1, ..., A_n$ are local functions. We propose for a composite operator of the most general form, a local functional written as $V(\varphi) = \sum_{k=1}^{n} V_k \varphi^k / k! = S(V, \varphi)$, and denote $A' = \{A_1 + \rho V_1, \ldots, A_n + \rho V_n\} = A + \rho V$.

The functional $\mathcal{G}(\mathcal{A})$ satisfies the equations

$$\frac{\delta}{\delta A_k} \mathcal{G}(A) = \frac{1}{k!} \frac{\delta^k}{\delta A_1^k} \mathcal{G}(A),\ k = 2, \ldots, n, \tag{2}$$

$$\sum_{k=2}^{n} A_k \frac{\delta}{\delta A_{k-1}} \mathcal{G}(A) + A_1 \mathcal{G}(A) = 0. \tag{3}$$

where the relations (2) express the relationship between derivatives with respect to different potentials, and (3) is the Schwinger equation [6,7].

In terms of perturbation theory, $\mathcal{G}(A)$ is the sum of all Feynman diagrams with lines $\Delta = -A_2^{-1}$ and vertices A_k, for $1 \le k \le n$, $k \ne 2$ multiplied by $c^{-1} = \exp(\frac{1}{2} Tr \ln \Delta)$. The functional $\mathcal{W}(A) = \ln \mathcal{G}(A)$ is the connected part of $\mathcal{G}(A)$, i.e., the sum of all connected diagrams $\mathcal{G}(A)$ with the usual symmetry coefficients and $\frac{1}{2} Tr \ln \Delta$. The same holds for the diagrams of $\mathcal{W}(A') = \ln \mathcal{G}(A')$ with vertices A'_k, $k \ne 2$ and lines $\Delta' = -(A'_2)^{-1}$.

Using the Wick-Hory theorem, one can write $\mathcal{G}(A)$ in the form

$$\mathcal{G}(A) = \exp\left\{\frac{1}{2} Tr \ln \Delta + \frac{1}{2} \frac{\delta}{\delta\varphi} \Delta \frac{\delta}{\delta\varphi}\right\} e^{-S_I(A,\varphi)}\Big|_{\varphi=0},$$

where $S_I(A, \varphi) = S(A, \varphi)|_{A_2=0}$ is the action of interaction [6,7]. Thus,

$$\begin{aligned}
\mathcal{G}(V, A) = -\frac{\partial}{\partial\rho} \mathcal{G}(A + \rho V)\Big|_{\rho=0} &= -\frac{\partial}{\partial\rho} \exp\left\{\frac{1}{2} Tr \ln \Delta' + \frac{1}{2} \frac{\delta}{\delta\varphi} \Delta' \frac{\delta}{\delta\varphi}\right\} e^{-S_I(A+\rho V, \varphi)}\Big|_{\rho=0, \varphi=0} = \\
&= \left(\Delta_V \frac{\delta}{\delta\Delta} + \sum_{k=1, k\ne 2}^{n} V_k \frac{\delta}{\delta A_k}\right) \exp\left\{\frac{1}{2} Tr \ln \Delta + \frac{1}{2} \frac{\delta}{\delta\varphi} \Delta \frac{\delta}{\delta\varphi}\right\} e^{-S_I(A+\rho V, \varphi)}\Big|_{\rho=0, \varphi=0} = \\
= \left(\Delta_V \frac{\delta}{\delta\Delta} + \sum_{k=1, k\ne 2}^{n} V_k \frac{\delta}{\delta A_k}\right) \mathcal{G}(A) &= \left(\Delta_V \frac{\delta \mathcal{W}(A)}{\delta\Delta} + \sum_{k=1, k\ne 2}^{n} V_k \frac{\delta \mathcal{W}(A)}{\delta A_k}\right) \exp\{\mathcal{W}(A)\},
\end{aligned}$$

where

$$\Delta_V = \frac{\partial}{\partial\rho} \Delta'\Big|_{\rho=0} = -\frac{\partial}{\partial\rho} (A_2 + \rho V_2)^{-1}\Big|_{\rho=0} = \Delta V_2 \Delta.$$

Hence, the functional

$$\mathcal{W}(V, A) = e^{-\mathcal{W}(A)} \mathcal{G}(V, A) = \Delta_V \frac{\delta \mathcal{W}(A)}{\delta\Delta} + \sum_{k=1, k\ne 2}^{n} V_k \frac{\delta \mathcal{W}(A)}{\delta A_k}$$

is the sum of $\frac{1}{2} Tr \Delta_V$ and connected diagrams $\mathcal{W}(A)$, with one line Δ replaced by Δ_V and one vertex A_k, $k \ne 2$ replaced by V_k in all possible ways. With this replacement, the diagrams remain connected. Since all diagrams $\mathcal{W}(V, A)$ are connected, as follows from the equality $\mathcal{W}(V, A) = \exp\{-\mathcal{W}(A)\} \mathcal{G}(V, A)$ they have the same symmetry coefficients as the connected diagrams of $\mathcal{G}(V, A)$.

For the considered model, $\mathcal{W}(A)$ is the generating functional of the connected Green's functions and we will call $\mathcal{W}(V,A)$ the generating functional of $V(\varphi)$-composite connected Green's functions. For their derivatives, connected and connected composite Green's functions we will use the notation

$$\mathcal{W}_k(A) = \frac{\delta^k}{\delta A_1^k}\mathcal{W}(A),\ \mathcal{W}_{Vk}(A) = \frac{\delta^k}{\delta A_1^k}\mathcal{W}(V,A).$$

We suppose that the asymptotic of connected and connected V-composite Green's functions at the critical point are generalized homogeneous. Knowing their form at $k=2$, one can easily obtain the anomalous dimensions of the main field φ and the composite operators $V(\varphi)$. Therefore, we construct equations for the full propagator $\mathcal{W}_2$ and V-composite full propagator $\mathcal{W}_{V2}$ for finding their asymptotic at the critical point, and we will use these equations to calculate the characteristics of critical phenomena of interest.

2. Functional Legendre Transforms

To construct equations that are satisfied by the Green's functions that characterize composite operators, it is convenient to use the formalism of functional Legendre transforms. In statistical physics it was used firstly in [8–10] and then began to be applied to the problems of quantum field theory as well.

The functional Legendre transform of order m of the generating functional of the connected Green's functions $\mathcal{W}(A)$ is the functional

$$\Gamma^{(m)}(\alpha,\bar{A}) = \mathcal{W}(A) - \sum_{k=1}^{m}\int \alpha_k(x_1,\ldots,x_k)A_k(x_1,\ldots,x_k)dx_1\ldots dx_k, \quad (4)$$

whose arguments are a set of functions $\alpha=(\alpha_1,\ldots,\alpha_m)$ and potentials $\bar{A}=(A_{m+1},\ldots,A_n)$. It is assumed that $m\le n$ and the potentials $A_1,\ldots,A_m$ on the right side (4) are expressed in terms of arguments $(\alpha,\bar{A})$ so that the equalities

$$\alpha_k = \frac{\delta W(A)}{\delta A_k},\ k=1,\ldots,m,$$

are fulfilled. Thereby,

$$\frac{\delta\Gamma^{(m)}(\alpha,\bar{A})}{\delta\alpha_k(x_1,\ldots,x_k)} = -A_k(x_1,\ldots,x_k),\ \ k=1,\ldots,m,$$

and the functional

$$\Phi^{(m)}(\alpha,A) = \Gamma^{(m)}(\alpha,\bar{A}) + \sum_{k=1}^{m}\int \alpha_k(x_1,\ldots,x_k)A_k(x_1,\ldots,x_k)dx_1\cdots dx_k$$

can be considered as an effective action, the stationarity point of which is the set of complete unconnected Green's functions without vacuum loops

$$\alpha_k = \frac{\delta}{\delta A_k}\mathcal{W}(A) = \frac{1}{\mathcal{G}(A)}\frac{\delta}{\delta A_k}\mathcal{G}(A) = \frac{1}{k!\mathcal{G}(A)}\frac{\delta^k}{\delta A_1^k}\mathcal{G}(A),\ k=1,\ldots,m,$$

in the considered model. They are solutions of the system of stationarity equations

$$\frac{\delta}{\delta\alpha_k}\Phi^{(m)}(\alpha,A)=0,\ k=1,\ldots,m$$

for $\Phi^{(m)}(\alpha,A)$. By substituting them into $\Phi^{(m)}(\alpha,A)$, we obtain the generating functional of connected Green's functions $\mathcal{W}(A)$.

To calculate the anomalous dimension of the operator $V(\varphi)$, we obtain self-consistency equations for the Green's functions of the form

$$\mathcal{G}(V,x,x') = \int e^{-S(\varphi)} V(\varphi)\varphi(x)\varphi(x')D\varphi. \tag{5}$$

We assume that the system is at a critical point and the asymptotics of its Green's functions at large distances are scale invariant. We also assume the translational invariance. In this case, the mean values of the local operators are equal to zero, and the two-point Green's function is nonzero only for fields of the same dimension, and up to a constant amplitude, it is a power function of the distance between the points. Thus, for the function (5) we get

$$\mathcal{G}(V,x,x') = \frac{C_V}{(x-x')^{2(\Delta_\varphi+\Delta_V/2)}}, \tag{6}$$

where Δ_φ is the total dimension of the field φ and Δ_V is the total dimension of the functional $V(\varphi)$.

Now, it should be said that in the most general case by introducing auxiliary fields, the calculation of the integral in (5) can be presented as a problem in the model of triple interaction of fields and the quadratic form of the functional $V(\varphi)$. So, for example, for $V(\varphi) = \varphi^6$ with $S(\varphi) = \frac{1}{2}(\partial\varphi)^2 + \frac{g}{4!}\varphi^4$,

$$\begin{aligned} &\int e^{-\frac{1}{2}(\partial\varphi)^2 - \frac{g}{4!}\varphi^4} \varphi^6 \varphi(x)\varphi(x')D\varphi = \\ &= \int e^{-\frac{1}{2}(\partial\varphi)^2 - \frac{g}{4!}\psi_1^2} \psi_1\psi_2 \delta(\psi_1 - \varphi^2)\delta(\psi_1^2 - \psi_2)\varphi(x)\varphi(x')D\varphi D\psi_1 D\psi_2 = \\ &= (c')^2 \int e^{-\frac{1}{2}(\partial\varphi)^2 - \frac{g}{4!}\psi_1^2 + i\chi_1(\psi_1-\varphi^2) + i\chi_2(\psi_1^2-\psi_2)} \psi_1\psi_2\, \varphi(x)\varphi(x')D\varphi D\psi_1 D\psi_2 D\chi_1 D\chi_2. \end{aligned} \tag{7}$$

where $(c')^{-1} = \int \exp\{i\psi\chi\}D\psi D\chi$.

Thus, as the most general, we can consider the problem of calculating the anomalous dimensions of functions of the form

$$\int e^{\frac{1}{3!}A_3\varphi^3 + \frac{1}{2!}A_2\varphi^2 + A_1\varphi^1} \varphi V \varphi\, \varphi(x)\varphi(x')D\varphi = \frac{\partial}{\partial\rho} \int e^{\frac{1}{3!}A_3\varphi^3 + \frac{1}{2!}A_2\varphi^2 + A_1\varphi^1 + \rho\varphi V\varphi} \varphi(x)\varphi(x')D\varphi \Big|_{\rho=0}.$$

To describe critical phenomena in this model, it is convenient to use the second Legendre transform.

3. Model with Cubic Interaction Potential

For functional Legendre transforms of an arbitrary order, there are representations in the form of an infinite sum of Feynman skeleton diagrams. In the theory with A_k-potentials of order $k \le 4$ for Legendre transforms of no higher than the fourth order it was developed in [10]. Generalization of these results to the case of models with an arbitrary number of potentials A_k for the third and fourth Legendre transforms based on the analysis of their equations of motion is presented in [11–13]. The solution to the problem of constructing a diagrammatic technique for Legendre transforms of any high order in the theory with any number of potentials in a polynomial action is given in [14,15].

The diagrammatic technique for $\Gamma^{(2)}(\alpha, A_3)$ turns out to be the simplest if instead of the arguments α_1, α_2 of this functional we use the functions β_1, β_2 connected with α_1, α_2 by the relations [7]

$$\alpha_1 = \beta_1,\ \alpha_2 = \frac{1}{2}(\beta_2 + \beta_1^2).$$

Since $\mathcal{G}_2(x,x') = \frac{1}{2}(\mathcal{W}_2(x,x') + \mathcal{W}_1(x)\mathcal{W}_1(x'))$, the solutions of the stationarity equations for $\Phi^{(2)}(\beta, \bar{A})$ are the full connected Green's functions: $\beta_k = \frac{\delta^k \mathcal{W}(A)}{\delta A_1^k}$ for $k = 1,2$ in the theory with action $S(\varphi) = \sum_{k=1}^{3} \frac{1}{k!} A_k \varphi^k$. These functions are the average field $\mathcal{W}_1$ and the full propagator $\mathcal{W}_2$.

The second Legendre transform for the considered model has the form

$$\Gamma^{(2)}(\beta, A_3) = \frac{1}{2} Tr \ln \beta_2 + \frac{1}{2}\beta_1 A_3 \beta_2 + \tilde{\Gamma}^{(2)}(\beta_2, A_3).$$

The diagrams of $\tilde{\Gamma}^{(2)}(\beta_2, A_3)$ contain an even number of A_3-vertices. In them, lines correspond to β_2, they have the usual symmetry coefficients and are 2-irreducible. A diagram with third-order vertices is 2-irreducible, if it is connected and remains connected when no more than two lines are removed from it. In the lowest approximation in terms of the number of vertices, $\tilde{\Gamma}^{(2)}(\beta_2, A_3)$ has one diagram that looks like two points connected to each other by three lines. Its contribution, including the symmetry coefficient is $\frac{1}{2\cdot 3!} A_3 \beta_2^3 A_3$.

Thus, we obtain for the functional $\Phi^{(2)}(\beta, A)$ the expression

$$\Phi^{(2)}(\beta, A) = \frac{1}{2} Tr \ln \beta_2 + \tilde{S}(\beta, A) + \tilde{\Gamma}^{(2)}(\beta_2, A_3),$$

where $\tilde{S}(A,\beta) = S_0(A,\beta_1) + \tilde{S}_{int}(A,\beta)$, $S_0(A,\beta_1) = \frac{1}{2}\beta_1 A_2 \beta_1 + A_1\beta_1$ and $\tilde{S}_{int}(A,\beta) = \frac{1}{2}\beta_1 A_3 \beta_2 + \frac{1}{2} A_2 \beta_2$.

Stationarity equations for the functional $\Phi^{(2)}(\beta, A)$ have the form

$$\frac{\delta \tilde{S}}{\delta \beta_1} = A_1 + A_2\beta_1 + \frac{1}{2} A_3 \beta_1^2 + \frac{1}{2} A_3 \beta_2 = 0, \tag{8}$$

$$\frac{1}{2}\beta_2^{-1} + \frac{1}{2} A_2 + \frac{1}{2} A_3 \beta_1 + \frac{\delta}{\delta \beta_2} \tilde{\Gamma}^{(2)}(\beta_2, A_3) = 0. \tag{9}$$

It follows from the assumption of scale invariance of their solutions that in this case $A_1 = \beta_1 = A_3\beta_2 = 0$ and the Equation (8) is satisfied in a trivial way. On the left side (9), in addition to $\frac{1}{2} A_3 \beta_1$, the term A_2 should also be discarded if we assume that the inverse full propagator has a non-trivial anomalous dimension, which A_2 does not. Thus, we obtain the equation

$$\beta_2^{-1} + 2\frac{\delta}{\delta \beta_2} \tilde{\Gamma}^{(2)}(\beta_2, A_3) = 0. \tag{10}$$

This is the skeleton Dyson-Schwinger equation

$$D^{-1} = \Delta^{-1} - \Sigma(D)$$

for the full propagator D, in which the contribution of the bare propagator Δ is discarded due to the difference in its dimension from the dimension of the asymptotics D and the self-mass operator $\Sigma(D)$ at the critical point.

The result is the equation

$$D^{-1} + \Sigma(D) = 0, \tag{11}$$

describing the manifestation of self-consistency of the critical behavior of the system in its pair correlations. It follows from (10) that the self-mass operator $\Sigma(D)$ has the form

$$\Sigma(D) = 2\frac{\delta}{\delta \beta_2} \tilde{\Gamma}^{(2)}(D, A_3),$$

and the knowledge of the diagram technique for the second Legendre transform makes it possible to obtain an approximation for it with an arbitrarily high accuracy.

The Dyson-Schwinger self-consistency equation enables one to find the critical dimension Δ_φ of the field φ. Knowing it, one can obtain equations for the functions (6) characterizing composite operators. To do this, denote $S'(\varphi,\rho)$ the action functional of the form $S'(\varphi,\rho) = S(\varphi) + \rho V(\varphi)$, where $S(\varphi)$ is the action of the model, in which the propagator and the mean field are solutions of the Equations (8) and (9). For quadratic with respect to the field φ functional $V(\varphi) = \frac{1}{2}\varphi V_2 \varphi$ and given function V_2, the equations

$$A_1 + (A_2 + \rho V_2)\beta_1 + \frac{1}{2}A_3\beta_1^2 + \frac{1}{2}A_3\beta_2 = 0, \tag{12}$$

$$\frac{1}{2}\beta_2^{-1} + \frac{1}{2}(A_2 + \rho V_2) + \frac{1}{2}A_3\beta_1 + \frac{\delta}{\delta\beta_2}\tilde{\Gamma}^{(2)}(\beta_2, A_3) = 0, \tag{13}$$

obtained by replacing $A_2 \to A_2 + \rho V_2$ in (8) and (9), describe the full propagator $\beta_2(\rho)$ and the mean field $\beta_1(\rho)$ in models with action $S'(\varphi,\rho)$. The main approximations $\beta_i' = \frac{\partial}{\partial\rho}\beta_i(\rho)|_{\rho=0}$, $i = 1,2$, for deviations of function $\beta_i(\rho)$ from their values at $\rho = 0$ satisfy the equations that are obtained by differentiating (12) and (13) with respect to ρ and then putting $\rho = 0$:

$$V_2\beta_1 + A_2\beta_1' + A_3\beta_1\beta_1' + \frac{1}{2}A_3\beta_2' = 0, \tag{14}$$

$$\frac{1}{2}\beta_2^{-1}\beta_2'\beta_2^{-1} - \frac{1}{2}V_2 - \frac{1}{2}A_3\beta_1' - \frac{\delta^2\tilde{\Gamma}^{(2)}(\beta_2, A_3)}{\delta\beta_2^2}\beta_2' = 0. \tag{15}$$

If we look for solutions to this system of equations at the critical point, assuming that the translational and scale invariance are not violated by the functions β_1', β_2', then all dimensional mean local characteristics of the system must be equal to zero. Therefore, all terms in (14) are equal to zero, and this equation is fulfilled in a trivial way. In the Equation (15), one should discard the term containing β_1' and the function V_2 without anomalous dimension, assuming that it has $\frac{1}{2}\beta_2^{-1}\beta_2'\beta_2^{-1}$.

Thus, under the assumptions made, the functions β_2' satisfy the equation

$$\beta_2^{-1}\beta_2'\beta_2^{-1} = 2\frac{\delta^2\tilde{\Gamma}^{(2)}(\beta_2, A_3)}{\delta\beta_2^2}\beta_2'.$$

Its solution for a specifically chosen functional $V(\varphi)$ is a correction D_V to the critical asymptotics D of a propagator satisfying Equation (12). It is the solution to the equation

$$D^{-1}D_V D^{-1} = \frac{\delta\Sigma(D, A_3)}{\delta D}D_V. \tag{16}$$

It is universal for all $V(\varphi)$ and does not depend on the function V_2, which plays the role of a bare term for D_V, just as the Equation (11) does not contain the potential A_2, which determines the leading approximation for the propagator in the framework of perturbation theory. Thus, to find the anomalous dimensions of the field φ and compound operators, we have the self-consistency Equations (11) and (16). Now, we demonstrate methods that can be used to solve them.

4. Calculation of Anomalous Dimensions in a Simple Model

As an example, let us apply the self-consistency equations to calculate the anomalous dimensions in the $O_n\varphi^4$- theory of a massless scalar n-component field $\varphi = (\varphi, \ldots, \varphi_n)$ in d-dimensional Euclidean space with action functional

$$S(\varphi) = \frac{1}{2}\sum_{\mu=1}^{d}(\partial_\mu\varphi)^2 + \frac{1}{8}g(\varphi^2)^2$$

where the following notations are used: $(\partial_\mu\varphi)^2 = \sum_{k=1}^{n}\left(\frac{\partial\varphi_k}{\partial x^\mu}\right)^2$, $\varphi^2 = \sum_{k=1}^{n}\varphi_k^2$.

4.1. Derivation of Basic Equations

In this model, Feynman diagrams with vertices of the fourth order are used to perform calculations within the framework of the standard perturbation theory. Therefore, we use the equality

$$\exp\left\{-\frac{g}{8}(\varphi^2)^2\right\} = \bar{c}\int \exp\left\{-\frac{1}{2g}\psi^2 + i\frac{1}{2}\varphi^2\psi\right\}D\psi,$$

where $\bar{c} = \left(\int \exp\left\{-\frac{1}{2g}\psi^2\right\}D\psi\right)^{-1}$, to go to a theory of interaction of the field φ and scalar field ψ, which is described by an action of the form

$$S(\varphi,\psi) = \frac{1}{2}(\partial\varphi)^2 + \frac{1}{2g}\psi^2 + \frac{i}{2}\varphi^2\psi. \quad (17)$$

The Green's functions of the field φ of this model and of the original $O_n\varphi^4$-theory coincide. Therefore, using the self-consistent equations constructed for the fields φ and ψ basing on (11), (16) and action $S(\varphi,\psi)$ (17), we can find their anomalous dimensions and anomalous dimensions of the quadratic on φ or ψ composite operators.

We assume that the system is homogeneous, isotropic and invariant with respect to O_n-rotations of the field φ. In this case, the propagator of field φ, which depends on two numbers k,l of its components and coordinates x,y of two points of space, has the form of an identity $n\times n$-matrix in k,l-components of φ multiplied by the scalar function $D_\varphi((x-y)^2)$, and the propagator of field ψ is a scalar function $D_\psi((x-y)^2)$.

The arguments of the functional $\Gamma^{(2)}$, which defines the self-mass operator Σ, in our case are the functions D_φ, D_ψ and local potential $A_3(x,y,z) = i\delta(x-y)\delta(y-z)$. The contribution of $Tr\ln\beta_2$ to the functional $\Phi^{(2)}(\beta,A)$ has the form $nTr\ln D_\varphi + Tr\ln D_\psi$. So we have 2 equations for D_φ, D_ψ

$$D_\varphi^{-1} + \Sigma_\varphi(D_\varphi,D_\psi) = 0,\ D_\psi^{-1} + \Sigma_\psi(D_\varphi,D_\psi) = 0 \quad (18)$$

and two equations for the corresponding functions $D_{\varphi V}, D_{\psi V}$

$$D_\varphi^{-1}D_{\varphi V}D_\varphi^{-1} = \frac{\delta\Sigma_\varphi(D_\varphi,D_\psi)}{\delta D_\varphi}D_{\varphi V} + \frac{\delta\Sigma_\varphi(D_\varphi,D_\psi)}{\delta D_\psi}D_{\psi V}, \quad (19)$$

$$D_\psi^{-1}D_{\psi V}D_\psi^{-1} = \frac{\delta\Sigma_\psi(D_\varphi,D_\psi)}{\delta D_\varphi}D_{\varphi V} + \frac{\delta\Sigma_\psi(D_\varphi,D_\psi)}{\delta D_\psi}D_{\psi V}. \quad (20)$$

Here, the self-emergy operators $\Sigma_\varphi, \Sigma_\psi$ are defined by the functional $\tilde{\Gamma}^{(2)}(D_\psi, D_\psi, A_3)$

$$\Sigma_\varphi(D_\varphi,D_\psi) = \frac{2}{n}\frac{\delta}{\delta D_\varphi}\tilde{\Gamma}^{(2)}(D_\varphi,D_\psi,A_3),$$

$$\Sigma_\psi(D_\varphi,D_\psi) = 2\frac{\delta}{\delta D_\psi}\tilde{\Gamma}^{(2)}(D_\psi,D_\psi,A_3).$$

In the leading two-vertex approximation, $\tilde{\Gamma}^{(2)}(D_\varphi,D_\psi,A_3) = -\frac{1}{2}nD_\varphi^2D_\psi$ self-mass operators and their derivatives have the form

$$\Sigma_\varphi = -D_\psi D_\varphi,\ \Sigma_\psi = -\frac{1}{2}nD_\varphi^2,\ \frac{\delta}{\delta D_\varphi}\Sigma_\varphi = -D_\psi,\ \frac{\delta}{\delta D_\psi}\Sigma_\varphi = -D_\varphi,\ \frac{\delta}{\delta D_\varphi}\Sigma_\psi = -nD_\varphi. \quad (21)$$

We will show that, for large n, this approximation makes it possible to find the leading term of the asymptotic of order $1/n$ for the anomalous dimensions of the fields φ, ψ, as well as all composite operators, and obtain explicit analytic expressions for them. Using the form of the action $S(\varphi,\psi)$ (17), we find the canonical dimensions d_φ, d_ψ, d_g

$$d_\varphi = \varpi - 1,\ d_\psi = 2,\ d_g = 2(2-\varpi),$$

where we introduced the notation $\varpi = d/2$ for half of the dimension of the space, which we will use in what follows. Now, choosing the traditional expression $\eta/2$ for the anomalous dimension of field φ and denoting σ the anomalous dimensions of field ψ , we present the solutions of Equation (18) with self-mass operators (21) in the form

$$D_\varphi(x,y) = \frac{C_\varphi}{(x-y)^{2(\varpi-1+\eta/2)}},\ D_\psi(x,y) = \frac{C_\psi}{(x-y)^{2(2+\sigma)}}$$

where C_φ, C_ψ are dimensionless constants.

We will carry out all calculations in the coordinate representation, using the well-known relations for integrals in a space of dimension d

$$\int \frac{e^{ipx}dx^d}{x^{2\alpha}} = \pi^{\varpi} H(\alpha)\left(\frac{4}{p^2}\right)^{\alpha'}, \tag{22}$$

$$\int \frac{dz^d}{(x-z)^{2\alpha}(z-y)^{2\beta}} = \frac{\pi^{\varpi} H(\alpha,\beta,d-\alpha-\beta)}{(x-y)^{2(\alpha+\beta-\varpi)}}. \tag{23}$$

Here, p,x,y,z are vectors of d-dimensional space, $\varpi = d/2$, $\alpha' = \varpi - \alpha$. If it does not lead to any confusion, we will give this meaning to the stroke in the notation in what follows. The function $H(\alpha)$ is the ratio of two gamma-functions $H(\alpha) = \Gamma(\alpha')/\Gamma(\alpha)$, and $H(\alpha_1,\ldots,\alpha_n) = H(\alpha_1)\cdots H(\alpha_n)$. These notations will be useful in the future for simplifying formulas. We will also use the abbreviated notation for products of gamma functions $\Gamma(\alpha_1,\ldots,\alpha_n) = \Gamma(\alpha_1)\cdots\Gamma(\alpha_n)$.

From (22) we get the representation for delta-function

$$\delta(x) = \frac{1}{(2\pi)^d}\int e^{ipx}dp^d = \frac{1}{(4\pi)^{\varpi}}\lim_{\alpha\to 0} H(\alpha)\left(\frac{4}{x^2}\right)^{\varpi-\alpha} = \frac{1}{\pi^{\varpi}}\lim_{\alpha\to 0}\frac{H(\alpha)}{x^{2\alpha'}}.$$

Using (23), for the integral operators $K, \bar{K}$ in the coordinate representation with kernels

$$K(x,y) = \frac{c}{(x-y)^{2\alpha}},\ \bar{K}(x,y) = \frac{1}{c(x-y)^{2(d-\alpha-\epsilon)}},$$

where c,α,ϵ are the given numbers, we obtain

$$\lim_{\epsilon\to 0}\int K(x,y)\bar{K}(y,z)dy^d = \pi^{\varpi} H(\alpha, d-\alpha)\lim_{\epsilon\to 0}\frac{H(\epsilon)}{(x-z)^{2\epsilon'}} = \pi^d H(\alpha,d-\alpha)\delta(x-z).$$

Therefore, the kernel of the inverse to K operator has the form

$$K^{-1}(x,y) = p(\alpha)\bar{K}(x,y) = \frac{p(\alpha)}{c(x-y)^{2(d-\alpha)}},\ p(\alpha) = \frac{1}{\pi^d H(\alpha,d-\alpha)} = \frac{H(\alpha',-\alpha')}{\pi^d}.$$

Because $\Gamma(x,-x) = \Gamma(x)\Gamma(-x) = -\pi/(x\sin(\pi x))$, the function $p(\alpha)$ can be written also as

$$p(\alpha) = -\frac{\alpha'\sin(\pi\alpha')\Gamma(\alpha,d-\alpha)}{\pi^{d+1}}. \tag{24}$$

In the leading one-loop approximation (21), the Equation (18) in the coordinate representation read

$$\frac{p(\varpi-1+\eta/2)}{C_\varphi (x-y)^{2(d-(\varpi-1+\eta/2))}} = \frac{C_\varphi C_\psi}{(x-y)^{2(\varpi-1+\eta/2+2+\sigma)}}, \tag{25}$$

$$\frac{p(2+\sigma)}{C_\psi (x-y)^{2(d-2-\sigma)}} = \frac{n}{2}\frac{C_\varphi^2}{(x-y)^{4(\varpi-1+\eta/2)}}. \tag{26}$$

From the equality of the exponents $(x-y)^2$ in the denominators (25) and (26), it follows that

$$\sigma=-\eta,\ C_\varphi^2 C_\psi = p(\varpi-1+\eta/2),\ nC_\varphi^2 C_\psi = 2p(2+\sigma),\ np(\varpi-1+\eta/2) = 2p(2+\sigma). \tag{27}$$

Writing σ as $\sigma=-\eta$ and using (24), (27), we get the equation

$$n(1-\eta/2)\sin(\pi(1-\eta/2))\Gamma(\varpi-1+\eta/2,\varpi+1-\eta/2) = \\ = 2(\varpi-2+\eta)\sin\pi(\varpi-2+\eta)\Gamma(2-\eta,d-2+\eta).$$

For large n, its solution with respect to η tends to zero and we obtain the following asymptotic estimates by powers of $1/n$

$$\eta = \eta_1\frac{1}{n} + \mathcal{O}\left(\frac{1}{n^2}\right),\ \ \sigma = -\eta_1\frac{1}{n} + \mathcal{O}\left(\frac{1}{n^2}\right),\ C_\varphi^2 C_\psi = \frac{\eta_1\Gamma(\varpi-1,\varpi+1)}{2}\frac{1}{n} + \mathcal{O}\left(\frac{1}{n^2}\right)$$

where

$$\eta_1 = \frac{4(\varpi-2)\sin(\pi\varpi)\Gamma(d-2)}{\pi\Gamma(\varpi-1,\varpi+1)} = \frac{2(\varpi-2)\sin(\pi\varpi)\Gamma(2\varpi)}{(2\varpi-1)\pi\varpi\Gamma(\varpi)^2}.$$

With this result, we can find leading approximations of $1/n$-expansion for anomalous dimensions of composite operators.

4.2. Anomalous Dimensions of Composite Operators

In the one-loop approximation (21) for the self-mass operators, the Equations (19) and (20) have the form

$$D_\varphi^{-1} D_{\varphi V} D_\varphi^{-1} = D_\varphi D_{\psi V} + D_\psi D_{\varphi V}, \tag{28}$$

$$D_\psi^{-1} D_{\psi V} D_\psi^{-1} = n D_\varphi D_{\varphi V}. \tag{29}$$

Not limited to scalar composite operators, we will also consider tensor ones. We will obtain equations for composite operators of rank k in a Euclidean space of dimension d, for which the functions $D_{\varphi V}(x), D_{\psi V}(x)$ have the form

$$C\frac{t^{(k)}_{\mu_1\ldots\mu_k}(x)}{x^{2\gamma}},$$

where C is a constant factor and $t^{(k)}_{\mu_1\ldots\mu_k}(x)$ is a symmetric tensor of rank k in d-dimensional Euclidean space defined as follows. It has dimension $-k$, and can be represented as a polynomial in x^2

$$t^{(k)}_{\mu_1\ldots\mu_k}(x) = x_{\mu_1}\cdots x_{\mu_k} + \sum_{i=1}^{\{k/2\}} x^{2i} t^{(k,i)}_{\mu_1\ldots\mu_k}(x),$$

where $\{n/2\}$ is the integer part of $n/2$. In addition, it is assumed that the trace of the tensor $t^{(k)}_{\mu_1 \dots \mu_k}(x)$ on any pair of indices with the numbers i, j is equal to zero:

$$Tr^{(ij)} t^{(k)}_{\mu_1 \dots \mu_k}(x) = \sum_{\mu_i, \mu_j = 1}^{d} \delta_{\mu_i \mu_j} t^{(k)}_{\mu_1 \dots \mu_k}(x) = 0$$

where $\delta_{\mu_i \mu_j}$ is the Kronecker symbol. These conditions define the tensor $t^{(k)}_{\mu_1 \dots \mu_k}(x)$ uniquely.

If ∂_x is a vector $\partial_x = \{\partial_1, \dots \partial_d\}$, where $\partial_\mu = \frac{\partial}{\partial x_\mu}$, then

$$t^{(k)}_{\mu_1 \dots \mu_k}(x) = \frac{(-1)^k x^{2(\gamma+k)} \Gamma(\gamma)}{2^k \Gamma(\gamma+k)} t^{(k)}_{\mu_1 \dots \mu_k}(\partial_x) \frac{1}{x^{2(\gamma)}} = \frac{(-1)^{k+1} x^{2k}}{2^k \Gamma(k)} t^{(k)}_{\mu_1 \dots \mu_k}(\partial_x) \ln x^2. \tag{30}$$

It is also easy to verify, using (23) and (30), that

$$\begin{aligned} &\int\int dx_1^{(d)} dx_2^{(d)} \frac{1}{(x-x_1)^{2a}} \frac{t^{(k)}_{\mu_1 \dots \mu_k}(x_1 - x_2)}{(x_1 - x_2)^{2b}} \frac{1}{(x_2 - x')^{2c}} = \\ &= \frac{(-1)^k \Gamma(b-k)}{2^k \Gamma(b)} t^{(k)}_{\mu_1 \dots \mu_k}(\partial_x) \int\int dx_1^{(d)} dx_2^{(d)} \frac{1}{(x-x_1)^{2a}} \frac{1}{(x_1-x_2)^{2(b-k)}} \frac{1}{(x_2-x')^{2c}} = \\ &= \frac{(-1)^k \pi^d H(a, b-k, c, 3\varpi - a - b - c + k) \Gamma(b-k)}{2^k \Gamma(b)} t^{(k)}_{\mu_1 \dots \mu_k}(\partial_x) \frac{1}{(x-x')^{2(a+b+c-k-d)}} = \\ &= \frac{\pi^d H(a, b-k, c, 3\varpi - a - b - c + k) \Gamma(b-k, a+b+c-d)}{\Gamma(b, a+b+c-k-d)} \frac{t^{(k)}_{\mu_1 \dots \mu_k}(x-x')}{(x-x')^{2(a+b+c-d)}}. \end{aligned} \tag{31}$$

We consider the Equations (28) and (29) in the coordinate representation and are looking for a solution of them in the form

$$D_{\varphi V \mu_1, \dots, \mu_k}(x, y) = D^{(k)}_{\varphi V}(x-y) t^{(k)}_{\mu_1 \dots \mu_k}(x-y), \; D^{(k)}_{\varphi V}(x) = \frac{C^{(k)}_\varphi}{x^{2\gamma_\varphi}}, \tag{32}$$

$$D_{\psi V \mu_1, \dots, \mu_k}(x, y) = D^{(k)}_{\psi V}(x-y) t^{(k)}_{\mu_1 \dots \mu_k}(x-y), \; D^{(k)}_{\psi V}(x) = \frac{C^{(k)}_\psi}{x^{2\gamma_\psi}}, \tag{33}$$

where $C^{(k)}_\varphi, C^{(k)}_\psi, \gamma_\varphi, \gamma_\psi$ are constant parameters. Due to (31), substituting the contribution of tensor composite operators (32), (33) into (28), (29), we get in both parts of them expressions that are proportional to the tensor $t^{(k)}_{\mu_1 \dots \mu_k}$. By comparing its coefficients, we obtain the equations

$$\frac{C^{(k)}_\varphi p(\alpha) q^{(k)}(\alpha, \gamma_\varphi)}{C^2_\varphi (x-y)^{2(d-2\alpha+\gamma_\varphi)}} + \frac{C_\psi C^{(k)}_\varphi}{(x-y)^{2(2+\sigma+\gamma_\varphi)}} + \frac{C_\varphi C^{(k)}_\psi}{(x-y)^{2(\alpha+\gamma_\psi)}} = 0, \tag{34}$$

$$\frac{C^{(k)}_\psi p(2+\sigma) q^{(k)}(2+\sigma, \gamma_\psi)}{C^2_\psi (x-y)^{2(d-2\sigma-4+\gamma_\psi)}} + \frac{n C_\varphi \delta C^{(k)}_\varphi}{(x-y)^{2(\alpha+\gamma_\varphi)}} = 0, \tag{35}$$

where $\alpha = \varpi - 1 + \eta/2$ is the critical dimension of the field φ. Here, we have used the notation $q^{(k)}(\zeta, \chi)$ for the function

$$\begin{aligned} q^{(k)}(\zeta, \chi) &= \frac{H(\varpi - \zeta, d - \zeta, \chi - k, 2\zeta - \chi + k - \varpi) \Gamma(\chi - k, d - 2\zeta + \chi)}{\Gamma(\chi, d - 2\zeta + \chi - k)} = \\ &= \frac{\Gamma(\zeta, \zeta - \varpi, \varpi - \chi + k, d - 2\zeta + \chi)}{\Gamma(\varpi - \zeta, d - \zeta, 2\zeta - \chi + k - \varpi, \chi)}. \end{aligned}$$

For the Equations (34) and (35) to have a solution, the following conditions must be fulfilled

$$\gamma_\psi = \gamma_\varphi + 2 + \sigma - \alpha = \gamma_\varphi + d - 3\alpha \tag{36}$$

$$C_\varphi^{(k)} p(\alpha) q^{(k)}(\alpha, \gamma_\varphi) + C_\varphi^2 C_\psi C_\varphi^{(k)} + C_\varphi^3 C_\psi^{(k)} = 0, \tag{37}$$

$$C_\psi^{(k)} p(2+\sigma) q^{(k)}(2+\sigma, \gamma_\psi) + n C_\varphi C_\psi^2 C_\varphi^{(k)} = 0. \tag{38}$$

Here, the equalities (36) ensure that the exponents of $(x-y)^2$ in (34) and (35) coincide, and (37), (38) is a system of homogeneous linear equations for constants $C_\varphi^{(k)}, C_\psi^{(k)}$. The condition for its solvability is written as

$$p(2+\sigma) q^{(k)}(2+\sigma, \gamma_\psi)(p(\alpha) q^{(k)}(\alpha, \gamma_\varphi) + C_\varphi^2 C_\psi) - n C_\varphi^4 C_\psi^2 = 0.$$

Using (27) and substituting here $p(\alpha) = C_\varphi^2 C_\psi$, $p(2+\sigma) = n C_\varphi^2 C_\psi / 2$, we get the equation

$$q^{(k)}(2+\sigma, \gamma_\psi)(q^{(k)}(\alpha, \gamma_\varphi) + 1) = 2. \tag{39}$$

Denoting $\alpha - \gamma_\varphi = 2 + \sigma - \gamma_\psi = \rho = -\Delta_V/2$, and substituting into (39) $\gamma_\varphi = \alpha - \rho$, $\gamma_\psi = \sigma - \rho$, we write (39) as an equation for dimension ρ of composite operator

$$Q^{(k)}(2+\sigma, \rho)(Q^{(k)}(\alpha, \rho) + 1) = 2, \tag{40}$$

where we used the notation

$$Q^{(k)}(\zeta, \rho) = q^{(k)}(\zeta, \zeta - \rho) = \frac{\Gamma(\zeta, -\zeta', d - \zeta - \rho, \rho + k + \zeta')}{\Gamma(\zeta', d - \zeta, \zeta - \rho, \rho + k - \zeta')}.$$

For $k = 0$ we get

$$Q^{(0)}(\zeta, \rho) = H(\varpi - \zeta, d - \zeta, \zeta - \rho, \rho + \zeta - \varpi), Q^{(0)}(\zeta, 0) = 1,$$

and (40) is fulfilled by $k = 0$, $\rho = 0$.

To analyze solutions of the Equation (40), it is convenient for us to write it in the form

$$Q^{(k)}(\alpha, \rho) - \frac{2}{Q^{(k)}(2+\sigma, \rho)} + 1 = 0, \tag{41}$$

and to represent $Q^{(k)}(\alpha, \rho)$, $2/Q^{(k)}(\sigma, \rho)$ as products of two functions:

$$Q^{(k)}(\alpha, \rho) = u_1^{(k)}(\alpha, \rho)\, u_2^{(k)}(\alpha, \rho), \quad \frac{2}{Q^{(k)}(2+\sigma, \rho)} = v_1^{(k)}(\sigma, \rho)\, v_2^{(k)}(\sigma, \rho),$$

where for $\alpha = \varpi - 1 + \eta/2$, $\sigma = -\eta$

$$u_1^{(k)}(\alpha, \rho) = \frac{(\eta - 2\rho + 2\varpi - 2)(\eta - 2\rho + 2\varpi)(\eta + 2(\rho + k - 1))(\eta + 2(\rho + k))}{(\eta + 2\varpi - 2)(\eta + 2\varpi)(\eta - 2)\eta}, \tag{42}$$

$$u_2^{(k)}(\alpha, \rho) = \frac{\Gamma(1 + \eta/2, 1 - \eta/2 + \rho + k, 1 + \eta/2 + \varpi, 1 - \eta/2 - \rho + \varpi)}{\Gamma(1 - \eta/2, 1 + \eta/2 + \rho + k, 1 - \eta/2 + \varpi, 1 + \eta/2 - \rho + \varpi)}, \tag{43}$$

$$v_1^{(k)}(\sigma, \rho) = \Gamma(2 - \eta - \rho, 2 - \eta + \rho + k - \varpi), \tag{44}$$

$$v_2^{(k)}(\sigma, \rho) = \frac{2\Gamma(\eta + \varpi - 2, \eta + 2\varpi - 2)}{\Gamma(2 - \eta, 2 - \eta - \varpi, \eta + \rho + k + \varpi - 2, \eta - \rho + 2\varpi - 2)}. \tag{45}$$

We are looking for a solution to the Equation (41) in the form $\rho = \rho(\eta) = \rho_0 + \rho_1 \eta + \mathcal{O}(\eta^2)$. Here, two situations are possible: at $\eta = 0$, the first and the second terms in the left part of (41) are singular or both are finite. In the first case, as we will see, the values of ρ_0 are determined by the singularity points of $1/Q^{(k)}(2,\rho_0)$; in the second case, the canonical values of ρ can be found from the condition that $Q^{(k)}(\alpha,\rho_0)$ is finite at $\eta = 0$.

For $\eta = 0$, the function $u_1^{(k)}(\alpha,\rho)$ can be singular, and $u_2^{(k)}(\alpha,\rho)$ for small η is finite:

$$u_1^{(k)}(\alpha,\rho) = \frac{2(\rho_0+k-1)(\rho_0+k)(\rho_0-\varpi)(1+\rho_0-\varpi)}{\eta(1-\varpi)\varpi} + \mathcal{O}(\eta^0),\ u_2^{(k)}(\alpha,\rho) = 1+\mathcal{O}(\eta).$$

The function $v_1^{(k)}(\alpha,\rho)$ is singular at the point $\eta = 0$ for $\rho_0 = 1+l$ and $\rho_0 = \varpi - 1 - l - k$, where $l \geq 1$ is an integer number:

$$v_1^{(k)}(\alpha,\rho)\big|_{\rho_0=1+l} = \frac{(-1)^l\Gamma(3+k+l-\varpi)}{\eta\,(l-1)!(1+\rho_1)} + \mathcal{O}(\eta^0),$$
$$v_1^{(k)}(\alpha,\rho)\big|_{\rho_0=\varpi-1-l-k} = \frac{(-1)^l\Gamma(3+k+l-\varpi)}{\eta\,(l-1)!(1-\rho_1)} + \mathcal{O}(\eta^0).$$

The function $v_2^{(k)}(\alpha,\rho)$ is finite for these values ρ_0 and $\eta = 0$ and

$$Q^{(k)}(\alpha,\rho)\Big|_{\rho_0=1+l} = \frac{2F_1^{(k)}(\varpi,l)}{\eta} + \mathcal{O}(\eta^0),\ Q^{(k)}(\alpha,\rho)\Big|_{\rho_0=\varpi-1-l-k} = \frac{2F_1^{(k)}(\varpi,l)}{\eta} + \mathcal{O}(\eta^0),$$
$$\frac{1}{Q^{(k)}(2+\sigma,\rho)}\Bigg|_{\rho_0=1+l} = \frac{F_2^{(k)}(\varpi,l)}{\eta(1+\rho_1)} + \mathcal{O}(\eta^0),\ \frac{1}{Q^{(k)}(2+\sigma,\rho)}\Bigg|_{\rho_0=\varpi-1-l-k} = \frac{F_2^{(k)}(\varpi,l)}{\eta(1-\rho_1)} + \mathcal{O}(\eta^0),$$

where

$$F_1^{(k)}(\varpi,l) = \frac{(k+l)(1+k+l)(1+l-\varpi)(2+l-\varpi)}{(1-\varpi)\varpi},$$
$$F_2^{(k)}(\varpi,l) = \frac{(-1)^l\Gamma(3+k+l-\varpi,\varpi-2,2\varpi-2)}{(l-1)!\,\Gamma(2-\varpi,l+k+\varpi-1,2\varpi-l-3)}.$$

From the cancellation of the singularity in η for $\rho_0 = 1+l$ and $\rho_0 = \varpi-1-l-k$, we get two equations for ρ_1

$$F_1^{(k)}(\varpi,l) - \frac{F_2^{(k)}(\varpi,l)}{1+\rho_1} = 0,\ \text{for}\ \rho_0 = 1+l,$$
$$F_1^{(k)}(\varpi,l) - \frac{F_2^{(k)}(\varpi,l)}{1-\rho_1} = 0,\ \text{for}\ \rho_0 = \varpi-1-l-k.$$

Solving them, we obtain the following result for the dimension ρ of the composite operator within the $1/n$-expansion

$$\rho^{(k)}(\varpi,l) = 1+l-\frac{1}{n}\rho_1^{(k)}(\varpi,l)\eta_1 + \mathcal{O}\left(\frac{1}{n^2}\right),$$
$$\rho^{(k)}(\varpi,l) = \varpi-1-l-k+\frac{1}{n}\rho_1^{(k)}(\varpi,l)\eta_1 + \mathcal{O}\left(\frac{1}{n^2}\right),$$

where

$$\rho_1^{(k)}(\varpi,l) = 1 + \frac{(-1)^l\Gamma(\varpi-2,2\varpi-2,3+k+l-\varpi)}{(l-1)!(k+l)(k+l+1)(1+l-\varpi)(2+l-\varpi)\Gamma(k+l+\varpi-1,2\varpi-3-l,-\varpi)}.$$

If $\rho_0 = 1-k,\ \varpi-1,\ -k,\ \varpi$, then $u_1^{(k)}(\alpha,\rho)$ is finite by $\eta \to 0$ and ρ_1 is defined by substitution $\rho = \rho_0 + \rho_1\eta$ in (41) and the requirement that the obtained equation is fulfilled in this way up to $\mathcal{O}(\eta)$. Using (42)–(45), it is easy to verify that if $\rho = \rho_0 + \rho_1\eta$, and $\rho_0 = 1-k,\ \varpi-1,\ -k,\ \varpi$, then $Q^{(k)}(\alpha,\rho) = u_1^{(k)}(\alpha,\rho)$ is non-singular by $\eta = 0$ and $1/Q^{(k)}(\sigma,\rho) = 1/Q^{(k)}(2,\rho_0) + \mathcal{O}(\eta)$. Taking into account these functions in the approximation $\eta = 0$, we obtain 4 equations for ρ_1

$$\frac{(1+2\rho_1)(k+\varpi-2)(k+\varpi-1)}{(1-\varpi)\varpi} + \frac{2k!\Gamma(2\varpi-2)}{\Gamma(2\varpi+k-3)} + 1 = 0, \text{ by } \rho_0 = 1-k,$$
$$\frac{(1-2\rho_1)(k+\varpi-2)(k+\varpi-1)}{(1-\varpi)\varpi} + \frac{2k!\Gamma(2\varpi-2)}{\Gamma(2\varpi+k-3)} + 1 = 0, \text{ by } \rho_0 = \varpi-1,$$
$$\frac{(1+2\rho_1)(1-k-\varpi)(k+\varpi)}{(1-\varpi)\varpi} - \frac{2(k+1)!\Gamma(2\varpi-2)}{\Gamma(k+2\varpi-2)} + 1 = 0, \text{ by } \rho_0 = -k,$$
$$\frac{(1-2\rho_1)(1-k-\varpi)(k+\varpi)}{(1-\varpi)\varpi} - \frac{2(k+1)!\Gamma(2\varpi-2)}{\Gamma(2\varpi+k-2)} + 1 = 0, \text{ by } \rho_0 = \varpi.$$

Solving them, we get the following result:

$$\rho = 1-k+\frac{1}{n}\rho_1'\eta_1 + \mathcal{O}\left(\frac{1}{n^2}\right),\ \rho = -k+\frac{1}{n}\rho_1''\eta_1 + \mathcal{O}\left(\frac{1}{n^2}\right),$$
$$\rho = \varpi-1-\frac{1}{n}\rho_1'\eta_1 + \mathcal{O}\left(\frac{1}{n^2}\right),\ \rho = \varpi-\frac{1}{n}\rho_1''\eta_1 + \mathcal{O}\left(\frac{1}{n^2}\right),$$

where

$$\rho_1' = \frac{1}{2(k+\varpi-2)(k+\varpi-1)}\left((1-k)(k+2\varpi-2) + \frac{\varpi k!\,\Gamma(2\varpi-1)}{\Gamma(2\varpi+k-3)}\right),$$
$$\rho_1'' = \frac{1}{2(k+\varpi-1)(k+\varpi)}\left((1-k)k + 2\varpi(1-k-\varpi) + \frac{\varpi(k+1)!\,\Gamma(2\varpi-1)}{\Gamma(2\varpi+k-2)}\right),$$

and $k = 0$ or a positive integer number. Note that if $\rho^{(k)}(\varpi,l)$ is a solution to Equation (41), then $\bar{\rho}^{(k)}(\varpi,l) = \varpi + k - \rho^{(k)}(\varpi,l)$ is also its solution. The characteristic feature of the solution is that itis a rational function whose numerator and denominator are polynomials on ϖ with integer coefficients.

Thus, our calculations made it possible to supplement the currently available list of results obtained in the framework of $1/n$ -expansions on the characteristics of critical behavior in the $O_n\varphi^4$- model [6,16–18] with anomalous dimensions of composite operators quadratic in the field φ. Their approximations of order $1/n$ are uniquely determined by the canonical dimensions of operators, which are found in the process of solving the self-consistency equations.

In the model with action (17), not all possible canonical dimensions of the composite quadratic on the fields φ and ψ operators can be obtained in such a way. The reason for this is that we used the one-loop approximation of the skeleton Equations (18)–(20). It is unsuitable for the calculations of the anomalous dimension of the composite operator if it is small compared to 1/n when n is large. In this case, the self-consistent equation is unsuitable for finding both anomalous and canonical dimensions.

4.3. Ultraviolet Divergences and Renormalizations

When carrying out calculations within the one-loop approximation in the skeleton equations for full propagators, the problem of ultraviolet divergences does not arise. If we impose on the asymptotics of the renormalized propagators normalization conditions of the form

$$D_\varphi(x) = \frac{1}{x^{2\Delta_\varphi}},\ D_\psi(x) = \frac{1}{x^{2\Delta_\psi}},$$

then the problem of matching the degrees of homogeneity of propagators and self-energy operators $\Sigma_\varphi(x), \Sigma_\varphi(x)$ is solved in the one-loop approximation by the simple relationship $\Delta_\psi = d - 2\Delta_\varphi$ between the dimensions of the fields φ and ψ. This results in equations that do not contain the coordinates x

$$p(\varpi - 1 + \eta/2) = g,\ 2p(2-\eta)/n = g,$$

where $g = C_\varphi^2 C_\psi$ can be considered as a non-renormalized interaction constant. The consequence of these equations is the equality $p(\varpi - 1 + \eta/2) = 2p(2-\eta)/n$, which links the anomalous dimension η of the field φ, the number of its components n and the dimension of the space d.

In solving these equations within the $1/n$ -expansion, there are no mathematical problems that could be considered to be a manifestation of ultraviolet divergences. Therefore, there is also no need to eliminate them by a nontrivial renormalization of the interaction constant g. It means that in this approximation we can consider it as renormalized: $g = g_r$. In contrast to the situation outside the critical point, g_r cannot be arbitrary, since by virtue of $g_r = p(\varpi - 1 + \eta/2)$, it is a function of only n and d.

In a passing to approximation in skeleton equations with a number of vertices greater than two, direct substitution of single-loop results for propagators $D_\varphi(x),\ D_\psi(x)$ turns out to be impossible since this leads to ultraviolet divergences of every non-point triple $\varphi\varphi\psi$ subdiagram in any d-space dimension.

However, if $\kappa = d - 2\Delta_\varphi - \Delta_\psi \neq 0$ the contributions of all diagrams of the self-energy operators $\Sigma_\varphi, \Sigma_\psi$ are finite. The divergences arising at $\kappa = 0$ appear in them as poles on κ. They can be eliminated by renormalizing $g = g_r Z_g(\kappa, g_r)$ the interaction constant g within the minimal subtraction scheme with renormalization constant

$$Z_g(\kappa, g_r) = 1 + \sum_{i,j>0} c_{ij} \frac{g_r^i}{\kappa^j}$$

where the κ and g_r-independent coefficients c_{ij} are chosen such that all singularities over κ in the self-energy operators are canceled. Due to renormalizability of the model, such a procedure is feasible. Multiplying by $z^{-2\kappa} = \exp\{-2\kappa \ln z\}$ the equation for D_ψ we obtain the equations of the form

$$p(\varpi - 1 + \eta/2) = -\Sigma_{\varphi r}(g_r, \kappa, \eta, \ln x),\ \ 2p(2-\eta-\kappa)/n = -\Sigma_{\psi r}(g_r, \kappa, \eta, \ln x). \tag{46}$$

both the left and the right parts can be expanded in series on κ in the neighborhood of $\kappa = 0$. Taking this into account, and assuming that

$$\kappa = \kappa(g_r) = \sum_{n=0}^{\infty} \kappa_n g_r^n,$$

we choose the coefficients κ_n so that the right-hand sides of Equation (46) do not depend on $\ln x$. Thus, for this concrete function $\kappa(g)$ we obtain self-consistent equations of the form

$$p(\varpi - 1 + \eta/2) = -\Sigma_\varphi^{(r)}(g_r, \eta, n, d),\ 2p(2-\eta-\kappa)/n = -\Sigma_\psi^{(r)}(g_r, \eta, n, d). \tag{47}$$

where $\Sigma_\varphi^{(r)}(g_r, \eta, n, d) = \Sigma_{\varphi r}(g_r, \kappa(g), \eta, \ln z),\ \Sigma_\psi^{(r)}(g_r \eta, n, d) = \Sigma_{\psi r}(g_r, \kappa(g), \eta, \ln z)$. Thus, The anomalous dimension of the field φ and the interaction constant g_r at the critical point depend only on the number of components of the field φ and the dimension of space.

In order to construct $\Sigma_\varphi^{(r)}(g_r, \eta, n, d), \Sigma_\psi^{(r)}(g_r, \eta, n, d)$ in the skeleton representation of the self-energy operators with not more than 2n vertices, one has to carry out the above procedure using polynomials on g_r of order $n-1$ for the vertex renormalization constant $Z_r(g_r)$ and the function $\kappa(g_r)$, discarding all values exceeding the precision g_r^n in the renormalized self-energy operators. The obtained results are also valid to a precision of

g_r^n.The first correction to the one-loop self-consistency equations constructed in this way was used to calculate η_2 and ν_2 [6,16,17].

The presented method of constructing the self-consistency equations for the fields φ and ψ makes it possible to obtain similar equations for the composite operators as well. They are written as a system of homogeneous linear equations for constants A and B

$$A(Q^{(k)}(\varpi-1+\eta/2,\rho)+\Xi_{\varphi\varphi}^{(kr)}(g_r,\rho,n,d))+\Xi_{\varphi\psi}^{(kr)}(g_r,\rho,n,d)B=0 \quad (48)$$

$$B(Q^{(k)}(2-\eta-\kappa,\rho)+\Xi_{\psi\psi}^{(kr)}(g_r,\rho,n,d))+\Xi_{\psi d}^{(kr)}(g_r,n,d)A=0 \quad (49)$$

with functions $\Xi_{\varphi\varphi}^{(kr)}, \Xi_{\varphi\psi}^{(kr)}, \Xi_{\psi\varphi}^{(kr)}, \Xi_{\psi\psi}^{(kr)}$ similar to renormalized operators of self-energy in (46). The solvability condition of Equations (48) and (49) is written as

$$(Q^{(k)}(\varpi-1+\eta/2,\rho)+\Xi_{\varphi\varphi}^{(kr)})(Q^{(kr)}(2-\eta-\kappa,\rho)+\Xi_{\psi\psi}^{(kr)})=\Xi_{\psi\varphi}^{(kr)}\Xi_{\varphi\psi}^{(kr)} \quad (50)$$

In the one loop approximation $\Xi_{\varphi\varphi}^{(kr)}=\Xi_{\varphi\psi}^{(kr)}=1$, $\Xi_{\psi\varphi}^{(kr)}=2$, $\Xi_{\psi\psi}^{(r)}=0$.

The nearest to the leading approximation for (48-50) allows us to calculate directly the indices $\kappa_1=d(2d-5)\eta_1/(4-d)$ (anomalous dimension of the operator φ^2) and $\omega_1=-(d-1)^2\eta_1$ (anomalous dimension of the operator ψ^2). For κ_1, it is possible to calculate it using single-loop results:

$$\kappa_1=2\rho_1-\eta_1=-2\rho_1'-\eta_1=((d-1)(d-2)/(d-4)-1)\eta_1=d(2d-5)\eta_1/(4-d).$$

For the exponent ω_1, this is impossible. To calculate it, one must take into account the contributions to the self-consistency equation of the skeleton diagrams with four and six vertices [6].

In this paper we present only the results of one-loop calculations for the anomalous dimensions of the composite operators. In the cases when it is necessary to calculate corrections for them in the main $1/\mathrm{n}$ - approximation, they can be obtained by solving the corresponding modified self-consistency equations derived on the basis of the additional contributions of multiloop diagrams.

5. Discussion

Using an approach based on the formalism of functional Legendre transforms, we derived self-consistent equations that can be applied for calculations of anomalous dimensions of composite operators in models of quantum field theory and statistical physics.

This made it possible in the $O_n(\varphi^2)^2$- model to calculate the anomalous dimensions of both scalar and tensor composite operators quadratic in the fields φ. Results are obtained up to corrections of order $1/n^2$. The methods developed and used for this purpose can be applied to other models as well.

It is important that the values of anomalous dimensions of different composite operators are solutions of the same equations. The diagrammatic technique of the second Legendre transform in the theory with cubic interaction was used to construct the self-consistent equations. By introducing auxiliary fields, similar to what we did in the $O_n-(\varphi^2)^2$-model and shown in (7), one can obtain corresponding modifications of the self-consistency equations for other models with interactions and composite operators of polynomial form.

In this case, the number of fields will increase, but the interaction will be cubic and anomalous dimensions of quadratic on field-composed operators can be calculated by solving the self-consistency equations. To determine the anomalous dimensions of polynomial composite operators of degree $n>2$, one needs multi-loop calculations; so, one should expect a value of order $1/n^m$ at $m>1$ as a result in the main approximation. If this is true, then the results we have obtained represent the anomalous dimensions of all composite operators, which in the main approximation at large n are quantities of the order of $1/n$.

We hope that the proposed method will find application in studies of behavior of quantum field models at a critical point. It would be interesting to apply the proposed approach to the gauge field theory, where, as it is known [19,20], at a critical point there appears a dependence of anomalous dimensions of basis fields on the choice of gauge and it is important to reveal the gauge invariant characteristics of critical phenomena.

Author Contributions: All authors (Y.P. and A.P.) contributed equally to this paper. All authors have read and agreed to the published version of the manuscript.

Funding: This research was funded by Russian Science Foundation grant number 22-22-00803.

Institutional Review Board Statement: Not applicable.

Informed Consent Statement: Not applicable.

Data Availability Statement: Not applicable.

Acknowledgments: The authors would like to thank the NRC "Kurchatov Institute" Petersburg Nuclear Physics Institute for administrative support of research.

Conflicts of Interest: The authors declare no conflict of interest.

Abbreviations

The following abbreviations are used in this manuscript:

NRC National Research Center

References

1. Wilson, K.G.; Kogut, J. The renormalization group and the ϵ-expansion. *Phys. Rep. (Sect. C Phys. Lett.)* **1974**, *12*, 75–200.
2. Bresin, E.; Le Guillou, J.; Zinn-Justin, J. Field Theoretical Approach to Critical Phenomena. In *Phase Transition and Critical Phenomena*; Domb, C., Green, M.S., Eds.; Academic Press: London, UK, 1976; Volume 6.
3. Amit, D.J. *Field Theory, the Renormalization Group and Critical Phenomena*; McGraw-Hill: New York, NY, USA, 1978.
4. Ma, S.-K. *Modern Theory of Critical Phenomena*; Sarat Book House: New York, NY, USA, 1982.
5. Zinn-Justin, J. *Quantum Field Theory and Critical Phenomena*; Clarendon Press: Oxford, UK, 1989.
6. Vasil'ev, A.N.*The Field Theoretic Renormalization Group in Critical Behavior Theory and Stochastic Dynamics*; Chapman & Hall /CRS, A CRS Press Company: Boca Raton, FL, USA; London, UK; New York, NY, USA; Washington, DC, USA, 2004.
7. Vasiliev, A.N. *Functional Methods in Quantum Field Theory and Statistical Physics*; CRS Press, Taylor & Francis Group: Boca Raton, FL, USA; London, UK; New York, NY, USA, 1998.
8. De Dominicis, C. Variational Formulations of Equilibrium Statistical Mechanics. *J. Math. Phys.* **1962**, *3*, 983–1003. [CrossRef]
9. De Dominicis, C.; Martin, P.C. Stationary Entropy Principle and Renormalization in Normal and Superfluid Systems. I. Algebraic Formulation. *J. Math. Phys.* **1964**, *5*, 14–31. [CrossRef]
10. De Dominicis, C.; Martin, P.C. Stationary Entropy Principle and Renormalization in Normal and Superfluid Systems. II. Diagrammatic Formulation. *J. Math. Phys.* **1964**, *5*, 31–60. [CrossRef]
11. Pis'mak, Y.M. Proof of the 3-irreducibility of the third Legendre transform. *Theor. Math. Phys.* **1974**, *18*, 299–309. [CrossRef]
12. Vasil'ev, A.N.; Kazanskii, A.K.; Pis'mak, Y.M. Equations for higher Legendre transforms in terms of 1-irreducible vertices. *Theor. Math. Phys.* **1974**, *19*, 186–200. [CrossRef]
13. Vasil'ev, A.N.; Kazanskii, A.K.; Pis'mak, Y.M. Diagrammatic analysis of the fourth Legendre transform. *Theor. Math. Phys.* **1974**, 20, 181–193. [CrossRef]
14. Pis'mak, Y.M. Combinatorial analysis of the overlapping problem for vertices with more than four legs. *Theor. Math. Phys.* **1975**, *24*, 34–48. [CrossRef]
15. Pis'mak, Y.M. Combinatorial analysis of the overlapping problem for vertices with more than four legs. II. Higher Legendre transforms. *Theor. Math. Phys.* **1975**, *24*, 177–194.
16. Vasiliev, A.N.; Pis'mak, Y.M.; Khonkonen, Y.R. Simple method of calculating the critical indices in the 1/n-expansion. *Teor. Mat. Fiz.* **1981**, *46*, 157–171. [CrossRef]
17. Vasiliev, A.N.; Pis'mak, Y.M.; Khonkonen, Y.R. 1/n-Expansion: Calculation of the exponents η and ν in the order $1/n^2$ for arbitrary number of dimensions. *Teor. Mat. Fiz.* **1981**, *47*, 291–306. [CrossRef]
18. Vasiliev, A.N.; Pis'mak, Y.M.; Khonkonen, Y.R. 1/n-Expansion: Calculation of the exponent η in the order $1/n^3$ by the conformal bootstrap method. *Teor. Mat. Fiz.* **1982**, *50*, 195–206. [CrossRef]

19. Vasil'ev, A.N.; Perekalin, M.M.; Pis'mak, Yu.M. On the possibility of conformal infrared asymptotic behavior in non-Abelian Yang–Mills theory. *Teor. Mat. Fiz.* **1983**, *55*, 323–334.
20. Vasil'ev, A.N.; Perekalin, M.M.; Pis'mak, Y.M. Conformal bootstrap equations for Yang–Mills type interaction. *Teor. Mat. Fiz.* **1984**, *60*, 317–319.