Surface Operators, holography and BPS equations

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(1) **Surface operators, chiral rings and localization in $\mathscr{N} = 2$ gauge theories**

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a. **Holographic Surface Operators**

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b. **BPS strings in $\mathscr{N} = 4$ SYM theory**

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To my Grandfather...

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Abstract

In this thesis, we do investigations related to surface operators in supersymmetric gauge theories in four dimensions. In the first part of the thesis, we study half-BPS surface operators in $\mathcal{N} = 2$ pure $SU(N)$ gauge theory following two different approaches. In the first approach we analyze the chiral ring equations for certain quiver theories in two dimensions coupled to four-dimensional gauge theory. The chiral ring equations, which arise from extremizing a twisted chiral superpotential, are solved as power series in the infrared scales of the quiver theories. In the second approach we use equivariant localization and obtain the twisted chiral superpotential as a function of the Coulomb moduli of the four-dimensional $SU(N)$ gauge theory, and find a perfect match with the results obtained from chiral ring equations.

In the second-half of the thesis, we study singular time-dependent $\frac{1}{8}$-BPS configurations in the abelian sector of $\mathcal{N} = 4$ Yang-Mills theory that represent BPS string-like defects in $\mathbb{R} \times S^3$ spacetime. Such BPS strings can be described as intersections of the zeros of holomorphic functions in two complex variables with a 3-sphere. We argue that these BPS strings map to $\frac{1}{8}$-BPS surface operators under state-operator correspondence of the conformal field theory. We show that the string defects are holographically dual to non-compact probe D3 branes in global $AdS_5 \times S^5$ that share supersymmetries with a class of dual-giant gravitons. For simple configurations, we demonstrate how to define a good variational problem for the associated action principle and propose a regularization scheme that leads to finite energy and global charges on both sides of the holographic correspondence.

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Synopsis

Introduction

Surface operators are non-local BPS operators in the Supersymmetric gauge theories. They are a higher dimensional generalization of one-dimensional Wilson and 't Hooft line operators. Surface operators were first defined by Gukov and Witten in [8], [9] as a part of the geometric Langlands program using supersymmetric gauge theories. They are a co-dimension two singular solution to the generalized Bogomolny equations in [11]. Analogous to the line operators, surface operators provide essential information about the vacuum structure of supersymmetric gauge theories [18].

Surface operators have a dual description in gauge theories: on the one hand, they can be described as monodromy defects in a four-dimensional theory, and on the other hand, they can be described as coupled 2d/4d systems. In the context of $\mathcal{N} = 4$ SYM, surface operators were shown to be holographically dual to probe D3 branes in the ten-dimensional $AdS_5 \times S^5$ geometry in [12], [13].

The goal of this thesis is two-fold: first, we will study the instanton partition function of the $\mathcal{N} = 2$ gauge theory in presence of surface operators from which, we extract the low energy effective-action that governs the dynamics of surface operators. In particular, we will calculate the effective-action from the two approaches and relate the parameters associated with surface operators in the dual descriptions. Second, we will classify $\frac{1}{8}$-BPS surface defects in $\mathcal{N}=4$ $SU(N)$ Super Yang-Mills theory and obtain their holographic duals in the $AdS_5\times S^5$ geometry. We will compute the energies and global charges of a subclass of solutions that we classify.

Background

Surface operators as Monodromy defects

Surface operators in the $SU(N)$ Susy gauge theory are described by a set of data of M integers: $\{n_1,n_2,\ldots,n_M\}$ such that $n_1+n_2+\ldots+n_M=N$ and a set of real parameters: $\{\alpha_1,\ldots,\alpha_M,\ldots,\beta_M,\ldots,\gamma_M,\ldots,\eta_M\}$. The real parameters are associated with the singularities of the bosonic fields in the 4-dimensional gauge theory that occur at a codimension-2 surface. In a theory defined on $\mathbb{R}^4$ manifold, surface defects induce a singularity in the profile of gauge fields in the following way

$$A=\begin{pmatrix}\alpha_1\otimes 1_{n_1} & 0 & \ldots & 0\\ 0 & \alpha_2\otimes 1_{n_2} & \ldots & 0\\ \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & \ldots & \alpha_M\otimes 1_{n_M}\end{pmatrix} d\psi. \tag{1}$$

where ψ is the polar angle in the plane transverse to the defect, 1_{n_i} are unit matrices of rank n_i. Near the location of a surface defect the gauge group $G=SU(N)$ breaks into a Levi subgroup: $L=S[U(n_1)\otimes\ldots\otimes U(n_M)]$. In the presence of surface defects, one can also introduce two-dimensional θ-angles that couple to the unbroken $U(1)$s at the defect. In the path integral this is represented by the following insertion

$$\exp\left(i\sum_{l=1}^{M}\eta_l\int_{2d}\mathrm{Tr}\,F^{(l)}\right)\,. \tag{2}$$

In the maximally supersymmetric $\mathcal{N}=4$ $SU(N)$ theories, surface defects also induce a singularity in the profile of one of the complex scalar fields:

$$\Phi=\frac{1}{z}\begin{pmatrix}(\beta_1+i\gamma_1)\otimes 1_{n_1} & 0 & \dots & 0\\ 0 & (\beta_2+i\gamma_2)\otimes 1_{n_2} & \dots & 0\\ \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & \dots & (\beta_M+i\gamma_M)\otimes 1_{n_M}\end{pmatrix}\,, \tag{3}$$

z is complex coordinate in the transverse plane.

Surface operators as flavor defects

There is a dual description of surface operators in supersymmetric gauge theories where, a surface defect supports on its world-volume a two-dimensional gauge theory with a flavour $SU(N)$ symmetry which is identified with the gauge symmetry group of four-dimensional theory.

In $\mathcal{N}=2$ $SU(N)$ theory, when the vacuum solution is obtained from its Coulomb branch, this combined 2d/4d system is described by two holomorphic functions: the prepotential F and the twisted-chiral superpotential W. The prepotential governs the dynamics of the bulk theory and depends on the Coulomb vev's and the infra-red (IR) scale of the gauge theory in four dimensions. The twisted-chiral superpotential controls the two-dimensional dynamics on the surface operator, and is a function of the continuous parameters labeling the defect, the two-dimensional IR scales, and also of the Coulomb vev's and the strong-coupling scale of the bulk gauge theory. The twisted superpotential thus describes the coupled 2d/4d system.

Gauged Linear sigma models in two dimensions

From the two-dimensional perspective, the effective dynamics is described by a non-linear sigma model with target-space equal to a coset space: $SU(N)/L$ where L is the Levi subgroup associated with a surface operator, defined earlier. The 2d sigma model is a Gauged linear Sigma model with $\mathcal{N} = (2,2)$ supersymmetry [36], [37]. In a simpler setup when $L = S[U(1) \otimes U(N-1)]$ the target space is CP^{N-1}, the 2d sigma model in the UV limit of energy scale, has a $U(1)$ gauge symmetry with chiral multiplets Q having some $SU(N)$ flavor symmetry. The kinetic term of the lagrangian is

$$\frac{1}{4}\int d^4\theta \left(Q^\dagger e^{2V} Q - \frac{1}{2e^2}\Sigma^\dagger\Sigma \right), \tag{4}$$

Σ is the twisted chiral superfield associated with the $U(1)$ gauge vector multiplet V. In addition, we have the Fayet-Iliopoulos term: $\frac{i\tau}{4}\int d^2\tilde{\theta}\,\Sigma + h.c.$ with the complexified coupling τ.

In two dimensions there is a special mass-like term which is considered for Q, it is introduced by first gauging the flavor symmetry $SU(N)$ and giving a background value to the scalar component of the vector superfield, and then setting the fields to be vanishing. This can be written as

$$\int d^4\theta\, Q^\dagger e^{2V_1} Q, \tag{5}$$

where $V_1 = \theta^R \bar{\theta}^L \widetilde{m} + h.c.$ This preserves $\mathcal{N} = (2,2)$ supersymmetry only when $\widetilde{m}$ is diagonalizable, the diagonal components are called twisted masses. The effective low energy Lagrangian has twisted superpotential:

$$W_{eff} = \frac{1}{4}\left[i\tau\Sigma - \frac{1}{2\pi}\sum_{i=1}^{n}(\Sigma - \widetilde{m}_i)\left(\log\left(\frac{\Sigma - \widetilde{m}_i}{\mu}\right) - 1\right)\right] \tag{6}$$

To make connection with the low energy effective-action associated with surface operators [16, 17], we identify, the flavor $SU(N)$ symmetry group with the four-dimensional gauge symmetry group and twisted masses $\widetilde{m}_i$ with the vevs of the scalar in the 4d $\mathscr{N} = 2$ vector multiplet.

Holographic description of Surface operators

The $\mathscr{N} = 4$ Yang-Mills theory is known to be dual to the ten-dimensional Type IIB supergravity from the AdS/CFT correspondence. The type IIB supergravity theory is described by the vielbein, a complex Weyl gravitino, a real four-form $C^{(4)}$ with self-dual field strength $F^{(5)}$, a complex two-form $C^{(2)}$, a complex spinor Λ and a complex scalar Φ. In the $AdS_5 \times S^5$ background most of the component fields except the vielbein and the four-form $C^{(4)}$ vanish. The variation of the gravitino takes the form

$$\delta\Psi_\mu = D_\mu\varepsilon - \frac{i}{480}\Gamma_\mu^{\ \nu\rho\alpha\beta\lambda} F_{\nu\rho\alpha\beta\lambda}\varepsilon\,. \tag{7}$$

Demanding $\delta\Psi_\mu = 0$ leads to the Killing spinor equations.

κ-symmetry of D3 branes in $AdS_5 \times S^5$

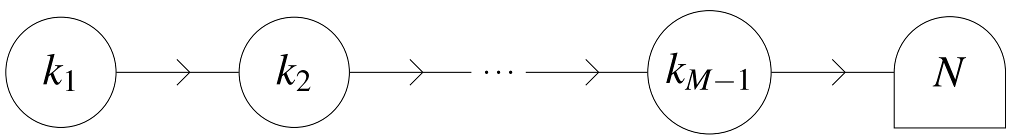
We now consider the action for D3-brane configurations, the action with 'Dirac-Born-Infeld term'+'Wess-Zumino term' is invariant under the local fermionic kappa-symmetry transformations. The necessary constraints are obtained when the ten-dimensional superspace spinor coordinates Θ are set to zero, and their variation $\delta\Theta$ are also set to zero [30]. D3-branes for which the world-volume gauge fields vanish, kappa-symmetry constraint is of the form: $\Gamma_\kappa\varepsilon = \pm i\sqrt{-\det h}\,\varepsilon$. This constraint is imposed on the background Killing spinors ε near the worldvolume, and Γ_κ is the product of four gamma-matrices on the D3 worldvolume.

Surface operators in $\mathcal{N} = 4$ SYM theories have holographic duals in $AdS_5 \times S^5$ geometry as probe D3 branes with non-compact worldvolumes that reach AdS boundary. In the Poincare patch of coordinates, the D3 branes that reach the AdS boundary and end in $\mathbb{R}^{1,1}$ submanifold, in the boundary limit, have their world-volume identified with the 'surface of support' of half-BPS surface operators. The symmetries and parameters of the 4-dimensional gauge theory in presence of surface operators are exactly identified with those of the holographic D3 branes. The complex parameters $\beta + i\gamma$ are mapped to the constants that appear in the embedding equation of the non-compact D3 worldvolume solutions. The parameters α and η are identified with the holonomies of the gauge field A and the dual gauge field $\tilde{A}$ living on the D3-brane worldvolume. The fermionic symmetry of the half-BPS D3 brane solutions can be verified by doing the κ-symmetry analysis.

We use the preserved supersymmetry of several half-BPS branes and find the common supersymmetries among them. Then we use the common supersymmetries in the κ-symmetry constraint to determine the most general $\frac{1}{16}$-BPS equations. From these equations, we obtain the classical solutions of the holographic duals to $\frac{1}{8}$-BPS defects in $\mathcal{N}$=4 SYM theory.

Surface operators in $\mathcal{N} = 2$ theories

A generic co-dimension two surface operator in pure $\mathcal{N} = 2$ $SU(N)$ gauge theory [16], [17] has a microscopic description as a quiver gauge theory of the type shown in the figure below



Here the round nodes, labeled by index I, correspond to $U(k_I)$ gauge theories in two dimensions whose field strength is described by a twisted chiral field $\Sigma^{(I)}$. The rightmost node represents the four-dimensional $\mathcal{N}=2$ gauge theory whose $SU(N)$ gauge group acts as a flavor group for the last two-dimensional node. The arrows correspond to (bi-)fundamental matter multiplets that are generically massive. Integrating out these fields leads to an effective action for the twisted chiral fields which, because of the two-dimensional $(2,2)$-supersymmetry, is encoded in a twisted chiral superpotential W. The contribution to W coming from the massive fields attached to the last node depends on the four-dimensional dynamics of the SU(N) theory and in particular on its resolvent [16]. The vacuum structure can be determined by the twisted chiral ring equations, which take the form

$$\exp\left(\frac{\partial W}{\partial \sigma_s^{(I)}}\right) = 1 \tag{8}$$

The main idea is that by evaluating W on the solutions to the twisted chiral ring equations one should reproduce precisely the superpotential calculated using localization.

In the thesis we extended this analysis in the following manner: first of all, we showed that in the classical limit there is a very specific choice of solutions to the twisted chiral ring equations that allows us to make contact with the twisted chiral superpotential calculated using localization. We established the correspondence between the continuous parameters labeling the monodromy defect and the dynamically generated scales of the two-dimensional quiver theory. We then showed that quantum corrections in the quiver gauge theory are mapped directly to corrections in the twisted superpotential(due to ramified instantons), extracted from the instanton partition function of the four-dimensional theory.

The expression for the effective twisted chiral superpotential associated with a general surface operator is the following

$$
\begin{aligned}
W = & -\sum_{I=1}^{M-2}\sum_{s=1}^{k_I}\sum_{t=1}^{k_{I+1}} \sigma_s^{(I)}\left(\log\frac{\sigma_s^{(I)}-\sigma_t^{(I+1)}}{\Lambda_I}-1\right)\\
& +\sum_{I=2}^{M-1}\sum_{s=1}^{k_I}\sum_{r=1}^{k_{I-1}} \sigma_s^{(I)}\left(\log\frac{\sigma_r^{(I-1)}-\sigma_s^{(I)}}{\Lambda_I}-1\right)\\
& -\sum_{s=1}^{k_{M-1}}\left\langle \mathrm{Tr}\left[\left(\sigma_s^{(M-1)}-\Phi\right)\left(\log\frac{\sigma_s^{(M-1)}-\Phi}{\Lambda_{M-1}}-1\right)\right]\right\rangle,
\end{aligned} \tag{9}
$$

$\sigma^{(I)}$ is the adjoint valued scalar in the vector multiplet of the I^{th} gauge node, subscript indices s,r and t denote the diagonal components in a rank k_I matrix, Λ_I is the complexified I.R. scale of the I^{th} node, and Φ is the adjoint scalar of the 4d $SU(N)$ gauge theory. The angular brackets account for the four-dimensional dynamics of the SU(N) theory. The quantum corrected vacuum expectation value of Φ appear in the twisted mass parameters associated with chiral matter between the last two nodes.

We solve the 2d twisted chiral ring relations to evaluate W on the vacuum solution $\sigma_\star$. In order to solve the chiral ring equations for each node in the quiver that are coupled to each other, we need to make some ansatz. First, we consider a generic point in the 4d Coulomb branch parameterized by the classical vev of Φ and take the diagonal components to be equal to $\mathrm{diag}\,(a_1,a_2,\ldots,a_N)$. Then, we express the vevs of 2d scalars $\sigma_s^{(I)}$ power series in Λ_I parameters around a chosen classical vacuum, with the help of following definitions:

$$
\begin{aligned}
q_I =& (-1)^{k_{I-1}}\Lambda_I^{n_I+n_{I-1}}\\
q_M =& (-1)^N\Lambda^{2N}\Big(\prod_{I=1}^{M-1} q_I\Big)^{-1}
\end{aligned} \tag{10}
$$

for $I=1,\ldots,M-1$. Here Λ is the four dimensional scale parameter.

Our ansatz chosen for the 2d scalars is the following:

$$
\begin{aligned}
\sigma_s^{(1)} &= a_s + \mathscr{O}(q_I) + \ldots \qquad \text{for } s = 1, \ldots, k_1 \ , \\
\sigma_t^{(2)} &= a_t + \mathscr{O}(q_I) + \ldots \qquad \text{for } t = 1, \ldots, k_2 \ , \\
&\vdots \\
\sigma_w^{(M-1)} &= a_w + \mathscr{O}(q_I) + \ldots \qquad \text{for } w = 1, \ldots, k_{M-1} \ .
\end{aligned}
\tag{11}
$$

This choice of ansatz corresponds to a partition of the classical vev of Φ given by

$$
\{ \underbrace{a_1, \cdots, a_{n_1}}_{n_1}, \underbrace{a_{n_1+1}, \cdots, a_{n_1+n_2}}_{n_2}, \cdots, \underbrace{a_{k_{M-1}+1}, \cdots, a_N}_{n_M} \}
$$

For $[1,1,1]$ surface operator in the $SU(3)$ theory, we calculate the 2d scalars on the vacuum, and from the twisted superpotential $(W_\star)$ evaluated on the vacuum, we obtain the following

$$
\begin{aligned}
\frac{\Lambda_1}{2} \frac{dW_\star}{d\Lambda_1} = \sigma_\star^{(1)} = a_1 + \frac{1}{a_{12}} \Lambda_1^2 + \frac{1}{a_{13}} \frac{\Lambda^6}{\Lambda_1^2 \Lambda_2^2} - \frac{1}{a_{12}^3} \Lambda_1^4 - \frac{1}{a_{13}^3} \frac{\Lambda^{12}}{\Lambda_1^4 \Lambda_2^4} \\
- \frac{1}{a_{12} a_{13} a_{23}} \Big(\Lambda_1^2 \Lambda_2^2 - \frac{\Lambda^6}{\Lambda_1^2} \Big) + \ldots \ ,
\end{aligned}
\tag{12}
$$

$$
\begin{aligned}
\frac{\Lambda_2}{2} \frac{dW_\star}{d\Lambda_2} = \operatorname{Tr} \sigma_\star^{(2)} = a_1 + a_2 - \frac{1}{a_{23}} \Lambda_2^2 + \frac{1}{a_{13}} \frac{\Lambda^6}{\Lambda_1^2 \Lambda_2^2} - \frac{1}{a_{23}^3} \Lambda_2^4 - \frac{1}{a_{13}^3} \frac{\Lambda^{12}}{\Lambda_1^4 \Lambda_2^4} \\
- \frac{1}{a_{12} a_{13} a_{23}} \Big(\Lambda_1^2 \Lambda_2^2 + \frac{\Lambda^6}{\Lambda_2^2} \Big) + \ldots \ .
\end{aligned}
\tag{13}
$$

these expressions are identified, with the q1- and q2-logarithmic derivatives of the twisted superpotential extracted from the instanton partition function of the $SU(3)$ theory.

Localization in 4d

In this subsection, we treat surface operators as monodromy defects. The partition function for $\mathcal{N} = 2$ theories with surface operators was first derived in [20], using the equivariant localization technique developed in [24], [25] for $\mathcal{N} = 2$ $SU(N)$ theory. The instanton partition function for the generic surface operator is given by

$$Z_{\text{inst}}[\vec{n}] = \sum_{\{d_I\}} Z_{\{d_I\}} \quad \text{with} \quad Z_{\{d_I\}}[\vec{n}] = \prod_{I=1}^{M} \left[\frac{(-q_I)^{d_I}}{d_I!} \int \prod_{\sigma=1}^{d_I} \frac{d\chi_{I,\sigma}}{2\pi i} \right] z_{\{d_I\}}. \tag{14}$$

And

$$\begin{aligned} z_{\{d_I\}} \;=\; & \prod_{I=1}^{M} \prod_{\sigma,\tau=1}^{d_I} \frac{\chi_{I,\sigma} - \chi_{I,\tau} + \delta_{\sigma,\tau}}{\chi_{I,\sigma} - \chi_{I,\tau} + \varepsilon_1} \times \prod_{I=1}^{M} \prod_{\sigma=1}^{d_I} \prod_{\rho=1}^{d_{I+1}} \frac{\chi_{I,\sigma} - \chi_{I+1,\rho} + \varepsilon_1 + \hat{\varepsilon}_2}{\chi_{I,\sigma} - \chi_{I+1,\rho} + \hat{\varepsilon}_2} \\ & \times \prod_{I=1}^{M} \prod_{\sigma=1}^{d_I} \prod_{s=1}^{n_I} \frac{1}{a_{I,s} - \chi_{I,\sigma} + \frac{1}{2}(\varepsilon_1 + \hat{\varepsilon}_2)} \prod_{t=1}^{n_{I+1}} \frac{1}{\chi_{I,\sigma} - a_{I+1,t} + \frac{1}{2}(\varepsilon_1 + \hat{\varepsilon}_2)} \,. \end{aligned} \tag{15}$$

$\chi_{I,\sigma}$ are parameters on the instanton moduli space. ε_1, $\hat{\varepsilon}_2$ specify the Ω-deformed background which is introduced to localize the integrals over the instanton moduli space. The M variables q_I are ramified instanton weights which, have a one-to-one map with energy scales in the coupled 2d/4d system, that we also use to show the equivalence of the effective twisted superpotential computed from the two descriptions.

In the partition function formula (14), M positive integers d_I count the number of ramified instantons in various sectors, with the convention that $d_{M+1} = d_1$. The notation $a_{I,s}$ indicate how 4d scalar vev are partitioned in the presence of a surface defect, as shown below:

$$a_{I,s} \subset \{ \underbrace{a_1, \cdots, a_{n_1}}_{n_1}, \underbrace{a_{n_1+1}, \cdots, a_{n_1+n_2}}_{n_2}, \cdots, \underbrace{a_{k_{M-1}+1}, \cdots, a_N}_{n_M} \} \tag{16}$$

In the vanishing limit of Ω-deformation, Z_{inst} gives two holomorphic functions: $\mathscr{F}_{\text{inst}}$, non-perturbative part of the prepotential of 4d gauge theory, and W_{inst}, non-perturbative correction to the effective twisted superpotential of the 2d worldvolume theory coupled to 4d $SU(N)$ theory. In this limit logarithmic value of Z_{inst} has following expansion:

$$\log Z_{\text{inst}} = -\frac{\mathscr{F}_{\text{inst}}}{\varepsilon_1 \hat{\varepsilon}_2} + \frac{W_{\text{inst}}}{\varepsilon_1} + \text{ regular terms} \tag{17}$$

For the $[1,1,1]$ surface operator in $SU(3)$ theory, the quantities: $q_1 \frac{\partial W_{\text{inst}}}{\partial q_1}$ and $q_2 \frac{\partial W_{\text{inst}}}{\partial q_2}$ are identified with expressions in (12) and (13), respectively, after considering the relation between the parameters q_I and the scaling parameters Λ_I and Λ in the 2d/4d quiver theory in equation (10).

Surface operators in the $\mathscr{N} = 4$ SYM theories

We take a Hamiltonian perspective and study (at a classical level) two-dimensional defects in the maximally supersymmetric $\mathscr{N} = 4$ Yang-Mills theory on $S^3 \times \mathbb{R}$ spacetime as classical singular solutions that preserve some supersymmetry. We focus our attention on a particularly interesting class of BPS strings that preserve four supersymmetries. Our goal here on the field theory side is twofold. Firstly, to show that there is a general characterization of these BPS strings by describing the equations defining their worldvolume in a compact way. Secondly, to show that these BPS strings are solutions to the same variational problem as other non-singular supersymmetric solutions in the theory and to calculate their (regularized) energies and charges.

We choose a suitable set of supersymmetries that we would like our solutions to preserve. For this we adopted a bottom-up approach by proposing simple classical half-BPS string solutions and determine their supersymmetries as projection conditions on the conformal Killing spinors of $S^3 \times \mathbb{R}$. These half-BPS strings are static configurations, with topology $S^1 \times \mathbb{R}$. By using the state operator correspondence, we show that these BPS strings are the states that correspond to the half-BPS Gukov-Witten surface operators in $\mathbb{R}^4$. By using global symmetries, we find more such half-BPS string solutions and observe that all these defects have two supersymmetries in common. We use the common supersymmetries to derive a set of non-abelian BPS equations whose solutions are at least $\frac{1}{16}$-BPS. It turns out that these BPS equations coincide with the one obtained earlier in the literature in the study of the gauge theory duals of giant gravitons and dual-giant gravitons in $AdS_5 \times S^5$ [32], [33]. In fact, we find that the time dependent non-singular classical configurations dual to half-BPS dual-giants share a common set of four supersymmetries with the half-BPS strings supported by one complex scalar field. We then go on to derive the general non-abelian $\frac{1}{8}$-BPS equations that bosonic configurations have to satisfy in order to preserve these four supersymmetries.

The resulting $\frac{1}{8}$-BPS equations that the complex scalar fields Z_i and the gauge connection field A satisfy are what we focus on and they are the following

$$\begin{aligned} (D_0+D_3+i)Z_1=0\,, \quad (D_1+iD_2)Z_1=0\,, \quad F_{12}+2\left[Z_1,Z_1^\dagger\right]=0\,,\\ F_{03}=0 \qquad F_{01}+F_{31}=0\,, \qquad F_{02}+F_{32}=0\,, \end{aligned} \tag{18}$$

along with $Z_2 = Z_3 = 0$. D_a are the gauge covariant and local Lorentz covariant derivatives. For most of the work, we set all the field strength $F_{ab} = 0$ and focus on nontrivial abelian scalar profiles of the Z_1 component. In terms of coordinate system[1] of choice on S^3, the most general solution of the BPS equations in (18), can be written in terms of a local Laurent series given below

$$Z_1 = \sum_{m,n} a_{m,n} e^{-i(m+n+1)\tau} \left(\cos\theta\, e^{i\phi_1}\right)^m \left(\sin\theta\, e^{i\phi_2}\right)^n . \tag{19}$$

One of our main results is a simple characterisation of the world-volumes of the time-dependent

[1]here the metric under consideration is: $ds^2 = -d\tau^2 + \left(d\theta^2 + \cos^2\theta\, d\phi_1^2 + \sin^2\theta\, d\phi_2^2\right)$

$\frac{1}{8}$-BPS strings, which we shall also refer to as wobbling strings. We show that at any given time the spatial configuration of the wobbling string is obtained as the intersection of the zeros of a holomorphic function $F(z_1, z_2) = 0$ with the 3-sphere defined by $|z_1|^2 + |z_2|^2 = 1$ with its time evolution obtained by (z_1, z_2) by multiplying with a τ dependent exponential factor. This is the general characterization we were after.

Next, we address two problematic issues that arise when one considers such singular BPS solutions on par with the regular ones: (i) they do not belong to the same variational problem $\delta S_{action} = 0$ and (ii) they have divergent energies, and other global charges. We overcome these hurdles by cutting off the spacetime arbitrarily close to the singularities of these solutions and adding appropriate boundary terms. In particular, we show that it is possible to make $\delta S = 0$ as we vary along the space of solutions that include the regular ones by adding boundary terms. And demanding that the global charges are rendered finite provides infinitely many conditions on the allowed set of boundary terms with $\delta S = 0$ that essentially fixes them uniquely.

The theory reduces essentially to a conformally coupled complex scalar field on $S^3 \times \mathbb{R}$, described by the lagrangian:

$$\mathscr{L} = -\frac{1}{g_{YM}^2}\sqrt{-g}\left[g^{\mu\nu}\partial_\mu Z \partial_\nu \bar{Z} + \bar{Z}Z\right] \tag{20}$$

When the solutions are singular we cut-off a region around (and arbitrarily close to) the singularities and add to the Lagrangian the boundary term $\mathbf{L}_{bdy}$.

The addition of the boundary term $\mathbf{L}_{bdy}$ to the action leaves the equation of motion unchanged, and no new special constraints in the boundary arise as long as we are in a configuration space which is, as big as the space, where Z satisfies the BPS conditions asymptotically close to the boundary region.

For the remaining analysis, we study the global charges for the scalar solutions of the following form:

$$Z = r_0 e^{i(\xi_0-\tau)} \left(\cos\theta e^{i(\phi_1-\tau)}\right)^m \left(\sin\theta e^{i(\phi_2-\tau)}\right)^n \tag{21}$$

All the solutions in the current $\frac{1}{8}$-BPS sector satisfy the following equations:

$$\begin{aligned}\mathscr{C}_1 &= \Pi_Z^\theta + i\cos\theta\sin\theta\left(\Pi_Z^{\phi_1} - \Pi_Z^{\phi_2}\right) = 0\,,\\ \mathscr{C}_2 &= \left(\Pi_Z^{\phi_1}\cos^2\theta + \Pi_Z^{\phi_2}\sin^2\theta\right) - \Pi_Z^\tau - \frac{i}{2}\cos\theta\sin\theta\,\bar{Z} = 0\,.\end{aligned} \tag{22}$$

One of the main results of our work in the thesis, is that we use the freedom allowed by the BPS constraints (22) in phase-space variables to add another set of boundary terms ($\mathbf{L}'_{bdy}$) uniquely that regularize the values of global symmetry charges for the following two subclasses of solutions in (21):

- when $m \geq 0$ and $n < 0$, we add the boundary term near $\theta = 0$

$$\mathbf{L}'_{bdy} = -\frac{i}{g_{YM}^2}\left(Z\mathscr{C}_2 - \bar{Z}\bar{\mathscr{C}}_2\right)\frac{m+n+1}{n+1}\tan\theta\, F\left(1,-m,2+n,-\tan^2\theta\right) \tag{23}$$

- when $m < 0$ and $n \geq 0$, we add the boundary term near $\theta = \frac{\pi}{2}$

$$\mathbf{L}'_{bdy} = \frac{i}{g_{YM}^2}\left(Z\mathscr{C}_2 - \bar{Z}\bar{\mathscr{C}}_2\right)\frac{m+n+1}{m+1}\cot\theta\, F\left(1,-n,2+m,-\cot^2\theta\right) \tag{24}$$

Here F denotes the hypergeometric function ${}_2F_1(a,b,c;z)$.

Holography of $\frac{1}{8}$-BPS strings

We then turn to the holographic approach to the study of these string solutions, by studying probe D3-branes in $AdS_5 \times S^5$. The analysis has been generalized to defects that preserve fewer number of supersymmetries in [14]. We consider various classes of half-BPS probe D3-branes in global $AdS_5 \times S^5$: the equations that define the worldvolume of these probes are largely inspired by the profiles of the scalar fields of the half-BPS strings in the boundary gauge theory on $S^3 \times \mathbb{R}$. These are noncompact probe branes that end on the boundary in $S^1 \times \mathbb{R}$. The intersection of the D3-brane probe with the boundary is essentially the half-BPS string of the $\mathcal{N} = 4$ theory.

We then do an analysis similar to that of the boundary theory and perform a κ-symmetry analysis to find the projections on the bulk Killing spinor for the various half-BPS probes. Remarkably, we find that the set of supersymmetries common to all these static defects coincides precisely with those preserved by the most general giant and dual-giant configurations in $AdS_5 \times S^5$. The worldvolumes of such probe branes are known to be described in terms of zeros of holomorphic functions [32, 33] and they are given below

$$F^{(I)}\left(\Phi_i, Z_j\right) \text{ for } I = 1,2,3\,, \tag{25}$$

here the complex coordinates: (Φ_0, Φ_1, Φ_2) represent the embedding of AdS_5 in the complex space $\mathbb{C}^{1,2}$ and the coordinates (Z_1, Z_2, Z_3) describe S^5 in $\mathbb{C}^3$. For the holographic duals of the $\frac{1}{8}$-BPS strings we focus on the solution

$$Z_2 = Z_3 = 0\,, \quad f(Z_1\Phi_0, Z_1\Phi_1, Z_1\Phi_2) = 0\,. \tag{26}$$

We show that near the boundary of AdS_5, the zero locus of the holomorphic function coincides with the location of the BPS string of the boundary theory and proceed to derive the singular boundary scalar field profiles from the D3-brane solutions. We thereby recover the general characterization of the BPS strings from a probe analysis in the bulk dual.

Finally we restrict attention to the D3 branes dual to the monomial type BPS strings of the CFT. By adding an appropriate set of boundary terms we define a variational problem that admits all such brane configurations as allowed solutions. We then carry out the holographic renormalization of energies and charges in an expansion around the large energy limit of the probe brane. We are able to match the expected boundary results in the leading approximation and we go on to obtain the first order correction to the Yang-Mills results. The holographic renormalization we carry out in the bulk closely resembles the analogous calculation in the boundary theory and provides a justification for the regularization we carry out in the boundary theory.

We give the values of the regularized energy for the following subcases:

- The static case, $m+n+1=0$, as expected the values of energy are zero.
- $m=0$ and $n<0$: $E_{0,n} = \frac{2N}{l}\left((n+1)\frac{R_0^2}{2l^2} + \frac{1}{2}(n-1)(n+1)^2 + O(\frac{l}{R_0})\right)$
- $m+n=0$ and $n<0$: $E_{-n,n} = \frac{2N}{l}\left(\Gamma(1-n)\Gamma(1+n)\frac{R_0^2}{2l^2} + \frac{n-1}{2} + O(\frac{l}{R_0})\right)$

Upon using the map for the parameters $R_0 = \frac{2\pi}{\sqrt{\lambda}} l\, r_0$ and considering $T_{D3} = \frac{N}{2\pi^2 l^4}$, in the limit $\frac{l}{R_0} \to 0$ the leading order terms match with the answers for energies from the boundary field theory.

Plan of the Thesis

We will survey various aspects of Surface operators in the Supersymmetric gauge theories in this thesis. It will contain the following chapters:

- Chapter 1 will provide an introduction of Surface opertors in Supersymmetric gauge theories.
- Chapter 2 will review various approaches to study Surface operators in Supersymmetric gauge theories.
- Chapter 3 will discuss half-BPS Surface operators in $\mathcal{N}=2$ $SU(N)$ gauge theories.

- Chapter 4 will discuss BPS Strings solutions associated with Surface operators in $\mathscr{N} = 4$ $SU(N)$ gauge theories.

- Chapter 5 will have the conclusion with a discussion of the results.

Chapter 1

Introduction

This thesis is an exploration of various low energy properties of two dimensional surface defects in supersymmetric gauge theories. Supersymmetric theories have many special properties and have been of interest to physicists and mathematicians for almost half a century. Supersymmetry is a symmetry that relates bosonic and fermionic degrees of freedom and it leads to strong constraints on quantum corrections to observables in the form of non-renormalization theorems. For instance, with sufficient supersymmetry one can show that the perturbation expansion truncates at one loop [1]. As a result it is sometimes possible to calculate physical observables exactly (in the coupling constant) in these theories, often including non-perturbative effects. Supersymmetric theories often have a rich vacuum structure that was first studied by Seiberg and Witten for gauge theories in four dimensions with eight supercharges [2, 3]. They showed that on the Coulomb branch of the gauge theory, the low energy effective action can be completely solved for by solving for the period integrals on an auxiliary Riemann surface.

Traditionally non-local operators have played an important role in providing valuable information about the vacuum structure of the theory. For example, Wilson line operators and 't Hooft line operators distinguish phases of the gauge theory, and they give information about the phase structure of the gauge theory [5, 6]. These operators are much better studied in the

the context of supersymmetric gauge theories. Similarly, it is hoped that higher dimensional defects such as surface defects will shed additional light on the non-perturbative aspects of the susy gauge theories [7].

Surface operators were first defined in the work of Gukov and Witten [8–10] as codimension-two singular solutions to the Kapustin-Witten equations [11]. They have many different avatars and they have been studied from various points of view in supersymmetric gauge theories. Originally they were described as "monodromy defects", at whose locations some of the bosonic fields in the gauge vector multiplet become singular. For instance, their presence would lead to a non-trivial holonomy for a loop in the transverse plane that encircles the defect. One can also describe "flavour surface defects", which have a low energy description as a coupled 2d-4d system. Typically in this case, the gauge group of the four dimensional theory appears as the flavour group of the 2d theory, which is typically a quiver gauge theory. Lastly, for gauge theories that admit a holographic dual they have also been studied using the holographic AdS/CFT correspondence [13, 14] in which certain probe-branes that end on the boundary on a 2-surface serve as the duals of surface defects on the gravity side.

In this thesis, we study various aspects of surface operators using all three different approaches stated above. We will look into the coupled 2d-4d description in the context of $\mathcal{N} = 2$ $SU(N)$ gauge theory, the two-dimensional world-volume theory is a gauged linear sigma model with $\mathcal{N} = (2,2)$ supersymmetry. Due to supersymmetry the low energy effective action of the combined 2d/4d system is governed by two holomorphic functions: the prepotential which governs the bulk 4d physics and the twisted chiral superpotential, that governs the 2d/4d physics. Following [16], we calculate the low-energy effective action that governs the dynamics of the coupled 2d-4d system [17], on the vacuum solution and relate it to the twisted superpotential extracted from the partition function of the $\mathcal{N} = 2$ $SU(N)$ theory in presence of the monodromy defect. We establish a one-to-one map between the dynamical two and four-dimensional coupling parameters and the continuous parameters that label the monodromy defect realization of surface operators. Doing this is an important step towards showing that the 4d theories with the two types of surface operators in the UV regime are described by the same physics in the IR regime, as [18, 19] discuss. In the 2d/4d setup, the two-dimensional world-volume theory has a flavor $SU(N)$ symmetry which is the gauge symmetry group in the four dimensions. The classical vacua, which are obtained by extremizing the low-energy effective action associated with the 2d-4d system, also referred to as the flavor defect, are chosen using the four-dimensional Coulomb branch of the $SU(N)$ gauge theory. In the presence of a surface operator, the gauge symmetry group $SU(N)$ breaks into a Levi subgroup: $S[U(n_1) \otimes U(n_2) \otimes \ldots \otimes U(n_M)]$ where the integer set $[n_1, n_2, \ldots, n_M]$ indicates how N is partitioned, so that $n_1 + n_2 + \ldots + n_M = N$. Each choice of the integer set gives rise to a different surface operator, and with the help of which, the choice of classical vacua are made. A further essential ingredient here is the use of the resolvent [47] in the 4d gauge theory that allows to incorporate quantum corrections from the Coulomb branch.

In the monodromy defect description, we focus on the instanton partition function of the 4d gauge theory in presence of the surface operator. The instanton partition function gets contribution from the additional ramified instanton weighting parameters, apart from the 4d-instanton weights. In $\mathcal{N} = 2$ $SU(N)$ theory, the gauge field gets a prescribed singularity near the defect which, is defined by introducing a certain set of continuous parameters. The ramified instanton weights are related to the parameters that label the surface defects as described in [20, 21]. The partition function of the $\mathcal{N} = 2$ theory is calculated using the equivariant localization technique. The calculation is done on the so-called omega deformed background, as the integral over the moduli space of instanton for the theory defined on $\mathbb{R}^4$ is divergent. This method for the $\mathcal{N} = 2$ $SU(N)$ theory was developed by Nekrasov and collaborators in [24,25], and for the theory with surface operators, the formula for the ramified instanton partition function was given by Kanno and Tachikawa. Our main goal in this part of the analysis is to describe the relationship between the ramified instanton weights and dynamical parameters in the 2d-4d description [17], which leads us to match the non-perturbative corrections to the low-energy effective actions computed from the dual-descriptions of surface operators.

In the second part of the thesis we adopt a Hamiltonian perspective and the study codimension-2 defects in the maximally supersymmetric $\mathcal{N} = 4$ Yang-Mills theory on $\mathbb{R} \times S^3$ spacetime as classical singular solutions. We refer to these solutions as BPS strings. We will focus our attention on a particularly interesting class of BPS strings that preserve four supersymmetries. Our goal in this work on the field theory side is twofold. Firstly, to find a general characterization of these BPS strings by describing the equations defining their worldvolume in a compact way. Secondly, to show that these BPS strings are solutions to the same variational problem as other non-singular supersymmetric solutions in the theory and to calculate their (regularized) energies and charges.

In [27] authors study surface operators of nontrivial topology (other than being defined on an $\mathbb{R}^2$ submanifold) in the 4-dimensional supersymmetric gauge theory. In [27] authors study these mondromy defects as junctions of surface operators. The surface defects of Gukov and Witten in [8] were half-BPS and were defined on an $\mathbb{R}^2$ subspace of $\mathbb{R}^4$. It is natural to assume that the junctions/networks of defects of [27] would preserve lesser amounts of supersymmetries since the additional translational symmetries are broken. Surface operators preserving lower amounts of supersymmetries were also looked upon by the authors in [10], [14]. In those $\frac{1}{4}$-BPS surface defects the nature of singularities at their location, in the transverse 2d plane, were more non-trivial than that of simple pole type ($\sim \frac{1}{z}$). The singularities of higher order poles ($\sim \frac{1}{z^n}$) were considered for the bosonic fields [10, 14]. In this regard, a complete classification of surface operators is an important problem to solve and that is what we do in this thesis, by studying BPS strings on $\mathbb{R} \times S^3$, in the $\mathcal{N} = 4$ SYM theory.

We begin by choosing a suitable set of supersymmetries that we would like our solutions to preserve. For this we adopt a bottom-up approach by proposing simple classical half-BPS string solutions and determine their supersymmetries as projection conditions on the conformal Killing spinors of $\mathbb{R} \times S^3$. These half-BPS strings are static configurations, with topology

$\mathbb{R} \times S^1$. By using the state operator correspondence, we show that these BPS strings are the states that correspond to the half-BPS Gukov-Witten surface operators in $\mathbb{R}^4$. By using global symmetries, we find more such half-BPS string solutions and observe that all these defects have two supersymmetries in common. The common supersymmetries can be used to derive a set of non-abelian BPS equations whose solutions are at least $\frac{1}{16}$-BPS. It turns out that these BPS equations coincide with those of [28, 29] obtained in the study of the gauge theory duals of giant gravitons and dual-giant gravitons in $AdS_5 \times S^5$ [30]. In fact, we find that the time dependent non-singular classical configurations dual to half-BPS dual-giants share a common set of four supersymmetries with the half-BPS strings supported by one complex scalar field. We then go on to derive the general non-abelian $\frac{1}{8}$-BPS equations that bosonic configurations have to satisfy in order to preserve these four supersymmetries.

In the remaining part we work only with the resulting $\frac{1}{8}$-BPS equations and we restrict our analysis to abelian solutions in the scalar sector. One of our main results is a simple characterisation of the world-volumes of the time-dependent $\frac{1}{8}$-BPS strings, which we shall also refer to as wobbling strings. We show that at any given time the spatial configuration of the wobbling string is obtained as the intersection of the zeros of a holomorphic function $F(z_1, z_2) = 0$ with the 3-sphere defined by $|z_1|^2 + |z_2|^2 = 1$ with its time evolution obtained by $(z_1, z_2) \to (z_1 e^{-i\tau}, z_2 e^{-i\tau})$. This is the general characterization that we derive in chapter 4. Then we show that these BPS strings can be obtained as solutions to a well-defined variational problem, by adding particular boundary terms at the location of the string. We then focus on a sub-class of solutions that correspond to functions $F(z_1, z_2)$ that are of the monomial type. The energy and global charges of these singular configurations appear to diverge if we directly use the found solutions in the $\mathcal{N} = 4$ gauge theory Lagrangian. However we show that by adding further boundary terms (that do not affect the variational problem), the energy and other global charges of these wobbling string solutions can be made finite. The second set of boundary terms that we add to remove the divergences in the global charges are unique. They are directly related to the BPS constraints our BPS strings solutions satisfy.

We then turn to the holographic approach to the study of these string solutions, by studying probe D3-branes in $AdS_5 \times S^5$. For half-BPS defects in $\mathcal{N} = 4$ SYM in $\mathbb{R}^4$, the holographic duals have been obtained in [12, 13] as both bubbling geometries as well as probe D3-branes. This has been generalized to defects that preserve fewer number of supersymmetries in [14]. We consider various classes of $\frac{1}{2}$-BPS probe D3-branes in global $AdS_5 \times S^5$: the equations that define the worldvolume of these probes are largely inspired by the profiles of the scalar fields of the half-BPS strings in the boundary gauge theory on $\mathbb{R} \times S^3$. These are noncompact probe branes that end on the boundary in $\mathbb{R} \times S^1$. The intersection of the D3-brane probe with the boundary is essentially the half-BPS string of the $\mathcal{N} = 4$ theory.

We then mirror the analysis of the boundary theory and perform a κ-symmetry analysis to find the projections on the bulk Killing spinor for the various $\frac{1}{2}$-BPS probes. Remarkably, we find that the set of supersymmetries common to all these static defects coincides precisely with those preserved by the most general giant and dual-giant configurations in $AdS_5 \times S^5$ derived in [32, 33]. The worldvolumes of such probe branes are known to be described in terms of zeros of holomorphic functions. For the holographic duals of the $\frac{1}{8}$-BPS wobbling strings, we show that near the boundary of AdS_5, the zero locus of the holomorphic function coincides with the location of the BPS string of the boundary theory and proceed to derive the singular boundary scalar field profiles from the D3-brane solutions. We thereby recover the general characterization of the wobbling strings from a probe analysis in the bulk dual.

Finally we restrict attention to the D3 branes dual to the monomial type BPS strings of the CFT. By adding an appropriate set of boundary terms we define a variational problem that admits all such brane configurations as allowed solutions. We then carry out the holographic renormalization of energies and charges in an expansion around the large energy limit of the probe brane. We are able to match the expected boundary results in the leading approximation and we go on to obtain the first order correction to the Yang-Mills results. The holographic renormalization we carry out in the bulk closely resembles the analogous calculation in the boundary theory and provides a justification for the regularization we carry out in the boundary theory.

The coming chapters of the thesis are organised in the following way: In the second chapter we review the background for studying surface operators from various points of view. We look into definition of the flavor defects and their construction from the 2d gauge linear sigma models. We also give the definition from the monodromy defect point of view, we write and describe the formula of the instanton partition function of the $\mathcal{N} = 2$ theory. And lastly, we review about the probe D3 branes, holographic duals of surface operators in $AdS_5 \times S^5$ spacetime, and describe the associated κ-symmetry. In the third chapter, we analyze surface operators in $\mathcal{N} = 2$ gauge theory as both flavor defects and monodromy defects and match the associated low energy effective actions obtained from the two descriptions [17]. In chapter four, we study the BPS string configurations in the $\mathcal{N} = 4$ gauge theory defined on $S^3 \times \mathbb{R}$, we define a consistent variational problem which includes them alongside with the regular BPS solutions [26]. We also study the holographic duals of the BPS string solutions in $AdS_5 \times S^5$ space and mirror the analysis done in the boundary theory. For simple configurations, we demonstrate how to define a good variational problem and propose a regularization scheme that leads to finite energy and global charges on both sides of the holographic correspondence. In chapter five, we conclude with a discussion of the results.

Chapter 2

Surface operators and D3 branes

In this chapter, we will review surface operators studied from various points of view. We will discuss the definition in the monodromy defect description and in the $\mathcal{N}=2$ theory we will write down the formula for the ramified instanton partition function. We will discuss the coupled 2d-4d theory in the context of surface operators in $\mathcal{N}=2$ theory. And in the last part of this chapter, we will review about the holographically dual description in terms of D3 branes in $AdS_5 \times S^5$ background space.

2.1 Surface operators as Monodromy defects

A surface operator in four-dimensional gauge theory is an operator supported on a 2- dimensional submanifold $D \subset M$ in the space-time manifold M [18,34]. Surface operators are somewhat special in the context of 4d gauge theory since the degree of the 2-form F matches the dimension of the tangent as well as normal space to D. We can either write an integral

$$\exp\left(i\eta\int_D F\right) \tag{2.1}$$

which defines an electric surface operator analogous to the definition of a Wilson loop in abelian $U(1)$ gauge theory, or write

$$F = 2\pi\,\alpha\,\delta_D + \ldots \tag{2.2}$$

where δ_D is a 2-form delta-function Poincaré dual to D and give the definition of magnetic surface operators. When the gauge group is $G = SU(N)$, the parameters α and η are in the Lie algebra of the maximal-torus subgroup of $SU(N)$. Therefore, the gauge field gets the prescribed singularity of the form given below

$$A = \begin{pmatrix} \alpha_1 \otimes 1_{n_1} & 0 & \ldots & 0 \\ 0 & \alpha_2 \otimes 1_{n_2} & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & \alpha_M \otimes 1_{n_M} \end{pmatrix} d\psi\,, \tag{2.3}$$

where ψ is the polar angle in the plane transverse to D. In the presence of surface operators, the $SU(N)$ gauge group breaks near the location of D, into a Levi-subgroup $\mathbb{L} = S[U(n_1) \otimes \ldots \otimes U(n_M)]$ so that $n_1 + n_2 + \ldots + n_M = N$, as clear from (2.3). And it is also possible to introduce two-dimensional θ-angles that couple to the unbroken $U(1)$s at the defect. In the path integral this is represented by the following insertion

$$\exp\left(i \sum_{l=1}^{M} \eta_l \int_D \mathrm{Tr}\, F^{(l)} \right)\,. \tag{2.4}$$

In the maximally supersymmetric $\mathcal{N} = 4$ $SU(N)$ theory, one of the complex scalar fields of the $\mathcal{N}=4$ vector multiplet also becomes singular near the location of the defect

$$\Phi = \frac{1}{z}\begin{pmatrix} (\beta_1+i\gamma_1)\otimes 1_{n_1} & 0 & \dots & 0 \\ 0 & (\beta_2+i\gamma_2)\otimes 1_{n_2} & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & (\beta_M+i\gamma_M)\otimes 1_{n_M} \end{pmatrix}, \tag{2.5}$$

where z is the complex coordinate in the transverse plane. Surface operators in $SU(N)$ gauge theory are labelled by the following data: a discrete set of integers $[n_1, n_2, \dots, n_M]$ which label the breaking of the gauge group and a set of $4M$ real parameters: $\{\alpha_1, \dots, \alpha_M, \dots, \beta_M, \dots, \gamma_M, \dots, \eta_M\}$.

In $\mathcal{N}=2$ theories with surface operators, quantum effects may not only renormalize the values of various associated parameters but can also change their nature. They are introduced in the 4d theory by giving a prescribed singularity for the gauge field, as shown above in equation (2.3), at a given energy scale. It can be a UV theory, or an IR theory, or some effective theory at intermediate energy scale, the interesting question is to study about the surface operators at other energy scales and / or regimes of the parameters. To answer such questions, surface operators are studied using the other definition where they are supported on D by introducing additional 2d degrees of freedom, with their own Lagrangian and a flavor symmetry group $SU(N)$ that becomes gauged upon coupling to 4d degrees of freedom. Upon integrating out the 2d degrees of freedom we get a singularity, supported on D, for the fields A (and Φ in the maximally supersymmetric) of the four-dimensional theory[1].

To discuss the equivalence between the two descriptions in $\mathcal{N}=2$ theories we will obtain the expressions for the effective action, at the respective vacuua. In the monodromy defect side,

[1] [35], [18] may be referred for a longer discussion and a detailed classification. In [35] authors also demonstrate in detail the invariance of super-conformal index, in the coupled 2d-4d gauge theory description of surface operators in $\mathcal{N}=2$ SQCD, under action of 2d Seiberg duality. We cite these here to indicate the necessity of studying surface operators from various point of views.

we focus on the instanton partition function in presence of surface operators. The formula for the partition function was first obtained in [20], also described in [21], is given below

$$Z_{\text{inst}}[\vec{n}] = \sum_{\{d_I\}} Z_{\{d_I\}} \quad \text{with} \quad Z_{\{d_I\}}[\vec{n}] = \prod_{I=1}^{M} \left[\frac{(-q_I)^{d_I}}{d_I!} \int \prod_{\sigma=1}^{d_I} \frac{d\chi_{I,\sigma}}{2\pi i} \right] z_{\{d_I\}} \,. \tag{2.6}$$

And

$$\begin{aligned} z_{\{d_I\}} \;=\; & \prod_{I=1}^{M} \prod_{\sigma,\tau=1}^{d_I} \frac{\chi_{I,\sigma} - \chi_{I,\tau} + \delta_{\sigma,\tau}}{\chi_{I,\sigma} - \chi_{I,\tau} + \varepsilon_1} \times \prod_{I=1}^{M} \prod_{\sigma=1}^{d_I} \prod_{\rho=1}^{d_{I+1}} \frac{\chi_{I,\sigma} - \chi_{I+1,\rho} + \varepsilon_1 + \hat{\varepsilon}_2}{\chi_{I,\sigma} - \chi_{I+1,\rho} + \hat{\varepsilon}_2} \\ & \times \prod_{I=1}^{M} \prod_{\sigma=1}^{d_I} \prod_{s=1}^{n_I} \frac{1}{a_{I,s} - \chi_{I,\sigma} + \frac{1}{2}(\varepsilon_1 + \hat{\varepsilon}_2)} \prod_{t=1}^{n_{I+1}} \frac{1}{\chi_{I,\sigma} - a_{I+1,t} + \frac{1}{2}(\varepsilon_1 + \hat{\varepsilon}_2)} \,, \end{aligned} \tag{2.7}$$

$\chi_{I,\sigma}$ are parameters on the instanton moduli space. ε_1, $\hat{\varepsilon}_2 (= \frac{\varepsilon_2}{M})$ specify the Ω-deformed background which is introduced to localize the integrals over the instanton moduli space. The M variables q_I are ramified instanton weights that are related to the continuous parameters α_i and η_j that label the surface operator, as described in [21]. In the formula (2.6), M positive integers d_I count the number of ramified instantons in various sectors, with the convention that $d_{M+1} = d_1$. The derivation of the formula for the partition function is outside the scope of this thesis, it was derived in [20] by Kanno and Tachikawa, by considering open strings excitations of the D3/D(-1)-brane systems in the type IIB string theory where four of the spatial directions undergo Z_M-orbifold projection. Here moduli space of instantons is captured by those open strings between the D(-1) branes. The parameters $\chi_{I,\sigma}$ represent open strings that begin and end on D(-1) branes and correspond to open string moduli. Additionally, the Ω-deformed background of Nekrasov [24, 25] is introduced in these calculations to make the moduli space compact, as the integration over the moduli with the undeformed $\mathbb{R}^4$ space is divergent. The Coulomb vevs $\langle a_i \rangle$ of the adjoint scalar in $\mathcal{N} = 2$ multiplet are partitioned in the presence of surface operators and it is represented by the notation $a_{I,s}$ which we clarify with the help of the equation:

$$a_{I,s} \subset \{ \underbrace{a_1, \cdots, a_{n_1}}_{n_1}, \underbrace{a_{n_1+1}, \cdots, a_{n_1+n_2}}_{n_2}, \cdots, \underbrace{a_{k_{M-1}+1}, \cdots, a_N}_{n_M} \} \tag{2.8}$$

In the vanishing limit of Ω-deformation, Z_{inst} has the following expansion [20, 22, 23]

$$\log Z_{\text{inst}} = -\frac{\mathscr{F}_{\text{inst}}(a_i, \Lambda)}{\varepsilon_1 \hat{\varepsilon}_2} + \frac{W_{\text{inst}}(a_i, q_i, \Lambda)}{\varepsilon_1} + \text{regular terms}, \tag{2.9}$$

here $\mathscr{F}_{\text{inst}}$ is non-perturbative part of the prepotential of the 4d gauge theory that depends on Coulomb vevs a_i and the 4d scale Λ. The function W_{inst} that depends on both the 4d Coulomb branch parameters, as well as continuous parameters of the surface operator, has a simple physical interpretation: it is the effective twisted superpotential of the 2d $\mathcal{N} = (2,2)$ theory on D.

2.2 Surface operators as Flavor defects

The dual description of surface operators in the gauge theories that we discuss is a coupled 2d-4d system. The world-volume of a surface operator D supports two-dimensional degrees of freedom. The 2d theory has some flavor symmetry group that is identified with the gauge symmetry group in the 4-dimensions. The type of theory that exists in the two dimensions depends on the symmetries preserved by the surface operator. In this thesis, we are interested in the half-BPS surface operators in the $\mathcal{N} = 2$ theory, and therefore, we will be reviewing the theories with 2d $\mathcal{N} = (2,2)$ supersymmetry in this section. In particular, we will discuss the IR-physics in a generic massive vacuum of a two-dimensional $\mathcal{N} = (2,2)$ Gauge linear sigma model with flavor symmetry $SU(N)$, and we will obtain the low energy effective twisted superpotential and use it for our analysis of surface operators as flavor defects.

The two-dimensional model [36, 37] is a gauge theory of the unitary group $U(k)$ of rank

k, with a chiral matter multiplet Q and, with additional $SU(n)$ flavor symmetry. The fields associated with gauge multiplet V are a complex scalar σ, two complex fermions λ_L and λ_R, a one-form bosonic field A and an auxiliary real field D. Off-shell field content of the chiral multiplet Q are a complex scalar q, two complex fermions ψ_L and ψ_R and an auxiliary complex scalar F.

The kinetic term of the Lagrangian is given by

$$\mathscr{L}_{kin} = \frac{1}{4} \int d\theta^L d\theta^R d\overline{\theta}^L d\overline{\theta}^R \left(Q^\dagger e^{2V} Q - \frac{1}{e^2} \mathrm{Tr}\left(\Sigma^\dagger \Sigma\right)\right), \tag{2.10}$$

here Σ^a is the twisted-chiral superfield containing the field strength of the $U(k)$ gauge connection, and transforming in the adjoint representation of the gauge group. It is related to the gauge vector multiplet V by $\Sigma = \{\overline{\mathscr{D}}_L, \mathscr{D}_R\}/2$ where $\mathscr{D}_\alpha = e^{-V} D_\alpha e^V$ and $\overline{\mathscr{D}}_\alpha = e^V \overline{D}_\alpha e^{-V}$. And D_α, $\overline{D}_\alpha$ are the covariant derivatives on the 2d $(2,2)$ superspace. Σ is twisted-chiral in the sense that it satisfies

$$\overline{D}_L \Sigma = D_R \Sigma = 0\,, \tag{2.11}$$

in comparison to the chiral superfield Q which satisfies: $\overline{D}_L Q = \overline{D}_R Q = 0$. The lowest component of Σ is the complex scalar field σ.

Back in (2.10), $\mathscr{L}_{kin}$ has the gauge coupling parameter e with mass dimension. In addition to $\mathscr{L}_{kin}$ contribution, another term $\mathscr{L}_{FI,\theta}$ is considered, with the Fayet-Iliopoulos and the 2d theta parameter

$$\mathscr{L}_{FI,\theta} = \frac{i\tau}{4} \int d\theta^L d\overline{\theta}^R \,\mathrm{Tr}\Sigma + h.c. \tag{2.12}$$

where τ is the complexified parameter: $\tau = ir + \frac{\theta}{2\pi}$ and classically it has no mass dimension. For this two-dimensional model, there is a complex-valued mass coupling term which is considered by first gauging the flavor symmetry $SU(n)$ and giving a background value to the scalar component of the associated vector superfield $\widetilde{V}$, and then setting the fields to be vanishing. The massive term can be written as

$$\mathscr{L}_{\widetilde{m}} = \int d\theta^L d\theta^R d\overline{\theta}^L d\overline{\theta}^R Q^\dagger e^{2\widetilde{V}} Q, \tag{2.13}$$

here $\widetilde{V} = \theta^R \overline{\theta}^L \widetilde{m} + h.c.$

$\widetilde{m}$ is a diagonalizable $n \times n$ matrix and called the twisted mass parameter. The real FI paramater r is renormalized according to the relation [36]

$$r(\mu) = r(\mu') - (n_1 - n_2) \log \frac{\mu'}{\mu}, \tag{2.14}$$

where μ & μ' are dimension-full scale parameters. n_1 is related to the sum of flavor charges of the chiral multiplet, and for the case of our current disscussion $n_2 = 0$. The action that we want to consider consists of the following contribution

$$\mathbf{S}_{action} = \int d^2x \left(\mathscr{L}_{kin} + \mathscr{L}_{FI,\theta} + \mathscr{L}_{\widetilde{m}} \right) \tag{2.15}$$

Following [37], we now discuss the space of vacua of the 2d model of interest. Here auxilliary fields F and D in the multiplets Q and V respectively, do not have kinetic terms in the action $\mathbf{S}_{action}$ and can be integrated out. After integrating out those the potential energy of this system is

$$\begin{aligned} U =& \frac{e^2}{2} \mathrm{Tr} \left(q q^\dagger - r \right)^2 + \frac{1}{8e^2} \mathrm{Tr} [\sigma, \sigma^\dagger]^2 \\ &+ \frac{1}{2} \| \sigma q - q \widetilde{m} \|^2 + \frac{1}{2} \left\| \sigma^\dagger q - q \widetilde{m}^\dagger \right\|^2 . \end{aligned} \tag{2.16}$$

The space of classical vacua is the space of zeros of U modulo the action of gauge transformations. And for the second term to be vanishing σ must be a diagonalizable $k \times k$ matrix.

The quantum fluctuations around each classical vacuum consist of massless modes and massive modes which are tangent to and transverse to the space of classical vacua, respectively. In the limit of IR physics where $e^2 \to \infty$, the massive modes decouple from the massless degrees of freedom(we refer to section 2 of [37] for a detailed case by case discussion) and the system approaches to a supersymmetric non-linear sigma model with target space equivalent to the space of classical solutions from the equation:

$$qq^\dagger = r \tag{2.17}$$

modulo the gauge transformations. When twisted masses are turned on, the values of diagonal components of σ must also be tuned at the values of $\widetilde{m}$ so that terms in the second line in the RHS of (2.16) vanish.

When the gauge group is $U(1)$, this target space is the $(n-1)$-dimensional complex projective space $\mathbb{CP}^{n-1}$ which is the space of vectors in $\mathbb{C}^n$ of length$^2 = r$ modulo the $U(1)$ phase rotation.

When the gauge group is $U(k)$, the target space is the generalized complex Grassmannian manifold $G(k,n)$, it is space of $k-$planes in $\mathbb{C}^n$ where there are k vectors $v_1, v_2, \ldots, v_k$ in $\mathbb{C}^n$. For the sigma model of our interest, these vectors are given by v_a= $\left(q^{a,i}\right)_{i=1,\ldots,n}$, they are orthogonal to each other and have length^2= r. The modulo action of the $U(k)$ group from the 2d gauge symmetry is considered as the change of basis for the orthogonal k vectors in $\mathbb{C}^n$ such that the k-plane the vectors span in $\mathbb{C}^n$ does not change.

In the low-energy effective action, the massive chiral superfield Q can be integrated out and the effective Lagrangian remains as a functional of the vector superfield. The low energy dynamics is encoded in the effective twisted-chiral superpotential [38, 39]

$$W_{eff} = \frac{1}{4}\sum_{a=1}^{k}\left[i\tau\,\Sigma_a - \frac{1}{2\pi}\sum_{i=1}^{n}(\Sigma_a - \widetilde{m}_i)\left(\log\left(\frac{\Sigma_a - \widetilde{m}_i}{\mu}\right) - 1\right)\right], \tag{2.18}$$

where Σ_a are the diagonal components of the adjoint valued $k \times k$ matrix Σ, here the values of the scalar components σ_a are well separated from each other. GLSMs with $k < n$ are going to be relevant in order to make the connection to the effective action associated with flavor defects. Also, the flavor symmetry group $SU(n)$ is identified with the $SU(N)$ 4-dimensional gauge symmetry group, and twisted masses $\widetilde{m}_i$ are identified with the vevs of the scalar in the 4d $\mathscr{N} = 2$ vector multiplet.

It is possible to consider a much more general Gauged Linear Sigma Model where the target space is equivalent to some flag variety, which is a space of flags $\mathbb{C}^{k_1} \subset \mathbb{C}^{k_2} \ldots \subset \mathbb{C}^{k_{M-1}} \subset \mathbb{C}^n$, discussed in [16]. The GLSM with this target space can be represented in terms of 2d linear quiver with gauge groups $U(k_1) \times U(k_2) \times \ldots \times U(k_{M-1})$, bifundamentals chiral matter multiplets between each consecutive pairs of nodes, and n fundamentals at the last node are represented as the arrowed lines in the quiver diagram, shown below

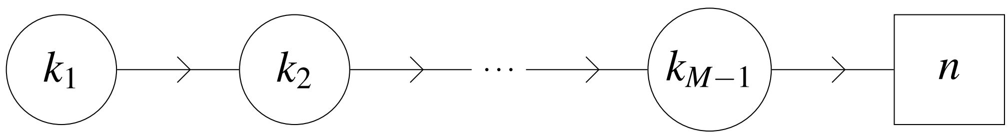


Figure 2.1: The general quiver describing a 2d Gauge Linear Sigma Model.

And to make the connection with effective action associated with the flavor defect with discrete data: $[n_1, n_2, \ldots, n_M]$, the flavor $SU(n)$ symmetry is identified with the 4-d gauge symmetry group and twisted masses due to flavor $SU(n)$ symmetry with the vevs of the scalar in the 4d $\mathscr{N} = 2$ vector multiplet. Here the rank k_I of each 2d gauge group is related to the integers in the discrete data as

$$k_I = n_1 + n_2 + \ldots + n_I \tag{2.19}$$

such that

$$n_1 + n_2 + \ldots + n_M = n = N\,. \tag{2.20}$$

The target space of the relevant GLSM is always equal to the coset space: $SU(N)/\mathbb{L}$ where $\mathbb{L}$ is Levi subgroup, associated with the surface operator, equal to $S[U(n_1) \otimes \ldots \otimes U(n_M)]$. The low-energy effective action after integrating out the massive bifundamentals matter multiplets, for the generic quiver theory in figure 2.1, is given in terms of the twisted-chiral fields $\Sigma^{(I)}$ from each 2d gauge node in the quiver. And because of the two-dimensional $(2,2)$-supersymmetry, it is encoded in a twisted-chiral superpotential W_{eff}. The expression for W_{eff} of a GLSM with a generic 2d quiver theory description is

$$\begin{aligned} W_{eff} = 2\pi i \sum_{I=1}^{M-1} \tau_I(\mu) \Big(\sum_{s=1}^{k_I} \sigma_s^{(I)} \Big) - \sum_{I=1}^{M-2} \sum_{s=1}^{k_I} \sum_{t=1}^{k_{I+1}} \varpi\big(\sigma_s^{(I)} - \sigma_t^{(I+1)}\big) \\ - \sum_{s=1}^{k_{M-1}} \sum_{i=1}^{N} \varpi\big(\sigma_s^{(M-1)} - \Phi_i\big)\,, \end{aligned} \tag{2.21}$$

here we have introduced the function

$$\varpi(x) = x \left(\log \frac{x}{\mu} - 1 \right), \tag{2.22}$$

and μ is an arbitrary common UV cut-off scale of the theory. Φ_i are the twisted masses from the flavor symmetry of the right-most node which will be identified with the scalar vevs of the $\mathcal{N} = 2$ vector multiplet when the coupled 2d-4d dynamics is turned on.

Given the twisted superpotential in (2.21), the massive vacua of the theory are obtained by solving the twisted chiral ring equations

$$\exp\left(\frac{\partial W_{eff}}{\partial \Sigma_s^{(I)}} \right) = 1\,. \tag{2.23}$$

When the quantum dynamics of the 4d theory is taken into account the terms in the last line of (2.21) are modified with the contribution from the chiral correlators in the $\mathcal{N}=2$ theory [16]. We describe this modification with some more details in the next coming chapter. We calculate W_{eff} on the vacuum of the coupled 2d-4d theory by solving the twisted-chiral ring equations of a generic quiver theory and obtain the values of the 2d scalars $\sigma_\star^{(I)}$ then, we evaluate W_{eff} on the solutions $\sigma_\star^{(I)}$.

2.3 Holographically dual D3 branes

Surface operators in $\mathcal{N}=4$ SYM theory are holographic dual to some non-compact D3 branes in $AdS_5\times S^5$ geometry, in the dictionary of AdS/CFT correspondence as desrcibed in [13, 14, 40], where authors contruct the probe brane solutions in $AdS_5\times S^5$ background that reach the boundary region of AdS and end there in two dimensional submanifold. The two-dimensional subspace region are identified with worldvolume of codimension-2 defects in the four-dimensional SYM gauge theory that lives on the boundary. The dual D3 branes are considered in the probe limit and do not backreact with the ten-dimensional geometry, they were first studied in this work [13], where authors also analyzed those dual objects which do gravitationally backreact in the ten-dimensional supergravity theory.

$AdS_5\times S^5$ geometry comes out as a solution to Equation of motion in type II B supergravity theory in which the complex Weyl-gravitino Ψ_M, complex spinor Λ, the complex two-form $C^{(2)}$ and the complex scalar Φ are zero. The only non-vanishing fields are the vielbein and the real four-form $C^{(4)}$ with self-dual five-form field strength $F^{(5)}$.

In type IIB supergravity theory, a stack of N number of D3 branes charged under the five-form field strength $F^{(5)}$ provides a solution to equations of motion, with the metric

$$ds^2 = H^{-\frac{1}{2}}dx_{1,3}^2 + H^{\frac{1}{2}}dr^2 + H^{\frac{1}{2}}r^2 d\Omega_5^2 \tag{2.24}$$

with the harmonic function $H = 1 + \frac{4\pi N g_s l_s^4}{r^4}$. Here, r is the radial coordinate transverse to the D3 branes, l_s and g_s are the string length and string coupling. In the AdS/CFT correspondence [41,42], $\frac{r}{l_s^2}$ is held fixed and string length l_s is taken to be small and the above metric becomes of $AdS_5 \times S^5$ geometry

$$\frac{ds^2}{l_s^2} = \frac{r^2}{L^2} dx_{1,3}^2 + \frac{dr^2}{r^2} + L^2 \, d\Omega_5^2 \,, \tag{2.25}$$

where we redefined $r \to \frac{r}{l_s^2}$ and L is the radius of S^5 equal to $(4\pi N g_s)^{\frac{1}{4}}$.

The supersymmetry of this background with only bosonic fields $C^{(4)}$ & e_μ^a turned on, is preserved by setting the Susy transformation of the fermionic fields to be zero. Here $\delta\Lambda$ is zero on account of the rest of the bosonic fields $C^{(2)}$ and Φ being zero, and the variation of the gravitino takes the form

$$\delta\Psi_\mu = D_\mu \varepsilon - \frac{i}{480} \Gamma_\mu^{\;\nu\rho\alpha\beta\lambda} F^{(5)}_{\nu\rho\alpha\beta\lambda} \varepsilon \,. \tag{2.26}$$

Demanding $\delta\Psi_\mu = 0$ leads to the Killing spinor equations which, can be written as

$$\begin{aligned} D_\mu \varepsilon - \frac{i}{2} \gamma \Gamma_\mu \varepsilon &= 0 \,, \\ D_m \varepsilon - \frac{i}{2} \tilde{\gamma} \Gamma_m \varepsilon &= 0 \end{aligned} \tag{2.27}$$

where μ denotes the AdS_5 coordinates and m denotes the S^5 coordinates. ε is the 32-component spinor in the 10 dimensions. γ and $\tilde{\gamma}$ are the products of matrices $\Gamma_0\Gamma_1\Gamma_2\Gamma_3\Gamma_4$ and $\Gamma_5\Gamma_6\Gamma_7\Gamma_8\Gamma_9$, with the tangent space indices.

The D3 branes preserve half of the 32 supersymmetries as world-volume κ-symmetry. Generally, the supersymmetry preserved by a D brane depends on its orientation and type, D3 branes that are parallel to each other will preserve the same Susy components and which are perpendicular in some of their directions would have different susy components. The supersymmetric world-volume brane action is derived by (super-)embedding the brane world-volume in superspace. In this case, the target-superspace is considered to have 10 bosonic spacetime coordinates $X^{\mu}(\sigma^m)$ and 32 fermionic coordinates $\Theta_{\alpha}(\sigma^m)$. In order to have world-volume supersymmetry, the number of on-shell bosonic[2] and fermionic degrees of freedom must be equal. The local fermionic κ-symmetry of the brane action is used to 'gauge' fix [30, 42, 43] half of the Θ coordinate components[3] with the help of the projection condition

$$(1+\Gamma)\,\Theta = 0\,, \tag{2.28}$$

where Γ is the pull-back of product of four 10-dimensional $\Gamma-$matrices:

$$\Gamma = \frac{1}{\sqrt{-h}}\frac{\varepsilon^{\mu_1\mu_2\mu_3\mu_4}}{4!}\Gamma_{\mu_1}\Gamma_{\mu_2}\Gamma_{\mu_3}\Gamma_{\mu_4}$$

and here h is determinant of the induced world-volume metric on D3. Since Θ transform under both the 10d susy transformation

$$\delta_{\varepsilon}\Theta = \varepsilon\,, \tag{2.29}$$

and κ-transformation as

$$\delta_{\kappa}\Theta = \frac{1}{2}(1+\Gamma)\,\kappa\,, \tag{2.30}$$

with κ being some arbitrary spinor parameter. The gauge fixing condition in (2.28) is used to compensate a supersymmetry transformation with a κ-symmetry transformation, with $\kappa = -\varepsilon$. And therefore, the condition for the preservation of world-volume supersymmetry is equivalent

[2] In case of the D3 branes, there are four world-volume gauge field components $A_m(\sigma)$ and six scalar fields $\Phi^a(\sigma)$ describing the transverse excitations, therefore, there are eight onshell bosonic degrees of freedom.

[3] and the equation of motion implies that half of the remaining 16 components are independent propagating degrees of freedom.

to the constraint

$$\Gamma \varepsilon = \varepsilon \,, \tag{2.31}$$

and this will leave 16 susy to be preserved on the D3 brane world-volume. In chapter 4, we will use the κ-symmetry constraint extensively to find the general world-volume solutions which will be holographic duals of $\frac{1}{8}$-BPS surface operators in 4-dimensional $\mathcal{N} = 4$ SYM theory. The bosonic part of the world-volume action consists of two terms: 'Dirac-Born-Infeld' term and 'Wess-Zumino' term

$$\mathscr{L} = \mathscr{L}_{DBI} + \mathscr{L}_{WZ} = -T_{D3}\sqrt{-h} + T_{D3}\, P\left[C^{(4)}\right] \,, \tag{2.32}$$

T_{D3} is the D3 brane tension and the notation $P\,[\,.\,]$ denotes that we consider the pullback of the argument. Here we have considered that no component of the D3 brane world-volume gauge field is turned on.

$\mathcal{N} = 4$ SYM theories with 2d defects were studied holographically in [40] using two orthogonal stacks of 'intersecting' D3 branes in type IIB supergravity where one of the stacks was assumed to be in the probe limit compared to the other stack. But the holographic setup of probe D3 branes in $AdS_5 \times S^5$ geometry in the context of surface operators was first considered in [13] by Drukker, Gomis and Matsuura and subsequently by Koh and Yamaguchi in [14], where authors analyzed the world-volume solutions which extend to the boundary of the AdS, ending in two-dimensional surfaces. Reference [14] had analyzed the world-volume solutions dual to the surface defects preserving less than half supersymmetries of the parent 4-dimensional theory. D3 branes with the world-volume topology of $AdS_3 \times S^1$, ending in the boundary in $\mathbb{R}^{1,1}$ or $S^1 \times \mathbb{R}$ submanifolds, are dual to the half BPS surface operators. The submanifold is $\mathbb{R}^{1,1}$ when the AdS metric is considered in the Poinareé coordinates and it is $S^1 \times \mathbb{R}$ in the global AdS coordinates. In the section 4.2.2 of chapter four, we explicitly compute the worldvolume metric for some half-BPS D3 branes solutions, that we find to be of topology

$AdS_3 \times S^1$, in $AdS_5 \times S^5$ background expressed in global coordinates.

Surface operators with 'wild ramifications' having higher-order singularities with the bosonic fields having the behavior $\Phi \sim \frac{1}{z^n}$, near the surface defects, were also studied by Witten in [10]. The type IIB supergravity duals of such defects, to some extent, were discussed in [14] and in general they are $\frac{1}{4}$-BPS. The ten-dimensional supergravity background probed by a stack of n number of parallel D3-branes, such that $n \ll N$, would be dual to the surface operator with the Levi-subgroup: $S[U(n) \otimes U(N-n)]$. And more generally, for M parallel stacks of branes will describe surface operators with Levi-subgroup: $S[U(n_1) \otimes U(n_2) \otimes \ldots \otimes U(n_M)]$, holographically, under the assumption that all the n_i are $\ll N$.

In the absence of surface operators, $\mathcal{N} = 4$ SYM is invariant under the $PSU(2,2|4)$ superconformal symmetry group. The half-BPS surface defect with 2d world-volume $\Sigma = \mathbb{R}^{1,1} \subset \mathbb{R}^{1,3}$ breaks the $SO(2,4)$ conformal group into a subgroup. The symmetries left unbroken by Σ generate an $SO(2,2) \times SO(2)_{23}$ subgroup of the $SO(2,4)$ conformal group, where $SO(2)_{23}$ rotates the plane transverse to Σ in $\mathbb{R}^{1,3}$. The singularity in the classical fields produced is also invariant under $SO(2,2)$. The $\mathcal{N} = 4$ scalar field Φ carries charge under an $SO(2)_R$ subgroup of the $SO(6)$ R-symmetry and is therefore $SO(4)$ invariant. The surface operator is therefore invariant under $SO(2,2) \times SO(2)_a \times SO(4)$, where $SO(2)_a$ is generated by some combination of $SO(2)_{23}$ and $SO(2)_R$. $\mathcal{N} = 4$ SYM has sixteen Poincare supersymmetries and sixteen conformal super-symmetries, generated by ten dimensional Majorana-Weyl spinors η_+ and η_- of opposite chirality. The surface operator for $\Sigma = \mathbb{R}^{1,1}$ preserves half of the Poincare and half of the conformal supersymmetries and is therefore half-BPS.

With the help of these symmetries of surface operators, we can determine the correct holographic-dual D3 branes in the ten-dimensional geometry.

The broken global bosonic symmetry group: $SO(2,2) \times SO(2) \times SO(4)$ is exactly the symmetry group associated with the half-BPS D3 brane solution in the $AdS_5 \times S^5$ background. And the fermionic symmetry can be verified by doing the κ-symmetry analysis. This leaves a single projection condition to be applied on the constant spinors $\eta_{\pm}$, in the Killing spinor

$$\varepsilon = h(\theta_i)\left(\sqrt{r}\,\eta_+ + \frac{\Gamma_4 - r x^\mu \Gamma_\mu}{\sqrt{r}}\,\eta_-\right), \tag{2.33}$$

r is the radial coordinate and $x^0, \ldots, x^3$ are the remaining coordinates in AdS_5 of the Poincaré patch. $h(\theta_i)$ is some function dependent on the S^5 coordinates.

The labeling parameters β, γ for surface defects can be recovered in the holographically dual side, their complex combinations $\beta_l + i\gamma_l$ are mapped to the constants that appear in the embedding equation of the non-compact D3 world-volume solutions [13]. And the other continuous parameters α, η are identified with the holonomies of the gauge field A and the dual gauge field $\tilde{A}$ living on the D3-brane world-volume. The holonomies are computed along the non-contractable S^1

$$\alpha = \oint \frac{A}{2\pi}, \qquad \eta = \oint \frac{\tilde{A}}{2\pi}. \tag{2.34}$$

In chapter four, we will discuss the holographic probe D3 branes in the global $AdS_5 \times S^5$ spacetime. The holographic dual D3 branes in the global coordinates always end in a $\Sigma^1 \times \mathbb{R}$ subspace in the boundary where Σ^1 is a closed 1d curve. For half-BPS solutions Σ^1 is a circle. We will show how to use the supersymmetry of several half-BPS branes by finding the common supersymmetries among them, and then substitute the common supersymmetries in the κ-symmetry constraint equation (2.31) to determine the most general $\frac{1}{16}$-BPS equations. From these equations, we obtain the classical solutions of the holographic duals to $\frac{1}{8}$-BPS defects in $\mathcal{N}$=4 SYM theory.

Chapter 3

Surface operators in $\mathscr{N} = 2$ theories

In this chapter we calculate the low-energy effective action for surface operators in pure $\mathscr{N} = 2$ $SU(N)$ supersymmetric gauge theories in four dimensions. First we focus on the two-dimensional world-volume theory on the surface operator and compute the associated effective twisted chiral superpotential W which governs the dynamics of the coupled 2d-4d system. With the vacuum structure determined by the twisted chiral ring equations in (2.23). In the second part we go to the localization part of the calculation where the ramified instanton partiton function is calculated. The effective twisted chiral superpotential is extracted from Z_{inst} and matched with the one obtained in the first description after establishing the map between various parameters in the two sides.

3.1 Twisted superpotential for coupled 2d/4d theories

3.1.1 $SU(2)$ gauge theory

In the beginning we work with the $SU(2)$ theory to illustrate our method and we consider the simple surface operator represented by the partition $[1,1]$. First, we assume the case in which the quantum effects of the $SU(2)$ theory are neglected. We consider a generic point in the Coulomb branch parameterized by the vev's $a_1 = a_2 = a$ of the adjoint $SU(2)$ scalar field Φ in the vector multiplet. The expression for W takes the form

$$W = 2\pi i\tau(\mu)\sigma - \mathrm{Tr}\left[(\sigma-\Phi)\left(\log\frac{\sigma-\Phi}{\mu}-1\right)\right]. \tag{3.1}$$

Using the RG running relation of the complexified 'FI' coupling[1], for convenience, we set the scale Λ_1 for our analysis at which $\tau(\Lambda_1) = 0$ and the expression for W becomes

$$W = -\mathrm{Tr}\left[(\sigma-\Phi)\left(\log\frac{\sigma-\Phi}{\Lambda_1}-1\right)\right]. \tag{3.2}$$

As pointed out in [16], the effect of 4d quantum dynamics can be accounted by considering the following superpotential:

$$W = -\left\langle \mathrm{Tr}\left[(\sigma-\Phi)\left(\log\frac{\sigma-\Phi}{\Lambda_1}-1\right)\right]\right\rangle. \tag{3.3}$$

The angular brackets signify taking the quantum corrected vev of the chiral observable in the four-dimensional $SU(2)$ theory. The twisted chiral ring equation using (3.3), is equivalent to

$$\exp\left\langle \mathrm{Tr}\log\frac{\sigma-\Phi}{\Lambda_1}\right\rangle = 1. \tag{3.4}$$

[1] $2\pi i\tau(\mu') = 2\pi i\tau(\mu) - 2\log\frac{\mu'}{\mu}$, here μ and μ' are complexified as well

As explained in [16], the left-hand side of (3.4) is simply the integral of the resolvent of the pure $\mathcal{N}=2$ $SU(2)$ theory in four dimensions which takes the form [47]

$$\exp\left\langle \mathrm{Tr}\log\frac{\sigma-\Phi}{\Lambda_1}\right\rangle = \log\left(\frac{P_2(\sigma)+\sqrt{P_2(\sigma)^2-4\Lambda^4}}{2\Lambda_1^2}\right). \tag{3.5}$$

Here Λ is the four-dimensional strong coupling scale of the SU(2) theory and

$$P_2(\sigma)=\sigma^2-u \tag{3.6}$$

is the characteristic polynomial appearing in the Seiberg-Witten solution where

$$u=\frac{1}{2}\langle\Phi^2\rangle = a^2+\frac{\Lambda^4}{2a^2}+\frac{5\Lambda^8}{32a^6}+\dots \tag{3.7}$$

We obtain the following two solutions for the scalar at the vacuum

$$\sigma_\star^\pm = \pm\sqrt{u+\Lambda_1^2+\frac{\Lambda^4}{\Lambda_1^2}} \tag{3.8}$$

The purely two-dimensional result can be recovered by taking the $\Lambda\to 0$ limit.

The proposal in [16] was that W evaluated at $\sigma_\star^\pm$ should reproduce the twisted superpotential calculated using localization methods and we shall explicitly verify this later in the chapter. Now we mention an essential simplification in this calculation that occurs when we are at vacuum, that is when $\frac{\partial W}{\partial\sigma}\Big|_{\sigma_\star^+}=0$. The logarithmic derivative of twisted superpotential simply becomes

$$\Lambda_1\frac{dW_\star^+}{d\Lambda_1}=\Lambda_1\frac{\partial W_\star^+}{\partial\Lambda_1}\Big|_{\sigma_\star^+}=2\sigma_\star^+. \tag{3.9}$$

And upon using the explicit form of the solution given in (3.8), in the the weak coupling limit expansion of u from (3.7) we obtain

$$\frac{1}{2}\Lambda_1 \frac{dW_\star^+}{d\Lambda_1} = a + \frac{1}{2a}\left(\Lambda_1^2 + \frac{\Lambda^4}{\Lambda_1^2}\right) - \frac{1}{8a^3}\left(\Lambda_1^4 + \frac{\Lambda^8}{\Lambda_1^4}\right) + \dots \tag{3.10}$$

we will show later that this result precisely matches the derivative of the twisted effective superpotential calculated using localization for the simple surface operator in the $SU(2)$ gauge theory, provided we suitably relate the dynamically generated scale Λ_1 of the two-dimensional theory to the ramified instanton counting parameter in presence of the monodromy defect.

3.1.2 Twisted chiral ring in quiver gauge theories

We will now show that the procedure described above can be generalized for any surface operator in the $SU(N)$ theory labelled by a partition of N. However, for surface operator other than of kind $[1, N-1]$ partition it will not be possible to exactly solve the twisted chiral ring equations. We develop a systematic perturbative approach in order to obtain a semi-classical expansion for the twisted chiral superpotential around a particular classical vacuum. As earlier in the case of $SU(2)$ theory, we again find that the derivatives of the twisted superpotential with respect to various scales have simple expression in terms of combinations of the twisted chiral field σ evaluated in the appropriately chosen vacuum.

The general two-dimensional quiver gauge theory is

$$U(k_1) \times U(k_2) \times \ldots \times U(k_{M-1}) \tag{3.11}$$

with (bi)-fundamental matter between successive nodes, coupled to a pure $\mathcal{N} = 2$ theory in four dimensions with gauge group $SU(N)$ acting as a flavor symmetry for the rightmost factor in (3.11). All this was represented in Fig. 3.1. We choose an ordering such that the ranks k_I follow

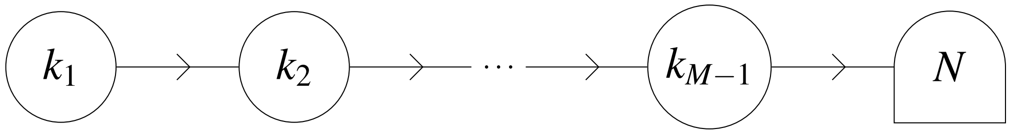


Figure 3.1: The general quiver describing a surface operator.

$$k_1 < k_2 < k_3 \ldots < k_{M-1} < N \tag{3.12}$$

which was also suggestive in (2.19). Our first goal is to obtain the twisted chiral ring of this 2d/4d system. Only the diagonal components of σ are relevant for this purpose [37], and thus for the I-th gauge group we take

$$\sigma^{(I)} = \text{diag}\left(\sigma_1^{(I)}, \sigma_2^{(I)}, \ldots, \sigma_{k_I}^{(I)}\right) \tag{3.13}$$

The (bi)-fundamental matter fields are massive and their (twisted) mass is proportional to the difference in the expectation values of the σ's in the two nodes connected by the matter multiplet. In order to minimize the potential energy, the twisted chiral field $\sigma^{(I)}$ gets a vev and this in turn leads to a non-vanishing mass for the (bi)-fundamental matter. Integrating out these massive fields, we obtain the following effective twisted superpotential

$$\begin{aligned} W = 2\pi i \sum_{I=1}^{M-1} \tau_I(\mu)\Big(\sum_{s=1}^{k_I} \sigma_s^{(I)}\Big) - \sum_{I=1}^{M-2} \sum_{s=1}^{k_I} \sum_{t=1}^{k_{I+1}} \varpi\big(\sigma_s^{(I)} - \sigma_t^{(I+1)}\big) \\ - \sum_{s=1}^{k_{M-1}} \Big\langle \text{Tr}\, \varpi\big(\sigma_s^{(M-1)} - \Phi\big)\Big\rangle , \end{aligned} \tag{3.14}$$

where $\varpi(x)$ is the function defined in (2.22). As done in the $SU(2)$ theory example, the RG running of the τ_I can be used again to set for the scale Λ_I where $\tau_I(\Lambda_I) = 0$ and the effective twisted superpotential becomes

$$
\begin{aligned}
W = & -\sum_{I=1}^{M-2}\sum_{s=1}^{k_I}\sum_{t=1}^{k_{I+1}}\sigma_s^{(I)}\left(\log\frac{\sigma_s^{(I)}-\sigma_t^{(I+1)}}{\Lambda_I}-1\right)\\
& +\sum_{I=2}^{M-1}\sum_{s=1}^{k_I}\sum_{r=1}^{k_{I-1}}\sigma_s^{(I)}\left(\log\frac{\sigma_r^{(I-1)}-\sigma_s^{(I)}}{\Lambda_I}-1\right)\\
& -\sum_{s=1}^{k_{M-1}}\left\langle \mathrm{Tr}\left[\left(\sigma_s^{(M-1)}-\Phi\right)\left(\log\frac{\sigma_s^{(M-1)}-\Phi}{\Lambda_{M-1}}-1\right)\right]\right\rangle,
\end{aligned}
\tag{3.15}
$$

The term with $I=1$ in the first line above is the result of integrating out the massive fundamental fields attached to the first node of the quiver; the terms with $I=2,\ldots,M-2$ in the first and in the second line of (3.15) are obtained by integrating out the (bi)-fundamental fields between the nodes I and $(I+1)$, and finally, the terms with $I=M-1$ in the second line and the last line of (3.15) are the result of integrating out the fundamental and anti-fundamental fields attached to the last gauge node of the quiver. The angular brackets account for the four-dimensional dynamics of the $SU(N)$ theory. One can easily verify that for $N=M=2$, the expression in (3.15) reduces to (3.3).

The twisted chiral ring

The twisted chiral ring relations are given by

$$
\exp\left(\frac{\partial W}{\partial\sigma_s^{(I)}}\right)=1\,. \tag{3.16}
$$

For $I=1,\ldots,M-2$, the equations are independent of the four-dimensional theory, and read

$$
Q_{I+1}(z)=(-1)^{k_{I-1}}\,\Lambda_I^{n_I+n_{I+1}}Q_{I-1}(z) \tag{3.17}
$$

with a new definiton of

$$Q_I(z) = \prod_{s=1}^{k_I} \left(z - \sigma_s^{(I)} \right) . \tag{3.18}$$

Here $z = \sigma_s^{(I)}$ for each s, and it is understood that $Q_0 = 1$ and $k_0 = 0$. For $I = M-1$, the presence of the four-dimensional $SU(N)$ gauge theory affects the last two-dimensional node of the quiver, and the corresponding chiral ring equation is

$$\exp\left\langle \mathrm{Tr}\log \frac{z-\Phi}{\Lambda_{M-1}} \right\rangle = (-1)^{k_{M-2}} \Lambda_{M-1}^{n_{M-1}+n_M-N} Q_{M-2}(z) \tag{3.19}$$

with $z = \sigma_s^{(M-1)}$ for each s. We now use the fact that the resolvent of the four-dimensional $SU(N)$ theory, which captures all information about the chiral correlators, is given by [2]

$$T(z) := \left\langle \mathrm{Tr} \frac{1}{z-\Phi} \right\rangle = \frac{P_N'(z)}{\sqrt{P_N(z)^2 - 4\Lambda^{2N}}} \tag{3.20}$$

where $P_N(z)$ is the characteristic polynomial of degree N encoding the Coulomb vev's of the $SU(N)$ theory. Since we are primarily interested in the semi-classical solution of the chiral ring equations, we exploit the fact that $P_N(z)$ can be written as a perturbation of the classical gauge polynomial in the following way:

$$P_N(z) = \prod_{i=1}^{N} (z - e_i) \tag{3.21}$$

[2]This result was established in [47] for the $\mathcal{N} = 2$ theory, where the topological property of the chiral correlators $\mathrm{Tr}\langle\Phi^k\rangle$ was used to derive it. In the appendix A.1 we compute the correlators $\mathrm{Tr}\langle\Phi^k\rangle$ for the $SU(2)$ and $SU(3)$ theories using the localization method.

where e_i are the quantum vev's of the pure $SU(N)$ theory given by [46]

$$e_i = a_i - \Lambda^{2N} \frac{\partial}{\partial a_i} \left(\prod_{j \neq i} \frac{1}{(a_i - a_j)^2} \right) + O\left(\Lambda^{4N}\right) \tag{3.22}$$

Integrating the resolvent (3.20) with respect to z and exponentiating the resulting expression, one finds

$$\exp\left\langle \mathrm{Tr} \log \frac{z - \Phi}{\Lambda_{M-1}} \right\rangle = \frac{P_N(z) + \sqrt{P_N(z)^2 - 4\Lambda^{2N}}}{2\Lambda_{M-1}^{N}} \tag{3.23}$$

Using this, we can rewrite the twisted chiral ring relation (3.19) associated to the last node of the quiver in the following form:

$$P_N(z) + \sqrt{P_N(z)^2 - 4\Lambda^{2N}} = 2\,(-1)^{k_{M-2}}\,\Lambda_{M-1}^{n_{M-1}+n_M-N} Q_{M-2}(z)\,, \tag{3.24}$$

where $z = \sigma_s^{(M-1)}$. With further simple manipulations, we obtain

$$P_N(z) = (-1)^{k_{M-2}}\,\Lambda_{M-1}^{n_{M-1}+n_M-N} Q_{M-2}(z) + \frac{\Lambda^{2N}}{(-1)^{k_{M-2}}\,\Lambda_{M-1}^{n_{M-1}+n_M-N} Q_{M-2}(z)}\,. \tag{3.25}$$

In the limit $\Lambda \to 0$ which corresponds to turning off the four-dimensional dynamics, we obtain the expected twisted chiral ring relation of the last two-dimensional node of the quiver. Equations (3.17) and (3.25) are the relevant chiral relations which we are going to solve order by order in the Λ_I's to obtain the weak-coupling expansion of the twisted chiral superpotential.

Solving the chiral ring equations

We now provide a systematic procedure to solve the twisted chiral ring equations we have just derived and to find the effective twisted superpotential of the 2d/4d theory. As illustrated in the simplest $SU(2)$ example, we shall do so by evaluating W on the solutions of the twisted chiral ring equations. Each choice of vacuum therefore corresponds to a different surface operator.

The last point can be clarified by solving the classical chiral ring equations, which are obtained by setting Λ_I and Λ to zero keeping their ratio fixed, *i.e.* by considering the theory at a scale much bigger than Λ_I and Λ. Thus, in this limit the right-hand sides of (3.17) and (3.25) vanish. A possible choice that solve chiral ring equations in this classical limit is:

$$\begin{aligned} \sigma_s^{(1)} &= a_s + \mathscr{O}(\Lambda_I) \qquad \text{for } s = 1,\ldots,k_1 \ , \\ \sigma_t^{(2)} &= a_t + \mathscr{O}(\Lambda_I) \qquad \text{for } t = 1,\ldots,k_2 \ , \\ &\vdots \\ \sigma_w^{(M-1)} &= a_w + \mathscr{O}(\Lambda_I) \qquad \text{for } w = 1,\ldots,k_{M-1} \ . \end{aligned} \tag{3.26}$$

This is equivalent to assuming that the classical expectation value of σ for the I-th node is

$$\sigma^{(I)} = \text{diag}\,(a_1, a_2, \ldots, a_{k_I}) \ . \tag{3.27}$$

This is also the choice appropriate to describe a surface defect that breaks the gauge group according to the Levi decomposition $SU(N) \to S[U(n_1) \otimes U(n_2) \otimes \ldots \otimes U(n_M)]$.

When we turn on the quantum dynamics, we make an ansatz for $\sigma^{(I)}$ as a power series in the various Λ_I's around the chosen classical vacuum. From the explicit expressions (3.17) and (3.25) of the chiral ring equations, it is easy to see that there is a natural set of parameters in terms of which these power series can be written; they are given by

$$q_I = (-1)^{k_{I-1}} \Lambda_I^{n_I + n_{I+1}} \tag{3.28}$$

for $I = 1, \ldots, M-1$. The chiral ring equations (3.25) of the last two-dimensional node of the quiver requires the definition of another parameter. It is related to the four-dimensional scale

Λ and hence to the four-dimensional instanton action. This remaining expansion parameter is

$$q_M = (-1)^N \Lambda^{2N} \left(\prod_{I=1}^{M-1} q_I \right)^{-1} . \tag{3.29}$$

In this thesis our proposal is to solve the chiral ring equations (3.17) and (3.25) as a simultaneous power series in all the q_I's, including q_M, which ultimately will be identified with the Nekrasov-like counting parameters in the ramified instanton computations which will be described later in the chapter.

We will explicitly illustrate these ideas in some examples in the next subsection, but first let us look in full generality how the logarithmic derivatives with respect to Λ_I are directly related to the solution $\sigma_\star^{(I)}$ of the twisted chiral ring equations (3.17) and (3.25). At on-shell value *i.e.* when $\frac{\partial W}{\partial \sigma}\Big|_{\sigma_\star} = 0$ the twisted superpotential $W_\star \equiv W(\sigma_\star)$ depends on Λ_I only explicitly. Using the expression of W in (3.15) it is easy to find

$$\Lambda_I \frac{dW_\star}{d\Lambda_I} = \Lambda_I \frac{\partial W_\star}{\partial \Lambda_I}\Big|_{\sigma_\star} = (n_I + n_{I+1}) \operatorname{tr} \sigma_\star^{(I)} . \tag{3.30}$$

This relation written in terms of the parameters q_I defined in (3.28), is as follows

$$q_I \frac{dW_\star}{dq_I} = \operatorname{tr} \sigma_\star^{(I)} . \tag{3.31}$$

The solution $\sigma_\star$ of the chiral ring equations can be expressed as the classical solution (3.26) plus quantum corrections, and we find

$$\begin{aligned} q_1 \frac{dW_\star}{dq_1} &= a_1 + \ldots + a_{n_1} + \text{corr.ns} , \\ q_2 \frac{dW_\star}{dq_2} &= a_2 + \ldots + a_{n_1+n_2} + \text{corr.ns} \end{aligned} \tag{3.32}$$

and so on. This corresponds to a partition of the classical vev's of the $SU(N)$ theory given by

$$\{ \underbrace{a_1, \cdots, a_{n_1}}_{n_1}, \underbrace{a_{n_1+1}, \cdots, a_{n_1+n_2}}_{n_2}, \cdots, \underbrace{a_{k_{M-1}+1}, \cdots, a_N}_{n_M} \}, \tag{3.33}$$

which is interpreted as a breaking of the gauge group $SU(N)$ according to the Levi decomposition.

We now illustrate our method in a few examples.

3.1.3 In $SU(3)$ theory

We consider the surface operators in the $SU(3)$ theory. There are two distinct partitions, namely $[1,2]$ and $[1,1,1]$, which we now discuss in detail.

SU(3)[1,2]

The two-dimensional theory is a $U(1)$ gauge theory with three flavors, represented by the quiver in figure below

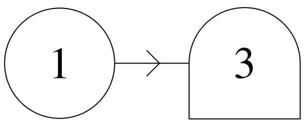


Figure 3.2: The quiver corresponding to the surface operator SU(3)[1,2].

The single chiral ring equation from (3.25) is given by

$$P_3(\sigma) = \Lambda_1^3 + \frac{\Lambda^6}{\Lambda_1^3} \tag{3.34}$$

where the gauge polynomial is defined in (3.21). We solve this equation order by order in Λ_1 and Λ, using the ansatz

$$\sigma_\star = a_1 + \sum_{\ell_1,\ell_2} c_{\ell_1,\ell_2}\, q_1^{\ell_1}\, q_2^{\ell_2} \tag{3.35}$$

where the expansion parameters, defined in (3.28) and (3.29), for the case in hand are

$$q_1 = \Lambda_1^3\,, \qquad q_2 = -\frac{\Lambda^6}{\Lambda_1^3}\,. \tag{3.36}$$

Inserting (3.35) into (3.34), we can recursively determine the coefficients $c_{\ell_1\ell_2}$ and, at the first orders, find the following result

$$\sigma_\star = a_1 + \frac{1}{a_{12}\,a_{13}}\Big(\Lambda_1^3 + \frac{\Lambda^6}{\Lambda_1^3}\Big) - \Big(\frac{1}{a_{12}^3\,a_{13}^2} + \frac{1}{a_{12}^2\,a_{13}^3}\Big)\Big(\Lambda_1^6 + \frac{\Lambda^{12}}{\Lambda_1^6}\Big) + \dots \tag{3.37}$$

where $a_{ij} = a_i - a_j$. According to (3.31), this solution coincides with the q_1-logarithmic derivative of the twisted superpotential. We will verify this statement by comparing (3.37) against the result obtained via localization methods.

SU(3)[1,1,1]

The two-dimensional theory is represented by the quiver below Since $M = 3$, there are now

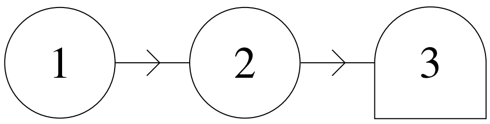


Figure 3.3: The quiver diagram representing the surface operator SU(3)[1,1,1].

two sets of twisted chiral ring equations. For the first node, from (3.17) we find

$$\prod_{s=1}^{2}\big(\sigma^{(1)} - \sigma_s^{(2)}\big) = \Lambda_1^2\,, \tag{3.38}$$

while for the second node, from (3.25) we get

$$P_3\big(\sigma_s^{(2)}\big) = -\Lambda_2^2\left(\sigma_s^{(2)} - \sigma^{(1)}\right) - \frac{\Lambda^6}{\Lambda_2^2\left(\sigma_s^{(2)} - \sigma^{(1)}\right)} \tag{3.39}$$

for $s = 1,2$. This configuration corresponds to a surface operator specified by the partition of the Coulomb vev's $\{\{a_1\},\{a_2\},\{a_3\}\}$. The ansatz for solving the quantum equations (3.38) and (3.39) takes the following form:

$$\begin{aligned}
\sigma_\star^{(1)} &= a_1 + \sum_{\ell_1,\ell_2,\ell_3} d_{\ell_1,\ell_2,\ell_3}\, q_1^{\ell_1} q_2^{\ell_2} q_3^{\ell_3}\ , \\
\sigma_{\star,1}^{(2)} &= a_1 + \sum_{\ell_1,\ell_2,\ell_3} f_{\ell_1,\ell_2,\ell_3}\, q_1^{\ell_1} q_2^{\ell_2} q_3^{\ell_3}\ , \\
\sigma_{\star,2}^{(2)} &= a_2 + \sum_{\ell_1,\ell_2,\ell_3} g_{\ell_1,\ell_2,\ell_3}\, q_1^{\ell_1} q_2^{\ell_2} q_3^{\ell_3}\ ,
\end{aligned} \tag{3.40}$$

with

$$q_1 = \Lambda_1^2\ , \quad q_2 = -\Lambda_2^2\ , \quad q_3 = \frac{\Lambda^6}{\Lambda_1^2\Lambda_2^2}\ . \tag{3.41}$$

Solving the coupled equations (3.38) and (3.39) order by order in q_I, we find the following result:

$$\begin{aligned}
\sigma_\star^{(1)} = a_1 &+ \frac{1}{a_{12}}\Lambda_1^2 + \frac{1}{a_{13}}\frac{\Lambda^6}{\Lambda_1^2\Lambda_2^2} - \frac{1}{a_{12}^3}\Lambda_1^4 - \frac{1}{a_{13}^3}\frac{\Lambda^{12}}{\Lambda_1^4\Lambda_2^4} \\
&- \frac{1}{a_{12}\,a_{13}\,a_{23}}\left(\Lambda_1^2\Lambda_2^2 - \frac{\Lambda^6}{\Lambda_1^2}\right) + \dots\ ,
\end{aligned} \tag{3.42}$$

$$\begin{aligned}
\mathrm{Tr}\sigma_\star^{(2)} = a_1 + a_2 &- \frac{1}{a_{23}}\Lambda_2^2 + \frac{1}{a_{13}}\frac{\Lambda^6}{\Lambda_1^2\Lambda_2^2} - \frac{1}{a_{23}^3}\Lambda_2^4 - \frac{1}{a_{13}^3}\frac{\Lambda^{12}}{\Lambda_1^4\Lambda_2^4} \\
&- \frac{1}{a_{12}\,a_{13}\,a_{23}}\left(\Lambda_1^2\Lambda_2^2 + \frac{\Lambda^6}{\Lambda_2^2}\right) + \dots\ .
\end{aligned} \tag{3.43}$$

According to (3.31) these expressions should be identified, respectively, with the q_1- and q_2-

logarithmic derivatives of the twisted superpotential. We will verify this relation later in the chapter using localization.

Above we have exhibited our method in detail for the SU(3) theory, similarly, surface defects can be analyzed in the higher rank $SU(N)$ theories. This method of solving the twisted chiral ring equations has proved to be very efficient and it quickly leads to very explicit results. One important feature of this approach is the choice of classical extrema of the twisted superpotential which has allowed to make direct contact with the localization calculations of the superpotential for Gukov-Witten defects in four-dimensional gauge theories. A further essential ingredient is the use of the quantum corrected resolvent in four dimensions, which plays a crucial role in obtaining the higher-order solutions of the twisted chiral ring equations of the two-dimensional quiver theory.

3.2 Localization in 4d

In this section we treat the surface operators as monodromy defects D. Four-dimensional manifold $\mathbb{R}^4 \simeq \mathbb{C}^2$ is parametrized by complex variables (z_1, z_2) and the location of D is at $z_2 = 0$. The presence of the surface operator induces a singular behavior in the gauge connection A as discussed in the previous chapter in (2.3). A detailed derivation of the localization results for a generic surface operator has been given in [20, 21], following some earlier mathematical work in [48–50]. The resulting formula of instanton partition function for a surface operator in (2.6) we had described in the previous chapter.

The contribution to the partition function coming from the one-instanton sector is of the form

$$Z_{1-\text{inst}} = -\sum_{I=1}^{M} \int \frac{d\chi_I}{2\pi i} \frac{q_I}{\varepsilon_1} \prod_{s=1}^{n_I} \frac{1}{\left(a_{I,s} - \chi_I + \frac{1}{2}(\varepsilon_1 + \hat{\varepsilon}_2)\right)} \prod_{t=1}^{n_{I+1}} \frac{1}{\left(\chi_I - a_{I+1,t} + \frac{1}{2}(\varepsilon_1 + \hat{\varepsilon}_2)\right)} . \quad (3.44)$$

This is a sum over M terms, each of which has $d_I = 1$ for $I = 1,\ldots,M$. Since the partition function is dimensionless and χ_I carries the dimension of a mass, we deduce that mass dimension of q_I is

$$\big[q_I\big] = n_I + n_{I+1} = b_I\,. \tag{3.45}$$

Another important dimensional constraint follows once we extract the non-perturbative contributions to the prepotential F and to the twisted effective superpotential W from Z_{inst} using

$$\log Z_{\text{inst}} = -\frac{\mathscr{F}_{\text{inst}}\,(a_i,\Lambda)}{\varepsilon_1\hat{\varepsilon}_2} + \frac{W_{\text{inst}}\,(a_i,q_i,\Lambda)}{\varepsilon_1} + \text{ regular terms}\,. \tag{3.46}$$

The prepotential extracted this way depends only on the product of all the q_I, and the instanton contributions to F are organized at weak coupling as a power series expansion in Λ^{2N} where Λ is the dynamically generated scale of the four-dimensional theory and $2N$ is the one-loop coeffcient of the gauge coupling β-function. Thus, it is natural to write

$$\prod_{I=1}^{M} q_I = (-1)^N \Lambda^{2N}\,. \tag{3.47}$$

The mass-dimensions (3.45) attributed to each of the q_I are perfectly consistent with this relation, since the integers n_I form a partition of N. We therefore find that we can use exactly the same parametrization used in the effective field theory and given in (3.28) and (3.29), which we rewrite here for convenience

$$\begin{aligned} q_I &= (-1)^{k_{I-1}}\,\Lambda_I^{n_I+n_{I+1}} \quad \text{for} \quad I = 1,\ldots,M-1\,,\\ q_M &= (-1)^N\,\Lambda^{2N}\Big(\prod_{I=1}^{M-1} q_I\Big)^{-1}\,. \end{aligned} \tag{3.48}$$

The integrations over χ_I in (2.6) and (3.44) have to be suitably defined and regularized, and we will describe this in detail, next.

3.2.1 Residues and contour prescriptions

The standard prescription to evaluate the integrals is to consider $a_{I,s}$ to be real and then close the integration contours in the upper-half $\chi_{I,\sigma}$-planes with the choice

$$\mathrm{Im}\,\hat{\varepsilon}_2 \gg \mathrm{Im}\,\varepsilon_1 > 0 \,. \tag{3.49}$$

With this prescription the multi-dimensional integrals receive contributions from a subset of poles of $z_{\{d_I\}}$, which are in one-to-one correspondence with a set of Young diagrams $Y = \{Y_{I,s}\}$, with $I = 1,\ldots,M$ and $s = 1,\ldots n$.

We briefly illustrate this for $SU(2)$, for which there is only one allowed partition, namely $[1,1]$, and hence one single surface operator to consider. In Tab. 3.1 we list the explicit results for this case, including the location of the poles and the contribution due to all the relevant Young tableaux configurations up to two boxes.

weight	poles	Y	Z_Y
q_1	$\chi_{1,1} = a + \frac{1}{2}(\varepsilon_1 + \hat{\varepsilon}_2)$	$(\square, \bullet)$	$\frac{1}{\varepsilon_1(2a+\varepsilon_1+\hat{\varepsilon}_2)}$
q_2	$\chi_{2,1} = -a + \frac{1}{2}(\varepsilon_1 + \hat{\varepsilon}_2)$	$(\bullet, \square)$	$\frac{1}{\varepsilon_1(-2a+\varepsilon_1+\hat{\varepsilon}_2)}$
$q_1 q_2$	$\chi_{1,1} = a + \frac{1}{2}(\varepsilon_1 + \hat{\varepsilon}_2)$ $\chi_{2,1} = -a + \frac{1}{2}(\varepsilon_1 + \hat{\varepsilon}_2)$	$(\square, \square)$	$-\frac{1}{\varepsilon_1^2\left(4a^2-\hat{\varepsilon}_2^2\right)}$
$q_1 q_2$	$\chi_{1,1} = a + \frac{1}{2}(\varepsilon_1 + \hat{\varepsilon}_2)$ $\chi_{2,1} = \chi_{1,1} + \hat{\varepsilon}_2$	$(\square\square, \bullet)$	$-\frac{1}{2\varepsilon_1\hat{\varepsilon}_2(2a+\hat{\varepsilon}_2)(2a+\varepsilon_1+\hat{\varepsilon}_2)}$
$q_1 q_2$	$\chi_{2,1} = -a + \frac{1}{2}(\varepsilon_1 + \hat{\varepsilon}_2)$ $\chi_{1,1} = \chi_{2,1} + \hat{\varepsilon}_2$	$(\bullet, \square\square)$	$-\frac{1}{2\varepsilon_1\hat{\varepsilon}_2(\hat{\varepsilon}_2-2a)(-2a+\varepsilon_1+\hat{\varepsilon}_2)}$
q_1^2	$\chi_{1,1} = a + \frac{1}{2}(\varepsilon_1 + \hat{\varepsilon}_2)$ $\chi_{1,2} = \chi_{1,1} + \varepsilon_1$	$\left(\begin{smallmatrix}\square\\\square\end{smallmatrix}, \bullet\right)$	$\frac{1}{2\varepsilon_1^2(2a+\varepsilon_1+\hat{\varepsilon}_2)(2a+2\varepsilon_1+\hat{\varepsilon}_2)}$
q_2^2	$\chi_{2,1} = -a + \frac{1}{2}(\varepsilon_1 + \hat{\varepsilon}_2)$ $\chi_{2,2} = \chi_{2,1} + \varepsilon_1$	$\left(\bullet, \begin{smallmatrix}\square\\\square\end{smallmatrix}\right)$	$\frac{1}{2\varepsilon_1^2(-2a+\varepsilon_1+\hat{\varepsilon}_2)(-2a+2\varepsilon_1+\hat{\varepsilon}_2)}$

Table 3.1: We list the weight factors, the locations of the poles, the corresponding Young diagrams, and the contribution to the partition function in all cases up to two boxes for the SU(2) theory. Here we have set $a_1 = -a_2 = a$.

The instanton partition function takes the following form

$$
\begin{aligned}
Z_{\text{inst}}[1,1] = 1 &+ \frac{q_1}{\varepsilon_1\,(2a+\varepsilon_1+\hat{\varepsilon}_2)} + \frac{q_2}{\varepsilon_1\,(-2a+\varepsilon_1+\hat{\varepsilon}_2)} \\
&+ \frac{q_1^2}{2\varepsilon_1^2\,(2a+\varepsilon_1+\hat{\varepsilon}_2)\,(2a+2\varepsilon_1+\hat{\varepsilon}_2)} + \frac{q_2^2}{2\varepsilon_1^2\,(-2a+\varepsilon_1+\hat{\varepsilon}_2)\,(-2a+2\varepsilon_1+\hat{\varepsilon}_2)} \\
&+ q_1 q_2\,\frac{\varepsilon_1+\hat{\varepsilon}_2}{\varepsilon_1^2\hat{\varepsilon}_2\,(-2a+\varepsilon_1+\hat{\varepsilon}_2)\,(2a+\varepsilon_1+\hat{\varepsilon}_2)} + \dots
\end{aligned}
\tag{3.50}
$$

The prepotential and the twisted effective superpotential are extracted according to (3.46) and using the map (3.48). For the twisted superpotential W_{inst}, we find the expression

$$
q_1\frac{dW_{\text{inst}}}{dq_1} = \frac{1}{2a}\Big(\Lambda_1^2+\frac{\Lambda^4}{\Lambda_1^2}\Big) - \frac{1}{8a^3}\Big(\Lambda_1^4+\frac{\Lambda^8}{\Lambda_1^4}\Big) + \dots\,. \tag{3.51}
$$

This precisely matches, up to two instantons, the non-perturbative part of the result (3.10) obtained by solving the twisted chiral ring equations for the quiver theory representing the surface defect in $SU(2)$. The agreement at higher instanton orders can also be checked.

However, there are many other possible choices of contours that one can make. One way to classify these distinct contours is using the Jeffrey-Kirwan (JK) prescription [51]. In this way of computation, the set of poles chosen for the residues is described by a JK parameter η, which is a particular linear combination of the $\chi_{I,s}$; the prescription chooses a set of factors $\mathscr{D}$ from the denominator of $z_{\{d_I\}}$ such that, if we only consider the $\chi_{I,s}$-dependent terms of these chosen factors, then, η can be written as a positive linear combination of these. The prescription in (3.49) corresponds to choosing

$$
\eta = -\sum_{I=1}^{M}\chi_I \tag{3.52}
$$

A detailed discussion of this method in the context of ramified instantons is given in [44] and [53] where different JK prescriptions are shown to be mapped to different quiver realizations(related to each other by Seiberg duality) of the surface operator.

For example the prescription corresponding to a JK parameter of the form

$$\eta = -\sum_{I=1}^{M-1} \zeta_I \chi_I + \zeta\, \chi_M\,, \tag{3.53}$$

where the first three ζ_I are the real part of the parameters $\mathrm{Re}(\tau_I(\mu))$ of the three 2d nodes which are related to the magnitude of the corresponding strong coupling scales Λ_I since

$$\left|\frac{\Lambda_I}{\mu}\right|^{n_I+n_{I+1}} = e^{-2\pi\zeta_I}\,, \tag{3.54}$$

and ζ is some positive large number. In this notation, this corresponds to closing the integration contours in the upper half-plane as before for the first $(M-1)$ variables, and in the lower half plane for χ_M . Applying this new prescription to the $SU(2)$ theory, we find a different set of poles that contribute. They are explicitly listed in Tab. 3.2.

weight	poles	Z_Y
q_1	$\chi_{1,1} = a + \frac{1}{2}(\varepsilon_1 + \hat{\varepsilon}_2)$	$\frac{1}{\varepsilon_1(2a+\varepsilon_1+\hat{\varepsilon}_2)}$
q_2	$\chi_{2,1} = a - \frac{1}{2}(\varepsilon_1 + \hat{\varepsilon}_2)$	$\frac{1}{\varepsilon_1(-2a+\varepsilon_1+\hat{\varepsilon}_2)}$
$q_1 q_2$	$\chi_{1,1} = \chi_{2,1} + \hat{\varepsilon}_2$ $\chi_{2,1} = -a - \frac{1}{2}(\varepsilon_1 + 3\hat{\varepsilon}_2)$	$-\frac{1}{2\varepsilon_1\hat{\varepsilon}_2(2a+\hat{\varepsilon}_2)(2a+\varepsilon_1+\hat{\varepsilon}_2)}$
$q_1 q_2$	$\chi_{1,1} = \chi_{2,1} + \hat{\varepsilon}_2$ $\chi_{2,1} = a - \frac{1}{2}(\varepsilon_1 + \hat{\varepsilon}_2)$	$\frac{\varepsilon_1+2\hat{\varepsilon}_2}{2\varepsilon_1^2\hat{\varepsilon}_2(2a+\hat{\varepsilon}_2)(-2a+\varepsilon_1+\hat{\varepsilon}_2)}$
q_1^2	$\chi_{1,1} = a_1 + \frac{1}{2}(\varepsilon_1 + \hat{\varepsilon}_2)$ $\chi_{1,2} = \chi_{1,1} + \varepsilon_1$	$\frac{1}{2\varepsilon_1^2(2a+\varepsilon_1+\hat{\varepsilon}_2)(2a+2\varepsilon_1+\hat{\varepsilon}_2)}$
q_2^2	$\chi_{2,1} = a - \frac{1}{2}(\varepsilon_1 + \hat{\varepsilon}_2)$ $\chi_{2,2} = \chi_{2,1} - \varepsilon_1$	$\frac{1}{2\varepsilon_1^2(-2a+\varepsilon_1+\hat{\varepsilon}_2)(-2a+2\varepsilon_1+\hat{\varepsilon}_2)}$

Table 3.2: We list the weight factors, the pole structure and the contribution to the partition function in all cases up to two boxes for the SU(2) theory using the contour prescription corresponding to the JK parameter (3.53).

Comparing with Tab. 3.1, we see that the only set of residues that give a seemingly different answer is the one with $d_1 = d_2 = 1$ with weight $q_1 q_2$. As opposed to the earlier case, where there were three contributions, now there are only two terms proportional to $q_1 q_2$. However, it is easy to see that if we sum these contributions, we find an exact match between the two prescriptions. This fact does not come as a surprise since it is a simple consequence of the residue theorem applied to the χ_2 integral. Therefore, all results that follow from the instanton partition function (and in particular the twisted superpotential) are the same in the two cases. But this feature play a fundamental role in the 3d/5d extension, which we also discuss later in [17], and in [52] with greater details.

Next we list our findings obtained by using the second residue prescription in (3.53) for the $SU(3)$ theory, we give the answer upto two-ramified-instanton terms. In the case of the surface operator corresponding to the partition $[1,2]$ we get

$$
\begin{aligned}
Z_{\text{inst}}[1,2] = 1 &+ \frac{q_1}{\varepsilon_1 (a_{12}+\varepsilon_1+\hat{\varepsilon}_2)(a_{13}+\varepsilon_1+\hat{\varepsilon}_2)} + \frac{q_2}{\varepsilon_1 (a_{21}+\varepsilon_1+\hat{\varepsilon}_2)(a_{31}+\varepsilon_1+\hat{\varepsilon}_2)} \\
&+ \frac{q_1^2}{2\varepsilon_1^2 (\varepsilon_1+\varepsilon_2+a_{12})(2\varepsilon_1+\varepsilon_2+a_{12})(\varepsilon_1+\varepsilon_2+a_{13})(2\varepsilon_1+\varepsilon_2+a_{13})} \\
&+ \frac{q_2^2}{2\varepsilon_1^2 (\varepsilon_1+\hat{\varepsilon}_2-a_{12})(2\varepsilon_1+\hat{\varepsilon}_2-a_{12})(\varepsilon_1+\hat{\varepsilon}_2-a_{13})(2\varepsilon_1+\hat{\varepsilon}_2-a_{13})} \\
&+ \frac{(\varepsilon_1+\hat{\varepsilon}_2)^2(3\varepsilon_1+4\hat{\varepsilon}_2) - 2\varepsilon_1 a_1^2 - (\varepsilon_1+\hat{\varepsilon}_2)\left(a_2^2+a_3^2\right)}{\varepsilon_1^2\hat{\varepsilon}_2 \left((\varepsilon_1+\hat{\varepsilon}_2)^2 - a_{12}^2\right)\left((\varepsilon_1+\hat{\varepsilon}_2)^2 - a_{13}^2\right)\left((\varepsilon_1+2\hat{\varepsilon}_2)^2 - a_{23}^2\right)} q_1 q_2 + \dots\,.
\end{aligned}
\tag{3.55}
$$

while for the surface operator described by the partition $[1,1,1]$ we obtain

$$
\begin{aligned}
Z_{\text{inst}}[1,1,1] = 1 &+ \frac{q_1}{\varepsilon_1 (a_{12}+\varepsilon_1+\hat{\varepsilon}_2)} + \frac{q_2}{\varepsilon_1 (a_{23}+\varepsilon_1+\hat{\varepsilon}_2)} + \frac{q_3}{\varepsilon_1 (a_{31}+\varepsilon_1+\hat{\varepsilon}_2)} \\
&+ \frac{q_1^2}{2\varepsilon_1^2 (\varepsilon_1+\varepsilon_2+a_{12})(2\varepsilon_1+\varepsilon_2+a_{12})} + \frac{q_2^2}{2\varepsilon_1^2 (\varepsilon_1+\varepsilon_2+a_{23})(2\varepsilon_1+\varepsilon_2+a_{23})} \\
&+ \frac{q_3^2}{2\varepsilon_1^2 (\varepsilon_1+\varepsilon_2+a_{31})(2\varepsilon_1+\varepsilon_2+a_{31})} + \frac{(2(\varepsilon_1+\varepsilon_2)+a_{13})\, q_1 q_2}{\varepsilon_1^2 (\varepsilon_1+\varepsilon_2+a_{12})(\varepsilon_1+2\varepsilon_2+a_{13})(\varepsilon_1+\varepsilon_2+a_{23})} \\
&+ \frac{(2(\varepsilon_1+\varepsilon_2)-a_{23})\, q_1 q_3}{\varepsilon_1^2 (\varepsilon_1+\varepsilon_2+a_{12})(\varepsilon_1+\varepsilon_2-a_{13})(\varepsilon_1+2\varepsilon_2-a_{23})} \\
&+ \frac{(2(\varepsilon_1+\varepsilon_2)-a_{12})\, q_2 q_3}{\varepsilon_1^2 (\varepsilon_1+2\varepsilon_2-a_{12})(\varepsilon_1+\varepsilon_2-a_{13})(\varepsilon_1+\varepsilon_2+a_{23})} + \dots\,.
\end{aligned}
\tag{3.56}
$$

W_{inst} can be extracted by applying (3.46) and we find that q_I-logarithmic derivatives of the twisted superpotential for the two partitions perfectly match the non-perturbative pieces of the solutions (3.37) and (3.42), (3.43), for the respective surface operators.

To summarize, in this chapter we looked at the two ways in which surface operators can be studied in 4d $\mathcal{N}=2$ $SU(N)$ pure gauge theories. In the first description surface operators are studied as a coupled 2d/4d systems (also referred as flavor defects) where a two dimensional gauge theory with $SU(N)$ flavor symmetry that is gauged in 4 dimensions is considered to be defined on a 2d submanifold. In the second description surface operators are introduced in the gauge theory by giving a prescribed singular behaviour to the gauge fields near the location of the defect. We study the ramified instanton partition function in this second part from [20, 21] and analyze it by considering various contour prescriptions for the integration over moduli parameters χ_I. We extract the effective twisted superpotential which governs the dynamics due to 2d defect. We show a map between the parameters in both the descriptions in (3.48), and in (3.26) we establish a one-to-one correspondence between the choice of massive vacua in the two dimensional theory and the monodromy defects of the $SU(N)$ gauge theory labelled by the partition $[n_1, \ldots, n_M]$ with $n_1 + \ldots + n_M = N$. And this helps us in getting a match of the effective twisted superpotential computed from the two descriptions of surface operator establishing their equivalence.

The equivalence of the effective-twisted superpotential was also discussed in the [19] where using $M5$ branes and $M2$ branes constructions it was proposed that $6d$ $(2,0)$ gauge theories, one with the codimension-2 defects and another codimension-4 defects (after dimensional reductions they become the two types of surface operators in the 4d theory) are described by the same physics in the IR regime, i.e. they are related by an IR duality.

Chapter 4

Surface operators in $\mathcal{N}=4$ theories

In this chapter we study some 1/8-BPS string solutions in the $\mathcal{N}=4$ Super Yang-Mills theory on $S^3\times\mathbb{R}$ manifold, which have co-dimension two singularity. We derive a simple characterisation for these solutions and make the variational problem for them well defined so that their treatment become on par with the regular solutions of BPS equations. In the second part of the chapter we look at the holographic duals of the strings solution in $\mathcal{N}=4$ SYM theory and do the similar analysis to complete.

4.1 $1/8$-BPS strings

We begin with the action of $\mathcal{N}=4$ Yang-Mills theory on $\mathbb{R}\times S^3$.

$$S=\frac{1}{g_{YM}^2}\int d^4x\,e\,\mathrm{Tr}\bigg(-\frac{1}{4}F_{ab}^2-\frac{1}{2}D_aX_{AB}D^aX^{AB}+\frac{1}{4}[X_{AB},X_{CD}][X^{AB},X^{CD}]\\ -\frac{1}{2}X_{AB}X^{AB}-i\bar{\lambda}_{+A}\Gamma^aD_a\lambda_+^A-\bar{\lambda}_{+A}[X^{AB},\lambda_{-B}]-\bar{\lambda}_-^A[X_{AB},\lambda_+^B]\bigg)\,. \tag{4.1}$$

Here F_{ab} are the field strenth for the gauge connection field A with tangent space indices and λ is the gaugino field. We use the convention in [54], so that X_{AB} denotes the three complex scalars where the notation is such that X is an anti-symmetric matrix given below

$$X = \begin{pmatrix} 0 & Z_3^\dagger & -Z_2^\dagger & Z_1 \\ -Z_3^\dagger & 0 & Z_1^\dagger & Z_2 \\ Z_2^\dagger & -Z_1^\dagger & 0 & Z_3 \\ -Z_1 & -Z_2 & -Z_3 & 0 \end{pmatrix} . \tag{4.2}$$

The entries of this matrix will be denoted X_{AB}, where the indices $A, B \in \{1,2,3,4\}$ are in the fundamental representaion of $\mathrm{SU}(4)$, the R-symmetry group. This is related to the matrix X^{AB} with raised indices by the relation

$$X^{AB} = \frac{1}{2}\varepsilon^{ABCD} X_{CD} \,, \tag{4.3}$$

where the ε is the completely anti-symmetric tensor. We have expressed all vector quantities in terms of the tangent space indices, using the vierbein

$$A_\mu = e^a_\mu A_a \ , \quad \gamma^\mu = e^\mu_a \Gamma^a \ , \quad D_\mu = e^a_\mu D_a \ \text{,etc.} \tag{4.4}$$

4.1.1 Geometry of $\mathbb{R} \times S^3$

We choose the following metric on $\mathbb{R} \times S^3$

$$ds^2 = -d\tau^2 + l^2 \left(d\theta^2 + \cos^2\theta \, d\phi_1^2 + \sin^2\theta \, d\phi_2^2 \right) \tag{4.5}$$

These are the natural coordinates that arise while taking the boundary limit of the bulk metric in (4.80). In our analysis we will also use another coordinates which will be of convenience for studying our solutions. We define the angles $\psi = \phi_1 + \phi_2$, $\varphi = \phi_1 - \phi_2$ and $\vartheta = 2\theta$, ,in terms of which the metric takes the form

$$ds^2 = -d\tau^2 + \frac{l^2}{4}\left((d\psi + \cos\vartheta\, d\varphi)^2 + d\vartheta^2 + \sin^2\vartheta\, d\varphi^2\right) . \tag{4.6}$$

In the above the three sphere is considered as a Hopf fibration over the two sphere where ψ is the Hopf-fibre coordinate and (ϑ, φ) specify the directions of the two sphere. We will use both these two coordinate systems interchangeably.

We use the following frame vielbein for the coordinate system we consider

$$\begin{aligned} e^1 &= \frac{l}{2}(-\sin\psi\, d\vartheta + \cos\psi \sin\vartheta\, d\varphi)\,,\ e^2 = \frac{l}{2}(\cos\psi\, d\vartheta + \sin\psi \sin\vartheta\, d\varphi)\ , \\ e^3 &= \frac{l}{2}(\cos\vartheta\, d\varphi + d\psi)\,, \qquad \text{and} \qquad e^0 = d\tau\,. \end{aligned} \tag{4.7}$$

For the remaining of this section we assume a scaling for the time-like coordinate $\tau \to \frac{\tau}{l}$ and take the length parameter l to be equal to 1. In the next section, where we do the holographic analysis we identify this l with the radius of sphere S^5, also denoted by l, when we reproduce the metric in (4.5) from the AdS_5 metric in the boundary limit.

The $SU(4)$ invariant form of the action and susy variations become apparent after decomposing the ten dimensional Majorana-Weyl spinor in an appropriate manner. We refer the reader to [54] for details and we merely present the results. The ten dimensional spinor is decomposed as follows:

$$\varepsilon = \begin{pmatrix} \varepsilon_+^A \\ \varepsilon_-^A \end{pmatrix} \tag{4.8}$$

where ε_-^A is the charge conjugate of ε_+^A. The $\pm$ subscript indicates the four-dimensional chirality of the spinors, $\gamma_5 \varepsilon_\pm = \pm\varepsilon_\pm$.

The action (4.1) is invariant under the following supersymmetry variations and we shall present these in our SU(4) notation:

$$\begin{aligned}
\delta A_a &= i(\bar{\lambda}_{+A}\Gamma_a\varepsilon_+^A - \bar{\varepsilon}_{+A}\Gamma_a\lambda_+^A) \\
\delta X^{AB} &= i(-\bar{\varepsilon}_-^A\lambda_+^B + \bar{\varepsilon}_-^B\lambda_+^A + \varepsilon^{ABCD}\bar{\lambda}_{+C}\varepsilon_{-D}) \\
\delta\lambda_+^A &= \frac{1}{2}F_{ab}\Gamma^{ab}\varepsilon_+^A + 2D_aX^{AB}\Gamma^a\varepsilon_{-B} + X^{AB}\Gamma^a\nabla_a\varepsilon_{-B} + 2i[X^{AC},X_{CB}]\varepsilon_+^B \\
\delta\lambda_{-A} &= \frac{1}{2}F_{ab}\Gamma^{ab}\varepsilon_{-A} + 2D_aX_{AB}\Gamma^a\varepsilon_+^B + X_{AB}\Gamma^a\nabla_a\varepsilon_+^B + 2i[X_{AC},X^{CB}]\varepsilon_{-B}\,.
\end{aligned} \tag{4.9}$$

The $\varepsilon_{\pm,A}$ are conformal Killing spinors on $\mathbb{R}\times S^3$. The subscript $\pm$ refers to the four dimensional chirality and the SU(4) index A indicates that there are four such spinors of each chirality. Each of the epsilons account for four independent real parameters and thus, the $\mathscr{N}=4$ gauge theory has 32 supersymmetries which can equivalently be encoded in the ten dimensional Majorana-Weyl spinor.

The conformal Killing spinor (CKS) of negative 4d chirality satisfies the following equation [54]

$$\nabla_a\varepsilon_{-A}^{(\pm)} = \pm\frac{i}{2}\Gamma_a\Gamma^0\varepsilon_{-A}^{(\pm)}\,, \tag{4.10}$$

where $\varepsilon_{-A}^{(\pm)}$ satisfy the chirality relation: $i\Gamma_{0123}\varepsilon_{-A}^{(\pm)} = \varepsilon_{-A}^{(\pm)}$. The expression for CKS that solve (4.10) for the frame in (4.7) are

$$\varepsilon_{-A}^{(-)} = e^{-\frac{i\tau}{2}}\eta_A^{(-)}\,, \tag{4.11a}$$

$$\varepsilon_{-A}^{(+)} = N\cdot\eta_A^{(+)} = e^{\frac{i\tau}{2}}\,e^{-\frac{\Gamma_{12}}{2}\psi}\,e^{-\frac{\Gamma_{31}}{2}\vartheta}\,e^{-\frac{\Gamma_{12}}{2}\varphi}\eta_A^{(+)}\,, \tag{4.11b}$$

here $\eta_A^{(\pm)}$ are the constant spinors that satisfy the 4d chirality relation $i\Gamma_{0123}\,\eta_A^{(\pm)} = \eta_A^{(\pm)}$.

4.1.2 $\frac{1}{2}$-BPS Configurations

The following is a (singular) classical configuration:

$$
\begin{aligned}
Z_1 &= \frac{c_1}{\cos\theta\, e^{i\phi_1}} = c_1 \sec\frac{\vartheta}{2}\, e^{-\frac{i}{2}(\varphi+\psi)} \\
F_{ab} &= Z_2 = Z_3 = 0 \,,
\end{aligned}
\tag{4.12}
$$

it preserves the half of the 32 supersymmetries of the four-dimensional Yang-Mills theory. Here c_1 is a Cartan generator of the gauge group, and we have expressed the solution in the two different sets of coordinates. These abelian solutions satisfies the following BPS equations

$$
D_0 Z_1 = 0\,, \quad (D_3 + i) Z_1 = 0\,, \quad (D_1 + i D_2) Z_1 = 0\,. \tag{4.13}
$$

For these purely bosonic and abelian solution, the number of supersymmetry preserved can be checked by putting the gaugino variation in (4.9) to zero.

The susy preserved by the half-BPS solution in (4.12) is determined in terms of the non-zero components of the killing spinor. We find the non-zero components by making use of the BPS constraint $\delta\lambda = 0$ in (4.9) and for the abelian solution in (4.12) it becomes

$$
2 D_a X^{AB} \Gamma^a \varepsilon_{-B} + X^{AB} \Gamma^a \nabla_a \varepsilon_{-B} = 0\,. \tag{4.14}
$$

For the first killing spinor solution in (4.11a), we find the following projections on the constant spinor:

$$
\begin{aligned}
(1 + \Gamma^{03})\, \eta_A^{(-)} &= 0 \quad \text{for} \quad A = 1,4 \\
(1 - \Gamma^{03})\, \eta_A^{(-)} &= 0 \quad \text{for} \quad A = 2,3\,.
\end{aligned}
\tag{4.15}
$$

And for the second killing spinor in (4.11b), we find the following projections on the constant spinor:

$$
\begin{aligned}
(1-\Gamma^{03})\,\eta_A^{(+)} &= 0 \quad \text{for} \quad A = 1,4 \\
(1+\Gamma^{03})\,\eta_A^{(+)} &= 0 \quad \text{for} \quad A = 2,3\ .
\end{aligned}
\tag{4.16}
$$

It can be counted that only half of the components in $\varepsilon^{\pm}_{-A}$ are non-zero and hence the solution (4.12) is half-BPS.

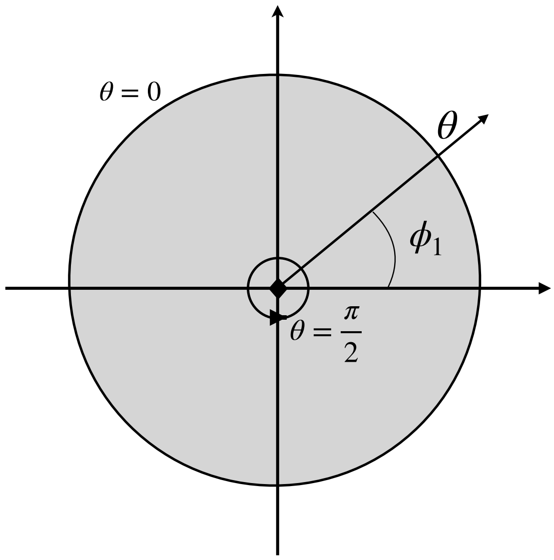


Figure 4.1: The topology of the space transverse to the defect is a disk. At the center of the disk we have $\theta = \frac{\pi}{2}$ and at the boundary of the disk we have $\theta = 0$.

The solution in (4.12) can be interpreted as a monodromy defect on $\mathbb{R} \times S^3$, analogous to the Gukov-Witten defect in $\mathbb{R}^4$. The defect is extended along the (τ, ϕ_2) directions while the two directions transverse to the defect are parametrized by (θ, ϕ_1) coordinates. The transverse space has the topology of a disk, as shown in Figure 4.1. The constant matrix c_1 that appears in the classical solution encodes the (β, γ) parameters that appears in the Gukov-Witten solution.

In the U(N) theory, we can write down the following generalized solution for the scalar profile:

$$Z_1 = \begin{pmatrix} c_{1,1}\,\mathbb{I}_{n_1} & 0 & \cdots & 0 \\ 0 & c_{1,2}\,\mathbb{I}_{n_2} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & c_{1,M}\,\mathbb{I}_{n_M} \end{pmatrix} \frac{1}{\cos\theta\, e^{i\phi_1}}\ . \tag{4.17}$$

In addition, it is possible to turn on an independent parameter for the gauge field that corresponds to a non-trivial holonomy for the gauge field, with $A = \alpha\, d\phi_1$, where α is an element of the Cartan subalgebra that breaks the U(N) to the subgroup U$(n_1)\times$U$(n_2)\times\ldots$U(n_M). The α-parameters encode the monodromy of the four dimensional gauge field around the location of the stringy defect. In the rest of our discussions in both the super-Yang-Mills theory and the holographic bulk theory, we shall not turn these parameters on and focus mostly on the scalar profiles.

More defects in the First class

One can get two more defects in the same class by using an $SU(3)$ rotation change the scalar Z_1 to one of the others, either Z_2 or Z_3. The derivation of the projection conditions follows along the same lines and we present these projection conditions on the killing spinor components below:

- For the defect corresponding to the scalar profile $Z_2 = c_2 \sec\frac{\vartheta}{2}\, e^{-\frac{1}{2}(\varphi+\psi)}$, the half-BPS projections are given by

$$\begin{aligned} (1\pm\Gamma^{03})\,\eta_A^{(\mp)} &= 0 \quad \text{for} \quad A = 2,4 \\ (1\mp\Gamma^{03})\,\eta_A^{(\mp)} &= 0 \quad \text{for} \quad A = 1,3\ . \end{aligned} \tag{4.18}$$

- For the defect corresponding to the scalar profile $Z_3 = c_3 \sec\frac{\vartheta}{2}\, e^{-\frac{1}{2}(\varphi+\psi)}$, we find the following projections:

$$\begin{aligned}(1 \pm \Gamma^{03})\, \eta_A^{(\mp)} &= 0 \quad \text{for} \quad A = 3,4 \\ (1 \mp \Gamma^{03})\, \eta_A^{(\mp)} &= 0 \quad \text{for} \quad A = 1,2\ .\end{aligned} \tag{4.19}$$

Second Class of half-BPS defects

There exist another class of solutions where the location of the singularity differs from the three solutions discussed above and the transverse direction are also different. The singularity profile is given by

$$\begin{aligned}Z_1 &= \frac{c_1}{\sin\theta\, e^{i\phi_2}} = c_1 \csc\frac{\vartheta}{2}\, e^{-\frac{i}{2}(\psi-\varphi)} \\ F_{ab} &= Z_2 = Z_3 = 0\ .\end{aligned} \tag{4.20}$$

As before, the non-zero components in the killing spinor due to the half-BPS property of this solution can be found by putting the constraint $\delta\lambda = 0$ in (4.9). The first set of projections on the spinor $\eta_A^{(-)}$, are identical to the projections in (4.15) for the first class of defects:

$$\begin{aligned}(1 + \Gamma^{03})\, \eta_A^{(-)} &= 0 \quad \text{for} \quad A = 1,4 \\ (1 - \Gamma^{03})\, \eta_A^{(-)} &= 0 \quad \text{for} \quad A = 2,3\ .\end{aligned} \tag{4.21}$$

However, on the $\eta_A^{(+)}$, a different set of supersymmetries is preserved and we find the following projections:

$$\begin{aligned}(1 + \Gamma^{03})\, \eta_A^{(+)} &= 0 \quad \text{for} \quad A = 1,4 \\ (1 - \Gamma^{03})\, \eta_A^{(+)} &= 0 \quad \text{for} \quad A = 2,3\ .\end{aligned} \tag{4.22}$$

	projection on η for $A = I, 4$	projection on η for $A \neq I, 4$
$Z_I = \frac{c_I}{\cos\theta\, e^{i\phi_1}}$	$(1 \pm \Gamma^{03})\, \eta_A^{(\mp)} = 0$	$(1 \mp \Gamma^{03})\, \eta_A^{(\mp)} = 0$
$Z_I = \frac{c_I}{\sin\theta\, e^{i\phi_2}}$	$(1 + \Gamma^{03})\, \eta_A^{(\mp)} = 0$	$(1 - \Gamma^{03})\, \eta_A^{(\mp)} = 0$

Table 4.1: We list the projection conditions on the constant spinor that make half of the components in the killing spinor $\varepsilon_{-A}^{(\pm)}$ zero for the six singular half-BPS solutions we discuss. Here index I takes values $1, 2$ or 3 so that we have three solutions in each class where only one scalar from Z_1, Z_2 or Z_3 is non-zero.

More defects in the Second Class

As before, one can obtain two more defects in the same class by performing an $\mathrm{SU}(3)$ rotation.

- For the defect corresponding to the scalar profile $Z_2 = c_2 \csc\frac{\vartheta}{2}\, e^{-\frac{i}{2}(\psi-\varphi)}$, we find that the half-BPS projections are given by

$$\begin{aligned}(1+\Gamma^{03})\,\eta_A^{(\mp)} &= 0 \quad \text{for} \quad A = 2,4 \\ (1-\Gamma^{03})\,\eta_A^{(\mp)} &= 0 \quad \text{for} \quad A = 1,3\,.\end{aligned} \tag{4.23}$$

- For the defect corresponding to the scalar profile $Z_3 = c_3 \csc\frac{\vartheta}{2}\, e^{-\frac{i}{2}(\psi-\varphi)}$, we find the following projections:

$$\begin{aligned}(1+\Gamma^{03})\,\eta_A^{(\mp)} &= 0 \quad \text{for} \quad A = 3,4 \\ (1-\Gamma^{03})\,\eta_A^{(\mp)} &= 0 \quad \text{for} \quad A = 1,2\,.\end{aligned} \tag{4.24}$$

We have also listed the supersymmetries preserved by all the six BPS solutions discussed so far at one place in a compact form, in the table (4.1) above.

4.1.3 Classical BPS equations

In the previous subsection we have discussed six half-BPS solutions. Together, these six singular solutions preserve two supersymmetries in common which correspond to the projection condition

$$(1+\Gamma^{03})\,\eta_4^{(-)}=0\;. \tag{4.25}$$

The remaining $\eta_A^{(\pm)}$ spinors are projected out. Substituting this projection into the supersymmetry variation of the gaugino $\delta\lambda_+^A$, we obtain the following BPS equations:

$$(D_0+D_3+i)Z_j=0\;,\qquad (D_1+iD_2)Z_j=0\quad\text{for}\quad j=1,2,3\;. \tag{4.26}$$

A similar calculation for the variation $\delta\lambda_{-A}$ leads to the BPS equations:

$$\begin{aligned}&F_{12}+2\sum_{j=1}^{3}\left[Z_j,Z_j^\dagger\right]=0\;,\qquad \left[Z_i,Z_j\right]=0\;,\\ &F_{03}=0\,,\quad F_{01}+F_{31}=0\;,\quad F_{02}+F_{32}=0\;.\end{aligned} \tag{4.27}$$

It is important to emphasize that the D_a are gauge and local Lorentz covariant derivatives on the $\mathbb{R}\times S^3$ background in the frame basis and these are the general non-abelian $\frac{1}{16}$-BPS equations. They were also obtained in [28] where the Bogomolny method of writing the energy of the Yang-Mills on $\mathbb{R}\times S^3$ as a sum of squares, to derive these equations. They were also derived in [29] by performing a supersymmetry analysis of $\frac{1}{16}$-BPS states

Time dependent solution and $\frac{1}{8}$-BPS Equations

We now introduce another class of half-BPS classical configurations, which were very well studied as dual of a dual-giant graviton and is given by

$$Z_1 = c\, e^{-i\tau} \ , \quad \text{and} \quad Z_2 = Z_3 = F_{ab} = 0 \ . \tag{4.28}$$

This time-dependent classical configuration satisfies the differential constraints

$$(D_0 + i)Z_1 = 0 \qquad D_a Z_1 = 0 \ . \tag{4.29}$$

The common supersymmetries preserved by this configuration along with defects in the first and second class in (4.12) and (4.20) that have non-trivial Z_1 profile, are given by the following projections:

$$(1 + \Gamma^{03})\, \eta_A^{(-)} = 0 \quad \text{for} \quad A = 1,4 \ , \tag{4.30}$$

with the other $\eta_A^{(-)}$, for $A = 2,3$ and all the $\eta_A^{(+)}$ set to zero. These leave 4 unbroken supercharges for a $\frac{1}{8}$-BPS configuration.

The $\frac{1}{8}$-BPS equations that can be derived for the supersymmetries left unbroken by the projections in (4.30) are

$$\begin{gathered} (D_0 + D_3 + i)\, Z_1 = 0 \ , \qquad (D_1 + iD_2)\, Z_1 = 0 \ , \\ Z_2 = Z_3 = 0 \ , \qquad F_{12} + 2\left[Z_1, Z_1^\dagger\right] = 0 \ , \\ F_{03} = 0 , \qquad F_{01} + F_{31} = 0 \ , \qquad F_{02} + F_{32} = 0 \ . \end{gathered} \tag{4.31}$$

Equations of Motion and Bianchi Identities

For half-BPS equations in (4.13), (4.29), it turns out that the equations of motion and the Bianchi identities are automatically satisfied. However for the $\frac{1}{8}$-BPS equations in (4.31) the agreement with these set of equations leads to some additional differential constraints.

The equation of motion for the single non-zero scalar field is given by

$$D_a D^a Z + 2[Z,[Z,Z^\dagger]] - Z = 0\ . \tag{4.32}$$

The above is satisfied with no additional constraint if the BPS equations in (4.31) are solved. For the gauge field the equations of motion and Bianchi identities are give by

$$D_a F^{ab} + 2i\left([Z^\dagger, D^b Z] - [D^b Z^\dagger, Z]\right) = 0\ , \qquad D_{[a} F_{bc]} = 0\ . \tag{4.33}$$

We show that the agreement with the above eight equations (four component: equations of motion and the Bianchi identities) require some additional constraints on the $\frac{1}{8}$-BPS solutions from (4.31). The constraint equations are given by

$$(D_0 + D_3 + i)(F_{01} - iF_{02}) = 0\ , \tag{4.34}$$

$$(D_1 + iD_2)\,(F_{01} - iF_{02}) = -4i[D_0 Z^\dagger, Z]\,. \tag{4.35}$$

For the remaining part we shall focus on abelian solutions in the scalar sector in which we set the gauge fields to zero. For these solutions, the differential constraints on F_{ab} which we derived in this section will not play any role. However, there are also defect-like solutions to the $\frac{1}{8}$-BPS equations involving only the gauge field in which we set the scalar field to zero. Such pure glue defects are outside the main focus of our work and we refer the reader to [26] where we discuss the classical solutions and their charges briefly in the Appendix.

4.1.4 Time dependent Wobbling Strings

In the $\frac{1}{8}$-BPS scalar sector, the BPS equations take the simplified form:

$$(D_0 + D_3 + i)Z = 0\ , \quad (D_1 + iD_2)Z = 0\ . \tag{4.36}$$

In terms of the coordinates on the sphere, these differential constraints are given by

$$\big(\partial_\tau + 2\partial_\psi + i\big)Z = 0 \quad \text{and} \quad \big(i\partial_\vartheta + (\csc\vartheta\,\partial_\phi - \cot\vartheta\,\partial_\psi)\big)Z = 0\ . \tag{4.37}$$

The solutions to the above can be written down as a local Laurent series given below

$$Z = \sum_{m,n} a_{m,n} e^{-i(m+n+1)\tau}\left(\cos\frac{\vartheta}{2}e^{\frac{i}{2}(\psi+\phi)}\right)^m\left(\sin\frac{\vartheta}{2}e^{\frac{i}{2}(\psi-\phi)}\right)^n\ . \tag{4.38}$$

We will also obtain a nice characterization of the most general solution to these equations in more general terms and to do so it will be convenient to define

$$v_0 = e^{i\tau}\ , \quad v_1 = \cos\frac{\vartheta}{2}\,e^{\frac{i}{2}(\psi+\phi)}\ , \quad v_2 = \cos\frac{\vartheta}{2}\,e^{\frac{i}{2}(\psi-\phi)}\ . \tag{4.39}$$

Now we can write the general solution in a compact form as follows:

$$Z v_0 = g\left(\frac{v_1}{v_0}, \frac{v_2}{v_0}\right)\ . \tag{4.40}$$

This includes both regular as well as singular solutions depending on the analytic properties of the function $g(z_1, z_2)$. The scalar field is singular at the location of the worldvolume of such a wobbling strings which means that for any given time τ, the scalar field should have a singularity along a one-dimensional path in S^3. To clarify our analysis in further steps it will be useful to introduce the following scale-invariant variables:

$$\zeta_0 = Z v_0 \ , \quad \zeta_1 = \frac{v_1}{v_0} \ , \quad \text{and} \quad \zeta_2 = \frac{v_2}{v_0} \ . \tag{4.41}$$

Thus time translation corresponds simply to scaling the v_i for $i = 1,2$ and Z by a phase. And now we can consider a solution of the BPS equations of the form

$$\zeta_0 F(\zeta_1, \zeta_2) - G(\zeta_1, \zeta_2) = 0 \ . \tag{4.42}$$

Here both F and G are analytic functions of their arguments. At the zeros of the function F, the scalar field has a singularity. The locus of such points is a set $\mathscr{K}$ given by the intersection of

$$F(\zeta_1, \zeta_2) = 0 \quad \text{and} \quad |\zeta_1|^2 + |\zeta_2|^2 = 1 \ . \tag{4.43}$$

Thus we have a simple characterization of the solutions to the $\frac{1}{8}$-BPS equations in the scalar sector that allows for a co-dimension two singularity in the solution for the Z-profile. The solutions to the equations in (4.43) are known to be algebraic links [55]. Thus at a given instant in time, the spatial configuration of the wobbling BPS string corresponds to a link in S^3.

Further, in order to address the singular behaviour of each diagonal entry of the adjoint valued $N \times N$ matrix Z we relax our assumption that ζ_0 is single-valued. Therefore, to obtain such a solution we consider zeros of functions in the scale-invariant variables $H(\zeta_0, \zeta_1, \zeta_2) = 0$. This holomorphic function can at most be of degree N in ζ_0, which is factorizable in ζ_0 and near each of its zeros, the general polynomial would factor into terms of the form in (4.42). In the holographic dual analysis we shall recover this general description of a wobbling string in a very natural way, later in the chapter.

Relation to Gukov-Witten defects on $\mathbb{R}^4$

The coordinate metric on the manifold $S^3 \times \mathbb{R} : -d\tau^2 + d\Omega_3^2$ is related to the metric on $\mathbb{R}^4 \simeq \mathbb{C}^2$: $ds^2_{\mathbb{C}^2} = |dz_1|^2 + |dz_2|^2$ by a wick rotation $\tau \to i\tau_E$ followed by a coordinate transformation so that $(z_1, z_2) = e^{\tau_E}(\cos\theta\, e^{i\phi_1}, \sin\theta\, e^{i\phi_2})$. Therefore the metric $ds^2_{\mathbb{C}^2}$ also has the following form

$$ds^2_{\mathbb{C}^2} = e^{2\tau_E}(d\tau_E^2 + d\Omega_3^2) \ . \tag{4.44}$$

The factor e^{τ_E} is known as the Weyl factor. The scalar fields $Z(z_i, \bar{z}_i)$ in $\mathbb{C}^2$ can be transformed into fields $Z(\tau, \theta, \phi_i)$ on $S^3 \times \mathbb{R}$ by using the fact that these scalars have Weyl weight 1:

$$Z'(x') = \Omega^{-1} Z(x) \ , \tag{4.45}$$

where Ω is the Weyl factor. In chapter 2, we define the Gukov-Witten defect with the topology of a complex plane $\mathbb{C} \subset \mathbb{C}^2$. It is extended along the complex plane parametrized by z_2 and the scalar field Z has a singular profile in the plane transverse to the defect, given by:

$$Z_{\mathbb{C}^2} = \frac{c}{z_1} \tag{4.46}$$

This scalar field solution can be transformed into a solution $Z_{S^3\times\mathbb{R}}(\tau, \theta, \phi_i)$ on $S^3 \times \mathbb{R}$ spacetime by following the steps outlined above. Here we have

$$Z_{S^3\times\mathbb{R}}(\tau, \theta, \phi_i) = (e^{-\tau_E})^{-1} Z_{\mathbb{C}^2}(z_i, \bar{z}_i) \tag{4.47}$$

$$= e^{\tau_E}\frac{c}{z_1} = \frac{c}{\cos\theta\, e^{i\phi_1}}. \tag{4.48}$$

Therefore, we see that our half-BPS string in SYM on $S^3 \times \mathbb{R}$ which we studied in (4.12) maps to the Gukov-Witten solution in SYM on $\mathbb{C}^2$.

Similarly our wobbling string solutions in the theory on $S^3 \times \mathbb{R}$ can be related to configurations in $\mathbb{C}^2$. The solutions in (4.40), $Z_{S^3\times\mathbb{R}} = \frac{1}{v_0} g\left(\frac{v_1}{v_0}, \frac{v_2}{v_0}\right) = \frac{1}{v_0} Z_{\mathbb{C}^2}$, therefore for the scalar profile on $\mathbb{C}^2$, we get:

$$Z_{\mathbb{C}^2} = g(z_1, z_2)\ . \tag{4.49}$$

Thus we conclude that our $\frac{1}{8}$-BPS configurations in SYM on $S^3 \times \mathbb{R}$ translate into $Z = g(z_1, z_2)$ in the Euclidean theory on $\mathbb{R}^4$. Such surface defects preserving less than half of the supersymmetries have been described previously in [14].

4.1.5 Some comments on Wobbling Strings

The $\frac{1}{8}$-BPS strings with relation to the algebraic link solutions in the equation (4.43) which include an important class of solutions when the function $F(z_1, z_2)$ has a singularity structure at the origin. This topological type of the link solution stabilizes near the origin and the intersection is known to give rise to a knot in S^3. For example, for the function

$$F(z_1, z_2) = z_1^p + z_2^q, \tag{4.50}$$

the solution to (4.43) is known (see for instance [59]) to be the torus knot $T_{p,q}$ (for $p, q \geq 2$ with p and q coprime). In [60] surface defects were studied in a four dimensional topological gauge theory with boundary. The surface defect had an embedding such that they end in the 3-dimensional boundary along a 1-d curve K. The curve K could be of non-trivial nature and may be a knot solution like given in (4.50). Therefore, surface operators were used to study the homology of knot invariants in such setup of TQFTs. Also in [61–63], the gauge theory on a four dimensional half space were shown to have solutions of the generalized Bogomolny equations [11] that correspond to codimension two defects with singular boundary conditions along a knot can be useful in studying topological invariants associated to the knot such as the Jones polynomial and play an important role in the programme of categorification [60, 61, 64].

It seems promising that the Hamiltonian analysis of the supersymmetric sector that includes these $\frac{1}{8}$-BPS strings in this physical $\mathcal{N} = 4$ theory on $\mathbb{R} \times S^3$ would also come to be useful in these efforts.

4.1.6 New variational problem with inclusion of Wobbling Strings

Now we discuss two problematic issues when the singular solutions are treated on par with the regular ones. Firstly, they do not belong to the same variational problem $\delta S = 0$. And at the second, they have divergent energies, angular momenta and R-charges.

We show that both these hurdles are handled by cutting off the spacetime arbitrarily close to the singularities of these solutions and adding appropriate boundary terms. In particular we show that for a generic class of singular BPS solutions:

- By adding boundary terms we can make $\delta S = 0$ as we vary along the space of solutions that include regular ones. But now this leaves a lot of ambiguity in the possible boundary terms.
- We show that by demanding that the global charges are rendered finite provides infinitely many conditions on the allowed set of boundary terms with $\delta S = 0$ that essentially fixes the charges uniquely.

On-shell Action and Boundary Terms

In an abelian sector of the theory with a single complex scalar field Z, the theory is described by a lagrangian of conformally coupled complex scalar field on $\mathbb{R} \times S^3$:

$$\mathscr{L} = -\frac{1}{g_{YM}^2}\sqrt{-g}\left[g^{\mu\nu}\partial_\mu Z \partial_\nu \bar{Z} + \bar{Z}Z\right] \ . \tag{4.51}$$

The line element on $S^3 \times \mathbb{R}$ we work with for the rest of the chapter is

$$ds^2 = -d\tau^2 + (d\theta^2 + \cos^2\theta\, d\phi_1^2 + \sin^2\theta\, d\phi_2^2) \ . \tag{4.52}$$

The Lagrangian evaluated on the solutions:

$$Z = e^{i\tau} g\left(\widehat{v}_1, \widehat{v}_2\right), \ \ \bar{Z} = e^{-i\tau}\bar{g}\left(\widehat{\bar{v}}_1, \widehat{\bar{v}}_2\right) \ , \tag{4.53}$$

with $\widehat{v}_i = v_i/v_0$, comes out to be

$$\mathscr{L}\Big|_{\text{onshell}} = \frac{1}{2}\partial_\mu\left(Z\Pi_Z^\mu + \bar{Z}\Pi_{\bar{Z}}^\mu\right) \tag{4.54}$$

where Π_Z^μ are the conjugate momenta: $\Pi_Z^\mu = \frac{\delta\mathscr{L}}{\delta(\partial_\mu Z)}$ and $\Pi_{\bar{Z}}^\mu = \frac{\delta\mathscr{L}}{\delta(\partial_\mu \bar{Z})}$. The argument in the RHS of (4.54) can be written in terms of the function g as follows

$$\begin{aligned} Z\Pi_Z^\theta + \bar{Z}\Pi_{\bar{Z}}^\theta &= -\frac{1}{g_{YM}^2}\cos\theta\sin\theta\left[g\partial_\theta\bar{g} + \bar{g}\,\partial_\theta g\right] \ , \\ Z\Pi_Z^{\phi_1} + \bar{Z}\Pi_{\bar{Z}}^{\phi_1} &= \frac{i}{g_{YM}^2}\tan\theta\left[g\widehat{\bar{v}}_1\partial_{\widehat{\bar{v}}_1}\bar{g} - \bar{g}\,\widehat{v}_1\partial_{\widehat{v}_1}g\right] \ , \\ Z\Pi_Z^{\phi_2} + \bar{Z}\Pi_{\bar{Z}}^{\phi_2} &= \frac{i}{g_{YM}^2}\cot\theta\left[g\widehat{\bar{v}}_2\partial_{\widehat{\bar{v}}_2}\bar{g} - \bar{g}\,\widehat{v}_2\partial_{\widehat{v}_2}g\right] \ , \\ Z\Pi_Z^\tau + \bar{Z}\Pi_{\bar{Z}}^\tau &= \cos^2\theta\,(Z\Pi_Z^{\phi_1} + \bar{Z}\Pi_{\bar{Z}}^{\phi_1}) + \sin^2\theta\,(Z\Pi_Z^{\phi_2} + \bar{Z}\Pi_{\bar{Z}}^{\phi_2}) \ . \end{aligned} \tag{4.55}$$

When the solutions are singular a region around(and arbitrarily close to) is cut-off and the following boundary term is added to the Lagrangian the

$$\mathbf{L}_{bdy} = -\frac{1}{2}\widehat{n}_\mu \left(Z\Pi_Z^\mu + \bar{Z}\Pi_{\bar{Z}}^\mu \right) , \tag{4.56}$$

where $\widehat{n}_\mu$ is the unit outward normal to the boundary. And now the Lagrangian evaluates to zero for all solutions – regular as well as the singular ones – thus making all the solutions belong to the same variational problem.

To fix the energies and other global charges we restrict ourselves to a class of solution where function $g(\widehat{v}_1, \widehat{v}_2) = c_{mn}\widehat{v}_1^m \widehat{v}_2^n$. These are natural (time-dependent) generalizations of the simple (static) surface defect. In terms of the coordinates on $\mathbb{R} \times S^3$ in (4.52), the scalar profile takes the following form:

$$Z = r_0\, e^{i(\xi_0 - \tau)} \left(\cos\theta\, e^{i(\phi_1 - \tau)} \right)^m \left(\sin\theta\, e^{i(\phi_2 - \tau)} \right)^n \tag{4.57}$$

For this 'monomial' class solutions, $Z\Pi_Z^\mu + \bar{Z}\Pi_{\bar{Z}}^\mu$ vanishes for $\mu = \tau, \phi_1, \phi_2$.

The general solutions in (4.53) also satisfy the following constraint equations

$$\begin{aligned} \Pi_Z^\theta + i\cos\theta\sin\theta\,(\Pi_Z^{\phi_1} - \Pi_Z^{\phi_2}) = 0 \,, \\ (\Pi_Z^{\phi_1}\cos^2\theta + \Pi_Z^{\phi_2}\sin^2\theta) - \Pi_Z^\tau - \frac{i}{2}\cos\theta\sin\theta\,\bar{Z} = 0 \,. \end{aligned} \tag{4.58}$$

These are nothing but the consequence of the BPS equations written in field-momentum phase space.

The solution (4.57) is singular at $\theta = 0$ ($\theta = \pi/2$) for negative values of n (m). The Lagrangian density (4.51) evaluated on (4.57) gives:

$$\mathscr{L}(r_0,m,n) = \frac{2r_0^2}{g_{YM}^2}\cos^{2m}\theta\,\sin^{2n}\theta\left[(m+n)(m+n+1)\cos\theta\,\sin\theta - m^2\tan\theta - n^2\cot\theta\right] \tag{4.59}$$

which can be written as

$$\mathscr{L}(r_0,m,n) = \frac{d}{d\theta}\left(\frac{r_0^2}{g_{YM}^2}\cos^{2m+1}\theta\,\sin^{2n+1}\theta\,(m\tan\theta - n\cot\theta)\right)\,. \tag{4.60}$$

This Lagrangian density in (4.59) when integrated over θ between 0 and $\pi/2$ vanishes for non-negative m and n, which in turn means that we have $\delta S = 0$ when we vary along the space of solutions (4.57) by changing the parameters $0 \le r_0, m, n < \infty$. But for $m < 0$ or $n < 0$ this integral here diverges. In particular for $n < 0$ ($m < 0$) the singularities come from $\theta = 0$ ($\theta = \frac{\pi}{2}$) region. As we have discussed in generality, to include these BPS defects into the same variational problem we made a proposal to cut-off the region around the surface defect and add the boundary term in (4.56). For the monomial solutions with $n < 0$, this corresponds to adding

$$\mathbf{L}_{bdy,0+} = \frac{1}{2}(Z\Pi_Z^\theta + \bar{Z}\Pi_{\bar{Z}}^\theta) \tag{4.61}$$

at $\theta = 0 + \varepsilon$, and for those solutions with $m < 0$

$$\mathbf{L}_{bdy,\frac{\pi}{2}-} = -\frac{1}{2}(Z\Pi_Z^\theta + \bar{Z}\Pi_{\bar{Z}}^\theta) \tag{4.62}$$

at $\theta = \frac{\pi}{2} - \varepsilon$.

There is still a lot of freedom left in the possible boundary terms even after making $\delta S = 0$ with a redefined variational probelm. For example, any term that is proportional to the constraints (4.58) can be added:

$$f_{\mathscr{C}}(Z, \Pi_Z^{\mu})\, \mathscr{C}(Z, \Pi_Z^{\mu}) + c.c\,, \tag{4.63}$$

where $\mathscr{C}(Z, \Pi_Z^{\mu})$ is one of the constraints in (4.58) and these terms vanish identically on-shell. Next, we show how this freedom is exploited to regularize the energy and other charges.

4.1.7 Regularization of Charges

The theoretical description of a single conformally coupled scalar we are considering on $S^3 \times \mathbb{R}$ has four conserved global charges E, S_1, S_2, J due to translational invariance under τ, ϕ_1, ϕ_2 coordinates and global $U(1)$ R-symmetry. The Hamiltonian density evaluated on the solution in (4.57) is

$$\mathscr{E}(r_0, m, n) = \frac{2r_0^2}{g_{YM}^2} \cos^{2m}\theta\, \sin^{2n}\theta\, \left[(m+n+1)\cos\theta\, \sin\theta + m^2 \tan\theta + n^2 \cot\theta\right] \tag{4.64}$$

For $m, n \geq 0$ this gives the energy E is equal to :

$$\begin{aligned} E(r_0, m, n) &= 4\pi^2 \frac{2r_0^2}{g_{YM}^2} \frac{(m+n+1)}{2} \frac{\Gamma(1+m)\Gamma(1+n)}{\Gamma(m+n+1)} \\ &= \frac{4\pi^2}{g_{YM}^2} r_0^2 (m+n+1)^2 B(m+1, n+1)\,, \end{aligned} \tag{4.65}$$

where $B(a, b)$ is the Euler Beta function. The other charges for $m, n \geq 0$ can be computed from the stress tensor

$$
\begin{aligned}
J(r_0,m,n) &= \frac{4\pi^2\, r_0^2}{g_{YM}^2}\,(m+n+1)\,B(m+1,n+1),\\
S_1(r_0,m,n) &= -\frac{4\pi^2\, r_0^2}{g_{YM}^2}\,(m+n+1)m\,B(m+1,n+1),\\
S_2(r_0,m,n) &= -\frac{4\pi^2\, r_0^2}{g_{YM}^2}\,(m+n+1)\,n\,B(m+1,n+1)\ .
\end{aligned}
\tag{4.66}
$$

It is easy to check that the charges satisfy the linear BPS relation: $E+S_1+S_2=J$.

For negative m or n we have divergent answers for the energy (E) and the other charges (S_1,S_2,J). In the coming steps we show how to remove the divergences from the energy in a way so that answer in (4.65) becomes valid for negative m or n. We next wish to highlight the nature of these singularities.

The energy density (4.64) can be rewritten in the form:

$$
\begin{aligned}
&\frac{g_{YM}^2}{2\,r_0^2}\mathscr{E}(r_0,m,n) = \cos^{2m}\theta\,\sin^{2n}\theta\,\left[(m+n+1)\,\cos\theta\,\sin\theta + m^2\,\tan\theta + n^2\cot\theta\right]\\
&= (m+n+1)^2\cos^{2m+1}\theta\,\sin^{2n+1}\theta - \frac{d}{d\theta}\left[\frac{1}{2}\cos^{2m}\theta\,\sin^{2n}\theta\,(m\,\sin^2\theta - n\cos^2\theta)\right]\ .
\end{aligned}
\tag{4.67}
$$

Upon integration this becomes

$$
\begin{aligned}
&\frac{g_{YM}^2}{2\,r_0^2}\int d\theta\;\mathscr{E}(r_0,m,n) + \frac{1}{2}\cos^{2m}\theta\,\sin^{2n}\theta\,(m\,\sin^2\theta - n\cos^2\theta)\\
=&(m+n+1)^2\int d\theta\,\cos^{2m+1}\theta\,\sin^{2n+1}\theta\;\;\equiv\;\mathscr{I}
\end{aligned}
\tag{4.68}
$$

which we define to be equal to the notation $\mathscr{I}$. The second term in the second line of (4.67) is responsible for the leading divergence in the bulk contribution to the energy as, near $\theta\to 0$. It diverges as θ^{2n} for $n<0$. A similar power law divergence appears for $m<0$ as $\theta\to\frac{\pi}{2}$. In fact, these divergences get cancelled by the energy contributions from the boundary terms in

(4.61) and (4.62) respectively.

Therefore, after taking into the contribution from the boundary terms: $\mathbf{L}_{bdy,0+}$ in (4.61) (when $n<0$) and $\mathbf{L}_{bdy,\frac{\pi}{2}-}$ in (4.62) (when $m<0$) we have the energy expression after evaluating the integral in (4.68)

$$\mathscr{I}=\begin{cases}-\frac{(m+n+1)^2}{2\,(m+1)}\cos^{2m+2}\theta\,F(1+m,-n,2+m,\cos^2\theta)\ , & \text{if } m\geq 0 \text{ and } n<0.\\ \frac{(m+n+1)^2}{2\,(n+1)}\sin^{2n+2}\theta\,F(1+n,-m,2+n,\sin^2\theta)\ , & \text{if } m<0 \text{ and } n\geq 0.\end{cases} \tag{4.69}$$

Here F denotes the hypergeometric function ${}_2F_1(a,b,c;z)$. We will consider the cases $(m>0,n<0)$ and $(m<0,n>0)$ differently in what follows.

For the $(m>0,n<0)$ cases, the contribution of $\mathscr{I}$ to the energy leads to power law (and in some special cases logarithmic) divergences near $\theta=0$. And for the $(m<0,n>0)$ cases, the contribution of $\mathscr{I}$ to the energy leads to such divergences near $\theta=\frac{\pi}{2}$. In order to cancel these divergences in the energy, in addition to the boundary terms $\mathbf{L}_{bdy,0+}$ in (4.61) or $\mathbf{L}_{bdy,\frac{\pi}{2}-}$ in (4.62), we add the second boundary terms of the form

$$\mathbf{L}'_{bdy}=f(m,n,\theta)\frac{i}{g_{YM}^2}(Z\mathscr{C}-\bar{Z}\bar{\mathscr{C}})\ , \tag{4.70}$$

where $\mathscr{C}$ is one of the constraint we saw in (4.58)

$$\mathscr{C}=(\Pi^{\phi_1}\cos^2\theta+\Pi^{\phi_2}\sin^2\theta)-\Pi^{\tau}-\frac{i}{2}\cos\theta\sin\theta\,\bar{Z}\ , \tag{4.71}$$

and $\bar{\mathscr{C}}$ its complex conjugate and $f(m,n,\theta)$ is a real function of $Z,\bar{Z},\Pi_Z^{\phi_i},\Pi_{\bar{Z}}^{\phi_i}$ obtained by the following replacements:

$$m \longrightarrow -i2g_{YM}^2 \cot\theta \left(\frac{\Pi_Z^{\phi_1}}{\bar{Z}} - \frac{\Pi_{\bar{Z}}^{\phi_1}}{Z} \right), \;\; n \longrightarrow -i2g_{YM}^2 \tan\theta \left(\frac{\Pi_Z^{\phi_2}}{\bar{Z}} - \frac{\Pi_{\bar{Z}}^{\phi_2}}{Z} \right) . \tag{4.72}$$

Such a term does not alter the property that the on-shell action vanishes, but it does contribute to the energy a term proportional to $\frac{2r_0^2}{g_{YM}^2}$ times

$$\frac{1}{2}(m+n+1)\cos^{2m+1}\theta \sin^{2n+1}\theta \, f(m,n,\theta) \tag{4.73}$$

to the energy. Now by using the Pfaff transformation

$$F(a,b,c;z) = (1-z)^{-b} F\left(c-a,b,c,;\frac{z}{z-1}\right) , \tag{4.74}$$

we can write the terms on the r.h.s in (4.69) as:

$$\mathscr{I} = \begin{cases} -\frac{(m+n+1)^2}{2(m+1)} \cos^{2m+2}\theta \sin^{2n}\theta \, F(1,-n,2+m,-\cot^2\theta) \, , & \text{if } m \geq 0 \text{ and } n < 0. \\ \frac{(m+n+1)^2}{2(n+1)} \cos^{2m}\theta \sin^{2n+2}\theta \, F(1,-m,2+n,-\tan^2\theta) \, , & \text{if } m < 0 \text{ and } n \geq 0. \end{cases} \tag{4.75}$$

Next we use the following for the factor $f(m,n,\theta)$ in the second boundary term $\mathbf{L}'_{bdy}$ in (4.70), with a valid justification

$$f(m,n,\theta) = \begin{cases} -\frac{m+n+1}{n+1} \tan\theta \, F(1,-m,2+n,-\tan^2\theta) \, , & \text{if } m \geq 0 \text{ and } n < 0. \\ \frac{m+n+1}{m+1} \cot\theta \, F(1,-n,2+m,-\cot^2\theta) \, , & \text{if } m < 0 \text{ and } n \geq 0. \end{cases} \tag{4.76}$$

One of the reasons for making use of the boundary terms in this form is that the hypergeometric terms have a sensible power series expansion near the respective boundaries. Combining now the bulk and boundary contributions we see that value, the energy of the wobbling string takes, is simply the analytic continuation of the energy value $E(r_0,m,n)$ in (4.65) to negative values of either n or m.

For other charges (S_1, S_2, J), the same set of boundary terms that allow us to include the defects to the class of dual-giant like solutions also regularize these charges, and maintain the linear BPS relation: $E + S_1 + S_2 = J$ between the charges are valid for these solutions as well. We collect our answers for a few cases in the appendix B.1.

This prescription to subtract away the (coordinate-dependent) divergences in the charges of the BPS string solutions is not very different to those methods adopted earlier in the literature which use renormalization procedure for the Euclidean action for Wilson line or surface defect operators to regularize the expectation values of those non-local operators, as done in [12, 13, 56]. And it will become clear from the dual holographic analysis that our prescription here is nothing more than the standard UV renormalization.

Recently, in [58] defect operators of various co-dimensions were constructed and analyzed in a topologically twisted version of $\mathcal{N} = 4$ SYM theory. The defect operators belonged to the cohomology of the chosen Q_{BRST} operator. The Q_{BRST} is nilpotent when restricted to a special sphere submanifold S^2_{YM}, as [58] describes.

4.2 D3-brane Probes in $AdS_5 \times S^5$

The holographic description of the wobbling strings can be given by considering probe D3-brane solutions in global $AdS_5 \times S^5$. In [13, 14] the holographic duals of surface operators that preserve some fraction of the supersymmetry with topology $\mathbb{R}^2 \subset \mathbb{R}^4$ were shown to be described by probe D3-branes ending on the boundary in two dimensional surfaces. In this section we will begin the discussion with the holographic duals of half-BPS strings. We will analyze the probe branes that end on the boundary on surfaces with topology $\mathbb{R} \times S^1 \subset \mathbb{R} \times S^3$.

4.2.1 The type IIB background

The $AdS_5 \times S^5$ background is defined as the following locus in the twelve-dimensional ambient space:

$$-|\Phi_0|^2 + |\Phi_1|^2 + |\Phi_2|^2 = -l^2 \quad \text{and} \quad |Z_1|^2 + |Z_2|^2 + |Z_3|^2 = l^2 . \tag{4.77}$$

Here we use the definition of the coordintes in the complex space:

$$(\Phi_0, \Phi_1, \Phi_2, Z_1, Z_2, Z_3) \in \mathbb{C}^{1,2} \times \mathbb{C}^3 . \tag{4.78}$$

We work with global coordinates in AdS_5 and this corresponds to the parametrization:

$$\begin{aligned} &\Phi_0 = l \cosh\rho \, e^{i\phi_0} \qquad \Phi_1 = l \sinh\rho \, \cos\theta \, e^{i\phi_1} \qquad \Phi_2 = l \sinh\rho \, \sin\theta \, e^{i\phi_2} . \\ &Z_1 = l \sin\alpha \, e^{i\xi_1} \qquad Z_2 = l \cos\alpha \, \sin\beta \, e^{i\xi_2} \qquad Z_3 = l \cos\alpha \, \cos\beta \, e^{i\xi_3} . \end{aligned} \tag{4.79}$$

The metric on $AdS_5 \times S^5$ is then simply inherited from the flat metric of the ambient space and takes the following form in global coordinates:

$$\begin{aligned} \frac{ds^2}{l^2} = &- \cosh^2\rho \, d\phi_0^2 + d\rho^2 + \sinh^2\rho \, (d\theta^2 + \cos^2\theta \, d\phi_1^2 + \sin^2\theta \, d\phi_2^2) \\ &+ d\alpha^2 + \sin^2\alpha \, d\xi_1^2 + \cos^2\alpha \, (d\beta^2 + \sin^2\beta \, d\xi_2^2 + \cos^2\beta \, d\xi_3^2) , \end{aligned} \tag{4.80}$$

where $\phi_0 = \frac{t}{l}$. We choose the frame vielbein that make $U(1)$ Hopf fibration over a Kaehler manifold $\mathbb{CP}^2$(or $\widetilde{\mathbb{CP}}^2$) for the S^5 and AdS_5 parts become apparent. The frame for the AdS_5

part is

$$
\begin{aligned}
&e^0 = l[\cosh^2\rho\, d\phi_0 - \sinh^2\rho\,(\cos^2\theta d\phi_1 + \sin^2\theta d\phi_2)],\\
&e^1 = l\, d\rho\,, \qquad e^2 = l \sinh\rho\, d\theta,\\
&e^3 = l\cosh\rho\sinh\rho\,(\cos^2\theta\ d\phi_{01} + \sin^2\theta\ d\phi_{02})\\
&e^4 = l\sinh\rho\ \cos\theta\sin\theta\ d\phi_{12}
\end{aligned}
\tag{4.81}
$$

where $\phi_{ij} = \phi_i - \phi_j$. For the S^5 part, we choose the frame

$$
\begin{aligned}
&e^5 = l\, d\alpha, \qquad e^6 = l\ \cos\alpha\, d\beta,\\
&e^7 = l\ \cos\alpha\sin\alpha\,(\sin^2\beta\, d\xi_{12} + \cos^2\beta\, d\xi_{13}),\\
&e^8 = l\ \cos\alpha\cos\beta\sin\beta\, d\xi_{23},\\
&e^9 = l\,(\sin^2\alpha\, d\xi_1 + \cos^2\alpha\sin^2\beta\, d\xi_2 + \cos^2\alpha\cos^2\beta\, d\xi_3)
\end{aligned}
\tag{4.82}
$$

where $\xi_{ij} = \xi_i - \xi_j$.

The Killing spinor for the $AdS_5 \times S^5$ background adapted to the above frame is given by [33]:

$$
\begin{aligned}
\varepsilon = {}& e^{-\frac{1}{2}(\Gamma_{79} - i\Gamma_5\tilde{\gamma})\,\alpha}\, e^{-\frac{1}{2}(\Gamma_{89} - i\Gamma_6\tilde{\gamma})\beta}\, e^{\frac{1}{2}\xi_1\Gamma_{57}}\, e^{\frac{1}{2}\xi_2\Gamma_{68}}\, e^{\frac{i}{2}\xi_3\Gamma_9\tilde{\gamma}}\\
&\times e^{\frac{1}{2}\rho\,(\Gamma_{03} + i\Gamma_1\gamma)}\, e^{\frac{1}{2}\theta\,(\Gamma_{12} + \Gamma_{34})}\, e^{\frac{i}{2}\phi_0\Gamma_0\gamma}\, e^{-\frac{1}{2}\phi_1\Gamma_{13}}\, e^{-\frac{1}{2}\phi_2\Gamma_{24}}\,\varepsilon_0 \equiv M\cdot\varepsilon_0\,,
\end{aligned}
\tag{4.83}
$$

where ε_0 is an arbitrary 32-component Weyl spinor satisfying $\Gamma_0\cdots\Gamma_9\varepsilon_0 = -\varepsilon_0$ and we have denoted $\gamma = \Gamma^{01234}$ and $\tilde{\gamma} = \Gamma^{56789}$.

4.2.2 $\frac{1}{2}$-BPS D3-brane Probes

We consider various classes of $\frac{1}{2}$-BPS probe D3-branes which end on the boundary in a two dimensional surface. In the first class, we consider D3-branes described by the equations:

$$\Phi_1 Z_1 = C_1\,, \quad Z_2 = Z_3 = 0\,. \tag{4.84}$$

The embedding equation is inspired by the profile of the complex scalar Z_1 in (4.12). The coefficient that appears in the probe equation and the constant c_1 in the profile of the scalar field (see equation (4.12)) are related in a following way

$$c_1 = \frac{\sqrt{\lambda}}{2\pi} C_1\ , \tag{4.85}$$

where $\lambda = g_{YM}^2 N$ is the 't Hooft coupling of the gauge theory. The relative factor in the normalization is due to an overall factor in the probe D3 brane action that is the tension of the D3 brane given by $T_{D3} = \frac{N}{2\pi^2 l^4}$. This map of parameters is essential to match the energies and charges computed in the bulk and boundary theories, in the leading order expansion in λ.

In terms of the real coordinates introduced in the previous subsection, the embedding of the probe D3-brane is given by the following real conditions:

$$\sinh\rho\,\cos\theta = \frac{R_0}{l}\,, \quad \alpha = \frac{\pi}{2}\,, \quad \phi_1 + \xi_1 = \xi_1^{(0)}\,. \tag{4.86}$$

We have chosen to write the complex constant $C_1 = R_0\, e^{i\xi_1^{(0)}}$ in a particularly convenient manner. We see that as $\rho \to \infty$, we have $\theta \to \frac{\pi}{2}$ so as to keep the first equation consistent, and the D3-brane ends on the circle parametrized by ϕ_1 on the boundary, while being extended along the ϕ_0-direction. The (θ, ϕ_2) coordinates parametrize the directions transverse to the boundary limit of the probe, exactly as for the corresponding string defect. We choose the static gauge in which the world-volume coordinates are identified as follows:

$$(\tau, \sigma_1, \sigma_2, \sigma_3) = (\phi_0, \theta, \phi_1, \phi_2)\,. \tag{4.87}$$

The induced metric on the world-volume, of topology $AdS_3 \times S^1$, is given by:

$$\frac{ds^2}{l^2}\bigg|_{D3} = -\left(\frac{R_0^2 + l^2\cos^2\theta}{l^2\cos^2\theta}\right)d\phi_0^2 + \frac{R_0^2(R_0^2+l^2)\sec^2\theta}{l^2(R_0^2+l^2\cos^2\theta)}d\theta^2 + \frac{R_0^2+l^2}{l^2}d\phi_1^2 + \frac{R_0^2}{l^2}\tan^2\theta d\phi_2^2\,. \tag{4.88}$$

The κ-symmetry analysis

We now classify the set of supersymmetries preserved by this probe D3-brane. The κ-symmetry equation that guarantees the supersymmetry of the worldvolume theory is given by

$$\gamma_{\tau\sigma_1\sigma_2\sigma_3}\varepsilon = \pm i\sqrt{-\det h}\;\varepsilon\ . \tag{4.89}$$

Here, the world-volume γ-matrices are defined by

$$\gamma_i = \mathfrak{e}_i^a\,\Gamma_a\,, \tag{4.90}$$

where the $\mathfrak{e}_i^a = e_\mu^a \partial_i X^\mu$ is obtained by the pullback of the one-form e_μ^a. For the probe D3-brane under consideration in (4.84), the world-volume gamma matrices are as follows:

$$\begin{aligned}
\gamma_\tau &= l\cosh^2\rho\,\Gamma_0 + l\sinh\rho\cosh\rho\,\Gamma_3\ ,\\
\gamma_{\sigma_1} &= l\tanh\rho\,\tan\theta\,\Gamma_1 + l\sinh\rho\,\Gamma_2\ ,\\
\gamma_{\sigma_2} &= -l\sinh^2\rho\cos^2\theta\,\Gamma_0 - l\sinh\rho\cosh\rho\cos^2\theta\,\Gamma_3 + l\sinh\rho\cos\theta\sin\theta\,\Gamma_4 - l\,\Gamma_9\ ,\\
\gamma_{\sigma_3} &= -l\sinh^2\rho\sin^2\theta\,\Gamma_0 - l\sinh\rho\cosh\rho\sin^2\theta\,\Gamma_3 - l\sinh\rho\cos\theta\sin\theta\,\Gamma_4\ .
\end{aligned} \tag{4.91}$$

The product of four γ matrices is

$$\frac{1}{l^4}\gamma_{\tau\sigma_1\sigma_2\sigma_3} = \sinh^2\rho\cosh\rho\left[\big(\sinh\rho(\Gamma_{0234}+\Gamma_{2349}) - \cosh\rho\,\Gamma_{0249}\big)\cos\theta\sin\theta\ -\Gamma_{0239}\sin^2\theta\right]$$
$$+\sinh^2\rho\sin^2\theta\left[\sinh\rho\,(\Gamma_{0134}+\Gamma_{1349}) - \cosh\rho\,\Gamma_{0149} - \tan\theta\,\Gamma_{0139}\right]. \tag{4.92}$$

In order to check the κ-symmetry equation, we need to commute the four-gamma products through the matrix M defined in (4.83). After doing some tedious Γ-matrix commutation algebra through the exponential factor M in (4.83), the κ-symmetry constraint reduces to the following simple expression:

$$\begin{aligned}\frac{1}{l^4}\gamma_{\tau\sigma_1\sigma_2\sigma_3}\cdot M\cdot\varepsilon_0 =& M\cosh\rho\sinh^3\rho\, e^{-i\phi_0\Gamma_0\gamma}e^{\phi_1\Gamma_{13}}(\Gamma_{0234}+\Gamma_{3968})\cdot\varepsilon_0\\ &-iM\sin^2\theta\sinh^4\rho\, e^{\phi_1\Gamma_{13}}e^{\phi_2\Gamma_{24}}(\Gamma_{12}+\Gamma_{014968})\cdot\varepsilon_0\\ &+iM\tan\theta\sinh^2\rho(1+\cos^2\theta\sinh^2\rho)\Gamma_{024968}\cdot\varepsilon_0\,.\end{aligned} \tag{4.93}$$

Using the embedding equation in (4.86) and the 10d chirality constraint, we find that the κ-symmetry constraint in (4.89) is satisfied with the choice of $(-)$ sign if the following projection constraint is imposed on the constant spinor ε_0:

$$\Gamma_{1357}\,\varepsilon_0 = \varepsilon_0\,. \tag{4.94}$$

We have thus shown that the probe D3-brane preserves half of the bulk supersymmetries.

More $\frac{1}{2}$-BPS Probes from $SU(3)$ Rotations

The choice of coordinates and frame in (4.80) and (4.81), (4.82) make possible for us to find other probe $D3$-branes that are closely related to the one we have analyzed so far, and whose supersymmetry can be checked by a minor modification of our previous analysis. These probes are obtained by using an $SU(3)$ rotation acting on the Z_i variables and as a result the induced metric remains the same as in (4.88).

From the κ-symmetry analysis we find that

$$\Phi_1 Z_2 = C_2 \quad \text{and} \quad Z_1 = Z_3 = 0 \tag{4.95}$$

is half-BPS and preserves the supersymmetries that survive the following projection:

$$\Gamma_{1368}\varepsilon_0 = \varepsilon_0\,. \tag{4.96}$$

Similarly the probe D3-brane

$$\Phi_1 Z_3 = C_3 \quad \text{and} \quad Z_1 = Z_2 = 0\,. \tag{4.97}$$

preserves half the supersymmetries if we impose the projection

$$\Gamma_{0924}\varepsilon_0 = i\,\varepsilon_0\,. \tag{4.98}$$

A second class of $\frac{1}{2}$-BPS D3 branes

There exist another set of D3 brane probes that are obtained by an $SU(2)$ rotation that acts on the complex Φ_i variables. The expression for the probe brane solution is

$$\Phi_2 Z_1 = D_1 \quad \text{and} \quad Z_2 = Z_3 = 0\,. \tag{4.99}$$

In terms of the real coordinates we now have the defining equations:

$$\sinh\rho\,\sin\theta = \frac{R_0}{l} \qquad \alpha = \frac{\pi}{2} \qquad \phi_2 + \xi_1 = \xi^{(0)}\,. \tag{4.100}$$

The induced metric on the worldvolume, of $AdS_3 \times S^1$ topology, is given by

$$\frac{ds^2}{l^2}\bigg|_{D3} = -\left(\frac{R_0^2 + l^2 \sin^2\theta}{l^2 \sin^2\theta}\right) d\phi_0^2 + \frac{R_0^2(R_0^2+l^2)\csc^2\theta}{l^2(R_0^2+l^2\sin^2\theta)} d\theta^2 + \frac{R_0^2+l^2}{l^2} d\phi_1^2 + \frac{R_0^2}{l^2}\cot^2\theta d\phi_2^2 \,. \tag{4.101}$$

We have shown the details of the κ-symmetry analysis in the appendix B.2. Here we write the result of the projection condition on constant spinor ε_0

$$\Gamma_{2457}\,\varepsilon_0 = \varepsilon_0 \,. \tag{4.102}$$

On doing an $SU(3)$ rotation on the Z_i variables as before the other two half-BPS probe D3-branes can be obtained. We collect this information in the table (4.2) below

	Probe Solution	projection on ε_0
1.	$\Phi_2 Z_1 = D_1$ and $Z_2 = Z_3 = 0$	$\Gamma_{2457}\,\varepsilon_0 = \varepsilon_0$
2.	$\Phi_2 Z_2 = D_2$ and $Z_3 = Z_1 = 0$	$\Gamma_{2468}\,\varepsilon_0 = \varepsilon_0$
3.	$\Phi_2 Z_3 = D_3$ and $Z_1 = Z_2 = 0$	$\Gamma_{0913}\,\varepsilon_0 = \varepsilon_0$

Table 4.2: We list the probe solutions in the second class along with the projection condition due to the κ-symmetry constraint.

4.2.3 $\frac{1}{16}$-BPS D3-brane Probes

Each of the six different probe D3-branes in the previous subsection that we have classified into two distinct classes depending on whether it wraps the ϕ_1 circle or the ϕ_2 circle on the boundary, preserve half of the ten-dimensional background supersymmetries. Following our analysis of the singular solutions on the boundary theory, we now give the projections that preserve the common set of supersymmetries amongst all these probe D3-branes.

$$\Gamma_{13}\varepsilon_0 = \Gamma_{24}\varepsilon_0 = -i\varepsilon_0\,, \qquad \Gamma_{09}\varepsilon_0 = -\varepsilon_0\,, \qquad \Gamma_{57}\varepsilon_0 = \Gamma_{68}\varepsilon_0 = i\varepsilon_0\,. \tag{4.103}$$

These projections preserve exactly two out of the 32 supersymmetries of the bulk background.

Remarkably, these projection conditions have been encountered previously in the context of studying giant-gravitons and dual giant-gravitons in $AdS_5 \times S^5$ [33]. Hence, we have shown that the set of two supersymmetries that the various probe branes share (and which are dual to stringy defects in the gauge theory), is the same set of supersymmetries shared by the D3-brane probes that describe giants and dual-giants.

In the appendix B.3, we review briefly the derivation of the constraints on the D3-brane worldvolume which was done in [33] from the projections in (4.103). The most general $\frac{1}{16}$-BPS solution to these constraint equations were given by Kim and Lee [32] in terms of three holomorphic functions:

$$F^{(I)}(\Phi_i, Z_j) = 0 \quad \text{for} \quad I = 1,2,3\,, \tag{4.104}$$

where the Φ_i and Z_j are defined in (4.79) and the functions each satisfy a scaling condition:

$$\sum_{i=0}^{2} \partial_{\phi_i} F^{(I)} - \sum_{i=1}^{3} \partial_{\xi_i} F^{(I)} = 0\,. \tag{4.105}$$

There are four sub-classes of solutions to these equations here that preserve $\frac{1}{8}$th of the bulk supersymmetry which were listed in [33], some of these were previously obtained in [31, 57]. They are either point-like in the AdS_5 directions (giants) or point-like in the S^5 (dual-giants) and they carry spins (J_1, J_2, J_3) only along the S^5 or they carried two spins along the AdS_5

directions and one spin along the S^5, which we denote (S_1, S_2, J).

All the particular solutions that were considered in [33] had compact world-volume and none of these extended to the boundary. What we show here is that the same set of BPS equations admit another completely different class of probe D3-branes that have an interpretation as holographic duals of string like defects, and whose world-volume ends on the conformal boundary $\mathbb{R} \times S^3$ along two directions, one of which is the time direction. In the coming part, we present a deeper analysis of the constraints that holomorphy places on the spatial direction of the boundary component of the probe brane.

4.2.4 Boundary profile of the bulk $\frac{1}{8}$-BPS Probes

The solutions from (4.104) which are the holographic duals of the wobbling stringy solutions in the boundary field theory have charges (S_1, S_2, J). Their world-volume which preserve $\frac{1}{8}$th of the bulk susy is described by

$$Z_2 = Z_3 = 0 \quad \text{and} \quad f(Z_1\Phi_0, Z_1\Phi_1, Z_1\Phi_2) = 0 \, . \tag{4.106}$$

The above is invariant under the scaling

$$\Phi_i \to \lambda\, \Phi_i \quad \text{and} \quad Z_1 \to \lambda^{-1} Z_1 \, .$$

In this subsection, we want to show that the zero locus in (4.106) precisely coincides with the locus in the boundary theory where the profile of the scalar field has a singularity. For this purpose, it is useful to use the following for the coordinates $\Phi_i \in \mathbb{C}^{1,2}$ of the ambient space:

$$\Phi_0 = \sqrt{r^2 + l^2}\, v_0, \;\; \Phi_1 = r\, v_1, \;\; \Phi_2 = r\, v_2 \tag{4.107}$$

where $v_0 = e^{i\phi_0}$, $v_1 = \cos\theta\, e^{i\phi_1}$ and $v_2 = \sin\theta\, e^{i\phi_2}$. Near the boundary region of AdS these take the form

$$\Phi_0 = r\, v_0, \;\; \Phi_1 = r\, v_1, \;\; \Phi_2 = r\, v_2\,, \tag{4.108}$$

and become coordinates on a null-cone $-|\Phi_0|^2 + |\Phi_1|^2 + |\Phi_2|^2 = 0$. The induced metric on this cone is of the form what we have seen in for the boundary field theory in (4.52) (of $\mathbb{R} \times S^3$ manifold) times the scaling factor r^2. Also near the boundary the function in (4.106) becomes $f(Z\, r\, v_0, Z\, r v_1, Z\, r\, v_2)$. Therefore, the worldvolume of the D3 brane intersects the boundary at the zeros of the functions

$$f(\lambda\, v_0, \lambda\, v_1, \lambda\, v_2) = 0\ ,$$

where $\lambda = r e^{i\xi}$ for arbitrary $\lambda \in \mathbb{C}^\star$. With such scaling properties, it is possible to show that above zero locus is equivalent to the holomorphic condition

$$F(\zeta_1, \zeta_2) = 0$$

where $\zeta_i = v_i / v_0 \in \mathbb{C}^2$. And thus we reproduce the characterising property of the BPS strings in the boundary theory using the constraining conditions of the D3 world-volume.

We have shown that if the worldvolume of the probe D3-brane is described as the zero locus of an arbitrary holomorphic function $f(Z\Phi_0, Z\Phi_1, Z\Phi_2)$, then we see that for those probes that reach the boundary, the world-volume, as it approaches the boundary is two dimensional and at a given instant in time, it is given by the locus $\mathscr{K}$, which is obtained by the intersection of a holomorphic function in $\mathbb{C}^2$ with the 3-sphere.

$$F(\zeta_1, \zeta_2) = 0 \quad \cap \quad |\zeta_1|^2 + |\zeta_2|^2 = 1\ . \tag{4.109}$$

The curve $\mathscr{K}$ is an algebraic link in S^3. To complete this part of the analysis in the bulk, we need to derive this boundary profile from the zeros of the holomorphic function. The holomorphic function in (4.106) could be written in a manner suggestive of the boundary solutions, as functions of the form $g\left(Z\Phi_0, \Phi_1/\Phi_0, \Phi_2/\Phi_0\right)$.

Since this function g is considered to be a polynomial in the variable $Z\Phi_0$ of degree, say, $p \leq N$, it can be factorised as

$$g\left(Z\Phi_0, \Phi_1/\Phi_0, \Phi_2/\Phi_0\right) = \prod_{r=1}^{p} \left[(Z\Phi_0) F_r^{(1)}\left(\Phi_1/\Phi_0, \Phi_2/\Phi_0\right) - F_r^{(0)}\left(\Phi_1/\Phi_0, \Phi_2/\Phi_0\right) \right] \tag{4.110}$$

From the discussion of the boundary limits of the coordinates Φ_i, we infer that near the boundary, this function becomes

$$g\left(Z\Phi_0, \Phi_1/\Phi_0, \Phi_2/\Phi_0\right) \longrightarrow (\lambda\, v_0)^p \prod_{r=1}^{p} F_r^{(1)}\left(v_1/v_0, v_2/v_0\right) \tag{4.111}$$

Here $\lambda = r e^{i\xi}$ is a field on the probe brane that determines the radial and angular profile of the probe brane and we will identify it with the complex scalar field denoted by Z in the boundary theory. The defects on the boundary are therefore given by zero-sets of $F_r^{(1)}\left(v_1/v_0, v_2/v_0\right)$. So far we reproduced the conclusion of the bulk analysis.

As a first step towards deriving the boundary profile, let us set $p = 1$ for simplicity. Then the bulk solution $g\left(Z\Phi_0, \Phi_1/\Phi_0, \Phi_2/\Phi_0\right)$ is of unit degree in Z:

$$g\left(Z\Phi_0, \Phi_1/\Phi_0, \Phi_2/\Phi_0\right) = Z\Phi_0 F^{(1)}\left(\Phi_1/\Phi_0, \Phi_2/\Phi_0\right) - F^{(0)}\left(\Phi_1/\Phi_0, \Phi_2/\Phi_0\right) = 0 \tag{4.112}$$

and very near (but not exactly at) the boundary this is equivalent to

$$\lambda\, v_0 = \frac{F_0\left(v_1/v_0, v_2/v_0\right)}{F_1\left(v_1/v_0, v_2/v_0\right)}\ . \tag{4.113}$$

From a single probe brane one therefore infers the boundary profile that corresponds to one of the eigenvalues of the scalar field Z of the boundary theory. For degree $p > 1$ and for generic polynomials, it follows that the resulting holomorphic function can be factorized, as in equation (4.110), and each of the linear factors lead to profiles for p of the eigenvalues of the matrix valued field Z. Given that Z is an $N \times N$ matrix, this leads to $p \leq N$ and is referred to as the stringy exclusion principle [33,57].

4.2.5 Holographic Wobbling Strings

In this part we focus on the holographic description of the monomial type defect solutions in the $\mathscr{N} = 4$ gauge theory and compute the holographically renormalized energies from the probe D3-brane point of view. It will be convenient to use the redefined radial coordinate $r = l \sinh \rho$, and work with the following metric on $AdS_5 \times S^5$:

$$ds^2_{AdS_5} = -V(r)\, dt^2 + \frac{dr^2}{V(r)} + r^2(d\theta^2 + \cos^2\theta\, d\phi_1^2 + \sin^2\theta\, d\phi_2^2) , \tag{4.114}$$

where $V(r) = 1 + \frac{r^2}{l^2}$. The Ramond-Ramond 4-form is given by

$$C_{(4)} = -\frac{r^4}{l} \cos\theta \sin\theta\, dt \wedge d\theta \wedge d\phi_1 \wedge d\phi_2 \; . \tag{4.115}$$

The Lagrangian density for a probe D3-brane is:

$$\mathscr{L} = -T_{D3}\sqrt{-h} + T_{D3}\, P[C^{(4)}] \tag{4.116}$$

where h is the determinant of the induced metric on the worldvolume and $P[\cdot]$ refers to the pullback of a spacetime differential form onto the worldvolume.

The probe D3-branes of interest are described by the following monomial type solution:

$$(Z_1\Phi_0) = \eta \left(\frac{\Phi_1}{\Phi_0}\right)^m \left(\frac{\Phi_2}{\Phi_0}\right)^n , \quad \text{and} \quad Z_2 = Z_3 = 0 \,. \tag{4.117}$$

The worldvolume coordinates are $(\phi_0 = \frac{t}{l}, \theta, \phi_1, \phi_2)$, and $r = r(\theta)$, $\xi = \xi(\phi_0, \phi_1, \phi_2)$ are the fluctuating fields on the worldvolume. The defining equation of the probe D3-brane in terms of the real 10-dimensional coordiantes are given by:

$$\begin{aligned}\left(\frac{l}{r}\right)^{m+n} \left(1 + \frac{r^2}{l^2}\right)^{\frac{1}{2}(m+n+1)} &= \frac{R_0}{l} \cos^m\theta \sin^n\theta, \\ \xi - \xi_0 = m\,\phi_1 + n\,\phi_2 - (m+n+1)\,\phi_0 \,, \end{aligned} \tag{4.118}$$

where we set the parameter $\eta = R_0\, e^{i\xi_0}$.

It is not possible to solve for $r(\theta)$ and get a closed form expression. But for a few cases it is possible to do so. For the remaining part we will be mostly discussing the following cases:

- The static case, for which $m+n+1 = 0$ and we shall consider $n < 0$. For this case we have

$$r(\theta) = R_0 \sec\theta \cot^{|n|}\theta \,. \tag{4.119}$$

- When $m = 0$ and $n < 0$, for this case $r(\theta)$ can be expanded order by order in $\frac{l}{R_0}$ as follows:

$$r(\theta) = R_0 \sin^{-|n|}\theta \left(1 - \frac{(n+1)l^2}{2R_0^2}\sin^{2|n|}\theta - \frac{(n+1)(3n+1)l^4}{8R_0^4}\sin^{4|n|}\theta + \ldots\right) \tag{4.120}$$

- When $m+n=0$ with $n<0$, for this case $r(\theta)$ has an exact expression from the defining equation in (4.118), given by

$$r(\theta) = \sqrt{R_0^2 \cot^{2|n|}\theta - l^2}\ . \tag{4.121}$$

On-shell Action and the Variational Problem

Now we address the complications in the variational problem when all the solutions in (4.117) are considered together. For the negative values of powers m and n, we have non-compact D3 branes and this will involve adding appropriate boundary terms near the boundary of AdS_5 (equivalently near $\theta=0$ or $\theta=\frac{\pi}{2}$).

From the general analysis of the κ-symmetry constraint presented in Appendix B.2 we have seen that the BPS constraint equations in (B.17) simplify the on-shell Lagrangian to take the following form (see equation (B.21) for the volume form on the D3-brane):

$$\mathscr{L}\big|_{\text{on-shell}} = -P\left[e^{09}\wedge(e^{13}+e^{24})\right] + P\left[C^{(4)}\right]\ . \tag{4.122}$$

We work with the ansatz $\xi=\xi(\phi_0,\phi_1,\phi_2)$ and $r=r(\theta)$ suitable for the monomial solution and denote the conjugate momenta as follows:

$$\Pi_r^\mu = \frac{\partial\mathscr{L}}{\partial(\partial_\mu r)}\ , \qquad \Pi_\xi^\mu = \frac{\partial\mathscr{L}}{\partial(\partial_\mu \xi)}\ . \tag{4.123}$$

The Lagrangian density (4.122) evaluated on these is given by

$$\mathscr{L} = l^2\, r\, \partial_\theta r(\sin^2\theta\, \partial_{\phi_1}\xi - \cos^2\theta\, \partial_{\phi_2}\xi) + r^2\, \left((l^2+r^2)(\partial_{\phi_1}\xi + \partial_{\phi_2}\xi) + r^2\partial_{\phi_0}\xi\right) \cos\theta\, \sin\theta$$
$$+ r^4 \cos\theta\, \sin\theta\ , \tag{4.124}$$

After taking the general expression for monomial solutions in (4.117) into consideration the lagrangian density becomes

$$\begin{aligned}\mathscr{L} &= l^2\, r\, \partial_\theta r(m\sin^2\theta - n\cos^2\theta) + (m+n)\, l^2\, r^2 \cos\theta\, \sin\theta \\ &= \partial_\theta\left[\frac{l^2}{2}(m\sin^2\theta - n\cos^2\theta)\, r^2\right] = \partial_\theta\left[\frac{1}{2} r\, \Pi_r^\theta\right] .\end{aligned} \tag{4.125}$$

Therefore the modified Lagrangian density $\mathscr{L} - \partial_\theta\left[\frac{1}{2} r\, \Pi_r^\theta\right]$ vanishes on-shell for any (m,n) locally. Equivalently, a boundary term can be added at near $\theta = 0$ region or at $\theta = \frac{\pi}{2}$ region

$$\mathscr{L}_{bdy}^{(1)} = \pm\frac{1}{2}\, r\, \Pi_r^\theta\ . \tag{4.126}$$

This complete our discussion of the variational problem associated with the classical solutions in the set given in (4.117). These solutions satisfy some constraints in the field-momentum phase space very similar to those encountered for the classical solutions in the field theory due to the BPS conditions. These are given by

$$\begin{aligned}&\mathscr{C}_1 : \quad \cos^2\theta\, \Pi_\xi^{\phi_1} + \sin^2\theta\, \Pi_\xi^{\phi_2} - \Pi_\xi^\tau - l^2 r^2 \sin\theta\cos\theta = 0\ , \\ &\mathscr{C}_2 : \quad r\, \Pi_r^\theta\left(1 + \frac{l^2}{r^2} - \frac{1}{r^4}(\cot\theta\, \Pi_\xi^{\phi_1} + \tan\theta\, \Pi_\xi^{\phi_2})\right) + \sin\theta\cos\theta(\Pi_\xi^{\phi_1} - \Pi_\xi^{\phi_2}) = 0\ .\end{aligned} \tag{4.127}$$

As in the boundary theory these are very useful in regularizing the energies.

Renormalized Energies

We now show how to perform the holographic renormalization procedure for the energy of the probe D3 branes. It is possible to perform for a subset of cases we listed earlier in the beginning of this subsection. In particular, we add some more boundary terms that are proportional to the phase space constraints (so that the vanishing result for the on-shell action is unaffected), in order to regulate the energies for a number of cases. This analysis, to some extent, is parallel to the one that we carried out in the Yang-Mills theory in the previous section.

The static case $m+n+1=0$

We begin with the static solution with $m+n+1=0$ and we set $m>0$ and $n<0$. For this time-independent case the energy coincides with the on-shell action and the value of energy is zero. This is perfectly consistent with the results of the boundary theory discussed in the appendix B.1.

The probes of type $(0,n)$

For the case: $(0,n)$ with $n<0$, the integral of the energy density is singular due to power divergent terms as $\theta \to 0$. In the limit of $\frac{l}{R_0} \to 0$, the probe brane profie $r(\theta)$ exactly coincides with the boundary profile $|Z|$ for the particular solution under consideration. Also, for the case of the $(0,n)$ BPS string, the boundary action was particularly simple (see equation (B.3)). Given these, the following boundary term is proposed for this case:

$$\mathscr{L}^{(2)}_{bdy} = \frac{1}{2} \tan\theta\, \mathscr{C}_1 \ , \tag{4.128}$$

where $\mathscr{C}_1$ is the constraint defined in (4.127). The vanishing of the action on-shell is unaffected by the addition of this term as the phase space combination in $\mathscr{C}_1$ vanishes on-shell. And this additional boundary term does modify the energy and, after including the overall factors of the tension of the D3-brane $T_{D3} = \frac{N}{2\pi^2 l^4}$ and the $4\pi^2$ coming from the angular integration the energy for the $(0,n)$ case takes the following value:

$$\frac{l\,E_{0,n}}{4\pi^2} = \frac{N}{2\pi^2}\left((n+1)\frac{R_0^2}{2l^2} + \frac{1}{2}(n-1)(n+1)^2 + O(\frac{l}{R_0})\right) \tag{4.129}$$

The map between the parameters of the probe brane and the gauge theory results that was discussed for the half-BPS case in (4.85). In this case of monomial solution we have the following

$$R_0 = \frac{2\pi}{\sqrt{\lambda}}\,l\,r_0\ , \tag{4.130}$$

And with the use of this map the value of the energy can be re-written in terms of gauge theory paramaters:

$$\frac{l\,E_{0,n}}{4\pi^2} = \frac{1}{g_{YM}^2}\left((n+1)r_0^2 + \frac{\lambda}{4\pi^2}(n-1)(n+1)^2 + \dots\right) \tag{4.131}$$

The leading term exactly matches with the result for the energy in (B.4) and in addition, there is the leading $O(\lambda)$ quantum correction.

The probes of type $(-n,n)$

For the $(-n,n)$ case $r(\theta)$ has the exact form given by:

$$r(\theta) = \sqrt{R_0^2 \cot^{2|n|}\theta - l^2} \, . \tag{4.132}$$

The energy contribution of the bulk after integration over θ is (upto the factor $4\pi^2\, T_{D3}$)

$$\begin{aligned} l\, E(-n,n,\theta) = & \frac{1}{2}\cos^2\theta_0^{(n)} - \frac{R_0^2}{2l^2(1-n)}\cos^{2-2n}\theta_0^{(n)}\, F(1-n,-n,2-n,\cos^2\theta_0^{(n)}) \\ & -\frac{1}{2}\cos^2\theta + \frac{R_0^2}{2l^2(1-n)}\cos^{2-2n}\theta\, F(1-n,-n,2-n,\cos^2\theta)\Big) \, . \end{aligned} \tag{4.133}$$

where $\theta_0^{(n)} = \arctan\left(\frac{R_0}{l}\right)^{\frac{1}{|n|}}$ with $0 < \theta < \theta_0$.

In the corresponding boundary theory problem, an additional boundary term contribution in (B.5) cancelled the power law divergences in this case. Here the proposal of the following boundary term takes care of the divergence problem

$$\mathscr{L}_{bdy}^{(2)} = \frac{1}{l^3} f(n,\theta)\, \mathscr{C}_1 \, , \tag{4.134}$$

where $\mathscr{C}_1$ is the constraint in (4.127) and the function $f(n,\theta)$ is of the following form (inspired largely by the corresponding boundary theory term in (4.76))

$$f(n,\theta) = \frac{1}{2(1+n)} \tan\theta\, F(1,n,2+n,-\tan^2\theta) \, . \tag{4.135}$$

The value of energy after taking the contribution from this additional boundary term is given by

$$
\begin{aligned}
l\,E_{-n,n} = \frac{n-1}{2} + \frac{R_0^2}{2l^2}\Gamma(1-n)\Gamma(1+n) \\
+ \frac{1}{2}\cos^2\theta_0^{(n)} - \frac{R_0^2}{2l^2(1-n)}\cos^{2-2n}\theta_0^{(n)}\,F(1-n,-n,2-n,\cos^2\theta_0^{(n)})\ . \quad (4.136)
\end{aligned}
$$

The above result tends to the value obtained for the result of the boundary theory in (B.6), after taking the limit in which $\frac{l}{R_0} \to 0$. In this limit, the profile of the D3 brane $r(\theta)$ coincides with boundary profile $|Z|$ of the boundary theory. This in turn corresponds to $\theta_0^{(n)} \to \frac{\pi}{2}$ which sets the terms in the second line of (4.136) to zero.

For the fractional values of n, we write the result in terms of gauge theory parameters by taking into account the map (4.130), and restoring the factor of $4\pi^2 T_{D3}$, as follows:

$$
\left(\frac{l\,E_{-n,n}}{4\pi^2}\right)_{n=-\frac{(2p+1)}{2}} = (-1)^p\frac{\pi(2p+1)}{2g_{YM}^2}r_0^2 + \frac{N}{4\pi^2}(n-1) \quad (4.137)
$$

$$
= \frac{1}{g_{YM}^2}\left((-1)^p\frac{\pi(2p+1)}{2}r_0^2 + \frac{\lambda}{4\pi^2}(n-1)\right)\ . \quad (4.138)
$$

Therefore, we have a perfect match at the leading order with the boundary gauge theory answer in (B.6) and in addition to it we have the first quantum correction to the energy of the BPS string.

Comments on the energy results

The holographic renormalization that we carried out for the probe brane provides a justification for the regularisation of the energies that was done on the CFT side in the previous section 4.1.7 . There are a few important questions that are left open. While our regularization does indeed cancel all coordinate dependent power law divergences there is a certain ambiguity in the finite part of the charges. The boundary terms are added such that the expression (4.65) that gives the energy answers for cases with $(m > 0, n > 0)$ also gives the energy after an analytic continuation in the (m, n) parameters that appear in the monomial defining the BPS string, for which one of m or n is negative. Further, the interpretation of the finite energy $E_{m,n}$ remains an open problem. Given that the sign of the energy can be either positive or negative it is possible that these can be interpreted as Casimir energies of some effective theory on the BPS string and more work would be needed to clarify this.

We next come to the limitations of our analysis. For the case of $m+n=0$, the analytic continuation of the energy in (4.65) to negative integer values of n leads to divergent results. On a careful examination, it turns out that, while the power law divergences do cancel, there are additional singularities that could be interpreted as logarithmic singularities. Similar divergences also appear for m and n both sufficiently negative. As it stands, for those cases for which the analytic continuation does not lead to a finite result, it would appear that more work needs to be done to completely regularize the energy and charges of the BPS strings.

Chapter 5

Conclusion

In this thesis, we discussed surface operators in the four-dimensional $\mathcal{N} = 2$ theory and the maximally supersymmetric $\mathcal{N} = 4$ SYM theory. In $\mathcal{N} = 2$ theory, we focused on the low energy dynamics of the theory with half-BPS surface defects and how distinct descriptions lead to identical low energy effective actions. Whereas, in $\mathcal{N} = 4$ SYM, we studied lower supersymmetric $\frac{1}{8}$-BPS monodromy defects in the Hamiltonian formalism and gave a general charectrization of such BPS strings.

In $\mathcal{N} = 2$ theory, our objective was to calculate the effective-twisted chiral superpotential associated with the surface operator from the two dual descriptions [16,19]. We extended in the line of approach taken in [16], we showed that there exists a precise correspondence between the choice of massive vacua in two dimensions and the Gukov-Witten defects of the $SU(N)$ gauge theory labeled by the partition $[n_1, ..., n_M]$. We had also described the relation between the $(M-1)$ dynamically generated scales Λ_I associated to the Fayet-Iliopoulos (FI) parameters for the two-dimensional nodes and the $(M-1)$ dimensionful parameters that naturally occur in the ramified instanton counting problem [20, 21, 23]. An important and non-trivial check of this proposal was provided by the perfect agreement in the quantum corrections in the quiver gauge theory and the corrections in the twisted superpotential due to ramified instantons of the four-dimensional theory.

In the second part of the thesis, we studied lower super-symmetric versions of monodromy defects in $\mathscr{N} = 4$ $SU(N)$ theory in $\mathbb{R} \times S^3$ manifold. The $\frac{1}{8}$-BPS wobbling strings on $\mathbb{R} \times S^3$ we discussed here are related by Wick rotation and Weyl transformation to $\frac{1}{8}$-BPS defects associated with surface operator in $\mathbb{R}^4$. One of the main result of the work in this part of the thesis is to show a characterization of the general BPS string solution that preserves four supercharges. The location of such a BPS string at any instant of time is obtained as the intersection of zeros of holomorphic functions in $\mathbb{C}^2$ with $S^3 \subset \mathbb{C}^2$. This description of the string solution can also be recovered from the holographic side by analyzing the worldvolume constraints on probe D3-branes in $AdS_5 \times S^5$ and studying the limiting behaviour near the boundary of AdS_5.

The holographic duals of the $\frac{1}{8}$-BPS strings preserve precisely the same supersymmetries as the (S_1, S_2, J) giants of [57] and the dual-giants of [33]. By the addition of appropriate boundary terms, we showed that the abelian solutions that are regular (and which are holographically dual to the dual-giants), as well as the singular string solutions of the CFT, can be made to belong to the same variational problem. We then showed that the singularities in the classical expressions of the energy and other charges can be systematically "renormalized" by including additional boundary terms on the CFT side. The holographic dual of this procedure was carried out for the monomial type D3-brane probes. The analysis of the probe brane theory paralleled that in the Yang-Mills theory and the leading order results for the energy and charges could be matched once the parameters of the solution were appropriately mapped to each other.

Our prescription to subtract away the (coordinate-dependent) divergences in the charges of the BPS string solutions were justified by the procedure in the dual holographic analysis for the monomial type D3-brane probes. It tells that this prescription here is nothing more than the standard UV renormalization. Similar prescriptions were discussed in [13, 56] where the renormalization of the Euclidean action for Wilson line or surface defect operators gives rise to finite expectation values of those non-local operators. In addition to this, the cases that we considered for the BPS strings to compute the energy have values either positive or negative. The interpretation of these values is unclear at this juncture. It is likely that these could be interpreted as a Casimir energies of the worldvolume theories on the stringy defects.

While we obtained a general characterization of the wobbling BPS strings in terms of the mathematical algebraic links, it would be important to have a more detailed understanding of the space of solutions to these equations as it could have interesting consequences for the physics of these defects. This is especially interesting in the context of work relating four dimensional gauge theory and knot theory in the Euclidean context. In the gauge theory on a four dimensional half space, the connections between the singular solutions of the generalized Bogomolny equations [11] and knot solutions that lives along the three dimensional boundary have been made in [61–63]. These knot solutions have been used to study associated topological invariants such as the Jones polynomial and play an important role in the programme of categorification [60, 61, 64].

Appendix A

A.1 Chiral correlators for the SU(N) theory

In this appendix, we briefly review the calculation of the chiral correlators using the equivariant localization technique [65–67], we use the notations set in [68]. The partition function of the pure $SU(N)$ $\mathcal{N}=2$ theory with multi-instanton contribution was calculated in [24, 25] (also refer to [65] for exposition with a few more details) in the omega-deformed background where the $SO(4)$ isometry group is deformed to the $SO(2)\times SO(2)$ group. This modification makes the parameter space over which the instanton moduli take their value compact and the value of partition function becomes without having any divergences. In the limit of vanishing omega-background parameters this calculation gives the partition function on the $\mathbb{R}^4$ space manifold. In the weak-coupling regime, the partition function with instanton contribution can be written as a power series expansion

$$Z_{\text{inst}} = \sum_{k=0} q^k Z_k \tag{A.1}$$

where the parameter

$$q = e^{2\pi i\tau} = (-1)^N \Lambda^{2N}, \tag{A.2}$$

here $\tau = \frac{\theta}{2\pi} + i\frac{4\pi}{g_{YM}^2}$, and Λ is the dynamically generated scale of the four-dimensional theory that we use in the third chapter. Z_k is the contribution from the k-th instanton sector and is obtained by doing the following multi-dimensional contour integral:

$$Z_k = \oint \prod_{I=1}^{k} \frac{d\chi_I}{2\pi i} z_k \tag{A.3}$$

the integrand is given by

$$z_k = \frac{(-1)^k}{k!} \left(\frac{\varepsilon_1 + \varepsilon_2}{\varepsilon_1 \, \varepsilon_2} \right)^k \frac{\Delta(0)\Delta(\varepsilon_1 + \varepsilon_2)}{\Delta(\varepsilon_1)\Delta(\varepsilon_2)} \prod_{I=1}^{k} \frac{1}{P\left(\chi_I + \frac{\varepsilon_1+\varepsilon_2}{2}\right) P\left(\chi_I - \frac{\varepsilon_1+\varepsilon_2}{2}\right)} \tag{A.4}$$

with

$$P(x) = \prod_{i=1}^{N} (x - a_i) \qquad\qquad \Delta(x) = \prod_{I<J}^{k} \left(x^2 - \chi_{IJ}^2\right) . \tag{A.5}$$

The contour integrals are computed by closing the contours in the upper half planes of the χ_I variables, assigning imaginary part to the ε's with prescription

$$\operatorname{Im}\varepsilon_2 \gg \operatorname{Im}\varepsilon_1 > 0 . \tag{A.6}$$

This method can be used to used to compute the chiral correlators, which are known to receive quantum corrections from all instanton sectors. The generating function of all chiral correlators of the form $\langle \operatorname{Tr}\Phi^l \rangle$ is

$$\langle \operatorname{Tr} e^{z\Phi} \rangle = \sum_{i=1}^{N} e^{z a_i} - \frac{1}{Z_{\text{inst}}} \sum_{k=1} q^k \int \prod_{I=1}^{k} \frac{d\chi_I}{2\pi i} z_k \, \mathscr{O}\left(z, \chi_I\right) \tag{A.7}$$

where $\mathscr{O}$ is the following observable

$$\mathscr{O}(z,\chi_I) = \sum_{I=1}^{k} e^{z\chi_I}\,(1-e^{z\varepsilon_1})\,(1-e^{z\varepsilon_2}) \ . \tag{A.8}$$

For the $SU(2)$ gauge theory, this calculation give the following answer for the correlator

$$\langle \mathrm{Tr}\,\Phi^2\rangle = 2a^2 - \frac{q}{a^2} + \frac{5}{16a^6}q^2 - \frac{9}{32a^{10}}q^3 + \dots, \tag{A.9}$$

where $a_1 = -a_2 = a$.

For the $SU(3)$ theory, we have the following answer for the correlators (with notation $a_{ij} = a_i - a_j$):

$$\begin{aligned}\langle \mathrm{Tr}\,\Phi^2\rangle =& a_1^2 + a_2^2 + a_3^2 - \left(\frac{1}{(a_{12})^2(a_{13})^2} + \frac{1}{(a_{12})^3 a_{13}} + \frac{1}{(a_{12})^2(a_{23})^2} - \frac{1}{(a_{12})^3 a_{23}}\right)4q \\ &+ \left(\frac{5}{(a_{12})^4(a_{13})^6} + \frac{10}{(a_{12})^5(a_{13})^5} + \frac{14}{(a_{12})^6(a_{13})^4}\right. \\ &\qquad \left. + \frac{17}{(a_{12})^7(a_{13})^3} + \frac{28}{(a_{12})^8(a_{13})^2} + \frac{56}{(a_{12})^9 a_{13}}\right)4q^2 \\ &+ \left(\frac{5}{(a_{21})^4(a_{23})^6} + \frac{10}{(a_{21})^5(a_{23})^5} + \frac{14}{(a_{21})^6(a_{23})^4}\right. \\ &\qquad \left. + \frac{17}{(a_{21})^7(a_{23})^3} + \frac{28}{(a_{21})^8(a_{23})^2} + \frac{56}{(a_{21})^9 a_{23}}\right)4q^2 + \dots\end{aligned} \tag{A.10}$$

$$
\begin{aligned}
\langle \mathrm{Tr}\,\Phi^3\rangle = & a_1^3+a_2^3+a_3^3-\left(\frac{2a_1}{(a_{12})^2(a_{13})^2}+\frac{2a_2}{(a_{21})^2(a_{23})^2}+\frac{a_3}{(a_{12})^2\,a_{13}\,a_{23}}\right)6q \\
& +\left(\frac{10a_1}{(a_{12})^4(a_{13})^6}+5\frac{2a_1-a_3}{(a_{12})^5(a_{13})^5}+2\frac{4a_1-5a_3}{(a_{12})^6(a_{13})^4}\right. \\
& \qquad \left.+2\frac{3a_1-7a_3}{(a_{12})^7(a_{13})^3}+\frac{6a_1-25a_3}{(a_{12})^8(a_{13})^2}-\frac{56a_3}{(a_{12})^9a_{13}}\right)6q^2 \\
& +\left(\frac{10a_2}{(a_{21})^4(a_{23})^6}+5\frac{2a_2-a_3}{(a_{21})^5(a_{23})^5}+2\frac{4a_2-5a_3}{(a_{21})^6(a_{23})^4}\right. \\
& \qquad \left.+2\frac{3a_2-7a_3}{(a_{21})^7(a_{23})^3}+\frac{6a_2-25a_3}{(a_{21})^8(a_{23})^2}-\frac{56a_3}{(a_{21})^9a_{23}}\right)6q^2+\dots
\end{aligned}
\tag{A.11}
$$

Appendix B

B.1 Energy values for some BPS string configurations

In this section we focus on a few cases and demonstrate our ideas discussed in the main section 4.1 by writing the energy values.

B.1.1 The static case $m+n+1=0$:

In this case, the solutions are time independent and for $m>0$ and $n<0$ profile for the scalar field takes the form

$$Z = r_0\, e^{i\xi_0}(\cos\theta\, e^{i\phi_1})^{-n-1}(\sin\theta\, e^{i\phi_2})^{n}\ . \tag{B.1}$$

The on-shell momenta Π_Z^{τ} vanishes and the requirement that the energy be finite reduces to requiring that the on-shell action be finite. The simple $\frac{1}{2}$-BPS defects, which correspond to the $(0,-1)$ and $(-1,0)$ cases fall into this category and we find that the on-shell action and energy vanish with the boundary terms.

B.1.2 The strings of type $(0,n)$

These are the time dependent solutions of the form

$$Z = r_0\, e^{i(\xi_0-\tau)} \left(\sin\theta\, e^{i(\phi_2-\tau)}\right)^n \ , \tag{B.2}$$

with $n<0$. The function $f(0,n,\theta)$ reduces to unity and the boundary action becomes of the form:

$$\mathbf{L}_{bdy,0+} + \mathbf{L}'_{bdy} = \frac{1}{2}(Z\Pi^\theta_Z + \bar{Z}\Pi^\theta_{\bar{Z}}) + \frac{i}{2}\tan\theta\,(Z\mathscr{C} - \bar{Z}\bar{\mathscr{C}}) \ , \tag{B.3}$$

where $\mathscr{C}$ is the constraint in (4.58). The renormalized energy in this case is equal to

$$\frac{E_{0,n}}{4\pi^2} = \frac{(n+1)}{g^2_{YM}}\, r_0^2 \ . \tag{B.4}$$

B.1.3 The strings of type $(-n,n)$

The boundary term has full contribution from both terms in (4.61) and (4.70), we give the expression in particular for when $n<0$:

$$\mathbf{L}_{bdy,0+} + \mathbf{L}'_{bdy} = \frac{1}{2}(Z\Pi^\theta_Z + \bar{Z}\Pi^\theta_{\bar{Z}}) + \frac{i}{2}\,(Z\mathscr{C} - \bar{Z}\bar{\mathscr{C}})\, f(-n,n,\theta) \ . \tag{B.5}$$

For fractional (rational) values of n, the finite result for the energy is

$$\left.\frac{E_{-n,n}}{4\pi^2}\right|_{n=-\frac{(2p+1)}{2}} = (-1)^p\, \frac{\pi\,(2p+1)}{2g^2_{YM}}\, r_0^2 \ . \tag{B.6}$$

B.2 κ-symmetry of the Second Class of probe branes

We discuss the κ-symmetry of second class of D3 probe branes, that are obtained by an SU(2) rotation that acts on the complex Φ_i variables. For the solution

$$\Phi_2 Z_1 = D_1 \quad \text{and} \quad Z_2 = Z_3 = 0\,, \tag{B.7}$$

the product of four world-volume γ matrices is

$$\begin{aligned}\frac{1}{l^4}\gamma_{\tau\sigma_1\sigma_2\sigma_3} =& -\cos^2\theta\sinh^2\rho\,(\cot\theta\Gamma_{0139} - \cosh\rho(\Gamma_{0149}+\Gamma_{0239}) + \sinh\rho(\Gamma_{0134}+\Gamma_{1349}))\\ &-\sin\theta\cos\theta\cosh\rho\sinh^2\rho(\cosh\rho\Gamma_{0249} - \sinh\rho(\Gamma_{0234}+\Gamma_{2349}))\,.\end{aligned} \tag{B.8}$$

In order to check the κ-symmetry equation, as before, we need to commute the four-gamma products through the matrix M defined in (4.83). After performing the relevant Γ-matrix algebra, we finally obtain

$$\begin{aligned}\frac{1}{l^4}\gamma_{\tau\sigma_1\sigma_2\sigma_3}\cdot M\cdot\varepsilon_0 =& iM\sinh^3\rho\cos\theta\Big[i\cosh\rho e^{-i\phi_0\Gamma_0\gamma}e^{\phi_2\Gamma_{24}}(\Gamma_{0134}-\Gamma_{4968}))\\ &+\sinh\rho\cos\theta e^{\phi_1\Gamma_{13}}e^{\phi_2\Gamma_{24}}(\Gamma_{12}-\Gamma_{023968})\Big]\cdot\varepsilon_0\\ &+iM\cot\theta\sinh^2\rho(1+\sin^2\theta\sinh^2\rho)\Gamma_{013968}\cdot\varepsilon_0\end{aligned} \tag{B.9}$$

We thus find that the D3-brane preserves one half of the bulk supersymmetries if the following projection is imposed on the constant spinor:

$$\Gamma_{2457}\,\varepsilon_0 = \varepsilon_0\,. \tag{B.10}$$

More $\frac{1}{2}$-BPS Probes in the second class

On doing an SU(3) rotation on the Z_i variables as before and we obtain two other half-BPS probe D3-branes in this same class. The following solutions

$$\Phi_2 Z_2 = D_2 \quad \text{and} \quad Z_3 = Z_1 = 0\,, \tag{B.11}$$

preserves half the supersymmetries if the following projection is imposed on the constant spinor:

$$\Gamma_{2468}\, \varepsilon_0 = \varepsilon_0\,. \tag{B.12}$$

Similarly, the D3-brane described by

$$\Phi_2 Z_3 = D_3 \quad \text{and} \quad Z_1 = Z_2 = 0\,, \tag{B.13}$$

preserves half the supersymmetries if the following projection is imposed on the constant spinor:

$$\Gamma_{0913}\, \varepsilon_0 = i\, \varepsilon_0\,. \tag{B.14}$$

B.3 Review: Constraints on the $\frac{1}{16}$-BPS D3-brane worldvolume

The five projection conditions in (4.103) leads to a simplification where the exponential factor M reduces to a mere phase in the Killing spinor, which now takes the form:

$$\varepsilon = e^{\frac{i}{2}(\phi_0+\phi_1+\phi_2+\xi_1+\xi_2+\xi_3)}\,\varepsilon_0\,. \tag{B.15}$$

We substitute the Killing spinor (B.15) into the κ-symmetry equation (4.89) and use the projection conditions (4.103) to reduce the l.h.s. into a linear combination of independent structures of the form $\Gamma_{a_1,a_2\ldots}\varepsilon_0$. The coefficient of each such structure is set to zero except the constant one, which is equated to the r.h.s.

In order to write these BPS constraint equations we make use of definition of the following complex 1-forms:

$$\mathbf{E}^1 = \mathfrak{e}^1 - i\mathfrak{e}^3 \qquad \mathbf{E}^2 = \mathfrak{e}^2 - i\mathfrak{e}^4 \qquad \mathbf{E}^5 = \mathfrak{e}^5 + i\mathfrak{e}^7 \qquad \mathbf{E}^6 = \mathfrak{e}^6 + i\mathfrak{e}^8\,, \tag{B.16}$$

The κ-symmmetry constraints that follow by setting to zero the coefficient of $\Gamma_{a_1,\ldots a_n}\varepsilon_0$ are equivalent to the vanishing of the pullback of the following 4-forms onto the D3 world-volume:

$$\begin{aligned} \mathbf{E}^{ABCD} &= 0 \\ \left(\mathfrak{e}^{09} + i\,(\tilde{\omega} - \omega)\right) \wedge \mathbf{E}^{AB} &= 0 \quad \text{for} \quad A,B = 0,1,2,5,6\,. \end{aligned} \tag{B.17}$$

Here we have also defined the following real 2-forms:

$$\tilde{\omega} = \mathfrak{e}^{13} + \mathfrak{e}^{24} = -\frac{i}{2}\left(\mathbf{E}^1 \wedge \overline{\mathbf{E}^1} + \mathbf{E}^2 \wedge \overline{\mathbf{E}^2}\right) \tag{B.18}$$

$$\omega = \mathfrak{e}^{57} + \mathfrak{e}^{68} = \frac{i}{2}\left(\mathbf{E}^5 \wedge \overline{\mathbf{E}^5} + \mathbf{E}^6 \wedge \overline{\mathbf{E}^6}\right) . \tag{B.19}$$

Substituting the equations in (B.17) into the κ-symmetry constraint and equating the coefficient of ε_0 on both sides, we find that for D3 probes that have a time-like world-volume we have

$$(\omega - \tilde{\omega}) \wedge (\omega - \tilde{\omega}) = 0 \tag{B.20}$$

$$\mathfrak{e}^{09} \wedge (\tilde{\omega} - \omega) = \pm \left|\mathfrak{e}^{09} \wedge (\tilde{\omega} - \omega)\right| = \pm \mathrm{dvol}_4 . \tag{B.21}$$

The above formula is an important one for the discussion of the on-shell actions for our probe brane solutions. We have been brief in reviewing the derivation of the constraints that give the general BPS solutions in (4.104), the complete step by step procedure is given in [33].

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