DECOMPOSITION METHOD FOR NEUTRON TRANSPORT EQUATION

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In this paper we are concerning with the integro-differential equation of 1-dimensional neutron transport problem in the stationary case by means of a vectorial variant of the decomposition method. The numerical test proves excellent agreement between the approximate solution and exact solution.

Key words: neutron transport problem, decomposition method, integro-differential equation.

1. INTRODUCTION

The nonlocal differential problems and *Nonlinear Evolution Equations* (NLEEs) are very important in Physics, Applied Mathematics, Mathematical Engineering and other domains. Therefore, different techniques designed to their analysis were developed in the past years, such as variational iteration method, Lie symmetry analysis, G'/G – expansion method, Painleve analysis, homotopy perturbation method, homotopy analysis method, Adomian decomposition and several others [1–13].

Among these problems, the neutron transport problem is very significant in nuclear physics. The fission of the nucleus in a reactor produces high speed neutrons which are subjected to a decreasing speed process until they become in an equilibrium state with the other atoms.

The density of neutrons describes their distribution in the reactor and it is the solution of an integro-differential equation. This problem – known as neutron transport problem – was investigated in many works.

The involved methods are [14–20]: homotopy perturbation method, finite elements, finite differences, the fictitious domain method, Fourier transform, Laplace transform, splitting technique, truncated Chebyshev series, spectral methods.

In this paper we examine a new technique for computation of the solution: vectorial decomposition method. The numerical test shows the efficiency of this procedure.

The paper is organized in the following way: section 2 contains a short description of the general decomposition method. The mathematical model of the problem is given in section 3. Section 4 deals with the resolution of the integro-differential using the new vectorial decomposition method. Section 5 includes numerical test and comparison between the approximate and the exact solution. Finally, some conclusions are presented in section 6.

2. BRIEF DESCRIPTION OF DECOMPOSITION METHOD

The Adomian decomposition method [13, 21–24] became a very efficient and reliable technique for solving linear and nonlinear ordinary and partial equations.
Some of the reasons are:

– it offers a competitive alternative to the Taylor series method and other series techniques;
– it has significant advantages comparing with other numerical methods in providing analytic, rapidly convergent approximation.

In this section we concern with a brief presentation of the method.
Consider the equation

$$Lu + Ru + Nu = g(x) \tag{1}$$

with prescribed conditions.
Here, u is the unknown function, L is the highest order derivative assumed to be easily invertible, R is the remainder linear operator, Nu represents the nonlinear terms and g is the source term.

Applying the inverse operator L^{-1} on both sides of (1) we deduce

$$u(x) = f(x) - L^{-1}(Ru) - L^{-1}(Nu) \tag{2}$$

The function $f(x)$ arises from the integration of $g(x)$ and taking into consideration the initial or boundary conditions.
The Adomian decomposition method computes the solution $u(x)$ as the series

$$u(x) = \sum_{n=0}^{\infty} u_n(x) \tag{3}$$

The nonlinear term series is calculated by

$$Nu = \sum_{n=0}^{\infty} A_n$$

A_n are the Adomian polynomials which are defined by the relation

$$A_n = \frac{1}{n!}\frac{\mathrm{d}^n}{\mathrm{d}\lambda^n}\left[N\left(\sum_{i=0}^{\infty} \lambda^i u_i \right) \right]\Bigg|_{\lambda=0}$$

It results $A_n = A_n(u_0,...,u_n)$. The components $u_0, u_1, u_2,...$ are recursively determined:

$$\begin{aligned} &u_0(x) = f(x) \\ &u_{n+1}(x) = -L^{-1}(Ru_n) - L^{-1}(A_n), \ n \geq 0 \end{aligned} \tag{4}$$

3. MATHEMATICAL MODEL

The 1-dimensional neutron transport equation in the stationary case is described by the relation [14]

$$t\frac{\partial\varphi}{\partial x}(x,t) + \varphi(x,t) = \frac{1}{2}\int_{-1}^{1} \varphi(x,\tau)\mathrm{d}\tau + f(x,t) \tag{5}$$

where $(x,t) \in [0,1]\times[-1,1]$. The prescribed conditions are

$$\varphi(0,t) = 0 \quad \text{for } t > 0, \quad \varphi(1,t) = 0 \quad \text{for } t < 0 \tag{6}$$

The notations are as follows:
$\varphi(x,t)$ is the density of neutrons, which migrate in a direction that makes an angle α with the x axis and $t = \cos\alpha$;
$f(x,t)$ is a given radioactive source function.
Similar to [14] we introduce the notations

$$\begin{aligned} &\varphi^+(x,t) = \varphi(x,t) \quad \text{for } t > 0 \\ &\varphi^-(x,t) = \varphi(x,-t) \quad \text{for } t > 0 \end{aligned} \tag{7}$$

$$u = \frac{1}{2}(\varphi^+ + \varphi^-), \quad v = \frac{1}{2}(\varphi^+ - \varphi^-) \tag{8}$$

$$g = \frac{1}{2}(f^+ + f^-), \ r = \frac{1}{2}(f^+ - f^-) \tag{9}$$

It results

$$-t^2\frac{\partial^2 u}{\partial x^2}(x,t)+u(x,t)=\int_0^1 u(x,\tau)\mathrm{d}\tau+g(x,t)-t\frac{\partial r}{\partial x}(x,t), \tag{10}$$
for $(x,t)\in[0,1]\times[0,1]$

$$\begin{aligned}&u(0,t)-t\frac{\partial u}{\partial x}(0,t)=-r(0,t)\\&u(1,t)+t\frac{\partial u}{\partial x}(1,t)=r(1,t)\end{aligned}\quad ,t\in[0,1] \tag{11}$$

We apply Simpson's formula of numerical integration in eq. (10), with the division points $t_k=kh, k=\overline{0,4}, h=0.25$ and denote $u_k(x)=u(x,t_k)$.
Taking into consideration that $u_0(x)=0$, which means in this case the neutrons move in a perpendicular direction toward the axis *Ox*, we obtain the differential system

$$\frac{\mathrm{d}^2U}{\mathrm{d}x^2}=M\ \ U(x)+S(x) \tag{12}$$

Here, we denoted

$$U(x)=\begin{bmatrix}u_1(x)\\u_2(x)\\u_3(x)\\u_4(x)\end{bmatrix},\quad M=\begin{bmatrix}\frac{32}{3} & -\frac{8}{3} & -\frac{16}{3} & -\frac{4}{3}\\ -\frac{4}{3} & \frac{10}{3} & -\frac{4}{3} & -\frac{1}{3}\\ -\frac{16}{27} & -\frac{8}{27} & \frac{32}{27} & -\frac{4}{27}\\ -\frac{1}{3} & -\frac{1}{6} & -\frac{1}{3} & \frac{11}{12}\end{bmatrix},\quad S(x)=\begin{bmatrix}\bar{S}_1(x)\\\bar{S}_2(x)\\\bar{S}_3(x)\\\bar{S}_4(x)\end{bmatrix} \tag{13}$$

where

$$\bar{S}_k(x)=\bar{S}(x,t_k)=\frac{1}{t_k^2}\left[t_k\frac{\partial r}{\partial x}(x,t_k)-g(x,t_k)\right] \tag{14}$$

We introduce the matrices

$$T=\mathrm{diag}(t_1,t_2,t_3,t_4)$$

$$R(x)=[r_1(x),r_2(x),r_3(x),r_4(x)]^t$$

with $r_k(x)=r(x,t_k), k=\overline{1,4}.$
The boundary conditions (11) can be written

$$\begin{aligned} U(0) - T \quad U'(0) &= -R(0) \\ U(1) + T \quad U'(1) &= R(1) \end{aligned} \tag{15}$$

4. RESOLUTION BY DECOMPOSITION METHOD

We proceed with the computation of the solution using the proposed method.
We rewrite the eq. (12) in operator form

$$LU = MU + S \tag{16}$$

where $L = \dfrac{\mathrm{d}^2}{\mathrm{d}x^2}$. Applying $L^{-1} = \int_0^x \int_0^y (.)\mathrm{d}t\mathrm{d}y$ on both sides of (16) we obtain

$$U(x) = U(0) + U'(0)x + L^{-1}\left(MU(x) + S(x)\right) \tag{17}$$

Implementing the vectorial decomposition method, we deduce

$$U(x) = \sum_{n=0}^{\infty} U_n(x) \tag{18}$$

with

$$U_0(x) = U(0) + U'(0)x + \int_0^x \int_0^y S(t)\mathrm{d}t\mathrm{d}y \tag{19}$$

$$U_n(x) = L^{-1}MU_{n-1}(x) = M\int_0^x \int_0^y U_{n-1}(t)\mathrm{d}t\mathrm{d}y \tag{20}$$

for $n \geq 1$.

Denote $S_0(x) = \int_0^x \int_0^y S(t)\mathrm{d}t\mathrm{d}y$.

The eqs. (19), (20) provide

$$U_n(x) = \frac{x^{2n}}{(2n)!}M^n U(0) + \frac{x^{2n+1}}{(2n+1)!}M^n U'(0) + M^n S_n(x), \; n \geq 0 \tag{21}$$

In the above relation,

$$S_n(x) = \int_0^x \int_0^y S_{n-1}(t)\mathrm{d}t\mathrm{d}y, \; n \geq 1 \tag{22}$$

At this stage, we introduce the square root matrix Q of the matrix M ($Q^2 = M$) which is given in Matlab by

$$Q=\begin{bmatrix} 1199/375 & -1102/1859 & -2563/1791 & -801/1991 \\ -551/1859 & 1868/1073 & -818/1187 & -757/3666 \\ -294/1849 & -217/1417 & 1168/1297 & -508/3757 \\ -296/2943 & -264/2557 & -1143/3757 & 1345/1491 \end{bmatrix} \tag{23}$$

Taking in consideration the equations (18)(21)(23), the solution $U(x)$ can be expressed as

$$U(x)=\left[\sum_{n=0}^{\infty} \frac{x^{2n}}{(2n)!}Q^{2n}\right]U(0)+\left[\sum_{n=0}^{\infty} \frac{x^{2n+1}}{(2n+1)!}Q^{2n}\right]U'(0)+\sum_{n=0}^{\infty} M^n S_n(x) \tag{24}$$

We introduce the matrix functions

$$\begin{aligned} \mathrm{ch}(A)&=\frac{\exp(A)+\exp(-A)}{2}=\sum_{n=0}^{\infty} \frac{1}{(2n)!}A^{2n} \\ \mathrm{sh}(A)&=\frac{\exp(A)-\exp(-A)}{2}=\sum_{n=0}^{\infty} \frac{1}{(2n+1)!}A^{2n+1} \end{aligned} \tag{25}$$

We deduce the function $U(x)$ in the form

$$U(x)=\mathrm{ch}(xQ)U(0)+\mathrm{sh}(xQ)Q^{-1}U'(0)+\sum_{n=0}^{\infty}M^n S_n(x) \tag{26}$$

The boundary conditions (15) yield

$$U(0)=TU'(0)-R(0) \tag{27}$$

and

$$\begin{aligned} &\mathrm{ch}(Q)U(0)+\mathrm{sh}(Q)Q^{-1}U'(0)+\sum_{n=0}^{\infty}M^n S_n(1)+ \\ &+T\left[\mathrm{sh}(Q)QU(0)+\mathrm{ch}(Q)U'(0)+\sum_{n=0}^{\infty}M^n S_n'(1)\right]=R(1) \end{aligned} \tag{28}$$

We substitute $U(0)$ in (28) and deduce the linear system

$$\begin{aligned} &\left[\mathrm{ch}(Q)T+\mathrm{sh}(Q)Q^{-1}+T\mathrm{sh}(Q)QT+T\mathrm{ch}(Q)\right]U'(0)= \\ &=R(1)+\mathrm{ch}(Q)R(0)-\sum_{n=0}^{\infty}M^n S_n(1)+T\mathrm{sh}(Q)QR(0)-T\sum_{n=0}^{\infty}M^n S_n'(1) \end{aligned} \tag{29}$$

After determining $U'(0)$, $U(0)$ can be found from (27). Finally, $U(x)$ is calculated through eq. (26).

The computation of $S_n(x)$ can be performed in a recursive manner, with the aid of Simpson or Gauss numerical integration.

Simpson's formula with 4 intervals provides

$$S_n(x)=\int_0^x\int_0^y S_{n-1}(t)\mathrm{d}t\mathrm{d}y\cong\int_0^x \frac{y}{12}\left[S_{n-1}(0)+S_{n-1}(y)+2S_{n-1}\left(\frac{y}{2}\right)+\right.$$
$$\left.+4S_{n-1}\left(\frac{y}{4}\right)+4S_{n-1}\left(\frac{3y}{4}\right)\right]\mathrm{d}y\cong S_{n-1}(0)\frac{x^2}{24}+\frac{x^2}{144}\left[S_{n-1}(x)+3S_{n-1}\left(\frac{x}{2}\right)+7S_{n-1}\left(\frac{x}{4}\right)+\right. \quad (30)$$
$$\left.+6S_{n-1}\left(\frac{x}{8}\right)+7S_{n-1}\left(\frac{3x}{4}\right)+10S_{n-1}\left(\frac{3x}{8}\right)+16S_{n-1}\left(\frac{3x}{16}\right)+12S_{n-1}\left(\frac{9x}{16}\right)+4S_{n-1}\left(\frac{x}{16}\right)\right]$$

for $n\ge 1$.

We observe that $S_{n-1}(0)=0,(\forall)n\ge 1$.

Similarly will be accomplished the calculation for $S_0(x)$.

On the other hand, $S_n'(x)$ will be computed by

$$S_n'(x)=\int_0^x S_{n-1}(y)\mathrm{d}y\cong\frac{x}{12}\left[S_{n-1}(x)+2S_{n-1}\left(\frac{x}{2}\right)+4S_{n-1}\left(\frac{x}{4}\right)+4S_{n-1}\left(\frac{3x}{4}\right)\right] \quad (31)$$

The other variant we take into consideration is the numerical integration by Gauss formula

$$\int_a^b f(x)\mathrm{d}x\cong\frac{b-a}{2}\sum_{i=1}^{3} A_i f\left(\frac{b-a}{2}t_i+\frac{a+b}{2}\right) \quad (32)$$

where $A_1=A_3=\frac{5}{9}, A_2=\frac{8}{9}, t_1=-\sqrt{\frac{3}{5}}, t_2=0, t_3=\sqrt{\frac{3}{5}}$. It follows

$$\begin{aligned}S_n(x)&=\int_0^x\int_0^y S_{n-1}(t)\mathrm{d}t\mathrm{d}y\approx\int_0^x \frac{y}{2}\sum_{i=1}^{3} A_i S_{n-1}\left(\frac{y}{2}t_i+\frac{y}{2}\right)\mathrm{d}y\approx\\ &\approx\frac{x^2}{8}\sum_{i,j=1}^{3} A_i A_j(t_j+1)S_{n-1}\left(\frac{x}{4}(t_i+1)(t_j+1)\right)\end{aligned} \quad (33)$$

For $S_n'(x)$ we have

$$S_n'(x)=\int_0^x S_{n-1}(y)\mathrm{d}y\cong\frac{x}{2}\sum_{i=1}^{3} A_i S_{n-1}\left(\frac{x}{2}(t_i+1)\right) \quad (34)$$

5. NUMERICAL RESULTS

We shall examine a numerical test for the vectorial decomposition method which was analyzed in the preceding sections.

Consider the integro-differential equation

$$t\frac{\partial\varphi}{\partial x}(x,t)+\varphi(x,t)=\frac{1}{2}\int_{-1}^{1}\varphi(x,\tau)\mathrm{d}\tau+f(x,t)\,,\;(x,t)\in[0,1]\times[-1,1] \tag{35}$$

where

$$f(x,t)=\begin{cases} t\left[\sin^2(tx)+(x-1)t\sin(2tx)\right]+(x-1)\sin^2(tx)+\dfrac{x-1}{2}\left[\dfrac{\sin(2x)}{2x}-1\right], & \text{for } x\neq 0\\ 0, \quad \text{for } x=0 \end{cases}$$

with the conditions $\varphi(0,t)=0 \quad \text{for } t>0, \quad \varphi(1,t)=0 \quad \text{for } t<0$.

The eqs. (9) provide

$$g(x,t)=\begin{cases} (x-1)\sin^2(tx)+\dfrac{x-1}{2}\left[\dfrac{\sin(2x)}{2x}-1\right], & \text{for } x\neq 0\\ 0, \quad \text{for } x=0 \end{cases}$$

$$r(x,t)=t\sin^2(tx)+(x-1)t^2\sin(2tx)$$

Also, we obtain

$$\bar{S}(x,t)=\frac{1}{t^2}\left[t\frac{\partial r}{\partial x}(x,t)-g(x,t)\right]=$$

$$=\begin{cases} 2t\sin(2tx)+2t^2(x-1)\cos(2tx)+\dfrac{1-x}{2t^2}\left[2\sin^2(tx)+\dfrac{\sin 2x}{2x}-1\right], & \text{for } x\neq 0\\ -2t^2, \quad \text{for } x=0 \end{cases}$$

The differential system (12) becomes

$$\begin{bmatrix} u_1'\\ u_2'\\ u_3'\\ u_4' \end{bmatrix}=\begin{bmatrix} \frac{32}{3} & -\frac{8}{3} & -\frac{16}{3} & -\frac{4}{3}\\ -\frac{4}{3} & \frac{10}{3} & -\frac{4}{3} & -\frac{1}{3}\\ -\frac{16}{27} & -\frac{8}{27} & \frac{32}{27} & -\frac{4}{27}\\ -\frac{1}{3} & -\frac{1}{6} & -\frac{1}{3} & \frac{11}{12} \end{bmatrix}\begin{bmatrix} u_1\\ u_2\\ u_3\\ u_4 \end{bmatrix}+\begin{bmatrix} \bar{S}(x,t_1)\\ \bar{S}(x,t_2)\\ \bar{S}(x,t_3)\\ \bar{S}(x,t_4) \end{bmatrix}$$

On the other hand, the conditions (15) are in this case

$$U(0)-TU'(0)=0,\ \ U(1)+TU'(1)=\begin{bmatrix} t_1\sin^2 t_1 \\ t_2\sin^2 t_2 \\ t_3\sin^2 t_3 \\ t_4\sin^2 t_4 \end{bmatrix}$$

Taking into account the eq.(26) we deduce

$$U(x)\approx \tilde{U}_n(x)=\begin{bmatrix} \tilde{u}_n(x,t_1) \\ \tilde{u}_n(x,t_2) \\ \tilde{u}_n(x,t_3) \\ \tilde{u}_n(x,t_4) \end{bmatrix}=\text{ch}(xQ)U(0)+\text{sh}(xQ)Q^{-1}U'(0)+\sum_{k=0}^{n}M^k S_k(x) \quad (36)$$

The computation was performed on a grid $x=0:0.05:1$ of the interval [0,1].
The exact solution of the eq. (35) is $\varphi(x,t)=(x-1)\sin^2(tx)$, while

$$u(x,t)=\frac{1}{2}[\varphi(x,t)+\varphi(x,-t)]=(x-1)\sin^2(tx),\ (x,t)\in[0,1]\times[0,1] \quad (37)$$

Table 1 contains the errors for different values of *n*:

$$E_n=\max\{|u_n(x_i,t_j)-u(x_i,t_j)|\ /\ x_i=i/20, i=\overline{0,20}, t_j=j/4, j=\overline{0,4}\}$$

There are two cases: employing Simpson’s integration (eqs. (30), (31)) and Gauss formula (eqs. (33), (34)).

Table 1

The errors for different values of *n*

n	computation based on eqs. (30), (31)	computation based on eqs. (33), (34)
2	0.01120332338819	0.01129740739601
3	0.00237282709821	0.00222986710352
4	0.000819120728469	0.000241840608410
5	0.000796459953999	0.000117534120375

The numerical solution $u_5(x_i,t_j), x_i=i/20, i=\overline{0,20}, t_j=j/4, j=\overline{0,4}$ can be interpolated on a 2-dimensional grid, for example $[x,t]=meshgrid$ $(0:0.05:1, 0:0.05:1)$.

In Fig.1 we obtained the graphical representation of the absolute error $|u_5(x,t)-u(x,t)|, (x,t)\in[0,1]\times[0,1]$, after interpolation.

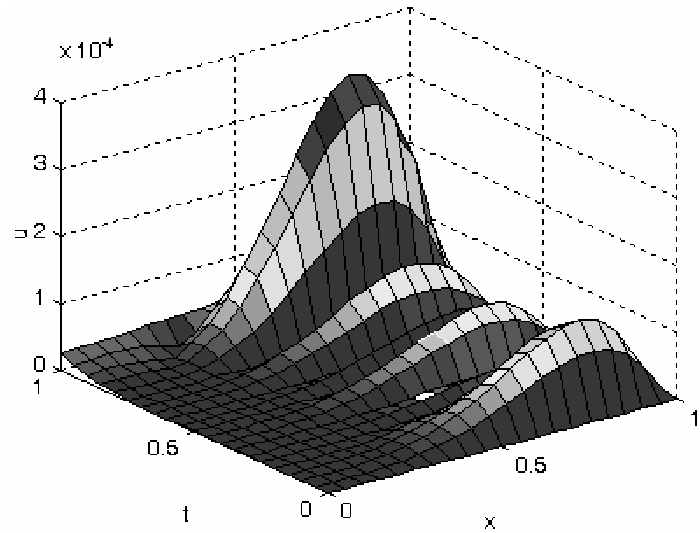

Fig.1 – Absolute error.

6. CONCLUSIONS

In this work, the steady 1-dimensional neutron transport equation was analyzed. The computation of the approximate solution is accomplished using a vectorial decomposition algorithm. Both Simpson and Gauss numerical integration formulas are involved. Comparison between exact and numerical solution proves very good agreement.

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