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Special Issue
New Advances in Quantum Geometry

Edited by
Prof. Dr. Shidong Liang, Prof. Dr. Tiberiu Harko and Dr. Matthew J. Lake

MDPI

https://doi.org/10.3390/physics4040086

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The Cosmology of a Non-Minimally Coupled $f(R,T)$ Gravitation

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Abstract: A non-minimally coupled cosmological scenario is considered in the context of $f(R,T) = f_1(R) + f_2(R)f_3(T)$ gravity (with R being the Ricci scalar and T the trace of the energy-momentum tensor) in the background of the flat Friedmann–Robertson–Walker (FRW) model. The field equations of this modified theory are solved using a time-dependent deceleration parameter for a dust. The behavior of the model is analyzed taking into account constraints from recent observed values the deceleration parameter. It is shown that the analyzed models can explain the transition from the decelerating phase to the accelerating one in the expansion of the universe, by staying true to the results of the observable universe. It is shown that the models are dominated by a quintessence-like cosmological dark fluid at the late universe.

Keywords: $f(R,T)$ theory; deceleration parameter; non-minimal coupling; dark energy

Citation: Sofuoğlu, D.; Tiwari, R.K.; Abebe, A.; Alfedeel, A.H.A.; Hassan, E.I. The Cosmology of a Non-Minimally Coupled $f(R,T)$ Gravitation. *Physics* **2022**, *4*, 1348–1358. https://doi.org/10.3390/physics4040086

Received: 12 August 2022
Accepted: 18 October 2022
Published: 7 November 2022

1. Introduction

The cosmological portrait of the universe has been changed by some indications of recent type Ia supernovae data [1–3] and the results of Planck Collaboration [4]. The revolutionary sign of these observations is that the expansion of the universe is currently accelerating. Various theories have been developed in the literature to explain this cosmic acceleration. It is believed that the cause of this acceleration is an energy called dark energy, which cannot be explained by the baryonic matter distribution. This dark energy is now known to have a large proportion of about 70% of the total energy distribution in the universe. In Λ-cold-dark-matter (ΛCDM) cosmology, this dark energy is usually explained by adding the cosmological constant Λ to the field equations of the general theory of relativity (GR).

However, such a cosmological scenario is pregnant with some cosmological problems [5], so some alternative models have been proposed [6–8]. In these alternative cosmologies, the model that behaves similar to the ΛCDM model is obtained without using the cosmological constant. The main reason here is the need for a cosmological model that can give the results of the observable universe, but on the other hand, will keep the problems brought by ΛCDM at bay. For example, it is one of the consequences of such a need to take the matter–energy content of the universe as the scalar field as exotic matter in Einstein's field equations that can produce enough negative pressure to accelerate the expansion of the universe; see Refs. [9–14].

In this context, modified gravity theories that serve this purpose are of great interest in current cosmological studies. The $f(R,T)$ theory of gravity is one of the most popular of these modified gravitational theories [15]. Here, the gravitational Lagrangian is given by an arbitrary function of the Ricci scalar, R, and the trace, T, of the energy–momentum tensor, the dependence of which can be induced by exotic imperfect fluids or quantum effects [15]. Since it was proposed in 2011, many researchers have performed a wide variety of research on this theory. Some to be mentioned are as follows. Xu et al. [16] studied quantum cosmology effects of the gravitational interaction described by the model. Friedmann–Robertson–Walker (FRW) model was examined by Myrzakulov [17]. Sharif and Zubair [18] obtained exact solutions of the field equations of $f(R,T)$ theory using the anisotropic behavior of spacetime for exponential and power expansion laws. Moraes et al. examined the transition from deceleration to acceleration in this theory [19]. Shamir [20] studied the locally rotationally symmetric (LRS) Bianchi type-I model. A string cosmological model was considered by Sharma and Singh [21] for Bianchi type-II universe. In order to understand the dynamic behavior of the anisotropic universe in $f(R,T)$ gravity, a large-scale search was made for the Bianchi-type $\mathrm{VI_h}$ model [22,23] by Mishra et al. Tiwari and Sofuoğlu [24] investigated the cosmological implications of a quadratically varying decelariton parameter in locally-rotationally-symmetric (LRS) Bianchi type-I model. Tiwari et al. studied Bianchi type-I universe taking into account time dependent gravitational and cosmological parameters [25]. Evolution of axially symmetric anisotropic sources was investigated in $f(R,T)$ theory by Zubair and Noureen [26]. Alfedeel and Tiwari showed that the generalized Friedman equation's exact solution for the average scale factor involves the hypergeometric function considering a novel approach [27]. An accelerating model studied in the presence of varying cosmological term by Tiwari et al. [28]. A cosmological model with variable deceleration parameter in $f(R,T)$ theory was constructed by Tiwari et al. [29]. Sahoo et al. [30] studied on Bianchi type-I universe taking bulk viscous fluid. Moraes and Sahoo [31] considered nonminimal coupling between geometry and matter in this theory. Sharma et al. [32] examined the existing of non-minimal matter–geometry interaction in Bianchi type-I model. Tiwari et al. [33] studied a non-minimal cosmological model in the presence of a varying deceleration parameter.

In the current study, inspired by the above discussion, we consider the $f(R,T)$ modified theory of gravity in the background of flat FRW universe by considering a variable deceleration parameter to investigate the phase change (from decelerating to accelerating expansion phase) in the expansion of the universe. For the choice of a particular case of the non-minimally function $f(R,T) = f_1(R) + f_2(R)f_3(T)$, exact solution of the field equations has been obtained. In Section 2, a basic formalism of $f(R,T)$ theory is presented, the solutions of the field equations are obtained in Section 3, and the conclusions are given in Section 4.

2. $f(R,T)$ Gravity

Throughout this section, we review the basic derivation of the $f(R,T)$ theory of gravity. Let us start by introducing the the action of $f(R,T)$ gravity is defined by Harko et al. [15]:

$$S = \int \sqrt{-g}\, d^4x \left(\frac{1}{16\pi G} f(R,T) + L_{\mathrm{m}} \right), \tag{1}$$

where $f(R,T)$ is an arbitrary function of R and $T = g^{ij}T_{ij}$ the trace of the energy–momentum tensor, T_{ij}; Latin letters $i, j, k, l, \ldots$ denote 4-dimensional tensor indices and take on the values 0 (time), 1, 2, and 3 (space); $g = \det|g_{ij}|$ is the determinant of the metric tensor, g_{ij}; G is Newtonian constant of gravity, and L_{m} is the matter Lagrangian. Accordingly, the energy–momentum tensor, T_{ij}, is defined as

$$T_{ij} = -\frac{2}{\sqrt{-g}} \frac{\delta(\sqrt{-g} L_{\mathrm{m}})}{\delta g^{ij}} = L_{\mathrm{m}} g_{ij} - 2\frac{\delta L_{\mathrm{m}}}{\delta g^{ij}}\,. \tag{2}$$

Here, we assume that $L_{\rm m}$ is a function the metric tensor, g_{ij}, rather than of its derivatives. By varying the action S in Equation (1) with respect to the metric tensor, g_{ij}, the modified gravitational field equations for $f(R,T)$ gravity reads:

$$f_R(R,T)R_{ij} - \frac{1}{2}f(R,T)g_{ij} - (\nabla_i\nabla_j - g_{ij}\Box)f_R(R,T) = 8\pi T_{ij} - f_T(R,T)(T_{ij} + \Theta_{ij})\,, \quad (3)$$

where ∇_i is the covariant derivative, $\Box \equiv \nabla^i\nabla_i$ is the d'Alembertian operator and the fractions moved to a linear form better visible in the text. Please confirm. $f_R(R,T) = \partial f(R,T)/\partial R$, $f_T(R,T) = \partial f(R,T)/\partial T$, $\Theta_{ij} = g^{ab}\delta T_{ab}/\delta g^{ij}$. Contracting Equation (3) with metric tensor g^{ij} produces

$$f_R(R,T)R + 3\Box f_R(R,T) - 2f(R,T) = 8\pi T - f_T(R,T)(T + \Theta)\,, \quad (4)$$

where $\Theta = g^{ij}\Theta_{ij}$. Upon using the matter Lagrangian $L_{\rm m}$, the energy–momentum tensor of matter is given by

$$T_{ij} = (\rho + p)u_i u_j - p g_{ij}\,, \quad (5)$$

where ρ, p, u_i are the fluid energy density, the pressure of the fluid and the fluid 4-velocity, respectively. Further, u_i is time-like quantity that satisfies $u^i u_i = 1$ and $u^i\nabla_j u_i = 0$. The variation of stress energy of perfect fluid is obtained by following Shamir [34] argument where the matter Lagrangian $L_{\rm m} = -p$ is assumed, thus

$$\Theta_{ij} = -2T_{ij} - p g_{ij}\,. \quad (6)$$

Substituting Equation (6) into Equation (3), the field equations take the form

$$f_R(R,T)R_{ij} - \frac{1}{2}f(R,T)g_{ij} - (\nabla_i\nabla_j - g_{ij}\Box)f_R(R,T) = 8\pi T_{ij} + f_T(R,T)(T_{ij} + p g_{ij})\,. \quad (7)$$

Equation (7) leads to the modified $f(R)$ and GR theories of gravity when $f(R,T) = f(R)$ and $f(R,T) = R$, respectively. Ref. [15] investigated three different functional forms of $f(R,T)$. More justification about the choice of $f(R,T)$ is given in [35]. These forms are given by

$$f(R,T) = \begin{cases} R + 2f(T), \\ f_1(R) + f_2(T), \\ f_1(R) + f_2(R)f_3(T). \end{cases}$$

In this paper, we adopt the last functional form,

$$f(R,T) = f_1(R) + f_2(R)f_3(T) = R + \lambda R T\,, \quad (8)$$

thus transforming Equation (7) into the following form:

$$R_{ij} - \frac{1}{2}g_{ij}R = +8\pi T_{ij} - \lambda[g_{ij}\Box - \nabla_i\nabla_j]T - \lambda T[R_{ij} - \frac{1}{2}g_{ij}R] + \lambda R[T_{ij} + p g_{ij}]\,, \quad (9)$$

or, alternatively,

$$R_{ij} - \frac{1}{2}g_{ij}R = \frac{8\pi + \lambda R}{(1+\lambda T)}T_{ij} - \frac{\lambda}{(1+\lambda T)}[g_{ij}\Box - \nabla_i\nabla_j]\,T + \frac{\lambda R}{(1+\lambda T)}p g_{ij}\,. \quad (10)$$

Here, λ is the coupling parameter of the model and vanishes automatically for GR.

The right-hand side of Equation (10) can be viewed as a total-effective energy momentum tensor, T^t_{ij},

$$T^t_{ij} = T_{ij} + T^f_{ij}\,, \quad (11)$$

with T^f_{ij} defined as

$$T^f_{ij} = \frac{\lambda}{1+\lambda T}\left(RT_{ij} + Rpg_{ij} - [g_{ij}\Box - \nabla_i\nabla_j]T\right), \tag{12}$$

showing the contribution term from $f(R,T)$. The limiting case $\lambda = 0$ in Equation (9) gives standard GR results.

The FRW Metric and Field Equations

The homogeneous and isotropic flat FRW universe is given as

$$ds^2 = dt^2 - a^2(t)(dx^2 + dy^2 + dz^2), \tag{13}$$

where $a(t)$ is the scale factor. In the flat FRW background, the non-minimally coupled $f(R,T)$ gravity for $T = \rho - 3p$, the time-time and space-space components of the modified field equations of Equation (9) gives the following Friedmann equations:

$$3H^2 = 8\pi\rho - 3\lambda H(\dot{\rho} - 3\dot{p}) - 3\lambda H^2(\rho - 3p) - 6\lambda(\dot{H} + 2H^2)(p+\rho), \tag{14}$$

$$2\dot{H} + 3H^2 = -8\pi p - \lambda(2\dot{H} + 3H^2)(\rho - 3p) - \lambda(\ddot{T} - 2H\dot{T}). \tag{15}$$

Here, H is the Hubble constant and the dot denotes time derivation.

As soon as $\dot{T} = \dot{\rho} - 3\dot{p}$ and $\ddot{T} = \ddot{\rho} - 3\ddot{p}$, Equation (15) becomes:

$$2\dot{H} + 3H^2 = -8\pi p - \lambda(2\dot{H} + 3H^2)(\rho - 3p) - \lambda[\ddot{\rho} - 3\ddot{p} - 3H(\dot{\rho} - 3\dot{p})]. \tag{16}$$

Equations (14) and (16) are the generalized Friedmann equation in $f(R,T)$ theory of gravity. These equations cannot be solved since they contain $\lambda, \dot{\rho}, \dot{p}, \ddot{\rho}$ and $\ddot{p}$. On the other hand, Equation (9) gives the Bianchi identity as

$$-8\pi\nabla^j T_{ij} = \frac{\lambda R}{2}(\nabla_i T) + \lambda(\nabla^j R)[T_{ij} + pg_{ij}] + \lambda R[\nabla^j T_{ij} + \nabla_j p], \tag{17}$$

which gives

$$(8\pi + \lambda R)\dot{\rho} + 3H(p+\rho) = -\frac{\lambda R}{2}(\dot{\rho} - \dot{p}) - \lambda\dot{R}(\rho + p). \tag{18}$$

Assuming that $p = w\rho$, Equation (17) can be re-arranged for $\dot{\rho}$ as

$$\dot{\rho} = -\frac{3H(1+w) + \lambda\dot{R}}{8\pi + \frac{1}{2}\lambda R(3-w)}\rho, \tag{19}$$

upon differentiating with respect to time yields:

$$\ddot{\rho} = \left\{-\frac{3\dot{H}(1+w) + \lambda\ddot{R}}{8\pi + \frac{1}{2}\lambda R(3-w)} + \frac{[3H(1+w) + \lambda\dot{R}(5-w)][3H(1+w) + \lambda\dot{R}]}{[8\pi + \frac{1}{2}\lambda R(3-w)]^2}\right\}\rho, \tag{20}$$

where for a flat FRW metric,

$$R = -6(\dot{H} + 2H^2), \tag{21}$$

$$\dot{R} = -6(\ddot{H} + 4H\dot{H}), \tag{22}$$

$$\ddot{R} = -6(\dddot{H} + 4\dot{H}^2 + 4H\ddot{H}) \tag{23}$$

are the Ricci scalar and its time deviatives. The generalized Friedmann Equations (14) and (16) now read:

$$3H^2 = 8\pi\rho - 3\lambda(1-3w)H[\dot{\rho} + H\rho] - 6\lambda(1+w)(\dot{H} + 2H^2)\rho, \tag{24}$$

$$2\dot{H}+3H^2=-8\pi w\rho-\lambda(1-3w)[(2\dot{H}+3H^2)\rho-2H\dot{\rho}+\ddot{\rho}]\,. \tag{25}$$

Subtracting Equations (24) and (25) one from another produces the generalized Raychaudhri equation,

$$\begin{aligned} 2\dot{H} \;\; &= \;\; -8\pi(1+w)\rho-\lambda(1-3w)[(2\dot{H}+3H^2)\rho-2H\dot{\rho}+\ddot{\rho}] \\ &\quad +3\lambda(1-3w)H[\dot{\rho}+H\rho]-6\lambda(1+w)(\dot{H}+2H^2)\rho\,. \end{aligned} \tag{26}$$

Having known the value of $\dot{\rho}$ and $\ddot{\rho}$ (from Equations (19) and (20), respectively) and H, Equation (26) is solved in Section 3 below to obtain an expression for the energy density ρ of the matter content of universe directly. In this model, the Hubble parameter, H, and deceleration parameter (DP) are defined, respectively, as

$$H\equiv\frac{\dot{a}}{a} \quad \text{and} \quad q\equiv-1-\frac{\dot{H}}{H^2}\,. \tag{27}$$

Using Equation (9), Equations (24) and (25) give the total-effective density, ρ^t, and the total-effective pressure, p^t:

$$\rho^t=\rho-3\lambda(1-3w)H(\dot{\rho}+H\rho)+6\lambda(1+w)(\dot{H}+2H^2)\rho\,, \tag{28}$$

$$p^t=w\rho-\lambda(1-3w)[(2\dot{H}+3H^2)\rho-2H\dot{\rho}+\ddot{\rho}]\,. \tag{29}$$

From Equations (28) and (29), with the help of Equation (11), one obtains the density ρ in terms of H, $\dot{H}$, $\dot{\rho}$, and $\ddot{\rho}$:

$$\rho=\frac{1}{4}\left[\frac{2\dot{H}-5\lambda H(1-3w)\dot{\rho}+\lambda(1-3w)\ddot{\rho}}{-2\pi(1+w)+\lambda(1+3w)\dot{H}+3\lambda(1+w)H^2}\right]. \tag{30}$$

3. Solutions of the $f(R,T)$ Field Equations

To solve the system of Equations (20)–(24) containing two equations and three unknowns (a, ρ, and p), one more equation is needed. Since the Type Ia supernova observations and various astronomical observations [1,2,36,37] indicated that the universe is accelerating, a time-dependent DP is needed that can explain the transition from deceleration expansion in the past at $z\geq 1$ to acceleration expansion at present. In concordance with this argument, many parametrization have proposed that DP is time-dependent to study various problems in cosmology [38–40]. For instance, for Berman [41] and Gomide [42], the law of variation for Hubble parameter that yields a constant DP. Ref. [43] introduced a linear function of the Hubble parameter, and well motivated by [44,45]. Motivated by the above discussion, in this paper, we adopt a generalization form of deceleration parameter that is introduced in Equation (27) as a function of Hubble parameter:

$$q=\alpha-\frac{\beta}{H^2}, \tag{31}$$

where α is a dimensionless constant, while the other constant, β, has the dimensions of H^2. Using this relation along with Equation (27) for solving the scale factor and the Hubble parameter, one obtains:

$$a=\left\{\sinh\left[\sqrt{(1+\alpha)\beta}\;t+c\right]\right\}^{\frac{1}{1+\alpha}}, \tag{32}$$

$$H=\sqrt{\frac{\beta}{1+\alpha}}\coth\left[\sqrt{(1+\alpha)\beta}\;t+c\right], \tag{33}$$

where c is the constant of integration.

Substituting the values of H, R, and $\dot{R}$ into Equation (19) gives:

$$\dot{\rho} = \frac{3A\coth(\tau)[1 + w + 4B\lambda(2A - B)\mathrm{cosech}^2(\tau)]}{8\pi + A\lambda(w-3)[A\coth^2(\tau) - 3B\mathrm{cosech}^2(\tau)]}\rho\,, \tag{34}$$

where $A = \sqrt{\frac{\beta}{1+\alpha}}$ and $B = \sqrt{(1+\alpha)\beta}$.

Integrating Equation (34) results into the following expression for energy density:

$$\rho = \rho_0 \exp\left\{\int \frac{3A\coth(\tau)[1 + w + 4B\lambda(2A - B)\mathrm{cosech}^2(\tau)]}{8\pi + \lambda(w-3)[A^2\coth^2(\tau) - 3AB\mathrm{cosech}^2(\tau)]}\, dt\right\}, \tag{35}$$

where $\tau \equiv \sqrt{(1+\alpha)\beta}\, t + c$, and ρ_0 is a constant of integration. It is worth mentioning that the processes of obtaining a simplified expression for the energy density ρ from Equations (20)–(25) is not straightforward as soon as it depends on $R, \dot{R}, \dot{\rho}$, and $\ddot{\rho}$. If $\lambda = 0$, Equation (35) give the GR limit as

$$\rho = \frac{3(1+w)\rho_0}{8\pi}\sqrt{\frac{\beta}{1+\alpha}}\sinh(\tau)\,. \tag{36}$$

It is always viable to write the explicit expression of ρ using its temporal derivatives, but the expression is large and complex. Instead, it is possible to calculate the integral of Equation (34) or Equation (35) for $w = 0$, which gives the following expression for the energy density ρ:

$$\rho = \rho_0 \left[(\sinh(\tau))^{-1}\right]^{\frac{3A}{4B(-9\lambda A^2 + 4\pi)}} \left[\frac{9\lambda A(-2A\cosh^2\tau + B)}{\sinh^2\tau} + 8\pi\right]^{\frac{-72\lambda A^2 B + 32\pi B + 9A}{-12B(-9\lambda A^2 + 4\pi)}} . \tag{37}$$

To show the graphical representation of the model, let us first write the expressions of the parameters in terms of redshift, z. Using the relation $a = (1+z)^{-1}$, one obtains:

$$q = \alpha - \frac{b}{h^2} = \alpha - \frac{b(1+\alpha)}{1 + (1+z)^{2+2\alpha}}\,, \tag{38}$$

$$h = \sqrt{(1+\alpha)^{-1}}\sqrt{(1+z)^{2(1+\alpha)} + 1}\,, \tag{39}$$

$$\rho = \rho_0 (1+z)^{\frac{3(1+\alpha)A}{4B(-9\lambda A^2 + 4\pi)}} \times \left[(1+z)^{2\alpha}\Big(9\lambda A\Big((-2A + B)[z^2 + z + 1] + (-2A + 7B)z - 2\Big) + 8\pi\Big)\right]^{\frac{-72\lambda A^2 B + 32\pi B + 9A}{-12B(-9\lambda A^2 + 4\pi)}}, \tag{40}$$

where $h = H/H_0$ is the normalized expansion rate and $b = \beta/H_0^2$ is a normalized constant with H_0 being the Hubble constant.

To plot the graphs, different values of the constants α and b were selected considering the observable universe as initial conditions $(z = 0)$. We take into account the results of three different observations for the current values of the deceleration parameter, namely $q_0 = -0.54$ [46], $q_0 = -0.73$ [47] and $q_0 = -0.81$ [48]; the models obtained for each of the observational values called Model 1 (M1), Model 2 (M2), and Model 3 (M3), respectively. In what follows, the graphs are plotted for the corresponding three different α and b pairs: $\alpha = 0.4761$, $b = 1.3903$ for M1, $\alpha = 0.5685$, $b = 1.6557$ for M2, and $\alpha = 0.6054$, $b = 1.7633$ for M3.

From Figure 1, one can see that the normalized Hubble parameter, h, has a larger value at high redshift zone and it is smaller at the low redshift zone for all the models. Figure 2 shows that while the sign of the deceleration parameter was initially positive, it became negative in the late universe for each model. This sign change indicates that the universe has moved from its decelerating expansion in the past to its current accelerating expansion. It is seen that the transition from slowing expansion to accelerating expansion takes place at almost $z = 1/2$ for three of the models.

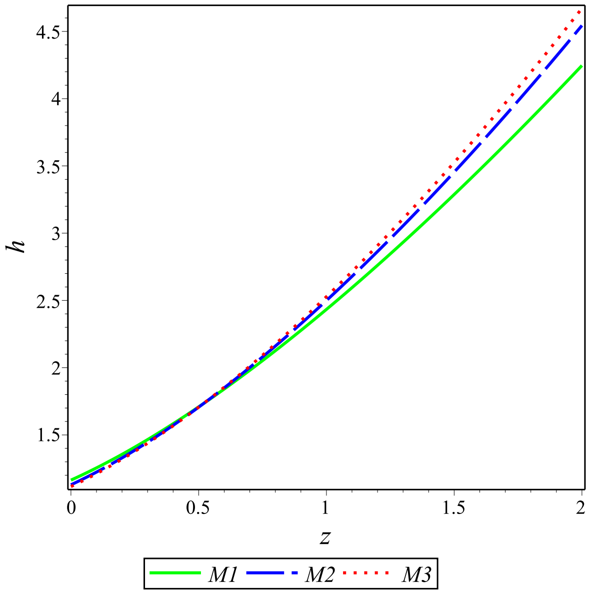

Figure 1. The normalized expansion rate, h (39), versus redshift, z, for three observational models M1, M2, and M3. See text for details.

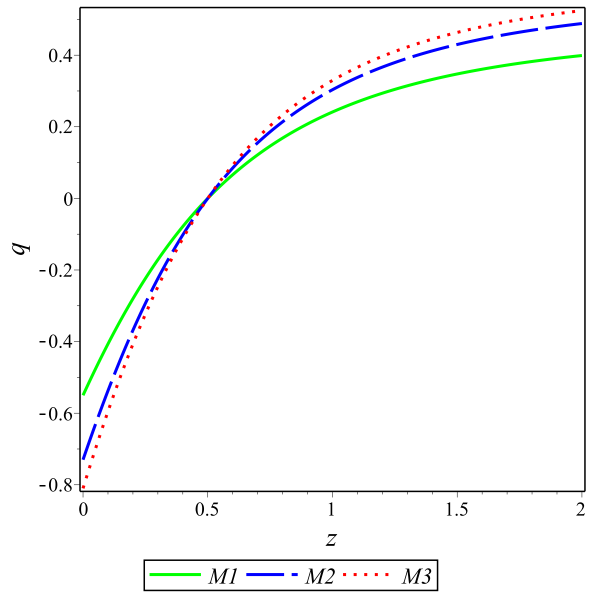

Figure 2. The deceleration parameter, q (38), versus redshift, z, for three observational models M1, M2, and M3. See text for details.

Figure 3 shows that the energy density ρ decreases from the high redshift region to the low redshift region and always remains positive.

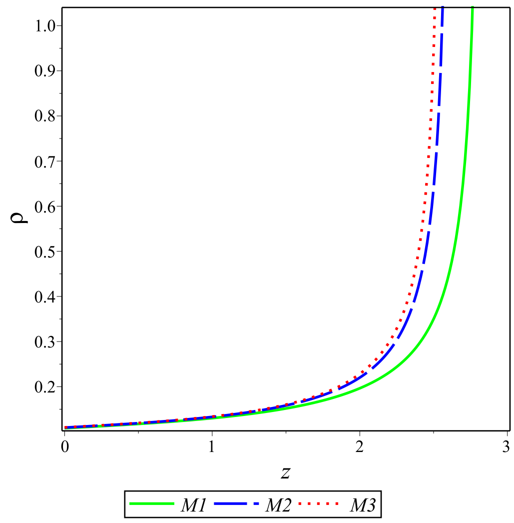

Figure 3. The energy density, ρ (40), versus redshift, z, with $\lambda = 0.1$ and $\rho_0 = 1$ for three observational models M1, M2, and M3. See text for details.

Now, let us use the relation

$$w^t = \frac{1}{3}(2q - 1) \tag{41}$$

between the deceleration parameter and total-effective equation of state (EoS) parameter, w^t, to obtain

$$w^t = \frac{1}{3}\left[2\alpha - \frac{2b(1+\alpha)}{(1+z)^{2+2\alpha}} - 1\right]. \tag{42}$$

Figure 4 shows the evolution of the total-effective EoS parameter in redshift for each model. It is seen that while the EoS parameter has positive values at high redshift regions, it decreases and takes negative values at low redshift regions. Current values of w^t indicate that the models are dominated by a quintessence-like dark fluid currently.

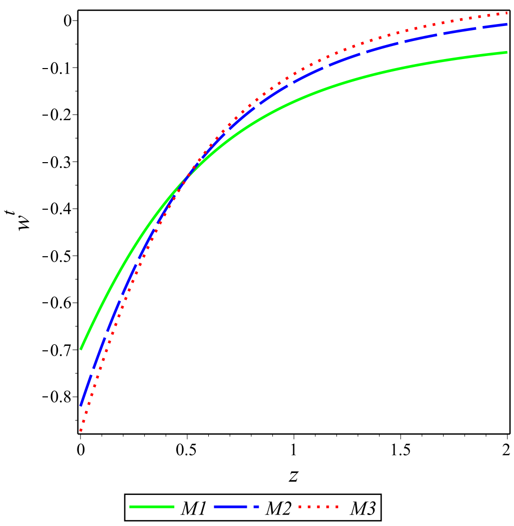

Figure 4. The The equation of state (EoS) parameter, w^t (42), versus redshift, z, for three observational models M1, M2, and M3. See text for details.

4. Conclusions

In this study, we investigated a non-minimally coupled cosmological model in the context of $f(R,T)$ theory for the flat Friedmann–Robertson–Walker (FRW) metric. For the choice of the function $f(R,T)$ in the form of $f(R,T) = f_1(R) + f_2(R)f_3(T)$ with $f_1(R) = f_2(R) = R$ and $f_3(T) = \lambda T$, the solutions of the field equations were solved under the assumption of a time-dependent deceleration parameter, which can explain the evolution of the expansion of the universe from beginning to current epoch. Results obtained were discussed by means of their graphs in redshift space, by taking into account three different observed values of the deceleration parameter as three different observational models (M1, M2, and M3).

The evolution of the deceleration parameters shows that the phase change in the expansion of the universe occurs at almost $z = 1/2$ redshift, and the deceleration parameter current values (q_0) are -0.55, -0.729, and -0.81 for M1, M2, and M3 models, respectively. These values of q_0 are consistent with the observational values of the deceleration parameter.

The total or effective equation of state parameter, w^t, is found to have positive values initially, but continue their evolution by taking negative values, for each model. The current values of w^t are less than $-1/3$ and greater than -1. These values tell us that each model is dominated by a quintessence-like dark fluid.

As a next step, it is worth constraining the parameter space of these models with current and upcoming astronomical data, as well as studying the cosmological perturbations in the context of these models to analyze large-scale structure formation scenarios.

Author Contributions: Conceptualization, D.S and R.K.T.; methodology, D.S., R.K.T. and A.A.; validation, D.S., A.A. and A.H.A.A.; formal analysis, D.S., A.A., A.H.A.A. and E.I.H.; writing—original draft preparation, D.S.; writing—review and editing, A.A. and A.H.A.A. All authors have read and agreed to the published version of the manuscript.

Funding: The authors extend their appreciation to the Deanship of Scientific Research at Imam Mohammad Ibn Saud Islamic University (Riyadh, Saudi Arabia) for funding this work through Research Group no. RG-21-09-18.

Data Availability Statement: The data used in the study are available from the corresponding references.

Conflicts of Interest: The authors declare no conflict of interest.

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