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Global geometry of the Vaidya spacetime

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Abstract. The transformation from initial coordinates (v, r) of the Vaidya metric, with light coordinate v, to the more physical diagonal coordinates (t, r) is obtained and analyzed from the point of view of global geometry. The exact solutions have been obtained in the case of a linear form of the mass function $m(v)$. In the diagonal coordinates under consideration, a narrow region has been revealed near the apparent horizon of the Vaidya accreting black hole, where the metric differs qualitatively from the Schwarzschild one and cannot be represented as a small perturbation of the Schwarzschild solution. The global geometry of the thick photon shell between the two (inner and outer) Schwarzschild regions is constructed and the corresponding matching conditions are discussed. The propagation of light beams in the Vaidya metric in the case of accretion is investigated and the time of the apparent horizon's crossings is calculated.

1. Introduction

The Vaidya metric describes the space-time created by a spherically symmetric radial flow of photons. This metric has the form [1-3]

$$ds^2 = \left[1 - \frac{2m(v)}{r}\right] dv^2 - 2dvdr - r^2 d\Omega, \tag{1}$$

where $m(v)$ is an arbitrary mass function, and (in the case of accretion) v is the advance light coordinate. If $m(v) = \text{const}$, the metric (1) describes the Schwarzschild black hole. We use the units $c = 1$, $G = 1$. The Vaidya metric (1) has a number of astrophysical and theoretical applications. For example, it is used to describe the quantum evaporation of black holes [4] or the emission from astrophysical objects [5]. The Vaidya metric was also considered in the studies of the gravitational collapse [6].

In this paper, coordinate transformation from standard coordinates of the Vaidya metric (v, r) to diagonal coordinates is investigated. The explicit form of the corresponding coordinate transformation is generally unknown [7]. However, in the case of a linear mass function, the solution was obtained in [8,9], and all the metric coefficients were calculated. The goal of this paper is to construct a global geometry for the Vaidya space-time, bounded by regions of pure Schwarzschild geometry. Therefore, we consider the thick photonic shell occupying a limited area along the radius.

After the transition to diagonal coordinates, we calculate the time by the clock of a static observer (staying at $r = \text{const}$), during which a photon reaches the apparent horizon, and we show that for some observers this time is finite.

2. Vaidya problem in diagonal coordinates

We write the metric in the curvature diagonal coordinates (t, r, θ, ϕ) as follows

$$ds^2 = f_0(t,r)dt^2 - \frac{dr^2}{f_1(t,r)} - r^2(d\theta^2 + \sin^2\theta d\phi^2), \tag{2}$$

where $f_0(t,r)$ and $f_1(t,r)$ are some new functions. The light coordinate v in (1) is a function of new coordinates

$$v = v(t, \tilde{r}), \ \ r = \tilde{r}. \tag{3}$$

Substituting $dv = \dot{v}dt + v'dr$ into (1) and equating the coefficients to the corresponding coefficients of (2), we get

$$f_0 = \left[1 - \frac{2m(v)}{r}\right]\dot{v}^2, \tag{4}$$

$$f_1 = 1 - \frac{2m(v)}{r}, \tag{5}$$

and

$$v' = \frac{1}{1 - \frac{2m(v)}{r}}. \tag{6}$$

We consider the model with linear mass function

$$m(v) = \alpha v + m_0, \tag{7}$$

where $\alpha > 0$ characterizes the accretion rate. From the above relations, one can find a solution in a parametric form. We present it here for the case $\alpha < 1/8$. Denote $y = 1 - 2m(v)/r$, then

$$r = 2\alpha\beta(t)\Psi(y), \tag{8}$$

where the function $\beta(t) \geq 0$ is determined by the boundary conditions,

$$\Psi(y) = |y - y_1|^{\frac{y_1}{y_2 - y_1}}|y - y_2|^{-\frac{y_2}{y_2 - y_1}}, \tag{9}$$

and

$$y_1 = \frac{1 - \sqrt{1 - 8\alpha}}{2}, \quad y_2 = \frac{1 + \sqrt{1 - 8\alpha}}{2}. \tag{10}$$

The function v is expressed through the parameter y as follows

$$v = \frac{r(1 - y) - 2m_0}{2\alpha}. \tag{11}$$

It is convenient to denote $M(t,r) = m(v)$.

Differentiating (8) by t and substituting in (4), we get

$$f_0 = \dot{\beta}^2(t)\frac{1}{y}|y - y_1|^{\frac{2y_2}{y_2 - y_1}}|y - y_2|^{\frac{-2y_1}{y_2 - y_1}}. \tag{12}$$

The expression (12) together with $f_1 = y$ and (8) gives the solution for the Vaidya metric in the curvature coordinates in parametric form, where y is the parameter. We see that there are coordinate singularities at $y = 0$, $y = y_1$, and $y = y_2$. This means that a single diagonal coordinate system is not enough to cover the entire space-time, and several such coordinate systems are required. At an arbitrarily weak but non-zero accretion rate, there is a region

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near the BH horizon in which there are coordinate singularities, and the metric is not only quantitatively, but even qualitatively different from the Schwarzschild one. This situation is described in more detail in [8-9].

The metric coefficient (12) takes the simplest form if one introduces the new time $d\tilde{t}^2 = \dot{\beta}^2 dt^2$. Then one has $\dot{\beta}^2 = 1$, $\beta = \pm t+$ const, where the sign should be chosen in each of the regions

$$-\infty < y < 0, \quad 0 < y < y_1, \quad y_1 < y < y_2, \quad y_2 < y < 1 \tag{13}$$

from the condition $\dot{v} > 0$. With this choice of $\tilde{t}$ as a time, the metric has the simplest form, but in this case, however, it is impossible to extend the time line from the Schwarzschild metric region to the region occupied by photons at all values of the radius r. Therefore, below we consider another choice of the function $\beta(t)$, with which we can match the Vaidya metric and the Schwarzschild metric along the boundary of the photonic shell. This will allow us to speak about the time when a photon crosses the horizon according to the clock of the external Schwarzschild observer.

3. Matching conditions

We now turn to the discussion of the boundary conditions for the partial differential equation (6), which otherwise can be considered as the matching condition. We consider the so-called "thick shell", when $m(v) =$ const at $v < v_1$ and $v > v_2$, and in the interval $v_1 \leq v \leq v_2$, the function $m(v)$ grows linearly according to (7). Denote $M_1 = m(v_1)$, $M_2 = m(v_2)$, and the special case with $M_1 = 0$ is also possible. In the coordinates (t, r), this configuration means that inside a certain radius r_1 there is a Schwarzschild metric with the mass function M_1, and outside some larger radius r_2 there is empty space and the Schwarzschild metric with the mass function $M_2 > M_1$. In the interval $r_1 < r < r_2$, the space is filled with radially moving photons.

For the Schwarzschild metric with $M_1 =$ const, one obtains the following expression for the light coordinate

$$v = t + r + 2M_1 \ln(r - 2M_1) + const. \tag{14}$$

We perform the matching of time lines along the inner boundary of the photon shell $v = v_1 = const$. Matching the time lines is equivalent to finding a solution of the equations (6) passing through the curve (14) with $v = v_1 = const$. Let us denote the parameter y along the line $v = v_1 = const$ as $y = \xi_1$. The radius is an invariant and it is expressed at the inner and at outer sides of the border as follows

$$r = \frac{2M_1}{1 - \xi_1} = 2\alpha\beta[t(\xi_1)]\Psi(\xi_1), \tag{15}$$

where time along the border is expressed from (14) as

$$t(\xi_1) = v_1 - \frac{2M_1}{1 - \xi_1} - 2M_1 \ln\left(\frac{2M_1\xi_1}{1 - \xi_1}\right) + C_1, \tag{16}$$

where $C_1 = const$. The expressions (15) and (16) define β as a function of t in a parametric form through the parameter ξ_1, which has the physical meaning as y along the inner border of the photon shell. The function $\beta(t)$, obtained by numerical solution of both (15) and (16), is shown at Figure 1.

Similarly, matching solutions along the outer boundary $v = v_2 = const$, one has

$$r = \frac{2M_2}{1 - \xi_2} = 2\alpha\beta[t(\xi_2)]\Psi(\xi_2), \tag{17}$$

$$t(\xi_2) = v_2 - \frac{2M_2}{1 - \xi_2} - 2M_2 \ln\left(\frac{2M_2\xi_2}{1 - \xi_2}\right) + C_2, \tag{18}$$

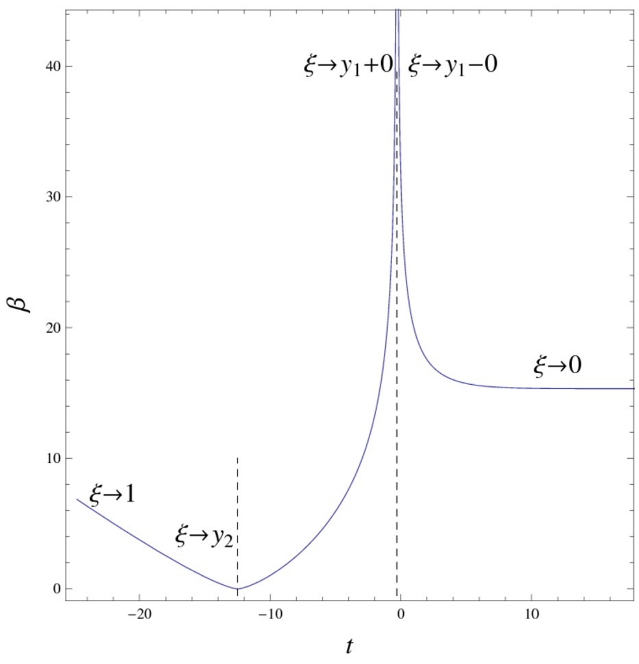

Figure 1. The function $\beta(t)$.

where $C_2 = const$. Moreover, the function $\beta(t)$ remains the same as in (15), since it is given by the metric inside the shell.

The compatibility of the matching conditions (15), (16) and (17), (18) is not trivial. Namely, it is impossible to select the static Schwarzschild metric simultaneously inside and outside the shell. In at least one of these areas, the metric should be non-static. This corresponds to the following time transformation:

$$g_{00}dt^2 = \left(1 - \frac{2M}{r}\right)\frac{dt^2}{d\tilde{t}^2}\tilde{t}^2 = \tilde{g}_{00}d\tilde{t}^2, \tag{19}$$

where the new $\tilde{g}_{00}$ depends on $\tilde{t}$.

The study of the conditions (15) and (17) shows that the lines $t = const$ inside the shell go along the lines $y = y_1$ and $y = y_2$, and for $y \neq y_{1,2}$ the parameters corresponding to the same line $t = const$ satisfy the condition $\xi_1 > \xi_2$. It is important that the line $t = const$ enters the line $y = 0$ with the infinite derivative $v' \to \infty$, which immediately follows from (6).

4. Photon infall time

Let us investigate the incoming light rays and find the time of the photon moving to the gravitational radius $y = 0$. For incoming rays from $ds^2 = 0$ it follows $v = const$, which in the coordinates (t, r) is written as $dt = -dr/\sqrt{f_0 f_1}$. We take into account that in this case $M = m(v) = const$. Substituting $f_1 = y$ and f_0, we find the change of β as the parameter y changes from some initial value y_i to the current value of y:

$$\Delta\beta = \beta - \beta_i = 2M\int_y^{y_i} \frac{dx|x-y_1|^{\frac{-y_2}{y_2-y_1}}|x-y_2|^{\frac{y_1}{y_2-y_1}}}{(1-x)^2} =$$
$$= M(1-x)^{-2}F_1\left[2; \frac{y_2}{y_2-y_1}, \frac{-y_1}{y_2-y_1}; 3, \frac{1-y_1}{1-x}, \frac{1-y_2}{1-x}\right]\Bigg|_{x=y}^{x=y_i}, \tag{20}$$

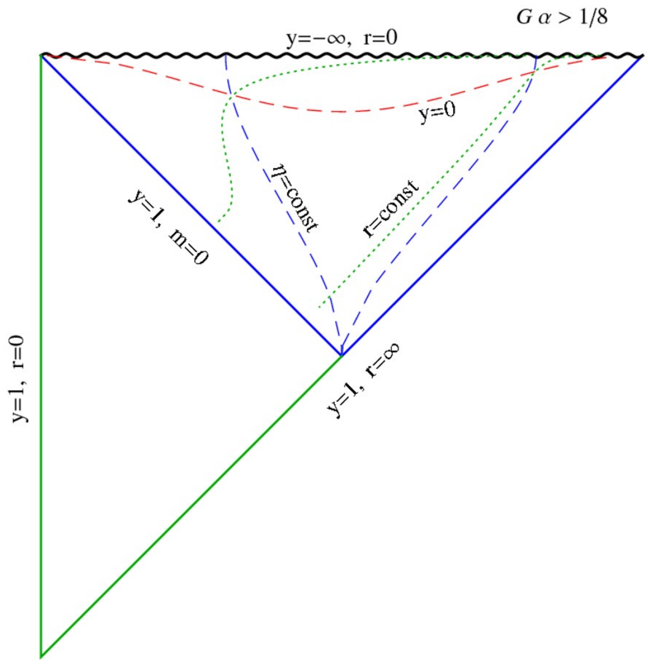

Figure 2. Global geometry of the Vaidya-Minkowski space-time.

where $F_1[...]$ is the Appell's hypergeometric function. From this expression it is clear that for $y_1 \neq 0$ the change in the function $\beta(t)$ is finite then the photon intersects the horizon. But we need to calculate the corresponding time interval Δt.

In the case of the simplest choice of a metric with $\beta = \pm t + const$, that was discussed earlier, the finiteness of Δt immediately follows from the finiteness of $\Delta\beta(t)$. This finite time is measured by an observer who is inside the shell, and the corresponding constant time lines cannot be continued outside the shell to the Schwarzschild region for all values of r.

Now we discuss the value of Δt when one matches the time lines inside the shell with time lines in the Schwarzschild metric outside the shell according to the conditions (15), (16) or (17), (18). An analysis done at the end of the previous section gives the following picture. If we match the time lines inside the shell with the static Schwarzschild metric inside it, and examine the approach of the photon inside the shell ($v_1 < v < v_2$) to the gravitational radius $y = 0$, then the time lines of the internal static observer will be stick into the $y = 0$ line. This means that the internal photons ($v_1 < v < v_2$) according to the clock of this observer will cross the gravitational radius (apparent horizon) at a finite time. Numerically, this time can be found from (20) and Figure 1. A boundary photon with $v = v_1$ will cross $y = 0$ at infinite time, and a boundary photon with $v = v_2$ will cross $y = 0$ in a finite time. If we choose the static Schwarzschild metric in the outer region $v > v_2$ and match the time lines, then according to the clock of the external observer all the photons will reach $y = 0$ at infinite time. Let us specify that here the internal static observer is the one that is at $r = const$ and to which the surrounding photon shell approaches from infinite radii. An external static observer is one that is located on $r = const$ after a spherically symmetric photon shell was passed through it.

5. Global geometry

We discussed above the matching conditions in the curvature coordinates (t, r). However, in these coordinates, the solution is obtained implicitly; therefore, the construction of the Carter–Penrose diagram is difficult. It is more convenient to construct such diagrams in other coordinates (η, y), where y is defined above, and η is a new temporary variable:

$$ds^2 = f_0(\eta, y)d\eta^2 - \frac{dy^2}{f_1(\eta, y)} - r^2(d\theta^2 + \sin^2\theta d\phi^2). \tag{21}$$

For the case of the pure Vaidya metric, an analytical solution was found in [8] and the corresponding diagrams were constructed. Here we will give an example of a similar diagram, but matched with the Minkowski $M_1 = 0$ region. For the case of superstrong accretion, see Figure 2. The paper with the full set of the diagrams for all types of accretions is under preparation.

6. Conclusions

This paper shows that for the Vaidya space-time of a thick photon shell in diagonal coordinates, a global geometry can be constructed in which the Vaidya metric is matched with the internal and external Schwarzschild metrics. At the same time, one of the Schwarzschild regions turns out to be non-static due to the transformation of the time required for the matching. There are static observers inside the shell and in the inner region, according to the clocks of which the photons cross the apparent horizon at a finite time.

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References

[1] Vaidya P 1943 *Current Sci.* **12** 183
[2] Vaidya P 1951 *Proc. Indian Acad. Sci.* A **33** 264
[3] Vaidya P 1953 *Nature* **171** 260
[4] Volovich I V, Zagrebnov V A and Frolov V P 1976 *Theor. Math. Phys.* **29** 1012
[5] Knutsen H 1984 *Astron. Space Sci.* **98** 207
[6] Dwivedi I H and Joshi P S 1995 *Commun. Math. Phys.* **166** 117
[7] Lindquist R W, Schwartz R A and Misner C W 1965 *Physical Review* **137** 1364
[8] Berezin V A, Dokuchaev V I and Eroshenko Yu N 2016 *Class. Quantum Grav.* **33** 145003
[9] Berezin V A, Dokuchaev V I and Eroshenko Yu N 2017 *J. of Exper. and Theor. Phys.* **124** 446