Radiative corrections in kaon decays

2017 J. Phys.: Conf. Ser. 800 012027

(http://iopscience.iop.org/1742-6596/800/1/012027)

Radiative corrections in kaon decays

M Knecht

Centre de Physique Théorique (CPT), UMR 7332 CNRS/Aix-Marseille Univ./Univ. du Sud Toulon-Var, Marseille, France

E-mail: knecht@cpt.univ-mrs.fr

Abstract. This account provides a brief overview of various non-perturbative methods that have been developed in order to address the issue of radiative corrections to kaon decays: chiral lagrangians, non-relativistic effective field theory and dispersive constructions. Only a small number of applications is mentioned, mainly for illustrative purposes.

1. Introduction

The experimental study of the kaon decay modes has made considerable progress during the last decade or so. This progress results from an increasing number of high-precision measurements, performed at several kaon facilities, like Fermilab with KTeV, DaΦne with KLOE, IHEP with ISTRA+, or the CERN SPS with NA48, NA48/2, and now NA62. In many "classical" (i.e. usually not considered as rare) decay channels, the number of collected events has multiplied the size of the existing statistical samples by several factors, sometimes even by one or several orders of magnitude. Many limits on lepton flavour violation, on violation of lepton flavour universality, on the violation of CKM unitarity, or on admixtures of non-standard components (e.g. right-handed or tensor currents) have already been obtained using these recent high-precision data on kaon decays (see the contributions of A. Buras, L. Tunstall, M. González-Alonso, E. Passemar to this conference for a survey of some of these aspects from a theoretical perspective).

Besides offering a window into possible new degrees of freedom, the study of kaon decays also provides information on low-energy constants, which can then be used in order to make predictions for other observables, or, more generally, to test our understanding of the properties of QCD in the non-perturbative, low-energy regime. This has been the original motivation for studying the most frequent decay modes, like the semi-leptonic $K_{\ell n}$ channels, with $n = 2, 3, 4$, and even 5. A rather complete description of these theoretical studies can be found in [1].

This very positive evolution allows today to look seriously for far more rare decay channels, which, due to both their extreme suppression and quite solid theoretical control in the standard model, offer excellent possibilities to unravel degrees of freedom that describe physics beyond the standard model. This is the motivation, for instance, for setting up experiments like NA62 at CERN or KOTO at J-PARC, in order to measure the branching fractions of the $K^\pm \to \pi^\pm \nu\bar{\nu}$ and $K_L \to \pi^0 \nu\bar{\nu}$ decay modes, respectively. See the contributions of K. Shiomi (KOTO) and of G. Ruggiero (NA62) to this conference.

But the increase of the experimental precision has also made it necessary to handle and to include effects that had so far been neglected. This is the case, for instance, of isospin-breaking corrections. Isospin-breaking effects have two origins: the difference $m_u - m_d$ of the masses of the light *up* and *down* quarks, and the electromagnetic interaction, $\alpha \neq 0$. Theoretical predictions are often made in a world where $\alpha = 0$ and $m_u = m_d$, so that one needs to connect this "theoretician's paradise" [2] to the real world, where experiments are made, and where isospin-breaking effects are definitely present. In particular, the inclusion of isospin-breaking corrections due to the mass difference between the charged and neutral pions (mainly an electromagnetic effect!), often turns out to be crucial in order to obtain agreement between theoretical prediction and experimental measurement [2].

Concerning radiative corrections to total decay rates Γ or to branching ratios, they usually turn out to be rather small, of the order of a few percents at most,

$$\Gamma = \Gamma_0 \left[1 + \frac{\Delta\Gamma}{\Gamma_0}\right] \qquad \left|\frac{\Delta\Gamma}{\Gamma_0}\right| \lesssim 1 - 2\%, \tag{1}$$

as expected from the size of the fine-structure constants α. On the other hand, radiative corrections to *differential* decay rates can become quite larger in certain regions of phase space,

$$\frac{d^2\Gamma(x,y)}{dxdy} = \frac{d^2\Gamma_0(x,y)}{dxdy}\left[1 + \delta(x,y)\right] \qquad |\delta(x,y)| \lesssim 10\%. \tag{2}$$

One then recovers the expected size of the correction $\Delta\Gamma/\Gamma_0$ to the total decay rate only after the integration over *all* phase space has produced a certain amount of cancellation between the values taken by $\delta(x,y)$ in different regions. For the same reason, when the decay rate is measured with experimental cuts on some phase space variables, radiative corrections can also be more important than mentioned above. It is thus even more important to have an evaluation, as accurate and as model-independent as possible, of isospin-breaking corrections if one wants to extract information from, for instance, the dependence of form factors with respect to some kinematic variable(s).

The remainder of this overview is devoted to the description of the main theoretical tools that have been developed in order to address the issue of radiative correction, or, more generally, of isospin-breaking effects, in kaon decays, and to illustrate them with a few applications. Section 2 is devoted to the low-energy effective framework using chiral lagrangians. Section 3 addresses the use of non-relativistic effective theories for the treatment of the cusp effect in $K \to \pi\pi\pi$ decays, and Section 4 describes the use of more dispersive-oriented approaches. The focus will definitely be on the effective theory techniques. Another way to deal with the issue of radiative corrections would be to rely on calculations from lattice QCD. This aspect is covered by the contribution of G. Martinelli to this conference.

2. Computing radiative corrections to kaon decay modes with chiral lagrangians

At energies well below the electroweak scale, the $\Delta S = 1$ non-leptonic transitions are described by an effective lagrangian involving a set of four-fermion operators $Q_i(\mu)$, modulated by Wilson coefficients $C_i(\mu)$. The latter sum up perturbative QCD corrections from M_W down to a scale $\mu \lesssim m_c$. In the absence of radiative corrections ($\alpha = 0$), the description of the semi-leptonic transitions involves a similar structure, but since there are no short-distance QCD corrections, it actually takes a simple factorized form, (leptonic current)$\times$(hadronic current).

2.1. Structures of the chiral lagrangians

The description of semi-leptonic decays thus amounts to the evaluation of form factors of the relevant hadronic currents. This evaluation can be addressed in the following way. At the energy scale $\mu << \Lambda_{\text{had}} \sim 1\text{GeV}$, where kaon physics takes place, the relevant degrees of freedom are no longer quarks, but the lightest pseudo-scalar mesons that become the Goldstone bosons of the spontaneous breaking of chiral symmetry in the limit $m_{u,d,s} \to 0$ of massless light quarks. It is possible to construct an effective lagrangian that describes the interactions among these pseudo-scalar mesons in a systematic low-energy expansion. The structure of the effective lagrangian involved in the computation of the hadronic currents is fixed by the symmetry properties of the QCD lagrangian in the chiral limit, and is given by [3, 4, 5, 6]

$$\mathcal{L}_{\text{eff}}^{\text{str}} = \mathcal{L}_2^{\text{str}}(2+0) + \mathcal{L}_4^{\text{str}}(10+0) + \mathcal{L}_6^{\text{str}}(90+23) + \cdots . \tag{3}$$

It describes the coupling of the pseudo-scalar mesons to the hadronic currents, as well as the strong interaction effects among the mesons themselves. The low-energy expansion starts at order $\mathcal{O}(E^2)$ and proceeds systematically to higher orders [6, 7, 8, 9, 10, 11, 12, 13, 14]. At each order of the expansion, a certain number of low-energy constants, which are not fixed through symmetry considerations, and which contain the true information on the dynamical aspects of the strong interactions at long distances, need to be introduced. The number of these low-energy constants is indicated between parentheses, the first (second) number being the number of low-energy constants describing the sector of even (odd, from NLO onwards) intrinsic parity. The effective description of the $\Delta S = 1$ non-leptonic transitions of the kaons takes a similar form [15, 16, 17, 18]:

$$\mathcal{L}_{\text{eff}}^{\Delta S=1} = \mathcal{L}_2^{\Delta S=1}(1+1) + \mathcal{L}_4^{\Delta S=1}(22+28) + \cdots . \tag{4}$$

The numbers between parentheses now indicate, at each order, the number of independent low-energy constants with flavour quantum numbers corresponding to the octet or the 27-plet representations. At lowest order, there are just two such constants, G_8 and G_{27}. At order $\mathcal{O}(E^4)$, the number of counterterms corresponds to the sector of even intrinsic parity only. The contribution from the sector of odd intrinsic parity to $\mathcal{L}_4^{\Delta S=1}$ is discussed in Refs. [19, 20, 21].

The description of strong interactions and non-leptonic transitions among the light pseudo-scalar mesons given so far does not account for radiative corrections. Adding electromagnetic interactions requires to also include the photon as a low-energy degree of freedoms in the effective lagrangian [22, 23]. But this is not yet sufficient in order to provide an adequate description of semi-leptonic transitions. Since virtual photons can connect the quark and lepton currents, the factorized structure mentioned above is in fact lost. The light charged leptons, $e^\pm$ and $\mu^\pm$, and the corresponding neutrinos must then be included as well among the light degrees of freedom [24]. At the level of the Fermi interaction between quarks and leptons, these modifications have been investigated and described in Refs. [25, 26, 27]. At the level of the low-energy effective description in terms of pseudo-scalar mesons, the low-energy effective lagrangian for semi-leptonic processes becomes more involved,

$$\mathcal{L}_{\text{eff}}^{\text{str}} \longrightarrow \mathcal{L}_{\text{eff}}^{\text{str}} + \mathcal{L}_{\text{eff}}^{\text{str+EM}} + \mathcal{L}_{\text{eff}}^{\text{lept}} . \tag{5}$$

Loops involving virtual photons and/or leptons will then produce their own divergences, which in turn require, at each order in the low-energy expansion, an additional set of low-energy constants. For the electromagnetic interactions of the pseudo-scalar mesons only, the structure at leading and at next-to-leading orders reads

$$\mathcal{L}_{\text{eff}}^{\text{str+EM}} = \mathcal{L}_2^{\text{str+EM}}(1) + \mathcal{L}_4^{\text{str+EM}}(13+0) + \cdots \tag{6}$$

where [28, 22, 23]

$$\mathcal{L}_2^{\text{str+EM}} = e^2 C \langle Q U^\dagger Q U \rangle \qquad \mathcal{L}_4^{\text{str+EM}}(13+0) = \sum_{i=1}^{13} K_i \mathcal{O}_i^{\text{str;EM}} . \tag{7}$$

Finally, adding the light leptons requires some additional structures [24]:

$$\mathcal{L}_{\text{eff}}^{\text{lept}} = \mathcal{L}_2^{\text{lept}}(0) + \mathcal{L}_4^{\text{lept}}(5) + \cdots \qquad \mathcal{L}_4^{\text{lept}} = \sum_{i=1}^{5} X_i \mathcal{O}_i^{\text{lept}} . \tag{8}$$

The same thing happens for the description of electromagnetic corrections to the $\Delta S = 1$ non-leptonic transitions,

$$\mathcal{L}_{\rm eff}^{\Delta S=1} \longrightarrow \mathcal{L}_{\rm eff}^{\Delta S=1} + \mathcal{L}_{\rm eff}^{\Delta S=1+EM}, \tag{9}$$

with [29, 30]

$$\mathcal{L}_{\rm eff}^{\Delta S=1+EM} = \mathcal{L}_2^{\Delta S=1+EM}(1) + \mathcal{L}_4^{\Delta S=1;EM}(14+?) + \cdots \tag{10}$$

$$\mathcal{L}_2^{\Delta S=1+EM} = e^2 G_8 F_0^6 \,{\rm g_{weak}} \langle \lambda_{23} U^\dagger Q U \rangle \qquad \mathcal{L}_4^{\Delta S=1+EM} = e^2 G_8 F_0^4 \sum_{i=1}^{14} Z_i \mathcal{O}_i^{\Delta S=1;EM}. \tag{11}$$

2.2. Determination of the low-energy constants

As we have just seen, including virtual photons and leptons in the low-energy description of semi-leptonic and non-leptonic transitions involves new low energy constants. We briefly discuss how these new low-energy constants have been determined. Those already present in $\mathcal{L}_{\rm eff}^{\rm str}$ and in $\mathcal{L}_{\rm eff}^{\Delta S=1}$, i.e. in the absence of electromagnetic interactions, will not be addressed here. A recent update of the determination of the low-energy constants in the mesonic sector, for instance, can be found in Ref. [31].

The low-energy constant C appearing in $\mathcal{L}_2^{\rm str;EM}(1)$ can be determined by a sum rule involving the spectral densities of the vector-vector and axial-axial two-point correlators [32],

$$C = -\frac{1}{16\pi^2}\frac{3}{2\pi}\int_0^\infty ds\, s\, \ln\frac{s}{\mu^2}\left[\rho_V(s) - \rho_A(s)\right]. \tag{12}$$

Saturating this sum rule with narrow-width vector and axial resonances, and using the Weinberg sum rules [33], gives $C = 3/(32\pi^2) M_V^2 M_A^2/(M_A^2 - M_V^2)\ln(M_A^2/M_V^2)$. The low-energy constants K_i appearing in $\mathcal{L}_4^{\rm str;EM}(13+0)$ have been determined along similar lines in Refs. [34, 35]. One first needs to express the K_i's in terms of suitable QCD correlators in the chiral limit (in this case one needs also to consider three- and four-point functions), convoluted with the free photon propagator. From the study of the short-distance properties of these correlators, one can then establish spectral sum rules similar to the one given above for C, and saturate them with the lowest-lying resonances in the various relevant channels.

The low-energy constants X_i appearing in $\mathcal{L}_4^{\rm lept}$ have been determined in Ref. [27] using a two-step matching procedure. One first computes the radiative corrections to the process $\bar{q}q' \to \ell\nu$ in the SM and in the four-fermion theory. Then one matches the radiatively corrected four-fermion theory to the chiral lagrangian, by identifying the QCD correlators (convoluted with the free photon propagator) that describe the X_i's. Finally, the resulting spectral sum rules can be saturated with the lowest-lying resonance states.

Finally, the low-energy constants ${\rm g_{weak}}$ and Z_i have been estimated in the large-N_c limit in Ref. [36]. In this approximation, one finds, for instance,

$$(g_8 e^2 {\rm g_{weak}})^\infty = -\left(\frac{\langle\bar\psi\psi\rangle}{F_0^3}\right)^2 \left[3C_8(\mu) + \frac{16}{3}e^2 C_6(\mu)(K_9 - 2K_{10})\right]. \tag{13}$$

The dependence on the short-distance scale μ vanishes at leading-order in the large-N_c limit. A scale dependence remains at subleading order in $1/N_c$, but the subleading contributions to ${\rm g_{weak}}$ and Z_i cannot be computed in general. In the case of ${\rm g_{weak}}$, the subleading contribution induced by Q_7 has however been obtained in Ref. [37],

$$(g_8 e^2 {\rm g_{weak}})^{1/N_c;Q_7} = -\frac{9}{8\pi^2} C_7(\mu)\frac{M_\rho^2}{F_0^2}\left[\ln\frac{\mu^2}{M_\rho^2} + \frac{1}{3} - 2\ln 2\right], \tag{14}$$

but this does not completely remove the residual scale dependence.

One may worry about the uncertainties on the values of the low-energy constants K_i, X_i and Z_i that come from saturation by lowest-lying narrow-width resonances, or from using the large-N_c limit (the former being itself an approximation to the latter). Typically, one should ascribe an uncertainty of about 30% to the values obatined as described above. This might not look as very precise, and it would certainly not be precise enough in order to make sufficiently accurate predictions for the dominant contributions, involving the low-energy constants of, say, $\mathcal{L}_4^{\rm str}$ or even $\mathcal{L}_6^{\rm str}$. However, radiative corrections represent only small effects, for which a precision of 30% in the corresponding low-energy constants is most of the time sufficient.

Radiative and isospin breaking corrections have therefore been computed within the low-energy expansion described above for many semi-leptonic and non-leptonic processes, for instance: $\pi \to \ell\nu_\ell(\gamma)$ and $K \to \ell\nu_\ell(\gamma)$ [24, 38, 39, 40], $K \to \pi\ell\nu_\ell(\gamma)$ [41, 42, 43, 44], $\pi^+ \to \pi^0 e\nu_e$ [45], $K^+ \to \pi^+\pi^-\ell\nu_\ell$ [46, 47, 48, 49], $K \to \pi\pi$ [50, 36, 51], $K \to \pi\pi\pi$ [52, 53, 54]. This list is certainly not extensive, and a more complete account can be found in Ref. [55].

3. Non-relativistic effective field theory

The high-precision experimental study of the decay mode $K \to \pi\pi^0\pi^0$ has revealed an important experimental feature, namely a cusp at $M_{00} = 2M_\pi$ (M_π denotes the mass of the charged pion) in the invariant mass distribution M_{00} of the two neutral pions. This cusp was first observed by NA48/2 [56] in a sample of $2.3 \cdot 10^7$ $K^\pm \to \pi^\pm\pi^0\pi^0$ events. It was promptly interpreted [57] as a final-state rescattering effect [58] $\pi^+\pi^- \to \pi^0\pi^0$ ($M_\pi \neq M_{\pi^0}$) corresponding to the combination $a_0 - a_2$ of S-wave scattering lengths. But simple phenomenological parameterizations [59, 60], or one-loop calculations in the low-energy expansion including isospin breaking [52, 53, 54], either do not reproduce the correct analyticity properties, or do not give a sufficiently accurate description of the cusp. A better description can be obtained by combining a non relativistic EFT framework, where $|\mathbf{p}|/M_\pi \sim \mathcal{O}(\epsilon)$, and a systematic expansion in powers of the scattering lengths, treated as free parameters, including orders ϵ^2, $a\epsilon^3$ and $a^2\epsilon^2$ [61, 62]. Radiative corrections were also treated within this non-relativistic framework [63]. This theoretical work, when applied to the data, allows for a very accurate determination of the relevant combination of scattering lengths, $a_0 - a_2 = 0.2571 \pm 0.0056$ [65]. This theoretical study was also extended to the $K_L \to \pi^0\pi^0\pi^0$ mode [64], for which the KTeV collaboration had collected a sample of $6.8 \cdot 10^7$ $K_L \to \pi^0\pi^0\pi^0$ events. But the rescattering effect is quite smaller than in the $K^\pm \to \pi^\pm\pi^0\pi^0$ mode, so that the combination of scattering lengths can only be determined with significantly less accuracy, $a_0 - a_2 = 0.215 \pm 0.031$ [66].

4. Dispersive construction of form factors and amplitudes

The systematic use of general properties like relativistic invariance, unitarity, crossing and analyticity, combined with the low-energy counting allows to construct, in a dispersive way, representations of scattering amplitudes or of form factors that include up to next-to-next-to leading order effects in the low-energy expansion. This "reconstruction theorem" was first proven in Ref. [67] for the $\pi\pi$ scattering amplitude in the isospin limit, and was subsequently implemented in order to obtain an explicit two-loop representation of the $\pi\pi$ amplitude [68]. But it actually is of a more general validity and, in particular, it is also applicable when isospin symmetry is explicitly broken, as has been discussed, for instance, in Ref. [69]. This "reconstruction theorem" also furnishes a convenient representation of amplitudes and form factors, which in turn provides a convenient starting point for studies that go beyond the strict framework of the low-energy expansion, when it becomes necessary to account for important final-state rescattering effects in $\pi\pi$ final states. For an illustration of the later aspect in the case of the $K_{\ell 4}$ form factors in the isospin limit, see the contribution of P. Stoffer to this conference, as well as [70].

The example that will be used as an illustration here concerns $M_\pi \neq M_{\pi^0}$ effects in the phases of form factors describing the amplitude for the $K_{\ell 4}$ decay mode $K^\pm \to \pi^+\pi^-e^\pm\nu_e$. Standard angular analysis of these form factors [71, 72] provides information on low-energy $\pi\pi$ scattering (thanks to Watson's theorem) through the difference $[\delta_S(s) - \delta_P(s)]_{\rm exp}$ of the phases of the $\pi\pi$ amplitude in the S and P waves, which is measurable in the interference of the F^{+-} and G^{+-} form factors (for the notation, see [1]). Comparison with solutions of the Roy equations [73]

$$[\delta_S(s) - \delta_P(s)]_{\rm exp} = f_{\rm Roy}(s; a_0^0, a_0^2), \tag{15}$$

allows to extract the values of the $\pi\pi$ S-wave scattering lengths in the isospin channels $I = 0, 2$. The Roy equations themselves follow from dispersion relations (analyticity, unitarity, crossing, the Froissard bound), $\pi\pi$ data at energies $\sqrt{s} \geq 1$ GeV, *and* isospin symmetry. Solutions of these equations can been constructed [74, 75] as long as the scattering lengths a_0^0 and a_0^2 lie within some range called the Universal Band.

Once standard radiative corrections have been taken care of, it is still important to take isospin-breaking corrections due to $M_\pi \neq M_{\pi^0}$ into account before analyzing data [2]. These effects were evaluated to order one loop in the low-energy expansion in Ref. [76], and they amount to replace the equation above by

$$[\delta_S(s) - \delta_P(s)]_{\rm exp} = f_{\rm Roy}(s; a_0^0, a_0^2) + \delta f_{\rm IB}(s; (a_0^0)^{\rm LO}_{\rm ChPT}, (a_0^2)^{\rm LO}_{\rm ChPT}). \tag{16}$$

Since the correction factor results from a one-loop calculation, it depends on the scattering lengths only through their lowest-order expressions, whereas they occur as free parameters in the solutions to the Roy equations. This could possibly constitute a drawback: if the radiative corrections depend stronly on the actual values of the scattering lengths, analyzing the data with the above determination of the correction factor could introduce a bias. This drawback is shared by other studies devoted to isospin breaking in $K_{\ell 4}$ decays [46, 47, 48, 49]. The issue whether it is actually possible to compute the correction factor in a way that it also involves the scattering lengths as free parameters,

$$[\delta_S(s) - \delta_P(s)]_{\rm exp} = f_{\rm Roy}(s; a_0^0, a_0^2) + \delta f_{\rm IB}(s; a_0^0, a_0^2), \tag{17}$$

was successfully addressed in Ref. [77]. A reanalysis of the NA48/2 data [78, 79] using the correction factor obtained in Ref. [77] leads to the following determination of the scattering lengths:

$$a_0^0 = 0.221 \pm 0.018 \qquad a_0^2 = -0.0453 \pm 0.0106. \tag{18}$$

Apart from slightly larger error bars, these values show no significant difference from those obtained [78, 79] using the correction factor with scattering lengths fixed at their lowest-order values [76],

$$a_0^0 = 0.2220(128)_{\rm stat}(50)_{\rm syst}\ (37)_{\rm th} \qquad a_0^2 = -0.0432(86)_{\rm stat}(34)_{\rm syst}(28)_{\rm th}. \tag{19}$$

5. Conclusions

The high precision reached by the data concerning non-leptonic and semi-leptonic decay modes of the kaons has made the treatment of isospin-breaking effects ($m_u \neq m_d$ and $\alpha \neq 0$) an unavoidable issue. A lot of activity has been going on in this field, extending the scope of the low-energy effective field theory in order to meet this necessity (inclusion of photons, leptons). Only a tiny fraction of the many applications has been mentioned in this overview. The good news is that the issue of additional low-energy constants that this extension brings with it has been dealt with in a rather satisfactory manner, although some progress on estimates of the Z_i's would be welcome. The effects due to $M_\pi \neq M_{\pi^0}$ were found to be important, especially for the description of the unitarity cusp in the $K \to \pi\pi\pi$ mode, or for the extraction of the $\pi - \pi$ scattering lengths from K_{e4}. The low-energy expansion at next-to-leading order is then not always sufficient, and more elaborate or better adapted approaches, like non-relativistic effective field theory or dispersive representations, need to be implemented in order to achieve the required precision. Finally, although it has been checked not to be the case for the determination of the phase difference $\delta_S(s) - \delta_P(s)$ from K_{e4} data, one should in general watch out for possible biases if the radiative corrections to form factors and/or decay distributions are computed for fixed values of the parameters one actually wants to extract from data.

Acknowledgments

I would like to thank the organizers for the kind invitation to participate to the KAON 2016 conference, and my colleagues V. Bernard and S. Descotes-Genon for a pleasant and fruitful collaboration on some of the topics reviewed here.

References

[1] Bijnens J, Colangelo G, Ecker G and Gasser J 1995 *The Second DAPHNE Physics Handbook* Vol 1, ed Maiani L, Pancheri G and Paver N (Frascati: INFN) p 315
[2] Gasser J 2008 *PoS KAON* 033
[3] Weinberg S 1979 *Physica* A **96** 327
[4] J. Gasser and H. Leutwyler 1983 *Phys. Lett.* B **125** 321
[5] J. Gasser and H. Leutwyler 1984 *Annals Phys.* **158** 142
[6] J. Gasser and H. Leutwyler 1985 *Nucl. Phys.* B **250** 465
[7] Wess J and Zumino B 1971 *Phys. Lett.* B **37** 95
[8] Witten E 1983 *Nucl. Phys.* B **223** 422
[9] Issler D 1990 Nonrenormalization of the chiral anomaly in chiral perturbation theory *Preprints* SLAC-PUB-4943-REV- and SLAC-PUB-4943.
[10] Akhoury R and Alfakih A 1991 *Annals Phys.* **210** 81
[11] Fearing H W and Scherer S 1996 *Phys. Rev.* D **53** 315
[12] Bijnens J, Colangelo G and Ecker G 1999 *JHEP* **9902** 020
[13] Ebertshauser T, Fearing H W and Scherer S 2002 *Phys. Rev.* D **65** 054033
[14] Bijnens J, Girlanda L and Talavera P 2002 *Eur. Phys. J.* C **23** 539
[15] Cronin J A 1967 Phys. Rev. **161** 1483
[16] Kambor J, Missimer J M and Wyler D 1990 *Nucl. Phys.* B **346** 17
[17] Esposito-Farese G 1991 *Z. Phys.* C **50** 255
[18] Ecker G, Kambor J and Wyler D 1993 *Nucl. Phys.* B **394** 101
[19] Ecker G, Neufeld H and Pich A 1992 *Phys. Lett.* B **278** 337
[20] Bijnens J, Ecker G and Pich A, *Phys. Lett.* B **286** 341
[21] Ecker G, Neufeld H and Pich A 1994 *Nucl. Phys.* B **413** 321
[22] Urech R 1995 *Nucl. Phys.* B **433** 234
[23] Neufeld H and Rupertsberger H 1996 *Z. Phys.* C **71** 131
[24] Knecht M, Neufeld H, Rupertsberger H and Talavera P 2000 *Eur. Phys. J.* C **12** 469
[25] Braaten E and Li C S 1990 *Phys. Rev.* D **42** 3888
[26] Braaten E and Li C S 1993 *J. Phys.* G **19** 1825
[27] Descotes-Genon S and Moussallam B 2005 *Eur. Phys.* J. C **42** 403
[28] Ecker G, Gasser J, Pich A and de Rafael E 1989 *Nucl. Phys.* B **321** 311
[29] Bijnens J and Wise M B 1984 *Phys. Lett.* **137B** 245
[30] Ecker G, Isidori G, Müller G, Neufeld H and Pich A 2000 *Nucl. Phys.* B **591** 419
[31] Bijnens J and Ecker G 2014 *Ann. Rev. Nucl. Part. Sci.* **64** 149
[32] Das T, Guralnik G S, Mathur V S, Low F E and Young J E 1967 *Phys. Rev. Lett.* 18 759
[33] Weinberg S 1967 *Phys. Rev. Lett.* **18** 507
[34] Moussallam B 1997 *Nucl. Phys.* B **504** 381
[35] Ananthanarayan B and Moussallam B 2004 *JHEP* **0406** 047
[36] Cirigliano V, Ecker G, Neufeld H and Pich A 2004 *Eur. Phys. J.* C **33** 369
[37] Knecht M, Peris S and de Rafael E 1999 *Phys. Lett.* B **457** 227
[38] Cirigliano V and Rosell I 2007 *JHEP* **0710** 005

IOP Conf. Series: Journal of Physics: Conf. Series **800** (2017) 012027 doi:10.1088/1742-6596/800/1/012027

[39] Gasser J and Zarnauskas G R S 2010 *Phys. Lett.* B **693** 122
[40] Cirigliano V and Neufeld H 2011 *Phys. Lett.* B **700** 7
[41] Cirigliano V, Knecht M, Neufeld H, Rupertsberger H and Talavera P 2002 *Eur. Phys. J.* C **23** 121
[42] Kastner A and Neufeld H 2008 *Eur. Phys. J.* C **57** 541
[43] Cirigliano V, Giannotti M and Neufeld H 2008 *JHEP* **0811** 006
[44] Gasser J, Kubis B, Paver N and Verbeni M 2005 *Eur. Phys. J.* C **40** 205
[45] Cirigliano V, Knecht M, Neufeld H and Pichl H 2003 *Eur. Phys. J.* C **27** 255
[46] Cuplov V 2004 Brisure d'Isospin et Corrections Radiatives au Processus $K_{\ell 4}$ *PhD thesis* Université de la Méditerranée - Aix-Marseille II
[47] Cuplov V and Nehme A 2003 Isospin breaking in K(l4) decays of the charged kaon *Preprint* hep-ph/0311274
[48] Nehme A 2004 *Phys. Rev.* D **69** 094012
[49] Stoffer P 2014 *Eur. Phys. J.* C **74** 2749
[50] Cirigliano V, Pich A, Ecker G and Neufeld H 2003 *Phys. Rev. Lett.* **91** 162001
[51] Cirigliano V, Ecker G and Pich A 2009 *Phys. Lett.* B **679** 445
[52] Bijnens J and Borg F 2004 *Nucl. Phys.* B **697** 319
[53] Bijnens J and Borg F 2005 *Eur. Phys. J.* C **39** 347
[54] Bijnens J and Borg F 2005 *Eur. Phys. J.* C **40** 383
[55] Cirigliano V, Ecker G, Neufeld H, Pich A and Portoles J 2012 *Rev. Mod. Phys.* **84** 399
[56] Batley J R *et al* [NA48/2 Collaboration] 2006 *Phys. Lett.* B **633** 173
[57] Cabibbo N 2004 *Phys. Rev. Lett.* **93** 121801
[58] Budini P, Fonda L 1961 *Phys. Rev. Lett.* **6** 419
[59] Cabibbo N and Isidori G 2005 *JHEP* **0503** 021
[60] Gamiz E, Prades J and Scimemi I 2007 *Eur. Phys. J.* C **50** 405
[61] Colangelo G, Gasser J, Kubis B and Rusetsky A 2006 *Phys. Lett.* B **638** 187
[62] Gasser J, Kubis B and Rusetsky A 2011 *Nucl. Phys.* B **850** 96
[63] Bissegger M, Fuhrer A, Gasser J, Kubis B and Rusetsky A 2009 *Nucl. Phys.* B **806** 178
[64] Bissegger M, Fuhrer A, Gasser J, Kubis B and Rusetsky A 2008 *Phys. Lett.* B **659** 576
[65] Batley J R *et al* 2009 *Eur. Phys. J.* C **64** 589
[66] Abouzaid E *et al* [KTeV Collaboration] 2008 *Phys. Rev.* D **78** 032009
[67] Stern J, Sazdjian H and Fuchs N H 1993 *Phys. Rev.* D **47** 3814
[68] Knecht M, Moussallam B, Stern J and Fuchs N H 1995 *Nucl. Phys.* B **457** 513
[69] Descotes-Genon S and Knecht M 2012 *Eur. Phys. J.* C **72** 1962
[70] Colangelo G, Passemar E and Stoffer P 2015 *Eur. Phys. J.* C **75** 172
[71] Cabibbo N and Maksymowicz A 1968 *Phys. Rev.* **137** B438; Erratum-ibid. **168** 1926
[72] Berends F A, Donnachie A and Oades G C 1968 *Phys. Rev.* **171** 1457
[73] Roy S M 1971 *Phys. Lett.* B **36** 353
[74] Ananthanarayan B, Colangelo G, Gasser J and Leutwyler H 2001 *Phys. Rept.* **353** 207
[75] Descotes-Genon S, Fuchs N H, Girlanda L and Stern J 2002 *Eur. Phys. J.* C **24** 469
[76] Colangelo G, Gasser J and Rusetsky A 2009 *Eur. Phys. J.* C **59** 777
[77] Bernard V, Descotes-Genon S and Knecht M 2013 *Eur. Phys. J.* C **73** 2478
[78] Batley J R *et al* [NA48/2 Collaboration] 2008 *Eur. Phys. J.* C **54** 411
[79] Batley J R *et al* [NA48-2 Collaboration] 2010 *Eur. Phys. J.* C **70** 635