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Morphisms and regularization of moduli spaces of pseudoholomorphic discs with Lagrangian boundary conditions

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MORPHISMS AND REGULARIZATION OF MODULI SPACES OF PSEUDOHOLOMORPHIC DISCS WITH LAGRANGIAN BOUNDARY CONDITIONS

by

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In the desert
I saw a creature, naked, bestial,
Who, squatting upon the ground,
Held his heart in his hands,
And ate of it.
I said, "Is it good, friend?"
"It is bitter—bitter," he answered;

"But I like it
"Because it is bitter,
"And because it is my heart."

In the Desert, Stephen Crane

And in twenty years they all came back,
  In twenty years or more,
And every one said, 'How tall they've grown!'
For they've been to the Lakes, and the Torrible Zone,
  And the hills of the Chankly Bore;
And they drank their health, and gave them a feast
Of dumplings made of beautiful yeast;
And everyone said, 'If we only live,
We too will go to sea in a Sieve,—
  To the hills of the Chankly Bore!'
    Far and few, far and few,
      Are the lands where the Jumblies live;
    Their heads are green, and their hands are blue,
      And they went to sea in a Sieve.

The Jumblies, Edward Lear

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ABSTRACT

We begin developing a theory of morphisms of moduli spaces of pseudoholomorphic curves and discs with Lagrangian boundary conditions as Kuranishi spaces, using a modification of the procedure of Fukaya-Oh-Ohta-Ono [15]. As an example, we consider the total space of the line bundles $\mathscr{O}(-n)$ and $\mathscr{O}$ on $\mathbb{P}^1$ as toric Kähler manifolds, and we construct isomorphic Kuranishi structures on the moduli space of holomorphic discs in $\mathscr{O}(-n)$ on $\mathbb{P}^1$ with boundary on a moment map fiber Lagrangian L and on a moduli space of holomorphic discs subject to appropriate tangency conditions in $\mathscr{O}$. We then deform this latter Kuranishi space and use this deformation to define a Lagrangian potential for L in $\mathscr{O}(-n)$, and hence a superpotential for $\mathscr{O}(-n)$. With some conjectural assumptions regarding scattering diagrams in $\mathbb{P}^1 \times \mathbb{P}$, this superpotential can then be calculated tropically analogously to a bulk-deformed potential of a Lagrangian in $\mathbb{P}^1 \times \mathbb{P}^1$.

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List of Abbreviations

$\mathbb{F}_n$	....	the compactification of $\mathscr{O}(-n)$, a Hirzebruch surface
$\mathbb{C}$	....	the complex numbers
$\mathbb{C}^*$	....	the complex numbers without 0
$\mathbb{P}^1$	....	the complex projective line
$\mathbb{P}^2$	....	the complex projective plane
$\mathscr{O}(-n)$	....	the degree $-n$ complex line bundle on $\mathbb{P}^1$
$\mathbb{R}$	....	the real numbers

Chapter 1

Introduction

1.1 Background and motivation

Moduli spaces of stable pseudoholomorphic curves and of stable pseudoholomorphic discs with Lagrangian boundary conditions are central objects of study in symplectic geometry, and have been since pioneering work of Gromov in 1985 [26]. They are crucial to Floer theory and lie at the heart of much of the interplay between theoretical physics and mathematics in the last forty years, including string theory, the study of instantons, and mirror symmetry. However, these moduli spaces are in general extremely difficult to work with, as they are not in general smooth and may not be of the expected dimension, due to disc and sphere bubbling phenomena.

The original development of Floer theory was largely guided by the following conjecture of Arnold [2] in 1978:

Conjecture 1.1.1 (Arnold Conjecture)**.** *The number of periodic trajectories of period 1 of a Hamiltonian vector field on a symplectic manifold* (X, ω) *is greater than or equal to* $\sum_k h_k(X; \mathbb{Z}/2)$.

See e.g. [3], [51]. Andreas Floer [12], for whom Floer theory is named, proved the theorem under the assumption that X is monotone, i.e. that the first Chern class $c_1(X)$ is a positive multiple of $[\omega]$. Floer's proof involved an analogue of Morse theory using an action functional whose gradient flow trajectories are related to pseudoholomorphic curves. Similar results were obtained in the cases where $X = T^{2n}$ by Conley-Zehnder [10], where X is a Riemann surface by Floer [13] and by Sikorav [47], where X is semi-positive by Hofer-Salamon [31] and by Ono [43], and then for any compact X by Fukaya-Ono [24].

The primary purpose of the original monotonicity assumption of Floer and later geometric assumptions was to control sphere bubbling, the phenomenon wherein the compactification of the moduli space of Floer trajectories includes limiting trajectories consisting of multiple components, including at least one spherical component. Fukaya-Ono where able to eliminate these assumptions by understanding the moduli space of pseudoholomorphic curves in X in greater detail, as will be discussed below.

One of the primary motivators for the specific work of this dissertation is mirror symmetry. Originating in the 1980s within supersymmetric string theory, mirror symmetry captured the attention of the mathematical community in 1991 after physicists Candelas, de la Ossa, Greene, and Parkes [7] used the theory to make dramatic predictions of the number of rational curves of a given degree contained in a quintic threefold. Their result was eventually proven mathematically through the work of Givental [25] and Lian-Liu-Yau [38].

Mathematically, mirror symmetry for a Calabi-Yau variety amounts to finding another Calabi-Yau variety, called the mirror, whose complex structure corresponds to the Kähler structure of the original variety, and vice versa. One approach to realizing and understanding this symmetry is the Strominger-Yau-Zaslow (SYZ) approach [48] from 1996, which posits that Calabi-Yau varieties should be equipped with special Lagrangian torus fibrations, and that the mirror Calabi-Yau is then obtained by taking the dual torus fibration, at least near the large complex structure limit. This idea was modified by Kontsevich-Soibelman [36] in 2006 to suggest that, approaching the large complex structure limit, a Calabi-Yau collapses to an integral affine manifold with singularities (intuitively, the base of an SYZ fibration), and the information necessary to construct the mirror Calabi-Yau from this integral affine manifold is contained in a scattering diagram within the manifold that encodes information about holomorphic discs in the original collapsing family of Calabi-Yau manifolds.

Moving beyond Calabi-Yau manifolds, the mirror of a Fano variety is generally understood to be a Landau-Ginzburg model, which is a holomorphic function $W : M \to \mathbb{C}$ for some mirror manifold M. For example, the Landau-Ginzburg mirror of $\mathbb{P}^2$ is the superpotential $W : (\mathbb{C}^*)^2 \to \mathbb{C}$ with $W(x, y) = x+y+\frac{q}{xy}$, where q is a parameter encoding information about the symplectic form on $\mathbb{P}^2$. The superpotential for an arbitrary toric Fano manifold has been calculated by Cho-Oh [9]. More generally, it has been proposed by Auroux [4] that the mirror of a pair (X, D) of a compact Kähler manifold X and a choice of anticanonical cycle will also be a Landau-Ginzburg model that is constructed by taking an SYZ fibration on the non-compact Calabi-Yau $X \setminus D$. In the case of general toric manifolds, a version of this conjecture has been proven by Fukaya-Oh-Ohta-Ono [18], in which they define a superpotential and show that, if it is Morse, the Jacobian ring of the superpotential is isomorphic to the quantum cohomology ring of the toric manifold. In my paper with Man-Wai Mandy Cheung, Hansol Hong, and Yu-Shen Lin [6], we give a method for calculating superpotentials for semi-Fano compactifications of log Calabi-Yau surfaces and confirm the Jacobian ring-quantum cohomology isomorphism for the degree five del Pezzo surface.

The chief difficulty in all of these situations, and the primary reason for all assumptions of monotonicity and positivity, is the irregularity of the relevant moduli spaces of pseudoholomorphic curves and discs. In symplectic geometry, several approaches have been developed for understanding and using these moduli spaces, with many positive results. Many, many people have contributed to the field, and the following account is far from exhaustive. These techniques include perturbing almost complex structures, using the polyfold theory of Hofer-Wysocki-Zehnder, and giving the moduli space a Kuranishi structure and using abstract perturbed multisections to define virtual fundamental chains. Perturbation of the almost complex structure has been in use since the field's inception, see e.g. McDuff-Salamon [41], while the polyfold approach was developed in the 2000s. The last approach was pioneered by Fukaya-Ono and Fukaya-Oh-Ohta-Ono (FOOO) in the late 1990s and the 2000s, see [24], [15], and has been extremely fruitful, being the basis for Fukaya-Ono's proof of the Arnold Conjecture and serving as the technical bedrock for many mirror symmetry results. This FOOO approach is extremely rich, and there is much to be gained from further expanding and developing it.

In the last decade, there have also been other versions of the Kuranishi machinery and virtual chain technique, including the Kuranishi atlases of McDuff-Wehrheim [40], the implicit atlases of Pardon [44], the axiomatic approach to virtual fundamental chains of Abouzaid [1], and the categorical work of Joyce [34]. Again, this list is far from exhaustive.

However, there are import aspects of the theory that need to be developed, and the central difficulties in working with and understanding these moduli spaces remain unresolved in general. In this thesis, we adapt the machinery of FOOO to begin to address two of the most major open issues: developing natural morphisms between moduli spaces that respect all of their relevant structures, and understanding regularizations of the moduli spaces concretely without highly restrictive assumptions on the geometry of the ambient spaces. To demonstrate the utility of these developments, we use them to define and explicitly calculate superpotentials for a family of highly non-Fano Kähler surfaces.

Viewing the total space of the line bundle $\mathscr{O}(-n)$ over $\mathbb{P}^1$ as an open subset of the corresponding Hirzebruch surface $\mathbb{F}_n$, we define potentials for moment map fiber Lagrangians in $\mathscr{O}(-n)$ in a way that allows them to be calculated tropically, allowing for direct construction of Landau-Ginzburg mirrors. These potential functions essentially involve calculating integrals over regularizations of the moduli spaces of holomorphic discs with boundary on a given Lagrangian, which in general depend on the choice of regularization.

The importance of this particular example is that it is among the simplest where the relevant moduli spaces of discs have excess dimension, and hence where the full power of the various regularization techniques is necessary. This is the root of a substantial portion of the difficulties in Floer theory and symplectic topology generally. As in the case of my past work, restrictive geometric conditions have previously been necessary to get satisfactory results, allowing people to avoid most moduli spaces with excess dimension altogether. For instance, the potential functions of moment fiber Lagrangians in smooth compact toric Fano and semi-Fano surfaces have been completely classified (FOOO [16], [18], [19], [20], and Chan-Lau [8], respectively) but there has been little success with other compact toric surfaces. It should be noted that there are only finitely many such toric surfaces satisfying the Fano or semi-Fano conditions, while there are infinitely many smooth compact toric surfaces in general. The only full result for any other compact toric surface is that of Auroux [5] on the Hirzebruch surface $\mathbb{F}_3$ compactifying the bundle $\mathscr{O}(-3)$ over $\mathbb{P}^1$, and the techniques used there do not extend to more general settings. Outside the surface case, FOOO were able to define potential functions for moment fiber Lagrangians in general compact toric manifolds, but the potential functions cannot be calculated without the Fano/semi-Fano condition.

The overarching procedure we use to define and calculate the desired potential functions is structured as follows. We need to understand one moduli space of pseudoholomorphic discs, so we adapt the Kuranishi structure machinery of FOOO [15], [17] to construct an isomorphism of Kuranishi spaces between the moduli space of interest and a moduli space of discs in the simpler manifold $\mathscr{O} \cong \mathbb{P}^1 \times \mathbb{C}$. We then realize this second moduli space as the central fiber in a deformation family of Kuranishi spaces, with the general fibers being regular, in the sense that the $\bar{\partial}$ map is transversal to 0. The general fibers will be bulk-deformed moduli spaces, which we understand as being deformations of the original moduli space and which we use to define the desired potential. Finally, we give a conjectural description of a tropical method, adapting work of Hong-Lin-Zhao [32], for calculating this potential for all n.

Theorem 1.1.2. *The Lagrangian potential function for a moment map fiber Lagrangian L in the non-compact toric surface $\mathscr{O}(-n)$ can be defined using bulk-deformed moduli spaces of holomorphic discs in $\mathscr{O} \cong \mathbb{P}^1 \times \mathbb{C}$.*

Conjecture 1.1.3. *This potential can be calculated by counting broken lines in a tropical scattering diagram extending the diagrams of HLZ* [32].

For example, the following are the expected superpotentials for $\mathscr{O}(-1)$, $\mathscr{O}(-2)$, $\mathscr{O}(-3)$, and $\mathscr{O}(-4)$, with Novikov variable set equal to 1:

$$
\begin{aligned}
W_{\mathscr{O}(-1)} &= y + x + \frac{y}{x}, \\
W_{\mathscr{O}(-2)} &= y + x + y + \frac{y^2}{x}, \\
W_{\mathscr{O}(-3)} &= y + x + 2y^2 + \frac{y^4}{x} + \frac{y^3}{x}, \\
W_{\mathscr{O}(-4)} &= y + x + 3y^3 + 3\frac{y^6}{x} + \frac{y^9}{x^2} + \frac{y^4}{x}.
\end{aligned}
$$

The first three of these have been calculated previously, as described above, while $W_{\mathscr{O}(-4)}$ is new.

The isomorphism construction and deformation family of Kuranishi spaces approach to defining an A_∞-structure are novel and represent significant advances in our understanding of morphisms of moduli spaces and their perturbations.

1.2 Lagrangian Floer theory preliminaries

We give a brief review of the Lagrangian Floer theory we will use, primarily following Fukaya-Oh-Ohta-Ono [15], [23] and Fukaya [14].

Let X be a $2n$-dimensional symplectic manifold with tame almost complex structure J and a compact, relatively spin Lagrangian submanifold L. We do not assume that X is compact, but we will assume that all moduli spaces of discs and curves we consider are compact. Given a class $\beta \in H_2(X, L; \mathbb{Z})$, we let $\mathcal{M}_{k+1,\ell}(X, L, \beta)$ be the moduli space of stable, nodal, pseudo holomorphic discs in class β with $k+1$ boundary marked points and ℓ interior marked points. The domain of every element of $\mathcal{M}_{k+1,\ell}(X, L, \beta)$ is a connected, nodal, genus 0, bordered Riemann surface with a single boundary component. These moduli spaces have natural stratifications based on the combinatorial type of the source (see 2.1.2). The marked points give the following evaluation maps

$$
\begin{aligned}
(\mathrm{ev}_1, \ldots, \mathrm{ev}_k, \mathrm{ev}_1^+, \ldots, \mathrm{ev}_\ell^+) : \mathcal{M}_{k+1,\ell}(X, L; \beta) &\to L^k \times X^\ell \\
\mathrm{ev}_0 : \mathcal{M}_{k+1,\ell}(X, L; \beta) &\to L.
\end{aligned}
$$

Note that, if X is compact, then so is $\mathcal{M}_{k+1,\ell}(X, L; \beta)$. Hence, this is usually referred to as the compactified moduli space. Again, we will be assuming that $\mathcal{M}_{k+1,\ell}(X, L, \beta)$ is compact, even if X is not.

Crucially, these moduli spaces are not in general an orbifold with boundary and corners. In light of this, we will put a smooth Kuranishi structure on $\mathcal{M}_{k+1,\ell}(X, L, \beta)$ with respect to which these evaluation maps will be well behaved. In particular, the maps will be smooth in a sense that allows us to make sense of pulling back forms by $(\mathrm{ev}_1, \ldots, \mathrm{ev}_k, \mathrm{ev}_1^+, \ldots, \mathrm{ev}_\ell^+)$, and the map ev_0 will be weakly submersive in a sense that allows us to push forward forms by ev_0 (intuitively, by integrating along fibers of ev_0). The virtual dimension of $\mathcal{M}_{k+1,\ell}(X, L; \beta)$ as a Kuranishi space is $MI(\beta)+n-3+k+1+2\ell$, where $MI(\beta)$ is the Maslov index of β and n is half of the (real) dimension of X, i.e. the dimension of L. In the case of X being a complex surface, this virtual dimension is $MI(\beta)+k+2\ell$.

We use the following Novikov ring Λ_0 over $\mathbb{R}$, along with its maximal ideal Λ_+ and fraction field Λ:

$$\Lambda_0 := \left\{ \sum_{i=1}^{\infty} a_i T^{\lambda_i} \mid a_i \in \mathbb{R}, \lambda_i \in \mathbb{R}_{\geq 0}, \lim_{i\to\infty} \lambda_i = \infty \right\}, \tag{1.2.1}$$

$$\Lambda_+ := \left\{ \sum_{i=1}^{\infty} a_i T^{\lambda_i} \mid a_i \in \mathbb{R}, \lambda_i \in \mathbb{R}_{>0}, \lim_{i\to\infty} \lambda_i = \infty \right\}, \tag{1.2.2}$$

$$\Lambda := \left\{ \sum_{i=1}^{\infty} a_i T^{\lambda_i} \mid a_i \in \mathbb{R}, \lambda_i \in \mathbb{R}, \lim_{i\to\infty} \lambda_i = \infty \right\}. \tag{1.2.3}$$

We can then define an A_∞-algebra structure on $\Omega^*(L;\Lambda_0)$, known as the de Rham model, using the following diagram

$$\begin{array}{ccc} \mathcal{M}_{k+1}(X,L,\beta) & \xrightarrow{(\mathrm{ev}_1,\ldots,\mathrm{ev}_k)} & L^k \\ {\scriptstyle \mathrm{ev}_0}\downarrow & & \\ L & & \end{array}$$

That is, we define $\mathfrak{m}_{k,\beta} : \Omega^*(L;\Lambda_0)^{\otimes k} \to \Omega^*(L;\Lambda_0)$ by

$$\mathfrak{m}_{k,\beta}(h_1,\ldots,h_k) := (\mathrm{ev}_0)_!\left(\mathrm{ev}_1^* h_1 \wedge \ldots \wedge \mathrm{ev}_k^* h_k\right).$$

We then define

$$\mathfrak{m}_k := \sum_{\beta\in H_2(X,L;\mathbb{Z})} \mathfrak{m}_{k,\beta} T^{\omega(\beta)}.$$

For the definition of the term A_∞-algebra structure, see for instance Definition 21.21 of FOOO [23]. It can be thought of as an algebraic codification of the compatibility conditions that arise between these operations $\mathfrak{m}_k$ from studying the boundaries of the moduli spaces $\mathcal{M}_{k+1}(X,L,\beta)$. We draw direct attention to the following A_∞ relation:

$$\mathfrak{m}_2(\mathfrak{m}_0(1),x)+(-1)^{\deg x+1}\mathfrak{m}_2(x,\mathfrak{m}_0(1))+\mathfrak{m}_1(\mathfrak{m}_1(x))=0.$$

If $\mathfrak{m}_0=0$, then $\mathfrak{m}_1^2=0$, and we can define the Floer homology of L to be the homology of $\Omega^*(L;\Lambda_0)$ with respect to $\mathfrak{m}_1$.

Given $b\in\Omega^{odd}(L,\Lambda_+)$, we can also consider the deformed A_∞-algebra structure $\mathfrak{m}_k^b$ given by

$$\mathfrak{m}_k^b(x_1,\ldots,x_k)=\sum_{\ell_0,\ldots,\ell_k}\mathfrak{m}_{k+\sum\ell_i}(b^{\otimes\ell_0},x_1,b^{\otimes\ell_1},\ldots,b^{\otimes\ell_{k-1}},x_k,b^{\otimes\ell_k}).$$

We say that b is a bounding cochain if $\mathfrak{m}_0^b=0$, in which case $\mathfrak{m}_1^b$ is a boundary operator. If there exists a bounding cochain b, we say that L is unobstructed.

Rather than working with the de Rham model, we will work with the associated canonical model, see Fukaya [14] for details. We fix a Riemannian metric on L and represent $H^*(L;\mathbb{R})$ as the subspace of harmonic forms in $\Omega^*(L;\mathbb{R})$. There is an associated A_∞-structure on $H^*(L;\Lambda_0)$, whose operations we denote by $\mathfrak{m}_k^{can}$, which is quasi-isomorphic to the de Rham model. The canonical model has the advantage of being unital, in the sense that the Poincaré dual $PD[L]$ of L satisfies the following relations:

$$\mathfrak{m}_{k+1}(x_1,\ldots,PD[L],\ldots,x_k)=0 \text{ for } k\geq 2 \text{ or } k=0,$$
$$\mathfrak{m}_2(PD[L],x)=(-1)^{\deg x}\mathfrak{m}_2(x,\mathfrak{m}_0(1))=x.$$

We then define a weak bounding cochain to be an element $b\in H^{odd}(L,\Lambda_+)$ such that

$$\mathfrak{m}_0^b(1)=W(b)PD[L]$$

for some constant $W(b) \in \Lambda_+$. We call the set of all such b the weak Maurer-Cartan space, which we write $\widehat{\mathcal{MC}}_+(L)$. If $\widehat{\mathcal{MC}}_+(L)$ is non-empty, then we say L is weakly unobstructed, and we again have that $\mathfrak{m}_1^b$ is a boundary operator and can be used to define a Floer homology.

Definition 1.2.1. *We call the function*

$$W : \widehat{\mathcal{MC}}_+(L) \to \Lambda_+$$

the Lagrangian potential function.

One is often interested in the moduli space of Maurer-Cartan solutions $\mathcal{MC}_+(L)$ which is obtained from $\widehat{\mathcal{MC}}_+(L)$ by modding out by gauge equivalence. The potential W factors through this equivalence giving a map $\widehat{\mathcal{MC}}_+(L) \to \Lambda_+$ we will also call W.

The final general Floer theoretic topic we will consider before getting into more specific geometric settings is pseudo-isotopy of A_∞-algebras, introduced in Fukaya [14]. One can also consult Tu [49]. We will avoid describing it in algebraic generality, focusing instead on giving a sketch of the geometric manifestation we will be using. Let J_0 and J_1 be tame almost complex structures on X connected by a path J_t of tame almost complex structures. For $\beta \in H_2(X, L)$, we consider the moduli space

$$\mathcal{M}_{k+1}(\beta; \mathcal{J}) = \bigsqcup_{t\in[0,1]} \{t\} \times \mathcal{M}_{k+1}(X, L, \beta; J_t)$$

This moduli space can be given an appropriate Kuranishi structure, though we shall not describe it in detail. See Fukaya [14]. We have the following evaluation maps,

$$\begin{aligned} \mathrm{ev} = (\mathrm{ev}_1, \ldots, \mathrm{ev}_k) : \mathcal{M}_{k+1}(\beta; \mathcal{J}) &\to L^k, \\ \mathrm{ev}_0 : \mathcal{M}_{k+1}(\beta; \mathcal{J}) &\to L, \\ \mathrm{ev}_t : \mathcal{M}_{k+1}(\beta; \mathcal{J}) &\to [0,1], \end{aligned}$$

and we will assume $\mathrm{ev}_0 \times \mathrm{ev}_t : \mathcal{M}_{k+1}(\beta; \mathcal{J}) \to L \times [0,1]$ is weakly submersive. On the level of the de Rham model, we take $h_1, \ldots, h_k \in \Omega^*(L, \Lambda_0)$ and define $\mathfrak{m}^t_{k,\beta}$ and $\mathfrak{c}^t_{k,\beta}$ by

$$\mathfrak{m}^t_{k,\beta}(h_1, \ldots, h_k) + dt \wedge \mathfrak{c}^t_{k,\beta}(h_1, \ldots, h_k) = (\mathrm{ev}_0 \times \mathrm{ev}_t)_!(\mathrm{ev}_1^* h_1 \wedge \cdots \wedge \mathrm{ev}_k^* h_k).$$

Here the superscript t indicates that $\mathfrak{m}^t_{k,\beta}$ and $\mathfrak{c}^t_{k,\beta}$ depend on t. We have an associated A_∞-homomorphism

$$\mathfrak{c}_k = \sum_{\beta \in H_2(X,L)} \int_{[0,1]_t} \mathfrak{c}^t_{k,\beta} T^{\omega(\beta)}$$

between the two A_∞-structures on $\Omega^*(L; \Lambda_0)$ coming from the almost complex structures J_0 and J_1. Using the natural inclusion and projection $H^*(L) \hookrightarrow \Omega^*(L) \twoheadrightarrow H^*(L)$ coming from the Hodge-Kodaira decomposition, this gives an A_∞-homomorphism $\mathfrak{c}^{can}_k$ on the canonical model. This then induces a map between the two Maurer-Cartan spaces

$$F : \mathcal{MC}_{+,J_0}(L) \to \mathcal{MC}_{+,J_1}(L)$$
$$F(b) = \sum_{k=0}^{\infty} \mathfrak{c}^{can}_k (b^{\otimes k})$$

associated with the two almost complex structures. This map F is independent of the choice of path J_t of almost complex structures, a fact we will refer to as Fukaya's trick. Furthermore, letting W_0 and W_1 be the potential functions associated with the two Maurer-Cartan spaces, we have the following equation:

$$W_1 \circ F = W_0.$$

This map F will manifest as a wall-crossing map in our geometric situation.

We finish this section with a loose discussion of bulk-deformed superpotentials for toric Fano surfaces, as developed by Hong-Lin-Zhao [32]. This will be a useful picture to have in mind, since it serves as the inspiration for the deformation of Kuranishi spaces approach to defining superpotentials. However, the underlying technicalities relating to moduli spaces and A_∞-structures will be rather different in our setting than in the setting of HLZ. We give some conjectural description of these differences in 3.4.

Let X_Σ be a toric Fano surface with fan Σ, and let L be a moment fiber Lagrangian. We let $q_1, \dots, q_n \in (\mathbb{C}^*)^2$ be chosen generically (after fixing L), let $\ell \leq n$, take an injection $f : \{1, \dots, \ell\} \to \{1, \dots, n\}$, and consider the fiber product

$$\begin{array}{ccc} \mathcal{M}_{k+1,\ell}(X_\Sigma, L, \beta)\,{}_{\mathrm{ev}^+}\times_i \{(q_{f(1)}, \dots, q_{f(\ell)})\} & \longrightarrow & \{(q_{f(1)}, \dots, q_{f(\ell)})\} \\ \downarrow & & \downarrow{\scriptstyle i} \\ \mathcal{M}_{k+1,\ell}(X_\Sigma, L, \beta) & \xrightarrow[\mathrm{ev}^+]{} & X_\Sigma^\ell \end{array}$$

We let $\mathcal{M}_{k+1,\ell}(X_\Sigma, L, \beta; \mathbf{q})$ be the disjoint union of these fiber products over all choices of the function f. The virtual dimension of these moduli spaces is $k + MI(\beta) - 2\ell$, inspiring the definition of the term generalized Maslov index defind as $GMI(u) = MI(\beta) - 2\ell$, where u is a holomorphic disc in this moduli space. We can use these moduli spaces to define a bulk-deformed A_∞-structure on $H^*(L; \Lambda_0)$. As shown in HLZ [32] by a dimension argument, we have that $H^1(L, \Lambda_+) = \mathcal{MC}_+^{\mathbf{q}}(L) = \widetilde{\mathcal{MC}}_+^{\mathbf{q}}(L)$, so our bulk-deformed potential $W^{\mathbf{q}}$ can be written as

$$W^{\mathbf{q}} : H^1(L; \Lambda_+) \to \Lambda_+.$$

It should be noted that in HLZ they use the coefficient ring $\mathbb{C}[t_1, \dots, t_n]/(t_1^2, \dots, t_n^2)$, with each point insertion q_i associated with a nilsquared element t_i. This kills contributions from discs going through a single point multiple times, which greatly simplifies a number of arguments and is the reason why they refer to their potential as the "nth order" bulk-deformed potential. For our present discussion, we will reap the benefits of this choice of nilpotent coefficients without grappling with the complications they introduce.

We now turn our attention to the special Lagrangian torus fibration given by the moment map $\mu : X_\Sigma \to B_\Sigma \subseteq \mathbb{R}^2$ on the open $(\mathbb{C}^*)^2$ orbit. As this is a trivial fibration, we can identify $H_1(L;\mathbb{Z})$ for all fibers L. We let e_1, e_2 be a basis of $H_1(L;\mathbb{Z})$ and e_1^*, e_2^* be the corresponding dual basis of $H^1(L;\mathbb{Z})$. When writing out $W^{\mathbf{q}}$ explicitly, we will make use of "exponential coordinates" on $H^1(L;\mathbb{Z})$, that is the functions $x = e^{e_1^*} : H_1(L;\mathbb{Z}) \to \mathbb{C}$ and $y = e^{e_2^*} : H_1(L;\mathbb{Z}) \to \mathbb{C}$. For example, with appropriate choices of e_1, e_2 and setting the Novikov variable T to 1, the potential for any moment fiber Lagrangian in $\mathbb{P}^2$ without bulk insertions is

$$W_{\mathbb{P}^2} = x + y + \frac{1}{xy},$$

which is the superpotential found by Hori-Vafa [33] and Cho-Oh [9].

If we fix these coordinates and include fixed bulk insertions, we find that the potential is no longer independent of the choice of fiber L. Instead, there are real codimension 1 regions in B_Σ, which we refer to as walls, separating B_Σ into open regions on which $W^{\mathbf{q}}$ is independent (up to Novikov scaling) of the choice of fiber L. We refer to the phenomenon of these potentials changing as we cross the codimension 1 locus in B_Σ as "wall crossing," and we refer to the walls together with the wall-crossing transformations as a "scattering diagram."

In more detail, the fibers lying inside the walls are precisely those that bound generalized Maslov index 0 discs, which lie in some moduli space $\mathcal{M}_{0,\ell}(X_\Sigma, L, \beta; \mathbf{q})$ of virtual dimension -1. We study the effect of crossing this wall using a corresponding pseudo-isotopy. Intuitively, given two special Lagrangian fibers L_0, L_1 on "opposite sides" of a wall,[1] we take an isotopy $\phi : [0,1] \times X_\Sigma$ with $\phi_1(L_0) = L_1$ and consider

[1] This intuitive notion of "opposite sides" is not a-priori well-defined, but we will find that these walls are in fact sufficiently well-behaved for this to make sense.

the pseudo-isotopy arising from the moduli spaces

$$\bigsqcup_{t\in[0,1]} \{t\} \times \mathcal{M}_{k+1,\ell}(X_\Sigma, L_0, \beta; \mathbf{q}; \phi_t^* J),$$

where J is the standard complex structure on X_Σ. This induces a wall crossing map

$$F_\phi : H^1(L_0; \Lambda_+) \to H^1(L_0; \Lambda_+).$$

These moduli spaces are isomorphic (even after being given appropriate Kuranishi structures) to the spaces

$$\bigsqcup_{t\in[0,1]} \{t\} \times \mathcal{M}_{k+1,\ell}(X_\Sigma, \phi_t(L_0), \beta; \mathbf{q}; J),$$

where the complex structure is fixed but the Lagrangian is changing. However, if we want to get an A_∞-homomorphism without varying the symplectic structure ω, we must use the former moduli spaces with fixed Lagrangian and varying almost complex structure.

The map F_ϕ is independent of the choice of isotopy ϕ, up to homotopy avoiding the bulk insertion points, by Fukaya's trick. After adjusting for changes in symplectic area, F_ϕ allows us to find the bulk-deformed potential at L_1 from the potential at L_0.

1.3 Motivating example: Log Calabi-Yau surfaces and mirror symmetry

We now discuss in some detail my paper with Man-Wai Mandy Cheung, Hansol Hong, and Yu-Shen Lin [6] as it pertains to the present topic. The geometric setting is that of a Looijenga pair (Y, D), a smooth projective rational surface Y with an anticanonical cycle D that is a reduced rational curve with at least one singular point. Letting Ω_Y be the meromorphic volume form on Y with simple poles on D, we see that the restriction Ω_X of Ω_Y to $X = Y \setminus D$ is a non-vanishing holomorphic volume form on X. In this sense, X is a log Calabi-Yau surface.

Inspired by the SYZ conjecture, Gross-Hacking-Keel [28] constructed a purely algebraic mirror family for (Y, D). Their procedure involves constructing an integral affine manifold B with singularity analogous to the base of an SYZ fibration. They then study the canonical scattering diagram $\mathfrak{D}^{GHK}$ in B encoding relative Gromov-Witten invariants, which correspond to the quantum corrections from holomorphic discs of Maslov index 0 in X. The scattering diagram essentially describes how to glue together local charts to form the mirror family for X.

In [6], we carry out the symplectic counterpart of this procedure, constructing a special Lagrangian fibration on X and showing that an associated Lagrangian Floer scattering diagram coincides with a scattering diagram $\mathfrak{D}^{GPS}$ of Gross-Pandharipande-Siebert [29], which then recovers the scattering diagram of GHK [28].

Theorem 1.3.1. [6] *Given a log Calabi-Yau surface, the associated Lagrangian Floer scattering diagram $\mathfrak{D}^{LF}$ recovers the scattering diagram $\mathfrak{D}^{GPS}$ and the canonical scattering diagram $\mathfrak{D}^{GHK}$.*

In more detail, we follow the Strominger-Yau-Zaslow (SYZ) [48] approach to mirror symmetry and its refinement by Kontsevich-Soibelman [36], and we construct a family of special Lagrangian fibrations on X with respect to a family of symplectic forms ω_ϵ, where ϵ indicates the symplectic area of some particular exceptional divisors. The bases of these fibrations are integral affine manifolds B_ϵ. The scattering diagram $\mathfrak{D}^{LF}_\epsilon$ then consists of a collection of decorated affine lines and rays in B_ϵ encoding information about which fibers bound holomorphic discs, or, more precisely, encoding the open Gromov-Witten invariants of the fibers. We then show that this coincides with the tropical scattering diagram of Gross-Pandharipande-Siebert on a large open subset of B_ϵ that embeds into $\mathbb{R}^2$ as an integral affine manifold. This can then be used to recover the canonical scattering diagram of Gross-Hacking Keel, which is key for constructing the mirror of the log Calabi-Yau surface. Our result thus provides a direct link between the SYZ framework for mirror symmetry and the tropical work of Gross-Pandharipande-Siebert [29] and Gross-Hacking-Keel [28] and their mirror construction.

This result has several applications:

1. We apply the result to get a version of mirror symmetry for rank 2 cluster varieties.

2. The result shows that, in this geometric context, the symplectic open Gromov-Witten invariants agree with the corresponding algebro-geometric log Gromov-Witten invariants, which count $\mathbb{A}^1$ curves in a log Calabi-Yau surface.

3. The result provides a method for explicitly calculating Lagrangian potential functions for certain compactifications of log Calabi-Yau surfaces, which can then be used to construct the Landau-Ginzburg mirrors of those compactifications. Furthermore, we use this procedure to verify a conjecture of Sheridan [46], previously confirmed in different ways by Pascaleff-Tonkonog [45] and Vanugopalan-Woodward [50], that a cubic surface contains a Lagrangian such that there are 21 holomorphic discs with boundary on the Lagrangian. This is an open analogue of the classical result that a cubic surface contains 27 lines.

Thinking more broadly, the tropicalization procedure we employ here provides an important tool for understanding moduli spaces with excess dimension, specifically those with virtual dimension -1 and actual dimension 0 that arise when studying wall-crossing. We use a Floer theoretic technique known now as "Fukaya's Trick," introduced in [14], which resolves the issue of excess dimension by essentially replacing the standard Kuranishi structure on each moduli space with another of a higher virtual dimension, matching the actual dimension. We then see that the Floer theoretic scattering matches the tropical scattering, which can then be understood explicitly. Unfortunately, Fukaya's trick cannot resolve the outstanding issues with moduli spaces of excess dimension generally, and when we turn to calculating potential functions of the compactifications of log Calabi-Yau surfaces, we have had to restrict our attention to Fano (and some semi-Fano) compactifications. Running into this limitation drew my attention to the gaps in the present theory of moduli spaces of discs, prompting me to pursue the following work in preparation.

1.4 The line bundles $\mathscr{O}(-n)$ and $\mathscr{O}$ on $\mathbb{P}^1$

Fix $n > 0$. We conclude the introduction by considering the total spaces of the line bundles $\mathscr{O}(-n)$ and $\mathscr{O}$ over $\mathbb{P}^1$. In particular, we will establish coordinates and notation to be used throughout.

We obtain the total space of $\mathscr{O}(-n)$ by gluing two charts $U_0 = \mathbb{C} \times \mathbb{C}$ and $U_1 = \mathbb{C} \times \mathbb{C}$ together by identifying $(x_0, y_0)_0 \in U_0$ with $(x_0^{-1}, x_0^n y_0)_1 \in U_1$ (for $x_0 \in \mathbb{C}^*$). We similarly obtain the total space of $\mathscr{O}$ by gluing U_0, U_1 by identifying $(x_0, y_0)_0$ with $(x_0^{-1}, y_0)_1$. We let D_{-n} (respectively D_0) be the self-intersection $-n$ (resp. 0) divisor in $\mathscr{O}(-n)$ (resp. $\mathscr{O}$) given by $y_0 = 0$ and $y_1 = 0$.

With these shared charts for $\mathscr{O}(-n)$ and $\mathscr{O}$, we have a natural map $\underline{\psi} : \mathscr{O}(-n) \to \mathscr{O}$.

$$\underline{\psi}_0(x_0, y_0)_0 = (x_0, y_0)_0$$
$$\underline{\psi}_1(x_1, y_1)_1 = (x_1, x_1^n y_1)_1.$$

We have an associated map from functions into $\mathscr{O}(-n)$ to functions into $\mathscr{O}$. Let Σ be a nodal disc, i.e. a genus zero bordered nodal Riemann surface with connected boundary, let $w : \Sigma \to \mathscr{O}(-n)$ be a continuous function, and let $\Sigma_i := w^{-1}(U_i)$. In coordinates, we write

$$w_0(z) = (u_0(z), v_0(z))_0$$
$$w_1(z) = (u_1(z), v_1(z))_1$$

where w_i is $w|_{\Sigma_i}$. Here $z \in \Sigma$, but we are not making any choice of coordinates on Σ. We get a new continuous function $\psi(w) : \Sigma \to \mathscr{O}$ defined by

$$\psi(w)_0(z) = (u_0(z), v_0(z))_0$$
$$\psi(w)_1(z) = (u_1(z), u_1(z)^n v_1(z))_1.$$

We equip $\mathscr{O}(-n)$ and $\mathscr{O}$ with the usual toric structures. They are toric open subsets of the Hirzebruch surfaces of degree n and 0, and we equip them with the restrictions of the associated symplectic forms. Let L be a standard moment map fiber Lagrangian in $\mathscr{O}(-n)$, given in U_0 coordinates by $|x_0| = r_{x_0} \neq 0, |y_0| = r_{y_0} \neq 0$ constant. We observe that, given $w : (\Sigma, \partial\Sigma) \to (\mathscr{O}(-n), L)$ mapping the boundary of Σ into L, the map $\psi(w)$ maps the boundary of Σ into a corresponding Lagrangian $\underline{\psi}(L)$ in $\mathscr{O}$, given in U_0 coordinates by $|x_0| = r_{x_0}, |y_0| = r_{x_0}^n r_{y_0}$. Since our map of maps ψ is induced directly induced by the underlying continuous (holomorphic) map $\underline{\psi}$, we have that if w is in homology class $\beta \in H_2(\mathscr{O}(-n), L)$, then $\psi(w)$ is in class $\underline{\psi}_*(\beta) := \underline{\psi}_* \beta \in H_2(\mathscr{O}, \underline{\psi}(L))$. Furthermore, if w is holomorphic, then so is $\psi(w)$. Thus, our map ψ restricts to a map

$$\psi : \mathcal{M}_{k+1,\ell}(\mathscr{O}(-n), L, \beta) \to \mathcal{M}_{k+1}(\mathscr{O}, \underline{\psi}(L), \underline{\psi}_*(\beta))$$

This ψ is stratawise smooth and injective, but far from surjective. Let $F_0 = \{(x_1, y_1)_1 \mid x_1 = 0\}$ be the fiber of $\mathscr{O}$ over $(0,0)_1$ in U_1. The image of ψ consists precisely of those maps $w' \in \mathcal{M}_{k+1}(\mathscr{O}, \underline{\psi}(L), \underline{\psi}_*(\beta))$ whose image intersects F_0 only at the point $(0,0)_1$ and whose order of intersection with D_0 is n times its order of intersection with the fiber F_0. Here a transversal intersection has order 1, the disc given by $(z,z^2)_1$ in $\mathscr{O}$ intersects F_0 with order 1 and intersects D_0 with order 2, and so on. We let $\mathcal{M}_{k+1}(\mathscr{O}, \underline{\psi}(L), \underline{\psi}_*(\beta); (F_0, D_0, n))$ denote the moduli space of discs in $\mathcal{M}_{k+1}(\mathscr{O}(-n), \underline{\psi}(L), \underline{\psi}_*(\beta))$ satisfying this condition:

Definition 1.4.1. *We let*

$$\mathcal{M}_{k+1}(\mathscr{O}, \underline{\psi}(L), \underline{\psi}_*(\beta); (F_0, D_0, n))$$

be the subspace of the moduli space $\mathcal{M}_{k+1}(\mathscr{O}, \underline{\psi}(L), \underline{\psi}_*(\beta))$ consisting of all stable marked discs $u : (\Sigma, \partial\Sigma) \to (\mathscr{O}, \underline{\psi}(L))$ with $k+1$ marked points, in class $\underline{\psi}_*\beta \in H_2(\mathscr{O}, \underline{\psi}(L))$, such that for each point $z_0 \in u^{-1}(F_0)$, the order of intersection of u with D_0 at z_0 is n times the order of intersection of u with F_0 at z_0.

Note that in particular every disc in $\mathcal{M}_{k+1}(\mathscr{O}, \underline{\psi}(L), \underline{\psi}_*(\beta); (F_0, D_0, n))$ passes through the point $(0,0)_1 = F_0 \cap D_0$.

We thus have the map

$$\psi : \mathcal{M}_{k+1}(\mathscr{O}(-n), L, \beta) \to \mathcal{M}_{k+1}(\mathscr{O}, \underline{\psi}(L), \underline{\psi}_*(\beta); (F_0, D_0, n)).$$

We can define an inverse map as follows. For any $(\Sigma, w) \in \mathcal{M}_{k+1}(\mathscr{O}, \underline{\psi}(L), \underline{\psi}_*(\beta); n)$, we have that $w^{-1}(U_1) \setminus w^{-1}(U_0)$ consists of isolated points and trees of constant spheres. Combining this observation with the condition defining $\mathcal{M}_{k+1}(\mathscr{O}, \underline{\psi}(L), \underline{\psi}_*(\beta); (F_0, D_0, n))$, we see that the partial inverse $\psi^{-1}(w|_{w^{-1}(U_0)})$ given below extends uniquely to a full holomorphic map $\psi^{-1}(w)$:

$$\psi^{-1}(w)_0(z) = (u_0(z), v_0(z))_0$$
$$\psi^{-1}(w)_1(z) = (u_1(z), u_1(z)^{-n} v_1(z))_1.$$

We thus have the following lemma:

Lemma 1.4.2. *The map ψ is a homeomorphism between $\mathcal{M}_{k+1}(\mathscr{O}(-n), L, \beta)$ and*

$\mathcal{M}_{k+1}(\mathscr{O}, \underline{\psi}(L), \underline{\psi}_*(\beta); (F_0, D_0, n))$.

In Section 3.1 we will build Kuranishi structures on both of these spaces so that an extension of ψ gives an isomorphism of Kuranishi spaces between them.

We note that $\mathscr{O}(-n)$ and $\mathscr{O}$ are not compact, so the following lemma establishing compactness of all relevant moduli spaces is crucial.

Lemma 1.4.3. *For all n, all toric moment fiber Lagrangians $L \subseteq \mathscr{O}(-n)$, and all effective disc classes $\beta \in H_2(\mathscr{O}(-n), L)$, the moduli space $\mathcal{M}_{k+1}(\mathscr{O}(-n), L, \beta)$ is compact.*

Proof. We consider the inclusion $\underline{\iota} : \mathscr{O}(-n) \hookrightarrow \mathbb{F}_n$ of the line bundle $\mathscr{O}(-n)$ into the Hirzebruch surface obtained compactifying the fibers, which induces a continuous injection

$$\iota : \mathcal{M}_{k+1}(\mathscr{O}(-n), L, \beta) \to \mathcal{M}_{k+1}(\mathbb{F}_n, \underline{\iota}(L), \underline{\iota}_*\beta).$$

To show that this map is a homeomorphism, it suffices to show that the image of every disc in $\mathcal{M}_{k+1}(\mathbb{F}_n, \underline{\iota}(L), \underline{\iota}_*\beta)$ is contained in $\underline{\iota}(\mathscr{O}(-n))$. For disc components, this follows from the Cho-Oh [9] classification of holomorphic discs with boundary on L. For $n \geq 1$, the only non-constant sphere components have image contained in the 0-section. For $n = 0$, the only non-constant sphere components have image contained in a constant section with value at most r_{y_0}. Thus, ι is a homeomorphism.

Since $\mathbb{F}_n$ is compact, the desired result follows.

□

In Section 3.2, we will modify the Kuranishi structures, then use the notion of deformation of Kuranishi structures to relate the moduli space $\mathcal{M}_{k+1}(\mathscr{O}, \underline{\psi}(L), \underline{\psi}_*(\beta); (F_0, D_0, n)))$ to the following new moduli space:

Definition 1.4.4. *Consider n distinct points $(a_1, b_1), \ldots, (a_n, b_n) \in U_1 \subseteq \mathscr{O}$. We have n corresponding fibers $F_{a_i} = \{(x_1, y_1)_1 \in U_1 \mid x_1 = a_i\}$ of $\mathscr{O}$ and n sections $S_{b_i} = \{([x_0, x_1], b_i) \mid [x_0, x_1] \in D_0\}$ of $\mathscr{O}$. We assume $|a_i| < r_{x_0}^{-1}$ and $|b_i| < r_{y_0}$ for all i.*

We let

$$\mathcal{M}_{k+1}(\mathscr{O}, \underline{\psi}(L), \underline{\psi}_*(\beta); (F_{a_1}, S_{b_1}, 1), \ldots, (F_{a_n}, S_{b_n}, 1))$$

be the subspace of the moduli space $\mathcal{M}_{k+1}(\mathscr{O}, \underline{\psi}(L), \underline{\psi}_*(\beta))$ *consisting of all stable marked discs* $u : (\Sigma, \partial\Sigma) \to (\mathscr{O}, \underline{\psi}(L))$ *with* $k+1$ *marked points, in class* $\underline{\psi}_*\beta \in H_2(\mathscr{O}, \underline{\psi}(L))$*, such that for each point* $z_i \in u^{-1}(F_{a_i})$*, the order of intersection of* u *with* S_{b_i} *at* z_i *equals the order of intersection of* u *with* F_{a_i} *at* z_i*.*

This moduli space is closely related to, though in general different from, a moduli space with n (possibly repeated) point insertions at points $(a_1, b_1), \ldots, (a_n, b_n)$.

Chapter 2

Moduli spaces: Kuranishi structures, morphisms, deformation families

2.1 Kuranishi structure construction

Unless otherwise specified, throughout this section X is a symplectic manifold with symplectic form ω and fixed ω-tame almost complex structure J, and L is a compact, relatively spin Lagrangian in X. Note that we do not require that X be compact, but we will require that the moduli spaces of discs and curves we consider be compact. Our primary examples, the total spaces of line bundles $\mathscr{O}(-n)$ on $\mathbb{P}^1$, are in fact non-compact, but the moduli spaces we consider are compact, see Lemma 1.4.3.

2.1.1 Kuranishi structure preliminaries

Let $\mathcal{X}$ be a compact metrizable space and let $p \in \mathcal{X}$. The following definitions are generalizations of those in FOOO [15], that essentially amount to forgetting the linear structure of the obstruction fibers.

Definition 2.1.1. *A (smooth) Kuranishi neighborhood of* p *in* $\mathcal{X}$ *consists of the data* $(V_p, E_p, \Gamma_p, s_p, \psi_p)$ *such that*

1. V_p *is a finite dimensional smooth manifold with corners*[1].
2. E_p *is a finite dimensional smooth manifold diffeomorphic to an open ball in a finite dimensional Euclidean space, together with a distinguished point, which we will call* $0 \in E_p$*. We call* E_p *the obstruction fiber.*

[1] We use the definition of smooth manifold with corners appearing in Joyce [35].

3. Γ_p *is a finite group acting smoothly and effectively on* V_p *and smoothly on* E_p.

4. s_p *is a* Γ_p *equivariant smooth map* $V_p \to E_p$ *called the Kuranishi map.*

5. ψ_p *is a homeomorphism from* $s_p^{-1}(0)/\Gamma_p$ *to a neighborhood of* p *in* $\mathcal{X}$. *Here* 0 *is the distinguished point of* E_p.

In FOOO [15], they strictly speaking refer to $U_p := V_p/\Gamma_p$ as the Kuranishi neighborhood, but we will use the term to refer to either V_p or the whole quintuple.

For a point $x \in V_p$, we let $(\Gamma_p)_x$ denote the isotropy subgroup at x, i.e. the subgroup of Γ_p fixing x.

Definition 2.1.2. *Let* $(V_p, E_p, \Gamma_p, s_p, \psi_p)$ *and* $(V_q, E_q, \Gamma_q, s_q, \psi_q)$ *be Kuranishi neighborhoods of points* $p \in \mathcal{X}$ *and* $q \in \psi_p(s_p^{-1}(0)/\Gamma_p) \subseteq \mathcal{X}$ *respectively. We say a triple* $(\phi_{pq}, \hat{\phi}_{pq}, h_{pq})$ *is a coordinate change or transition map if*

1. $h_{pq} : \Gamma_q \to \Gamma_p$ *is an injective group homomorphism.*

2. $\phi_{pq} : V_{pq} \to V_p$ *is an* h_{pq} *equivariant smooth embedding from a* Γ_q *invariant open neighborhood* V_{pq} *of* o_q *to* V_p, *such that the induced map* $\underline{\phi}_{pq} : V_{pq}/\Gamma_q \to V_p/\Gamma_p$ *is injective.*

3. $(\phi_{pq}, \hat{\phi_{pq}})$ *is an* h_{pq} *equivariant smooth embedding of (trivial) fiber bundles* $V_{pq} \times E_q \to V_p \times E_p$ *with* $\hat{\phi}_{pq}(0) = 0$

4. $\hat{\phi}_{pq} \circ s_q = s_p \circ \phi_{pq}$

5. $\psi_q = \psi_p \circ \underline{\phi}_{pq}$ *on* $(s_q^{-1}(0) \cap V_{pq})/\Gamma_q$

6. *The map* h_{pq} *restricts to an isomorphism on isotropy groups* $(\Gamma_q)_x \to (\Gamma_p)_{\phi_{pq}(x)}$ *for any* $x \in V_{pq}$.

Note that this transition map is asymmetrical in p and q and is in general only defined in one direction.

Definition 2.1.3. *A Kuranishi structure on* $\mathcal{X}$ *assigns a Kuranishi neighborhood* $(V_p, E_p, \Gamma_p, s_p, \psi_p)$ *for each* $p \in \mathcal{X}$ *and a coordinate change* $(\phi_{pq}, \hat{\phi}_{pq}, h_{pq})$ *for each* $q \in \psi_p(s_p^{-1}/\Gamma_p)$ *such that the following holds*

1. $\dim V_p - \dim E_p$ *is independent of* p*. This is called the virtual dimension of the Kuranishi structure.*

2. $\underline{\phi}_{pq} \circ \underline{\phi}_{qr} = \underline{\phi}_{pr}$.

We now introduce a preliminary notion of morphism of Kuranishi spaces. It is likely more restrictive than is naturally necessary, and will be developed further in future work. This notion is, in some sense, very "hands on," in contrast with the more abstract morphisms of Kuranishi spaces of Joyce [34].

Definition 2.1.4. *Let* $\mathcal{X}$ *and* $\mathcal{Y}$ *be compact metrizable spaces with Kuranishi structures given by charts* $(V_{\mathcal{X},p}, E_{\mathcal{X},p}, \Gamma_{\mathcal{X},p}, s_{\mathcal{X},p}, \psi_{\mathcal{X},p})$ *and* $(V_{\mathcal{Y},q}, E_{\mathcal{Y},q}, \Gamma_{\mathcal{Y},q}, s_{\mathcal{Y},q}, \psi_{\mathcal{Y},q})$ *respectively for each point* $p \in \mathcal{X}$ *and* $q \in \mathcal{Y}$*, along with transition maps* $(\phi_{\mathcal{X},p,p'}, \hat{\phi}_{\mathcal{X},p,p'}, h_{\mathcal{X},p,p'})$ *and* $(\phi_{\mathcal{Y},q,q'}, \hat{\phi}_{\mathcal{Y},q,q'}, h_{\mathcal{Y},q,q'})$ *respectively.*

A Kuranishi morphism $(f, \{(f_p, (f_p)_*)\}_{p\in\mathcal{X}})$ *is a continuous function* $f : \mathcal{X} \to \mathcal{Y}$ *along with smooth map* $f_p : V_{\mathcal{X},p} \to V_{\mathcal{Y},f(p)}$ *and diffeomorphism onto its image* $(f_p)_* : E_{\mathcal{X},p} \to E_{\mathcal{Y},f(p)}$ *with* $(f_p)_*(0) = 0$*, such that the following diagrams commute:*

$$\begin{array}{ccc} E_{\mathcal{X},p} & \xrightarrow{(f_p)_*} & E_{\mathcal{Y},f(p)} \\ {\scriptstyle s_{\mathcal{X},p}}\uparrow & & \uparrow{\scriptstyle s_{\mathcal{Y},f(p)}} \\ V_{\mathcal{X},p} & \xrightarrow[f_p]{} & V_{\mathcal{Y},f(p)} \end{array} \qquad \begin{array}{ccc} V_{\mathcal{X},p} & \xrightarrow{f_p} & V_{\mathcal{Y},f(p)} \\ {\scriptstyle \phi_{\mathcal{X},p,p'}}\uparrow & & \uparrow{\scriptstyle \phi_{\mathcal{Y},f(p),f(p')}} \\ V_{\mathcal{X},p,p'} & \xrightarrow[f_{p'}]{} & V_{\mathcal{Y},f(p),f(p')} \end{array}$$

The morphism f *is an isomorphism if* f *is a homeomorphism, every* f_p *is a diffeomorphism, and every* $(f_p)_*$ *is a diffeomorphism.*

We also note here that the specific situation of this dissertation does not require the detailed treatment of good coordinate systems or the associated construction of virtual fundamental chains, so we will not discuss them here.

2.1.2 Universal family with coordinate at infinity

For each $\mathfrak{p} = [(\Sigma, \vec{z}, \vec{z}^{int}), u] \in \mathcal{M}_{k+1,\ell}(\beta)$, we have an associated graph $\mathcal{G}_{\mathfrak{p}}$ with some extra data, called the combinatorial type of $\mathfrak{p}$. A vertex v of $\mathcal{G}_{\mathfrak{p}}$ corresponds to an irreducible component Σ_v of Σ (either a disc or a sphere). We decorate v with the information of which marked points are contained in Σ_v, and also with $\beta_v = [u|_{\Sigma_v}]$ in either $H_2(X, L; \mathbb{Z})$ or $H_2(X; \mathbb{Z})$. An edge e between v_1 and v_2 corresponds to a singular point in the intersection of two components $\Sigma_{v_1}, \Sigma_{v_2}$. We also orient the edges and can assign each a length $T_e \in \mathbb{R}_{>0}$. If a directed edge e is the ordered pair (v, v'), we say that e is an "outgoing" edge of v and an "incoming" edge of v'. Since we are considering only the genus zero case, our graph is always a tree. We choose one of the disc vertices to be the root of the tree and orient all edges so that they point toward the root. That is, each non-root vertex will have one outgoing edge, with all other edges incoming, and following the unique outgoing edge gives a path to the root vertex.

Definition 2.1.5 (Combinatorial Type, [17] 15.6)**.** *A graph $\mathcal{G}$ equipped with the data described above is the combinatorial type of $\mathfrak{p}$, and $\mathcal{M}_{k+1,\ell}(\beta; \mathcal{G})$ is the set of $\mathfrak{p}$ with combinatorial type $\mathcal{G}$.*

Let $\mathcal{G}$ be a combinatorial type, and consider the following process. Shrink an edge e of $\mathcal{G}$ and identify its vertices v_1, v_2, to get a new vertex v. We put $\beta_v = \beta_{v_1} + \beta_{v_2}$ and the marked points of v_1, v_2 are assigned to v. For combinatorial types $\mathcal{G}, \mathcal{G}'$, we say $\mathcal{G} > \mathcal{G}'$ if $\mathcal{G}'$ is obtained from $\mathcal{G}$ in this way.

We let $C^0(\mathcal{G})$ denote the set of vertices of $\mathcal{G}$, and we let $C^0_d(\mathcal{G})$ and $C^0_s(\mathcal{G})$ be the set of disc vertices and sphere vertices respectively. Similarly, we let $C^1(\mathcal{G})$ denote the set of edges of $\mathcal{G}$, and we let $C^1_o(\mathcal{G})$ and $C^1_c(\mathcal{G})$ be the set of boundary singular point edges and interior singular point edges respectively.

We let $\Gamma_{\mathfrak{p}}$ be the group of automorphisms of $\mathfrak{p} \in \mathcal{M}_{k+1,\ell}(\beta)$, where the automorphism is required to fix the interior marked points. We let $\Gamma^+_{\mathfrak{p}}$ be the (larger) group of automorphisms of $\mathfrak{p}$ where the automorphisms may permute the interior marked points instead of fixing them.

Now, consider a disc $\mathfrak{x} = [\Sigma, \vec{z}, \vec{z}^{int}] \in \mathcal{M}_{k+1,\ell}$ of combinatorial type $\mathcal{G}$. We have that $\mathcal{M}_{k+1,\ell}$ is an effective orbifold with corners with local model $\mathfrak{V}(\mathfrak{x})/\Gamma_{\mathfrak{x}}$.

For each $v \in C^0_d(\mathcal{G})$, the element $\mathfrak{x}$ determines a marked disc $\mathfrak{x}_v \in \mathcal{M}_{k_v+1,\ell_v}$, consisting of a single component. Likewise, for each $v \in C^0_s(\mathcal{G})$, the element $\mathfrak{x}$ determines a marked sphere $\mathfrak{x}_v \in \mathcal{M}^{cl}_{\ell_v}$, consisting of a single component. Let $\mathfrak{V}(\mathfrak{x}_v)/\Gamma_{\mathfrak{x}_v}$ be a local orbifold model of the appropriate moduli space at $\mathfrak{x}_v$ such that every disc/sphere in the local chart consists of a single component.

We now define a universal family with coordinate at infinity, which we usually refer to just as a coordinate at infinity.

Definition 2.1.6 (Coordinate at infinity, Def 16.2 [17]). *Let $\pi : \mathfrak{M}_{\mathfrak{x}_v} \to \mathfrak{V}(\mathfrak{x}_v)$ be a fiber bundle, whose fibers are two (real) dimensional manifolds with fiberwise complex structure. This bundle is a universal family with coordinate at infinity (or simply a coordinate at infinity) if it satisfies the following conditions:*

1. *$\mathfrak{M}_{\mathfrak{x}_v}$ has a fiberwise biholomorphic $\Gamma^+_{\mathfrak{x}}$ action and π is is $\Gamma^+_{\mathfrak{x}}$ equivariant.*
2. *For $\mathfrak{y} \in \mathfrak{V}(\mathfrak{x}_v)$ the fiber $\pi^{-1}(\mathfrak{y})$ is biholomorphic to $\Sigma_{\mathfrak{y}}$ minus marked points corresponding to the singular points of $\mathfrak{y}$.*
3. *As a part of the data we fix a closed subset $\mathfrak{K}_{\mathfrak{x}_v} \subseteq \mathfrak{M}_{\mathfrak{x}_v}$ such that the restriction of π to $\mathfrak{K}_{\mathfrak{x}_v}$ is proper.*
4. *We consider the product of $\mathfrak{V}(\mathfrak{x}_v)$ with the union*

$$\left(\bigsqcup_{\substack{e\in C^1_o(\mathcal{G})\\ e \text{ is an outgoing edge of } v}} (0,\infty)\times[0,1]\right) \sqcup \left(\bigsqcup_{\substack{e\in C^1_o(\mathcal{G})\\ e \text{ is an incoming edge of } v}} (-\infty,0)\times[0,1]\right)$$
$$\sqcup \left(\bigsqcup_{\substack{e\in C^1_c(\mathcal{G})\\ e \text{ is an outgoing edge of } v}} (0,\infty)\times S^1\right) \sqcup \left(\bigsqcup_{\substack{e\in C^1_c(\mathcal{G})\\ e \text{ is an incoming edge of } v}} (-\infty,0)\times S^1\right) \tag{2.1.1}$$

As a part of the data we fix a diffeomorphism between $\mathfrak{M}_{\mathfrak{x}_v} \setminus \mathfrak{K}_{\mathfrak{x}_v}$ and 2.1.1 that commutes with the projection to $\mathfrak{V}(\mathfrak{x}_v)$ and is a fiberwise biholomorphic map.

Moreover, the diffeomorphism sends each end corresponding to a singular point z_e *to the end in 2.1.1 corresponding to the edge* e

5. *The diffeomorphism in (4) extends to a fiber preserving diffeomorphism* $\mathfrak{M}_{\mathfrak{x}_v} \cong \mathfrak{V}(\mathfrak{x}_v) \times (\Sigma_{\mathfrak{x}_v} \setminus \{\textit{singular points}\})$*. This diffeomorphism sends each of the interior or boundary marked points of the fiber of* $\mathfrak{y}$ *to the corresponding marked point of* $\{\mathfrak{y}\} \times \Sigma_{\mathfrak{x}_v}$*. However, this diffeomorphism does not preserve fiberwise complex structure. We fix this extension of the diffeomorphism as part of the data.*

6. *The action of an element of* $\Gamma^+_{\mathfrak{x}_v}$ *on 2.1.1 is given using the fixed neck biholomorphism by exchanging the factors associated to the edges* e *and by rotation of the* S^1 *factors.*

7. *We assume the coordinate at infinity is invariant under the action of* Γ^+_x *(the whole group) in the sense described below.*

We also fix a family of metrics on the fibers of $\pi : \mathfrak{M}_{\mathfrak{x}_v} \to \mathfrak{V}(\mathfrak{x}_v)$ that coincide with the standard flat metric on the neck regions. See Remark 16.13 in FOOO [17].

In order to define invariance of a coordinate at infinity under the action of $\Gamma^+_{\mathfrak{x}}$ (the whole group, not just the portion for a single component), we need to consider the following fiber bundle, which is essentially just a combination of the universal family for each component of $\mathfrak{x}$. Take $\prod_{v \in C^0(\mathcal{G})} \mathfrak{V}(\mathfrak{x}_v)$ and pull back the coordinate at infinity for each component by the projection map. The fiberwise disjoint union of these fiber bundles over the product of the bases is then our desired bundle:

$$\bigodot_{v \in C^0(\mathcal{G})} \mathfrak{M}_{\mathfrak{x}_v} \to \prod_{v \in C^0(\mathcal{G})} \mathfrak{V}(\mathfrak{x}_v)$$

That is, the base of the new bundle is the product $\prod_{v \in C^0(\mathcal{G})} \mathfrak{V}(\mathfrak{x}_v)$ of the bases of the individual bundles, and the fiber of this new bundle over a point $(\mathfrak{x}_v)_{v \in C^0(\mathcal{G})}$ is the disjoint union of the original fibers over each $\mathfrak{x}_v$. In particular, each fiber of this new bundle still has real dimension 2.

This bundle has a $\Gamma^+_{\mathfrak{x}_v}$-action for each v. Furthermore, the group $\Gamma^+_{\mathfrak{x}}$ acts on the sum of the second factors in 2.1.1 by exchanging edge factors and rotating S^1 factors.

We require that this gives a $\Gamma_{\mathfrak{x}}^{+}$-action on the restricted bundle

$$\bigodot_{v\in C^0(\mathcal{G})} (\mathfrak{M}_{\mathfrak{x}_v} \setminus \mathfrak{K}_{\mathfrak{x}_v}) \to \prod_{v\in C^0(\mathcal{G})} \mathfrak{V}(\mathfrak{x}_v).$$

Each $\gamma \in \Gamma_{\mathfrak{x}}^{+}$ is biholomorphic as a map between fibers of this bundle.

Remark 2.1.7. *We also have an extension of this action to the entire bundle $\bigodot_{v\in C^0(\mathcal{G})} \mathfrak{M}_{\mathfrak{x}_v}$ using the diffeomorphism between fibers fixed by the coordinate at infinity. That is, given $\gamma \in \Gamma_{\mathfrak{x}}^{+}$ and $\mathfrak{y}, \mathfrak{y}' \in \prod_{v\in C^0(\mathcal{G})} \mathfrak{V}(\mathfrak{x}_v)$ with $\gamma\mathfrak{y} = \mathfrak{y}'$, we have a diffeomorphism $\gamma : \Sigma_{\mathfrak{y}} \to \Sigma_{\mathfrak{y}'}$ obtained by mapping $\Sigma_{\mathfrak{y}}$ diffeomorphically to $\Sigma_{\mathfrak{x}}$, applying γ to $\Sigma_{\mathfrak{x}}$, then mapping diffeomorphically to $\Sigma_{\mathfrak{y}'}$.*

We now fix a coordinate at infinity for each $\mathfrak{x}_v$ that is invariant under the $\Gamma_{\mathfrak{x}}^{+}$ action. We use it to model a neighborhood of $\mathfrak{x}$ in $\mathcal{M}_{k+1,\ell}$.

Let $\mathfrak{y} \in \mathfrak{V}(\mathfrak{x}; \mathcal{G})$, let $T_e \in \mathbb{R}$ for each edge $e \in C^1(\mathcal{G})$ be a large number to be chosen later, and let $\vec{T}_o = (T_e; e \in C_o^1(\mathcal{G}))$ and $(\vec{T}_c, \vec{\theta}) = (T_e, \theta_e; e \in C_c^1(\mathcal{G}))$ in

$$\left(\prod_{e\in C_o^1(\mathcal{G})} (T_{e,0}, \infty]\right) \times \left(\prod_{e\in C_c^1(\mathcal{G})} ((T_{e,0}, \infty] \times S^1)/\sim\right),$$

where the equivalence relation $\sim$ identifies (T, θ) and (T', θ') if both coordinates are equal or if $T = T' = \infty$ (essentially, it closes the cylinder at infinity). The $\vec{T}_o$ and $(\vec{T}_c, \vec{\theta})$ are gluing parameters, and we are performing a straightforward gluing (not to be confused with the gluing of maps that appears later) to obtain a new Riemann surface.

Take a representative $\Sigma_{\mathfrak{y}}$ of $\mathfrak{y}_v$ and let $K_{\mathfrak{y}_v} = \Sigma_{\mathfrak{y}_v} \cap \mathfrak{K}_{\mathfrak{x}_v}$. We call the union $\bigcup_{v\in C^0(\mathcal{G})} K_{\mathfrak{y}_v}$ the *core* of $\mathfrak{y}$. Our coordinate at infinity gives a biholomorphic map between the complement $\bigcup_{v \in C^0(\mathcal{G})} \Sigma_{\mathfrak{y}_v} \setminus K_{\mathfrak{y}_v}$ of the core and

$$\left(\bigcup_{\substack{e \in C^1_o(\mathcal{G}) \\ e \text{ an outgoing edge of } v}} (0,\infty) \times [0,1] \right) \cup \left(\bigcup_{\substack{e \in C^1_o(\mathcal{G}) \\ e \text{ an incoming edge of } v}} (-\infty,0) \times [0,1] \right)$$
$$\cup \left(\bigcup_{\substack{e \in C^1_c(\mathcal{G}) \\ e \text{ an outgoing edge of } v}} (0,\infty) \times S^1 \right) \cup \left(\bigcup_{\substack{e \in C^1_c(\mathcal{G}) \\ e \text{ an incoming edge of } v}} (-\infty,0) \times S^1 \right).$$

We call the coordinates of each summand above $(\tau'_e, t_e), (\tau''_e, t_e), (\tau'_e, t'_e), (\tau''_e, t''_e)$ respectively (identifying S^1 with $\mathbb{R}/\mathbb{Z}$).

For each $T_e \in \vec{T}_o$ or $(T_e, \theta_e) \in (\vec{T}_c, \vec{\theta})$ with $T_e \neq \infty$, we identify portions of each summand above by equating

$$\tau'_e - 5T_e = \tau''_e + 5T_e =: \tau_e, \tag{2.1.2}$$

$$t'_e = t''_e - \theta_e =: t_e, \tag{2.1.3}$$

getting the union

$$\bigcup_{v\in C^0(\mathcal{G})} K_{\mathfrak{y}_v} \cup \left(\bigcup_{\substack{e\in C^1_o(\mathcal{G}) \\ T_e\neq\infty}} [-5T_e, 5T_e]_{\tau_e} \times [0,1]_{t_e} \right) \cup \left(\bigcup_{\substack{e\in C^1_c(\mathcal{G}) \\ T_e\neq\infty}} [-5T_e, 5T_e]_{\tau_e} \times S^1_{t_e} \right) \tag{2.1.4}$$

$$\cup \left(\bigcup_{\substack{e\in C^1_o(\mathcal{G}) \\ e \text{ an outgoing edge of } v \\ \text{with } T_e=\infty}} (0,\infty)_{\tau'_e} \times [0,1]_{t_e} \right) \cup \left(\bigcup_{\substack{e\in C^1_o(\mathcal{G}) \\ e \text{ an incoming edge of } v \\ \text{with } T_e=\infty}} (-\infty,0)_{\tau''_e} \times [0,1]_{T_e} \right) \tag{2.1.5}$$

$$\cup \left(\bigcup_{\substack{e\in C^1_c(\mathcal{G}) \\ e \text{ an outgoing edge of } v \\ \text{with } T_e=\infty}} (0,\infty)_{\tau'_e} \times S^1_{t'_e} \right) \cup \left(\bigcup_{\substack{e\in C^1_c(\mathcal{G}) \\ e \text{ an incoming edge of } v \\ \text{with } T_e=\infty}} (-\infty,0)_{\tau''_e} \times S^1_{t'_e} \right), \tag{2.1.6}$$

where the subscripts on the various intervals and copies of S^1 indicate the coordinate being used. Adding in a finite number of points corresponding to the edges with infinite length, we obtain a singular stable bordered Riemann surface. We let $\overline{\Phi}(\mathfrak{y}, \vec{T}_o, (\vec{T}_c, \vec{\theta}))$ be the element of $\mathcal{M}_{k+1,\ell}$ represented by this Riemann surface.

Definition 2.1.8. *The above discussion defines a map* $\overline{\Phi}$

$$\left(\prod_{v\in C^0(\mathcal{G})} \mathfrak{V}(\mathfrak{x}_v) \right) \times \left(\prod_{e\in C^1_d(\mathcal{G})} (T_{e,0}, \infty] \right) \times \left(\prod_{e\in C^1_c(\mathcal{G})} ((T_{e,0},\infty] \times S^1)/\sim \right) \xrightarrow{\overline{\Phi}} \mathcal{M}_{k+1,\ell}.$$

The map $\overline{\Phi}$ is continuous, open, and stratawise smooth. Furthermore, $\overline{\Phi}$ is $\Gamma^+_{\mathfrak{x}}$ equivariant (lemma 16.9 in [17]), and we have $\overline{\Phi}(\mathfrak{y}, \vec{T}^o, (\vec{T}^c, \vec{\theta})) = \overline{\Phi}(\mathfrak{y}', \vec{T}^{o'}, (\vec{T}^{c'}, \vec{\theta}'))$ if and only if there exists $\gamma \in \Gamma_{\mathfrak{x}}$ sending $(\mathfrak{y}, \vec{T}^o, (\vec{T}^c, \vec{\theta}))$ to $(\mathfrak{y}', \vec{T}^{o'}, (\vec{T}^{c'}, \vec{\theta}'))$ (see Remark 9.5 in Fukaya-Ono [24]). Thus, $\overline{\Phi}$ induces a map $\overline{\Phi}/\Gamma_{\mathfrak{x}}$, which is an open homeomorphism onto its image. The map $\overline{\Phi}$ depends on the choice of coordinate at infinity, but we have good control over how $\overline{\Phi}$ changes with a change in coordinate at infinity (see proposition 16.11, 16.15, corollary 16.16, and lemma 16.18 in [17]), which is important for establishing smoothness of Kuranishi transitions.

2.1.3 Choice of connection, parallel transport, Sobolev spaces, and Hilbert manifolds

There are several places in the FOOO construction where a choice of connection on X is used. It is, for example, used to define the linearized $\bar{\partial}$ operator, it is used to give local coordinates for gluing, and it is used to extend a choice of obstruction vector space to an obstruction bundle by parallel transport. FOOO make a single choice of connection (see [22] Section 2), the Levi-Civita connection of a certain metric, and then use that for all purposes. However, for our case, it will be essential to extend our obstruction fibers to obstruction bundles in a more general way. We describe this process and our other choices related to the connection here.

As in FOOO [22], we take a metric g on X that is Hermitian with respect to the almost complex structure J such that the Lagrangian L is totally geodesic and satisfies $JT_pL \perp T_pL$ for all $p \in L$. We then let ∇ be the Levi-Civita connection of this metric. We use ∇ to define an exponential map $\mathrm{Exp} : TX \to X \times X$ and its local inverse $E : U \to TX$ given by

$$\mathrm{Exp}(x, v) = (x, \exp_x v),$$
$$E(x, y) = (x, \exp_x^{-1}(y)).$$

Here $U = \{(x, y) \in X \times X \mid d(x, y) < \iota_X\}$ where $d(x, y)$ is the Riemannian distance between x and y and ι_X is the injectivity radius of X with our metric. Given a map $u : \Sigma \to X$ and v a section of u^*TX, we will write $\mathrm{Exp}(u, v) : \Sigma \to X$ for the map $z \mapsto \mathrm{Exp}(u(z), v(z))$.

For $x, y \in X$ with $d(x, y) < \iota_X$, we have a unique geodesic of length $d(x, y)$ between

x and y, and we use ∇ to define a parallel transport map $\mathrm{Pal}_x^y : T_xX \to T_yX$ along this geodesic. We define $(\mathrm{Pal}_x^y)^J$ to be the complex linear part of Pal_x^y. Given two maps $u, w : \Sigma \to X$, we use $(\mathrm{Pal}_x^y)^J$ to get the map $(\mathrm{Pal}_x^y)^{(0,1)} : T_{u(x)}X \otimes \Lambda_x^{0,1} \to T_{w(x)}X \otimes \Lambda_x^{0,1}$.

In order to define appropriate notions of parallel transport for maps, we first need to define the following Hilbert spaces and manifolds. For an introduction to Banach manifolds, see for instance Lang [37]. For an introduction to viewing spaces of maps as Banach manifolds see Eliasson [11].

We fix an element $\mathfrak{x} = [\Sigma_{\mathfrak{x}}, \vec{z}, \vec{z}^{int}] \in \mathcal{M}_{k+1,\ell}$ of combinatorial type $\mathcal{G}$ and we fix a coordinate at infinity for $\mathfrak{x}$. Here $\Sigma_{\mathfrak{x}}$ is the specific Riemann surface in the equivalence class $\mathfrak{x}$ given by the coordinate at infinity. We let $\mathfrak{y} \in \prod_{v \in C^0(\mathcal{G})} \mathfrak{V}(\mathfrak{x}_v)$ and let $\mathfrak{Y} = \overline{\Phi}(\mathfrak{y}, \vec{T}_o, (\vec{T}_c, \vec{\theta}))$. Our coordinate at infinity and the construction of the $\overline{\Phi}$ map gives specific Riemann surfaces $\Sigma_{\mathfrak{y}}$ and $\Sigma_{\mathfrak{Y}}$ representing $\mathfrak{y}$ and $\mathfrak{Y}$ respectively.

We will need the following smooth exponential weight function $\Sigma_{\mathfrak{y}_v} \to [0, \infty)$ in order to define appropriate weighted Sobolev norms

$$e_{v,\delta}(\tau_e, t_e) = \begin{cases} = 1 & \text{on } K_{\mathfrak{y}_v}, \\ = e^{\delta|\tau_e + 5T_e|} & \text{if } \tau_e > 1 - 5T_e, \text{ and } e \text{ is an outgoing edge of } v, \\ \in [1, 10] & \text{if } \tau_e < 1 - 5T_e, \text{ and } e \text{ is an outgoing edge of } v, \\ = e^{\delta|\tau_e - 5T_e|} & \text{if } \tau_e < 5T_e - 1, \text{ and } e \text{ is an incoming edge of } v, \\ \in [1, 10] & \text{if } \tau_e > 5T_e - 1, \text{ and } e \text{ is an incoming edge of } v. \end{cases}$$

See FOOO [17] (19.15).

We now define our first Hilbert manifold.

Definition 2.1.9. *The Hilbert manifold $W^2_{m+1,\delta}((\Sigma_{\mathfrak{y}_v}, \partial\Sigma_{\mathfrak{y}_v}); X, L)$ is the space of maps $w_v : (\Sigma_{\mathfrak{y}_v}, \partial\Sigma_{\mathfrak{y}_v}) \to (X, L)$ locally of L^2_{m+1} class with the following expression* *finite:*

$$\sum_{k=1}^{m+1} \sum_{edges\ e\ of\ v} \int_{e\text{-}th\ neck} e_{v,\delta} |\nabla^k w_v|^2 vol_{\Sigma_{\mathfrak{y}_v}} + \sum_{edges\ e\ of\ v} \int_{e\text{-}th\ neck} e_{v,\delta} |d(w_v(z), w_v(z_e))|^2 vol_{\Sigma_{\mathfrak{y}_v}}.$$

The smooth structure is determined by our connection ∇*.*

As in FOOO [17], we choose m large. In particular, we take m large enough that every function we consider is continuous.

The tangent space to $W^2_{m+1,\delta}((\Sigma_{\mathfrak{y}_v}, \partial\Sigma_{\mathfrak{y}_v}); X, L)$ at a point u_v is the following Sobolev space.

Definition 2.1.10 (Def 19.6 in [17], Def 3.4 in [22])**.** *The Sobolev space*

$$W^2_{m+1,\delta}((\Sigma_{\mathfrak{y}_v}, \partial\Sigma_{\mathfrak{y}_v}); u_v^*TX, u_v^*TL)$$

is the vector space of pairs $(V, \vec{v})$*, where* $\vec{v} = (v_e)_{e \in edges\ of\ v}$ *with* $v_e \in T_{u_v(z_e)}X$ *for* $e \in C^1_c(\mathcal{G})$ *and* $v_e \in T_{u_v(z_e)}L$ *for* $e \in C^1_o(\mathcal{G})$*, and where* V *is a section* $(\Sigma_{\mathfrak{y}_v}, \partial\Sigma_{\mathfrak{y}_v}) \to (u_v^*TX, u_v^*TL)$ *with the following norm finite:*

$$\begin{aligned}||(V, \vec{v})||^2_{L^2_{m+1,\delta}} = &\sum_{k=0}^{m+1} \int_{K_v} |\nabla^k V|^2 vol_{\Sigma_{\mathfrak{y}_v}} + \sum_{edges\ e\ of\ v} ||v_e||^2 \\ &+ \sum_{k=0}^{m+1} \sum_{edges\ e\ of\ v} \int_{e\text{-}th\ neck} e_{v,\delta} |\nabla^k (V(z) - Pal^{u_v(z)}_{u_v(z_e)} v_e)|^2 vol_{\Sigma_{\mathfrak{y}_v}}.\end{aligned}$$

With this norm, $W^2_{m+1,\delta}((\Sigma_{\mathfrak{y}_v}, \partial\Sigma_{\mathfrak{y}_v}); u_v^*TX, u_v^*TL)$ is a separable Hilbert space.

We also need the following Sobolev space, which will be the codomain of the linearized $\bar{\partial}$ equation.

Definition 2.1.11. *For* $u_v \in W^2_{m+1,\delta}((\Sigma_{\mathfrak{y}_v}, \partial\Sigma_{\mathfrak{y}_v}); X, L)$*, the Sobolev space*

$$L^2_{m,\delta}(\Sigma_{\mathfrak{y}_v}; u_v^*TX \otimes \Lambda^{0,1})$$

is the vector space of sections κ of $u_v^*TX \otimes \Lambda^{0,1}$ with the following norm finite:

$$||\kappa||^2_{L^2_{m,\delta}} = \sum_{k=0}^{m} \int_{K_v} |\nabla^k \kappa|^2 vol_{\Sigma_{\mathfrak{y}_v}} + \sum_{k=0}^{m} \sum_{edges\ e\ of\ v} \int_{e\text{-}th\ neck} e_{v,\delta} |\nabla^k \kappa|^2 vol_{\Sigma_{\mathfrak{y}_v}}.$$

With this norm, $L^2_{m,\delta}(\Sigma_{\mathfrak{y}_v}; u_v^*TX \otimes \Lambda^{0,1})$ is a separable Hilbert space. We have an associated Hilbert bundle over $W^2_{m+1,\delta}((\Sigma_{\mathfrak{y}_v}, \partial\Sigma_{\mathfrak{y}_v}); X, L)$.

Definition 2.1.12. *The Hilbert bundle $\mathcal{E}^2_{m,\delta}((\Sigma_{\mathfrak{y}_v}, \partial\Sigma_{\mathfrak{y}_v}); X, L)$ is the bundle over $W^2_{m+1,\delta}((\Sigma_{\mathfrak{y}_v}, \partial\Sigma_{\mathfrak{y}_v}); X, L)$ consisting of pairs (u_v, κ) with $\kappa \in L^2_{m,\delta}(\Sigma_{\mathfrak{y}_v}; u_v^*TX \otimes \Lambda^{0,1})$.*

$\bar{\partial}$ gives a section of this bundle. We will discuss linearization of $\bar{\partial}$ in Section 2.1.4. The spaces in Defintions 2.1.9, 2.1.10, 2.1.11, and 2.1.12 will be used to define Fredholm regularity and to reduce from the infinite dimensional setting to the finite dimensional setting.

Remark 2.1.13. *Given any two $\mathfrak{y}_v, \mathfrak{y}'_v \in \mathfrak{V}(\mathfrak{x}_v)$, the coordinate at infinity gives a diffeomorphism between $\Sigma_{\mathfrak{y}_v}$ and $\Sigma_{\mathfrak{y}'_v}$, in such a way that the Hilbert manifold in Definition 2.1.9, the Hilbert space in Definition 2.1.10, and the Hilbert bundle in Definition 2.1.12 are all independent of $\mathfrak{y}_v$ up to diffeomorphism. However, since the coordinate at infinity does not give a biholomorphism between $\Sigma_{\mathfrak{y}_v}$ and $\Sigma_{\mathfrak{y}'_v}$, the diffeomorphisms between Hilbert manifolds, spaces, and bundles do not commute with $\bar{\partial}$, and we need to treat them as distinct.*

It will also be convenient to collect the Hilbert bundles $\mathcal{E}^2_{m,\delta}((\Sigma_{\mathfrak{y}_v}, \partial\Sigma_{\mathfrak{y}_v}); X, L)$ for different $\mathfrak{y}_v$ into a single object.

Definition 2.1.14. *We let $W^2_{m+1,\delta}(\mathfrak{V}(\mathfrak{x}_v); X, L)$ be the union*

$$\bigcup_{\mathfrak{y}_v \in \mathfrak{V}(\mathfrak{x}_v)} W^2_{m+1,\delta}((\Sigma_{\mathfrak{y}_v}, \partial\Sigma_{\mathfrak{y}_v}); X, L).$$

Following the observation in Remark 2.1.13, we give $W^2_{m+1,\delta}(\mathfrak{V}(\mathfrak{x}_v); X, L)$ the smooth structure of $\mathfrak{V}(\mathfrak{x}_v) \times W^2_{m+1,\delta}((\Sigma_{\mathfrak{x}_v}, \partial\Sigma_{\mathfrak{x}_v}); X, L)$ using our coordinate at infinity.

We then let $\mathcal{E}^2_{m,\delta}(\mathfrak{V}(\mathfrak{x}_v); X, L)$ denote the Hilbert bundle over $W^2_{m+1,\delta}(\mathfrak{V}(\mathfrak{x}_v); X, L)$ with fiber over $(\mathfrak{y}_v, u_v)$ equal to $L^2_{m,\delta}(\Sigma_{\mathfrak{y}_v}; u_v^*TX \otimes \Lambda^{0,1})$.

We have several more spaces to define, which will be used during the gluing procedure and for establishing smoothness of our Kuranishi structure. We define the following Sobolev space, which is closely related to the one in Definition 2.1.10 but is used instead for gluing. Our source for the sections is no longer a single component, and we assume that the map along which we pull back TX, TL is smooth componentwise. We first need another weight function $e_{\vec{T},\delta} : \Sigma_{\mathfrak{Y}} \to [1, \infty)$ to define the norm.

$$e_{\vec{T},\delta}(\tau_e, t_e) = \begin{cases} = 1 & \text{on } K_{\mathfrak{y}_v}, \\ = e^{\delta|\tau_e - 5T_e|} & \text{if } 1 < \tau_e < 5T_e - 1, \\ = e^{\delta|\tau_e + 5T_e|} & \text{if } -1 > \tau_e > 1 - 5T_e, \\ \in [1, 10] & \text{if } |\tau_e - 5T_e| < 1 \text{ or } |\tau_e + 5T_e| < 1, \\ \in [e^{5T_e\delta}/10, e^{5T_e\delta}] & \text{if } |\tau_e| < 1. \end{cases}$$

See FOOO [17] (19.16).

Definition 2.1.15 (Def 19.9 in [17])**.** *For* $\mathfrak{Y} = \Phi(\mathfrak{y}, \vec{T}_o, (\vec{T}_c, \vec{\theta}))$*, let* $u : (\Sigma_{\mathfrak{Y}}, \partial\Sigma_{\mathfrak{Y}}) \to (X, L)$ *be a smooth map. The Sobolev space*

$$W^2_{m+1,\delta}((\Sigma_{\mathfrak{Y}}, \partial\Sigma_{\mathfrak{Y}}); u^*TX, u^*TL)$$

is the vector space of pairs $(V, \vec{v})$*, where* $\vec{v} = (v_e)_{e \text{ with } T_e = \infty}$ *with* $v_e \in T_{u(z_e)}X$ *for* $e \in C^1_c(\mathcal{G})$ *and* $v_e \in T_{p_e}L$ *for* $e \in C^1_o(\mathcal{G})$*, and where* V *is a section* $(\Sigma_{\mathfrak{Y}}, \partial\Sigma_{\mathfrak{Y}}) \to$

(u^*TX, u^*TL) with the following norm $||(V,\vec{v})||^2_{L^2_{m+1,\delta}}$ finite:

$$\sum_{v\in C^0(\mathcal{G})}\sum_{k=0}^{m+1}\int_{K_v}|\nabla^k V|^2 vol_{\Sigma_{\mathfrak{Y}}} + \sum_{e \text{ with } T_e=\infty}||v_e||^2$$
$$+\sum_{v\in C^0(\mathcal{G})}\sum_{k=0}^{m+1}\sum_{\text{edge } e \text{ of } v \text{ with } T_e=\infty}\int_{e\text{-th neck } (v \text{ side})} e_{v,\delta}|\nabla^k(V(z)-Pal_{u(z_e)}^{u(z)}v_e)|^2 vol_{\Sigma_{\mathfrak{Y}}}$$
$$+\sum_{k=0}^{m+1}\sum_{\text{edge } e \text{ of } v \text{ with } T_e\neq\infty}\int_{e\text{-th neck } (v \text{ side})} e_{\vec{T},\delta}|\nabla^k(V(z)-Pal_{u(z_e)}^{u(z)}v_e)|^2 vol_{\Sigma_{\mathfrak{Y}}}$$
$$+\sum_{e \text{ with } T_e\neq\infty}\left|\left|V\left(0,\frac{1}{2}\right)_e\right|\right|^2,$$

where $\left(0,\frac{1}{2}\right)_e$ is a point in the e-th neck.

This is again a separable Hilbert space with this norm. We also have the following Sobolev space, which is closely related to that in Definition 2.1.11 and will be used in gluing.

Definition 2.1.16. *For u smooth and $\mathfrak{Y}$ as above, the Sobolev space*

$$L^2_{m,\delta}(\Sigma_{\mathfrak{Y}}; u^*TX\otimes\Lambda^{0,1})$$

is the vector space of sections κ of $u^*TX\otimes\Lambda^{0,1}$ with the following norm finite:

$$||\kappa||^2_{L^2_{m,\delta}} = \sum_{v\in C^0(\mathcal{G})}\sum_{k=0}^{m}\int_{K_v}|\nabla^k\kappa|^2 vol_{\Sigma_{\mathfrak{Y}}}$$
$$+\sum_{v\in C^0(\mathcal{G})}\sum_{k=0}^{m}\sum_{\text{edge } e \text{ of } v \text{ with } T_e=\infty}\int_{e\text{-th neck } (v \text{ side})} e_{v,\delta}|\nabla^k\kappa|^2 vol_{\Sigma_{\mathfrak{Y}}}$$
$$+\sum_{e \text{ with } T_e\neq\infty}\int_{e\text{-th neck}} e_{\vec{T},\delta}|\nabla^k\kappa|^2 vol_{\Sigma_{\mathfrak{Y}}}.$$

Finally, we need the following Hilbert manifolds, Sobolev spaces, and Hilbert bundle. Let $K'_{\mathfrak{x}_v}$ be a compact subset of the core $K_{\mathfrak{x}_v}$ such that the interior $\operatorname{Int} K'_{\mathfrak{x}_v}$ is non-empty. We use the diffeomorphism $K_{\mathfrak{y}_v}\cong K_{\mathfrak{x}_v}$ to define $K'_{\mathfrak{y}_v}$.

Definition 2.1.17. *The Hilbert manifold* $W^2_{m+1}((K'_{\mathfrak{y}_v}, K'_{\mathfrak{y}_v} \cap \partial\Sigma_{\mathfrak{y}_v}); X, L)$ *is the space of maps* $w : (K'_{\mathfrak{y}_v}, K'_{\mathfrak{y}_v} \cap \partial\Sigma_{\mathfrak{y}_v}) \to (X, L)$ *of* L^2_{m+1} *class. The smooth structure is determined by our connection* ∇.

The tangent space at $u \in W^2_{m+1}((K'_{\mathfrak{y}_v}, K'_{\mathfrak{y}_v} \cap \partial\Sigma); X, L)$ is the following separable Hilbert space.

Definition 2.1.18. *The Sobolev space* $W^2_{m+1}((K'_{\mathfrak{y}_v}, K'_{\mathfrak{y}_v} \cap \partial\Sigma_{\mathfrak{y}_v}); u_v^*TX, u_v^*TL)$ *is the space of sections* $V : (K'_{\mathfrak{y}_v}, K'_{\mathfrak{y}_v} \cap \partial\Sigma_{\mathfrak{y}_v}) \to (u_v^*TX, u_v^*TL)$ *of* L^2_{m+1} *class.*

We also have the following separable Hilbert space.

Definition 2.1.19. *The Sobolev space* $L^2_m(K'_{\mathfrak{y}_v}; u_v^*TX \otimes \Lambda^{0,1})$ *is the space of sections* $\kappa : K'_{\mathfrak{y}_v} \to u_v^*TX \otimes \Lambda^{0,1}$ *of* L^2_m *class. Explicitly, the norm is*

$$||\kappa||^2_{L^2_{m,\delta}} = \sum_{k=0}^{m} \int_{K'_v} |\nabla^k \kappa|^2 vol_{\Sigma_{\mathfrak{y}_v}}.$$

These Hilbert spaces then fit together to give the following Hilbert bundle.

Definition 2.1.20. *The Hilbert bundle* $\mathcal{E}^2_m((K'_{\mathfrak{y}_v}, K'_{\mathfrak{y}_v} \cap \partial\Sigma_{\mathfrak{y}_v}); X, L)$ *is the bundle over* $W^2_{m+1}((K'_{\mathfrak{y}_v}, K'_{\mathfrak{y}_v} \cap \partial\Sigma_{\mathfrak{y}_v}); X, L)$ *consisting of pairs* (u, κ) *with* $\kappa \in L^2_m(K'_{\mathfrak{y}_v}; u_v^*TX \otimes \Lambda^{0,1})$.

We again have a section $\bar{\partial}$ of this Hilbert bundle.

As in Definition 2.1.14, we collect these Hilbert bundles into a single object.

Definition 2.1.21. *We let* $W^2_{K_{\mathfrak{x}_v}, m+1}(\mathfrak{V}(\mathfrak{x}_v); X, L)$ *be the union*

$$\bigcup_{\mathfrak{y}_v \in \mathfrak{V}(\mathfrak{x}_v)} W^2_{m+1}((K'_{\mathfrak{y}_v}, K'_{\mathfrak{y}_v} \cap \partial\Sigma_{\mathfrak{y}_v}); X, L).$$

Following the observation in Remark 2.1.13, we give $W^2_{K_{\mathfrak{x}_v}, m+1}(\mathfrak{V}(\mathfrak{x}_v); X, L)$ *the smooth structure of* $\mathfrak{V}(\mathfrak{x}_v) \times W^2_{m+1}((K'_{\mathfrak{x}_v}, K'_{\mathfrak{x}_v} \cap \partial\Sigma_{\mathfrak{x}_v}); X, L)$.

We then let $\mathcal{E}^2_{K_{\mathfrak{x}_v}, m}(\mathfrak{V}(\mathfrak{x}_v); X, L)$ *denote the Hilbert bundle over* $W^2_{K_{\mathfrak{x}_v}, m+1}(\mathfrak{V}(\mathfrak{x}_v), X)$ *with fiber over* $(\mathfrak{y}_v, u_v)$ *equal to* $L^2_m(K'_{\mathfrak{y}_v}; u_v^*TX \otimes \Lambda^{0,1})$*. Finally,* *we let*

$$W^2_{K'_{\mathfrak{x}},m+1}(\mathfrak{V}(\mathfrak{x});X,L) := \prod_{v\in C^0(\mathcal{G})} W^2_{K'_{\mathfrak{x}_v},m+1}(\mathfrak{V}(\mathfrak{x}_v);X,L),$$
$$\mathcal{E}^2_{K'_{\mathfrak{x}},m}(\mathfrak{V}(\mathfrak{x});X,L) := \bigodot_{v\in C^0(\mathcal{G})} \mathcal{E}^2_{K'_{\mathfrak{x}_v},m+1}(\mathfrak{V}(\mathfrak{x}_v);X,L).$$

With our various Hilbert spaces, manifolds, and bundles defined, we can now return to discussing parallel transport.

Given two maps $u_v, w_v \in W^2_{m+1,\delta}((\Sigma_{\mathfrak{y}_v}, \partial\Sigma_{\mathfrak{y}_v}); X, L)$ such that $\sup\{d(u_v(z), w_v(z)) \mid z \in \Sigma_v\} < \iota_X$ (the injectivity radius of X), our pointwise parallel transport maps give the following maps of sections.

$$\mathrm{Pal}_{u_v}^{w_v} : W^2_{m+1,\delta}((\Sigma_{\mathfrak{y}_v}, \partial\Sigma_{\mathfrak{y}_v}); u_v^*TX, u_v^*TL) \to W^2_{m+1,\delta}((\Sigma_{\mathfrak{y}_v}, \partial\Sigma_{\mathfrak{y}_v}); w_v^*TX, w_v^*TL),$$
$$(\mathrm{Pal}_{u_v}^{w_v})^J : W^2_{m+1,\delta}((\Sigma_{\mathfrak{y}_v}, \partial\Sigma_{\mathfrak{y}_v}); u_v^*TX, u_v^*TL) \to W^2_{m+1,\delta}((\Sigma_{\mathfrak{y}_v}, \partial\Sigma_{\mathfrak{y}_v}); w_v^*TX, w_v^*TL),$$
$$(\mathrm{Pal}_{u_v}^{w_v})^{(0,1)} : L^2_{m,\delta}(\Sigma_{\mathfrak{y}_v}, u_v^*TX \otimes \Lambda^{0,1}) \to L^2_{m,\delta}(\Sigma_{\mathfrak{y}_v}, w_v^*TX \otimes \Lambda^{0,1}).$$

Similarly, given two smooth maps $u, w : (\Sigma_{\mathfrak{Y}}, \partial\Sigma_{\mathfrak{Y}}) \to (X, L)$, we have the following parallel transport maps of sections.

$$\mathrm{Pal}_{u}^{w} : W^2_{m+1,\delta}((\Sigma_{\mathfrak{Y}}, \partial\Sigma_{\mathfrak{Y}}); u^*TX, u^*TL) \to W^2_{m+1,\delta}((\Sigma_{\mathfrak{Y}}, \partial\Sigma_{\mathfrak{Y}}); w^*TX, w^*TL),$$
$$(\mathrm{Pal}_{u_v}^{w_v})^J : W^2_{m+1,\delta}((\Sigma_{\mathfrak{Y}}, \partial\Sigma_{\mathfrak{Y}}); u^*TX, u^*TL) \to W^2_{m+1,\delta}((\Sigma_{\mathfrak{Y}}, \partial\Sigma_{\mathfrak{Y}}); w^*TX, w^*TL),$$
$$(\mathrm{Pal}_{u_v}^{w_v})^{(0,1)} : L^2_{m,\delta}(\Sigma_{\mathfrak{Y}}, u^*TX \otimes \Lambda^{0,1}) \to L^2_{m,\delta}(\Sigma_{\mathfrak{Y}}, w^*TX \otimes \Lambda^{0,1}).$$

Finally, given two maps $u, w \in W^2_{m+1}((K_{\mathfrak{y}_v}, K_{\mathfrak{y}_v} \cap \partial\Sigma_{\mathfrak{y}_v}); X, L)$, we have the following parallel transport maps of sections.

$$\mathrm{Pal}_u^w : W^2_{m+1}((K'_{\mathfrak{y}_v}, K'_{\mathfrak{y}_v} \cap \partial\Sigma_{\mathfrak{y}_v}); u_v^*TX, u_v^*TL) \to W^2_{m+1}((K'_{\mathfrak{y}_v}, K'_{\mathfrak{y}_v} \cap \partial\Sigma_{\mathfrak{y}_v}); w_v^*TX, w_v^*TL),$$

$$(\mathrm{Pal}_{u_v}^{w_v})^J : W^2_{m+1}((K'_{\mathfrak{y}_v}, K'_{\mathfrak{y}_v} \cap \partial\Sigma_{\mathfrak{y}_v}); u_v^*TX, u_v^*TL) \to W^2_{m+1}((K_{\mathfrak{y}_v}, K_{\mathfrak{y}_v} \cap \partial\Sigma_{\mathfrak{y}_v}); w_v^*TX, w_v^*TL),$$

$$(\mathrm{Pal}_{u_v}^{w_v})^{(0,1)} : L^2_m(K'_{\mathfrak{y}_v}, u_v^*TX \otimes \Lambda^{0,1}) \to L^2_m(K'_{\mathfrak{y}_v}, w_v^*TX \otimes \Lambda^{0,1}).$$

We now depart from FOOO [17], [22]. Let $\mathfrak{p} = (\mathfrak{x}, u)$ where $u : (\Sigma_{\mathfrak{x}}, \partial\Sigma_{\mathfrak{x}}) \to (X, L)$ is pseudoholomorphic.

Remark 2.1.22. *There is an important but subtle point to make here. When we write an element $\mathfrak{p} = (\mathfrak{x}, u)$, we are choosing an equivalence class of disc maps up to automorphisms of the disc map. That is, $\mathfrak{x}$ is an isomorphism class of marked bordered Riemann surfaces, and if we are given a specific representative Σ of $\mathfrak{x}$, there is a uniquely determined map u_Σ making $(\Sigma \cup \vec{z} \cup \vec{z}^{int}, u_\Sigma)$ a representative of $(\mathfrak{x}, u)$.*

We will often have multiple different universal families with coordinate at infinity, and we will need to take great care in these situations. See Lemma 2.1.26 and Section 2.1.5. In particular, in this section $(\mathfrak{x}, u)$ is always source stable, but this will not always be the case in later sections.

Let $E_{(\mathfrak{x},u),v}$ be a finite dimensional complex submanifold of $L^2_m(K'_{\mathfrak{x}_v}; u_v^*TX \otimes \Lambda^{0,1})$ containing 0 such that every element of $E_{(\mathfrak{x},u),v}$ is smooth and supported in $\operatorname{Int} K'_{\mathfrak{x}_v}$. Note that we are now only assuming that u_v is defined on $K'_{\mathfrak{x}_v}$, and *not* on all of $\Sigma_{\mathfrak{x}_v}$. We call this vector space an *obstruction fiber*. We define an *extension of the obstruction fiber* as follows.

Definition 2.1.23. *Given an obstruction fiber $E_{(\mathfrak{x},u),v}$, an extension of the obstruction fiber consists of a choice of the following map, which may be completely unrelated to ∇:*

$$\mathrm{Triv}_{K'_{\mathfrak{x}_v}, E_{(\mathfrak{x},u),v}} : \mathcal{U}_{K_{\mathfrak{x}_v},(\mathfrak{x},u),v} \times E_{(\mathfrak{x},u),v} \to \mathcal{E}^2_{K_{\mathfrak{x}_v},m}(\mathfrak{V}(\mathfrak{x}_v); X, L), \tag{2.1.7}$$

where $\mathcal{U}_{K'_{\mathfrak{r}_v},(\mathfrak{r},u),v}$ *is a neighborhood of* $(\mathfrak{r}_v, u_v|_{K'_{\mathfrak{r}_v}})$ *in* $W^2_{K_{\mathfrak{r}_v},m+1}(\mathfrak{V}(\mathfrak{r}_v); X, L)$*, such that the following properties are satisfied:*

1. *The map* $\mathrm{Triv}_{K'_{\mathfrak{r}_v},E_{(\mathfrak{r},u),v}}$ *is a diffeomorphism onto its image.*

2. *The map* $\mathrm{Triv}_{K'_{\mathfrak{r}_v},E_{(\mathfrak{r},u),v}}$ *defines a trivial sub-bundle* $\mathcal{E}_{K_{\mathfrak{r}_v},(\mathfrak{r},u),v}$ *of* $\mathcal{E}^2_{K'_{\mathfrak{r}_v},m}(\mathfrak{V}(\mathfrak{r}_v); X, L)$ *over* $\mathcal{U}_{K'_{\mathfrak{r}_v},(\mathfrak{r},u),v}$*. The fiber of this bundle over* $(\mathfrak{r}_v, u_v|_{K'_{\mathfrak{r}_v}})$ *must be* $(E_{(\mathfrak{r},u),v})$*.*

Recall that we take m "large." In particular, we take m large enough that

$$W^2_{m+1}((K'_{\mathfrak{y}_v}, K'_{\mathfrak{y}_v} \cap \partial\Sigma_{\mathfrak{y}_v}); X, L) \subseteq C^{10}((K'_{\mathfrak{y}_v}, K'_{\mathfrak{y}_v} \cap \partial\Sigma_{\mathfrak{y}_v}); X, L) \tag{2.1.8}$$

We thus have that our neighborhood $\mathcal{U}_{K_{\mathfrak{r}_v},(\mathfrak{r},u),v}$ is open if we use the C^{10} topology instead of the L^2_{m+1} topology, which we will use in the proof of Proposition 2.1.33.

Recall from Remark 2.1.7 that we have an action of the group $\Gamma^+_{\mathfrak{r}}$ on the fiber bundle

$$\bigodot_{v \in C^0(\mathcal{G})} \mathfrak{M}_{\mathfrak{r}_v} \to \prod_{v \in C^0(\mathcal{G})} \mathfrak{V}(\mathfrak{r}_v)$$

as part of our coordinate at infinity data. Here the fiber over $\mathfrak{y} \in \prod_{v\in C^0(\mathcal{G})} \mathfrak{V}(\mathfrak{r}_v)$ is $\Sigma_{\mathfrak{y}}$ minus singular points, and the action preserves the cores of the fibers (although it does not give a biholomorphism between them). That is, given $\mathfrak{y}, \mathfrak{y}' \in \prod_{v\in C^0(\mathcal{G})} \mathfrak{V}(\mathfrak{r}_v)$ and $\gamma \in \Gamma^+_{\mathfrak{r}}$ with $\gamma\mathfrak{y} = \mathfrak{y}'$, we have a diffeomorphism $\gamma : \Sigma_{\mathfrak{y}} \to \Sigma_{\mathfrak{y}'}$ that restricts to a diffeomorphism $\gamma : K_{\mathfrak{y}} \to K_{\mathfrak{y}'}$. This will give us an action of $\Gamma^+_{(\mathfrak{r},u)}$ on all of our spaces of functions and sections (after taking appropriate products to account for the domains) by pullback, where $\Gamma^+_{(\mathfrak{r},u)} \subseteq \Gamma^+_{\mathfrak{r}}$ is the group of automorphisms γ of $\Sigma_{\mathfrak{r}}$ fixing boundary marked points such that we have $u \circ \gamma = u$ on $K'_{\mathfrak{r}}$, recalling that our map u is only assumed to be defined on $K'_{\mathfrak{r}}$ in this case.

For instance, assuming the union $K'_{\mathfrak{y}} = \bigcup_{v\in C^0(\mathcal{G})} K'_{\mathfrak{y}_v}$ is $\Gamma^+_{(\mathfrak{r},u)}$-invariant (in the sense that the action on $\Sigma_{\mathfrak{y}}$ restricts to an action on this subspace), we have that

$\Gamma^+_{(\mathfrak{x},u)}$ acts on the Hilbert manifold $W^2_{K'_\mathfrak{x},m+1}(\mathfrak{V}(\mathfrak{x});X,L)$ as follows: an element of $W^2_{K'_\mathfrak{x},m+1}(\mathfrak{V}(\mathfrak{x});X,L)$ is a pair $(\mathfrak{y},w)$ with $w:(K'_\mathfrak{y},K'_\mathfrak{y}\cap\partial\Sigma_\mathfrak{y})\to(X,L)$, and $\gamma(\mathfrak{y},w)=(\gamma\mathfrak{y},w\circ\gamma^{-1})$. Similarly, we have that $\Gamma^+_{(\mathfrak{x},u)}$ acts on the Hilbert manifold $\mathcal{E}^2_{K'_\mathfrak{x},m}(\mathfrak{V}(\mathfrak{x});X,L)$ by $\gamma(\mathfrak{y},w,\kappa)=(\gamma\mathfrak{y},w\circ\gamma^{-1},(\gamma^{-1})^*\kappa)$. This leads us to the following definition.

Definition 2.1.24. *Given obstruction fibers $E_{(\mathfrak{x},u),v}$ for all $v\in C^0(\mathcal{G})$, we say that the total obstruction fiber $E_{(\mathfrak{x},u)}=\prod_{v\in C^0(\mathcal{G})}E_{(\mathfrak{x},u),v}$ is invariant with respect to $\Gamma^+_{(\mathfrak{x},u)}$ and that the extension of the obstruction fibers is equivariant with respect to $\Gamma^+_{(\mathfrak{x},u)}$ if the sets $K'_\mathfrak{y}$, and $\prod_{v\in C^0(\mathcal{G})}\mathcal{U}_{K',(\mathfrak{x},u),v}$, and $E_{(\mathfrak{x},u)}$ are invariant under the action of $\Gamma^+_{(\mathfrak{x},u)}$, and the following map $\mathrm{Triv}_{K',E_{(\mathfrak{x},u)}}=\prod_{v\in C^0(\mathcal{G})}\mathrm{Triv}_{K',E_{(\mathfrak{x},u),v}}$ is equivariant:*

$$\prod_{v\in C^0(\mathcal{G})}\mathrm{Triv}_{K',E_{(\mathfrak{x},u),v}}:\prod_{v\in C^0(\mathcal{G})}\left(\mathcal{U}_{K',(\mathfrak{x},u),v}\times E_{(\mathfrak{x},u),v}\right)\to\prod_{v\in C^0(\mathcal{G})}\mathcal{E}^2_{K',m}(\mathfrak{V}(\mathfrak{x}_v);X,L).$$

We define extension of the obstruction fiber using our core Hilbert manifolds, but it induces extensions for our other sources, which we will make more direct use of.

Definition 2.1.25. *Consider a given $\Gamma^+_{(\mathfrak{x},u)}$-equivariant extension $\mathrm{Triv}_{K',E_{(\mathfrak{x},u)}}$ of a $\Gamma^+_{(\mathfrak{x},u)}$-invariant obstruction fiber $E_{(\mathfrak{x},u)}$ and maps $w_v:(\Sigma_{\mathfrak{y}_v},\partial\Sigma_{\mathfrak{y}_v})\to(X,L)$ and $w':(\Sigma_\mathfrak{Y},\partial\Sigma_\mathfrak{Y})\to(X,L)$ where $\mathfrak{Y}=\overline{\Phi}(\mathfrak{y}',\vec{T}_o,(\vec{T}_c,\vec{\theta}))$, such that $(\mathfrak{y}_v,w_v|_{K'_{\mathfrak{y}_v}})\in\mathcal{U}_{K',(\mathfrak{x},u),v}$ and $(\mathfrak{y}'_v,w'|_{K'_{\mathfrak{y}'_v}})\in\mathcal{U}_{K',(\mathfrak{x},u),v}$. We have induced extension maps*

$$(\mathrm{Triv}_{E_{(\mathfrak{x},u),v}})^{w_v}_{u_v}:E_{(\mathfrak{x},u),v}\to L^2_{m,\delta}(\Sigma_{\mathfrak{y}_v};w_v^*TX\otimes\Lambda^{0,1}),$$
$$(\mathrm{Triv}_{E_{(\mathfrak{x},u)}})^{w'}_{u}:E_{(\mathfrak{x},u)}\to L^2_{m,\delta}(\Sigma_\mathfrak{Y};(w')^*TX\otimes\Lambda^{0,1}).$$

The first map is obtained by sending the section $\kappa\in E_{(\mathfrak{x},u),v}$ to $\mathrm{Triv}_{E_{(\mathfrak{x},u),v}}(\mathfrak{y}_v,w_v|_{K'_{\mathfrak{y}_v}},\kappa)\in L^2_m(K'_{\mathfrak{y}_v};(w_v)|^*_{K'_{\mathfrak{y}_v}}TX\otimes\Lambda^{0,1})$ then extending by 0. The second map is obtained similarly.

Here we run into a situation where me must be careful to be aware of our choices of representatives in a given class $(\mathfrak{y},w')$. We assume in the above definition that $\mathfrak{Y}=\overline{\Phi}(\mathfrak{y}',\vec{T}_o,(\vec{T}_c,\vec{\theta}))$, giving a representative $\Sigma_\mathfrak{Y}$, but in general $\mathfrak{Y}$ will not be *uniquely* expressible as $\overline{\Phi}(\mathfrak{y}',\vec{T}_o,(\vec{T}_c,\vec{\theta}))$. This is why we require $\Gamma^+_{(\mathfrak{x},u)}$ equivariance in the above definition.

Lemma 2.1.26. *Fix a class* $(\mathfrak{Y}, w) \in \mathcal{M}_{k+1,\ell}(\beta)$ *(assumed to be source stable), and let* $\mathfrak{Y} = \overline{\Phi}(\mathfrak{y}, \vec{T}_o, (\vec{T}_c, \vec{\theta})) = \overline{\Phi}(\mathfrak{y}', \vec{T}'_o, (\vec{T}'_c, \vec{\theta}'))$*. This gives two representatives* $(\Sigma_{\mathfrak{Y}}, w)$ *and* $(\Sigma'_{\mathfrak{Y}}, w')$ *of* $(\mathfrak{Y}, w)$*. By definition, we have a bilohomorphism* $\alpha : \Sigma'_{\mathfrak{Y}} \xrightarrow{\sim} \Sigma_{\mathfrak{Y}}$ *over* w*,* w'*.*

Given a $\Gamma^+_{(\mathfrak{x},u)}$*-equivariant extension* $\mathrm{Triv}_{K',E_{(\mathfrak{x},u)}}$ *of a* $\Gamma^+_{(\mathfrak{x},u)}$*-invariant obstruction fiber* $E_{(\mathfrak{x},u)}$*, the following diagram commutes:*

$$\begin{array}{ccc} E_{(\mathfrak{x},u)} & \xrightarrow{(\mathrm{Triv}_{E_{(\mathfrak{x},u)}})^{w}_{u}} & L^2_{m,\delta}(\Sigma_{\mathfrak{Y}}; w^*TX \otimes \Lambda^{0,1}) \\ & {\scriptstyle (\mathrm{Triv}_{E_{(\mathfrak{x},u)}})^{w'}_{u}} \searrow & \downarrow {\scriptstyle \alpha^*} \\ & & L^2_{m,\delta}(\Sigma'_{\mathfrak{Y}}; (w')^*TX \otimes \Lambda^{0,1}). \end{array}$$

Proof. We recall that $\overline{\Phi}(\mathfrak{y}, \vec{T}_o, (\vec{T}_c, \vec{\theta})) = \overline{\Phi}(\mathfrak{y}', \vec{T}'_o, (\vec{T}'_c, \vec{\theta}'))$ implies that there exists $\gamma \in \Gamma_{\mathfrak{x}}$ with $(\mathfrak{y}, \vec{T}_o, (\vec{T}_c, \vec{\theta})) = \gamma(\mathfrak{y}', \vec{T}'_o, (\vec{T}'_c, \vec{\theta}'))$ (see the end of Section 2.1.2). Thus the bilholomorphism α must be induced by γ. It follows that $\gamma \in \Gamma^+_{(\mathfrak{x},u)}$. The desired result then follows directly from the definition of a $\Gamma^+_{(\mathfrak{x},u)}$-equivariant extension of a $\Gamma^+_{(\mathfrak{x},u)}$-invariant obstruction fiber. □

In the FOOO program, they use the parallel transport maps induced by ∇ to extend their obstruction vector spaces, which can be phrased in terms of our definition without difficulty. The $\Gamma^+_{(\mathfrak{x},u)}$ invariance of the obstruction vector spaces and $\Gamma^+_{(\mathfrak{x},u)}$ of the extensions is covered in Lemmas 17.11 and 17.16 in FOOO [17]. Using this more general definition changes very little about the FOOO construction, but the added flexibility will be essential for our construction of compatible Kuranishi structures on different moduli spaces.

We come now to one of the two main applications of our increased flexibility in defining the obstruction bundle. Let (Y, ω', J') be another compact symplectic manifold (of the same dimension as X) with compatible almost complex structure J' and embedded Lagrangian L'. Let D_X and D_Y be closed subsets of X and Y respectively not intersecting L or L'. Consider a bi-pseudoholomorphic map ψ : $X \setminus D_X \to Y \setminus D_Y$ such that $\underline{\psi}(L) = L'$. Observe that $\mathcal{E}^2_{K'_{\mathfrak{x}},m}(\mathfrak{V}(\mathfrak{x}); X \setminus D_X, L)$ is an open subset of $\mathcal{E}^2_{K'_\mathfrak{x},m}(\mathfrak{V}(\mathfrak{x}); X, L)$, and likewise $\mathcal{E}^2_{K'_\mathfrak{x},m}(\mathfrak{V}(\mathfrak{x}); Y \setminus D_Y, L')$ is an open subset of $\mathcal{E}^2_{K'_\mathfrak{x},m}(\mathfrak{V}(\mathfrak{x}); Y, L')$. We have induced isomorphisms

$$\psi_* : L^2_m((K'_{\mathfrak{y}_v}, K'_{\mathfrak{y}_v} \cap \partial\Sigma_{\mathfrak{y}_v}); X \setminus D_X, L) \to L^2_m((K'_{\mathfrak{y}_v}, K'_{\mathfrak{y}_v} \cap \partial\Sigma_{\mathfrak{y}_v}); Y \setminus D_Y, L'),$$

which gives the diffeomorphism

$$\psi_* : \mathcal{E}^2_{K'_\mathfrak{x},m}(\mathfrak{V}(\mathfrak{x}); X \setminus D_X, L) \to \mathcal{E}^2_{K'_\mathfrak{x},m}(\mathfrak{V}(\mathfrak{x}); Y \setminus D_Y, L')$$

mapping $(\mathfrak{y}_v, w_v, \kappa_v) \mapsto (\mathfrak{y}_v, \psi \circ w_v, d\psi(\kappa_v))$.

We then have the following proposition.

Proposition 2.1.27. *Let $E_{(\mathfrak{x},u)} \subseteq \prod_{v \in C^0(\mathcal{G})} L^2_m(K'_{\mathfrak{x}_v}; u_v^* TX \otimes \Lambda^{0,1})$ be a $\Gamma^+_{(\mathfrak{x},u)}$-invariant obstruction fiber such that $K'_\mathfrak{x}$ is disjoint from $u^{-1}(D_X)$. Assume further that $\psi_* E_{(\mathfrak{x},u)}$ is $\Gamma^+_{(\mathfrak{x},\psi\circ u)}$-invariant.*

Let $\mathrm{Triv}_{K',\psi_* E_{(\mathfrak{x},u)}}$ be a $\Gamma^+_{(\mathfrak{x},\psi\circ u)}$-equivariant extension of the obstruction fiber $\psi_* E_{(\mathfrak{x},u)}$ on Y, and let $\mathcal{U}_{Y,v}$ be an open neighborhood in $\mathcal{U}_{K',(\mathfrak{x},\psi\circ u),v} \subseteq W^2_{K'_{\mathfrak{x}_v},m+1}(\mathfrak{V}(\mathfrak{x}_v); Y, L')$ such that for all $(\mathfrak{y}, w) \in \mathcal{U}_{K',(\mathfrak{x},\psi\circ u),v}$ we have $w(K'_\mathfrak{y}) \cap D_Y = \emptyset$. We then let $\mathcal{U}_{X,v}$ be $(\psi_*)^{-1}(\mathcal{U}_{Y,v})$, that is the open neighborhood of $(\mathfrak{x}_v, u_v)$ in $W^2_{K'_{\mathfrak{x}_v},m+1}(\mathfrak{V}(\mathfrak{x}_v); X, L)$ consisting of all $(\mathfrak{y}_v, w_v)$ with $(\mathfrak{y}_v, \psi \circ w_v) \in \mathcal{U}_{Y,v}$.

Then there exists a $\Gamma^+_{(\mathfrak{x},u)}$-equivariant extension of the obstruction fiber $E_{(\mathfrak{x},u)}$ on X (the top row of the following diagram) such that the following diagram commutes.

$$\begin{array}{ccc} \prod_{v \in C^0(\mathcal{G})} \left(\mathcal{U}_{X,v} \times E_{(\mathfrak{x},u),v}\right) & \xrightarrow{\mathrm{Triv}_{K',(\mathfrak{x},u)}} & \prod_{v \in C^0(\mathcal{G})} \mathcal{E}^2_{K'_{\mathfrak{x}_v},m}(\mathfrak{V}(\mathfrak{x}_v); X \setminus D_X, L) \\ \downarrow \psi_* & & \downarrow \psi_* \\ \prod_{v \in C^0(\mathcal{G})} \left(\mathcal{U}_{Y,v} \times \psi_* E_{(\mathfrak{x},u),v}\right) & \xrightarrow{\mathrm{Triv}_{K',(\mathfrak{x},\psi\circ u)}} & \prod_{v \in C^0(\mathcal{G})} \mathcal{E}^2_{K'_{\mathfrak{x}_v},m}(\mathfrak{V}(\mathfrak{x}_v); Y \setminus D_Y, L') \end{array}$$

Proof. Given $(\mathfrak{y}_v, w_v, \kappa_v) \in \mathcal{U}_{X,v} \times E_{(\mathfrak{x},u),v}$, we define

$$\mathrm{Triv}_{K',(\mathfrak{x},u),v}(\mathfrak{y}_v, w_v, \kappa_v) = (\psi_*)^{-1}_{(\mathfrak{y}_v, w_v)} \left(\mathrm{Triv}_{K',(\mathfrak{x},\psi\circ u),v}(\mathfrak{y}_v, \psi \circ w_v, (\psi_*)_{(\mathfrak{x}_v,u_v)} \kappa_v)\right).$$

Each of the three conditions for this map to be an extension of the obstruction fiber $E_{(\mathfrak{x},u)}$ is clear.

To see that $\mathrm{Triv}_{K',(\mathfrak{x},u)}$ is $\Gamma^+_{(\mathfrak{x},u)}$-equivariant, let $\gamma \in \Gamma^+_{(\mathfrak{x},u)}$ and note that we have a natural homomorphism $\psi_* : \Gamma^+_{(\mathfrak{x},u)} \to \Gamma^+_{(\mathfrak{x},\psi\circ u)}$, since γ acts entirely by pre-composition. Furthermore, $\psi_*(\gamma(\mathfrak{y},w,\kappa)) = (\psi_*\gamma)\psi_*(\mathfrak{y},w,\kappa)$ and $(\psi_*)^{-1}_{\mathfrak{y},w}(\psi_*(\gamma)(\mathfrak{y},\psi\circ w,\kappa')) = \gamma(\psi_*)^{-1}_{\mathfrak{y},w}(\mathfrak{y},\psi\circ w,\kappa')$. Given

$$(\mathfrak{y},w,\kappa) \in \prod_{v\in C^0(\mathcal{G})} \left(\mathcal{U}_{X,v} \times E_{(\mathfrak{x},u),v}\right),$$

we thus have that

$$\begin{aligned}
\mathrm{Triv}_{K,(\mathfrak{x},u)}\left(\gamma(\mathfrak{y},w,\kappa)\right) &= (\psi_*)^{-1}_{(\mathfrak{y},w)}\left(\mathrm{Triv}_{K,(\mathfrak{x},\psi\circ u)}\left(\psi_*(\gamma)(\mathfrak{y},\psi\circ w,(\psi_*)_{(\mathfrak{x},u)}\kappa)\right)\right)\\
&= (\psi_*)^{-1}_{(\mathfrak{y},w)}\left(\psi_*(\gamma)\mathrm{Triv}_{K,(\mathfrak{x},\psi\circ u)}\left(\mathfrak{y},\psi\circ w,(\psi_*)_{(\mathfrak{x},u)}\kappa\right)\right)\\
&= \gamma\left((\psi_*)^{-1}_{(\mathfrak{y},w)}\left(\mathrm{Triv}_{K,(\mathfrak{x},\psi\circ u)}\left(\mathfrak{y},\psi\circ w,(\psi_*)_{(\mathfrak{x},u)}\kappa\right)\right)\right)\\
&= \gamma\left(\mathrm{Triv}_{K,(\mathfrak{x},u)}\left(\mathfrak{y},w,\kappa\right)\right).
\end{aligned}$$

□

We close this subsection with the following important technical point, which is necessary to show that we can still define the Kuranishi map in a manner compatible with the increased flexibility of our definition of the obstruction fiber.

Lemma 2.1.28. *Given an extension*

$$\prod_{v\in C^0(\mathcal{G})} \mathrm{Triv}_{K',E_{(\mathfrak{x},u),v}} : \prod_{v\in C^0(\mathcal{G})} \left(\mathcal{U}_{K',(\mathfrak{x},u),v} \times E_{(\mathfrak{x},u),v}\right) \to \prod_{v\in C^0(\mathcal{G})} \mathcal{E}^2_{K',m}(\mathfrak{V}(\mathfrak{x}_v);X,L).$$

of the obstruction fiber $E_{(\mathfrak{x},u)}$, we have a smooth map

$$\mathfrak{s} : \prod_{v\in C^0(\mathcal{G})} \mathcal{U}'_{K',(\mathfrak{x},u),v} \to \prod_{v\in C^0(\mathcal{G})} L^2_m(K'_{\mathfrak{x}_v}, u^*TX \otimes \Lambda^{0,1}),$$

where $\mathcal{U}'_{K',(\mathfrak{x},u),v}$ is an open subset of $\mathcal{U}_{K',(\mathfrak{x},u),v}$, such that

$$\mathfrak{s}(\mathfrak{y},w) \in E_{(\mathfrak{x},u)}$$

if and only if

$$\bar\partial w \in \prod_{v\in C^0(\mathcal{G})} \mathrm{Triv}_{K',E_{(\mathfrak{x},u),v}}\left((\mathfrak{y},w)\times E_{(\mathfrak{x},u)}\right).$$

Proof. Recall that $\mathcal{E}_{K'_{\mathfrak{x}_v},(\mathfrak{x},\mathfrak{u}),v}$ is the (trivial) obstruction fiber sub-bundle of $\mathcal{E}^2_{K'_{\mathfrak{x}_v},m}(\mathfrak{V}(\mathfrak{x}_v);X,L)$ over $\mathcal{U}_{K',(\mathfrak{x},u),v}$. We will construct a tubular neighborhood in $\mathcal{E}^2_{K'_{\mathfrak{x}_v},m}(\mathfrak{V}(\mathfrak{x}_v);X,L)$ of a neighborhood of $((\mathfrak{x},u),0)$ in $\mathcal{E}_{K'_{\mathfrak{x}_v},(\mathfrak{x},\mathfrak{u}),v}$.

For each point $p \in E_{\mathfrak{y}_v,w_v}$ we have the (finite dimensional) tangent space $T_pE_{\mathfrak{y}_v,w_v} \subseteq L^2_m(K'_{\mathfrak{y}_v}, w^*TX \otimes \Lambda^{0,1})$, and the corresponding normal space $N_pE_{\mathfrak{y}_v,w_v} \subseteq L^2_m(K'_{\mathfrak{y}_v}, w^*TX \otimes \Lambda^{0,1})$ induced by the L^2 pairing is closed. We thus have a map (fiberwise an isometric embedding) $i_{(\mathfrak{y}_v),w_v)} : NE_{\mathfrak{y}_v,w_v} \to L^2_m(K'_{\mathfrak{y}_v}, w^*TX \otimes \Lambda^{0,1})$ from the normal bundle into the ambient space, sending $(p,\kappa) \mapsto p+\kappa$. Combining these maps over points $(\mathfrak{y}_v, w_v)$ in the base $\mathcal{U}_{K',(\mathfrak{x},u),v}$ for various extends to a map (again, fiberwise over $\mathcal{E}_{K'_{\mathfrak{x}_v},(\mathfrak{x},\mathfrak{u}),v}$ an isometric embedding)

$$i : N\mathcal{E}_{K'_{\mathfrak{x}_v},(\mathfrak{x},\mathfrak{u}),v} \to \mathcal{E}^2_{K'_{\mathfrak{x}_v},m}(\mathfrak{V}(\mathfrak{x}_v);X,L).$$

The differential $(Di)_{(\mathfrak{x}_v,u_v),0,0}$ of i at $((\mathfrak{x}_v,u_v),0,0) \in N\mathcal{E}_{K'_{\mathfrak{x}_v},(\mathfrak{x},\mathfrak{u}),v}$ is an isomorphism from $T_{\mathfrak{x}_v,u_v}\mathcal{U}_{K',(\mathfrak{x},u),v} \oplus T_0E_{\mathfrak{x}_v,u_v} \oplus N_0E_{\mathfrak{x}_v,u_v}$ to $L^2_m(K'_{\mathfrak{x}_v}, u_v^*TX \otimes \Lambda^{0,1})$. Thus, the restriction of i to an open neighborhood of $((\mathfrak{x}_v,u_v),0,0)$ in $N\mathcal{E}_{K'_{\mathfrak{x}_v},(\mathfrak{x},\mathfrak{u}),v}$ is a diffeomorphism onto its image $\mathcal{V} \subseteq \mathcal{E}^2_{K'_{\mathfrak{x}_v},m}(\mathfrak{V}(\mathfrak{x}_v);X,L)$. From this, we get a smooth map $j : \mathcal{V} \to \mathcal{E}_{K'_{\mathfrak{x}_v},(\mathfrak{x},\mathfrak{u}),v}$. We then get a smooth map

$$\pi_{K'_{\mathfrak{x}_v},(\mathfrak{x},\mathfrak{u}),v} : \mathcal{V} \to L^2_m(K'_{\mathfrak{x}_v}, u_v^*TX \otimes \Lambda^{0,1})$$

sending

$$((\mathfrak{y}_v,w_v),\kappa) \mapsto \mathrm{Triv}^{-1}_{K',E_{(\mathfrak{x},u)}}(j((\mathfrak{y}_v,w_v),\kappa)) + (\mathrm{Pal}^{u_v}_{w_v})^{(0,1)}(\kappa - j((\mathfrak{y}_v,w_v),\kappa)).$$

Taking $\mathcal{U}'_{K',(\mathfrak{x},u),v}$ to be an open subset of $\bar{\partial}^{-1}(\mathcal{V}) \subseteq \mathcal{U}_{K',(\mathfrak{x},u),v}$, we get that $\mathfrak{s} = \pi_{K'_{\mathfrak{x}_v},(\mathfrak{x},\mathfrak{u}),v} \circ \bar{\partial}$ is the desired map. □

2.1.4 Obstruction bundle data

We now return to FOOO [17]. There are several things we need before defining obstruction bundle data. We start with the following definition:

Definition 2.1.29 (Def 17.5 in [17])**.** *A symmetric stabilization of an element* $(\mathfrak{x},u)$ *is a choice of additional interior marked points* $\vec{w}$ *such that*

1. $\vec{w} \cap \vec{z}^{int} = \emptyset$.

2. *u is an immersion at each point of $\vec{w}$.*

3. *$(\Sigma_{\mathfrak{x}}, \vec{z}, \vec{w} \cup \vec{z}^{int})$ is stable.*

4. *Each element of $\Gamma^+_{(\mathfrak{x},u)}$ permutes the points of $\vec{w}$.*

Much of our machinery only works when our discs are source stable, which is why we need to be able to stabilize. Every time we introduce a new stabilization, we will also introduce a codimension 2 submanifold for each added point, which will eventually be used to forget the added points.

Let $\mathfrak{Y} = \overline{\Phi}(\mathfrak{y}, \vec{T}^o, (\vec{T}^c, \vec{\theta}))$ be an element of $\mathcal{M}_{k+1,\ell+\ell'}$ represented by $(\Sigma_{\mathfrak{Y}}, \vec{z}_{\mathfrak{Y}}, \vec{z}^{int}_{\mathfrak{Y}} \cup \vec{w}_{\mathfrak{Y}})$. We use the Levi-Civita connection of our metric g from Section 2.1.3 to define the linearized $\bar{\partial}$ operator at $(\mathfrak{y}_v, w_v)$:

$$D_{\mathfrak{y}_v,w_v}\bar{\partial} : W^2_{m+1,\delta}((\Sigma_{\mathfrak{y}_v}, \partial\Sigma_{\mathfrak{y}_v}); w_v^*TX, w_v^*TL) \to L^2_{m,\delta}(\Sigma_{\mathfrak{y}_v}; w_v^*TX \otimes \Lambda^{0,1})$$

$$D_{\mathfrak{y}_v,w_v}\bar{\partial}(V, \vec{v}) = \frac{d}{dt}\bigg|_{t=0} \Big(((\mathrm{Pal}_{w_v}^{\mathrm{Exp}(w_v,tV)})^{(0,1)})^{-1}\bar{\partial}\,\mathrm{Exp}(w_v, tV)\Big).$$

We also have the following linearized $\bar{\partial}$ operator at $(\mathfrak{Y}, w')$:

$$D_{\mathfrak{Y},w'}\bar{\partial} : W^2_{m+1,\delta}((\Sigma_{\mathfrak{Y}}, \partial\Sigma_{\mathfrak{Y}}); (w')^*TX, (w')^*TL) \to L^2_{m,\delta}(\Sigma_{\mathfrak{Y}}; (w')^*TX \otimes \Lambda^{0,1})$$

$$D_{\mathfrak{Y},w'}\bar{\partial}(V, \vec{v}) = \frac{d}{dt}\bigg|_{t=0} \Big(((\mathrm{Pal}_{w'}^{\mathrm{Exp}(w',tV)})^{(0,1)})^{-1}\bar{\partial}\,\mathrm{Exp}(w', tV)\Big).$$

For each pair (v, e) consisting of a vertex and adjacent edge of $\mathcal{G}$, we have the following evaluation maps. If e corresponds to a boundary singular point, that is $e \in C^1_o(\mathcal{G})$, we have the map

$$\mathrm{ev}_{v,e} : W^2_{m+1,\delta}((\Sigma_{\mathfrak{y}_v}, \partial\Sigma_{\mathfrak{y}_v}); u^*TX, u^*TL) \to T_{u(z_e)}L$$

mapping $(s, \vec{v}) \mapsto \pm v_e$ (the e component of $\vec{v}$), where the sign is positive if e is an outgoing edge of v and negative if e is an incoming edge of v. If e corresponds to an interior singular point, so $e \in C^1_c(\mathcal{G})$, we have the map

$$\mathrm{ev}_{v,e} : W^2_{m+1,\delta}((\Sigma_{\mathfrak{y}_v}, \partial\Sigma_{\mathfrak{y}_v}); u^*TX, u^*TL) \to T_{u(z_e)}X$$

mapping $(s, \vec{v}) \mapsto \pm s(z_e)$, where the sign is again positive if e is an outgoing edge of v and negative if it is an incoming edge of v. We combine these evaluation maps into the following total evaluation map:

$$\mathrm{ev}_{\mathcal{G}} : \bigoplus_{v \in C^0(\mathcal{G})} W^2_{m+1,\delta}((\Sigma_{\mathfrak{y}_v}, \partial\Sigma_{\mathfrak{y}_v}); u^*TX, u^*TL) \to \bigoplus_{e \in C^1_o(\mathcal{G})} T_{u(z_e)}L \oplus \bigoplus_{e \in C^1_o(\mathcal{G})} T_{u(z_e)}X \tag{2.1.9}$$

Finally, for each boundary marked point z_i, we have the following evaluation map

$$\mathrm{ev}_{z_i} : W^2_{m+1,\delta}((\Sigma_{\mathfrak{y}_v}, \partial\Sigma_{\mathfrak{y}_v}); u^*TX, u^*TL) \to T_{u(z_i)}L$$

mapping $s \mapsto s(z_i)$. We also get a corresponding total evaluation map

$$\mathrm{ev}_{\vec{z}} : W^2_{m+1,\delta}((\Sigma_{\mathfrak{y}_v}, \partial\Sigma_{\mathfrak{y}_v}); u^*TX, u^*TL) \to \bigoplus_{z_i \in \vec{z}} T_{u(z_i)}L.$$

We can now define obstruction bundle data.

Definition 2.1.30 (See Def 17.7 in [17])**.** *We call the following data obstruction bundle data $\mathfrak{E}_{\mathfrak{p}}$ centered at $\mathfrak{p} = (\mathfrak{x}, u) = [(\Sigma, \vec{z}, \vec{z}^{int}), u] \in \mathcal{M}_{k+1,\ell}(\beta; \mathcal{G})$:*

1. *A symmetric stabilization $\vec{w}$ of $(\mathfrak{x}, u)$. We let $\mathcal{G}_{\vec{w} \cup \mathfrak{x}}$ denote the combinatorial type of the stabilized map.*

2. *A neighborhood $\mathfrak{V}(\mathfrak{x}_v \cup \vec{w}_v)$ of $\mathfrak{x}_v \cup \vec{w}_v$ in $\mathcal{M}_{k_v+1,\ell_v+\ell'_v}$ or $\mathcal{M}_{k_v+1,\ell_v+\ell'_v}$. We choose $\mathfrak{V}(\mathfrak{x}_v \cup \vec{w}_v)$ so that every point is an irreducible disc or sphere.*

3. *A universal family with coordinate at infinity of $\mathfrak{x}_v \cup \vec{w}_v$ defined on $\mathfrak{V}(\mathfrak{x}_v \cup \vec{w}_v)$. We require this coordinate at infinity to be invariant under the $\Gamma^+_{(\mathfrak{x} \cup \vec{w}, u)}$ action in the sense given following Definition 2.1.6. With this coordinate at infinity chosen, we now have a particular choice of representative $[(\Sigma_{\mathfrak{x}}, \vec{z}, \vec{z}^{int} \cup \vec{w}), u] \in \mathcal{M}_{k+1,\ell+\ell'}(\beta; \mathcal{G})$ and hence $[(\Sigma_{\mathfrak{x}}, \vec{z}, \vec{z}^{int}), u] \in \mathcal{M}_{k+1,\ell}(\beta; \mathcal{G})$.*

4. *A finite dimensional,* $\Gamma^+_{(\mathfrak{x},u)}$*-invariant submanifold* $\prod_{v\in C^0(\mathcal{G})} E_{(\mathfrak{x},u),v}$ *of* $\prod_{v\in C^0(\mathcal{G})} L^2_m(K_{\mathfrak{x}_v}; u^*TX\otimes\Lambda^{0,1})$ *containing the point* 0*, such that every section in* $E_{(\mathfrak{x},u),v}$ *is smooth with compact support in* $\operatorname{Int} K'_{\mathfrak{x}_v}$ *for some compact* $K'_{\mathfrak{x}_v}\subseteq K_{\mathfrak{x}_v}$ *with non-empty interior. We called this an obstruction fiber in Section 2.1.3.*

5. *Extensions* $\operatorname{Triv}_{K',E_{(\mathfrak{x},u),v}} : \mathcal{U}_{K',(\mathfrak{x},u),v}\times E_{(\mathfrak{x},u),v}\to \mathcal{E}^2_{K'_{\mathfrak{x}_v},m}(\mathfrak{V}(\mathfrak{x}_v); X, L)$ *of the obstruction fibers* $E_{(\mathfrak{x},u),v}$*, as in Definition 2.1.23 such that they are equivariant with respect to the action of* $\Gamma^+_{\mathfrak{p}}$ *in the sense of Definition 2.1.24.*

6. *We require that* $(\mathfrak{x},u)$ *be Fredholm regular with respect to* $\mathfrak{E}_{\mathfrak{p}}$ *in the sense that the sum of the image of* $D_{\mathfrak{x}_v,u_v}\bar\partial$ *and* $T_0E_{(\mathfrak{x},u),v}$ *is* $L^2_{m,\delta}(\Sigma_{\mathfrak{x}_v}; u_v^*TX\otimes\Lambda^{0,1})$*.*

7. *We require that* $(\mathfrak{x},u)$ *is evaluation map transversal with respect to* $\mathfrak{E}_{\mathfrak{p}}$*, in the sense that the restriction of* $\operatorname{ev}_{\mathcal{G}}$ *to* $\bigoplus_{v\in C^0(\mathcal{G})}(D_{u,v}\bar\partial)^{-1}(T_0E_{(\mathfrak{x},u),v})$ *is surjective.*

8. *We require that* $(\mathfrak{x},u)$ *is evaluation map transversal at the* 0*th boundary marked point, in the sense that the restriction of* ev_{z_0} *to* $\bigoplus_{v\in C^0(\mathcal{G})}(D_{u,v}\bar\partial)^{-1}(T_0E_{(\mathfrak{x},u),v})$ *is surjective.*

9. *For each* $w_i\in\Sigma_v$ *we take a codimension* 2 *submanifold* $\mathcal{D}_i$ *of* X *such that* $u(w_i)\in\mathcal{D}_i$ *and* $u_*T_{w_i}\Sigma_v + T_{u(w_i)}\mathcal{D}_i = T_{w_i}X$*. Moreover, given* $v\in\Gamma^+_{\mathfrak{p}}$ *and* $v(w_i)=w_j$*, then* $\mathcal{D}_i=\mathcal{D}_j$*.*

This differs slightly from the definition given in FOOO [17] in order to accommodate our more general approach to the obstruction bundle, but it is essentially the same.

Remark 2.1.31. *Evaluation map transversality at the* 0*th boundary marked point (condition (8) above) is not necessary for constructing a Kuranishi structure following the FOOO program, but will be necessary for our applications. We could also ask for a similar transversality at interior marked points, but in our case it will be simpler not to.*

The following definition provides the appropriate notion of one map being "close" to another, for our purposes.

Definition 2.1.32 (Def 17.12 in [17]). *Let* $\mathfrak{Y} = \overline{\Phi}(\mathfrak{y}, \vec{T}^o, (\vec{T}^c, \vec{\theta}))$ *for* $\mathfrak{y} \in \mathfrak{V}(\mathfrak{x} \cup \vec{w}_{\mathfrak{p}})$, *and let* $w' : (\Sigma_{\mathfrak{Y}}, \partial\Sigma_{\mathfrak{Y}}) \to (X, L)$ *be a* C^{10} *map*[2] *in homology class* β. *We say that* $(\mathfrak{Y}, w')$ *is* ϵ*-close to* $\mathfrak{p}$ *with respect to* $\mathfrak{E}_{\mathfrak{p}}$ *if the following conditions hold.*

1. *The map* $\overline{\Phi}$ *gives an identification between* $K_{\mathfrak{y}_v}$ *and a subset of* $K_{\mathfrak{Y}}$. *We require* $|u - w'|_{C^{10}(K_{\mathfrak{Y}})} < \epsilon$.

2. *The map* w' *is holomorphic on each neck region of* $\Sigma_{\mathfrak{Y}}$.

3. *The diameter of the* w' *image of each connected component of the neck region is smaller than* ϵ.

4. $T_e > \epsilon^{-1}$ *for each* e.

Note that, although we use a particular choice of representative $\Sigma_{\mathfrak{Y}}$ in this definition, any other choice will differ only by an element of $\Gamma_{\mathfrak{p}}$, which does not affect any of the conditions.

This ϵ-closeness condition only becomes useful after extending the core in the following sense. Given a choice of coordinate at infinity, we define the extended core of $\mathfrak{y}_v$ as

$$\begin{aligned} K_{\mathfrak{y}_v}^{+\vec{R}} = K_{\mathfrak{y}_v} &\cup \bigcup_{e \in C_o^1(\mathcal{G}) \text{ an outgoing edge of } v} (0, R_{(v,e)}] \times [0,1] \\ &\cup \bigcup_{e \in C_o^1(\mathcal{G}) \text{ an incoming edge of } v} [-R_{(v,e)}, 0) \times [0,1] \\ &\cup \bigcup_{e \in C_c^1(\mathcal{G}) \text{ an outgoing edge of } v} (0, R_{(v,e)}] \times S^1 \\ &\cup \bigcup_{e \in C_c^1(\mathcal{G}) \text{ an incoming edge of } v} [-R_{(v,e)}, 0) \times S^1. \end{aligned}$$

See FOOO [17] Def 17.21. When our coordinate at infinity is given by obstruction bundle data $\mathfrak{E}_{\mathfrak{p}}$, we will let ${\mathfrak{E}_{\mathfrak{p}}}^{+\vec{R}}$ denote the obstruction bundle data together with the extended core.

[2]The C^{10} norm used here is induced by our metric g on X and the metric on the source. Unless otherwise stated, this will be the case for all norms.

Without extending the core, $\mathfrak{p}$ may not even be ϵ-close to itself. However, allowing for extension of the core, we have the following proposition.

Proposition 2.1.33 (Modification of Prop 17.22 in [17]). *Let $\mathfrak{p} \in \mathcal{M}_{k+1,\ell}(\beta)$ and let $\mathfrak{E}_\mathfrak{p}$ be a choice of obstruction bundle data centered at $\mathfrak{p}$. Then there exists $\epsilon > 0$ and $\vec{R}$ such that:*

1. *If $(\mathfrak{Y}, u')$ is ϵ-close to $\mathfrak{p}$ with respect to ${\mathfrak{E}_\mathfrak{p}}^{+\vec{R}}$, then $(\mathfrak{y}_v, u'|_{K_{\mathfrak{y}_v}}) \in \mathcal{U}_{K,(\mathfrak{x},u),v}$ for all v.*
2. *If $(\mathfrak{Y}, u')$ is ϵ-close to $\mathfrak{p}$ with respect to ${\mathfrak{E}_\mathfrak{p}}^{+\vec{R}}$, then $(\mathfrak{Y}, u')$ is Fredholm regular with respect to ${\mathfrak{E}_\mathfrak{p}}^{+\vec{R}}$ in the sense that the sum of the image of $D_{\mathfrak{Y},u'}\bar{\partial}$ and $T_0(\mathrm{Triv}_{E_{(\mathfrak{x},u)}})_u^{u'}(E_{(\mathfrak{x},u)})$ is all of $L^2_{m,\delta}(\Sigma_\mathfrak{Y}; (w')^*TX \otimes \Lambda^{0,1})$.*
3. *If $(\mathfrak{Y}, u')$ is ϵ-close to $\mathfrak{p}$ with respect to ${\mathfrak{E}_\mathfrak{p}}^{+\vec{R}}$, then $(\mathfrak{Y}, u')$ is evaluation map transversal with respect to ${\mathfrak{E}_\mathfrak{p}}^{+\vec{R}}$ in the sense that the restriction of $\mathrm{ev}_\mathcal{G}$ to $(D_{\mathfrak{Y},w'}\bar{\partial})^{-1}T_0(\mathrm{Triv}_{E_{(\mathfrak{x},u)}})_u^{u'}(E_{(\mathfrak{x},u)}))$ is surjective.*
4. *$\mathfrak{p}$ is ϵ-close to ${\mathfrak{E}_\mathfrak{p}}^{+\vec{R}}$.*

Proof. Parts (2), (3), and (4) are proven in exactly the same way as in FOOO [17], using an exponential decay estimate on the neck regions and a Mayer-Vietoris argument due originally to Mrowka in his thesis [42]. Part (1) follows from the first condition in Definition 2.1.32 and the fact that we chose m large, see expression (2.1.8). □

2.1.5 Stabilization data

Now, for each $\mathfrak{p} = (\mathfrak{x}, u) \in \mathcal{M}_{k+1,\ell}(\beta)$ we fix obstruction bundle data $\mathfrak{E}_\mathfrak{p}$. Note that this data includes a choice of coordinate at infinity, a symmetric stabilization with $\ell_\mathfrak{p}$ marked points $\vec{w}_\mathfrak{p}$ of $\mathfrak{p}$, and a choice of codimension 2 submanifold $\mathcal{D}_{\mathfrak{p},i}$ for each point $w_{\mathfrak{p},i}$ such that $u(w_{\mathfrak{p},i}) \in \mathcal{D}_{\mathfrak{p},i}$ and $\mathcal{D}_{\mathfrak{p},i}$ is transversal to u. We will take finitely many points $\mathfrak{p}_c$ and use only the obstruction bundle data at those points. To this end, we need the following lemma, unchanged from FOOO [17].

Lemma 2.1.34 (Lemma 18.2 in [17]). *For each $\mathfrak{p} = (\mathfrak{x}, u) \in \mathcal{M}_{k+1,\ell}(\beta)$ with its fixed obstruction bundle data $\mathfrak{E}_\mathfrak{p}$, the following holds for sufficiently small $\epsilon_\mathfrak{p}$.*

Letting $\mathfrak{q} = (\mathfrak{Y}, u') \in \mathcal{M}_{k+1,\ell}(\beta)$, the set of symmetric stabilizations $\vec{w}'_\mathfrak{q}$ of $\mathfrak{Y}$ with $\ell_\mathfrak{p}$ points such that the following holds is either empty or consists of a single $\Gamma_\mathfrak{p}$ orbit:

1. $\mathfrak{Y} \cup \vec{w}'_{\mathfrak{q}} = \overline{\Phi}(\mathfrak{y}, \vec{T}^o, (\vec{T}^c, \vec{\theta}))$ *for some* $\mathfrak{y} \in \mathfrak{V}(\mathfrak{x} \cup \vec{w}_{\mathfrak{p}})$ *and* $(\vec{T}^o, (\vec{T}^c, \vec{\theta}))$.
2. *The pair* $(\mathfrak{Y} \cup \vec{w}'_{\mathfrak{q}}, u')$ *is* $\epsilon_{\mathfrak{p}}$*-close to* $\mathfrak{p}$.
3. *We have that* $u'(w'_{\mathfrak{q},i}) \in \mathcal{D}_{\mathfrak{p},i}$ *for all* i.

Remark 2.1.35. *The above Lemma 2.1.34 supplies a number* $\epsilon_{\mathfrak{p}} > 0$ *which we will treat as fixed so we can choose our finitely many points* $\mathfrak{p}_c$*. However, we will continue making statements about taking* $\epsilon_{\mathfrak{p}}$ *sufficiently small, even after we have supposedly fixed* $\epsilon_{\mathfrak{p}}$*. We are secretly carrying out a kind of induction. We can take a single point* $\mathfrak{p}$ *with its obstruction bundle data* $\mathfrak{E}_{\mathfrak{p}}$ *and obtain our desired results locally without having to fix* $\epsilon_{\mathfrak{p}}$*. We then use the local result to prove the above lemma, actually fix* $\epsilon_{\mathfrak{p}}$*, and then go through the entire proof process again.*

In FOOO [17]*, they carry this out more explicitly. However, this makes the notation substantially bulkier. Since our changes to the FOOO program have no impact on this point, we will suppress it for the remainder of the paper in the interest of readibility.*

For each $\mathfrak{p}$ we fix $\epsilon_{\mathfrak{p}}$ such that Lemma 2.1.34 and Proposition 2.1.33 both hold. We let $\mathfrak{W}^+(\mathfrak{p})$ be the set of all $\mathfrak{q} \in \mathcal{M}_{k+1,\ell}(\beta)$ such that the symmetric stabilization $\vec{w}'_{\mathfrak{q}}$ in Lemma 2.1.34 exists. This set is open in $\mathcal{M}_{k+1,\ell}(\beta)$ (this is not obvious, see Definition 18.3 in FOOO [17]). We choose a sequence of sets $\operatorname{Int}\mathfrak{W}^0_{\mathfrak{p}} \subseteq \mathfrak{W}^0_{\mathfrak{p}} \subseteq \operatorname{Int}\mathfrak{W}_{\mathfrak{p}} \subseteq \mathfrak{W}_{\mathfrak{p}} \subseteq \mathfrak{W}^+_{\mathfrak{p}}$, with $\mathfrak{p} \in \operatorname{Int}\mathfrak{W}^0_{\mathfrak{p}}$ and both $\mathfrak{W}^0_{\mathfrak{p}}$ and $\mathfrak{W}_{\mathfrak{p}}$ compact. We take and fix a finite set $\{\mathfrak{p}_c | c \in \mathfrak{C}\} \subseteq \mathcal{M}_{k+1,\ell}(\beta)$ such that $\bigcup_{c\in\mathfrak{C}} \operatorname{Int}\mathfrak{W}^0_{\mathfrak{p}_c} = \mathcal{M}_{k+1,\ell}(\beta)$. Only obstruction bundle data at $\mathfrak{p}_c$ for $c \in \mathfrak{C}$ is used for the remainder of the construction. For $\mathfrak{p} \in \mathcal{M}_{k+1,\ell}(\beta)$, we define $\mathfrak{C}(\mathfrak{p}) = \{c \in \mathfrak{C} \mid \mathfrak{p} \in \mathfrak{W}_{\mathfrak{p}_c}\}$. For each $c \in \mathfrak{C}(\mathfrak{p})$, we take additional marked points $\vec{w}^{\mathfrak{p}}_c$ for $\mathfrak{p}$ as given by Lemma 2.1.34.

We have one further requirement to impose on our choices of obstruction bundle data.

Condition 2.1.36. *For each* $\mathfrak{p} = (\mathfrak{x}, u) \in \mathcal{M}_{k+1,\ell}(\beta)$*, we require that the obstruction fibers* $(\mathrm{Triv}_{E_{\mathfrak{p}_c}})^u_{u_c}(E_{\mathfrak{p}_c})$ *are independent. That is, we need the sum space*

$$\sum_{c\in\mathfrak{C}(\mathfrak{p})} (\mathrm{Triv}_{E_{\mathfrak{p}_c}})^u_{u_c}(E_{\mathfrak{p}_c}) \subseteq L^2_m(\Sigma_{\mathfrak{x}}; u^*TX \otimes \Lambda^{0,1})$$

to be a smooth manifold of dimension equal to the sum of the dimensions of the individual spaces $(\mathrm{Triv}_{E_{\mathfrak{p}_c}})^u_{u_c}(E_{\mathfrak{p}_c})$*.*

We take this condition so that we may identify $\bigoplus_{c\in\mathfrak{C}(\mathfrak{p})}(\mathrm{Triv}_{E_{\mathfrak{p}_c}})^u_{u_c}(E_{\mathfrak{p}_c})$ with $\sum_{c\in\mathfrak{C}(\mathfrak{p})}(\mathrm{Triv}_{E_{\mathfrak{p}_c}})^u_{u_c}(E_{\mathfrak{p}_c})$.

In the original FOOO setting, Lemma 18.8 in FOOO [17] shows that we can impose this condition on our choices of obstruction bundle data. In explicit cases, such as the one considered in this paper, this condition is easy to arrange.

Definition 2.1.37 (Def 18.9 in FOOO[17]). *We call the following "stabilization data" at* $\mathfrak{p}$*:*

1. *A symmetric stabilization* $\vec{w}_{\mathfrak{p}} = (w_{\mathfrak{p},1}, \ldots, w_{\mathfrak{p},\ell_{\mathfrak{p}}})$ *of* $\mathfrak{p} = (\mathfrak{x}, u)$*. Let* $\ell_{\mathfrak{p}} = \#\vec{w}_{\mathfrak{p}}$*.*
2. *For each* $w_{\mathfrak{p},i}$ $(i = 1, \ldots, \ell_{\mathfrak{p}})$*, we take codimension two submanifolds* $\mathcal{D}_{\mathfrak{p},i}$ *of* X *transversal to* $u_{\mathfrak{p}}$ *at* $u_{\mathfrak{p}}(w_{\mathfrak{p},i})$ *and with* $u_{\mathfrak{p}}(w_{\mathfrak{p},i}) \in \mathcal{D}_{\mathfrak{p},i}$*. We assume these* $\mathcal{D}_{\mathfrak{p},i}$ *are invariant under the* $\Gamma_{\mathfrak{p}}$ *action in the same sense as in Definition 2.1.30.*
3. *A new coordinate at infinity for* $\mathfrak{p} \cup \vec{w}_{\mathfrak{p}}$*.*
4. $\vec{w}_{\mathfrak{p}} \cap \vec{w}^{\mathfrak{p}}_c = \emptyset$ *for any* $c \in \mathfrak{C}(\mathfrak{p})$*.*
5. *We require that the support of the obstruction bundle of* $\mathfrak{E}_{\mathfrak{p}_c}$ *at* $\mathfrak{p}$ *be contained in the core of the new coordinate at infinity, in the sense described below.*

The new coordinate at infinity gives a representative $\Sigma_{\mathfrak{x}\cup\vec{w}_{\mathfrak{p}}}$ of $\mathfrak{x}\cup\vec{w}_{\mathfrak{p}}$. The obstruction bundle data $\mathfrak{E}_{\mathfrak{p}_c}$ gives a representative $\Sigma_{\mathfrak{x}\cup\vec{w}^{\mathfrak{p}}_c}$ of $\mathfrak{x}\cup\vec{w}^{\mathfrak{p}}_c$. We have that $\Sigma_{\mathfrak{x}\cup\vec{w}_{\mathfrak{p}}}$ and $\Sigma_{\mathfrak{x}\cup\vec{w}^{\mathfrak{p}}_c}$ are biholomorphic, as their classes differ only by marked points. The biholomorphism preserves all original marked points, but not $\vec{w}_{\mathfrak{p}}$ and $\vec{w}'_c$. It follows that there is a uniquely determined map u_c making $(\Sigma_{\mathfrak{x}\cup\vec{w}^{\mathfrak{p}}_c}\cup\vec{z}\cup\vec{z}^{int}, u_c)$ a representative of $\mathfrak{p}$ (see Remark 2.1.22). The biholomorphism is thus an element of $\Gamma_{(\mathfrak{x},u)}$. However, because our extension of the obstruction fiber is $\Gamma_{(\mathfrak{x},u)}$-equivariant, we can identify the support of the obstruction vector space at $(\Sigma_{\mathfrak{x}\cup\vec{w}^{\mathfrak{p}}_c}\cup\vec{z}\cup\vec{z}^{int}, u_c)$ with a subset of $\Sigma_{\mathfrak{x}\cup\vec{w}_{\mathfrak{p}}}$ in a way independent of the choice of this element of $\Gamma_{(\mathfrak{x},u)}$ by Lemma 2.1.26. It is in the sense of this identification that we need the support of the obstruction bundle of $\mathfrak{E}_{\mathfrak{p}_c}$ at $\mathfrak{p}$ be contained in the core of $\Sigma_{\mathfrak{x}\cup\vec{w}_{\mathfrak{p}}}$.

Remark 2.1.38. *From here on, we will be simultaneously using multiple choices of universal family with coordinate at infinity, namely one for each $c \in \mathfrak{C}(\mathfrak{p})$ coming from our choices of obstruction bundle data, and another coming from the stabilization data at $\mathfrak{p}$. Keeping careful track of the different choices will be crucial for following the remainder of the construction and the proofs of various technical points. To this end, we will write $\overline{\Phi}_c$ for the $\overline{\Phi}$ map coming from the corresponding choice of obstruction bundle data, and we will write $\overline{\Phi}_{\mathfrak{p}}$ for the map coming from the choice of stabilization data.*

2.1.6 Thickened moduli space and Kuranishi chart

Fix $\vec{T}_0 = (\vec{T}_0^o, \vec{T}_0^c)$ and let $\epsilon_0 > 0$. We fix metrics on all the Deligne-Mumford moduli spaces. We fix a stabilization data at $\mathfrak{p} = (\mathfrak{x}, u)$ and let $\mathfrak{V}_{\epsilon_0}(\mathfrak{p}\cup\vec{w}_{\mathfrak{p}})$ be the ϵ_0 neighborhood of $\mathfrak{p}\cup\vec{w}_{\mathfrak{p}}$ in $\mathcal{M}_{k+1,\ell+\ell_{\mathfrak{p}}}(\mathcal{G}_{(\mathfrak{p}\cup\vec{w}_{\mathfrak{p}})})$. We consider the set of all $(\mathfrak{Y}, u', (\vec{w}_c'))$ such that the following holds for some $\vec{R}$:

1. There exists $\mathfrak{y} \in \mathfrak{V}_{\epsilon_0}(\mathfrak{p}\cup\vec{w}_{\mathfrak{p}})$ and $(\vec{T}^o, (\vec{T}^c, \vec{\theta})) \in (\vec{T}_0^o, \infty] \times ((\vec{T}_0^c, \infty] \times S^1)$ such that $\mathfrak{Y} = \overline{\Phi}_{\mathfrak{p}}(\mathfrak{y}, \vec{T}^o, (\vec{T}^c, \vec{\theta}))$.

2. $(\mathfrak{Y}, u')$ is ϵ_0-close to u with respect to the $\vec{R}$ extended core coming from the coordinate at infinity given by the stabilization data.

3. We let $\Sigma_{\mathfrak{Y}}$ be the (unmarked) representative of $\mathfrak{Y}$ given by $\overline{\Phi}_{\mathfrak{p}}(\mathfrak{y}, \vec{T}^o, (\vec{T}^c, \vec{\theta}))$, and we let $\vec{z}\cup\vec{z}^{int}$ be marked points such that $\Sigma_{\mathfrak{Y}}\cup\vec{z}\cup\vec{z}^{int}\cup\vec{w}_{\mathfrak{p}}$ is in $\mathfrak{Y}$. We require that the pair $(\Sigma_{\mathfrak{Y}}\cup\vec{z}\cup\vec{z}^{int}\cup\vec{w}_c^{\mathfrak{p}}, u')$ is ϵ_0-close to u with respect to the extended core obstruction bundle data $\mathfrak{E}_{\mathfrak{p}_c}^{+\vec{R}}$ for all $c \in \mathfrak{C}(\mathfrak{p})$.

We say that $(\mathfrak{Y}^{(1)}, u^{(1)}, (\vec{w}_c^{(1)}))$ is *equivalent* to $(\mathfrak{Y}^{(2)}, u^{(2)}), (\vec{w}_c^{(2)})$ if there exists a biholomorphic map $v : \Sigma_{\mathfrak{Y}^{(1)}} \to \Sigma_{\mathfrak{Y}^{(2)}}$ such that $u^{(1)} = u^{(2)} \circ v$ and such that v fixes all marked points. That is, v maps every marked point on $\Sigma_{\mathfrak{Y}^{(1)}}$ to the corresponding marked point on $\Sigma_{\mathfrak{Y}^{(2)}}$, for the marked points coming from $\mathfrak{p}$, the additional marked points coming from the stabilization data we fixed, and the additional marked points coming from the obstruction bundle data $\mathfrak{E}_{\mathfrak{p}_c}$ for all $c \in \mathfrak{C}(\mathfrak{p})$ (in the sense $v(w^{(1)}_{c,i}) = w^{(2)}_{c,i}$).

Definition 2.1.39 (Def 18.10 in [17]). *We let $\mathfrak{U}_{k+1,(\ell;\ell_p,(\ell_c))}(\beta,\mathfrak{p})_{\epsilon_0,\vec{T}_0}$ be the set of equivalence classes $(\mathfrak{Y}, u', (\vec{w}'_c))$ satisfying (1)-(3) above.*

Taking ϵ_0 and $\epsilon_{\mathfrak{p}_c}$ sufficiently small, we can then define:

Definition 2.1.40. *Let $\mathfrak{q}^+ = (\mathfrak{Y}, u', (\vec{w}'_c)) \in \mathfrak{U}_{k+1,(\ell;\ell_p,(\ell_c))}(\beta,\mathfrak{p})_{\epsilon_0,\vec{T}_0}$. We define $E_c(\mathfrak{q}^+) = (\mathrm{Triv}_{E_{(\mathfrak{x}_c,u_c)}})^{u'}_{u_c}(E_{\mathfrak{p}_c}) \subseteq L^2_m(\Sigma_{\mathfrak{Y}};(u')^*TX \otimes \Lambda^{0,1})$. We also define $\mathcal{E}(\mathfrak{q}^+) = \sum_{c\in\mathfrak{C}(\mathfrak{p})} E_c(\mathfrak{q}^+)$.*

This is our extension of the obstruction vector space given by $\mathfrak{E}_{\mathfrak{p}_c}$ to the representative $(\Sigma_{\mathfrak{Y}} \cup \vec{z} \cup \vec{z}^{int} \cup \vec{w}'_c, u')$ of $(\mathfrak{Y}, u', (\vec{w}'_c))$, see Definition 2.1.25. Note that this involves taking an expression $\mathfrak{Y} = \overline{\Phi}_c(\mathfrak{y}_c, \vec{T}^{o\prime}, (\vec{T}^{c\prime}, \vec{\theta}'))$. Such an expression exists by our requirement (3) above, and the space $E_c(\mathfrak{q}^+)$ is independent of all choices of representative because our obstruction bundle data is taken to be $\Gamma^+_{\mathfrak{p}_c}$ equivariant.

Note also that $E_c(\mathfrak{q}^+, \Sigma_{\mathfrak{Y}})$ does not depend on $\epsilon_0, \vec{T}_0$, or the stabilization data we fixed, and that for appropriate choices of $\vec{R}_c$, we can extend the cores given by each $\mathfrak{E}_{\mathfrak{p}_c}$ so that they all agree, and we then have that the support of $\mathcal{E}(\mathfrak{q}^+)$ is contained in the extended core with respect to all of the different $\mathfrak{E}_{\mathfrak{p}_c}$.

Definition 2.1.41 (Def 18.15 in [17]). *The thickened moduli space*

$$\mathcal{M}_{k+1,(\ell;\ell_{\mathfrak{p}},(\ell_c))}(\beta;\mathfrak{p})_{\epsilon_0,\vec{T}_0}$$

is the subset of $\mathfrak{U}_{k+1,(\ell;\ell_p,(\ell_c))}(\beta,\mathfrak{p})_{\epsilon_0,\vec{T}_0}$ consisting of equivalence classes of elements $\mathfrak{q}^+ = (\mathfrak{Y}, u', (\vec{w}'_c))$ such that $\bar{\partial}u' \in \mathcal{E}(\mathfrak{q}_+)$.

Again, by the equivariance of our obstruction bundle data, the statement $\bar{\partial}u' \in \mathcal{E}(\mathfrak{q}_+, \Sigma_{\mathfrak{Y}})$ does not depend on any choices of representatives.

Proposition 2.1.33 then gives the following lemma.

Lemma 2.1.42 (Lemma 18.16 in [17]). *By taking ϵ_0 and $\epsilon_{\mathfrak{p}_c}$ sufficiently small and $\vec{T}_0$ sufficiently large, we get the following statements.*

1. *If $\mathfrak{q}^+ = (\mathfrak{Y}, u', (\vec{w}'_c))$ is in $\mathfrak{U}_{k+1,(\ell;\ell_p,(\ell_c))}(\beta,\mathfrak{p})_{\epsilon_0,\vec{T}_0}$, then the condition $\bar{\partial}u' \in \mathcal{E}(\mathfrak{q}_+)$ is Fredholm regular, in the sense of Proposition 2.1.33.*

2. *If $\mathfrak{q}^+ = (\mathfrak{Y}, u', (\vec{w}'_c))$ is in $\mathfrak{U}_{k+1,(\ell;\ell_p,(\ell_c))}(\beta,\mathfrak{p})_{\epsilon_0,\vec{T}_0}$, then $\mathfrak{q}^+$ is evaluation map transversal in the sense of Proposition 2.1.33.*

3. *$\mathfrak{p} \cup (\vec{w}^{\mathfrak{p}}_c) = (\mathfrak{x}, u, (\vec{w}^{\mathfrak{p}}_c)) \in \mathfrak{U}_{k+1,(\ell;\ell_p,(\ell_c))}(\beta,\mathfrak{p})_{\epsilon_0,\vec{T}_0}$.*

Let $\mathcal{M}_{k+1,(\ell;\ell_{\mathfrak{p}},(\ell_c))}(\beta;\mathfrak{p};\mathcal{G}_{\mathfrak{p}})_{\epsilon_0,\vec{T}_0}$ denote the stratum of $\mathcal{M}_{k+1,(\ell;\ell_{\mathfrak{p}},(\ell_c))}(\beta;\mathfrak{p})_{\epsilon_0,\vec{T}_0}$ consisting of all elements with combinatorial type $\mathcal{G}_{\mathfrak{p}}$. These are all elements of the Hilbert manifold $\prod_{v\in C^0(\mathcal{G})} W^2_{m+1,\delta}(\mathfrak{V}(\mathfrak{x}_v);X,L)$. By Condition 2.1.36, we have an open neighborhood $\mathcal{U}_{\mathfrak{p}}$ of $\mathfrak{p}$ in $\prod_{v\in C^0(\mathcal{G}_{\mathfrak{p}})} W^2_{m+1,\delta}(\mathfrak{V}(\mathfrak{x}_v);X,L)$ on which we can combine the maps $\mathrm{Triv}_{K',E_{\mathfrak{p}_c}}$ to get a smooth trivialization

$$\mathcal{U}_{\mathfrak{p}} \times \left(\sum_{c\in\mathfrak{C}(\mathfrak{p})} (\mathrm{Triv}_{K'_c,E_{\mathfrak{p}_c}})^{u}_{u_c} E_{\mathfrak{p}_c} \right) \to \prod_{v\in C^0(\mathcal{G}_{\mathfrak{p}})} \mathcal{E}^2_{m,\delta}(\mathfrak{V}(\mathfrak{x}_v);X,L). \tag{2.1.10}$$

Combined with Lemma 2.1.42 and Lemma 2.1.28, we can use the implicit function theorem to get the following result.

Lemma 2.1.43 (Lemma 19.1 in [17]). *For ϵ_0 and $\epsilon_{\mathfrak{p}_c}$ sufficiently small and $\vec{T}_0$ sufficiently large, the stratum $\mathcal{M}_{k+1,(\ell;\ell_{\mathfrak{p}},(\ell_c))}(\beta;\mathfrak{p};\mathcal{G}_{\mathfrak{p}})_{\epsilon_0,\vec{T}_0}$ has the structure of a smooth manifold.*

Definition 2.1.44 (Def 20.6 in FOOO[17]). *An element $(\mathfrak{Y}, u', (\vec{w}'_c))$ of $\mathcal{M}_{k+1,(\ell,\ell_{\mathfrak{p}},(\ell_c))}(\beta;\mathfrak{p})_{\epsilon_0,\vec{T}_0}$ satisfies the transversal constraint at all additional marked points if for all marked points $\vec{w}_{\mathfrak{p}}$ of $\mathfrak{Y}$ from the stabilization data at $\mathfrak{p}$ we have that $w_{\mathfrak{p}_i} \in \mathcal{D}_{\mathfrak{p},i}$, and for all marked points $\vec{w}'_c$ we have that $w'_{c,i} \in \mathcal{D}_{c,i}$. We let $\mathcal{M}_{k+1,(\ell,\ell_{\mathfrak{p}},(\ell_c))}(\beta;\mathfrak{p})^{trans}_{\epsilon_0,\vec{T}_0}$ be the set of all such elements.*

Then, using the gluing techniques of FOOO [17], [22], which will be covered in detail in the Appendix, we have the following result.

Proposition 2.1.45 (FOOO [17]). *$\mathcal{M}_{k+1,(\ell,\ell_{\mathfrak{p}},(\ell_c))}(\beta;\mathfrak{p})^{trans}_{\epsilon_0,\vec{T}_0}$ has the structure of a smooth (C^∞) manifold with corners.*

We have a natural Kuranishi section

$$\mathfrak{s}:\mathcal{M}_{k+1,(\ell,\ell_{\mathfrak{p}},(\ell_c))}(\beta;\mathfrak{p})^{trans}_{\epsilon_0,\vec{T}_0}\to\prod_{v\in C^0(\mathcal{G}_{\mathfrak{p}})}\mathcal{E}^2_{m,\delta}(\mathfrak{V}(\mathfrak{x});X,L)$$

obtained by composing the $\bar{\partial}$ operator with the trivialization of the obstruction bundle in expression (2.1.10). We can then define a homeomorphism

$$\overline{\mathfrak{forget}}:\left(\mathcal{M}_{k+1,(\ell,\ell_{\mathfrak{p}},(\ell_c))}(\beta;\mathfrak{p};\mathfrak{A};\mathfrak{B})^{trans}_{\epsilon_0,\vec{T}_0}\cap\mathfrak{s}^{-1}(0)\right)/\Gamma_{\mathfrak{p}}\to\mathcal{M}_{k+1,\ell}(\beta). \tag{2.1.11}$$

See Appendix Section A.2

Our Kuranishi chart is then

Proposition 2.1.46 (Prop 21.14 in [17]). *Let $\mathfrak{p}\in\mathcal{M}_{k+1,\ell}(\beta)$. Then the smooth manifold with corners $\mathcal{M}_{k+1,(\ell,\ell_{\mathfrak{p}},(\ell_c))}(\beta;\mathfrak{p})^{trans}_{\epsilon_0,\vec{T}_0}$, together with the group $\Gamma_{\mathfrak{p}}$, the smooth Kuranishi section $\mathfrak{s}$, and the homeomorphism in Proposition A.2.4, gives a Kuranishi neighborhood of $\mathcal{M}_{k+1,\ell}(\beta)$ at $\mathfrak{p}$.*

We have that the evaluation maps at marked points

$$\mathrm{ev}:\mathcal{M}_{k+1,(\ell,\ell_{\mathfrak{p}},(\ell_c))}(\beta;\mathfrak{p})^{trans}_{\epsilon_0,\vec{T}_0}\to L^{k+1}\times X^{\ell}$$

are smooth, see Lemma 21.25 in FOOO [17].

After further shrinking the charts, this choice of charts admits appropriate transition maps to give a Kuranishi structure on the whole moduli space $\mathcal{M}_{k+1,\ell}(\beta)$.

2.1.7 Subspace Kuranishi structures

Having gone through the process for constructing a Kuranishi structure for $\mathcal{M}_{k+1,\ell}(X,L,\beta)$, we note that we will actually need a slightly more general treatment of Kuranishi structures on moduli spaces. Namely, we need to be able to talk about subspaces of these moduli spaces.

Definition 2.1.47. *Let $\mathcal{Y} \subseteq \mathcal{X}$ be compact metrizable spaces with Kuranishi structures and Kuranishi morphism $(f; \{(f_p, (f_p)_*)_p\})$ from $\mathcal{Y}$ to $\mathcal{X}$ such that f is the inclusion map, the maps f_p, are all injective, and the maps $(f_p)_*$ are all diffeomorphisms. We call this morphism $(f; \{(f_p, (f_p)_*)_p\})$ a Kuranishi inclusion, and we call $\mathcal{Y}$ a Kuranishi subspace of $\mathcal{X}$.*

Example 2.1.48 (Fibers of Weak Submersion)**.** *Let M be a smooth manifold and let $f : \mathcal{M}_{k+1,\ell}(X, L, \beta) \to M$ be a weak submersion (with respect to our constructed Kuranishi structure on $\mathcal{M}_{k+1,\ell}(X, L, \beta)$. For each point $y \in M$ the fiber $f^{-1}(y)$ naturally has the structure of a Kuranishi space, obtained by taking the Kuranishi charts (V', E) induced by the charts (V, E):*

$$\begin{array}{ccc} V' = f^{-1}(y) \cap V & \xrightarrow{\mathfrak{s}} & E \\ \downarrow & \nearrow_{\mathfrak{s}} & \\ V & & \end{array}$$

This is a special case of the material of Chapter 4 in FOOO [23]*, which handles fiber products in detail. In our terminology, each fiber is a Kuranishi subspace of $\mathcal{M}_{k+1,\ell}(X, L, \beta)$.*

Example 2.1.49. *Fix a point $p \in X \setminus L$ and say we have a Kuranishi structure on $\mathcal{M}_{k+1,1}(X, L, \beta)$ such that the interior point evaluation map ev^+ is weakly submersive. Then by the above discussion the moduli space of pseudoholomorphic maps in $\mathcal{M}_{k+1,1}(X, L, \beta)$ whose image contains the point p can be naturally viewed as the subspace $(\mathrm{ev}^+)^{-1}(p) \subseteq \mathcal{M}_{k+1,1}(X, L, \beta)$ with corresponding Kuranishi structure.*

2.2 Morphisms of moduli spaces induced by maps of ambient spaces

Now that we have reviewed the FOOO construction of a Kuranishi structure for a moduli space of pseudoholomorphic discs, we apply Proposition 2.1.27 to get some preliminary results on morphisms of moduli spaces of pseudoholomorphic discs induced by maps of ambient spaces.

Let X and Y be birational smooth Kähler varieties with relatively spin Lagrangians L_X and L_Y respectively, together with a birational holomorphic map $\underline{\psi}$: $X \to Y$ that maps L_X diffeomorphically onto L_Y. Let $D_X \subseteq X$ and $D_Y \subseteq Y$ be the minimal Zariski closed sets such that $\underline{\psi}|_{X \setminus D_X} : X \setminus D_X \to Y \setminus D_Y$ is biholomorphic, and assume that $L_X \cap D_X = \emptyset$ and $L_Y \cap D_Y = \emptyset$. Finally, let $\beta \in H_2(X, L_X)$ be an effective disc class such that for all nodal discs $u \in \beta$ every non-constant component of the map u intersects D_X transversally. That is, we assume that no element of β has a non-constant component contained in D_X.

We do not assume X or Y are compact, but we do assume that the usual "compactifications" of the moduli spaces of smooth holomorphic discs by stable nodal discs are genuinely compact. This would be guaranteed by X and Y being compact, but that condition is not necessary, as in our primary example in this dissertation.

Theorem 2.2.1. *In the above situation, we can construct compatible Kuranishi structures on $\mathcal{M}_{k+1,\ell}(X, L_X, \beta)$ and $\mathcal{M}_{k+1,\ell}(Y, L_Y, \underline{\psi}_*\beta)$ such that we have an induced morphism of Kuranishi spaces*

$$\mathcal{M}_{k+1,\ell}(X, L_X, \beta) \xrightarrow{\psi} \mathcal{M}_{k+1,\ell}(Y, L_Y, \underline{\psi}_*\beta).$$

Furthermore, the Kuranishi structure on $\mathcal{M}_{k+1,\ell}(Y, L_Y, \underline{\psi}_*\beta)$ induces a Kuranishi structure on the image moduli space $\psi(\mathcal{M}_{k+1,\ell}(X, L_X, \beta))$ with respect to which the morphism

$$\mathcal{M}_{k+1,\ell}(X, L_X, \beta) \xrightarrow{\psi} \psi\left(\mathcal{M}_{k+1,\ell}(X, L_X, \beta)\right),$$

is an isomorphism.

The moduli space $\psi(\mathcal{M}_{k+1,\ell}(X, L_X, \beta))$ consists exactly of those stable nodal holomorphic discs in Y with boundary on L_Y in class $\underline{\psi}_*\beta$ that lift to stable nodal holomorphic discs in X with boundary on L_X in class β.

Example 2.2.2. *The situation outlined in Section 1.4 is consistent with this setup for any class $\beta \in H_2(\mathscr{O}(-n), L)$. We go through the proof of Theorem 2.2.1 in this par-*

ticular case in Section 3.1. As alluded to above, neither $\mathscr{O}(-n)$ nor $\mathscr{O}$ is compact, but the moduli spaces of stable nodal discs $\mathcal{M}_{k+1}(\mathscr{O}(-n), L, \beta)$ and $\mathcal{M}_{k+1}(\mathscr{O}, \underline{\psi}(L), \underline{\psi}_*\beta)$ are compact.

Proof. We begin by choosing obstruction bundle data for every (holomorphic) point $\mathfrak{p} \in \mathcal{M}_{k+1,\ell}(X, L_X, \beta)$ and every (holomorphic) point $\mathfrak{p}' \in \mathcal{M}_{k+1,\ell}(Y, L_Y, \underline{\psi}_*\beta)$. For our obstruction bundle data, we need to choose a symmetric stabilization $\vec{w}$ and corresponding codimension 2 submanifolds $\mathcal{D}_i$ (items (1) and (9)), a universal family with coordinate at infinity (items (2) and (3)), a $\Gamma^+_{\psi(\mathfrak{p}\cup\vec{w})}$-invariant obstruction fiber $E_{\psi(\mathfrak{p})}$ (item (4)) such that items (6)-(8) are satisfied, and a $\Gamma^+_{\psi(\mathfrak{p}\cup\vec{w})}$-equivariant extension $\mathrm{Triv}_{K', E_{\psi(\mathfrak{p}\cup\vec{w})}}$.

Note that we do not require any non-trivial obstruction bundle data for constant components of maps, so we do not take any and do not need to worry about the possibility of compatibility issues arising from such components. They are already source stable and Fredholm regular, and all evaluation map transversality requirements can be achieved without non-trivial obstruction bundle data on the constant components. This latter point relies on the fact that our nodal discs are bordered nodal Riemann surfaces of genus 0. If we were considering higher genus Riemann surfaces, we would need to consider the contribution from constant components more carefully.

By the transversality assumptions of our setup, we can choose a symmetric stabilization $\vec{w}$ of each $\psi(\mathfrak{p})$ so that $\underline{\psi}(u(w_i)) \notin D_Y$ for all w_i, and we can choose the corresponding codimension 2 submanifolds $\mathcal{D}_i$ to be transversal to F_0. This then induces a compatible stabilization for $\mathfrak{p} \in \mathcal{M}_{k+1}(\mathscr{O}(-n), L, \beta)$. We then take a universal family with coordinate at infinity for each component together with its extra marked points $\mathfrak{x}_v \cup \vec{w}_v$.

We will need the following lemma, the proof of which is postponed until the end of Section 2.3.

Lemma 2.2.3. *Given any $\mathfrak{p} = [\mathfrak{x}, u] \in \mathcal{M}_{k+1,\ell}(X, L_X, \beta)$, and choices of items 1, 2, and 3 of Definition 2.1.30, we can take an obstruction fiber (item 4) $E_\mathfrak{p} \subseteq L^2_m(K_{\mathfrak{x};u^*TX\otimes\Lambda^{0,1}})$ at $\mathfrak{p}$ such that every element of $E_\mathfrak{p}$ is supported away from $u^{-1}(D_X)$ and such that, for any choice of extension of the obstruction fiber satisfying item 5 in Definition 2.1.30, items 6, 7, and 8 of the definition are satisfied as well.*

Note that this Lemma relies completely on the condition that D_X does not contain any non-constant components of u. An identical statement is true for a point $\psi(\mathfrak{p}) \in \mathcal{M}_{k+1,\ell}(Y, L_Y, \underline{\psi}_*\beta)$ and the closed set D_Y.

We choose an obstruction fiber $E_{\mathfrak{p}}$ at each point $\mathfrak{p} \in \mathcal{M}_{k+1,\ell}(X, L_X, \beta)$ as given by Lemma 2.2.3. We then push this obstruction fiber forward to get
$\underline{\psi}_* E_{\mathfrak{p}} \subseteq L^2_m(K_{\mathfrak{x};(\psi(u))^*TX\otimes\Lambda^{0,1}})$. Adding to this fiber using Lemma 2.2.3 if necessary (i.e. taking a higher dimensional obstruction fiber containing $\underline{\psi}_* E_{\mathfrak{p}}$), we get an obstruction fiber $E_{\psi(\mathfrak{p})}$ that we can extend to a choice of obstruction bundle data at $\psi(\mathfrak{p})$. Taking obstruction fiber $E'_{\mathfrak{p}} = (\psi_*)^{-1}(E_{\psi(\mathfrak{p})})$, we then apply Proposition 2.1.27 to get obstruction bundle data at $\mathfrak{p}$. We restrict the domains of all of our obstruction bundle data so that the support of any element of an obstruction fiber is always kept away from the preimage of D_X or D_Y.

Finally, for every point $\mathfrak{p}' \in \mathcal{M}_{k+1,\ell}(Y, L_Y, \underline{\psi}_*\beta)$ outside of the image of ψ, we take any choice of obstruction bundle data.

Having taken all necessary obstruction data, we next need to take stabilization data at each point $\mathfrak{p} \in \mathcal{M}_{k+1,\ell}(X, L_X, \beta)$ and every point $\mathfrak{p}' \in \mathcal{M}_{k+1,\ell}(Y, L_Y, \underline{\psi}_*\beta)$.

For each image point $\psi(\mathfrak{p}) \in \mathcal{M}_{k+1,\ell}(Y, L_Y, \underline{\psi}_*\beta)$, we take $\epsilon_{\psi(\mathfrak{p})}$ as in Section 2.1.5, and for each $\mathfrak{p}' \in \mathcal{M}_{k+1,\ell}(Y, L_Y, \underline{\psi}_*\beta) \setminus \psi(\mathcal{M}_{k+1,\ell}(X, L_X, \beta))$ we take $\epsilon_{\mathfrak{p}'}$ sufficiently small for the neighborhood $\mathfrak{W}^+(\mathfrak{p}')$ in $\mathcal{M}_{k+1,\ell}(Y, L_Y, \underline{\psi}_*\beta)$ to not intersect $\psi(\mathcal{M}_{k+1,\ell}(X, L_X, \beta))$. For each $\psi(\mathfrak{p}) \in \psi(\mathcal{M}_{k+1,\ell}(X, L_X, \beta))$, we choose the subsets $\operatorname{Int}\mathfrak{W}^0_{\psi(\mathfrak{p})} \subseteq \mathfrak{W}^0_{\psi(\mathfrak{p})} \subseteq \operatorname{Int}\mathfrak{W}_{\psi(\mathfrak{p})} \subseteq \mathfrak{W}_{\psi(\mathfrak{p})}$ of $\mathfrak{W}^+_{\psi(\mathfrak{p})} \subseteq \mathcal{M}_{k+1,\ell}(Y, L_Y, \underline{\psi}_*\beta)$ as in Section 2.1.5, and we take $\mathfrak{W}^0_{\mathfrak{p}} = \psi^{-1}(\mathfrak{W}^0_{\psi(\mathfrak{p})})$ and $\mathfrak{W}_{\mathfrak{p}} = \psi^{-1}(\mathfrak{W}_{\psi(\mathfrak{p})})$. For $\mathfrak{p}' \in \mathcal{M}_{k+1,\ell}(Y, L_Y, \underline{\psi}_*\beta) \setminus \psi(\mathcal{M}_{k+1,\ell}(X, L_X, \beta))$ we also choose subsets $\mathfrak{W}^0_{\mathfrak{p}'}, \mathfrak{W}_{\mathfrak{p}'}$ of $\mathfrak{W}^+_{\mathfrak{p}'}$, with the important point being that these sets again do not intersect
$\psi(\mathcal{M}_{k+1,\ell}(X, L_X, \beta))$.

We take a finite set $\{\mathfrak{p}'_c \mid c \in \mathfrak{C}'\} \subseteq \mathcal{M}_{k+1,\ell}(Y, L_Y, \underline{\psi}_*\beta)$ such that $\bigcup_{c\in\mathfrak{C}'} \operatorname{Int}\mathfrak{W}^0_{\mathfrak{p}'_c} = \mathcal{M}_{k+1,\ell}(Y, L_Y, \underline{\psi}_*\beta)$. Because of our choices of $\mathfrak{W}^0_{\mathfrak{p}'}$, this gives finite sets $\mathfrak{C} \subseteq \mathfrak{C}'$ and $\{\mathfrak{p}_c \mid c \in \mathfrak{C}\} \subseteq \mathcal{M}_{k+1,\ell}(X, L_X, \beta)$ such that $\bigcup_{c\in\mathfrak{C}} \operatorname{Int}\mathfrak{W}^0_{\mathfrak{p}} = \mathcal{M}_{k+1,\ell}(X, L_X, \beta)$ and $\bigcup_{c\in\mathfrak{C}} \operatorname{Int}\mathfrak{W}^0_{\psi(\mathfrak{p})} \supseteq \psi(\mathcal{M}_{k+1,\ell}(X, L_X, \beta))$, and such that for each $\psi(\mathfrak{p})$ the set $\mathfrak{C}'(\psi(\mathfrak{p})) \subseteq \mathfrak{C}$.

We now take stabilization data (Definition 2.1.37) at each $\mathfrak{p} \in \mathcal{M}_{k+1,\ell}(X, L_X, \beta)$ in a similar way to how we chose obstruction bundle data. That is, we choose a symmetric stabilization $\vec{w}_{\mathfrak{p}}$ of $\mathfrak{p}$ so that $u(w_{\mathfrak{p},i}) \notin D_X$ for all $w_{\mathfrak{p},i}$, and we choose the corresponding codimension 2 submanifolds $\mathcal{D}_{\mathfrak{p},i}$ of X to be transversal to D_X. We then take a universal family with coordinate at infinity for each component together with its extra marked points $\mathfrak{x}_v \cup \vec{w}_{\mathfrak{p}}$. By taking the same coordinate at infinity and symmetric stabilization, and taking the codimension 2 submanifolds $\psi(\mathcal{D}_{\mathfrak{p},i})$ of D_Y, we get a corresponding choice of stabilization data at $\psi(\mathfrak{p})$. We also choose stabilization data at $\mathfrak{p}'$ for each $\mathfrak{p}' \in \mathcal{M}_{k+1,\ell}(Y, L_Y, \underline{\psi}_*\beta) \setminus \psi\left(\mathcal{M}_{k+1,\ell}(X, L_X, \beta)\right)$.

We now take ϵ_0 and ϵ_0' small and consider the sets $\mathfrak{U}_{k+1,(\ell;\ell_\mathfrak{p},(\ell_c))}(X, L_X, \beta; \mathfrak{p})_{\epsilon_0,\vec{T}_0}$ and $\mathfrak{U}_{k+1,(\ell;\ell_\mathfrak{p},(\ell_c))}(Y, L_Y, \underline{\psi}_*\beta; \psi(\mathfrak{p}))_{\epsilon_0',\vec{T}_0}$ (see Definition 2.1.39). By taking ϵ_0 sufficiently small relative to ϵ_0', we get for all $(\mathfrak{Y}, u', (\vec{w}_c')) \in \mathfrak{U}_{k+1,(\ell;\ell_\mathfrak{p},(\ell_c))}(X, L_X, \beta; \mathfrak{p})_{\epsilon_0,\vec{T}_0}$ that we have that $(\mathfrak{Y}, \psi\circ u', (\vec{w}_c')) \in \mathfrak{U}_{k+1,(\ell;\ell_\mathfrak{p},(\ell_c))}(Y, L_Y, \underline{\psi}_*\beta; \psi(\mathfrak{p}))_{\epsilon_0',\vec{T}_0}$. We also have $\psi_*\bar{\partial}u' = \bar{\partial}(\psi\circ u')$, so, by the choices of our obstruction bundle data, $\bar{\partial}u' \in \mathcal{E}((\mathfrak{Y}, u', (\vec{w}_c')), \Sigma_\mathfrak{Y})$ if and only if $\bar{\partial}\psi\circ u' \in \mathcal{E}((\mathfrak{Y}, \psi\circ u', (\vec{w}_c')), \Sigma_\mathfrak{Y})$. That is, we have the following map of thickened moduli spaces (see Definition 2.1.41):

$$\psi : \mathcal{M}_{k+1,(\ell;\ell_\mathfrak{p},(\ell_c))}(X, L_X, \beta; \mathfrak{p})_{\epsilon_0,\vec{T}_0} \to \mathcal{M}_{k+1,(\ell;\ell_\mathfrak{p},(\ell_c))}(Y, L_Y, \underline{\psi}_*\beta; \psi(\mathfrak{p}))_{\epsilon_0',\vec{T}_0}.$$

From our various choices of codimension 2 submanifolds $\mathcal{D}$ and $\psi(\mathcal{D})$, this then gives a map

$$\psi : \mathcal{M}_{k+1,(\ell;\ell_\mathfrak{p},(\ell_c))}(X, L_X, \beta; \mathfrak{p})^{trans}_{\epsilon_0,\vec{T}_0} \to \mathcal{M}_{k+1,(\ell;\ell_\mathfrak{p},(\ell_c))}(Y, L_Y, \underline{\psi}_*\beta; \psi(\mathfrak{p}))^{trans}_{\epsilon_0',\vec{T}_0}$$

which is a smooth embedding. We take an open neighborhood $V_{\psi(\mathfrak{p})}$ of $\psi(\mathfrak{p})$ in $\mathcal{M}_{k+1,(\ell;\ell_\mathfrak{p},(\ell_c))}(Y, L_Y, \underline{\psi}_*\beta; \psi(\mathfrak{p}))_{\epsilon_0',\vec{T}_0}$ so that $V_{\psi(\mathfrak{p})} \cap \psi(\mathcal{M}_{k+1,(\ell;\ell_\mathfrak{p},(\ell_c))}(X, L_X, \beta; \mathfrak{p})^{trans}_{\epsilon_0,\vec{T}_0})$ is closed in $V_{\psi(\mathfrak{p})}$. We then take $V_\mathfrak{p} = \psi^{-1}(V_{\psi(\mathfrak{p})})$.

Each $(V_{\psi(\mathfrak{p})}, \mathcal{E}(\mathfrak{p}), \Gamma_\mathfrak{p}, \mathrm{Triv}\circ\bar{\partial})$ is a Kuranishi neighborhood of $\psi(\mathfrak{p})$ in $\mathcal{M}_{k+1,\ell}(Y, L_Y, \underline{\psi}_*\beta)$ and $(V_\mathfrak{p}, \mathcal{E}(\mathfrak{p}), \Gamma_\mathfrak{p}, \mathrm{Triv}\circ\bar{\partial})$ is a Kuranishi neighborhood of $\mathfrak{p}$ in $\mathcal{M}_{k+1,\ell}(X, L_X, \beta)$. The natural transition maps used in FOOO [17] for the Kuranishi structures on $\mathcal{M}_{k+1,\ell}(X, L_X, \beta)$ and $\mathcal{M}_{k+1,\ell}(Y, L_Y, \underline{\psi}_*\beta)$ are compatible in the sense of Definition 2.1.4. Thus, we have constructed the first morphism of Kuranishi spaces in the theorem statement:

$$\mathcal{M}_{k+1,\ell}(X, L_X, \beta) \xrightarrow{\psi} \mathcal{M}_{k+1,\ell}(Y, L_Y, \underline{\psi}_*\beta).$$

We can now take Kuranishi neighborhood $(\psi(V_\mathfrak{p}), \mathcal{E}(\mathfrak{p}), \Gamma_\mathfrak{p}, \mathrm{Triv}\circ\bar{\partial})$ of the point $\psi(\mathfrak{p})$ in the moduli space $\psi(\mathcal{M}_{k+1,\ell}(X, L_X, \beta)$. With this Kuranishi structure, we have the Kuranishi isomorphism from the theorem statement:

$$\mathcal{M}_{k+1,\ell}(X, L_X, \beta) \xrightarrow{\psi} \psi\left(\mathcal{M}_{k+1,\ell}(X, L_X, \beta)\right).$$

Note that the Kuranishi structure on the space $\psi(\mathcal{M}_{k+1,\ell}(X, L_X, \beta))$ is induced from the Kuranishi structure on $\mathcal{M}_{k+1,\ell}(Y, L_Y, \underline{\psi}_*\beta)$ in the sense of Section 2.1.7. Each chart of $\psi(\mathcal{M}_{k+1,\ell}(X, L_X, \beta))$ is the fiber of a submersion on the corresponding chart of $\mathcal{M}_{k+1,\ell}(Y, L_Y, \underline{\psi}_*\beta)$, and these can be glued together to give a global weak submersion some fiber of which gives $\psi(\mathcal{M}_{k+1,\ell}(X, L_X, \beta))$. □

2.3 Deformation family regularization of moduli spaces

We now introduce the concept of deformation family regularization of Kuranishi spaces. The most natural definition of a deformation family of Kuranishi spaces is in fact simply a weakly submersive map $\pi : \mathcal{M} \to B$, as each fiber is then a Kuranishi space.

Definition 2.3.1. *Let $\mathcal{M}$ be a Kuranishi space, let B be a smooth manifold, and let $\pi : \mathcal{M} \to B$ be weakly submersive. For each $t \in B$, the fiber $\mathcal{M}_t$ is a Kuranishi space. We call $\pi : \mathcal{M} \to B$ a deformation family of Kuranishi spaces.*

Consider the following simple example

Example 2.3.2. *Fix L a moment fiber Lagrangian in $\mathbb{P}^1 \times \mathbb{P}^1$ and $\beta_1 + \beta_2 = \beta \in H_2(\mathbb{P}^1 \times \mathbb{P}^1, L)$ an effective Maslov index 4 disc class containing no multiply covered discs. The moduli space $\mathcal{M}_{k+1,1}(\mathbb{P}^1 \times \mathbb{P}^1, L, \beta)$ is regular, in the sense that we can take trivial obstruction bundle everywhere and still have a valid Kuranishi structure. The interior $\mathrm{Int}\mathcal{M}_{k+1,1}(\mathbb{P}^1 \times \mathbb{P}^1, L, \beta)$ of the moduli space is a (non-compact) manifold without boundary, and the restriction of ev^+, the evaluation map at the interior marked point, to $\mathrm{Int}\mathcal{M}_{k+1,1}(\mathbb{P}^1 \times \mathbb{P}^1, L, \beta)$ is a submersion. Letting $\gamma : (-1, 1) \to \mathbf{P}^1 \times \mathbf{P}^1$ be a path whose image is contained in the image of ev^+, the pullback $\mathrm{Int}\mathcal{M}_{k+1,1}(\mathbb{P}^1 \times \mathbb{P}^1, L, \beta)\ {}_{\mathrm{ev}^+}\times_\gamma (-1, 1)$ is a deformation family of Kuranishi spaces over $(-1, 1)$.*

For our purposes, we will be interested in deformation families of Kuranishi spaces where the generic fibers are regular. We can then think of the regular fibers as being deformations of the irregular fibers.

Definition 2.3.3. *Let $\pi : \mathcal{M} \to B$ be a deformation family of Kuranishi spaces. For each $t \in B$, the fiber $\mathcal{M}_t$ is a Kuranishi space. Fix a privileged element $t_0 \in B$. Assume further that, for all $t \neq t_0$, the Kuranishi section of $\mathcal{M}_t$ is transversal to* *the* 0*-section without any perturbation. We refer to this deformation family as a deformation family regularization of* $\mathcal{M}_{t_0}$*.*

This provides an alternative approach to perturbing a moduli space that can be more explicitly calculable in practice. In our primary example studying $\mathscr{O}(-n)$, we will construct a deformation family over an open interval that allows us to deform our moduli spaces of interest to ones that are regular without perturbation. We can then use these regular moduli spaces to define our A_∞ structure on $H^\bullet(L;\Lambda)$.

More precisely, in order to define our A_∞ structure, we need to deform our moduli spaces in families so that the deformations respect the stratification structure of the moduli spaces, since it is this relationship that gives rise to the A_∞-structure. This is a stronger condition than that appearing in Definition 2.3.3. In light of this, we take the following definition of a stratified deformation family regularization of a moduli space.

Definition 2.3.4 (Stratified deformation family)**.** *Let* $\mathcal{M}$ *be a moduli space of pseudoholomorphic discs with natural stratification induced by the stratification of the Deligne-Mumford space* $\mathcal{M}_{k+1,\ell}$ *of stable marked genus 0 Riemann surfaces with at most one boundary component. Let* B *be a smooth manifold with privileged element* $t_0 \in B$*, and let* $\pi : \mathcal{M} \to B$ *be a deformation family of Kuranishi spaces. Assume further that, for each Kuranishi chart* V_α *of* $\mathcal{M}$*, the restriction of the submersion* $\pi_\alpha : V_\alpha \to B$ *to any stratum of* V_α *is also a submersion. We then call* $\pi : \mathcal{M} \to B$ *a stratified deformation family of* $\mathcal{M}_{t_0}$*.*

If this deformation family also gives rise to a regularization of $\mathcal{M}_{t_0}$, we call it a stratified deformation regularization:

Definition 2.3.5 (Stratified deformation regularization)**.** *Let* $\pi : \mathcal{M} \to B$ *be a stratified deformation family of moduli spaces of pseudoholomorphic discs. If this family is also a deformation family regularization of* $\mathcal{M}_{t_0}$ *for some* $t_0 \in B$*, we call this a stratified deformation regularization of* $\mathcal{M}_{t_0}$*.*

Remark 2.3.6. *In their recent book, FOOO* [23] *introduce the notion of a "system of K-spaces," which axiomatizes a number of the important features of the moduli spaces* *we study, including the boundary compatibility property mentioned above, but we will not work directly with this notion here.*

We now describe a method for constructing stratified deformation regularizations. Assume we have already constructed Kuranishi structures on the moduli spaces $\mathcal{M}_{k+1,\ell}(X,L,\beta)$ for all β following FOOO, as outlined in this chapter. We will build a Kuranishi structure on the space $\mathcal{M}_{k+1,\ell}(X,L,\beta)\times(-\epsilon,\epsilon)$ that will give rise to a deformation family over $(-\epsilon,\epsilon)$.

Let $U\subseteq X$ be an open set such that every element of the obstruction fiber $E_{\mathfrak{p}}$ is supported away from $u^{-1}(U)$ for all $\mathfrak{p}$. Let $\eta:(-\epsilon,\epsilon)\times(-\epsilon',\epsilon')\to \mathrm{Diff}(U)$ be a family of difeomorphisms of U. Assume $\eta_{s,0}=id_X$ and that the composition relation

$$\eta_{s+t,t'}\circ\eta_{s,t}=\eta_{s,t+t'}$$

holds for all $s\in(-\epsilon,\epsilon)$ and $t,t'\in(-\epsilon',\epsilon')$ such that $s+t\in(-\epsilon,\epsilon)$ and $t+t'\in(-\epsilon',\epsilon')$.[3] Furthermore, assume, for all $(s,t)\in(-\epsilon,\epsilon)\times(-\epsilon',\epsilon')$, that $\eta_{s,t}$ is the identity on a neighborhood of L and is pseudoholomorphic on an open neighborhood of $U\cap\overline{X\setminus U}$. For each pseudoholomorphic $\mathfrak{p}=(\mathfrak{x},u)\in\mathcal{M}_{k+1,\ell}(X,L,\beta)$ we have a Kuranishi chart $V_{\mathfrak{p}}\xrightarrow{\mathfrak{s}}E_{\mathfrak{p}}$; assume finally that the derivative

$$\frac{d}{dt}\Big|_{t=0}\bar{\partial}\eta_{s,t}(u)\in L^2_{m,\delta}(\Sigma_{\mathfrak{x}};u^*TX\otimes\Lambda^{0,1}) \tag{2.3.1}$$

does not lie in $T_0(E_{\mathfrak{p}})$ for any $\mathfrak{p}$ and s.

We have chosen $\eta_{s,t}$ so that it gives smooth maps $(\eta_{s,t})_*$ from each Hilbert manifold of core maps

$$W^2_{m+1}((K'_{\mathfrak{x}},K'_{\mathfrak{x}}\cap\Sigma_{\mathfrak{x}});X,L)$$

to itself.

[3] This condition may seem somewhat strange at first, but it arises naturally in our primary example.

Lemma 2.3.7. *In the above situation, we can construct a Kuranishi structure on* $\mathcal{M}_{k+1,\ell}(X,L,\beta)\times(-\epsilon,\epsilon)$ *with the following properties:*

1. *We have one Kuranishi chart* $V'_{\mathfrak{p}}$ *for each* $\mathfrak{p}\in\mathcal{M}_{k+1,\ell}(X,L,\beta)$ *such that* $V'_{\mathfrak{p}}$ *covers* $V_{\mathfrak{p}}\times(-\epsilon,\epsilon)$.

2. *The projections* $V'_{\mathfrak{p}}\to(-\epsilon,\epsilon)$ *make a* $\mathcal{M}_{k+1,\ell}(X,L,\beta)\times(-\epsilon,\epsilon)$ *stratified deformation family of* $\mathcal{M}_{k+1,\ell}(X,L,\beta)\times\{0\}$ *over* $(-\epsilon,\epsilon)$.

3. *If, for every map* $(\mathfrak{Y},w)\in V_{\mathfrak{p},s}$ *in the fiber of* $V'_{\mathfrak{p}}$ *over* $s\in(-\epsilon,\epsilon)$*, we have that*

$$(\eta_{s,t})_*\circ res\circ w\in W^2_{m+1}((K_{\mathfrak{x}},K_{\mathfrak{x}}\cap\partial\Sigma_{\mathfrak{x}});X,L)$$

extends to a map

$$(\eta_{s,t})_!(w)\in W^2_{m+1,\delta}((\Sigma_{\mathfrak{Y}},\partial\Sigma_{\mathfrak{Y}});X,L),$$

then the map $(\eta_{s,t})_!$ *in the diagram below exists, is unique, and is a diffeomorphism between the Kuranishi neighborhood* $V_{\mathfrak{p},s}$ *of the point* $(\mathfrak{p},s)=((\mathfrak{x},u),s)$ *in the fiber over* s *and the Kuranishi neighborhood* $V_{\mathfrak{p},s+t}$ *of the point* $(\mathfrak{p},s)$ *in the fiber over* $s+t$.

$$\begin{array}{ccc} V_{\mathfrak{p},s} & \xrightarrow{(\eta_{s,t})_!} & V_{\mathfrak{p},s+t} \\ \downarrow{\scriptstyle res} & & \downarrow{\scriptstyle res} \\ W^2_{m+1}((K'_{\mathfrak{x}},K'_{\mathfrak{x}}\cap\partial\Sigma_{\mathfrak{x}});X,L) & \xrightarrow[(\eta_{s,t})_*]{} & W^2_{m+1}((K'_{\mathfrak{x}},K'_{\mathfrak{x}}\cap\partial\Sigma_{\mathfrak{x}});X,L) \end{array} \tag{2.3.2}$$

Proof. We let J be the ω-tame almost-complex structure we have been implicitly using for X, and we let $J_{s,t}:=\eta^*_{s,t}J$ be the pullback almost-complex structure. Each $J_{s,t}$ is well-defined on all of X, and agrees with J on $X\setminus U$, by the assumptions on η. For each J-holomorphic $\mathfrak{p}=(\mathfrak{x},u)\in\mathcal{M}_{k+1,\ell}(X,L,\beta)$, we have a Kuranishi chart

$$V_{\mathfrak{p}}\xrightarrow{\mathfrak{s}}E_{\mathfrak{p}},$$

where $\mathfrak{s}=\pi\circ\bar\partial_J$ is given by Lemma 2.1.28. We then consider the map

$$\bar{\mathfrak{s}}:W^2_{m+1}((K'_{\mathfrak{x}},K'_{\mathfrak{x}}\cap\Sigma_{\mathfrak{x}});X,L)\times(-\epsilon,\epsilon)\times(-\epsilon',\epsilon')\to L^2_m(K'_{\mathfrak{x}};u^*TX\otimes\Lambda^{0,1})$$

sending $((\mathfrak{Y},w),s,t)\mapsto\pi\circ\bar\partial_{J_{s,t}}(w)$. As before, $K'_{\mathfrak{x}}$ is a compact subset of the interior of the core $K_{\mathfrak{x}}$. Here we are using the coordinate at infinity given by the stabilization data at $\mathfrak{p}$. We define the Kuranishi neighborhood $V^0_{\mathfrak{p}}$ analogously to Section 2.1.6 using the condition $\bar{\partial}_{J_{s,t}} w \in E_{\mathfrak{p}}((\mathfrak{Y}, w)) = \pi^{-1}(E_{\mathfrak{p}})$. We get immediately that $V^0_{\mathfrak{p}}$ is stratawise smooth, and it is in fact a smooth manifold with corners, since the gluing argument is naturally carried out without changing the coordinates s, t. Furthermore, the projection to the s coordinate $V^0_{\mathfrak{p}} \to (-\epsilon, \epsilon)$ is a submersion, and for each fixed value of $s \in (-\epsilon, \epsilon)$ the fiber $V^0_{\mathfrak{p},s}$ over s is a smooth manifold with corners.

The space $V^0_{\mathfrak{p},s}$ consists entirely of the pairs $((\mathfrak{Y}, w)), t)$ where $\bar{\partial}_{J_{s,t}}(w) \in E_{\mathfrak{p}}(\mathfrak{Y}, w)$, which is equivalent to the condition that $\bar{\partial}_J(\eta_{s,t} \circ w|_{K''}) \in E_{\mathfrak{p}}(\mathfrak{Y}, w)$, where K'' is a compact subset of the interior of the core K such that K' is a compact subset of the interior of K''. This condition is equivalent to the condition

$$\bar{\partial}_J w|_{K'} \in E_{\mathfrak{p}}(\mathfrak{Y}, w) - \bar{\partial}_J(\eta_{s,t} \circ w|_{K'}) + \bar{\partial}_J w|_{K'},$$

and the condition that $((\mathfrak{Y}, w), t) \in V^0_{\mathfrak{p},s}$ for some choice of $t \in (-\epsilon', \epsilon')$ is equivalent to the condition

$$\bar{\partial}_J w|_{K'} \in E'_{\mathfrak{p}}((\mathfrak{Y}, w), s) = E_{\mathfrak{p}}(\mathfrak{Y}, w) + \bar{\partial}_J w|_{K'} - \bigcup_{t \in (-\epsilon', \epsilon')} \bar{\partial}_J(\eta_{s,t} \circ w|_{K'}).$$

By the condition in Equation 2.3.1, the spaces $E'_{\mathfrak{p}}((\mathfrak{Y}, w), s)$ give a trivialization of an obstruction fiber $E'_{\mathfrak{p}}(\mathfrak{p}, s)$. Thus, we can take $E'_{\mathfrak{p}}(\mathfrak{Y}, w) = \bigcup_{s \in (-\epsilon, \epsilon)} \big(E'_{\mathfrak{p}}((\mathfrak{Y}, w), s), s\big)$, and, by forgetting the t coordinate of $V^0_{\mathfrak{p}}$, we get Kuranishi charts $V'_{\mathfrak{p}} \to E'_{\mathfrak{p}}(\mathfrak{Y}, w)$ for $\mathcal{M}_{k+1,\ell}(X, L, \beta) \times (-\epsilon, \epsilon)$ such that the projection to $(-\epsilon, \epsilon)$ gives a stratified deformation family of $\mathcal{M}_{k+1,\ell}(X, L, \beta) \times \{0\}$. All coordinate changes are naturally induced from those of the original Kuranishi structure on $\mathcal{M}_{k+1,\ell}(X, L, \beta)$. Since $\eta_{s,0} = id$ for all s, we have that $E'_{\mathfrak{p}}(\mathfrak{Y}, w) \supseteq E_{\mathfrak{p}}(\mathfrak{Y}, w)$, so $V'_{\mathfrak{p}} \supseteq V_{\mathfrak{p}} \times (-\epsilon, \epsilon)$. We have thus shown items 1 and 2 of the lemma.

Now, as in item 3 of the lemma, assume that every map

$$(\eta_{s,t})_* \circ \mathrm{res} \circ w \in W^2_{m+1}((K'_{\mathfrak{x}}, K'_{\mathfrak{x}} \cap \partial\Sigma_{\mathfrak{x}}); X, L)$$

extends to a map

$$(\eta_{s,t})_!(w) \in W^2_{m+1,\delta}((\Sigma_{\mathfrak{Y}}, \partial\Sigma_{\mathfrak{Y}}); X, L).$$

The restriction maps in Diagram 2.3.2 are injective, since every map $(\mathfrak{Y}, w) \in V_{\mathfrak{p},s}$ is $J_{s,t}$-holomorphic on $K'' \setminus K'$ for some (unique) t. The map $(\eta_{s,t})_*$ gives a diffeomorphism from $\mathrm{res}(V_{\mathfrak{p},s})$ onto its image, so, since $\eta_{s+t,-t} = \eta_{s,t}^{-1}$, it suffices to show that this image is contained in $\mathrm{res}(V_{\mathfrak{p},s+t})$.

Since $(\eta_{s,t})_* \circ \mathrm{res} \circ w$ extends to $(\eta_{s,t})_!(w)$, we have that $\bar{\partial}_J(\mathrm{res}((\eta_{s,t})_!(w))) \in E'_{\mathfrak{p}}((\mathfrak{Y}, w), s+t)$, which implies $(\eta_{s,t})_!(w) \in V_{\mathfrak{p},s+t}$, as desired.

□

Remark 2.3.8. *The reason the above proof is more roundabout than just declaring* $E_{\mathfrak{p},s} = E_{\mathfrak{p}} - \bigcup_{t\in(-\epsilon',\epsilon')} \bar{\partial}(\eta_{s,t} \circ u)$ *is that these forms do not vanish on all neck regions, so we cannot just plug this choice of obstruction fiber into our usual process for constructing a Kuranishi structure. Note that the condition that the forms of the obstruction fiber vanish on the neck regions is necessary both for the extension of the obstruction fiber over the Hilbert manifold* $W^2_{m+1}(K'_{\mathfrak{r}}, K'_{\mathfrak{r}} \cap \partial\Sigma; X, L)$ *and for conducting the necessary gluing argument to show that our Kuranishi charts are smooth manifolds with corners.*

The proof of Theorem 2.2.1 can be adapted to show the following lemma:

Lemma 2.3.9. *Given diffeomorphisms* $\eta_{s,t}$ *as in Lemma 2.3.7, and given a Kuranishi inclusion*

$$\mathcal{M}_{k+1,\ell}(Y, L_Y, \beta) \xrightarrow{\psi} \mathcal{M}_{k+1,\ell}(X, L_X, \underline{\psi}_*\beta)$$

induced by a map

$$\underline{\psi} : Y \to X$$

as in Theorem 2.2.1, we can construct a Kuranishi structure on $\mathcal{M}_{k+1,\ell}(Y, L_Y, \underline{\psi}_*\beta) \times (-\epsilon, \epsilon)$ *and a Kuranishi structure on* $\mathcal{M}_{k+1,\ell}(Y, L_Y, \beta)$ *such that such that we have a Kuranishi inclusion*

$$\mathcal{M}_{k+1,\ell}(Y, L_Y, \beta) \xrightarrow{\psi} \mathcal{M}_{k+1,\ell}(X, L_X, \underline{\psi}_*\beta) \times \{0\}. \tag{2.3.3}$$

We now combine Lemmas 2.3.7 and 2.3.9 to arrive at our procedure for constructing well-behaved stratified deformation families. Assume that the condition in item 3 of Lemma 2.3.7 holds for $\eta_{0,t}$ for all t. We take the image $(\eta_{0,t})_!$ of each chart $V_{\psi(\mathfrak{p})}$ of image Kuranishi space $\psi(\mathcal{M}_{k+1,\ell}(Y, L_Y, \beta))$ of the map ψ in 2.3.3. This is a closed subset $(\eta_{0,t})_!(V_{\psi(\mathfrak{p})})$ of the Kuranishi chart $V'_{\psi(\mathfrak{p}),t}$ of the fiber over t of the stratified deformation family $\mathcal{M}_{k+1,\ell}(X, L_X, \underline{\psi}_*\beta) \times (-\epsilon, \epsilon)$.

Under the final assumption that the union for fixed t of the images of these $(\eta_{0,t})_!(V_{\psi(\mathfrak{p})})$ in the compact metrizable space underlying the Kuranishi space $\mathcal{M}_{k+1,\ell}(X, L_X, \underline{\psi}_*\beta) \times (-\epsilon, \epsilon)$ form a compact subspace, this then gives a stratified deformation family of $\psi(\mathcal{M}_{k+1,\ell}(Y, L_Y, \beta))$ with the property that the Kuranishi charts of the fiber over a given t are obtained from the Kuranishi charts of the central fiber $\psi(\mathcal{M}_{k+1,\ell}(Y, L_Y, \beta))$ by applying $(\eta_{0,t})_!$.

Chapter 3

Application to $\mathscr{O}(-n)$

3.1 Kuranishi structures and correspondence map between $\mathscr{O}(-n)$ and $\mathscr{O}$

We carry out the process behind Theorem 2.2.1 in our specific situation of interest. We need to build compatible Kuranishi structures for the moduli spaces $\mathcal{M}_{k+1}(\mathscr{O}(-n), L, \beta)$, and $\mathcal{M}_{k+1}(\mathscr{O}, \underline{\psi}(L), \underline{\psi}_*(\beta); (F_0, D_0, n))$. That is, recalling that the two moduli spaces are homeomorphic under the map ψ, we will build isomorphic Kuranishi structures on them. For readibility, we will actually construct two different pairs of isomorphic Kuranishi structures. The first, constructed in this section, is simpler. Notably, all obstruction fibers for these Kuranishi structures will be vector spaces, as in the original FOOO construction. The second, constructed in the following subsection 3.2, is similar to the first but with larger obstruction fibers, which will not naturally be vector spaces.

Remark 3.1.1. *Strictly speaking, we could use ψ as an identification between the two spaces, take any Kuranishi structure on one of the moduli spaces, and thus have isomorphic Kuranishi structures on both spaces.*[1] *However, as much as having the Kuranishi structures themselves, we are interested in embedding the Kuranishi charts into appropriate Hilbert manifolds of maps into $\mathscr{O}(-n)$ and $\mathscr{O}$, and this requires greater subtlety.*

We first consider $\mathcal{M}_{k+1}(\mathscr{O}, \underline{\psi}(L), \underline{\psi}_*\beta)$ and begin constructing a Kuranishi struc-

[1] Even if we were to take this approach, we would need to give more thought to the stratawise smooth structure on $\mathcal{M}_{k+1,\ell}(\mathscr{O}, \underline{\psi}(L), \underline{\psi}_*(\beta); (F_0, D_0, n))$.

ture, following FOOO [15], [17] as outlined in the previous section. We recall that $\underline{\psi}(L) \subseteq U_1 \subseteq \mathscr{O}$ is given by $|x_1| = r_{x_0}^{-1}$ and $|y_1| = r_{y_0}$ for fixed constants r_{x_0} and r_{y_0}. We choose a Riemannian metric on $\mathbb{P} \times \mathbb{P}$ such that it coincides with the Euclidean metric on the open subset $U_1' = \{(x_1, y_1) \in U_1 \mid |x_1| < 2r_{x_0}^{-1}, |y_1| < 2r_{y_0}\}$ of U_1.

We next need to take obstruction bundle data (Definition 2.1.30) and stabilization data (Definition 2.1.37) at each $\mathfrak{p}' \in \mathcal{M}_{k+1}(\mathscr{O}, \underline{\psi}(L), \underline{\psi}_* \beta)$. We will only take specific care in choosing obstruction bundle data and stabilization data at points $\psi(\mathfrak{p}) = (\mathfrak{x}, \psi(u)) \in \mathcal{M}_{k+1}(\mathscr{O}, \underline{\psi}(L), \underline{\psi}_* \beta)$. For our obstruction bundle data, we need to choose a symmetric stabilization $\vec{w}$ and corresponding codimension 2 submanifolds $\mathcal{D}_i$ (items (1) and (9)), a universal family with coordinate at infinity (items (2) and (3)), a $\Gamma^+_{\psi(\mathfrak{p} \cup \vec{w})}$-invariant obstruction fiber $E_{\psi(\mathfrak{p})}$ (item (4)) such that items (6)-(8) are satisfied, and a $\Gamma^+_{\psi(\mathfrak{p} \cup \vec{w})}$-equivariant extension $\mathrm{Triv}_{K', E_{\psi(\mathfrak{p} \cup \vec{w})}}$.

Recall that, since $\psi(\mathfrak{p})$ is holomorphic and $\beta \in H_2(\mathscr{O}, \underline{\psi}(L))$, we have that non-constant components of $\psi(\mathfrak{p})$ intersect the fibers $F_0 = \{(x_1, y_1)_1 \mid x_1 = 0\}$ at a finite number of isolated points. We can thus choose a symmetric stabilization $\vec{w}$ of $\psi(\mathfrak{p})$ so that $\underline{\psi}(u(w_i)) \notin F_0$ for all w_i, and we choose the corresponding codimension 2 submanifolds $\mathcal{D}_i$ to be transversal to F_0. This then induces a compatible stabilization for $\mathfrak{p} \in \mathcal{M}_{k+1}(\mathscr{O}(-n), L, \beta)$. We then take a universal family with coordinate at infinity for each component together with its extra marked points $\mathfrak{x}_v \cup \vec{w}_v$.

Next, we choose an obstruction fiber $E_{\psi(\mathfrak{p} \cup \vec{w}), v} \subseteq L^2_{m,\delta}(K_{\mathfrak{x}_v \cup \vec{w}}; \psi(u)_v^* T\mathscr{O} \otimes \Lambda^{0,1})$. With this fiber, we need $\psi(\mathfrak{p})$ to be Fredholm regular, evaluation map transversal (at singular points), and evaluation map transversal at the 0th boundary marked point. We also need the total obstruction fiber $E_{\psi(\mathfrak{p} \cup \vec{w})}$ to be invariant with respect to $\Gamma^+_{\psi(\mathfrak{p} \cup \vec{w})}$. We want to satisfy all of these conditions and keep the support of the obstruction bundle away from $\psi(u)^{-1}(F_0)$, so that we can then build a compatible structure on $\mathcal{M}_{k+1}(\mathscr{O}(-n), L, \beta)$. Actually, we have the following stronger fact.

Lemma 3.1.2. *Taking trivial obstruction fiber*

$$E_{\psi(\mathfrak{p}\cup\vec{w}),v} = \{0\} \subseteq L^2_{m,\delta}(K_{\mathfrak{x}_v\cup\vec{w}}; \psi(u)^*_v T\mathscr{O} \otimes \Lambda^{0,1})$$

at $\psi(\mathfrak{p}) = (\mathfrak{x}, \psi(u))$*, we have that* $\psi(\mathfrak{p})$ *is Fredholm regular, evaluation map transversal at singular points, and evaluation map transversal at the 0th boundary marked point.*

This trivial bundle is of course invariant with respect to $\Gamma^+_{\psi(\mathfrak{p}\cup\vec{w})}$. Note that the Fredholm regularity condition is as an element of $\mathcal{M}_{k+1}(\mathscr{O}, \underline{\psi}(L), \underline{\psi}_*\beta)$.

Proof. By Cho-Oh [9], all disc components are Fredholm regular without any obstruction bundle. It is straightforward to check that all sphere components are also Fredholm regular without any obstruction bundle.

The statements about evaluation map transversality rely crucially on the fact that our discs are genus zero, so the components form a tree. Each sphere component has a unique "outgoing" singular point (corresponding to an outgoing edge in its combinatorial type graph), so the restriction of the corresponding single component evaluation map

$$\mathrm{ev}_{v,e} : W^2_{m+1,\delta}((\Sigma_{\mathfrak{y}_v}, \emptyset); \psi(u)^*TX, \psi(u)^*TL) \to T_{\psi(u)(z_e)}X$$

to $(D_{\psi(u),v}\bar{\partial})^{-1}(0)$ is surjective. Note that this would not necessarily hold if our ambient space (in this case $\mathscr{O}$) contained negative self intersection rational curves.

Similarly, each disc component not containing the 0th marked point has a unique outgoing singular point, and the restriction of the corresponding single component evaluation map

$$\mathrm{ev}_{v,e} : W^2_{m+1,\delta}((\Sigma_{\mathfrak{y}_v}, \partial\Sigma_{\mathfrak{y}_v}); \psi(u)^*TX, \psi(u)^*TL) \to T_{\psi(u)(z_e)}L$$

to $(D_{\psi(u),v}\bar{\partial})^{-1}(0)$ is surjective. This can be seen by applying the T^2-action to the disc.

The evaluation map transversality at the 0th marked point follows similarly. □

However, to allow us to construct a compatible Kuranishi structure on $\mathcal{M}_{k+1}(\mathscr{O}(-n), L, \beta)$, we will add non trivial obstruction bundle. The only additional obstruction bundle we will need for this purpose will be to accommodate sphere components in multiples of the class $[D_{-n}]$, which are crucially not Fredholm regular without some obstruction bundle (this being the whole substance of the problem at hand). To this end, we make the following observation.

Lemma 3.1.3. *Let $\mathfrak{p} = (\mathfrak{x}, u) \in \mathcal{M}_{k+1}(\mathscr{O}(-n), L, \beta)$ with symmetric stabilization induced by that for $\psi(\mathfrak{p})$. For each component $\mathfrak{p}_v$ not in some class $a[D_{-n}]$ with $a > 0$, take trivial obstruction fiber $E_{\mathfrak{p}\cup\vec{w},v} = \{0\} \subseteq L^2_{m,\delta}(K_{\mathfrak{x}_v\cup\vec{w}}; u_v^*T\mathscr{O}(-n) \otimes \Lambda^{0,1})$.*

For each component $\mathfrak{p}_v$ in some class $a[D_{-n}]$ with $a > 0$, we can take an obstruction vector space $E_{\mathfrak{p}\cup\vec{w},v} = \{0\} \subseteq L^2_{m,\delta}(K_{\mathfrak{x}_v\cup\vec{w}}; u_v^*T\mathscr{O}(-n) \otimes \Lambda^{0,1})$ such that:

1. *$E_{\mathfrak{p}\cup\vec{w},v}$ is invariant with respect to $\Gamma^+_{\mathfrak{p}\cup\vec{w}}$*
2. *Each element of $E_{\mathfrak{p}\cup\vec{w},v}$ is supported away from $u^{-1}(F_0)$.*
3. *$\mathfrak{p}$ is Fredholm regular, evaluation map transversal at singular points, and evaluation map transversal at the 0th boundary marked point.*

Proof. Most of the work of this proof is contained in the following two lemmas.

Lemma 3.1.4. *Let $u : \mathbb{P}^1 \to \mathscr{O}(-n)$ be a holomorphic map belonging to homology class $a[D_{-n}]$ for some positive integer a. The pullback bundle $u^*T\mathscr{O}(-n)$ splits as $(T\mathbb{P}^1)^{\otimes a} \oplus \mathscr{O}(-an)$.*

Proof. First consider the map $f : \mathbb{P}^1 \to \mathscr{O}(-n)$ sending $[z_0 : z_1] \mapsto (0, \frac{z_0}{z_1}) \in U_0$ for $z_1 \neq 0$ and sending $[z_0 : z_1] \mapsto (0, \frac{z_1}{z_0}) \in U_1$ for $z_0 \neq 0$. We have an exact sequence

$$0 \to \mathscr{O}(-n) \xrightarrow{i} f^*T\mathscr{O}(-n) \xrightarrow{d\pi} T\mathbb{P}^1 \to 0$$

where the map i sends a section $v : \mathbb{P}^1 \to \mathscr{O}(-n)$ to the vector field $\lim_{t\to 0} \frac{tv}{t}$. Considering the inclusion $T\mathbb{P}^1 \xrightarrow{df} f^*T\mathscr{O}(-n)$, we see that this sequence splits.

Now, u factors as $f \circ u'$ where $u' : \mathbb{P}^1 \to \mathbb{P}^1$ is a degree a map, so $u^*T\mathscr{O}(-n)$ splits as $(u')^*T\mathbb{P}^1 \oplus (u')^*\mathscr{O}(-n) \cong (T\mathbb{P}^1)^{\otimes a} \oplus \mathscr{O}(-an)$. □

This then leads us to the following lemma.

Lemma 3.1.5. *Let $u : \mathbb{P}^1 \to \mathscr{O}(-n)$ be a holomorphic map belong to homology class $a[D_{-n}]$ for some positive integer a. The cokernel of the linearized $\bar{\partial}$ map*

$$D_u\bar{\partial} : W^2_{m+1,\delta}((\mathbb{P}^1, \emptyset); u^*T\mathscr{O}(-n), u^*TL) \to L^2_{m,\delta}(\mathbb{P}^1; u^*T\mathscr{O}(-n) \otimes \Lambda^{0,1})$$

is isomorphic to $H^{0,1}(\mathbb{P}^1, \mathscr{O}(-an))$.

Furthermore, we can choose representatives $e_1, \dots, e_{an-1}$ *of a basis of* $H^{0,1}(\mathbb{P}^1, (u')^*\mathscr{O}(-n)) \cong H^{0,1}(\mathbb{P}^1, \mathscr{O}(-an))$ *so that each* e_i *is supported away from a neighborhood of* $u^{-1}(F_0)$, *and so that* u *is Fredholm regular with respect to the obstruction vector space spanned by* $(0, e_i) \in L^2_{m,\delta}(\mathbb{P}^1; u^*T\mathscr{O}(-n) \otimes \Lambda^{0,1})$.

Proof. As in the proof of Lemma 3.1.4 we factor u as $u = f \circ u'$. The linearized $\bar\partial$ operator respects the splitting given by Lemma 3.1.4, and hence we have $D_u\bar\partial = D_{u'}\bar\partial \oplus D_0\bar\partial$ where

$$\begin{aligned} D_{u'}\bar\partial :& W^2_{m+1,\delta}(\mathbb{P}^1; (u')^*T\mathbb{P}^1) \to L^2_{m,\delta}(\mathbb{P}^1; (u')^*T\mathbb{P}^1 \otimes \Lambda^{0,1}), \\ D_0\bar\partial :& W^2_{m+1,\delta}(\mathbb{P}^1; \mathscr{O}(-an)) \to L^2_{m,\delta}(\mathbb{P}^1; \mathscr{O}(-an) \otimes \Lambda^{0,1}). \end{aligned}$$

$D_{u'}\bar\partial$ is surjective. The map

$$\bar\partial : W^2_{m+1,\delta}(\mathbb{P}^1; \mathscr{O}(-an)) \to L^2_{m,\delta}(\mathbb{P}^1; \mathscr{O}(-an) \otimes \Lambda^{0,1})$$

is linear. Thus, the cokernel of $D_u\bar\partial$ is isomorphic to the cokernel of $\bar\partial$ on $\mathscr{O}(-an)$. Since $\mathbb{P}^1$ has complex dimension 1, this cokernel is exactly $H^{0,1}(\mathbb{P}^1, \mathscr{O}(-an))$.

Assume WLOG that $u'([0:1]) = [0:1]$ and that $u'([1:0]) = [1:0]$. Let $\phi : \mathbb{P}^1 \to [0,1]$ be a smooth function equal to 0 on an open neighborhood of $[0:1]$ and equal to 1 on an open neighborhood of $[1:0]$. Let $c_1, \dots, c_a \in \mathbb{C} = \{[z_0 : z_1] \mid z_0 \neq 0\}$ be the points of $(u')^{-1}([1:0])$ taken with appropriate multiplicity (so the c_i are repeated if u' has higher multiplicity at c_i). Here $(u')^*\mathscr{O}(-n)$ is given by charts

$$\begin{aligned} V_0 &= (\{[z_0 : z_1] \mid z_1 \neq 0\} \setminus (u')^{-1}([1:0])) \times \mathbb{C} \\ V_1 &= (\{[z_0 : z_1] \mid z_0 \neq 0\} \setminus (u')^{-1}([0:1])) \times \mathbb{C}, \end{aligned}$$

each of which is a copy of $\mathbb{C}$ with finitely many points removed crossed with a $\mathbb{C}$ fiber. These charts are glued as follows for $u'([z_0 : z_1]) \neq [0:1], [1:0]$:

$$\left(\frac{z_0}{z_1}, y\right)_0 \sim \left(\frac{z_1}{z_0}, \prod_{j=1}^{a}\left(\frac{z_1}{z_0} - c_j\right)^{-n} y\right)_1.$$

For each positive integer $i \in [n(k-1)+1, nk]$ we define $h_i(z) = \prod_{j=1}^{k-1}(z - c_j)^n \cdot (z - c_k)^{i-n(k-1)}$. We then define sections $g_{0,i}$ and $g_{1,i}$ of V_0 and V_1 respectively as follows:

$$g_{0,i}\left(\frac{z_0}{z_1}\right) = \left(\frac{z_0}{z_1}, \phi(u'([z_0,z_1]))h\left(\frac{z_1}{z_0}\right)\right)_0$$
$$g_{1,i}\left(\frac{z_1}{z_0}\right) = \left(\frac{z_1}{z_0}, (\phi(u'([z_0,z_1]))-1)h\left(\frac{z_1}{z_0}\right)\prod_{j=1}^{a}\left(\frac{z_1}{z_0}-c_j\right)^{-n}\right)_1.$$

Note that on the intersection of $V_0 \cap V_1$ the difference between these two sections is a holomorphic (local) section.

We then define the section e_i of $(u')^*\mathscr{O}(-n)\otimes\Lambda^{0,1}$ as follows:

$$e_i([z_0:z_1]) = \begin{cases} \bar{\partial} g_{0,i} & \text{for } u'([z_0:z_1]) \neq [1:0] \\ \bar{\partial} g_{1,i} & \text{for } u'([z_0:z_1]) \neq [0:1]. \end{cases}$$

Each section e_i vanishes where $\phi\circ u'$ is constant, and the classes $[e_1],\ldots,[e_{an-1}]$ form a $\mathbb{C}$ basis of $H^{0,1}(\mathbb{P}^1,(u')^*\mathscr{O}(-n))$, so the desired result follows. □

By an argument similar to that in Lemma 3.1.2, this obstruction vector space is enough to guarantee the necessary evaluation map transversalities.

Finally, as in Lemma 17.11 in FOOO [17], we can take an average over the action of $\Gamma^+_{\mathfrak{p}\cup\vec{w}}$ to make $E_{\mathfrak{p}\cup\vec{w}}$ $\Gamma^+_{\mathfrak{p}\cup\vec{w}}$-invariant. This does not interfere with our conditions on the support of $E_{\mathfrak{p}\cup\vec{w}}$. This concludes the proof of Lemma 3.1.3.

□

For each point $\mathfrak{p}\in\mathcal{M}_{k+1}(\mathscr{O}(-n),L,\beta)$ we take the obstruction fiber $E_{\mathfrak{p}\cup\vec{w}}\subseteq L^2_{m,\delta}(K_{\mathfrak{r}\cup\vec{w}};u^*T\mathscr{O}(-n)\otimes\Lambda^{0,1})$ appearing in Lemma 3.1.3, and for each point $\psi(\mathfrak{p})\in\mathcal{M}_{k+1}(\mathscr{O},\underline{\psi}(L),\underline{\psi}_*\beta;(F_0,D_0,n))$ we choose obstruction fiber $E_{\psi(\mathfrak{p}\cup\vec{w})}=\psi_*E_{\mathfrak{p}\cup\vec{w}}\subseteq L^2_{m,\delta}(K_{\mathfrak{r}\cup\vec{w}};\psi(u)^*T\mathscr{O}\otimes\Lambda^{0,1})$. For each point

$$\mathfrak{p}'\in\mathcal{M}_{k+1}(\mathscr{O},\underline{\psi}(L),\underline{\psi}_*\beta)\setminus\mathcal{M}_{k+1}(\mathscr{O},\underline{\psi}(L),\underline{\psi}_*\beta;(F_0,D_0,n))$$

we take trivial obstruction fiber.

The only part of Definition 2.1.30 that we still need are a $\Gamma^+_{\mathfrak{p}\cup\vec{w}}$-equivariant extension $\mathrm{Triv}_{K',E_{\mathfrak{p}\cup\vec{w}}}$ of $E_{\mathfrak{p}\cup\vec{w}}$ and a $\Gamma^+_{\psi(\mathfrak{p}\cup\vec{w})}$-equivariant extension $\mathrm{Triv}_{K',E_{\psi(\mathfrak{p}\cup\vec{w})}}$ of $E_{\psi(\mathfrak{p}\cup\vec{w})}$.

For $\mathrm{Triv}_{K',E_{\psi(\mathfrak{p}\cup\vec{w})}}$, we use parallel transport along geodesics with respect to our chosen metric on $\mathscr{O}$, as is standard. For $\mathrm{Triv}_{K',E_{\mathfrak{p}\cup\vec{w}}}$, we employ Proposition 2.1.27 to pull back the extension $\mathrm{Triv}_{K',E_{\psi(\mathfrak{p}\cup\vec{w})}}$ along ψ. We have thus chosen all necessary obstruction bundle data.

For each $\psi(\mathfrak{p}) \in \mathcal{M}_{k+1}(\mathscr{O}, \underline{\psi}(L), \underline{\psi}_*\beta; (F_0, D_0, n))$ we take $\epsilon_{\psi(\mathfrak{p})}$ as in Section 2.1.5, and for each $\mathfrak{p}' \in \mathcal{M}_{k+1}(\mathscr{O}, \underline{\psi}(L), \underline{\psi}_*\beta) \setminus \mathcal{M}_{k+1}(\mathscr{O}, \underline{\psi}(L), \underline{\psi}_*\beta; (F_0, D_0, n))$ we take $\epsilon_{\mathfrak{p}'}$ sufficiently small for the neighborhood $\mathfrak{W}^+(\mathfrak{p}')$ in $\mathcal{M}_{k+1}(\mathscr{O}, \underline{\psi}(L), \underline{\psi}_*\beta)$ to not intersect $\mathcal{M}_{k+1}(\mathscr{O}, \underline{\psi}(L), \underline{\psi}_*\beta; (F_0, D_0, n))$.

For each $\psi(\mathfrak{p}) \in \mathcal{M}_{k+1}(\mathscr{O}, \underline{\psi}(L); \underline{\psi}_*\beta; (F_0, D_0, n))$, we choose the subsets

$$\mathrm{Int}\,\mathfrak{W}^0_{\psi(\mathfrak{p})} \subseteq \mathfrak{W}^0_{\psi(\mathfrak{p})} \subseteq \mathrm{Int}\,\mathfrak{W}_{\psi(\mathfrak{p})} \subseteq \mathfrak{W}_{\psi(\mathfrak{p})}$$

of $\mathfrak{W}^+_{\psi(\mathfrak{p})} \subseteq \mathcal{M}_{k+1}(\mathscr{O}, \underline{\psi}(L), \underline{\psi}_*\beta)$ as in Section 2.1.5, and we take $\mathfrak{W}^0_{\mathfrak{p}} = \psi^{-1}(\mathfrak{W}^0_{\psi(\mathfrak{p})})$ and $\mathfrak{W}_{\mathfrak{p}} = \psi^{-1}(\mathfrak{W}_{\psi(\mathfrak{p})})$. For

$$\mathfrak{p}' \in \mathcal{M}_{k+1}(\mathscr{O}, \underline{\psi}(L), \underline{\psi}_*\beta) \setminus \mathcal{M}_{k+1}(\mathscr{O}, \underline{\psi}(L); \underline{\psi}_*\beta; (F_0, D_0, n))$$

we also choose subsets $\mathfrak{W}^0_{\mathfrak{p}'}, \mathfrak{W}_{\mathfrak{p}'}$ of $\mathfrak{W}^+_{\mathfrak{p}'}$, with the important point being that these sets again do not intersect $\mathcal{M}_{k+1}(\mathscr{O}, \underline{\psi}(L); \underline{\psi}_*\beta; (F_0, D_0, n))$.

We take a finite set $\{\mathfrak{p}'_c \mid c \in \mathfrak{C}'\} \subseteq \mathcal{M}_{k+1}(\mathscr{O}, \underline{\psi}(L), \underline{\psi}_*\beta)$ such that $\bigcup_{c\in\mathfrak{C}'} \mathrm{Int}\,\mathfrak{W}^0_{\mathfrak{p}'_c} = \mathcal{M}_{k+1}(\mathscr{O}, \underline{\psi}(L), \underline{\psi}_*\beta)$. Because of our choices of $\mathfrak{W}^0_{\mathfrak{p}'}$, this gives a finite sets $\mathfrak{C} \subseteq \mathfrak{C}'$ and $\{\mathfrak{p}_c \mid c \in \mathfrak{C}\} \subseteq \mathcal{M}_{k+1}(\mathscr{O}(-n), L, \beta)$ such that $\bigcup_{c\in\mathfrak{C}} \mathrm{Int}\,\mathfrak{W}^0_{\mathfrak{p}} = \mathcal{M}_{k+1}(\mathscr{O}(-n), L, \beta)$ and $\bigcup_{c\in\mathfrak{C}} \mathrm{Int}\,\mathfrak{W}^0_{\psi(\mathfrak{p})} \supseteq \mathcal{M}_{k+1}(\mathscr{O}, \underline{\psi}(L); \underline{\psi}_*\beta; (F_0, D_0, n))$, and such that for each $\psi(\mathfrak{p})$ the set $\mathfrak{C}'(\psi(\mathfrak{p})) \subseteq \mathfrak{C}$.

We now take stabilization data (see Definition 2.1.37) at each $\mathfrak{p} \in \mathcal{M}_{k+1}(\mathscr{O}(-n), L; \beta)$ in a similar way to how we chose obstruction bundle data. That is, we choose a symmetric stabilization $\vec{w}_{\mathfrak{p}}$ of $\mathfrak{p}$ so that $u(w_{\mathfrak{p},i}) \notin F_1$ for all $w_{\mathfrak{p},i}$, and we choose the corresponding codimension 2 submanifolds $\mathcal{D}_{\mathfrak{p},i}$ of $\mathscr{O}(-n)$ to be transversal to F_1. We then take a universal family with coordinate at infinity for each component together with its extra marked points $\mathfrak{x}_v \cup \vec{w}_{\mathfrak{p}}$. By taking the same coordinate at infinity and symmetric stabilization, and taking the codimension 2 submanifolds $\psi(\mathcal{D}_{\mathfrak{p},i})$ of $\mathscr{O}$, we get a corresponding choice of stabilization data at $\psi(\mathfrak{p})$. We also choose stabilization data at $\mathfrak{p}'$ for each $\mathfrak{p}' \in \mathcal{M}_{k+1}(\mathscr{O}, \underline{\psi}(L), \underline{\psi}_* \beta) \setminus \mathcal{M}_{k+1}(\mathscr{O}, \underline{\psi}(L), \underline{\psi}_* \beta; (F_1, D_0, n))$.

We now take ϵ_0 and ϵ_0' small and consider the sets $\mathfrak{U}_{k+1,(0;\ell_{\mathfrak{p}},(\ell_c))}(\mathscr{O}(-n), L, \beta; \mathfrak{p})_{\epsilon_0, \vec{T}_0}$ and $\mathfrak{U}_{k+1,(0;\ell_{\mathfrak{p}},(\ell_c))}(\mathscr{O}, \underline{\psi}(L), \underline{\psi}_* \beta; \psi(\mathfrak{p}))_{\epsilon_0', \vec{T}_0}$ (see Definition 2.1.39). By taking ϵ_0 sufficiently small relative to ϵ_0', we get that for all

$$(\mathfrak{Y}, u', (\vec{w}_c')) \in \mathfrak{U}_{k+1,(0;\ell_{\mathfrak{p}},(\ell_c))}(\mathscr{O}(-n), L; \beta; \mathfrak{p})_{\epsilon_0, \vec{T}_0}$$

we have that $(\mathfrak{Y}, \psi \circ u', (\vec{w}_c')) \in \mathfrak{U}_{k+1,(0;\ell_{\mathfrak{p}},(\ell_c))}(\mathscr{O}, L'; \beta'; \psi(\mathfrak{p}))_{\epsilon_0', \vec{T}_0}$. We also have $\psi_* \bar{\partial} u' = \bar{\partial}(\psi \circ u')$, so, by the choices of our obstruction bundle data,
$\bar{\partial} u' \in \mathcal{E}((\mathfrak{Y}, u', (\vec{w}_c')), \Sigma_{\mathfrak{Y}})$ if and only if $\bar{\partial} \psi \circ u' \in \mathcal{E}((\mathfrak{Y}, \psi \circ u', (\vec{w}_c')), \Sigma_{\mathfrak{Y}})$. That is, we have the following map of thickened moduli spaces (see Definition 2.1.41):

$$\psi : \mathcal{M}_{k+1,(0;\ell_{\mathfrak{p}},(\ell_c))}(\mathscr{O}(-n), L; \beta; \mathfrak{p})_{\epsilon_0, \vec{T}_0} \to \mathcal{M}_{k+1,(0;\ell_{\mathfrak{p}},(\ell_c))}(\mathscr{O}, \underline{\psi}(L), \underline{\psi}_* \beta; \psi(\mathfrak{p}))_{\epsilon_0', \vec{T}_0}.$$

From our various choices of codimension 2 submanifolds $\mathcal{D}$ and $\psi(\mathcal{D})$, this then gives a map

$$\psi : \mathcal{M}_{k+1,(0;\ell_{\mathfrak{p}},(\ell_c))}(\mathscr{O}(-n), L; \beta; \mathfrak{p})^{trans}_{\epsilon_0, \vec{T}_0} \to \mathcal{M}_{k+1,(0;\ell_{\mathfrak{p}},(\ell_c))}(\mathscr{O}, L'; \beta'; \psi(\mathfrak{p}))^{trans}_{\epsilon_0', \vec{T}_0}$$

which is a smooth embedding. We take an open neighborhood $V_{\psi(\mathfrak{p})}$ of $\psi(\mathfrak{p})$ in $\mathcal{M}_{k+1,(0;\ell_{\mathfrak{p}},(\ell_c))}(\mathscr{O}, \underline{\psi}(L), \underline{\psi}_* \beta; \psi(\mathfrak{p}))_{\epsilon_0', \vec{T}_0}$ such that the set

$$V_{\psi(\mathfrak{p})} \cap \psi(\mathcal{M}_{k+1,(0;\ell_{\mathfrak{p}},(\ell_c))}(\mathscr{O}(-n), L; \beta; \mathfrak{p})^{trans}_{\epsilon_0, \vec{T}_0})$$

is closed in $V_{\psi(\mathfrak{p})}$. We then take $V_{\mathfrak{p}} = \psi^{-1}(V_{\psi(\mathfrak{p})})$.

Each $(V_{\psi(\mathfrak{p})}, \mathcal{E}(\mathfrak{p}), \Gamma_{\mathfrak{p}}, \mathrm{Triv} \circ \bar{\partial})$ is a Kuranishi neighborhood of $\psi(\mathfrak{p})$ in $\mathcal{M}_{k+1}(\mathscr{O}, \underline{\psi}(L), \underline{\psi}_* \beta)$ and $(V_{\mathfrak{p}}, \mathcal{E}(\mathfrak{p}), \Gamma_{\mathfrak{p}}, \mathrm{Triv} \circ \bar{\partial})$ is a Kuranishi neighborhood of $\mathfrak{p}$ in $\mathcal{M}_{k+1}(\mathscr{O}(-n), L, \beta)$. The natural transition maps used in FOOO [17] for the Kuranishi structures on $\mathcal{M}_{k+1}(\mathscr{O}(-n), L, \beta)$ and $\mathcal{M}_{k+1}(\mathscr{O}, \underline{\psi}(L), \underline{\psi}_* \beta)$ are compatible in the sense of Definition 2.1.4. Thus, we have constructed our first example of a Kuranishi morphism of moduli spaces:

Theorem 3.1.6. *With the Kuranishi structures constructed above, the map $\underline{\psi}$: $\mathscr{O}(-n) \to \mathscr{O}$ defined in Section 1.4 induces a Kuranishi morphism*

$$\psi : \mathcal{M}_{k+1}(\mathscr{O}(-n), L, \beta) \to \mathcal{M}_{k+1}(\mathscr{O}, \underline{\psi}(L), \underline{\psi}_* \beta)$$

in the sense of Definition 2.1.4.

Note that the virtual dimension of the target moduli space will be higher than that of the target if β includes any copies of the class D_{-n}. In particular, ψ is a smooth embedding on each Kuranishi neighborhood $V_{\mathfrak{p}}$ with image a closed submanifold of $V_{\psi(\mathfrak{p})}$. We can now take Kuranishi neighborhood $(\psi(V_{\mathfrak{p}}), \mathcal{E}(\mathfrak{p}), \Gamma_{\mathfrak{p}}, \mathrm{Triv} \circ \bar{\partial})$ of the point $\psi(\mathfrak{p})$ in the moduli space $\mathcal{M}_{k+1}(\mathscr{O}, \underline{\psi}(L), \underline{\psi}_* \beta; (F_0, D_0, n))$. With this Kuranishi structure, we have the following result:

Theorem 3.1.7. *With the Kuranishi structures constructed above, the map $\underline{\psi}$: $\mathscr{O}(-n) \to \mathscr{O}$ defined in Section 1.4 induces a Kuranishi isomorphism*

$$\psi : \mathcal{M}_{k+1}(\mathscr{O}(-n), L, \beta) \to \mathcal{M}_{k+1}(\mathscr{O}, \underline{\psi}(L), \underline{\psi}_* \beta; (F_0, D_0, n))$$

in the sense of Definition 2.1.4.

This is a specific case of Theorem 2.2.1.

3.2 Deformation of moduli space

We now apply the process described in Section 2.3 in our specific case. We will construct a stratified deformation family $\mathcal{M} \to (-\epsilon, \epsilon)$ such that the fiber over 0 is the moduli space $\mathcal{M}(\mathscr{O}, \underline{\psi}(L), \underline{\psi}_*\beta; (F_0, D_0, n))$ (with a new Kuranishi structure built by modifying the structure from Section 3.1) and such that a general fiber has Kuranishi section transversal to the 0 section, so that the underlying moduli space is smooth. In the following section, we will use these generic fibers to construct the desired A_∞-structure on $H^\bullet(\underline{\psi}(L), \Lambda)$.

Fix generic points $a_1, \ldots, a_n \in \mathbb{C}$. Let $U_c = \{|x_1| > c\} \times \mathbb{C} \subseteq \mathscr{O}$, the complement in U_1 of a closed neighborhood of the fiber F_0. We choose c so that $L \subseteq U_{3c}$. Let $\phi_1, \phi_2 : \mathbb{P}^1 \to [0,1]$ be two bump functions on the zero section of $\mathscr{O}$ both equal to 1 on $\mathscr{O} \setminus U_{2c}$ and equal to 0 on U_{3c}. We fix functions $f, g : \{1, \ldots, n\} \to \mathbb{Z}_{>0}$ with $f(j) < g(j) < f(j+1)$ for all j and we fix $p_t(z) = \prod_{j=1}^{n-1} \frac{z - t^{g(j)}}{1 - t^{g(j)} z}$. We define our diffeomorphisms $\eta_{s,t} : U_c \to U_c$ to be

$$\eta_{s,t}(x_1, y_1) =$$
$$(x_1, e^{\phi_1(x_1)\log(\prod_{j=1}^{n}((x_1 - s^{f(j)}a_j)^{-1}(x_1 - (s+t)^{f(j)}a_j)))}(y_1 - s\phi_2(x_1)p_s(x_1)) + (s+t)\phi_2(x_1)p_{s+t}(x_1))).$$

We choose s, t sufficiently small relative to c that the argument of log has positive real part for all $x_1 \geq c$, so we may choose a specific branch of log without any ambiguity arising from multiplying log by non-integer values. The choices of f, g, and p_t are made to be compatible with tropicalization, as will be discussed in Section 3.4.

These satisfy the composition rule $\eta_{s+t,t'} \circ \eta_{s,t} = \eta_{s,t+t'}$ prescribed in Section 2.3, and also have $\eta_{s,0}$ is the identity. By choosing ϕ_1, ϕ_2 generically, we also have the derivative condition given in statement 2.3.1. Furthermore, given any nodal disc map w in class $\underline{\psi}_*\beta \in H_2(\mathscr{O}, L)$ (pseudoholomorphic or otherwise) with $k+1$ boundary points such that it order of intersection with D_0 at each point is at least n times its order of intersection with F_0 at that point, we have that $(\eta_{0,t})_!(w)$ is well-defined, satisfying the tangency condition that the order of intersection with F_{ta_j} is at least equal to its order of intersection with the section $S_{t^{n+1}a_j^n}$ taking value $t^{n+1}a_j^n$.

It follows that the procedure given in Section 2.3 gives us Kuranishi structures on $\mathcal{M}_{k+1}(\mathscr{O}(-n), L, \beta)$ and $\mathcal{M}_{k+1}(\mathscr{O}, \underline{\psi}(L), \underline{\psi}_*\beta)$ so that we have a Kuranishi inclusion $\mathcal{M}_{k+1}(\mathscr{O}(-n), L, \beta) \to \mathcal{M}_{k+1}(\mathscr{O}, \underline{\psi}(L), \underline{\psi}_*\beta)$ and a stratified deformation family $\mathcal{M} \to (-\epsilon, \epsilon)$ of $\psi(\mathcal{M}_{k+1}(\mathscr{O}(-n), L, \beta))$ where the fiber over t is the moduli space

$$\mathcal{M}_{k+1}(\mathscr{O}, \underline{\psi}(L), \beta; (F_{a_1}, S_{b_1}, 1), \ldots, (F_{a_n}, S_{b_n}, 1))$$

with an appropriate Kuranishi structure.

3.3 Superpotential for $\mathscr{O}(-n)$

We now arrive at the business of defining our potential function for the Lagrangian L in $\mathscr{O}(-n)$. We will require the following three regularity results, which we present without proof:

Lemma 3.3.1. *For n generic points $q_j = (a_j, b_j) \in (\mathbb{C}^*)^2 \subseteq U_1 \subseteq \mathscr{O}$ and an effective disc class $\beta \in H_2(\mathscr{O}, \underline{\psi}(L))$ of Maslov index less than or equal to $(\beta \cdot F_0)n$, we have that the bulk-deformed moduli space*

$$\mathcal{M}_{k+1,(\beta\cdot F_0)n}(\mathscr{O}, \underline{\psi}(L), \beta; \underbrace{q_1, \ldots, q_1}_{\beta\cdot F_0 \text{ times}}, \ldots, \underbrace{q_n, \ldots, q_n}_{\beta\cdot F_0 \text{ times}})$$

is empty. For generic points q_j, it then follows that the following space is also empty:

$$\mathcal{M}_{k+1}(\mathscr{O}, \underline{\psi}(L), \beta; (F_{a_1}, S_{b_1}, 1), \ldots, (F_{a_n}, S_{b_n}, 1)).$$

Lemma 3.3.2. *For n generic points $q_j = (a_j, b_j) \in (\mathbb{C}^*)^2 \subseteq U_1 \subseteq \mathscr{O}$ and effective disc class $\beta \in H_2(\mathscr{O}, \underline{\psi}(L))$ of Maslov index $2 + (\beta \cdot F_0)n$, we have that the bulk-deformed* *moduli space*

$$\mathcal{M}_{1,(\beta\cdot F_0)n}(\mathscr{O},\underline{\psi}(L),\beta;\underbrace{q_1,\dots,q_1}_{\beta\cdot F_0\ times},\dots,\underbrace{q_n,\dots,q_n}_{\beta\cdot F_0\ times})$$

is a smooth dimension 0 manifold and is a $((\beta\cdot F_0)!\cdot n)$*-fold unramified covering of*

$$\mathcal{M}_1(\mathscr{O},\underline{\psi}(L),\beta;(F_{a_1},S_{b_1},1),\dots,(F_{a_n},S_{b_n},1)),$$

which is thus also a smooth dimension 0 manifold.

Lemma 3.3.3. *For* n *generic points* $q_j=(a_j,b_j)\in(\mathbb{C}^*)^2\subseteq U_1\subseteq\mathscr{O}$ *and effective disc class* $\beta\in H_2(\mathscr{O},\underline{\psi}(L))$ *of Maslov index strictly greater than* $2+(\beta\cdot F_0)n$*, we have that the bulk-deformed moduli space*

$$\mathcal{M}_{k+1,(\beta\cdot F_0)n}(\mathscr{O},\underline{\psi}(L),\beta;\underbrace{q_1,\dots,q_1}_{\beta\cdot F_0\ times},\dots,\underbrace{q_n,\dots,q_n}_{\beta\cdot F_0\ times})$$

is a smooth manifold with boundary and corners and is a $((\beta\cdot F_0)!\cdot n)$*-fold unramified covering of*

$$\mathcal{M}_{k+1}(\mathscr{O},\underline{\psi}(L),\beta;(F_{a_1},S_{b_1},1),\dots,(F_{a_n},S_{b_n},1)),$$

which is thus also a smooth manifold with boundary and corners.

There are no sphere bubbles, so its boundary and corners are all isomorphic to fiber products of the form

$$\mathcal{M}_{k_1+1}(\beta_1)\,{}_{\mathrm{ev}_{k_1+1}}\times_{\mathrm{ev}_0}\mathcal{M}_{k_2+1}(\beta_1)$$

where $k_1+k_2=k$*, and* $\beta_1+\beta_2=\beta$*, and*

$$\mathcal{M}_{k'+1}(\beta_j)=\mathcal{M}_{k'+1}(\mathscr{O},\underline{\psi}(L),\beta_j;(F_{a_1},S_{b_1},1),\dots,(F_{a_n},S_{b_n},1)).$$

With these regularity results in hand, the stratified deformation families constructed in Section 3.2 are in fact stratified deformation regularizations, in the sense that all fibers except the central fiber have transversal Kuranishi section without perturbation. The following proposition is then a straightforward consequence of standard Floer theoretic arguments.

Proposition 3.3.4. *We fix n generic points $q_j = (a_j, b_j) \in (\mathbb{C}^*)^2 \subseteq U_1 \subseteq \mathscr{O}$. The moduli spaces*

$$\mathcal{M}_{k+1}(\mathscr{O}, \underline{\psi}(L), \underline{\psi}_* \beta; (F_{a_1}, S_{b_1}, 1), \ldots, (F_{a_n}, S_{b_n}, 1))$$

for $\beta \in H_2(\mathscr{O}(-n), L)$, taken with trivial Kuranishi structure (as they are all already manifolds with boundary and corner) define an A_∞-structure on $H^*(L; \Lambda_0)$ as in Section 1.2.

We again have that $H^1(L; \Lambda_+)$ is the weak Maurer-Cartan space, so our A_∞-structure defines our Lagrangian potential function

$$W : H^1(L; \Lambda_+) \to \Lambda_+$$

for $L \subseteq \mathscr{O}(-n)$.

3.4 Conjectural calculation

The remainder of this chapter is devoted to discussing the calculation of these superpotentials, which will be worked out rigorously in upcoming work. Essentially, we can realize the terms of our superpotential as terms of a related bulk-deformed superpotential, which we expect to be calculable using tropical scattering diagrams.

3.4.1 Scattering diagram and wall crossing

We now turn to the problem of explicitly calculating this potential function. This will require the introduction of a new A_∞-structure on $H^*(\underline{\psi}(L); \Lambda_0)$. This is because the moduli spaces used to define the A_∞-structure on $H^*(L; \Lambda_0)$ in Proposition 3.3.4 do not behave appropriately under wall crossing. Intuitively, this is because the A_∞-structure in Proposition 3.3.4 really only knows the information contained in L as a Lagrangian in $\mathscr{O}(-n)$, and only one chamber of the scattering diagram in $\mathscr{O}$ that we will end up studying corresponds to this $\mathscr{O}(-n)$ information.

The A_∞-structure we consider will be the bulk-deformed A_∞-structure deformed by the point insertions q_j. The regularity results of the previous section then imply that the terms of our superpotential for $\mathscr{O}(-n)$ will all be terms of the associated bulk-deformed superpotential. There is a natural Floer theoretic scattering diagram associated with this bulk-deformation. This scattering diagram was studied for toric Fano surfaces by Hong-Lin-Zhao [32], using nilsquared coefficients to avoid contributions from repeated point insertions. Their scattering diagram can then be understood tropically and used to calculate the associated superpotential, which they call the n-th order bulk-deformed superpotential, for n the number of point insertions. We expect that a similar procedure is possible for the full bulk-deformed scattering diagram and superpotential, which we intend to prove in future work.

3.4.2 Tropical geometry

Recall that the modified SYZ conjecture of Kontsevich-Soibelman [36] predicts that, near the large complex structure limit, Calabi-Yau manifolds should collapse to integral affine manifolds with singularities. It is also expected that holomorphic curves in a Calabi-Yau manifold should collapse to affine 1-skeletons in the integral affine manifold. This brings us to the world of tropical geometry. Specifically, we will be considering tropical geometry as it appears in the Gross-Siebert program [30], inspired by work of Mikhalkin [39], which can be thought of as an algebraic analogue of the SYZ approach to mirror symmetry, as well as as a powerful combinatorial calculational tool, which is the primary way in which we will be using it.

Let $N \cong \mathbb{Z}^2$ be a lattice with dual lattice $M := \mathrm{Hom}_{\mathbb{Z}}(N, \mathbb{Z})$, and let $N_{\mathbb{R}} := N \otimes_{\mathbb{Z}} \mathbb{R}$ and $M_{\mathbb{R}} := M \otimes_{\mathbb{Z}} \mathbb{R}$. Let $\Sigma \subseteq M_{\mathbb{R}}$ be the toric fan of a toric surface Y. We let $\Sigma^{[1]}$ denote the set of 1-cones of Σ.

Definition 3.4.1. *A parametrized tropical disc of Y with stop at p in $M_{\mathbb{R}}$ is a triple (T, w, h) where*

1. *T is a rooted tree with root vertex x and non-compact edges (i.e. edges with only one adjacent vertex). The root x (which is univalent) is the only vertex that is not trivalent. In particular, x is the only leaf of the tree.*

2. *w is an assignment of a positive integer weight to each edge. We assume that each unbounded edge is assigned the weight 1.*

3. *$h : T \to M_{\mathbb{R}}$ is a proper map with $h(x) = p$ that is also an affine embedding on each edge. For each non-compact edge e, the affine ray $h(e)$ is of the form $m' + \mathbb{R}_{\geq 0} m$ for some $m' \in M_{\mathbb{R}}$ and $m \in M$ a primitive generator of a 1-cone of Σ. At each trivalent vertex, the following balancing condition must hold:*

$$w(e_1)v_(e_1) + w(e_2)v_(e_2) + w(e_3)v_(e_3) = 0.$$

Here $v(e_i)$ is the primitive vector tangent to $h(e_i)$ pointing away from the vertex.

We will often refer to the triple (T, w, h) simply as h.

We refer to the image of a parametrized tropical disc as a tropical disc.

We next define the degree of a tropical disc.

Definition 3.4.2. *The degree of a tropical disc (T, w, h) of Y is an element of $\mathbb{Z}_{\geq 0}^{\Sigma^{[1]}}$, where the value of the ρ coordinate for each $\rho \in \Sigma^{[1]}$ is the number of unbounded edges of (T, w, h) in the ρ direction.*

This degree is analogous to the intersection numbers of a holomorphic disc in Y with the components of the toric boundary. Fixing a moment fiber L, the homology class in $H_2(Y, L)$ of the holomorphic disc is uniquely determined by these intersection numbers. Likewise, for a fixed stop p, the degree of a tropical disc with stop at p functions like a homology class of the disc.

Definition 3.4.3. *We define the Maslov index $MI(h)$ of a parametrized tropical disc (T, w, h) to be twice the sum of the weights of its unbounded edges.*

After establishing more directly the relationship between tropical discs and holomorphic discs in Y with boundary on a moment fiber Lagrangian, this definition of the Maslov index of a tropical disc will correspond to Lemma 3.1 in Auroux [4] or Theorem 5.1 in Cho-Oh [9]. Note that the Maslov index is determined by the degree of h, so if Δ is the degree of h we can define $MI(\Delta)$ to be $MI(h)$. This is consistent with the fact that the Maslov index of a disc with boundary on a fixed Lagrangian L depends only on its homology class. Furthermore, the moduli space of tropical discs of degree Δ with stop at a point p has real dimension $MI(\Delta) - 2$, which equals the virtual dimension of the fiber of $ev_0 : \mathcal{M}_1(Y, L, \beta) \to L$ for $MI(\beta) = MI(\Delta)$.

We will later need to study tropical discs subject to point constraints, so we define the generalized Maslov index of a tropical disc.

Definition 3.4.4. *Given ℓ generic fixed point constraints $q_1, \ldots, q_\ell \in M_\mathbb{R}$ and a tropical disc h with every q_i contained in the image of h, we define the generalized Maslov index $GMI(h) := MI(h) - 2\ell$.*

This aligns with the generalized Maslov index of a holomorphic disc as defined in Hong-Lin-Zhao [32]. The moduli space of tropical discs of degree Δ subject to the generic point constraints $q_1, \ldots, q_\ell$ has dimension $GMI(\Delta) - 2$ if it is non-empty. We will thus be interested in counting generalized Maslov index 2 tropical discs, as these belong to a discrete moduli space. We will need to count them with an appropriate weight, defined as follows:

Definition 3.4.5. *For each trivalent vertex v with adjacent edges e_1, e_2, e_3 of a tropical disc (T, w, h), the Mikhalkin weight at v is*

$$Mult_v := w(e_1)w(e_2)|v(e_1) \wedge v(e_2)| \in \mathbb{Z}_{\geq 0}.$$

Because of the balancing condition of tropical discs, this value is independent of the labeling of the three edges.

The Mikhalkin weight $Mult(h)$ of the disc is the product of the Mikhalkin weights of its trivalent vertices.

Example 3.4.6. *Let $Y = \mathbb{P}^1 \times \mathbb{P}^1$ with complete toric fan Σ determined by the 1-cones generated by $(1,0), (0,1), (-1,0), (0,-1) \in M$. We fix three generic points*

$q_1, q_2, q_3 \in M_{\mathbb{R}}$. Three tropical discs with stop at p all of generalized Maslov index 2 are depicted in Figure 3·1. All edges have weight 1, and all three tropical discs have Mikhalkin weight 1.

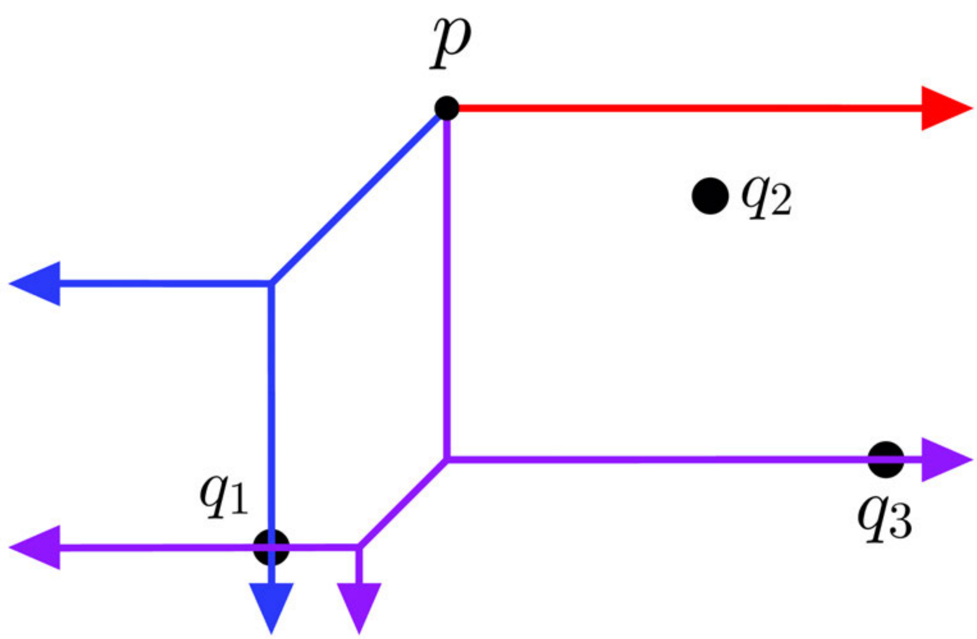

Figure 3·1: Three tropical discs of generalized Maslov index 2 with stop at p.

We turn now to tropical scattering diagrams, as defined by Gross-Pandharipande-Siebert [29]. As one might expect based on the name, these will be closely related to the Lagrangian Floer scattering diagrmams of Section 1.2. Let R be a complete local $\mathbb{C}$-algebra and let $\mathfrak{m}_R$ be the unique maximal ideal of R. Our primary example will be $R = \mathbb{C}[[t_1, \ldots, t_\ell]]$.

Definition 3.4.7. *A tropical scattering diagram $\mathfrak{D}$ is a collection of walls $\{(\mathfrak{d}, f_\mathfrak{d})\}$ where*

- *$\mathfrak{d} \subseteq M_{\mathbb{R}}$ is either a ray of the form $\mathfrak{d} = m'_\mathfrak{d} + \mathbb{R}_{\geq 0} m_\mathfrak{d}$ or a line of the form $\mathfrak{d} = m'_\mathfrak{d} + \mathbb{R} m_\mathfrak{d}$, for some $m'_\mathfrak{d} \in M_{\mathbb{R}}$ and $m_\mathfrak{d} \in M \setminus \{0\}$. The set $\mathfrak{d}$ is called the support of the line or ray.*
- *$f_\mathfrak{d} \in \mathbb{C}[[z^{m_\mathfrak{d}}]] \hat{\otimes}_{\mathbb{C}} R \subseteq \mathbb{C}[M] \hat{\otimes}_{\mathbb{C}} R$, called wall functions, satisfy $f_\mathfrak{d} \equiv 1 \mod z^{m_\mathfrak{d}} \mathfrak{m}_R$.*

such that for every power $k > 0$, there are only a finite number of $(\mathfrak{d}, f_{\mathfrak{d}})$ with $f_{\mathfrak{d}} \not\equiv 1$ mod $\mathfrak{m}_R^k$.

Given a wall $\mathfrak{d}$ and a path ϕ crossing this wall (once, transversally), there is an associated automorphism

$$\theta_{\phi,\mathfrak{d}}(z^m) = z^m f_{\mathfrak{d}}^{\langle n_0, m\rangle},$$

where $n_0 \in N$ is primitive normal to $\mathfrak{d}$, and $\langle n_0, \phi'(t_0)\rangle > 0$, where t_0 is the moment ϕ crosses the wall $\mathfrak{d}$.

Given a scattering diagram $\mathfrak{D}$, we write

$$\mathrm{Sing}(\mathfrak{D}) = \bigcup_{\mathfrak{d}\in\mathfrak{D}} \{m'_{\mathfrak{d}}\} \cup \bigcup_{\mathfrak{d}_1,\mathfrak{d}_2 \dim \mathfrak{d}_1\cap\mathfrak{d}_2=0} \mathfrak{d}_1 \cap \mathfrak{d}_2,$$

and we consider a smooth immersion $\phi : [0,1] \to M_{\mathbb{R}} \setminus \mathrm{Sing}(\mathfrak{D})$ such that endpoints lie outside of the support of the scattering diagram $\mathfrak{D}$ and such that ϕ is transversal to the support of the scattering diagram.

Now, for each power $k > 0$, ϕ will cross only a finite number s_k of walls with $f_{\mathfrak{d}} \not\equiv 1$ mod $\mathfrak{m}_R^k$. We label them by $\mathfrak{d}_j$, where $j = 1, \ldots s_k$ with respect to the order of the path intersecting the walls. The automorphism $\theta^k_{\phi,\mathfrak{D}} := \theta_{\mathfrak{d}_{s_k}} \circ \cdots \circ \theta_{\mathfrak{d}_1}$ is well-defined, so we can define the total wall-crossing automorphism as

$$\theta_{\phi,\mathfrak{D}} = \lim_{k\to\infty} \theta^k_{\phi,\mathfrak{D}}.$$

We then have the following theorem, due in various forms to Kontsevich-Soibelman [36] and Gross-Siebert [30].

Theorem 3.4.8. *Let $\mathfrak{D}'$ be a scattering diagram. Then there exists a scattering diagram $\mathfrak{D}$ containing $\mathfrak{D}'$ such that $\mathfrak{D}\setminus\mathfrak{D}'$ consists only of rays, and such that $\theta_{\phi,\mathfrak{D}} = \mathrm{Id}$ for any closed loop ϕ for which $\theta_{\phi,\mathfrak{D}}$ is defined. After combining $(\mathfrak{d}, f_{\mathfrak{d}}), (\mathfrak{d}', f_{\mathfrak{d}'})$ into $(\mathfrak{d}, f_{\mathfrak{d}} f_{\mathfrak{d}'})$ if $\mathfrak{d} = \mathfrak{d}'$, the resulting $\mathfrak{D}$ is unique.*

A scattering diagram with this property of having $\theta_{\phi,\mathfrak{D}}$ depend only on the homotopy class of ϕ is called a *consistent scattering diagram.*

The scattering diagrams we will be interested in are variants of the following diagrams due to Hong-Lin-Zhao [32]. Let $u \in \mathbb{R}^2$ be a point of the discrete Legendre transform of the moment polytope of Y and let L_u be the corresponding moment fiber Lagrangian. We will again be using nilsquared coefficients in the ring $\mathbb{C}[t_1, \ldots, t_\ell]/(t_1^2, \ldots, t_\ell^2)$.

Definition 3.4.9 (HLZ [32] Definition 3.10)**.** *Given generic* $q_1, \ldots, q_\ell, u \in \mathbb{R}^2$*, the bulk-deformed tropical superpotential* $W_\ell^{trop}(u)$ *is defined as*

$$W_\ell^{trop}(u) = \sum_h Mult(h) z^{\partial[h]} t_h$$

where $t_h = \prod_{q_j \in h} t_j$ *and the summation is over all (rigid) generalized Maslov index two tropical discs ending at* u*.*

We then define our scattering diagram inductively by considering the bulk-deformed tropical superpotential $W_{\ell-1}^{trop}(q_j)$ at each point with respect to the remaining $\ell - 1$ points q_i and taking initial walls $q_j + (-\partial[h])\mathbb{R}_{\geq 0}$ with function $(1 + \mathrm{Mult}(h) z^{\partial[h]} t_h)$ for each term of $W_{\ell-1}^{trop}(q_j)$. We then take the completion $\mathfrak{D}$ of this scattering diagram as above to one such that the function $\theta_{\phi,\mathfrak{D}}$ is the identity for any contractible loop in $\mathbb{R}^2 \setminus \{q_1, \ldots, q_\ell\}$.

Our holomorphic discs and Lagrangian Floer scattering diagram are related to the corresponding tropical objects via tropicalization. For our purposes, this is accomplished using the map

$$\begin{aligned} \mathrm{Log}_{t^{-1}} : (\mathbb{C}^*)^2 &\to \mathbb{R}^2 \\ (x, y) &\mapsto (\log_{t^{-1}} |x|, \log_{t^{-1}} |y|). \end{aligned}$$

Taking the limit $t \to 0$, the image of any holomorphic curve or disc converges to a tropical curve or disc. This leads to the following result:

Theorem 3.4.10 (HLZ [32] Theorem 5.10)**.** *Let Y be a toric Fano surface and let $p_1^t, \ldots, p_\ell^t \in (\mathbb{C}^*)^2 \subseteq X$ be points with $Log_{t^{-1}}(p_j^t) = q_j$. Then $W_\ell(u) = W_\ell^{trop}(u)$ if $u \in \mathcal{U}_t$ and $t << 1$.*

That is, sending the bulk-insertions in Y to the boundary as $t \to 0$, the tropical scattering diagram and superpotential recover their holomorphic counterparts.

We now recall the notion of the broken line, which are used for calculating tropical superpotentials.

Definition 3.4.11. *Let $\mathfrak{D}$ be a consistent scattering diagram on $M_\mathbb{R}$. A broken line with stop at $u \in M_\mathbb{R}$ is a continuous map*

$$\mathfrak{b} : (-\infty, 0] \to M_\mathbb{R}$$

such that $\mathfrak{b}(0) = u$ and with the below properties: there exist

$$-\infty = t_0 < t_1 < \cdots < t_n = 0$$

such that $\mathfrak{b}(t_i) \in Supp(\mathfrak{D}) \backslash Sing(\mathfrak{D})$, and such that $\mathfrak{b}|_{[t_{i-1}, t_i]}$ is affine with rational direction $\beta'(t)$ positively proportional to some primitive $m_i \in M$. For each $i = 1, \cdots, n$, we have a decoration $c_i z^{d_i m_i}$, with $d_i \in \mathbb{Z}_{>0}$, such that

1. *$c_1 = 1$, and $d_1 = 1$, and m_1 is a generator of a 1-cone of Σ.*
2. *If $\mathfrak{b}(t_i) \in \bigcap_j \mathfrak{d}_j$ for some collection of walls $\mathfrak{d}_j$, then $c_{i+1} z^{d_{i+1} m_{i+1}}$ is a term in*

$$\Big(\prod_i (\theta_{\mathfrak{b}, \mathfrak{d}_j})^{\epsilon_j}\Big)(c_i z^{d_i m_i}), \tag{3.4.1}$$

   where $\epsilon_j = sgn\langle m_i, \gamma_{\mathfrak{d}_j} \rangle$.

The following proposition of Gross [27] gives the connection between broken lines and generalized Maslov index 2 tropical discs.

Proposition 3.4.12 (Proposition 5.32 Gross [27])**.** *For generic u lying outside the support of our scattering diagram, there is a one-to-one correspondence between broken lines with endpoint u and generalized Maslov index two tropical discs ending at u.*

Furthermore, if the monomial associated with the last segment of a broken line corresponding to a disc h is cz^m, then this is the term of the superpotential corresponding to h.

3.4.3 Superpotential for $\mathscr{O}(-n)$

To calculate the desired potentials, we must extend the work of Hong-Lin-Zhao [32] to allow for contributions from repeated point insertions. In this section we give a conjectural description of the appropriate extension, to be treated completely in future work.

One natural idea is to simply relax the condition that $t_j^2 = 0$, that is to use the ring $\mathbb{C}[t_1, \ldots, t_\ell]$ instead of $\mathbb{C}[t_1, \ldots, t_\ell]/(t_1^2, \ldots, t_\ell^2)$, and construct a scattering diagram as in HLZ. However, the resulting tropical superpotential is not in general constant on chambers of the scattering diagram.

It seems that the appropriate solution will be to modify the notion of generalized Maslov index for tropical discs to accommodate discs sending a vertex to a point insertion. However, it will require more detailed Floer theoretic calculations to determine what precisely the appropriate modification will be. For instance, the most straightforward modification would be to treat a vertex mapping to a point insertion as reducing the generalized Maslov index by 4. This would then lead to a form of "internal scattering," wherein the associated scattering diagram would have initial walls of the form $q_j + (-\sum_m \partial[h_m])\mathbb{R}_{\geq 0}$ with function $(1 + (\prod_m \mathrm{Mult}(h_m))\, z^{\sum_m \partial[h_m]} t_h)$, where h_m ranges over any finite list of terms from $W_{\ell-1}^{trop}(q_j)$. This has the disadvantage that, even when $\ell = 1$, the scattering diagram has infinitely many initial walls, the support of which is dense in $M_{\mathbb{R}}$. However, for each fixed degree k, modding out by $(t_1^k, \ldots, t_\ell^k)$ produces a finite scattering diagram where the potential is constant on each chamber. This construction gives, in some sense, a maximal extension of the HLZ tropical bulk-deformed potential, which is tropically very natural but unlikely to correspond to a relevant Floer theoretic scattering diagram. We conjecture that the correct extension of the tropical bulk-deformed potential will be a weaker extension, in the sense of adding strictly fewer initial walls.

Furthermore, inspired by the generic regularity lemmas of Section 3.3, we conjecture that only tropical discs not mapping a vertex to a point insertion will contribute to the count matching our superpotential for $\mathscr{O}(-n)$. Combining these conjectures with our choice of deformation regularization and associated tropicalization of the bulk insertions, we are able to carry out some expected calculations, to be made rigorous in later work.

3.4.4 Examples of conjectured calculations

For each $\mathscr{O}(-n)$, we let $\beta_1 \in H_2(\mathscr{O}(-n), L)$ be the class of the unique Maslov index 2 holomorphic disc with boundary on L intersecting the fiber defined by $x_0 = 0$ in U_0, we let $\beta_2 \in H_2(\mathscr{O}(-n), L)$ be the class of the unique Maslov index 2 holomorphic disc with boundary on L intersecting D_{-n}, and we let $\beta_3 \in H_2(\mathscr{O}(-n), L)$ be the class of the unique Maslov index 2 holomorphic disc with boundary on L intersecting the fiber F_0 defined by $x_1 = 0$ in U_1. To compare the superpotentials we find to known superpotentials for Hirzebruch surfaces, we let $\beta_4 \in H_2(\mathbb{F}_n, L)$ be the class of the unique Maslov index 2 holomorphic disc with boundary on L intersecting the ∞-section of $\mathscr{O}(-n)$.

Superpotential of $\mathscr{O}(-1)$

The simplest non-trivial example of our procedure is the case where $n = 1$ and we are attempting to define and calculate a superpotential for $\mathscr{O}(-1)$. Since the compactification $\mathbb{F}_1$ is Fano, this superpotential can be defined without any sort of perturbation, so we should hope that our procedure produces a modification of this standard superpotential, as given by Hori-Vafa [33] and Cho-Oh [9]. Specifically, the superpotential for $\mathbb{F}_1$ is

$$W_{\mathbb{F}_1} = T^{\omega(\beta_1)}x + T^{\omega(\beta_2)}y + T^{\omega(\beta_3)}\frac{y}{x} + T^{\omega(\beta_4)}\frac{1}{y}.$$

We consider the scattering diagram defined in Section 3.4.3, which in this instance is determined by a single point. We expect this to correspond appropriately to the Floer theoretic scattering diagram without the addition of any further walls. This scattering diagram is pictured on the left in Figure 3·2. To calculate the superpotential, we consider the broken lines with stop in the top left chamber of the scattering diagram, pictured on the right in Figure 3·2. The choice of chamber is determined by our deformation regularization. Only the red broken lines contribute. The superpotential is thus

$$W_{\mathscr{O}(-1)} = T^{\omega(\beta_1)}x + T^{\omega(\beta_2)}y + T^{\omega(\beta_3)}\frac{y}{x}.$$

This is precisely $W_{\mathbb{F}_1}$ minus the term $T^{\omega(\beta_4)}\frac{1}{y}$ corresponding to the disc class β_4, as we would hope.

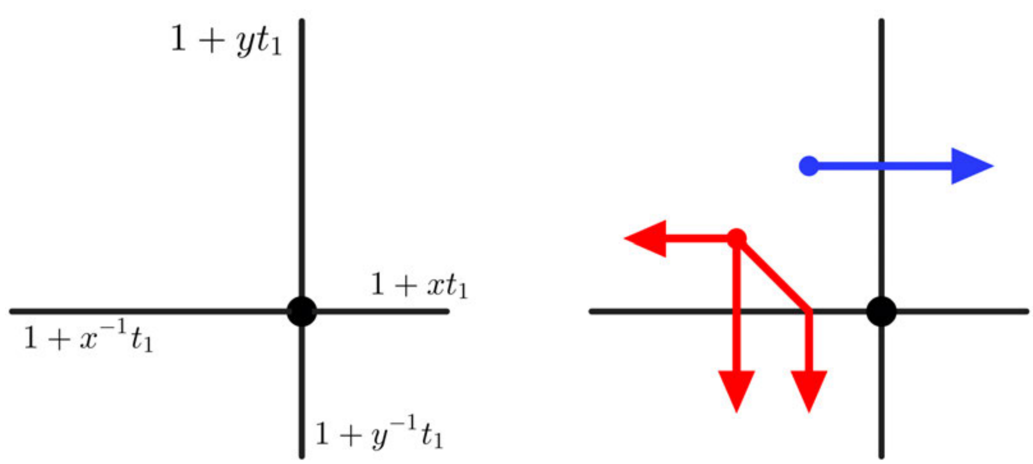

Figure 3·2: On the left is the scattering diagram associated with a single point insertion in $\mathbb{P}^1 \times \mathbb{P}^1$. On the right are the broken lines with stop in the chamber of interest (top left). Those in red contribute to the superpotential for $\mathscr{O}(-1)$.

Superpotential of $\mathscr{O}(-2)$

The next simplest case is $\mathscr{O}(-2)$. The compactification $\mathbb{F}_2$ is semi-Fano, and its superpotential has been calculated in a number of ways, see Chan-Lau [8], Auroux [5], and FOOO [21], all arriving at consistent answers. In our coordinates, their superpotential is

$$W_{\mathbb{F}_2} = T^{\omega(\beta_1)}x + \left(T^{\omega(\beta_2)} + T^{\omega(\beta_2+D_{-2})}\right)y + T^{\omega(\beta_3)}\frac{y^2}{x} + T^{\omega(\beta_4)}\frac{1}{y}.$$

To calculate $W_{\mathscr{O}(-2)}$, we consider the scattering diagram defined in Section 3.4.3, determined by two points. Again, we expect this to correspond appropriately to the Floer theoretic scattering diagram without the addition of any further walls. This scattering diagram (with a particular choice of locations of the points) is pictured in Figure 3·3. We consider the broken lines with stop in the top left chamber of the scattering diagram, see Figure 3·4. Again, the choice of chamber is determined by our deformation regularization, and only the red broken lines contribute. The superpotential is thus

$$W_{\mathscr{O}(-2)} = T^{\omega(\beta_1)}x + \left(T^{\omega(\beta_2)} + T^{\omega(\beta_2+D_{-2})}\right)y + T^{\omega(\beta_3)}\frac{y^2}{x},$$

which is $W_{\mathbb{F}_2}$ without the $T^{\omega(\beta_4)}\frac{1}{y}$ term, as expected.

Superpotential of $\mathscr{O}(-3)$

We next consider $\mathscr{O}(-3)$. To the best of my knowledge, its compactification $\mathbb{F}_3$ is the only surface that is neither Fano nor semi-Fano for which a superpotential has been explicitly calculated in the literature. This was found by Auroux [5] by deforming the complex structure of $\mathbb{F}_3$ to that of $\mathbb{F}_1$, while simultaneously deforming the special Lagrangian fibration and volume form. The problem then becomes one of explicit wall-crossing calculation in $\mathbb{F}_1$, which is Fano. The superpotential he found is, in our

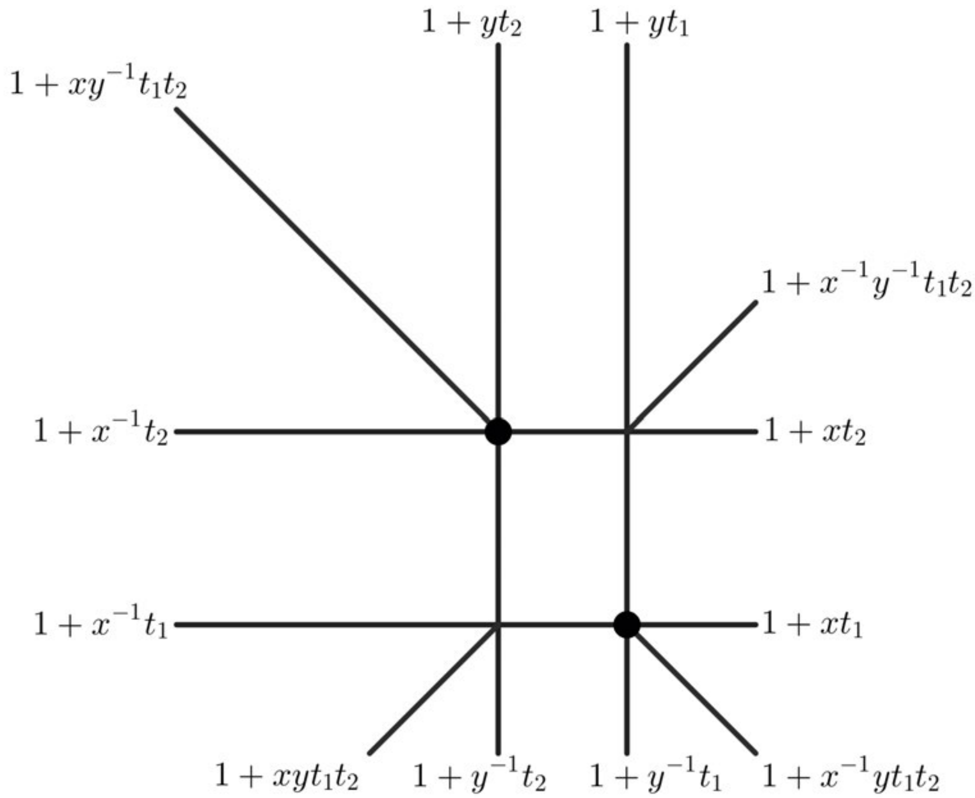

Figure 3·3: A scattering diagram associated with two points in $\mathbb{P}^1 \times \mathbb{P}^1$.

coordinates,

$$W_{\mathbb{F}_3} = T^{\omega(\beta_1)}x + 2T^{\omega(2\beta_2+D_{-3})}y^2 + T^{\omega(\beta_2+\beta_3+D_{-3})}\frac{y^4}{x} + T^{\omega(\beta_3)}\frac{y^3}{x} + T^{\omega(\beta_4)}\frac{1}{y}.$$

As Auroux points out, this superpotential is asymmetrical in β_1, β_3 on account of the coefficient $T^{\omega(\beta_2+\beta_3+D_{-3})}$. This asymmetry comes directly from the same asymmetry in the deformation procedure, and a different choice of deformation yields the superpotential

$$W_{\mathbb{F}_3} = T^{\omega(\beta_1)}x + 2T^{\omega(2\beta_2+D_{-3})}y^2 + T^{\omega(\beta_2+\beta_1+D_{-3})}\frac{y^4}{x} + T^{\omega(\beta_3)}\frac{y^3}{x} + T^{\omega(\beta_4)}\frac{1}{y}.$$

This superpotential only differs from the first by a change of toric fan exchanging the directions corresponding to β_1 and β_3.

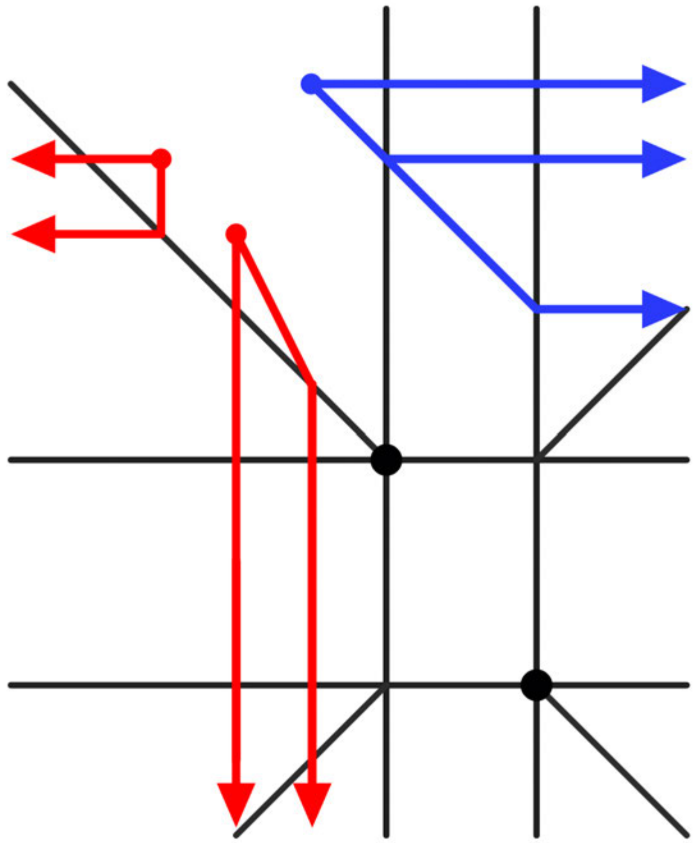

Figure 3·4: The broken lines with stop in the chamber of interest (top left). Those in red contribute to the superpotential for $\mathscr{O}(-2)$.

We consider the scattering diagram defined in Section 3.4.3, determined by three points, as is pictured in Figure 3·5(with a particular choice of locations of the points). We expect that this scattering diagram is "missing" initial walls, but that the superpotential we find is still correct. We consider the broken lines with stop in the top left chamber of the scattering diagram, arbitrarily close to the wall with direction $(-1, 2)$. This position is again given by our choice of deformation regularization. We only picture those broken lines expected to contribute to the superpotential, which is thus

$$W_{\mathscr{O}(-3)} = T^{\omega(\beta_1)}x + 2T^{\omega(2\beta_2+D_{-3})}y^2 + T^{\omega(\beta_2+\beta_3+D_{-3})}\frac{y^4}{x} + T^{\omega(\beta_3)}\frac{y^3}{x}.$$

This is again what we would expect given the known superpotential $W_{\mathbb{F}_3}$. The same asymmetry between β_1 and β_3 is present, resulting from the asymmetry between U_0 and U_1 in our construction. Exchanging their roles, we can arrive at the superpotential

$$W_{\mathscr{O}(-3)} = T^{\omega(\beta_1)}x + 2T^{\omega(2\beta_2+D_{-3})}y^2 + T^{\omega(\beta_2+\beta_3+D_{-3})}\frac{y^4}{x} + T^{\omega(\beta_3)}\frac{y^3}{x}.$$

Superpotential of $\mathscr{O}(-4)$

We consider the scattering diagram defined in Section 3.4.3, determined by four points, as is pictured in Figure 3·6 (with a particular choice of locations of the points). We expect that this scattering diagram is "missing" initial walls, but that the superpotential we find is still correct. We consider the broken lines with stop in the top left chamber of the scattering diagram, arbitrarily close to the wall with direction $(-1, 3)$. This position is again given by our choice of deformation regularization. We only picture those broken lines expected to contribute to the superpotential, which is thus

$$W_{\mathscr{O}(-4)} = T^{\omega(\beta_1)}x + 3T^{\omega(3\beta_2+D_{-4})}y^3 + 3T^{\omega(2\beta_2+2\beta_3+D_{-4})}\frac{y^6}{x} + T^{\omega(\beta_2+3\beta_3+D_{-4})}\frac{y^9}{x^2} + T^{\omega(\beta_3)}\frac{y^4}{x}.$$

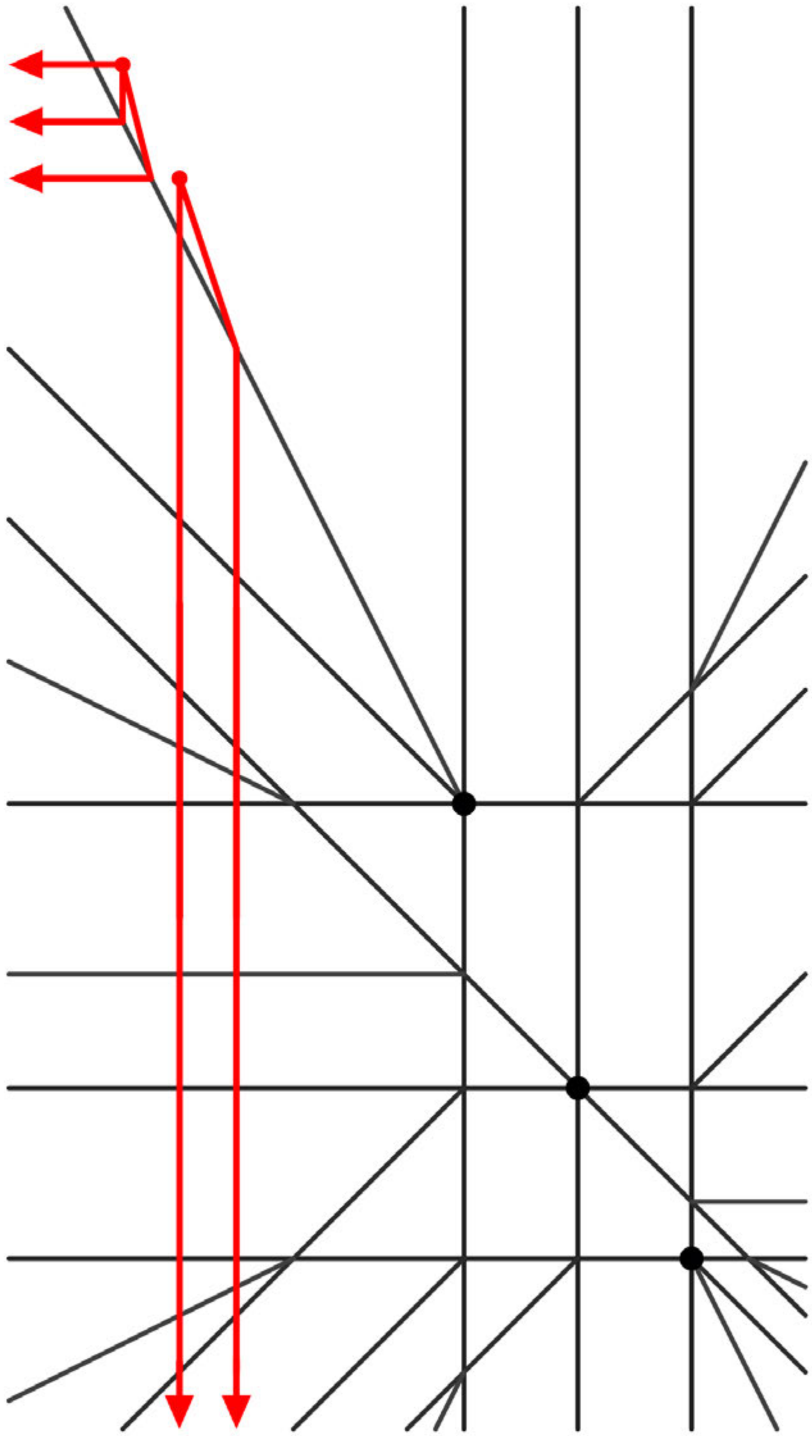

Figure 3·5: A scattering diagram associated with three points in $\mathbb{P}^1 \times \mathbb{P}^1$. Wall functions are no longer labeled. The broken lines in red are expected to be those contributing to $\mathscr{O}(-3)$.

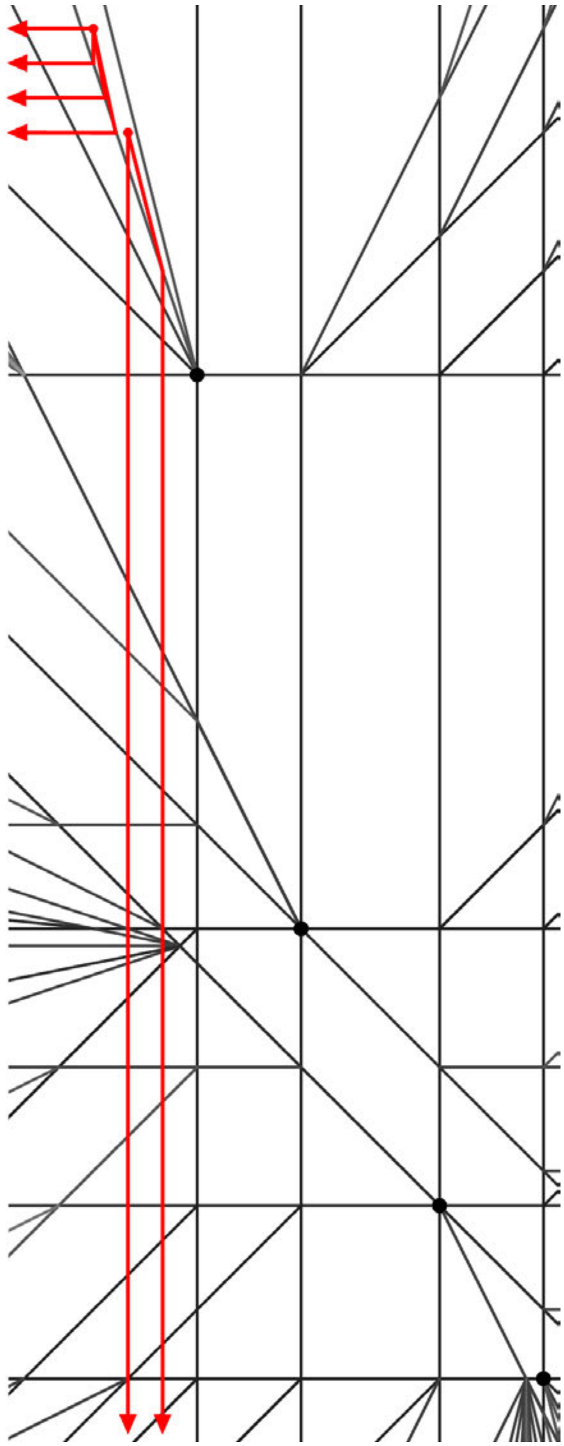

Figure 3·6: A scattering diagram associated with four points in $\mathbb{P}^1 \times \mathbb{P}^1$. Wall functions are not labeled. The broken lines in red are expected to be those contributing to $\mathscr{O}(-4)$.

Chapter 4

Discussion

4.1 Summary of results

We present here a summary of the main results from this dissertation.

Let X and Y be birational smooth Kähler varieties with relatively spin Lagrangians L_X and L_Y respectively, together with a birational holomorphic map $\underline{\psi}: X \to Y$ that maps L_X diffeomorphically onto L_Y. Let $\beta \in H_2(X, L_X)$ be an effective disc class such that for all nodal discs $u \in \beta$, every non-constant component of the map u yields a non-constant component of $\underline{\psi} \circ u =: \psi(u)$. Assume that the moduli spaces $\mathcal{M}_{k+1,\ell}(X, L_X, \beta)$ and $\mathcal{M}_{k+1,\ell}(Y, L_Y, \underline{\psi}_*\beta)$ of nodal holomorphic discs are compact.

Theorem 4.1.1 (Theorem 2.2.1)**.** *In the above situation, we can construct compatible Kuranishi structures on $\mathcal{M}_{k+1,\ell}(X, L_X, \beta)$ and $\mathcal{M}_{k+1,\ell}(Y, L_Y, \underline{\psi}_*\beta)$ such that we have an induced morphism of Kuranishi spaces*

$$\mathcal{M}_{k+1,\ell}(X, L_X, \beta) \xrightarrow{\psi} \mathcal{M}_{k+1,\ell}(Y, L_Y, \underline{\psi}_*\beta).$$

Furthermore, the Kuranishi structure on $\mathcal{M}_{k+1,\ell}(Y, L_Y, \underline{\psi}_*\beta)$ induces a Kuranishi structure on the image moduli space $\psi(\mathcal{M}_{k+1,\ell}(X, L_X, \beta))$ with respect to which the morphism

$$\mathcal{M}_{k+1,\ell}(X, L_X, \beta) \xrightarrow{\psi} \psi\left(\mathcal{M}_{k+1,\ell}(X, L_X, \beta)\right),$$

is an isomorphism.

In the case of our primary example, this theorem yields the following corollary.

Corollary 4.1.2 (Theorem 3.1.7). *For appropriate choices of Kuranishi structures, the map* $\underline{\psi} : \mathscr{O}(-n) \to \mathscr{O}$ *defined in Section 1.4 induces a Kuranishi isomorphism*

$$\psi : \mathcal{M}_{k+1}(\mathscr{O}(-n), L, \beta) \to \mathcal{M}_{k+1}(\mathscr{O}, \underline{\psi}(L), \underline{\psi}_* \beta; (F_0, D_0, n))$$

for all classes $\beta \in H_2(\mathscr{O}(-n), L)$*.*

There are in fact many examples of a birational map $\underline{\psi} : X \to Y$ where the assumptions of Theorem 2.2.1 are satisfied for *some* class $\beta \in H_2(X, L_X)$, but it is more unusual for them to be satisfied for *all* classes β, and it is this latter situation where we can hope to understand the Floer theory of L_X by understanding the Floer theory of L_Y.

These results provide a concrete example of an important principle for moduli spaces of pseudoholomorphic discs and curves, which is that they do not "know" the difference between failures of transversality arising directly from the almost-complex geometry of the ambient space and failures of transversality arising from the imposition of extra constraints, such as bulk-insertions or tangency conditions. This principle then lends itself to an alternative method for regularizing moduli spaces based in deforming these extra constraints, as opposed to deforming the almost-complex structure or $\bar{\partial}$ map more directly:

Proposition 4.1.3 (Proposition 3.3.4). *We can deform the moduli spaces*

$$\mathcal{M}_{k+1}(\mathscr{O}, \underline{\psi}(L), \underline{\psi}_* \beta; (F_0, D_0, n))$$

to regular moduli spaces, which can then be used to define an A_∞*-structure on* $H^1(L; \Lambda_+)$*.*

4.2 Conjectures and future work

Following immediately from the work on our primary example, we have some natural conjectures.

Conjecture 4.2.1. *For all $n \geq 1$, there exists a deformation regularization giving the following superpotential for $\mathscr{O}(-n)$:*

$$W_{\mathscr{O}(-n)} = T^{\omega(\beta_1)}x + T^{\omega(\beta_2)}y + T^{\omega(\beta_3)}\frac{y^n}{x} + T^{\omega(n\beta_2-\beta_3+D_{-n})}x\left(\left(1 + T^{\omega(\beta_3-\beta_2)}\frac{y^{n-1}}{x}\right)^n - 1\right).$$

This is consistent with our tentative calculations and would hold if the definition of $W_{\mathscr{O}(-n)}$ can be reduced to the tropical calculation in which only one wall, with function $\left(1 + t_1 \cdots t_n \frac{y^{n-1}}{x}\right)$, is relevant.

Given that much of the work on calculating superpotentials is motivated by mirror symmetry, we make the following conjecture.

Conjecture 4.2.2. *The superpotentials we have defined give Landau-Ginzburg B-side mirrors of the symplectic A-side manifolds $\mathscr{O}(-n)$, in the sense that the quantum cohomology of $\mathscr{O}(-n)$ is isomorphic to the Jacobian ring of the superpotential.*

It is also natural to ask how these superpotentials for $\mathscr{O}(-n)$ are related to the Hirzbruch surfaces, which have been more prominent objects of study in this context.

Conjecture 4.2.3. *For all $n \geq 1$, the superpotential for $\mathbb{F}_n$ can be defined so that*

$$W_{\mathbb{F}_n} = W_{\mathscr{O}(-n)} + T^{\omega(\beta_4)}\frac{1}{y}.$$

If the definition can be reduced to a similar tropical calculation of a bulk-deformed superpotential in $\mathbb{P}^1 \times \mathbb{P}^1$ instead of in $\mathscr{O}$, then this will hold. Adapting the specific construction herein will likely be infeasible. However, near term future work will involve studying the relationship between point blowup and superpotentials for toric surfaces, which should be able to either confirm or deny this conjecture.

More broadly, future work will involve applying the ideas of this thesis to understand the role that maps between ambient spaces play in Floer theory, as well as to develop further tools for answering direct questions about moduli spaces of pseudoholomorphic discs.

Appendix A

Kuranishi construction further details

We provide some further details of the Kuranishi structure construction regarding gluing and cutting by transversals. This appendix essentially rephrases content from Fukaya-Oh-Ohta-Ono [17] using our notation.

A.1 Gluing details

The following discussion of gluing assumes that the obstruction bundles are vector bundles, as in FOOO. The necessary generalization to accommodate our obstruction bundles without linear structure is straightforward.

Proposition A.1.1 (FOOO [17] Theorem 19.3)**.** *For each sufficiently small ϵ_3 and sufficiently large $\vec{T}$, there exist ϵ_2, ϵ_4 and a $\Gamma_{\mathfrak{p}}^+$ equivariant map*

$$Glu_{(\vec{T},\vec{\theta})} : \mathcal{M}_{k+1,(\ell,\ell_{\mathfrak{p}},(\ell_c))}(\beta;\mathfrak{p};\mathcal{G}_{\mathfrak{p}})_{\epsilon_4} \to \mathcal{M}_{k+1,(\ell,\ell_{\mathfrak{p}},(\ell_c))}(\beta;\mathfrak{p};(\vec{T},\vec{\theta}))_{\epsilon_2}$$

which is a diffeomorphism onto its image. The image of $Glu_{(\vec{T},\vec{\theta})}$ contains the space $\mathcal{M}_{k+1,(\ell,\ell_{\mathfrak{p}},(\ell_c))}(\beta;\mathfrak{p};(\vec{T},\vec{\theta}))_{\epsilon_3}$.

We start with an element $(\mathfrak{y}, u^{\mathfrak{y}}, (\vec{w}_c)) \in \mathcal{M}_{k+1,(\ell,\ell_{\mathfrak{p}},(\ell_c))}(\beta;\mathfrak{p};\mathcal{G}_{\mathfrak{p}})_{\epsilon_4}$ and construct the map $Glu_{\vec{T},\vec{\theta}}(u^{\mathfrak{y}}) : (\Sigma_{\mathfrak{Y}}, \partial\Sigma_{\mathfrak{Y}}) \to (X, L)$, where $\mathfrak{Y} = \overline{\Phi}_{\mathfrak{p}}(\mathfrak{y}, \vec{T}, \vec{\theta})$. We let $\alpha : K_{\mathfrak{y}} \to K_{\mathfrak{Y}}$ be the identification used to give the complex structure of $\Sigma_{\mathfrak{Y}}$ on $K_{\mathfrak{Y}}$. We will need to construct sequences $\hat{u}_{(i)}$ and $u_{(i)}$ of functions on $\Sigma_{\mathfrak{y}}$ and $\Sigma_{\mathfrak{Y}}$ respectively, and it will be helpful to refer to the identification α explicitly.

We will also need the following (monotone, smooth) bump functions on

$[-5T_e, 5T_e]_{\tau_e} \times [0,1]_{t_e}$ and $[-5T_e, 5T_e]_{\tau_e} \times S^1_{t_e}$:

$$\chi^{\leftarrow}_{e,r}(\tau_e, t_e) = \begin{cases} 1 & \tau_e < r-1 \\ 0 & \tau_e > r+1 \end{cases}$$

$$\chi^{\rightarrow}_{e,r} = 1 - \chi^{\leftarrow}_{e,r}$$

We proceed now to the gluing process. We first walk through the process without giving bounds on any of the quantities involved, and then present the necessary bounds in describing how this process yields the desired gluing map.

Pregluing, or Step 0:

Step 0-2:

We define an approximate solution $u_{(0)} : (\Sigma_{\mathfrak{Y}}, \partial\Sigma_{\mathfrak{Y}}) \to (X, L)$, where $\mathfrak{Y} = \overline{\Phi}_{\mathfrak{p}}(\mathfrak{y}, \vec{T}, \vec{\theta})$ using the coordinate at infinity associated with our choice of stabilization data at $\mathfrak{p}$. For $e \in C^1(\mathcal{G}_{\mathfrak{p}})$ we denote by $v_{\leftarrow}(e)$ and $v_{\rightarrow}(e)$ its two vertices. Here e is an outgoing edge of $v_{\leftarrow}(e)$ and is an incoming edge of $v_{\rightarrow}(e)$. We put:

$$u_{(0)} = \begin{cases} \chi^{\leftarrow}_{e,T_e}(u^{\mathfrak{y}}_{v_{\leftarrow}(e)} - p^{\mathfrak{y}}_e) + \chi^{\rightarrow}_{e,-T_e}(u^{\mathfrak{y}}_{v_{\rightarrow}(e)} - p^{\mathfrak{y}}_e) + p^{\mathfrak{y}}_e & \text{on the } e\text{th neck} \\ u^{\mathfrak{y}}_v \circ \alpha^{-1} & \text{on } K_{\mathfrak{Y}_v}. \end{cases}$$

Step 0-3: Since $(\mathfrak{y}, u^{\mathfrak{y}}, (\vec{w}_c)) \in \mathcal{M}_{k+1,(\ell,\ell_{\mathfrak{p}},(\ell_c))}(\beta; \mathfrak{p}; \mathcal{G}_{\mathfrak{p}})_{\epsilon_4}$, we have that $\bar{\partial} u^{\mathfrak{y}} \in \mathcal{E}(\mathfrak{y}, u^{\mathfrak{y}})$. However, in general $\bar{\partial} u_{(0)} \notin \mathcal{E}(\mathfrak{Y}, u_{(0)})$, since $\alpha^* \mathcal{E}(\mathfrak{Y}, u_{(0)}) \neq \mathcal{E}(\mathfrak{y}, u^{\mathfrak{y}})$. We are interested in understanding the relationship between $\bar{\partial} u_{(0)}$ and $\mathcal{E}(\mathfrak{Y}, u_{(0)})$, or equivalently between $\bar{\partial} u^{\mathfrak{y}}$ and $\alpha^* \mathcal{E}(\mathfrak{Y}, u_{(0)})$ so we consider the projection

$$\Pi_{\alpha^* \mathcal{E}(\mathfrak{Y}, u_{(0)})} : L^2_{m,\delta}(\Sigma_y; (u^{\mathfrak{y}})^* TX \otimes \Lambda^{0,1}) \to \alpha^* \mathcal{E}(\mathfrak{Y}, u_{(0)})$$

We define $\mathfrak{se}_{(0)} = \Pi_{\alpha^* \mathcal{E}(\mathfrak{Y}, u_{(0)})} \left(\bar{\partial} u^{\mathfrak{y}}\right)$.

Step 0-4:

We next define

$$\mathrm{Err}_{(0)} = \begin{cases} \chi^{\leftarrow}_{e,0}\bar{\partial} u_{(0)} & \text{on the } e\text{th neck if } e \text{ is outgoing} \\ \chi^{\rightarrow}_{e,0}\bar{\partial} u_{(0)} & \text{on the } e\text{th neck if } e \text{ is incoming} \\ \bar{\partial} u_{(0)} - \mathfrak{se}_{(0)} & \text{on } K_{\mathfrak{y}_v}. \end{cases}$$

Step j:

Step j-1:

We define $\hat{u}_{(j)} : (\Sigma_{\mathfrak{y}_v}, \partial\Sigma_{\mathfrak{y}_v}) \to (X, L)$ as follows:

$$\hat{u}_{(j)}(z) = \begin{cases} \chi^{\leftarrow}_{e,2T_e}(\tau_e, t_e) u_{(j-1)}(\tau_e, t_e) + \chi^{\rightarrow}_{e,2T_e}(\tau_e, t_e) p^{\mathfrak{y}}_e & \text{if } z = (\tau_e, t_e) \text{ is on the} \\ & e\text{th neck that is outgoing} \\ \chi^{\leftarrow}_{e,-2T_e}(\tau_e, t_e) u_{(j-1)}(\tau_e, t_e) + \chi^{\rightarrow}_{e,-2T_e}(\tau_e, t_e) p^{\mathfrak{y}}_e & \text{if } z = (\tau_e, t_e) \text{ is on the} \\ & e\text{th neck that is incoming} \\ u_{(j-1)}(\alpha(z)) & \text{if } z \in K_{\mathfrak{y}}. \end{cases}$$

Step j-2:

We define the following vector space:

$$\hat{\mathcal{E}}(\hat{u}_{(j)}) := \alpha^* \mathcal{E}(\mathfrak{Y}, u_{(j-1)}) \subseteq L^2_{m,\delta}(\Sigma_{\mathfrak{y}}; (u_{(j-1)})^* TX \otimes \Lambda^{0,1}).$$

That is, we take the obstruction vector space at $u_{(j-1)}$ given by our choices of obstruction bundle data and then use the identification (biholomorphism) $\alpha : K_{\mathfrak{y}_v} \to K_{\mathfrak{Y}_v}$ to “move” it to $\hat{u}_{(j)}$. Note in particular that $\hat{\mathcal{E}}(\hat{u}_{(j)})$ does not equal the obstruction vector space $\mathcal{E}(\mathfrak{y}, \hat{u}_{(j)})$ at $\hat{u}_{(j)}$ obtained directly from the obstruction bundle data.

Consider the $L^2_{m,\delta}$ projection

$$\Pi_{\hat{\mathcal{E}}(\hat{u}_{(j)})} : L^2_{m,\delta}(\Sigma_{\mathfrak{y}}; \hat{u}^*_{(j)}TX \otimes \Lambda^{0,1}) \to \hat{\mathcal{E}}(\hat{u}_{(j)}).$$

Given an element $A \in L^2_{m,\delta}(\Sigma_{\mathfrak{y}}; \hat{u}^*_{(j)}TX \otimes \Lambda^{0,1})$, we define the derivative of $\prod_{\hat{\mathcal{E}}(\hat{u}_{(j)})}$ at A_v with respect to $V \in L^2_{m,\delta}((\Sigma_{\mathfrak{y}}, \partial\Sigma_{\mathfrak{y}}); \hat{u}^*_{(j)}TX, \hat{u}^*_{(j)}TL)$ by

$$\begin{aligned} D\Pi_{\hat{\mathcal{E}}(\hat{u}_{(j)})}(A, \bullet) &: L^2_{m,\delta}((\Sigma_{\mathfrak{y}_v}, \partial\Sigma_{\mathfrak{y}}); \hat{u}^*_{(j)}TX, \hat{u}^*_{(j)}TL) \to L^2_{m,\delta}(\Sigma_{\mathfrak{y}}; \hat{u}^*_{(j)}TX \otimes \Lambda^{0,1}) \\ V &\mapsto \frac{d}{dt}\bigg|_{t=0} \Big(((\mathrm{Pal}_{\hat{u}_{(j)}}^{\mathrm{Exp}(\hat{u}_{(j)}, tV)})^{(0,1)})^{-1} \Pi_{\hat{\mathcal{E}}(\mathrm{Exp}(\hat{u}_{(j)}, tV))} \mathrm{Exp}(\hat{u}_{(j)}, tV) \Big) \end{aligned}$$

We recall our linearized $\bar{\partial}$ operator at $(\mathfrak{y}, \hat{u}_{(j)})$:

$$\begin{aligned} D_{\mathfrak{y}, \hat{u}_{(j)}}\bar{\partial} : W^2_{m+1,\delta}((\Sigma_{\mathfrak{y}}, \partial\Sigma_{\mathfrak{y}}); \hat{u}^*_{(j)}TX, \hat{u}^*_{(j)}TL) &\to L^2_{m,\delta}(\Sigma_{\mathfrak{y}}; \hat{u}^*_{(j)}TX \otimes \Lambda^{0,1}) \\ D_{\mathfrak{y}, \hat{u}_{(j)}}\bar{\partial}(V) &= \frac{d}{dt}\bigg|_{t=0} \Big(((\mathrm{Pal}_{\hat{u}_{(j)}}^{\mathrm{Exp}(\hat{u}_{(j)}, tV)})^{(0,1)})^{-1} \bar{\partial}\, \mathrm{Exp}(\hat{u}_{(j)}, tV) \Big). \end{aligned}$$

We then consider the operator $D_{(j)}$ sending

$$(V, \Delta p) \mapsto D_{\mathfrak{y}, \hat{u}_{(j)}}\bar{\partial}V - D\Pi_{\hat{\mathcal{E}}(\hat{u}_{(j)})}(\mathfrak{se}_{(j-1)}, V).$$

We have the following lemma.

Lemma A.1.2 (Lemma 19.15 in FOOO [17]). *The sum of the image of $D_{(j-1)}$ and the subspace $\hat{\mathcal{E}}(\hat{u}_{(j-1)})$ is $L^2_{m,\delta}(\Sigma_{\mathfrak{y}}, \hat{u}^*_{(j-1)}TX \otimes \Lambda^{0,1})$ if $\vec{T}$ is sufficiently large. We also have that the restriction of $\mathrm{ev}_{\mathcal{G}}$ to $D^{-1}_{(j-1)}(\hat{\mathcal{E}}(\hat{u}_{(j-1)}))$ is surjective for sufficiently large $\vec{T}$.*

It follows that the sum $D_{(j-1)}(\ker \mathrm{ev}_{\mathcal{G}}) + \hat{\mathcal{E}}(\hat{u}_{(j-1)})$ equals $L^2_{m,\delta}(\Sigma_{\mathfrak{y}}, \hat{u}^*_{(j-1)}TX \otimes \Lambda^{0,1})$ if $\vec{T}$ is sufficiently large.

The map $\mathrm{ev}_{\mathcal{G}}$ is defined in (2.1.9).

Proof. The first two statements follow from Lemma 2.1.42. To see the last statement, let $\kappa \in L^2_{m,\delta}(\Sigma_{\mathfrak{y}}, \hat{u}^*_{(j-1)}TX \otimes \Lambda^{0,1})$ and write $\kappa = D_{(j)}(V, \Delta p) + \gamma$, with

$$(V, \Delta p) \in \bigoplus_{v \in C^0(\mathcal{G})} W^2_{m+1,\delta}(\Sigma_{\mathfrak{y}_v}; \hat{u}^*_{(j),v}TX, \hat{u}^*_{(j),v}TL)$$

and with $\gamma \in \hat{\mathcal{E}}(\hat{u}_{(j)})$. Since the restriction of $\mathrm{ev}_{\mathcal{G}}$ to $D^{-1}_{(j)}(\hat{\mathcal{E}}(\hat{u}_{(j)}))$ is surjective, we can find $(V', \Delta p') \in D^{-1}_{(j)}(\hat{\mathcal{E}}(\hat{u}_{(j)}))$ with $\mathrm{ev}_{\mathcal{G}}(V', \Delta p') = \mathrm{ev}_{\mathcal{G}}(V, \Delta p)$. We thus have that

$$\begin{aligned} \kappa &= D_{(j)}(V, \Delta p) - D_{(j)}(V', \Delta p') + D_{(j)}(V', \Delta p') + \gamma \\ &= D_{(j)}(V - V', \Delta p - \Delta p') + D_{(j)}(V', \Delta p') + \gamma \end{aligned}$$

and $\mathrm{ev}(\mathcal{G})(V - V', \Delta p - \Delta p') = 0$. □

We consider $\ker \mathrm{ev}_{\mathcal{G}} \cap D^{-1}_{(j)}(\hat{\mathcal{E}}(\hat{u}_{(j)}))$ and let $\mathfrak{H}_{(j)}$ be its orthogonal complement in $\ker \mathrm{ev}_{\mathcal{G}}$. We then define $V_{(j),v}$ and $\Delta p_{(j),e}$ so that $(V_{(j),v}, \Delta p_{(j),e}) \in \mathfrak{H}_{(j)}$ is the unique element such that

$$D_{(j)}(V_{(j)}) + \mathrm{Err}_{(j-1),v} \in \hat{\mathcal{E}}(\hat{u}_{(j)})$$

and $\lim_{\tau_e \to \pm\infty} V_{(j),v}(\tau_e, t_e) = \Delta p_{(j),e}$ where $\pm\infty = +\infty$ if e is outgoing and $-\infty$ if e is incoming. This is definition 19.17 in FOOO [17].

We next define an approximate solution $u_{(j)} : (\Sigma_{\mathfrak{Y}}, \partial\Sigma_{\mathfrak{Y}}) \to (X, L)$. We put:

$$u_{(j)} = \begin{cases} u_{(j-1)} + \chi^{\leftarrow}_{e,T_e}(V_{(j),v_{\leftarrow}(e)} - \Delta p_{(j),e}) & \\ \qquad\qquad + \chi^{\leftarrow}_{e,-T_e}(V_{(j),v_{\rightarrow}(e)} - \Delta p_{(j),e}) + \Delta p_{(j),e} & \text{on the } e\text{th neck} \\ \mathrm{Exp}(u_{(j-1)}, V_{(j)} \circ \alpha^{-1}) & \text{on } K_{\mathfrak{Y}_v}. \end{cases}$$

Step j-3:

Define $\mathfrak{se}_{(j)} = \alpha^* \Pi_{\mathcal{E}(\mathfrak{Y}, u_{(j)})}(\bar{\partial} u_{(j)})$ and $\mathfrak{e}_{(j)} = \mathfrak{se}_{(j)} - \mathfrak{se}_{(j-1)}$.

Step j-4:

Define

$$\mathrm{Err}_{(j)} = \begin{cases} \chi^{\leftarrow}_{e,0}\bar{\partial}u_{(j)} & \text{on the } e\text{th neck if } e \text{ is outgoing} \\ \chi^{\rightarrow}_{e,0}\bar{\partial}u_{(j)} & \text{on the } e\text{th neck if } e \text{ is incoming} \\ \alpha^*\bar{\partial}u_{(j)} - \mathfrak{se}_{(j)} & \text{on } K_{\mathfrak{y}_v}. \end{cases}$$

All of the control we need over this process is captured in the following proposition.

Lemma A.1.3 (Prop 19.20 in FOOO [17]). *There exist constants $T_m, C_1, C_2, C_3, \epsilon_5 > 0$ and $0 < \mu < 1$ such that the following inequalities hold if $T_e > T_m$ for all e. We let $T_{\min} = \min\{T_e \mid e \in C^1(\mathcal{G}_{\mathfrak{p}})\}$.*

$$\left|\left|(V_{(j),v}, \Delta p_{(j),e})\right|\right|_{L^2_{m+1,\delta}(\Sigma_{\mathfrak{y}_v})} < C_1\mu^{j-1}e^{-\delta T_{\min}}, \tag{A.1.1}$$

$$\left|\left|\Delta p_{(j),e}\right|\right| < C_1\mu^{j-1}e^{-\delta T_{\min}}, \tag{A.1.2}$$

$$\left|\left|u_{(j)} - u_{(0)}\right|\right|_{L^2_{m+1,\delta}(K_{\mathfrak{y}_v})} < C_2 e^{-\delta T_{\min}}, \tag{A.1.3}$$

$$\left|\left|\mathfrak{e}_{(j)}\right|\right|_{L^2_{m+1,\delta}(K_{\mathfrak{y}_v})} < C_3\mu^{j-1}e^{-\delta T_{\min}}, \tag{A.1.4}$$

$$\left|\left|\mathrm{Err}_{(j),v}\right|\right|_{L^2_{m+1,\delta}(\Sigma_{\mathfrak{y}_v})} < \epsilon_5 C_3\mu^{j}e^{-\delta T_{\min}}, \tag{A.1.5}$$

where we assume $j \geq 1$ for the second to last inequality A.1.4.

Inequalities A.1.1 and A.1.2 imply that the sequence $u_{(j)}$ converges, so we can define $\mathrm{Glu}_{\vec{T},\vec{\theta}}(\mathfrak{y}, u) = \lim_{j\to\infty} u_{(j)}$. Inequalities A.1.4 and A.1.5 imply $\bar{\partial}\mathrm{Glu}_{\vec{T},\vec{\theta}}(\mathfrak{y}, u) = \sum_{j=0}^{\infty} \mathfrak{e}_{(j)} \in \mathcal{E}(\mathfrak{Y}, \mathrm{Glu}_{\vec{T},\vec{\theta}}(\mathfrak{y}, u))$. Thus, $\mathrm{Glu}_{\vec{T},\vec{\theta}}(\mathfrak{y}, u)) \in \mathcal{M}_{k+1,(\ell,\ell_{\mathfrak{p}},(\ell_c))}(\beta; \mathfrak{p}; (\vec{T}, \vec{\theta}))_{\epsilon_2}$.

A.2 Cutting by transversals

Recall that every time we chose an additional marked point as part of obstruction bundle data or stabilization data, we also chose a corresponding real codimension 2 submanifold of X. We describe below how we use these submanifolds to forget the additional marked points.

Definition A.2.1 (Def 20.6 in FOOO [17]). *An element $(\mathfrak{Y}, u', (\vec{w}'_c))$ of $\mathcal{M}_{k+1,(\ell,\ell_{\mathfrak{p}},(\ell_c))}(\beta; \mathfrak{p})_{\epsilon_0,\vec{T}_0}$ satisfies the transversal constraint at all additional marked* *points if for all marked points $\vec{w}_{\mathfrak{p}}$ of $\mathfrak{Y}$ from the stabilization data at $\mathfrak{p}$ we have that $w_{\mathfrak{p}_i} \in \mathcal{D}_{\mathfrak{p},i}$, and for all marked points $\vec{w}'_c$ we have that $w'_{c,i} \in \mathcal{D}_{c,i}$.*

We have the following lemma.

Lemma A.2.2 (Lemma 20.7 in FOOO [17]). *The set $\mathcal{M}_{k+1,(\ell,\ell_{\mathfrak{p}},(\ell_c))}(\beta;\mathfrak{p})^{trans}_{\epsilon_0,\vec{T}_0}$ is a closed subset of $\mathcal{M}_{k+1,(\ell,\ell_{\mathfrak{p}},(\ell_c))}(\beta;\mathfrak{p})_{\epsilon_0,\vec{T}_0}$ and is a stratawise smooth submanifold of codimension $\ell_{\mathfrak{p}} + 2\sum_{c\in\mathfrak{B}} \ell_c$.*

We then consider the subset of $\mathcal{M}_{k+1,(\ell,\ell_{\mathfrak{p}},(\ell_c))}(\beta;\mathfrak{p})^{trans}_{\epsilon_0,\vec{T}_0}$ consisting of pseudo-holomorphic maps.

Definition A.2.3 (Def 20.9 in FOOO [17]). *We denote by $\mathcal{M}_{k+1,(\ell,\ell_{\mathfrak{p}},(\ell_c))}(\beta;\mathfrak{p})^{trans}_{\epsilon_0,\vec{T}_0} \cap \mathfrak{s}^{-1}(0)$ the set of all $(\mathfrak{Y}, u', (\vec{w}_c)) \in \mathcal{M}_{k+1,(\ell,\ell_{\mathfrak{p}},(\ell_c))}(\beta;\mathfrak{p})^{trans}_{\epsilon_0,\vec{T}_0}$ such that u' is pseudo-holomorphic.*

Forgetting all additional marked points gives a map

$$\mathfrak{forget} : \mathcal{M}_{k+1,(\ell,\ell_{\mathfrak{p}},(\ell_c))}(\beta;\mathfrak{p})^{trans}_{\epsilon_0,\vec{T}_0} \cap \mathfrak{s}^{-1}(0) \to \mathcal{M}_{k+1,\ell}(\beta).$$

Modding out by the action of $\Gamma_{\mathfrak{p}}$ gives the induced map

$$\overline{\mathfrak{forget}} : \left(\mathcal{M}_{k+1,(\ell,\ell_{\mathfrak{p}},(\ell_c))}(\beta;\mathfrak{p})^{trans}_{\epsilon_0,\vec{T}_0} \cap \mathfrak{s}^{-1}(0)\right) / \Gamma_{\mathfrak{p}} \to \mathcal{M}_{k+1,\ell}(\beta). \tag{A.2.1}$$

Proposition A.2.4 (Prop 20.11 in FOOO [17]). *The map A.2.1 is a homeomorphism onto an open neighborhood of $\mathfrak{p}$.*

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CURRICULUM VITAE

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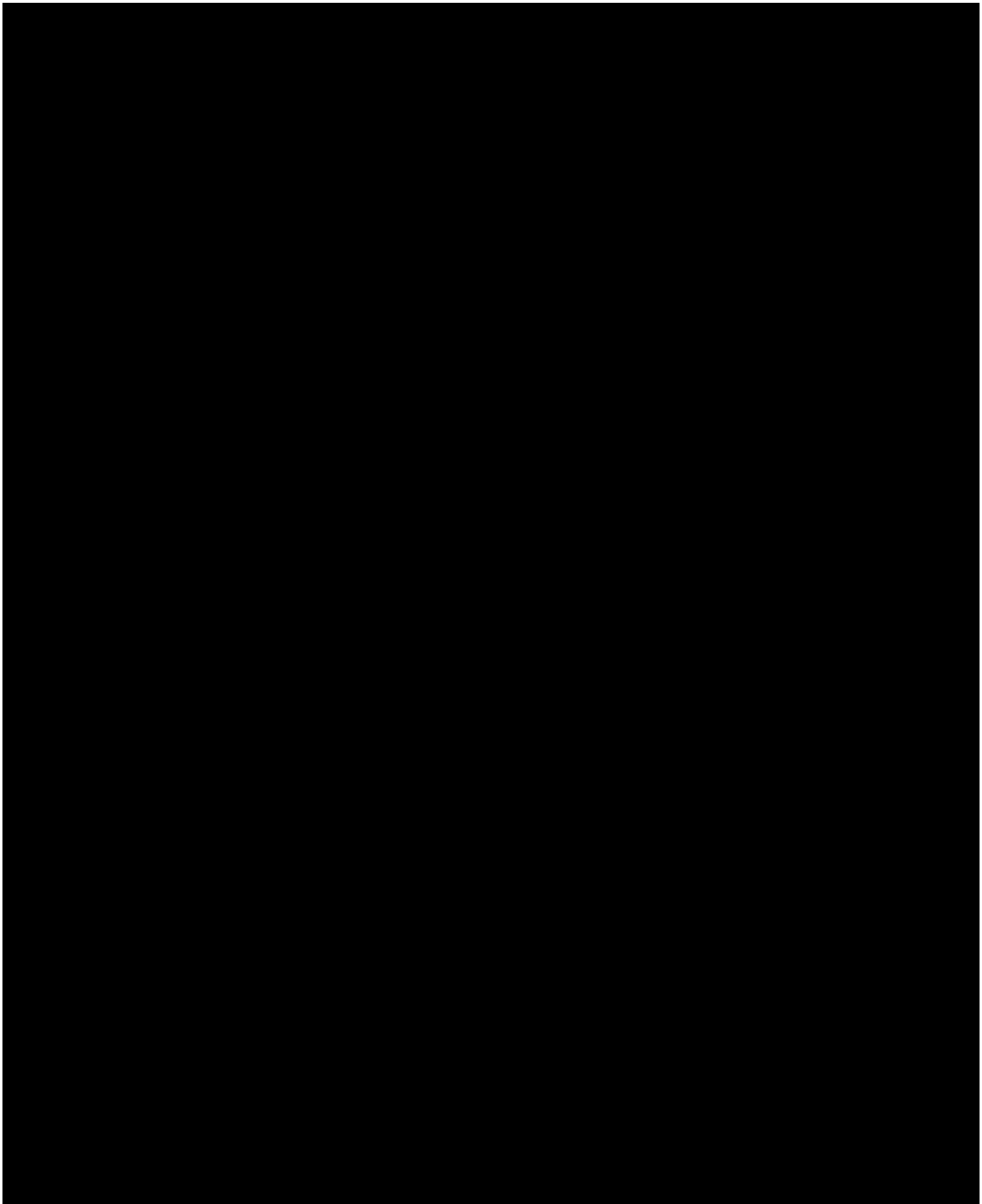

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