The Hamilton-Jacobi analysis by Peter Bergmann and Arthur Komar of classical general relativity

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Peter Bergmann initiated in 1966 an application of Hamilton-Jacobi techniques to general relativity. Little had been done by this time on extending this analysis to gauge theories. He proved that when, as in the case of Einstein's theory, the phase space generator of evolution consisted of a linear combination of constraints, the Hamilton principal function must be independent of spacetime coordinates. Also the Hamilton Jacobi equations that determined this functional of the 3-metric retained their form under phase space functionals that were invariant under the action of the spacetime diffeomorphism group. Komar followed up beginning in 1967 with a series of papers in which he proved that a complete solution of the Hamilton Jacobi equations was determined by a commuting set of diffeomorphism invariants. These invariants thereby labeled equivalence classes of solutions of Einstein's equations under the action of the full four-dimensional diffeomorphism group. Furthermore, this set satisfied canonical commutation relations with another invariant set. The hope and expectation was that these invariants could be promoted to quantum operators in a quantum theory of gravity. This framework will be contrasted with J. A. Wheeler's geometrodynamical program in which the only underlying covariance group is spatial diffeomorphisms. The full spacetime diffeomorphism symmetry is replaced by the notion of 'multi-fingered' time. A related dispute concerning the 'sandwich conjecture' will be discussed, relevant to the functional integral approach to quantum gravity. Two three geometries cannot determine a corresponding four geometry if they lie in distinct four dimensional diffeomorphism equivalence classes.

Keywords: Hamilton-Jacobi equations; geometrodynamics; quantum gravity.

1. Introduction

The following is a brief historical overview of work on a Hamilton-Jacobi approach to general relativity that was undertaken by Peter Bergmann and Arthur Komar in the 1960's and 1970's. A more detailed initial version, discussing both the relation of their research to previous and concurrent approaches, and to later progress, appears in Ref. 1. A further revision and expansion is in progress. The emphasis throughout their investigations was on the full four-dimensional diffeomorphism covariance of Einstein's theory. This lead to a divergence with the geometrodynamical approach of John Wheeler and company where only the spatial covariance is fully respected.

2. Bergmann's initial Hamilton-Jacobi analysis of general relativity

It is not widely recognized that it was Peter Bergmann who pointed out to Peres prior to the publication of his groundbreaking paper in Ref. 2 that his S appearing as an argument in the four general relativistic constraint equations should be interpreted "as the Hamilton-Jacobi functional for the gravitational field." Of course the following are now identified as the Wheeler-DeWitt equations,

$$\mathcal{H}_\mu\left(g_{ab}, \frac{\delta S}{\delta g_{cd}}\right) = 0, \tag{1}$$

where the $\mathcal{H}_\mu\left(g_{ab}, p^{cd}\right)$ are the secondary constraints in general relativity. Bergmann proved in Ref. 3 that in a theory in which the Hamiltonian is constrained to vanish S could not depend explicitly on the time. The argument applied equally well to spatial dependence, as noted first in Ref. 4. Thus $S = S[g_{ab}(\vec{x})]$. Bergmann also showed that the Hamilton-Jacobi equations were form invariant under canonical transformations generated by diffeomorphism invariants. The fact that the numerical value of S is altered under the action of $\mathcal{H}_0$ presented a puzzle. The question arose whether this could this be inconsistent with the accepted notion of 'frozen time'.

3. Komar's isolation of solution equivalence classes

Komar observed in Ref. 4 that although there were only four Hamilton-Jacobi equations the principal function S delivered $6 \times \infty^3$ expressions for the momenta,

$$p^{ab}(\vec{x}) = \frac{\delta S}{\delta g_{ab}(\vec{x})}, \tag{2}$$

and therefore the $p^{ab}(\vec{x})$ are not uniquely determined. Two additional constraints needed to be imposed, with $A = 1, 2$,

$$\alpha^0_A\left[g_{ab}(\vec{x}), \frac{\delta S}{\delta g_{cd}(\vec{x})}\right] - \alpha_A(\vec{x}) = 0. \tag{3}$$

From the fact that

$$\frac{\delta \alpha^0_A}{\delta g_{ab}} + \frac{\delta \alpha^0_A}{\delta p^{cd}} \frac{\delta^2 S}{\delta g_{cd} \delta g_{ab}} = 0, \tag{4}$$

and similarly for the $\mathcal{H}_\mu$ it follows that

$$\{H_\mu, \alpha^0_A\} = \frac{\delta^2 S}{\delta g_{ab} \delta g_{cd}}\left(-\frac{\delta H_\mu}{\delta p^{ab}}\frac{\delta \alpha^0_A}{\delta p^{cd}} + \frac{\delta H_\mu}{\delta p^{cd}}\frac{\delta \alpha^0_A}{\delta p^{ab}}\right) = 0. \tag{5}$$

In other words, the α^0_A must be diffeomorphism invariants (and they must also commute with each other.)

The constant values of $\alpha^0_A\left[g_{ab}(\vec{x}), p^{cd}(\vec{x})\right]$ identify equivalence classes under the action of the spacetime diffeomorphism group. In Ref. 5 he showed that there existed invariant functionals β^A_0 that were canonically conjugate to the α^0_A. However, as formulated at this stage by Komar, one cannot yet obtain solutions of Einstein's equations by setting $\beta^A(\vec{x}) = \frac{\delta S}{\delta \alpha_A(\vec{x})}$. One still requires a temporal coordinate - like the 'intrinsic' q^0 that appears in the free particle action.

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$$dS_p = p_\mu dq^\mu, \tag{6}$$

with constraint $p^2 + m^2 = 0$. This can be compared to vacuum general relativity where the non-vanishing contribution to the increment in the action takes the form

$$dS_{gr} = \int d^3x\, p_{ab} dg^{ab}, \tag{7}$$

with constraints $\mathcal{H}_\mu = 0$.

In the particle case one can choose the 'intrinsic time' $t = q^0$ as the evolution parameter and also solve for p_0 resulting in

$$dS_p = -\left(\vec{p}^2 + m^2\right)^{1/2} dt + p_a dq^a. \tag{8}$$

This yields the complete Hamilton principal function

$$S_p(q^a, t; \alpha^b) = -\left(\vec{\alpha}^2 + m^2\right)^{1/2} t + \alpha_a q^a. \tag{9}$$

The analogue of the gravitational α_0^A in this case is p^a. The analogue of the canonical conjugate β_B^0 would be the reparameterization constant $q^a - p^a q^0/p^0$. The general solution is obtained from

$$\beta^a = \frac{\partial S_p}{\partial \alpha^a}$$

Bergmann and Komar, Ref. 6, had explicitly recognized this type of emergence of intrinsic time evolution. Earlier, Komar in Ref. 7 had proposed that intrinsic curvature-based coordinates could be constructed using Weyl curvature scalars. He and Bergmann in Ref. 8 proved that these scalars depended only on g_{ab} and p^{cd}. The question naturally arise as to why Bergmann and Komar did not proceed with the use of intrinsic coordinates in their Hamilton-Jacobi treatment. A Bergmann quote in Ref. 9 from 1971 is revealing: "Although intrinsic coordinates lead, in principle, to a complete set of observables in general relativity, their defects, of which the most glaring is their deviation from Lorentz coordinates, render this procedure illusory. It appears preferable to retain coordinates that are approximately, or asymptotically Lorentzian and hence not to destroy one's intuition." Thus in spite of prolonged occupation with a Hamiltonian formulation of the the underlying general covariance of general relativity, Bergmann still seemed to have ceded undue importance to the more familiar Poincaré symmetry of conventional field theory.

However, as shown in Ref. 10 it is in principle possible to carry out a canonical change of variable in the non-vanishing increment dS_{GR} to intrinsic spacetime coordinates $x^\mu = X^\mu(g_{ab}, p^{cd})$, analogous to the parameter choice $t = q^0$ in the free relativistic particle model. This is a corrected version of Refs. 11 and 12. Current work with Kurt Sundermeyer and Jürgen Renn is in preparation. One makes a canonical change of variables such that

$$\begin{aligned} dS_{gr} &= \int d^3x\, p^{ab} dg_{ab} \\ &= \int d^3x \left(\pi_\mu dX^\mu + p^A dg_A + \frac{\delta G}{\delta g_{ab}} dg_{ab} + \frac{\delta G}{\delta g_A} dg_A + \frac{\delta G}{\delta X^\mu} dX^\mu \right). \end{aligned} \tag{10}$$

One must find a generator $G[g_{ab}, X_A, g_B]$ such that $p_{ab} = \frac{\delta G}{\delta g_{ab}}$. Then

$$dS'_{gr} := d(S_{gr} - G) = \int d^3x \left(\pi_\mu dX^\mu + p^A dg_A \right). \tag{11}$$

Next choose the X^μ as intrinsic coordinates, i.e., set $x^\mu = X^\mu$. Finally, one eliminates the canonical conjugates to X^μ, π_ν, by solving the constraints. Then we have the resulting intrinsic Hamilton-Jacobi equation

$$\pi_0 \left[g_A, \frac{\delta S'_{gr}}{\delta g_B}, x^\mu \right] + \frac{\partial S'_{gr}}{\partial x^0} = 0. \tag{12}$$

From the complete solutions $S'_{gr}\,[g_A, x^\mu; \alpha_B]$ one can obtain the full set of physically distinct solutions of Einstein's equations from

$$\beta^A = \frac{\delta S'_{gr}}{\delta \alpha_A}. \tag{13}$$

4. Contrast with geometrodynamics

The contrast of this program with Wheeler's geometrodynamics cannot be overstated. The multifingered time approach assumed that the full four-dimensional diffeomorphism symmetry had been lost. States should be labeled by the $2 \times \infty^3$ diffeomorphism invariants $\alpha_A(\vec{x})$, and not by three-geometries.

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