A New Class of Integrable Models of (1+1)-Dimensional Dilaton Gravity Coupled to Toda Matter

A.T. Filippov

Joint Institute for Nuclear Research, Dubna, Moscow Region RU-141980
Alexandre.Filippov@jinr.ru

Abstract

A new class of integrable two-dimensional dilaton gravity theories, in which scalar matter fields satisfy the Toda equations, is proposed. The simplest case of the Toda system is considered in some detail, and on this example we outline how the general solution can be obtained. We also demonstrate how the wave-like solutions of the general Toda systems can be simply derived. In the dilaton gravity theory, these solutions describe nonlinear waves coupled to gravity. A special attention is paid to making the analytic structure of the solutions of the Toda equations as simple and transparent as possible, with the aim to apply the idea of the separation of variables to non-integrable theories.

1 Introduction

The theories of (1+1)-dimensional dilaton gravity coupled to scalar matter fields are known to be reliable models for some aspects of higher-dimensional black holes, cosmological models, and waves. The connection between higher and lower dimensions was demonstrated in different contexts of gravity and string theory and, in several cases, has allowed finding the general solution or special classes of solutions in high-dimensional theories [1]. A generic example is the spherically symmetric gravity coupled to Abelian gauge fields and massless scalar matter fields. It exactly reduces to a (1+1)-dimensional dilaton gravity and can be explicitly solved if the scalar fields are constants independent of coordinates. These solutions can describe interesting physical objects – spherical static black holes and simplest cosmologies. However, when the scalar matter fields, which presumably play a significant cosmological role, are nontrivial, not many exact analytical solutions of high-dimensional theories are known[2]. Correspondingly, the two-dimensional models of dilaton gravity that nontrivially couple to scalar matter are usually not integrable.

To obtain integrable models of this sort one usually has to make serious approximations, in other words, to deform the original two-dimensional model obtained by direct dimensional reductions of realistic higher-dimensional theories[3]. Nevertheless, the de-

[1] See, e.g., [1]- [27] for a more detailed discussion of this connection, references, and solution of some integrable two-dimensional and one-dimensional models of dilaton gravity.

[2] See, e.g., [8], [11], [12], [17]- [23]; a review and further references can be found in [25], [26] and [23]

[3] We note that several important four-dimensional space-times with symmetries defined by two commuting Killing vectors may also be described by two-dimensional models of dilaton gravity coupled to scalar matter. For example, cylindrical gravitational waves can be described by a (1+1)-dimensional dilaton gravity coupled to one scalar field [28]- [30], [22]. The stationary axially symmetric pure gravity

formed models can qualitatively describe certain physically interesting solutions of higher-dimensional gravity or supergravity theories related to the low-energy limit of superstring theories.

In our previous work (see, e.g., [20] - [23] and references therein) we constructed and studied some explicitly integrable models based on the Liouville equation. Recently, we attempted to find solutions of some realistic two-dimensional dilaton gravity models (derived from higher-dimensional gravity theories by dimensional reduction) using a generalized separation of variables introduced in [21], [22]. These attempts showed that seemingly natural ansatzes for the structure of the separation, which proved a success in previously studied integrable models, do not give interesting enough solutions ('zero' approximation of a perturbation theory) in realistic nonintegrable models. Thus an investigation of more complex dilaton gravity models, which are based on the two dimensional Toda chains, was initiated. Here we briefly present some results of this investigation. In particular, we propose a simplest class of models, which can be explicitly solved in terms of the solutions of the Toda equations. These solutions as well as their moduli space will be presented in a very simple and convenient form that allows for a simple description of the analytic and asymptotic properties of the solutions. At the same time this representation is extremely convenient for their reductions to the wave solutions that include static and cosmological ones. This construction essentially and naturally generalizes the previous results, [20] - [23], and shows that essentially more complex structures of the separation of variables should be employed in realistic theories of gravity.

2 General model of (1+1)-dimensional dilaton gravity minimally coupled to scalar matter fields.

The effective Lagrangian of the (1+1)-dimensional dilaton gravity coupled to scalar fields ψ_n obtainable by dimensional reductions of a higher-dimensional spherically symmetric (super)gravity can usually be (locally) transformed to the form (see [20] - [23] for a detailed motivation and specific examples):

$$\mathcal{L} = \sqrt{-g}\left[\varphi R(g) + V(\varphi,\psi) + \sum_{mn} Z_{mn}(\varphi,\psi)\, g^{ij}\,\partial_i\psi_m\,\partial_j\psi_n\right]. \qquad (1)$$

Here $g_{ij}(x^0,x^1)$ is the (1+1)-dimensional metric with the signature (-1,1), $g \equiv \det|g_{ij}|$, and R is the Ricci curvature of the two-dimensional space-time,

$$ds^2 = g_{ij}\,dx^i\,dx^j\,, \quad i,j = 0,1\,. \qquad (2)$$

The effective potentials V and Z_{mn} depend on the dilaton $\varphi(x^0,x^1)$ and on $N-2$ scalar fields $\psi_n(x^0,x^1)$ (we note that the matrix Z_{mn} should be negative definite to exclude the so called 'phantom' fields). They may depend on other parameters characterizing the parent higher-dimensional theory (e.g., on charges introduced in solving the equations for the Abelian fields). Here we consider the 'minimal' kinetic terms with the diagonal and constant Z-potentials, $Z_{mn}(\varphi,\psi) = \delta_{mn}Z_n$. This approximation excludes the important

([31], [11]) is equivalent to a (0+2)-dimensional dilaton gravity coupled to one scalar field. Similar but more general dilaton gravity models were also obtained in string theory. Some of them can be solved by using modern mathematical methods developed in the soliton theory (see e.g. [1], [2], [11], [19]).

class of the sigma - model - like scalar matter discussed, e.g., in [27]; such models can be integrable if $V \equiv 0$ and $Z_{mn}(\varphi, \psi)$ satisfy certain rather stringent conditions. In (1) we also used the Weyl transformation to eliminate the gradient term for the dilaton.

To simplify derivations, we write the equations of motion in the light-cone metric,

$$ds^2 = -4f(u,v)\,du\,dv\,.$$

By first varying the Lagrangian in generic coordinates and then passing to the light-cone coordinates we obtain the equations of motion (Z_n are constants!)

$$\partial_u\partial_v\varphi + f\,V(\varphi,\psi) = 0, \tag{3}$$

$$f\partial_i(\partial_i\varphi/f) = \sum Z_n(\partial_i\psi_n)^2\,, \qquad i = u, v\,. \tag{4}$$

$$2Z_n\partial_u\partial_v\psi_n + fV_{\psi_n}(\varphi,\psi) = 0\,, \tag{5}$$

$$\partial_u\partial_v \ln|f| + fV_\varphi(\varphi,\psi) = 0\,, \tag{6}$$

where $V_\varphi = \partial_\varphi V$, $V_{\psi_n} = \partial_{\psi_n} V$. These equations are not independent. Actually, (6) follows from (3) – (5). Alternatively, if (3), (4), and (6) are satisfied, one of the equations (5) is also satisfied.

The higher-dimensional origin of the Lagrangian (1) suggests that the potential is the sum of the exponentials of linear combinations of the scalar fields, $q_n^{(0)}$, and of the dilaton φ [4]. In our previous work [23] we studied the constrained Liouville model, in which the system of the equations of motion (3), (5) and (6) is equivalent to the system of the independent Liouville equations for the linear combinations of the fields $q_n \equiv F + q_n^{(0)}$, where $F \equiv \ln|f|$. The easily derived solutions of these equations should satisfy the constraints (4), which was the most difficult part of the problem. The solution of the whole problem revealed an interesting structure of the moduli space of the solutions that allowed us to easily identify static, cosmological and wave-like solutions and effectively embed these essentially one-dimensional (in some broad sense) solutions into the set of all two-dimensional solutions and study their analytic and asymptotic properties.

Here we propose a natural generalization of the Liouville model to the model in which the fields are described by the Toda equations (or by nonintegrable deformations of them). To demonstrate that the model shares many properties with the Liouville one and to simplify a transition from the integrable models to nonintegrable theories we suggest a different representation of the Toda solutions, which is not directly related to their group - theoretical background.

Consider the theory defined by the Lagrangian (1) with the potential:

$$V = \sum_{n=1}^{N} 2g_n \exp q_n^{(0)}\,, \qquad Z_n = -1\,, \tag{7}$$

where

$$q_n^{(0)} \equiv a_n\varphi + \sum_{m=3}^{N} \psi_m a_{mn}\,. \tag{8}$$

[4] Actually, the potential V usually contains terms non exponentially depending on φ (e.g., linear in φ), and then the exponentiation of φ is only an approximation, see the discussion in [23].

In what follows we also use

$$q_n \equiv F + q_n^{(0)} \equiv \sum_{m=1}^{N} \psi_m a_{mn} \,, \tag{9}$$

where $\psi_1 + \psi_2 \equiv \ln|f| \equiv F$ ($f \equiv \varepsilon e^F$, $\varepsilon = \pm 1$), $\psi_1 - \psi_2 \equiv \varphi$ and hence $a_{1n} = 1 + a_n$, $a_{2n} = 1 - a_n$.

Rewriting the equations of motion in terms of ψ_n, we find that Eqs. (3) - (6) are equivalent to N equations of motion for N functions ψ_n,

$$\partial_u \partial_v \psi_n = \varepsilon \sum_{m=1}^{N} \epsilon_n a_{nm} g_m e^{q_m} \,; \quad \epsilon_1 = -1, \;\; \epsilon_n = +1, \;\text{if}\; n \geq 2 \,, \tag{10}$$

and two constraints,

$$\partial_i^2 \varphi \equiv \partial_i^2 (\psi_1 - \psi_2) = -\sum_{n=1}^{N} \epsilon_n (\partial_i \psi_n)^2, \quad i = u, v \,. \tag{11}$$

With arbitrary parameters a_{mn}, these equations of motion are not integrable. But as proposed in [16] - [18], [20] [23], Eqs.(10) are integrable and constraints (11) can be solved if the N-component vectors $v_n \equiv (a_{mn})$ are pseudo-orthogonal.

Now, consider more general nondegenerate matrices a_{mn} and define the new scalar fields x_n:

$$x_n \equiv \sum_{m=1}^{N} a_{nm}^{-1} \epsilon_m \psi_m \,, \qquad \psi_n \equiv \sum_{m=1}^{N} \epsilon_n a_{nm} x_m \,. \tag{12}$$

In terms of these fields, Eqs.(11) read as

$$\partial_u \partial_v x_m \equiv \varepsilon g_m \exp \sum_{k,n=1}^{N} \epsilon_n a_{nm} a_{nk} x_k \equiv \exp \sum_{k=1}^{N} A_{mk} x_k \,, \tag{13}$$

and we see that the symmetric matrix

$$A \equiv a^T \epsilon a \,, \quad \epsilon_{mn} \equiv \epsilon_m \, \delta_{mn} \,, \tag{14}$$

defines the main properties of the model.

If A is a diagonal matrix, we return to the N-Liouville model. If A is the Cartan matrix of a Lie algebra, the system (13) coincides with the corresponding Toda system, which is integrable and can be more or less explicitly solved (see, e.g., [32], [33]).[5] Here we mostly consider the A_N Toda systems having very simple solutions. However, the solutions have to satisfy the constraints that in terms of x_n are:

$$2 \sum_{n=1}^{N} a_n \, \partial_i^2 x_n \; = - \sum_{n,m=1}^{N} \partial_i x_m \; A_{mn} \; \partial_i x_n \,, \quad i = u, v \,. \tag{15}$$

[5] It can easily be seen that, due to the special structure of a_{mn} ($a_{1n} = 1 + a_n$, $a_{2n} = 1 - a_n$), the Cartan matrices of the simple algebras of rank 2 and 3 cannot be represented in the form (14). Further analysis shows that this probably is true also for any rank. As will be shown in a forthcoming publication, any symmetric matrix A_{mn}, which is the direct sum of a diagonal $L \times L$-matrix $\gamma_n^{-1} \delta_{mn}$ and of an arbitrary symmetric matrix $\bar{A}_{mn}$, can be represented in form (14). If $\bar{A}_{mn}$ is a Cartan matrix, the system (13) reduces to L independent Liouville (Toda A_1) equations and the higher-rank Toda system.

In the N-Liouville model the most difficult problem was to solve the constraints (15) but this problem was eventually solved. In the general nonintegrable case of an arbitrary matrix A we do not know even how to approach this problem. We hope that in the Toda case the solution can be somehow derived but this problem is not addressed here. Instead, in Section 4 we introduce a simplified model that can be completely solved.

Now, let us write the general equations in the form that is particularly useful for the Toda systems. Introducing notation

$$X_n \equiv \exp(-\frac{1}{2}A_{nn}x_n)\ , \quad \Delta_2(X) \equiv X\ \partial_u\partial_v X - \partial_u X\ \partial_v X, \quad \alpha_{mn} \equiv -2A_{mn}/A_{nn}\ , \quad (16)$$

it is easy to rewrite Eqs.(13) in the form:

$$\Delta_2(X_n) = -\frac{1}{2}\varepsilon\ g_n A_{nn} \prod_{m\neq n} X_m^{\alpha_{nm}}\ . \quad (17)$$

The multiplier $|-\frac{1}{2}\varepsilon\ g_n A_{nn}|$ can be removed by using the transformation $x_n \mapsto x_n + \delta_n$ and the final (standard) form of the equations of motion is

$$\Delta_2(X_n) = \varepsilon_n \prod_{m\neq n} X_m^{\alpha_{nm}}\ , \quad (18)$$

where $\varepsilon_n \equiv \pm 1$.

These equations are in general not integrable. However, when A_{mn} are the Cartan matrices, they simplify to integrable equations (see [32]). For example, for the Cartan matrix of A_N, only the near-diagonal elements of the matrix α_{mn} are nonvanishing, $\alpha_{n-1,n+1} = 1$. This allows one to solve Eq.(18) for any N. The parameters α_{mn} are invariant w.r.t. transformations $x_n \mapsto \lambda_n x_n + \delta_n$ and hence A_{mn} can be made non-symmetric while preserving the standard form of the equations (recall that the Cartan matrices of B_N, C_N, G_2, and F_4 are not symmetric). In this sense, α_{mn} are the fundamental parameters of the equations of motion. From this point of view, the characteristic property of the Cartan matrices is the simplicity of Eqs.(18) which allows one to solve them by a generalization of separation of variables. As is well known, when A_{mn} is the Cartan matrix of any simple algebra, this procedure gives the exact general solution (see [32]). In next Section we show how to construct the exact general solution for the A_N Toda system and write a convenient representation for the general solution that differs from the standard one given in [32].

3 Solution of the A_N Toda system

The A_N equations are extremely simple,

$$\Delta_2(X_n) = \varepsilon_n X_{n-1} X_{n+1}\ , \quad X_0 \mapsto 1\ , \quad X_{N+1} \mapsto 1, \quad n = 1, ..., N, \quad (19)$$

and can be reduced to one equation for X_1 by using the relation between $\Delta_2(X)$ and higher determinants, $\Delta_n(X)$ (see [32]):

$$\Delta_2(\Delta_n(X)) = \Delta_{n-1}(X)\ \Delta_{n+1}(X)\ , \quad \Delta_1(X) \equiv X, \quad n \geq 2\ . \quad (20)$$

From Eqs.(19), (20) we find that

$$\Delta_{N+1}(X_1) = \prod_n \varepsilon_n \,. \tag{21}$$

This equation looks horrible but is known to be soluble.

Let us start with the Liouville (A_1 Toda) equation $\Delta_2(X) = g$ (see [34], [35], [32], [23]). Calculating the derivatives of $\Delta_2(X)$ w.r.t. u and v, we find that

$$\partial_u^2\big(X^{-1}\,\partial_v^2 X\big) \;=\; 0\,, \qquad \partial_v\big(X^{-1}\,\partial_u^2 X\big) \;=\; 0\,. \tag{22}$$

It follows that if X satisfies (22) then there exist some 'potentials' $\mathcal{U}(u)$, $\mathcal{V}(v)$ such that

$$\partial_u^2 X \;-\; \mathcal{U}(u)\,X \;=\; 0\,, \qquad \partial_v^2 X \;-\; \mathcal{V}(v)\,X \;=\; 0\,. \tag{23}$$

Thus the Liouville solution can be written as ([23])

$$X(u,v) = \sum a_\mu(u)\; C_{\mu\nu}\; b_\nu(v)\,, \tag{24}$$

where $a_\mu(u)$ and $b_\nu(u)$ $(\mu,\nu = 1,2)$ are linearly independent solutions of the equations

$$a''(u) \;-\; \mathcal{U}(u)\,a(u) \;=\; 0, \qquad b''(v) \;-\; \mathcal{V}(v)\,b(v) \;=\; 0\,. \tag{25}$$

and $C_{\mu\nu}$ is a nonsingular matrix. As the potentials are unknown, the solutions a_1, b_1 can be taken arbitrary while a_2, b_2 then may be defined by the Wronskian first-order equations

$$W[a_1(u), a_2(u)] = 1\,, \qquad W[b_1(v), b_2(v)] = 1\,. \tag{26}$$

The matrix $C_{\mu\nu}$ should obviously satisfy the normalization condition $\det C = g$.

We have repeated this well known derivation at some length because it is completely applicable to the A_N Toda equation (21). By similar but rather cumbersome derivations it can be shown that X_1 satisfy the equations

$$\partial_u^{N+1} X + \sum_{n=0}^{N-1} \mathcal{U}_n(u)\;\partial_u^n X = 0\,, \qquad \partial_v^{N+1} X + \sum_{n=0}^{N-1} \mathcal{V}_n(v)\;\partial_v^n X = 0\,. \tag{27}$$

Thus the solution of (21) can be written in the same 'separated' form (24), where now $a_\mu(u)$ and $b_\nu(v)$ satisfy the ordinary linear differential equations of the order $N+1$ (corresponding to Eqs.(27)), with unit Wronskians,

$$W[a_1(u), ..., a_{N+1}(u)] = 1\,, \qquad W[b_1(v), ..., b_{N+1}(v)] = 1\,, \tag{28}$$

and $\det C = \prod \varepsilon_n$.

As an exercise, we suggest the reader to prove these statements for $N = 2$. The key relation that follows from the condition $\partial_u \Delta_3(X) = 0$ is the partial integral

$$\partial_v\left[\partial_v\left(\frac{X}{\partial_u X}\right) / \partial_v\left(\frac{\partial_u^3 X}{\partial_u X}\right)\right] = 0\,. \tag{29}$$

It follows that the expression in the square brackets is equal to an arbitrary function $A_0(u)$ and thus we have

$$\partial_v\left[\left(\frac{X}{\partial_u X}\right) \;+\; A_0(u)\left(\frac{\partial_u^3 X}{\partial_u X}\right)\right] = 0\,. \tag{30}$$

Denoting the expression in the square bracket by $-A_1(u)$ and introducing the notation $\mathcal{U}_1(u) = A_1/A_0$ and $\mathcal{U}_0(u) = A_1^{-1}$, we get Eq.(27) with $N = 2$.

Let us return to the general solution of Eq.(21). In fact, considering Eqs.(28) as inhomogeneous differential equations for $a_{N+1}(u)$, $b_{N+1}(v)$ with arbitrary chosen functions $a_n(u)$, $b_n(v)$ $(1 \leq n \leq N)$, it is easy to write the explicit solution of this problem:

$$a_{N+1}(u) = \sum_{n=1}^{N} a_n(u) \int^u d\bar{u}\ W_N^{-2}(\bar{u})\ M_{N,n}(\bar{u})\ . \tag{31}$$

Here W_N is the Wronskian of the arbitrary chosen functions a_n and $M_{N,n}$ are the complementary minors of the last row in the Wronskian. Replacing a by b and u by v we can find the expression for $b_{N+1}(v)$ from the same formula (31). To complete the solution we should derive the expressions for all X_n in terms of a_n and b_n. This can be done with simple combinatorics that allows one to express X_n in terms of the n-th order minors. For example, it is very easy to derive the expressions for X_2:

$$X_2 = \varepsilon_1 \Delta_2(X_1) = \varepsilon_1 \sum_{i<j} W[a_i(u), a_j(u)]\ W[b_i(v), b_j(v)]\ ,$$

which is valid for any $N \geq 1$ $(i, j = 1, ..., N+1)$. Note that expressions for all X_n have a similar separated form. This possibly hints that some rather complex separation of variables may give us a tool for (approximate) solving more general, nonintegrable equations (18).

Our simple representation of the A_N Toda solution is completely equivalent to what one can find in [32] but is more convenient for treating some problems. For example, it is useful in discussing asymptotic and analytic properties of the solutions of the original physics problems. It is especially appropriate for constructing wave-like solutions of the Toda system which is similar to the wave solutions of the N-Liouville model. In fact, quite like the Liouville model, the Toda equations support the wave-like solutions. To derive them let us first identify the moduli space of the Toda solutions. Recalling the N-Liouville case, we may try to identify the moduli space with the space of the potentials $\mathcal{U}_n(u)$, $\mathcal{V}_n(v)$. Possibly, this is not the best choice and, in fact, in the Liouville case we finally made a more cumbersome choice suggested by the solution of the constraints. For our present purposes the choice of the potentials is as good as any other because each choice of $\mathcal{U}_n(u)$ and $\mathcal{V}_n(v)$ defines some solution and, vice versa, any solution given by the set of the functions $(a_1(u), ..., a_{N+1}(u))$, $(b_1(v), ..., b_{N+1}(v))$ satisfying the Wronskian constraints (28) defines the corresponding set of the potentials $(\mathcal{U}_0(u), ..., \mathcal{U}_{N-1}(u))$, $(\mathcal{V}_0(v), ..., \mathcal{V}_{N-1}(v))$.

Now, as in the Liouville case, we may consider the reduction of the moduli space to the space of constant 'vectors' $(U_0, ..., U_{N-1})$, $(V_0, ..., V_{N-1})$. The fundamental solutions of the equations (27) with these potentials are exponentials (in the nondegenerate case)

$$a_n(u) = \exp(\mu_n u)\ , \quad b_n(v) = \exp(\nu_n v)\ , \quad \sum_{n=1}^{N+1} \mu_n = 0, \quad \sum_{n=1}^{N+1} \nu_n = 0. \tag{32}$$

In this reduced case we may regard the space of the parameters (μ_n, ν_n) the new moduli space, in complete agreement with the Liouville case. Of course, the constraints (11) define further restrictions on the moduli (μ_n, ν_n) but here we do not address this problem.

Note only that, as in the N-Liouville case, one can construct nonsingular waves. To show this is not much more difficult than in the Liouville case but requires more lengthy derivations. We hope to publish these in a forthcoming paper.

4 A simple integrable model of (1+1)-dimensional dilaton gravity coupled to Toda scalar matter

Let us suppose that the potential V is independent of φ, i.e. $V(\varphi,\psi)\equiv V(\psi)$[6]. Then Eq.(6) is simply the D'Alembert equation for $F(u,v)$. It follows that the metric can be written as

$$f=\varepsilon a'(u)\ b'(v).$$

Due to the residual freedom of the coordinate choice in the light-cone metric we can choose (a,b) as the new (local) coordinates and then denote them by (u,v). In this coordinates and notation we simply have $f=\varepsilon$ and $F=0$ in all the equations. We thus see that the equations (5) are independent of φ and can be solved independently of the equations (3), (4). Suppose that the matter fields ψ are known and first solve equations (3) and (4).

The general solution of Eq.(3) can be written as

$$\varphi=-\varepsilon\int_0^u\int_0^v d\bar{u}d\bar{v}\ V[\psi(\bar{u},\bar{v})]+A(u)+B(v), \tag{33}$$

where $A(u)$, $B(v)$ are arbitrary functions. The constraints (4) in this model have the form

$$\partial_i^2\varphi=-\sum_{n=1}^{N}(\partial_i\psi_n)^2,\quad i=u,v\,. \tag{34}$$

Using (5) we easily derive

$$\partial_i V=\varepsilon\partial_j\sum_1^N(\partial_i\psi_n)^2,\quad (i,j)=(u,v)\text{ or }(v,u)\,, \tag{35}$$

find $A(a)$, $B(b)$ in terms ψ, and finally obtain:

$$\varphi=-\varepsilon\int_0^u\int_0^v d\bar{u}d\bar{v}\ V[\psi(\bar{u},\bar{v})]-\int_0^u d\bar{u}\int_0^{\bar{u}}d\bar{u}\ \Phi_u(\bar{u})-\int_0^v d\bar{v}\int_0^{\bar{v}}d\bar{v}\ \Phi_v(\bar{v})+A'(0)u+B'(0)v\,, \tag{36}$$

where we omitted the unimportant arbitrary term $A(0)+B(0)=\varphi(0,0)$ and denoted

$$\Phi_u(u)\equiv\sum_1^N(\partial_u\psi_n(u,0))^2\,,\qquad \Phi_v(v)\equiv\sum_1^N(\partial_v\psi_n(v,0))^2\,.$$

Now, to get integrable equations for ψ we take the potential (7) with $q_n^{(0)}$ given by the r.h.s. of Eq.(9). Then, we can use for the scalar fields the equations (10) and (12) – (14). If we take the potential for which the ψ equations of motion can be reduced to integrable Toda equations we find an explicit solution for the nontrivial class of dilaton gravity minimally coupled to scalar matter fields. This model is a very complex generalization of the well studied CGHS model in which the scalar fields are free and $V=g$. In future, we plan a detailed study of the A_N case. The easiest case is $N=1$ (the Liouville equation

[6]In this model we suppose that there are N scalar matter fields ψ_n with $n=1,...,N$ while F is trivial and φ is treated separately

for one ψ). The first really interesting but simple theory is the case of two scalar fields satisfying the A_2 Toda equations. Taking, for example,

$$V = \exp(\sqrt{3}\,\psi_1 - \psi_2) + \exp(2\,\psi_2),$$

we find the simplest realization of the A_2 Toda dilaton gravity model the complete solution of which can be obtained by use of the above derivations.

As a simple exercise one may consider the reduction from dimension (1+1) both to the dimension (1+0) ('cosmological' reduction) and to the dimension (0+1) ('static' or 'black hole' reduction) as well as the moduli space reduction to waves. One of the most interesting problems for future investigations is the connection between these three objects. It was discovered in the N-Liouville theory but now we see that it can be found in a much more complex theory described by the Toda equations. It is not impossible that the connection also exists (in a weaker form?) in some nonintegrable theories, say, in theories close to the Toda models.

Note in conclusion, that the Toda equations (one-dimensional) were earlier employed mostly in connection with the cosmological and black hole solutions (see, e.g. [36] - [38]). To include into consideration the waves one has to step up at least on dimension higher. The principal aim of the present paper was to make the first step and explore this problem in a simplest two-dimensional Toda environment.

Acknowledgment:

The author appreciates financial support from the Department of Theoretical Physics of the University of Turin and INFN (Turin Section) and of the Theory Division of CERN, where some results were obtained. The useful discussions with V. de Alfaro and A. Sorin are kindly acknowledged. The author is especially grateful to V. de Alfaro for his support during many years and very fruitful collaboration.

This work was supported in part by the Russian Foundation for Basic Research (Grant No. 06-01-00627-a).

References

[1] V.A. Belinskii and V.E. Zakharov, Sov. Phys. JETP **48** (1978) 985.

[2] D. Maison, Phys. Rev. Lett. **41** (1978) 521.

[3] P. Thomi, B. Isaak, and P. Hajicek, Phys. Rev. **D 30** (1984) 1168.

[4] P. Breitenlohner and D. Maison, Ann. Inst. H. Poincare **46** (1987) 215.

[5] P. Breitenlohner, G. Gibbons and D. Maison, Commun. Math. Phys. **120** (1988) 295.

[6] M.O. Katanaev, J. Math. Phys. **31** (1990) 882.

[7] T. Banks and M. O'Loughlin, Nucl. Phys. **B 362** (1991) 649.

[8] C. Callan, S. Giddings, J. Harvey and A. Strominger, Phys. Rev. **D 45** (1992) 1005.

[9] J. Gegenberg, G. Kunstatter, and D. Louis-Martinez, Phys. Rev. **D 51** (1995) 1781.

[10] M. Cavaglià, V. de Alfaro and A.T. Filippov, Int. J. Mod. Phys. **D 4** (1995) 661; Int. J. Mod. Phys. **D 5** (1996) 227; Int. J. Mod. Phys. **D 6** (1996) 39.

[11] H. Nicolai, D. Korotkin, and H. Samtleben, Integrable classical and quantum gravity, hep-th/9612065 (Cargése lectures).

[12] A.T. Filippov, Mod. Phys. Lett. **A 11** (1996) 1691; Int. J. Mod. Phys. **A 12** (1997) 13.

[13] A. Lukas, B.A. Ovrut and D. Waldram, Phys. Lett. **B393** (1997) 65.

[14] F. Larsen and F. Wilczek, Phys. Rev. **D55** (1997) 4591.

[15] H. Lü, S. Mukherji and C.N. Pope, Int. J. Mod. Phys. **A14** (1999) 4121.

[16] A.T. Filippov, Phys.Part.Nucl. **32S1** (2001) 38; Yadernaya Fizika, **65** (2002) 997 (in English) (see Phys. At. Nucl. **65** (2002) 963).

[17] V. de Alfaro and A.T. Filippov, Integrable low dimensional theories describing higher dimensional branes, black holes, and cosmologies, hep-th/0307269.

[18] V. de Alfaro and A.T. Filippov, Integrable low dimensional models for black holes and cosmologies from high dimensional theories, **Mem. Acc. Sci. Torino**, in press, hep-th/0504101.

[19] G.A. Alekseev, Theor. Math. Phys. **143** (2005) 720.

[20] A.T. Filippov, Theor. Math. Phys. **146** (2006) 95; hep-th/0505060.

[21] A.T. Filippov, Many faces of dimensional reducton; in: Proceedings of the workshop 'Gribov-75' (May 22-24, 2005 , Budapest, Hungary), World. Sci.

[22] A.T. Filippov, Some unusual dimensional reductions of gravity: geometric potentials, separation of variables, and static - cosmological duality, hep-th/0605276.

[23] V. de Alfaro and A.T. Filippov, Theor. Math. Phys. **153** (2006) 1709; hep-th/0612258v2.

[24] J.E. Lidsey, D. Wands and E.J. Copeland, Phys. Rep. **337** (2000) 343.

[25] T. Strobl, Gravity in Two Spacetime Dimensions, hep-th/0011240.

[26] D. Grumiller, W. Kummer and D. Vassilevich, Phys. Rep. **369** (2002) 327.

[27] M. Gasperini and G. Veneziano, Phys. Rep. **373** (2003) 1.

[28] A. Einstein and N. Rosen, J.Franklin Inst. **223** (1937) 43;
N. Rosen, Phys. Zs. der Zowjetunion **12** (1937) 366.

[29] K. Kuhař, Phys. Rev. **D 4** (1971) 986.

[30] S. Chandrasekhar, Proc. Roy. Soc. London **A 408** (1986) 309.

[31] F.J. Ernst, Phys. Rev. **167** (1968) 1175.

[32] A.N. Leznov and M.V. Saveliev, Group - Theoretical Methods for Integration of Non Linear Dynamical Systems, Birkhäuser, Boston 1992.

[33] A.V. Razumov and M.V. Saveliev, Lie Algebras, Geometry, and Toda Type Systems, Cambridge University Press, Cambridge, 1996.

[34] G.D. Dzhordzhadze, A. Pogrebkov and M.K. Polivanov, On the solutions with singularities of the Liouville equation, preprint IC/78/126, ICTP, Trieste, 1978.

[35] J.L. Gervais, **IJMPA 6** (1991) 2805.

[36] L. Castellani, A. Ceresole, R. D'Auria, S. Ferrara, P. Fré and M. Trigiante, Nucl. Phys. **B 527** (1998) 142.

[37] V.D. Ivashchuk and V.N. Melnikov, Class. Quant. Grav. **18** (2001) R87.

[38] P. Fré, A. Sorin, Nucl. Phys. **B733** (2006) 33.