SISSA/ISAS, International School for Advanced Studies
via Beirut 4 - 34014 Trieste - Italy

Aspects of time-dependent backgrounds in String Theory

Thesis submitted for the degree of
"Doctor Philosophiæ"

In
Particle Theory

CANDIDATE

Mónica Borunda Pacheco

SUPERVISOR

Dr. Martin O'Loughlin
Dr. Marco Serone

Trieste, October 2004

Acknowledgments

First and foremost, I would like to dedicate this thesis to my parents **Hilda** and specially **Luis**, for their love, their never-ending support and the sense of security they have given when I needed it most. They have waited so long for this moment to come true and it has finally been arrived.

I would like to thank **Marco Serone** for his supervision of the first part of my thesis.

I am indebted to **Martin O'Loughlin** for his supervision, time, help, guidance and patience during the last half part of my doctorate studies.

I would also like to thank **Michele Trapletti**, **Bruce Bassett**, **Shinji Tsujikawa**, **Matthias Blau** and **George Papadopoulos** for collaborating with me on the various parts of my doctoral work.

I am indebted to **Matthias Blau** for useful discussions, for his guidance, time, support, help, friendship and encouragement throughout the last years.

I thank **Daniele Perini** for his company and invaluable friendship during all these years, for sharing with me smiles, laughs and tears, for becoming part of my family, for all the wonderful moments we had spent together and for teaching me Italian ;-)

I thank **Lotfi Boubekeur** for interesting and useful discussions, for his support in times of trouble, for his smiles in times of happiness, for his time, patience and encouragement, for believing in me and for his invaluable friendship.

I want to thank **Rodolfo Martinez** for his long distance encouragement during the whole period.

I would like to thank **CONACyT** for its help during my graduate studies.

Last but not least, lots of names come to my mind in this moment that I can not write them down so I also thank all **my beloved people** that was close to me during these years.

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Chapter 1

Introduction

In the last decades string theory has become the leading candidate for a theory that consistently unifies all fundamental interactions of nature, including gravity. The theory is supposed to predict gravity and gauge symmetry around flat space. Moreover, the theory is UV-finite. The elementary objects are one-dimensional strings whose vibration modes should correspond to the usual elementary particles. At distances large with respect to the size of the strings, the low energy excitations can be described by an effective field theory and therefore can be compared to quantum field theory, which successfully describes the real world dynamics at low energy. However, even though a lot of progress has been made, it has not yet been possible to confront string theory with real physics. One possibility to achieve this is through cosmology by studying the cosmological implications of string theory.

On the other hand, standard cosmology still faces the old basic questions such as the initial singularity, dimensionality of space-time, cosmological constant, horizon and flatness problem and the origin of the density perturbations in the cosmic microwave background. Moreover, standard cosmology has been fettered by the chains of the singularity theorems of General Relativity demonstrating the need for a more robust theoretical framework, and thus is usually only seen as understanding the evolution of our universe over the past 15 billion years or so since the Big Bang.

However since string theory may reveal to us a consistent theory of Quantum Gravity it should also be able to address the problems of standard cosmology. On the other hand string theory needs to be confronted with physics. All these reasons suggest that cosmology and string theory complement each other in several ways giving rise to string cosmology.

String cosmology is rapidly growing in importance for at least two reasons. First, unless there is a fortunate conspiracy of scales and one or more of the extra dimensions are $\sim O(\text{TeV}^{-1})$ [1], we cannot reasonably expect to detect any stringy effects in accelerators in the foreseeable future. Hence string cosmology may provide the only means for testing string or M-theory concretely, through *e.g.* non-commutative effects [2–4] or the AdS/CFT correspondence [5]. Second, string theory offers the exciting possibility of a resolution to the Big Bang singularity thereby opening up a potentially infinite pre-history of the Big Bang. Models constructed in this vein include the Pre Big Bang [6–8] (see also [9, 10]) and the recent ekpyrotic/cyclic models [11–14] which has lead to new work on string propagation in orbifold backgrounds with curvature singularities [15, 16].

A rigorous way to understand string cosmology is by studying time dependent backgrounds in string theory [15, 17–19], unfortunately this is a hard task due to cosmological singularities that arise in this context, and still much work has to be done.

On the other hand, a good approximation can be made by studying the adiabatic evolution of time independent backgrounds in string theory. In this direction a lot of work has been done within a low energy effective approach (see [9, 20] for recent reviews, discussions and references), like for example, the Brandenberger and Vafa (B&V) model [21] in 1988, continuing with the Pre Big Bang scenario[10], ekpyrotic universes[22], brane cosmology [23], and so on. More recently, there has also been a renewed surge of activity in applying string theory to issues relevant to cosmology through the advent of brane-world models [24–31], the focus on de Sitter space-time [32–38], the application of holographic ideas [39–42], and the controversial possibility that string theory (and quantum gravity more generally) might leave an observable imprint in the cosmic microwave background radiation [43–56].

In this thesis we are going to study aspects of the two different approaches to string cosmology. The first part of the thesis will deal with an aspect of one of the possible low energy effective approaches: we are going to study the adiabatic evolution of the spatial dimensions in the BV scenario [57]. In the second part, on the other hand, we will give a first possible step towards understanding string theory in the cosmological setting of non-trivial time-dependent backgrounds. We will see that the Penrose limits of cosmological singularities are singular homogeneous plane waves. Then we will show the exact string solution in this background and we will study its thermodynamics [58–60].

Since in both parts of the thesis we will deal with string thermodynamics let us motivate our interest in it. It has long been recognized that one requisite element of string cosmology is an understanding of string theory at finite temperature, and much work along these lines has been undertaken [61–74]. These works have also demonstrated that finite temperature studies have the capacity to shed light on critical foundational issues of string theory itself. As a prime example, compared with ordinary quantum field theory string theory is exceedingly well behaved in the ultraviolet, a property largely due to the exponential growth of states as a function of energy. Since the earliest days of string theory it has been recognized that such a growth of states leads to a Hagedorn temperature, a temperature above which the statistical partition function diverges because the exponential Boltzmann suppression is overcome by the exponential density of states. The existence of a Hagedorn temperature is particularly tantalizing because a long-standing, critical question has been: what are the true, fundamental degrees of freedom in string theory? While strings provide a natural and perturbatively useful set of constituents, the existence of a Hagedorn temperature is an indication that strings, somewhat like low energy/temperature hadrons in quantum chromodynamics, may not be the elementary degrees of freedom of string theory. Perhaps, many authors have speculated, just as quark and gluons emerge as the basic ingredients of quantum chromodynamics at high enough temperatures (and energies), the true degrees of freedom of string theory may also emerge at sufficiently high temperatures. In fact, if the specific heat at the Hagedorn temperature is finite this indicates that the Hagedorn temperature is not a limiting temperature but instead demarcates a phase transition, one that might well signal the appearance of the true, elementary degrees of freedom of string theory.

Over the years there have been a number of studies of the Hagedorn temperature in string theory; most have focused on string propagation in flat backgrounds since the calculation of the density of states requires an exactly solvable model. For example, in ref. [74] open bosonic, open and closed supersymmetric, and heterotic string theories were studied in flat noncompact space and it was found that the Hagedorn temperature is a limiting temperature for the open string models but not for the closed string models. In [62] closed strings were studied on a flat compact manifold (T^9) and it was found that the Hagedorn temperature is a limiting temperature. More recently, [75, 76] have studied D-brane thermodynamics and found, for example, that Dp-branes for $p < 5$ possess a non-limiting Hagedorn temperature and hence likely have a high temperature Hagedorn phase.

In spite of many studies a precise understanding of the properties that a Hagedorn phase in string theory would exhibit, is still lacking.[1] A number of the works cited did reveal one likely qualitative feature of strings near the Hagedorn temperature, namely, the coalescing of energy into one long string, but very little else is known. In light of the work of [61] this may not be particularly surprising. Through indirect reasoning, these authors showed that the Hagedorn phase is characterized by a drastic reduction in the number of degrees of freedom (the free energy behaves as T^2 instead of T^D) and hence

[1]Interesting recent progress on this question is reported in [77, 78].

it may well be a highly nontrivial task to identify and rigorously describe the correct high temperature constituents.

In light of the above discussion it is interesting to perform a finite temperature analysis of string cosmology scenarios. In particular, in the first part of the thesis, chapter 3, we will do just this in the context of the BV scenario. Standard cosmology, because of the limitations of general relativity, suffers a great lacking: it is unable to make any predictions about the number of dimensions we live in or about the spatial topology of the Universe. String theory and M-theory, in contrast, predict that we live in either 10 or 11 space-time dimensions. Perhaps the greatest challenge for string cosmology, after understanding the big bang singularity, is to explain why and how three spatial dimensions became observable and large while 6 or 7 are either small or unobservable for some other reason. Perhaps the only proposal in this direction so far is the so-called Brandenberger-Vafa (BV) scenario [21] where it is assumed that the universe is small and compact and that exactly three space dimensions become large because of the dynamics of winding modes which play a particularly important role. Subsequently it has been pointed out in [6] that the low energy effective action of a dilaton-gravity system, naturally emerging in string theory, has a duality symmetry that is a manifestation of the string T-duality $R \to \alpha'/R$ symmetry and which plays a crucial role in the analysis of [21]. In this respect the dilaton-gravity system is more suitable than standard general relativity for the BV scenario [21]. Although some works have already shown that the BV scenario can be realized and extended in a variety of ways [79–89] (see also [90]), a full string-theory analysis is very complicated and still not completed, even in the simplest toroidal compactification.

Winding states and *T-duality* are intrinsic features of string theory and play a fundamental role in the BV model. In this scenario, the universe starts as a 9 dimensional small torus filled with an ideal gas of strings and evolves such that 6 dimensions remain small and 3 dimensions become large as we see them today. How does it take place? Since we are dealing with a compact universe filled with a string gas the strings wind around the dimensions. When the system is out of thermal equilibrium there are interactions between these states, in particular, the winding states wrapping around one sense can annihilate with the states wrapping around the other sense. Now, the key ingredient is that a string describes a 2 dimensional world-sheet, therefore it is easy for the strings to see each other when the dimensionality of space is 3 or less. On the other hand, when the dimensions are more than 3, the strings generically miss each other being unable to annihilate. This, in principle, leads to 3 spatial dimensions free to expand (due to the Kaluza Klein (KK) contribution), and on the other hand, leave the remaining 6 spatial dimensions with winding strings around them preventing them from expansion. On the other hand, since string theory exhibits *T-duality* ($R \to 1/R$), it implies that neither the temperature nor the physical length are singular as $R \to 0$, avoiding the cosmological singularity.

Chapter 3 and [57] are inspired by this scenario. As opposed to the non-equilibrium evolution considered in the BV scenario we will be interested in the adiabatic evolution that takes place when the universe is in thermal equilibrium filled with an ideal gas of closed strings (so we are always dealing with weak string coupling). The analysis is performed in such a way that the "string ingredients", *T-duality* and the presence of winding states, are manifestly present. We study two regimes. First, we address the question of what happens if we start off with 9 dimensional small toroidal universe within a micro-canonical string treatment - the Hagedorn regime. It turns out that the dimensions remain nearly constant around the initial value. It is important to recall how to interpret this result and its connection to the original BV scenario. The above result is obtained by assuming thermal equilibrium and a free, ideal, string gas, whereas the dynamics and interactions of winding modes at very early times are crucial in the BV scenario. The results we get in case (i) in relation to the original BV proposal [21] should therefore be interpreted in the following way - unless string interactions are taken into account and/or thermal equilibrium is relaxed, no interesting dynamics emerges.

Second, we assume that by some mechanism, for example the BV mechanism, there are d large and $9-d$ small dimensions. We consider an almost radiation regime, and thus we study their evolution within a canonical string approach. We will show that when there is only pure radiation, the small dimensions can be stabilized and kept small relative to the large dimensions. Essentially, it is only required that the initial expansion rate of the large dimensions is bigger or of the same order as that of the small ones[2]. This mainly comes from the fact that the pressure in the small dimensions vanishes in the case of pure radiation.

When matter, in the form of KK and winding modes, is included, the choice of initial conditions becomes more relevant. The crucial point is played by winding modes that are able to distinguish large and small dimensions, leading respectively to a positive/negative contribution to the pressure along the large/small dimensions. This leads to the possibility of an expansion of the large dimensions and at the same time keeping the small ones almost constant. In fact we found that there exists a wide range of parameters for which the small dimensions actually remain small (see fig. 3.4), while the large ones expand as required in the presence of radiation and string matter (see fig. 3.5). This is actually achieved for the natural initial condition where the small dimensions

[2]It is actually quite hard to imagine initial conditions where the expansion rate of the small dimensions are bigger than that of the large ones.

are close to the the self-dual radius $\sqrt{\alpha'}$, together with the condition of the expansion rate mentioned in the pure radiation case.

Thus we found that there exist a wide range of initial conditions for which the small dimensions are stabilized around the self-dual radius before entering into a purely radiation dominated phase, regime in which they are asymptotically stabilized anyway to a nearly constant value [3].

In the second part of the thesis we will deal with the study of string theory in the cosmological setting of non-trivial time-dependent and possibly singular backgrounds. Two observations, that will be better explained latter on, are our main motivation for performing this work: *i)* string theory in pp waves backgrounds simplifies due to the existence of a natural light-cone gauge, and in many cases can be exactly solved and quantized [92]; *ii)* the Penrose limit metric gives us information about the rate of growth of curvature and geodesic deviation along a null geodesic of the original space-time [59]. Thus the study of Penrose limits of cosmological singularities becomes an interesting question.

The Penrose limit construction [93] associates to every space-time metric and choice of null geodesic in that space-time a plane wave metric,

$$ds^2 = 2dudv + A_{ab}(u)x^a x^b du^2 + d\vec{x}^2 \,, \tag{1.1}$$

which is characterized by the plane wave profile matrix $A_{ab}(u)$. The computation of the Penrose limit along a null geodesic γ amounts to determining the matrix $A_{ab}(u)$ from the metric of the original space-time.

This has recently been used to show [94] that the Penrose limit of the $AdS_5 \times S^5$ IIB superstring background is the maximally supersymmetric plane wave [92]. String theory in this RR background is exactly solvable [95, 96], giving rise to a novel explicit form of the AdS/CFT correspondence [97]. Following these developments many Penrose limits have been computed for various supergravity backgrounds and their applications have been explored.

In [59] a simple covariant characterization and definition of the Penrose limit wave profile matrix $A_{ab}(u)$ was obtained which does not require taking any limit and which shows that $A_{ab}(u)$ directly encodes diffeomorphism invariant information about the original space-time metric. The geometric significance of $A_{ab}(u)$ (and hence of the Penrose limit) turns out to be that it is the standard [98] *transverse null geodesic deviation matrix* of the original metric along the null geodesic $\gamma(u)$.

The relevance of this result lies in the fact that it tells us precisely which aspects

[3] See also [91] for a different approach leading to the same conclusion.

of the original background, namely covariant information about the rate of growth of curvature and geodesic deviation along a null geodesic, are detected by the Penrose limit and hence probed by, say, string theory in the resulting plane wave background. Thus since singularities of $A_{ab}(u)$ result from curvature singularities of the original space-time, it is of interest to analyze the nature of Penrose limits of space-time singularities, as they encode information about the rate of growth of curvature and geodesic deviation as one approaches the singularity of the original space-time along a null geodesic.

In particular, as a first step toward studying string propagation in singular (and perhaps time-dependent) space-time backgrounds, it is then of interest to determine the Penrose limits of space-time singularities in general. In [99–102] it was found that for a variety of particular brane and cosmological backgrounds the exact Penrose limit is characterized by a wave profile of the special form

$$A_{ab}(u) \sim \frac{c_{ab}}{u^2} \quad . \tag{1.2}$$

Plane wave metrics with precisely such a profile have the scale invariance $(u, v) \rightarrow (\lambda u, \lambda^{-1} v)$ and are thus *homogeneous* singular plane waves (HPWs) [99, 103, 104]. Without loss of generality, $A_{ab}(u)$ can be chosen to be diagonal and, anticipating the interpretation of the entries of A_{ab} as harmonic oscillator frequencies, $A_{ab}(u)$ can be parametrized as

$$A_{ab}(u) = -\omega_a^2 \delta_{ab} u^{-2} \quad , \tag{1.3}$$

where ω_a^2 can be positive or negative.

Moreover, in [59, 105] was observed that the Penrose limit of space-time singularities of cosmological FRW and Schwarzschild-like metrics, i.e. the leading behavior of the profile $A_{ab}(u)$ as one approaches the singularity, is also of the above form. In chapter 5, we will show that this is an universal behavior: the Penrose limits of a large class of black hole, cosmological and null singularities, indeed all the Szekeres-Iyer metrics [106, 107] with singularities of "power-law type", are plane wave metrics of the form (1.3) and the eigenvalues are bounded by 1/4 [58, 59]. More precisely we will find that Penrose limits of spherically symmetric space-like or time-like singularities of power-law type satisfying (but not saturating) the Dominant Energy Condition (DEC) are singular homogeneous plane waves of the type (1.3). The resulting frequency squares ω_a^2 are bounded from above by 1/4 unless one is on the border to an extremal equation of state.

In chapter 6 we will see that string theory in singular homogeneous plane wave backgrounds is exactly solvable [104, 108, 109]. It has also been shown that in a class of such models the string oscillator modes can be analytically continued across the singularity [104]. It is natural, then to consider thermal string theory in such backgrounds, since we retain the necessary analytical control to make explicit calculations even though the space is not flat.

In [110–114] the thermal partition function in the canonical ensemble for type IIB superstrings in plane wave backgrounds with constant null RR 5 form has been studied. It has been found that the Hagedorn temperature is higher than the Hagedorn temperature for strings in flat space, and that the corrections depend on the parameter f, where f is the Ramond-Ramond flux [111–114]. Furthermore, in [110] it was found that the free energy at the Hagedorn temperature is finite which may indicate a phase transition when the system reaches this regime. On the other hand in [113, 115] it was found that in the case of a plane wave supported by the NSNS 3 form flux, the Hagedorn temperature is the same as the one for flat space.

However, as we will see strings propagating in singular homogeneous plane wave backgrounds lead to a time-dependent Hamiltonian and then the concept of thermodynamical quantities requires some thought. A possibility to study such systems is by introducing a suitable analogue of the Boltzmann thermal state for time-independent Hamiltonian system,

$$\hat{\rho}_H = \frac{\mathrm{e}^{-\beta H}}{\mathrm{tr}\,\mathrm{e}^{-\beta H}} , \tag{1.4}$$

which can be constructed as a density operator of the form

$$\hat{\rho}_I = \frac{\mathrm{e}^{-\beta I}}{\mathrm{tr}\,\mathrm{e}^{-\beta I}} , \tag{1.5}$$

where I is an invariant of the system, i.e. $dI/dt = 0$ [116–118]. Therefore the density operator satisfies the quantum counterpart of the classical Liouville theorem for the phase space density, namely $\mathrm{tr}\hat{\rho}_I = 1$ and $d\hat{\rho}_I/dt = 0$. The invariant

$$I = H(t_0) \tag{1.6}$$

is such that the corresponding density operator reduces to the standard choice in the case of a time-independent system and provides an "adiabatic" approximation to the system provided that $H(t)$ varies sufficiently slowly with time near $t = t_0$.

One can then calculate the "thermal" partition function, $\mathrm{tr}\,\mathrm{e}^{-\beta I}$, and it turns out that it diverges when the parameter β takes on a value that is the same as the inverse of the Hagedorn temperature for strings propagating in flat space [60]. This may be an indication that strings propagating in singular homogeneous plane waves backgrounds, which are the Penrose limit in the near singularity limit of the cosmological backgrounds, behave in a similar way to strings propagating in flat space.

This thesis is organized as follows. Chapter 2 and 3 can be considered as the first part of the thesis since the goal is to make a study of the low effective energy scenarios of string cosmology. In chapter 2 we present the preliminaries of string theory and string thermodynamics in flat space. Section 2.1 shows the string theory solutions in the light-cone gauge. In section 2.2 we show the partition function of the system and show the exponential growth in the states. Section 2.3 remarks on the subtleties of thermodynamics in the presence of gravity. The thermal partition function and the appearance of the Hagedorn temperature are presented in Sections 2.4 and 2.5 respectively. The ideal gas of strings and its microcanonical and canonical treatment are introduced in section 2.6. In section 2.7 we study the ideal gas of strings in compact spaces, we study the singular behavior of the partition function and we introduce the complex temperature formalism to deal with the microcanonical ensemble. Finally in section 2.8 we deal with the question of imposing conservation laws to the thermal ensembles by introducing chemical potentials and see that the singularities depend on the chemical potentials.

In chapter 3 we study the fate of the spatial dimensions in the scenario of string gas cosmology at finite temperature. In section 3.2 some general aspects of dilaton gravity are briefly reviewed. Section 3.3 presents the computation of the free energy that will be used in the next chapters. In section 3.4 we analyze the dynamics of the system in the extreme Hagedorn regime of high energy densities considering that all the dimensions are of the same size. Finally, in section 3.5 the dilaton-gravity equations are solved in an almost-radiation dominated regime considering large and small dimensions, as well as in the presence of some massive stringy matter.

The second part of the thesis can be considered chapter 4, 5 and 6 since the main goal is to make a first step into the study of string theory in time dependent backgrounds.

In chapter 4 we show the basic concepts of plane waves and Penrose limits. We start in section 4.1 by introducing the plane wave metrics and showing them in different coordinates systems. In section 4.2 we study the geodesics. Then we proceed to section 4.3 where we will focus on a special type of plane wave, namely homogeneous plane waves, which will be required for the next two chapters. Section 4.4 shows us the traditional approach to computing Penrose limits and in section 4.4 some of their properties are presented. Some explicit examples are in section 4.6 while a covariant presentation of the Penrose limit is given in section 4.5. We end this chapter by showing solvable string models in some of these backgrounds.

In chapter 5 we present the Penrose limits in the near singularity limit of spacetime singularities. We begin section 5.1 by explicitly calculating the Penrose limits of static spherically symmetric metrics in terms of the geodesic deviation approach used in section 4.7, in particular we focus on the Schwarzschild and the FRW metric. In section

5.2 we calculate the Penrose limits of other more general metrics. In section 5.3 we introduce the Szekeres-Iyer metrics (section 5.3.1), analyze their null geodesics (section 5.3.2) and their Penrose limits (section 5.3.3). In section 5.4 we supplement this by an analysis of the Dominant Energy Condition in these models.

The study of string theory and string thermodynamics in the singular homogeneous plane wave backgrounds is performed in chapter 6. We begin by showing the solution of string model in these backgrounds in section 6.1. Then some implications of the scale invariance are shown in section 6.2. In section 6.3 we describe the basic concepts when dealing with thermodynamics of time-dependent systems. Finally in section 6.4 we show the calculation of the "thermal" partition function in these backgrounds and show that it diverges at a parameter which corresponds to the same value of the inverse of the Hagedorn temperature in the case of strings propagating in flat space.

The last chapter gives some conclusions and in appendix A we present the definitions and modular transformations of the Jacobi theta functions. Appendix B shows in some detail the saddle point approximation method for solving integrals. The construction of the partition function of an orbifold is explicitly shown in appendix C. Finally the Ricci and Einstein tensors of the Szekeres-Iyer metrics are presented in appendix D.

Chapter 2

Strings & Thermodynamics in flat space

In the present chapter we will introduce the basic notions of string theory in flat space as well as string thermodynamics. This notions will be essential for chapter 3, where they will be applied to a gas of strings. On the other hand, once we know the ingredients to do some string thermodynamics in flat space it will be easy to do some string thermodynamics in an homogeneous singular plane wave background, chapter 6. Since the objective of this chapter is by no means an introduction to string theory, but to present the concepts that are necessary to be able to consistently follow this thesis, we will not be as complete as one could in this chapter and only show the approaches that will be useful for us in the next chapters. If the reader is interested in a complete introduction on string theory he/she can refer to [119–124] and references therein.

2.1 Solution of bosonic string theory

Let us consider a fixed space-time background $\mathcal{M}$ of dimension D, with coordinates $X = (X^\mu)$, $\mu = 0, \cdots, D-1$, described by the metric $G_{\mu\nu}(X)$ with signature $(-, +, \cdots, +)$.

In analogy to the point particle case, the motion of a relativistic string in this background is described by its generalized world-line, a two-dimensional surface Σ, namely the world-sheet with coordinates $\xi = (\tau, \sigma)$. The background metric induces a metric

$$g_{\alpha\beta} = \frac{\partial X^\mu}{\partial \xi^\alpha}\frac{\partial X^\nu}{\partial \xi^\beta} G_{\mu\nu}\,, \tag{2.1}$$

where $\alpha, \beta = 0, 1$, are world sheet indexes.

In the point particle case, point particles classically follow an extremal path when travelling from one point in space-time to another. In this case, the natural action is proportional to the length of the world-line between some initial and final points. In analogy, the natural action for a relativistic string is its area, measured with the induced metric:

$$S_{NG} = -\frac{1}{2\pi\alpha'}\int_\Sigma d^2\xi |\det g_{\alpha\beta}|^{1/2}\,. \tag{2.2}$$

This is the Nambu-Goto action, which is the direct generalization of the action for a massive relativistic particle. The prefactor $(2\pi\alpha')^{-1}$ is the energy per length or tension of the string and has dimension $[\text{length}]^{-2}$ or $[\text{mass}]^2$.

Even though this action provides a direct geometrical meaning, it is inconvenient for calculating the equations of motion, Hamiltonian, etc. The Polyakov action, which is equivalent to the Nambu Goto action, is used instead. In this approach one introduces an intrinsic metric on the world-sheet $h_{\alpha\beta}(\xi)$ and then the action takes the form of a non-linear sigma-model on the world-sheet [125],

$$S_P = -\frac{1}{4\pi\alpha'}\int_\Sigma d^2\xi\sqrt{h}h^{\alpha\beta}\partial_\alpha X^\mu\partial_\beta X^\nu G_{\mu\nu}(X)\,, \tag{2.3}$$

where $h = |\det h_{\alpha\beta}|$. The energy momentum tensor is

$$T_{\alpha\beta} \equiv \frac{4\pi\alpha'}{\sqrt{h}}\frac{\delta S_P}{\delta h^{\alpha\beta}} = \partial_\alpha X^\mu\partial_\beta X_\mu - \frac{1}{2}h_{\alpha\beta}\partial_\gamma X^\mu\partial^\gamma X_\mu\,, \tag{2.4}$$

and is traceless $h^{\alpha\beta}T_{\alpha\beta} = 0$. Therefore $T_{\alpha\beta}$ has only two independent components. The variation of (2.3) vanishes and the equations of motion reduce to $T_{\alpha\beta} = 0$. The two independent components of $T_{\alpha\beta}$ vanish on shell. Since the trace of the energy-momentum tensor is the Noether current of scale transformations, the two-dimensional field theory, eq (2.3), is scale invariant, indeed it is a conformal field theory.

The Polyakov action has three local symmetries:

- two reparametrizations of the world-sheet:

$$\xi^\alpha \to \bar{\xi}^\alpha(\tau,\sigma) \tag{2.5}$$

- conformal or Weyl transformation:

$$h_{\alpha\beta}(\xi) \to e^{\Lambda(\xi)}h_{\alpha\beta}(\xi) \tag{2.6}$$

These three local symmetries are used to gauge-fix the metric $h_{\alpha\beta}$, with the standard choice corresponding to the conformal gauge,

$$h_{\alpha\beta}(\xi) = \eta_{\alpha\beta}, \quad \text{where} \quad \eta_{\alpha\beta} = \text{Diag}(-1,1)\,. \tag{2.7}$$

However (2.7) exhibits a residual gauge invariance. One can still perform reparametrizations under which the metric only changes by a local, positive scale factor, because this factor can be absorbed by a Weyl transformation. Thus by introducing light cone coordinates,

$$\xi^{\pm} = \tau \pm \sigma \tag{2.8}$$

the conformal reparametrizations are those reparametrizations which do not mix the light cone coordinates:

$$\xi^{+} \to \bar{\xi}^{+}(\xi^{+}) , \qquad \xi^{-} \to \bar{\xi}^{-}(\xi^{-}) . \tag{2.9}$$

The Minkowski world-sheet metric tensor $h_{\alpha\beta} = \eta_{\alpha\beta}$ becomes

$$\eta_{+-} = \eta_{-+} = -\frac{1}{2} , \qquad \eta_{++} = \eta_{--} = 0 . \tag{2.10}$$

2.1.1 Classical equations and quantization

If we consider Minkowski space as the background $G_{\mu\nu} = \eta_{\mu\nu}$, and rewriting (2.3) in light cone coordinates the variation of the Polyakov action with respect to X^{μ} leads to

$$\delta S = T \int d^2\xi \, (\delta X^{\mu} \partial_{+} \partial_{-} X_{\mu}) - T \int_{\tau_0}^{\tau_1} d\tau X'_{\mu} \delta X^{\mu} , \tag{2.11}$$

where $T = (2\pi\alpha')$ is the string tension and

$$\partial_{\pm} = \frac{1}{2}(\partial_{\tau} \pm \partial\sigma) . \tag{2.12}$$

Using periodic boundary conditions for the closed string, and $X'|_{\sigma=0,\bar{\sigma}} = 0$ [1] for the open string, we find the equations of motion

$$\partial_{+}\partial_{-} X^{\mu} = 0 . \tag{2.13}$$

From now on and for the purposes of this thesis we will focus on closed strings.

On the other hand, the Virasoro constraints which were obtained before, namely, $T_{\alpha\beta} = 0$ or

$$T_{10} = T_{01} = \frac{1}{2}\dot{X}X' = 0 , \tag{2.14}$$

$$T_{00} = T_{11} = \frac{1}{4}(\dot{X}^2 + X'^2) = 0 , \tag{2.15}$$

$$\Rightarrow (\dot{X} \pm X')^2 = 0 , \tag{2.16}$$

[1] The prime denotes a derivative with respect to σ and the dot will denote a derivative with respect to τ.

must be imposed as well.

The most general solution to the equation of motion, eq (2.13), that also satisfies the periodicity condition

$$X^\mu(\tau,\sigma+2\pi) = X^\mu(\tau,\sigma)\,, \tag{2.17}$$

can be written as

$$X^\mu(\tau,\sigma) = X^\mu_L(\tau+\sigma) + X^\mu_R(\tau-\sigma)\,, \tag{2.18}$$

where

$$\begin{aligned} X^\mu_L(\tau+\sigma) &= \frac{x^\mu}{2} + \frac{\alpha' p^\mu}{2}(\tau+\sigma) + i\sqrt{\frac{\alpha'}{2}}\sum_{k\neq 0}\frac{\bar\alpha^\mu_k}{k}e^{-ik(\tau+\sigma)}\,,\\ X^\mu_R(\tau-\sigma) &= \frac{x^\mu}{2} + \frac{\alpha' p^\mu}{2}(\tau-\sigma) + i\sqrt{\frac{\alpha'}{2}}\sum_{k\neq 0}\frac{\alpha^\mu_k}{k}e^{-ik(\tau-\sigma)}\,, \end{aligned} \tag{2.19}$$

x^μ and p^μ corresponds to the center of mass position and momentum, X^μ_L describes "left-moving" modes and X^μ_R describes "right-moving" modes. The α^μ_k and $\bar\alpha^\mu_k$ are arbitrary Fourier modes, and k runs over the integers. Since the function $X^\mu(\tau,\sigma)$ must be real it implies that x^μ and p^μ are real and that

$$(\alpha^\mu_k)^\dagger = \alpha^\mu_{-k}\,, \quad (\bar\alpha^\mu_k)^\dagger = \bar\alpha^\mu_{-k}\,. \tag{2.20}$$

By imposing the Poisson brackets on X^μ and $\dot X^\mu$,

$$\begin{aligned} \{X^\mu(\sigma,\tau),\dot X^\nu(\sigma',\tau)\}_{P.B.} &= 2\pi\alpha'\delta(\sigma-\sigma')\eta^{\mu\nu}\,,\\ \{X^\mu(\sigma,\tau),X^\nu(\sigma',\tau)\}_{P.B.} &= \{\dot X^\mu(\sigma,\tau),\dot X^\nu(\sigma',\tau)\}_{P.B.} = 0\,, \end{aligned} \tag{2.21}$$

the Poisson brackets for the oscillators and center of mass position and momentum follow

$$\begin{aligned} \{\alpha^\mu_m,\alpha^\nu_n\}_{P.B.} &= \{\bar\alpha^\mu_m,\bar\alpha^\nu_n\}_{P.B.} = im\delta_{m+n,0}\eta^{\mu\nu}\,,\\ \{\alpha^\mu_m,\bar\alpha^\nu_n\}_{P.B.} &= 0\,, \quad \{x^\mu,p^\nu\}_{P.B.} = \eta^{\mu\nu}\,. \end{aligned} \tag{2.22}$$

The Hamiltonian, given by

$$H = \int d\sigma(\dot X\Pi - L) = \frac{1}{4\pi\alpha'}\int d\sigma(\dot X^2 + X'^2)\,, \tag{2.23}$$

where Π is the momentum conjugate to X^μ

$$\Pi^\mu = \frac{\delta L}{\delta \dot X^\mu}\,, \tag{2.24}$$

can be written as well in terms of oscillators

$$H = \frac{1}{2}\sum_{n\in Z}(\alpha_{-n}\alpha_n + \bar{\alpha}_{-n}\bar{\alpha}_n) \tag{2.25}$$

On the other hand, the Virasoro operators are defined by

$$L_m = 2T\int d\sigma\, T_{--}e^{im(\tau-\sigma)}\,, \qquad \bar{L}_m = 2T\int d\sigma\, T_{++}e^{im(\tau+\sigma)}\,, \tag{2.26}$$

and can be expressed in terms of the oscillators:

$$L_m = \frac{1}{2}\sum_n \alpha_{m-n}\alpha_n\,, \qquad \bar{L}_m = \frac{1}{2}\sum_n \bar{\alpha}_{m-n}\bar{\alpha}_n\,. \tag{2.27}$$

By comparing with (2.25) we see that

$$H = L_0 + \bar{L}_0\,. \tag{2.28}$$

The other operator $L_0 - \bar{L}_0$, which comes from the level matching condition $1/4\pi\alpha' \int d\sigma \dot{X}X'$, is the generator of translations in σ. So $L_0 - \bar{L}_0 = 0$ means that there is no preferred point on the string.

There are different approaches to quantize strings which agree among themselves:

- Covariant Canonical Quantization: the classical variables of the string motion become operators and then impose the Virasoro constraints, in the quantum theory as conditions on states in the Hilbert space.

- Light-Cone Quantization: solve the Virasoro constraints at the level of the classical theory and then quantize. The solution of the classical constraints is achieved in the so-called *light cone gauge*.

- Path Integral Quantization: as the name indicates one should quantize by means of the path integral however it works in an extended Hilbert space that also contains ghost fields.

In what follows we will focus on the *light-cone quantization*.

The invariance we were left with is eq (2.9). For closed strings ξ^+ and ξ^- are reparametrized independently and they transform $\tau = \frac{1}{2}(\xi^+ + \xi^-)$ and $\sigma = \frac{1}{2}(\xi^+ - \xi^-)$ into

$$\begin{aligned} \tau' &= \frac{1}{2}[\bar{\xi}^+(\tau+\sigma) + \bar{\xi}^-(\tau-\sigma)] \\ \sigma' &= \frac{1}{2}[\bar{\xi}^+(\tau+\sigma) - \bar{\xi}^-(\tau-\sigma)]\,. \end{aligned} \tag{2.29}$$

The first equation tell us that τ' is a solution to the free massless wave equation, and once chosen σ' is completely determined. On the other hand, since X^μ also satisfies the free massless wave equation, it is then a natural choice to reparametrize τ' such that it is equal to one of the X^μ, $\tau' \sim X^+$. In other words,

$$X^+ = x^+ + \alpha' p^+ \tau \,. \tag{2.30}$$

The X^+ component of the string coordinates corresponds to the time coordinate as seen in a frame in which the string is traveling at infinite momentum. The light-cone coordinates are defined as

$$X^\pm = X^0 \pm X^1 \,. \tag{2.31}$$

Imposing the Virasoro constraints we can solve for X^- in terms of the transverse coordinates X^i, which means that we can eliminate both X^+ and X^- and only work with the transverse directions. Thus we are left with all positions and momenta of the string but only the transverse oscillators.

We can now quantize the system replacing the Poisson brackets by commutator brackets via the substitution

$$\{\cdots\}_{P.B.} \to -i[\cdots] \tag{2.32}$$

and replace x^μ, p^μ, α^i_n and $\bar{\alpha}^i_n$ by operators.

In order to generalize to superstrings one simply introduces a $D-$plet of Majorana fermions $\psi^\mu_A(\sigma,\tau)$ transforming in the vector representation of the Lorentz group $SO(D-1,1)$. The fermion equation of motion is simply the two-dimensional Dirac equation $\rho^\alpha \partial_\alpha \psi = 0$, where ρ^α represents two-dimensional Dirac matrices. Then, in the light cone coordinates the equations of motion are

$$\begin{aligned} 0 &= \partial_+ \psi^\mu_- = \partial_+(\partial_- X^\mu) \\ 0 &= \partial_- \psi^\mu_+ = \partial_-(\partial_+ X^\mu) \end{aligned} \tag{2.33}$$

and one can proceed as in the bosonic case, by imposing either periodic, Ramond, or antiperiodic, Neveu-Schwarz, boundary conditions one can find the mode expansion and so on. Concerning the light cone quantization we have to take care of the freedom in applying local supersymmetry transformations that preserve the gauge choices. It turns out that they are sufficient to gauge away ψ^+ completely so that we may make the gauge choice

$$\psi^+ = 0 \,. \tag{2.34}$$

2.1.2 Light-cone Hamiltonian

The Virasoro constraint eq (2.16) become

$$(\dot{X}^- \pm X'^-) = (\dot{X}^i \pm X'^i)^2/2p^+ \,, \tag{2.35}$$

giving an expression for the oscillators of X^- in terms of the α^i:

$$\alpha_n^- = \frac{1}{\sqrt{2\alpha' p^+}} \left(\sum_{m\in Z} : \alpha^i_{n-m}\alpha^i_m : -2a\delta_{n,0} \right) \,, \tag{2.36}$$

and a similar expression for $\bar{\alpha}^-$, a is a normal ordering constant in α_0^{-2}.

In the light-cone gauge the identification of α_0^- with p^- is the mass-shell condition, therefore

$$M^2 = (2p^+p^- - p^ip^i) \Rightarrow \alpha' M^2 = 4(N-a) \tag{2.37}$$

where N is the level number operator

$$N = \sum_{m=1}^{\infty} \alpha^i_{-m}\alpha^i_m \,. \tag{2.38}$$

In order that the theory is Lorentz invariant it is required that $a = 1$ and that the dimensions are 26.

The ground state is $|p^\mu\rangle$, for which we have the mass-shell condition $\alpha' M^2 = -4$, which leads to a tachyon. The first excited level will be $\alpha^i_{-1}\bar{\alpha}^j_{-1}|p\rangle$ (imposing the level matching mentioned above)[3]. We can decompose this into irreducible representations of the transverse rotation group $SO(24)$ in the following manner

$$\alpha^i_{-1}\bar{\alpha}^j_{-1}|p\rangle = \alpha^{[i}_{-1}\bar{\alpha}^{j]}_{-1}|p\rangle + [\alpha^{\{i}_{-1}\bar{\alpha}^{j\}}_{-1} - \frac{1}{d-2}\delta^{ij}\alpha^k_{-1}\bar{\alpha}^k_{-1}]|p\rangle + \frac{1}{d-2}\delta^{ij}\alpha^k_{-1}\bar{\alpha}^k_{-1}|p\rangle \,. \tag{2.39}$$

These states can be interpreted as a spin-2 particle $G_{\mu\nu}$ (graviton), an antisymmetric tensor $B_{\mu\nu}$ and a scalar Φ, which are representations of the $SO(24)$ group.

2.2 The partition function

The partition function is

$$Z(\tilde{\tau}) = \mathrm{Tr}[\exp(2\pi i\tilde{\tau}_1 P - 2\pi\tilde{\tau}_2 H)] = (q\bar{q})^{-d/24}\mathrm{Tr}(q^{L_0}\bar{q}^{\tilde{L}_0}) \,, \tag{2.40}$$

[2] The same happens if one wishes to generalize to superstrings. The oscillators of Ψ^- are given in terms of oscillators of the transverse fields.

[3] In the supersymmetric case the spectrum must be additionally truncated by means of the Gliozzi, Scherk and Olive (GSO) projection to get rid of the tachyon and unphysical states.

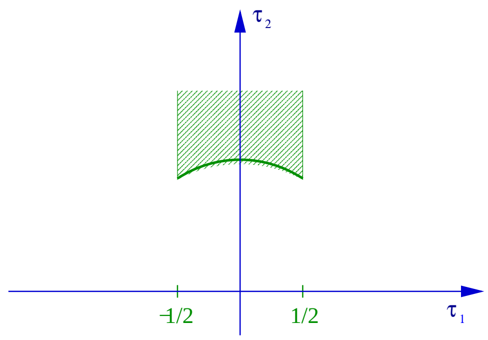

Figure 2.1: Fundamental domain of the torus.

where $\tilde{\tau} = \tilde{\tau}_1 + i\tilde{\tau}_2$ is the modular parameter of the torus, $q = \exp(2\pi i\tilde{\tau})$, the momentum $P = L_0 - \bar{L}_0$ generates translations in σ, and the Hamiltonian $H = L_0 + \bar{L}_0$ generates translations in τ. Explicitly

$$Z(\tilde{\tau}) = V_d (q\bar{q})^{-d/24} \int \frac{d^d k}{(2\pi)^d} \exp(-\pi\tilde{\tau}_2\alpha' k^2) \prod_{\mu,n} \sum_{N_{\mu n},\bar{N}_{\mu n}=0}^{\infty} q^{nN_{\mu n}} \bar{q}^{n\bar{N}_{\mu n}} \,. \tag{2.41}$$

For bosonic string theory the sums are geometric

$$\sum_{N=0}^{\infty} q^{nN} = (1-q^n)^{-1} \,, \tag{2.42}$$

and the partition function for the transverse fields becomes

$$Z(\tilde{\tau}) = iV_d Z_X(\tilde{\tau})^D \tag{2.43}$$

where D stands for the transverse dimensions, which in the case of the $d = 26$ dimensional bosonic theory is $D = 24$, and the contribution from every bosonic field is given by

$$Z_X(\tilde{\tau}) = (4\pi^2\alpha'\tilde{\tau}_2)^{-1/2} |\eta(\tilde{\tau})|^{-2} \,, \tag{2.44}$$

where $\eta(\tilde{\tau})$ is the Dedekind eta function [4].

Thus one-loop partition function for bosonic string theory is

$$Z_T = \int_{\mathcal{F}} \frac{d\tilde{\tau}_2 d\tilde{\tau}_1}{8\pi^2\alpha'\tilde{\tau}_2^2} Z(\tilde{\tau}) \tag{2.45}$$

where $\mathcal{F}$ is the fundamental region of the modular group $PSL(2, Z)$, corresponding to integrating over tori not equivalent under diffeomorphisms, figure 2.1.

[4] Refer to appendix A.

Likewise one can calculate the partition function for closed superstrings, type IIA/B which can be written in terms of the Jacobi theta functions[5] as

$$Z(\tilde{\tau}) = V_{10}(4\pi^2\alpha'\tilde{\tau}_2)^{-5}\eta^{-12}(\tilde{\tau})\bar{\eta}^{-12}(\bar{\tilde{\tau}})\frac{1}{4}|\theta_3^4(\tilde{\tau}) - \theta_4^4(\tilde{\tau}) - \theta_2^4(\tilde{\tau})|^2 . \tag{2.46}$$

2.2.1 Asymptotic formulas for level densities

In this section we are interested in the asymptotic behavior of the level density for very highly excited states. For these purposes we can write the D dimensional partition function in terms of the degeneracy factor $d_N^{(D)}$ at every string level N. Thus let us consider the one loop partition function

$$Z = \int_{\mathcal{F}} \frac{d^2\tilde{\tau}}{\tilde{\tau}_2^2}\frac{1}{\tilde{\tau}_2^{\frac{D-2}{2}}} Z(\tilde{\tau},\bar{\tilde{\tau}}) \tag{2.47}$$

where $\tilde{\tau}_2^{(D-2)/2}$ is the contribution of the non-compact D-dimensional momenta, and of course Z is modular covariant,

$$Z(\tilde{\tau},\bar{\tilde{\tau}}) = |\tilde{\tau}|^{D-2} Z(-\frac{1}{\tilde{\tau}},\frac{1}{\bar{\tilde{\tau}}}) . \tag{2.48}$$

In the case of non-compact bosonic or type IIA/B theory the partition function factorizes as $Z(q,\bar{q}) = Z_D^{\mathrm{L}}(q)\bar{Z}_D^{\mathrm{R}}(\bar{q})$, in other cases this behavior is the asymptotic behavior as $q \to 1$ [63]. Then, the partition function is such that

$$Z_D^{\mathrm{L}}(q) = \sum_{N=0}^{\infty} d_N^{(D)} q^N , \qquad \bar{Z}_D^{\mathrm{R}}(\bar{q}) = \sum_{\bar{N}=0}^{\infty} d_{\bar{N}}^{(D)} q^{\bar{N}} . \tag{2.49}$$

Thus $d_N^{(D)}$ will tell us the asymptotic behavior of the level density and one can calculate it by a contour integral on a small circle about the origin [119, 126]

$$\begin{aligned} d_N^{(D)} &= \frac{1}{2\pi i}\oint \frac{dq}{q^{N+1}} Z_D^{\mathrm{L}}(q) \\ d_{\bar{N}}^{(D)} &= \frac{1}{2\pi i}\oint \frac{d\bar{q}}{\bar{q}^{\bar{N}+1}} \bar{Z}_D^{\mathrm{R}}(\bar{q}) \end{aligned} \tag{2.50}$$

For large N one can evaluate the integral by using a saddle point approximation[6]. $Z_D^{\mathrm{L}}(q)$ vanishes rapidly for $q \to 1$, while if N is very large, q^{N+1} is very small for $q < 1$. Therefore for large N there is a defined saddle point for q near 1. The behavior of $Z_D^{\mathrm{L}}(q)$ and $\bar{Z}_D^{\mathrm{R}}(\bar{q})$ as $q \to 1$, can be obtained from the $\tilde{\tau} \to 1/\tilde{\tau}$ modular transformations of the

[5] Refer to appendix A.

[6] Refer to appendix B.

Dedekind eta and Jacobi beta functions, respectively[7], which relates these two limits by $q \to \exp(4\pi^2/\ln q)$,

$$\begin{aligned} Z_D^{\rm L}(q) &\sim \left(-\frac{\ln q}{2\pi}\right)^{\frac{D-2}{2}} e^{-\frac{4\pi^2 a_L}{\ln q}} , \\ \bar{Z}_D^{\rm R}(\bar{q}) &\sim \left(-\frac{\ln \bar{q}}{2\pi}\right)^{\frac{D-2}{2}} e^{-\frac{4\pi^2 a_R}{\ln \bar{q}}} , \end{aligned} \tag{2.51}$$

where $a_{L,R}$ are the zero-point energies of the various string theories, $a_L = a_R = 1$ (bosonic), $a_L = a_R = 1/2$ (superstring).

One can then solve (2.50) by comparing it with (B.1). And by means of (B.6) we have

$$\begin{aligned} d_N^{(D)} &\sim e^{4\pi\sqrt{a_L N}} N^{-\frac{D+1}{4}} , \\ d_{\bar{N}}^{(D)} &\sim e^{4\pi\sqrt{a_R \bar{N}}} \bar{N}^{-\frac{D+1}{4}} . \end{aligned} \tag{2.52}$$

Using the mass formula $M^2 \sim 4N/\alpha'$, and imposing the level matching condition $N = \bar{N}$, the mass density function $\rho(M)$ follows [63]

$$\rho(M) = d_N \frac{dN}{dM} \sim e^{2\pi\sqrt{\alpha'}(\sqrt{a_L}+\sqrt{a_R})M} M^{-D} . \tag{2.53}$$

Thus the level density grows so rapidly with mass that the partition function of the free theory cannot be defined beyond a maximum temperature $T = T_H$, which is known as the Hagedorn temperature [127].

This exponential growth in the states is a pure *stringy* effect and as we will see, it will lead to a divergence of the canonical partition function which may indicate either a limiting temperature in the theory or a phase transition.

2.3 Thermodynamics in the presence of Gravity

One must be careful when studies the thermodynamics in these systems. Statistical mechanics is rigorously valid only in the infinite volume limit.

On the other hand, any gravitating system with a nonzero mass density per unit volume and sufficiently large volume suffers from the Jeans instability, which is the tendency for the system to collapse. The Jeans instability, in any D spatial dimensions, occurs at a length scale

$$R \geq \left(\frac{\pi v_s}{G\rho}\right)^{\frac{1}{2}} , \tag{2.54}$$

[7] Refer to appendix A.

where v_S is the speed of sound, ρ the density of the system and G the Newton's constant. For a relativistic system this gives a constraint on R for the system to lie inside its Schwarzschild radius R_S, which is reached when $GM/R^{D-2} \sim 1$. Thus, in the case of a relativistic medium, the Jeans instability of the thermal ensemble is the statement that a thermal ensemble of sufficiently large volume should collapse into a black hole.

The Jeans instability tell us that a gravitating system of infinite volume can not be static. Since temperature is an equilibrium concept, it is not a rigorously defined notion in the presence of gravity, however it is a good approximate concept at sufficiently low temperatures: The energy density must be low enough that one can consider an ensemble with a volume that is

- small enough to be unaffected by the Jeans instability, which in ten dimensional string theory is
$$G\rho R^2 << 1\,, \tag{2.55}$$
- large enough to contain many degrees of freedom, for string theory
$$R >> \sqrt{\alpha'}\,. \tag{2.56}$$

They can be simultaneously obeyed if and only if

$$g^2\rho\alpha' << 1\,, \tag{2.57}$$

where g is the closed string coupling ($G \sim g^2$)[61] .

Near the Hagedorn temperature and in ten dimensions the energy density of a gas of free strings is $\rho \sim (\alpha')^{-5}$. Thus (2.57) is satisfied if $g^2\alpha'^{-4} << 1$ which is the weak coupling condition.

2.4 Canonical partition function

The canonical partition function $Z(\beta)$, where $\beta = 1/(kT)$ with k the Boltzmann constant, which from now on we will set $k = 1$ and T the temperature, is defined as the Laplace transform of the density of states

$$Z(\beta) = \int_0^\infty dE\,\rho(E)\,e^{-\beta E}\,. \tag{2.58}$$

By using $E = 2N$ it can be written as

$$Z(\beta) = \int_\mu^\infty dE\,E^{-\frac{D+1}{2}}\,e^{[2\pi(\sqrt{a_L}+\sqrt{a_R})\sqrt{\alpha'}-\beta]E}\,, \tag{2.59}$$

where μ is a cutoff in order to make (2.53) applicable. Thus one can see that the partition function $Z(\beta)$ is defined provided that

$$\beta > \beta_H \,, \tag{2.60}$$

with $\beta_H = 4\pi\sqrt{\alpha'}$, or $2\pi\sqrt{2\alpha'}$ being respectively the Hagedorn temperature for the bosonic or supersymmetric closed string.

2.4.1 Quantum Field approach

As in ordinary Statistical Mechanics, the Helmholtz free energy $F(\beta)$ is defined as

$$F(\beta) = -\frac{1}{\beta}\log Z(\beta)\,. \tag{2.61}$$

In Quantum Field Theory, the canonical partition function in $R^{1,d-1}$ is equal to the vacuum energy of the same theory in $R^{d-1}\times S$ with the length of the compactified circle equal to β. Likewise the free energy will be calculated by computing the one loop world-sheet path integrals for string propagation on $R^{d-1}\times S$, where S corresponds to the euclidean time compactified on a circle of radius β. Therefore one must sum over winding modes in which X^0 is periodic only modulo a multiple of β:

$$X^0 = x^0 + n\sigma + m\tau + \cdots\,. \tag{2.62}$$

where σ and τ are the world sheet coordinates. If we also want to describe fermionic string theories in the RNS formalism, then in addition to summing over m and n, it is necessary to sum over spin structures. Thus we must require

- space-time bosons to be periodic under $X^0 \to X^0 + \beta$.
- space-time fermions to be antiperiodic under $X^0 \to X^0 + \beta$,

which can be implemented by considering a transformation $X^0 \to X^0 + \beta$ accompanied by a 2π spatial rotation [61].

This calculation corresponds to computing the partition function of type IIA/B string theory compactified on S_1/Z_2, where Z_2 is generated by g, the product of a translation $\pi\beta$ along the circle and $(-1)^F$, with F the space-time fermion operator.

The standard calculation[8] gives

$$\begin{aligned} F(\beta) \;=\; & -\frac{1}{\beta}\frac{V_{25}}{(4\pi\alpha')^{25/2}}\int_{\mathcal{F}}\frac{d^2\tilde{\tau}}{2\tilde{\tau}_2^{11/2}}\frac{1}{2|\eta|^{24}} \\ & \times\sum_{n,w\in\mathbb{Z}}\left\{|\theta_2^4|^2\Lambda_{n,w}(-1)^n + |\theta_4^4|^2\Lambda_{n,w+\frac{1}{2}} + |\theta_3^4|^2\Lambda_{n,w+\frac{1}{2}}(-)^n\right\} \end{aligned} \tag{2.63}$$

[8] Refer to appendix C.

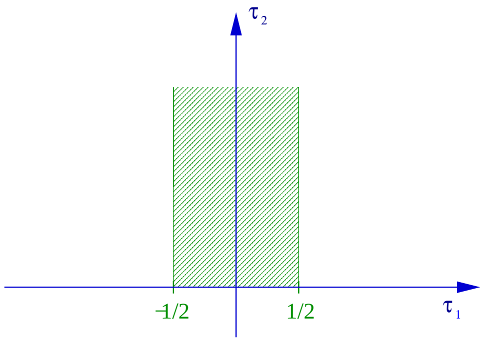

Figure 2.2: S region.

where $\tilde{\tau}$ is the modular parameter of the torus, $\theta_3(0|\tilde{\tau})$ is the Jacobi theta function and $\Lambda_{n,m}$ is the lattice contribution from S_1/Z_2. By means of the unfolding technique [128, 129] one can unfold the $\mathcal{F}$ region into S [9]

$$\begin{aligned} F(\beta) &= -\frac{V_{25}}{(2\pi\sqrt{\alpha'})^{26}} \int_S \frac{d\tilde{\tau}_2}{2\tilde{\tau}_2^6} \frac{1}{2} d\tilde{\tau}_1 \frac{|\theta_2^4|^2}{|\eta|^{24}} \sum_{p\in\mathbb{Z}} \left[\frac{1-(-)^p}{2}\right] e^{-p^2 \frac{\pi\beta^2}{4\tilde{\tau}_2\alpha'}} \\ &= -\frac{V_{25}}{(2\pi\sqrt{\alpha'})^{26}} \int_S \frac{d\tilde{\tau}_2}{2\tilde{\tau}_2^6} \frac{1}{4} d\tilde{\tau}_1 \frac{|\theta_2^4|^2}{|\eta|^{24}} \left[\theta_3\left(0|\frac{i\beta^2}{2\tilde{\tau}_2\alpha'}\right) - \theta_4\left(0|\frac{i\beta^2}{2\tilde{\tau}_2\alpha'}\right)\right] \end{aligned} \quad (2.64)$$

and the integral is evaluated over the strip $S = \{\tilde{\tau}|\tilde{\tau}_2 > 0, -1/2 < \tilde{\tau}_1 < 1/2\}$ shown in figure 2.2.

Likewise one can find the Helmholtz free-energy per unit volume for the bosonic string at one loop [128]

$$F(\beta) = -\pi^{-26} 2^{-14} \int_S \frac{d^2\tilde{\tau}}{\tilde{\tau}_2^2} \tilde{\tau}_2^{-12} |\eta(\tilde{\tau})|^{-48} \left[\theta_3\left(0|\frac{i\beta^2}{2\alpha'\pi^2\tilde{\tau}_2}\right) - 1\right]. \quad (2.65)$$

2.4.2 Counting degrees of freedom

On the other hand one could also calculate the free energy by counting the degrees of freedom which may give a more physical feeling. In order to illustrate this let us consider for simplicity the bosonic string.

The free energy per physical degree of freedom and per unit volume of a bosonic (fermionic) quantum field in $D = d + 1$ dimensions is

$$F_{B,F}(\beta, m) = \pm\frac{1}{\beta} \int \frac{d^{D-1}k}{(2\pi)^{D-1}} \ln(1 \mp e^{-\beta w_k}), \quad (2.66)$$

[9] Refer to appendix C.

where $w_k = \sqrt{k^2+m^2}$. After a bit of manipulation we can rewrite, for example, the bosonic case of (2.66) in terms of the Jacobi theta function:

$$\begin{aligned}
F_B(\beta,m) &= -\frac{\Omega_d}{\beta(2\pi)^d}\sum_{l=1}^{\infty}\frac{1}{l}\int_m^{\infty}(p^2-m^2)^{\frac{d-2}{2}}p\;dp\;e^{-\beta lp} \\
&= -\frac{\Omega_d}{\beta(2\pi)^d}\sum_{l=1}^{\infty}\frac{1}{l}\frac{1}{\sqrt{\pi}}\Gamma[d/2]\left(\frac{2m}{\beta}\right)^{(d-1)/2} mK_{(d+1)/2}(\beta lm) \\
&= -\frac{\Omega_d}{\beta(2\pi)^d}\sum_{l=1}^{\infty}\frac{m}{\sqrt{2l\pi}}\Gamma[d/2]\left(\frac{2m}{\beta}\right)^{(d-1)/2}\left(\frac{\beta lm}{2}\right)^{(d+1)/2} \\
&\quad\times\int_0^{\infty}\frac{dt}{t^{1+(d+1)/2}}e^{-t-\beta^2l^2m^2/(4\alpha' t)} \\
&= -\pi^{D/2}2^{D/2-1}\int_0^{\infty}dss^{-1-D/2}e^{-m^2s/2}\left[\theta_3\left(0|\frac{i\beta^2}{2\pi\alpha' s}\right)-1\right]
\end{aligned} \quad (2.67)$$

where K is the modified Bessel function of second kind and $\Omega_d = 2\pi^{d/2}/\Gamma(d/2)$ is the area of the S^D sphere. Here we expanded $\log(1+x)$, made the change of variables $p = \sqrt{k^2+m}$ and used two different integral representations of the modified Bessel function. Likewise we can find a expression for the fermion fields:

$$F_F(\beta,m) = -\pi^{D/2}2^{D/2-1}\int_0^{\infty}dss^{-1-D/2}e^{-m^2s/2}\left[\theta_4\left(0|\frac{i\beta^2}{2\pi\alpha' s}\right)\right]. \quad (2.68)$$

In order to get the Helmholtz free energy (2.65) we need to include the contribution of all strings states, which is done by taking into consideration the degeneracy factor at every string level N, $d_N^{(D)}$, and impose the level matching condition by means of a delta function. Thus for example in the case of bosonic string theory:

$$\begin{aligned}
F_B &= \sum_{N,\bar{N}=0}^{\infty} d_N^{(D)}\bar{d}_N^{(D)}\int_{-1/2}^{1/2}d\tilde{\tau}_1 e^{i2\pi(N-\bar{N})\tilde{\tau}_1}F_B(\beta,M) \\
&= -\pi^{-D}2^{-(D+2)/2}\int_S\frac{d^2\tilde{\tau}}{\tilde{\tau}_2^2}\tilde{\tau}_2^{-(D-2)/2}|\eta(\tilde{\tau})|^{-2(D-2)}\left[\theta_3\left(0|\frac{i\beta^2}{2\pi^2\alpha'\tilde{\tau}_2}\right)-1\right]
\end{aligned} \quad (2.69)$$

where we have used $M^2 \sim 4N$, $s = \pi\tilde{\tau}_2$ and recall (2.49). For $D = 26$, eq (2.69) corresponds to (2.65) [74].

Likewise by considering a set of relativistic free gas of bosons and fermions in D spatial dimensions, the free energy for the type IIA/B superstring theory can be computed by the supersymmetric system, $\rho_B(M) = \rho_F(M)$:

$$F = -\frac{1}{\beta}\frac{V}{(2\pi)^D}\int d^Dk\int dMg(M)\log\frac{1+e^{-\beta\sqrt{k^2+M^2}}}{1-e^{-\beta\sqrt{k^2+M^2}}}, \quad (2.70)$$

which simplifies to eq (2.64).

2.5 Hagedorn temperature

As anticipated in eq (2.60) the thermal partition function diverges due to the exponential growth of the density of states of highly excited strings. This divergence can be seen directly as well from eq (2.69) (for simplicity we consider the bosonic case), which can be written as

$$F(T) = -\int_S \frac{d\tilde{\tau} d\bar{\tilde{\tau}}}{2\tilde{\tau}_2}(4\pi^2\alpha'\tilde{\tau}_2)^{-13}|\eta(\tilde{\tau})|^{-48}\sum_{r=1}^{\infty} exp(-\frac{r^2}{4\pi T^2\alpha'\tilde{\tau}_2})\,. \tag{2.71}$$

The integral can diverge when $\tilde{\tau}_2 \to 0$, so in order to look at this divergence we focus on $\tilde{\tau} = i\tilde{\tau}_2$. The asymptotic behavior of the Dedekind eta function is obtained by means of its $\tilde{\tau} \to 1/\tilde{\tau}$ modular transformation[10],

$$\eta(-1/i\tau_2) = \eta(i/\tau_2) \to exp(-\pi/12\tau_2)\,, \tag{2.72}$$

we obtain

$$F(T) \to -\int_S \frac{d\tilde{\tau} d\bar{\tilde{\tau}}}{2\tilde{\tau}_2}(4\pi^2\alpha'\tilde{\tau}_2)^{-13}[\tilde{\tau}_2^{-1/2} exp(-\pi/12\tilde{\tau}_2)]^{-48}\sum_{r=1}^{\infty} exp(-\frac{r^2}{4\pi T^2\alpha'\tilde{\tau}_2})\,. \tag{2.73}$$

At $r = 1$ the divergence at $\tilde{\tau}_2 \to 0$ comes from the term

$$\exp\left(-\frac{1}{\tilde{\tau}_2}\left[-4\pi + \frac{1}{4\pi T^2\alpha'}\right]\right) \tag{2.74}$$

which implies that

$$T_H = \frac{1}{4\pi\alpha'^{1/2}}\,. \tag{2.75}$$

The same procedure applies when dealing with the supersymmetric case obtaining as the Hagedorn temperature

$$T_H = \frac{1}{2\pi\sqrt{2\alpha'}}\,. \tag{2.76}$$

Since the free energy is finite at the transition point, it suggests a first order phase transition rather than a limiting temperature [61].

In the canonical ensemble description while approaching the Hagedorn temperature from below the system undergoes violent fluctuations [130], which can be interpreted as the breakdown of the canonical ensemble. Therefore in order to study the system beyond the Hagedorn temperature one must use the microcanonical ensemble [62, 64–66, 130].

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[10] Refer to appendix A.

2.6 The ideal gas of strings

If string theory describes the real world, the very early universe should be modeled by a hot gas of strings. Thus it is of interest to see the general features a stringy early universe might have, as opposed to a universe of elementary particles. The assumption of the ideal gas approximation is that the system is in equilibrium.

2.6.1 The microcanonical ensemble

The microcanonical ensemble is the starting point for the discussion of the thermodynamics of a system. In this ensemble one considers the system at a fixed total energy E. The basic quantity is the total density of states $\Omega(E)$ which is defined to be the total number of energy eigenstates of the system in the energy range E to $E + dE$,

$$\Omega(E) \equiv \sum_{\alpha} \delta(E - E_\alpha)\,, \tag{2.77}$$

where the sum is over all states α of the system and E_α denotes the energy of the state α. The importance of $\Omega(E)$ follows from the fundamental postulate of statistical mechanics which states that in equilibrium, an isolated system with total energy E samples all its eigenstates at that energy with equal probability (provided there are no conserved quantities other than the energy). Then, the entropy is defined as

$$S(E) \equiv \ln \Omega(E)\,, \tag{2.78}$$

and the thermodynamical properties such as the temperature T, the pressure P and the specific heat C_V are defined as

$$\frac{1}{T} \equiv \left(\frac{\partial S}{\partial E}\right)_V , \qquad P \equiv \left(\frac{\partial S}{\partial V}\right)_E , \qquad C_V \equiv -[T^2 \left(\frac{\partial^2 S}{\partial E^2}\right)_V]^{-1} . \tag{2.79}$$

2.6.2 The canonical ensemble

For point like particles (usual systems) an equivalent description of the thermodynamics is given by the canonical ensemble, which basic quantity is the canonical partition function:

$$Z(\beta) \equiv \sum_{\alpha} \mathrm{e}^{-\beta E_\alpha}\,. \tag{2.80}$$

The canonical partition function is the Laplace transform of $\Omega(E)$:

$$Z(\beta) = \int_0^\infty dE\, \mathrm{e}^{-\beta E}\, \Omega(E)\,. \tag{2.81}$$

For usual systems the microcanonical distribution function $\Omega(E)$ and the canonical partition function $Z(\beta)$ provide equivalent descriptions since in that case the integral in (2.81) is dominated by a saddle point located at the value E_0 which is a solution of $\beta = \frac{\partial S(E_0)}{\partial E_0}$. The specification of a β thus picks out an energy $E_0(\beta)$ in the integral. This $E_0(\beta)$ coincides, up to small corrections, with the average energy as defined in the canonical ensemble $\langle E\rangle \equiv Z(\beta)^{-1}\sum_\alpha E_\alpha e^{-\beta E_\alpha}$. For usual systems it is also true that the energy fluctuation $\delta E \equiv [\langle E^2\rangle - \langle E\rangle^2]^{1/2}$ is small compared to $\langle E\rangle$ ($\delta E/\langle E\rangle \sim 1/\sqrt{\langle E\rangle}$). Therefore there is a physical correspondence between the energy E_0 of the microcanonical description and the inverse temperature β of the canonical description. Thus the thermodynamic quantities are essentially the same whether they are computed in the microcanonical ensemble with the system held at a fixed energy E_0 or in the canonical ensemble at a fixed value $\beta = \beta(E_0)$.

For the case of the string gas the physical correspondence between the microcanonical and canonical ensembles fails at high energies. The reason is that at high energies the behavior of $\Omega(E)$ as a function of energy is such that the saddle point approximation to the integral in (2.81) is no longer valid. At a given β (2.81) receives contributions from a large range of energies. Therefore at high energy densities the microcanonical ensemble must be used.

2.7 The ideal gas of strings in compact spaces

The canonical ensemble for free strings has been described in [61, 62, 64–67, 74, 110, 128–133]. The multi-string grand canonical ensemble for free strings can be defined in terms of a sum over single string partition functions for bosonic modes Z_1^B and fermionic modes Z_1^F,

$$\log Z(\beta,\mu,\nu) = \sum_{r=1}^{\infty} \frac{1}{r}\left(Z_1^B(r\beta, r\mu, r\nu) - (-1)^r Z_1^F(r\beta, r\mu, r\nu)\right) . \tag{2.82}$$

If we have non-compact spatial dimensions, the behavior near the Hagedorn temperature will be dominated by the single string $r=1$ term, which we denote $Z_1 = Z_1^B + Z_1^F$, therefore the free energy is $\beta F = -\log Z \sim -Z_1$. However when we are dealing with compact dimensions one should be more careful in relating the single string thermodynamics potentials to the full thermodynamic potentials.

Let us consider a system of closed strings in D spatial dimensions, $\bar{d}$ dimensions are compactified on a $\bar{d}$-torus, each with radius R_i and the remaining d directions are uncompactified, i.e. $D = d + \bar{d}$. The single closed string free energy density is given by

$$Z_1(\beta) = \frac{\beta V_D}{(2\pi\sqrt{\alpha'})^{D+1}} \int_S \frac{d\tilde{\tau}_2 d\tilde{\tau}_1}{\tilde{\tau}_2^2} \mathrm{e}^{-\frac{\beta^2}{4\pi\alpha'\tilde{\tau}_2}} K(q,\bar{q}) W(q,\bar{q},\{R\}) \,, \tag{2.83}$$

where V_D is the spatial volume, $K(q,\bar{q})$ is the standard modular invariant integrand for the uncompactified one-loop partition function and $W(q,\bar{q},\{R\},\{\lambda\},\{\nu\})$ is the contribution due to momenta and windings in the compactified directions, the lattice term,

$$\begin{aligned} W(q,\bar{q},\{R\}) &= \prod_{i=1}^{\bar{d}} W_0(q,\bar{q},R_i) \,, \\ W_0(q,\bar{q},R_i) &= \frac{\sqrt{\alpha'\tilde{\tau}_2}}{R_i} \sum_{p_i,w_i=-\infty}^{\infty} q^{\frac{\alpha'}{4}\left(\frac{p_i}{R_i}-\frac{w_i R_i}{\alpha'}\right)^2} \bar{q}^{\frac{\alpha'}{4}\left(\frac{p_i}{R_i}+\frac{w_i R_i}{\alpha'}\right)^2} . \end{aligned} \tag{2.84}$$

2.7.1 Hagedorn singularity

To extract the singularity of eq (2.83) at the Hagedorn temperature, we must first perform the integral over $\tilde{\tau}_1$ to obtain the leading order contribution in the limit $\tilde{\tau}_2 \to 0$. Again, the saddle point approximation technique becomes our main tool. The relevant factors, for example in type IIA/B, come from $K(q,\bar{q}) \to (\sqrt{\tilde{\tau}_2}/|\tilde{\tau}|)^{-D+1}\exp(\frac{\beta_H^2\tilde{\tau}_2}{4\pi\alpha'|\tilde{\tau}|^2})$ and $W \to V_{\bar{d}}^{-1}(2\pi\sqrt{\alpha'})^{\bar{d}}(\sqrt{\tilde{\tau}_2}/|\tilde{\tau}|)^{\bar{d}}$ leading to

$$\begin{aligned} &\tilde{\tau}_2^{-2} \int_{-1/2}^{1/2} d\tilde{\tau}_1 \left(\frac{|\tilde{\tau}|}{\sqrt{\tilde{\tau}_2}}\right)^{d-1} \exp\left(\frac{\beta_H^2\tilde{\tau}_2}{4\pi\alpha'|\tilde{\tau}|^2}\right) \\ &\sim \tilde{\tau}_2^{(d-3)/2} \int_{-\infty}^{\infty} dx (1+x^2)^{(d-1)/2} \exp\left(\frac{\beta_H^2}{4\pi\alpha'\tilde{\tau}_2}\frac{1}{(1+x^2)}\right) , \end{aligned} \tag{2.85}$$

where the change of variable $x = \tilde{\tau}_1/\tilde{\tau}_2$ was used. The saddle point is at $x = 0$, thus (2.83) becomes

$$Z_1 \sim \frac{\beta V_D}{(2\pi\sqrt{\alpha'})^d \beta_H V_{\bar{d}}} \int_0^{\Lambda} \frac{d\tilde{\tau}_2}{\tilde{\tau}_2} \tilde{\tau}_2^{d/2} \exp\left(\frac{\beta_H^2 - \beta^2}{4\pi\alpha'\tilde{\tau}_2}\right) , \tag{2.86}$$

where the integral is cut off at the upper limit. As $T \to T_H$, (2.86) takes the integral representation of the incomplete Gamma function, which can be written as

$$Z_1 = \begin{cases} c_d(\beta-\beta_H)^{d/2} + \text{regular} & : d\,\text{odd} \\ c_d(\beta-\beta_H)^{d/2}\log(\beta-\beta_H) + \text{regular} & : d\,\text{even}\,, \end{cases} \tag{2.87}$$

with $c_d(\beta_H) \sim (-1)^{(d+1)/2}$ and $c_d(\beta_H) \sim (-1)^{d/2+1}$ for d odd and even respectively. The free energy always remains bounded for $d \neq 0$.

As we approach the Hagedorn temperature, the energy fluctuations become large and there is no smooth thermodynamic limit for a fixed temperature anymore. Thus we must look at the behavior of $\Omega(E,V)$ as E becomes large. When space has d non compact directions, one studies the thermodynamic limit $V \equiv R^d$ large, and if $d=0$, one studies the behavior of $\Omega(E,V)$ at large E as a function of V. The usual approach to perform this study is by the complex temperature formalism [131].

2.7.2 Complex temperature formalism

$Z(\beta)$ is an analytic function of β obtained by a Laplace transform (2.81). In the region of β where this integral converges, $Z(\beta)$ equals the canonical partition function. For strings at large E, $\Omega(E) = O(\mathrm{e}^{\beta_H E})$, hence in general (2.81) converges only in the domain $\mathrm{Re}\beta > \beta_H$. By staying within this region, one can recover $\Omega(E)$ by an inverse Laplace transform

$$\Omega(E) = \int_{L-i\infty}^{L+i\infty} \frac{d\beta}{2\pi i} Z(\beta)\mathrm{e}^{\beta E} \tag{2.88}$$

with $L > \beta_H$. By analytically continuing the partition function $Z(\beta)$ to the whole complex β-plane and deforming the contour of integration one could get a density of states valid in regimes different from the Hagedorn one.

To study the singularity structure of $Z(\beta)$ one writes (2.63) as

$$\log Z(T) = \frac{V_d}{(4\pi^2\alpha')^{d/2}} \int_{\mathcal{F}} \frac{d^2\tilde{\tau}}{2\tilde{\tau}_2^{\frac{(d+2)}{2}}} W(q,\bar{q},R_i) \sum_{\sigma,\rho=\pm} K_{\sigma,\rho}(q,\bar{q})\Lambda_{\sigma,\rho}(\beta,\tilde{\tau}) \tag{2.89}$$

where

$$\Lambda_{\sigma,\rho}(\beta,\tilde{\tau}) = \sum_{L,J} \mathrm{e}^{-\frac{\pi\tilde{\tau}_2}{2}\left(\frac{2\pi^2\alpha' J^2}{\beta^2}+\frac{L^2\beta^2}{2\pi^2\alpha'}\right)} \mathrm{e}^{i\pi\tilde{\tau}_1 LJ} \tag{2.90}$$

where L and J run over integers such that $(-1)^L = \sigma$ and $(-1)^J = \rho$. By expanding $K_{\sigma\rho}$ as $K_{\sigma\rho} = q^{(\sigma-1)/2}\sum_N d_N^{(\sigma\rho)} q^N$, we have

$$\log Z(T) = \frac{V_d}{(4\pi^2\alpha')^{d/2}} \int_{\mathcal{F}} \frac{d^2\tilde{\tau}}{2\tilde{\tau}_2^{\frac{(d+2)}{2}}} \sum_{\sigma\rho=\pm} \sum_{N,\bar{N},m_i,n_i} d_N^{(\sigma\rho)} d_{\bar{N}}^{(\sigma\rho)} \sum_{L,J} \mathrm{e}^{-2\pi[B_{\sigma\rho(\beta)}\tilde{\tau}_2 - iC_{\sigma\rho}\tilde{\tau}_1]}, \tag{2.91}$$

where

$$
\begin{aligned}
B_{\sigma\rho}(\beta) &= \bar{m}_L^2+\bar{m}_R^2+\frac{1}{4}\left(\frac{J(\rho)^2}{\bar{\beta}^2}+L(\sigma)^2\bar{\beta}^2\right)\,, \\
C_{\sigma\rho} &= \bar{m}_L^2-\bar{m}_R^2+\frac{1}{2}L(\sigma)J(\rho)\,, \\
\bar{m}_L^2 &= N+\frac{1}{4}\sum_i\left(\frac{m_i}{\bar{R}_i}+n_i\bar{R}_i\right)^2+\frac{\sigma-1}{4} \\
\bar{m}_R^2 &= \bar{N}+\frac{1}{4}\sum_i\left(\frac{m_i}{\bar{R}_i}-n_i\bar{R}_i\right)^2+\frac{\sigma-1}{4}
\end{aligned}
\tag{2.92}
$$

where $\bar{R}^2\equiv R^2/\alpha'$ and $\bar{\beta}^2\equiv\beta^2/(2\pi^2\alpha')$. The singular behavior arises as $\tilde{\tau}_2\to\infty$ and is given by

$$
\int_{1/2\pi}^{\infty}d\tilde{\tau}_2 \mathrm{e}^{-2\pi B_{\sigma\rho}\tilde{\tau}_2}\tilde{\tau}_2^{-d/2-1}=B_{\sigma\rho}^{d/2}\Gamma(-\frac{d}{2},B_{\sigma\rho})\,, \tag{2.93}
$$

where Γ is the incomplete Gamma function and the lower bound in the $\tilde{\tau}_2$ integral is chosen for convenience. The singular points arise at $B_{\sigma\rho}=0$, which by the level matching ($C_{\sigma\rho}=0$), are

$$
\begin{aligned}
\bar{\beta} &= \pm i\frac{\sqrt{2}}{J(\rho)}(\bar{m}_L\pm\bar{m}_R)\,,\quad (J,L\neq 0)\,, \\
\bar{\beta} &= \pm 2i\frac{\sqrt{2\bar{m}_L}}{L(\rho)}\,,\quad (J=0)\,, \\
\bar{\beta} &= \pm i\frac{J(\rho)}{2\sqrt{2\bar{m}_L}}\,,\quad (L=0)\,.
\end{aligned}
\tag{2.94}
$$

The first leading singularity β_H, that arises approaching the origin from the right is at $\bar{\beta}=2$ ($\beta_H=2\pi\sqrt{2\alpha'}$) coming from $L=\pm1$. The next singularities are all confined to the region $-1\le\mathrm{Re}\bar{\beta}\le 1$ and depend on the radius of the compact dimensions. Thus we see that close to β_H the singularity structure of $\log Z$ is

$$
\log Z\sim[h(\beta,d)(\beta-\beta_H)^{d/2}+\Lambda(\beta)]V_d\,, \tag{2.95}
$$

where $\Lambda(\beta)$ denotes all the regular terms at β_H, and

$$
h(\beta,d)=\left\{\begin{array}{cc} (2\pi^2\alpha'\beta_H)^{-d/2}\frac{\pi(-1)^{(d+1)/2}}{\Gamma[d/2]+1} & : d\,\mathrm{odd} \\ (2\pi^2\alpha'\beta_H)^{-d/2}\frac{(-1)^{(d+1)/2}}{\Gamma[d/2]+1}\log(\beta-\beta_H) & : d\,\mathrm{even}\end{array}\right. . \tag{2.96}
$$

We see that we find the same singularity structure we previously found in (2.87), however by the complex temperature formalism we are able to explicitly calculate all the singularities of the analytically continued $Z(\beta)$.

Given this singularity structure we are able to calculate $\Omega_d(E)$, by using (2.88). However let us notice that the singularities of $\log Z$ are in general branch points. For $\beta_1 < \beta \leq \beta_H$ the discontinuity of $\log Z$ across the cut beginning at β_H is given by

$$\Delta = \begin{cases} -2V_d h(\beta-\beta_H)^{d/2} & : d\,\text{odd} \\ -2iV_d\pi h_0(\beta-\beta_H)^{d/2} & : d\,\text{even} \end{cases}, \tag{2.97}$$

where $h = h(\beta, d)$ as above, and h_0 is the same as $h(\beta, d)$ but without the $\log(\beta - \beta_H)$ term. On the other hand, as we just saw there is an isolated singularity at β_H, all other singularities are to the left of β_H. Since the integral in (2.88) converges as $\text{Im}\beta \to \pm\infty$ for $\text{Re}\beta \geq L_1$ we can deform the contour in (2.88) to C_1 as shown in figure 2.3. Hence

$$\Omega(E, V) = \int_{C_1} \frac{d\beta}{2\pi i} Z(\beta, V) \mathrm{e}^{\beta E}, \tag{2.98}$$

where $Z(\beta, V)$ now stands for the analytical continuation of the canonical partition function. C_1 is the sum of two contours $C_1 = C_0 + C_1'$: C_0 is the part that goes from L_1 to β_H, encircles β_H and goes back to L_1 which gives a contribution $O(\mathrm{e}^{\beta_H E})$ to $\Omega(E, V)$, and C_1' is the vertical part of C_1 ($\text{Re}\beta = L_1$) which gives a contribution $O(\mathrm{e}^{L_1 E})$. While performing the C_0 integral we must be careful and take care of the branch cut which makes this term proportional to $\sin(-\Delta/2i)$. Thus the density of states is

$$\Omega_d(E) = \begin{cases} \frac{1}{\pi}\int_{\beta_1}^{\beta_H} d\beta \mathrm{e}^{\beta E + \Lambda(\beta)V_d} \sin[-iV_d h(\beta-\beta_H)^{d/2}] + O(e^{L_1 E}) & : d\,\text{odd} \\ \frac{1}{\pi}\int_{\beta_1}^{\beta_H} d\beta \mathrm{e}^{\beta E + [\Lambda(\beta) + (\beta-\beta_H)^{d/2} h]V_d} \sin[V_d \pi h_0(\beta-\beta_H)^{d/2}] + O(e^{L_1 E}) & : d\,\text{even} \end{cases} \tag{2.99}$$

where β_1 is the next-to-leading singularity in the real axis. We can expand $\Lambda(\beta) = \Lambda(\beta_H) + \Lambda'(\beta_H)(\beta - \beta_H)$. The sine can also be expanded provided that $\eta\bar{E} >> 1$, where $\eta = \beta_H - \beta_1$ and $\bar{E} = (E - \Lambda'(\beta_H)V_d)$. Thus retaining only the first term and neglecting the terms proportional to $(\beta - \beta_H)^{d/2}$ in the exponent of the integrand for d even one finds

$$\Omega_d(E) \sim \mathrm{e}^{\beta_H E + \Lambda_0 V_d} \frac{V_d(2\pi^2\alpha'\beta_H)^{-d/2}}{\bar{E}^{\frac{d+2}{2}}}, \qquad d > 2, \tag{2.100}$$

with $\Lambda(\beta_H) \equiv \Lambda_0$.

The case $d = 0$ is different since $\log Z$ has a pole rather than a branch cut. In this case $\log(Z) \sim -\log(\beta - \beta_H) + \Lambda(\beta)$ and then the density of states is given by

$$\Omega_0(E) \sim \mathrm{e}^{\beta_H E + \Lambda_0 V_d}. \tag{2.101}$$

However in this case, which is the one we will consider in more detail, the leading singularity β_H is not sufficient to study the thermodynamical properties of the system since

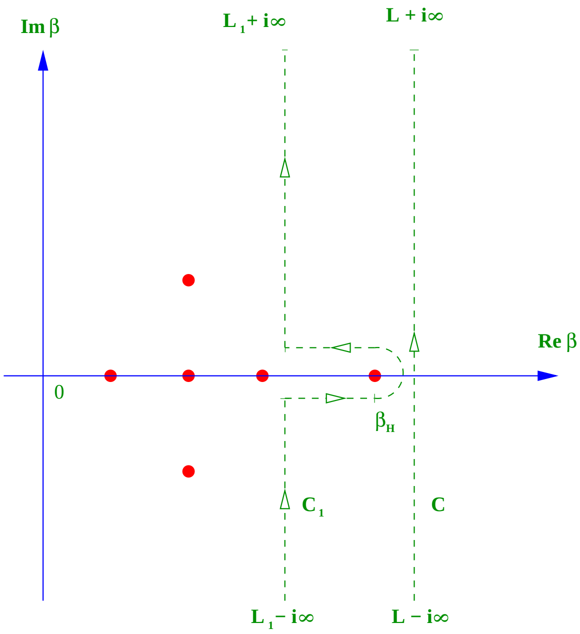

Figure 2.3: Contour integration to calculate $\Omega_d(E)$.

the specific heat $c_V = -\beta^2(\partial^2 S/\partial E^2)|_V^{-1}$ is negative. So let us consider the subleading singularity. In the case where all compact dimensions have the same radius $\bar{R} > 1$, the subleading singularity appears at $\beta_1 = \sqrt{4 - 2\alpha'/R^2}$, and it has a degeneracy of 36 ($L = \pm$, and 18 different KK momenta $m_i = \pm 1$). Hence β_1 is an order 18 pole of $\log Z$. Since there are no zeros of Z between β_1 and β_H, Z can be reparametrized as

$$Z(\beta) \sim \mathrm{e}^{\Lambda(R,\beta)} \frac{1}{\beta - \beta_H} \frac{\eta}{(\beta - \beta_1)^{18}}\,, \tag{2.102}$$

where $\Lambda(R, \beta)$ is a regular term. Using the residue theorem the density of states is then

$$\Omega_0(E) \sim \mathrm{e}^{\beta_H E + \Lambda_0(R)} \frac{1}{\eta^{17}} \left[1 - \frac{(E\eta)^{17}}{\Gamma(18)} \mathrm{e}^{\eta E}\right]\,, \tag{2.103}$$

where $\Lambda_0(R) = \Lambda(R, \beta_H)$ is constant along the integration region.

For $d = 1, 2$, eq (2.100) is not valid since the corrections are large in the thermodynamical limit V_d, $\bar{E} \to \infty$ with $\bar{E}/V_d$ constant. Hence one should solve (2.88) by the saddle point approximation. The density of states in this case is

$$\Omega_1(E) = \mathrm{e}^{\beta_H E + \Lambda_0 V_1} \int \frac{dz}{2\pi i} \mathrm{e}^{f_1(z)} \tag{2.104}$$

where $z = \beta - \beta_H$ and

$$f_1(z) = \bar{E}z + h(\beta, 1)\sqrt{z}V_1 + O(z^2)\,. \tag{2.105}$$

The saddle point of f_1 is at $z_0 = \pi V_1^2/\bar{E}^2(2\pi^2\alpha'\beta_H)^{-1}$ and thus

$$\Omega_1(E) \sim \frac{V}{\bar{E}^{3/2}\sqrt{2\pi^2\alpha'\beta_H}} e^{\beta_H E + \Lambda_0 V_1 - \pi V_1^2/(\bar{E}2\pi^2\alpha'\beta_H)} . \tag{2.106}$$

The corrections are of order $f_1'''(z_0)/(f_1''(z_0))^{3/2} \sim \sqrt{\bar{E}}/V_1$ arising from higher orders in the saddle point approximation. These are small provided that $\sqrt{\bar{E}} < V_1$ which is always valid in the above thermodynamical limit. The corrections of order $z_0^2 V_1 \sim V_1^5/\bar{E}^4$ arise from the $O(z^2)$ in (2.105). These require that $\bar{E}/V_1 < 1$ or that the coefficients associated to them are small.

In a similar way we can calculate Ω_2 by using the expression (2.104) replacing $V_1 \to V_2$ and $f_1 \to f_2$. In this case the saddle point of f_2 is located at $-\log z_0 = 1 + 2\pi^2\alpha'\beta_H\bar{\rho}$ with $\rho = \bar{E}/V_2$.

$$\Omega_2(E) \sim e^{\beta_H E + \Lambda_0 V_2 - \frac{1}{2}(2\pi^2\alpha'\beta_H\bar{\rho}+1)} e^{-\frac{V_2}{2\pi^2\alpha'\beta_H} e^{-(2\pi^2\alpha'\beta_H\bar{\rho}+1)}} . \tag{2.107}$$

The corrections are of order $f_2'''(z_0)/(f_2''(z_0))^{3/2} \sim (V_2 z_0)^{-1/2}$ and then the saddle point approximation is valid if $\bar{\rho} < \log V_2$ which is valid in the thermodynamic limit. If $\bar{\rho} > \log V_2$ we can use (2.99) without neglecting the factor $h \sim \log z$ in the exponent. Since this factor is slowly varying we can make a good approximation by fixing $-\log z_0 \sim 2\pi^2\alpha'\beta_H \log V_2$. Hence one gets

$$\Omega_2(E) \sim e^{\beta_H E + \Lambda_0 V_2} \frac{V_2(2\pi^2\alpha'\beta_H)^{-1}}{(\bar{E} - V_2 \log V_2)^2} . \tag{2.108}$$

2.8 Conservation laws and chemical potentials

If there are conserved quantities other than the energy an isolated system at fixed energy E will not sample all of its energy eigenstates at that energy. It will only sample a subset that corresponds to a fixed value of each of the conserved quantities, because by definition all processes that take place inside the system will keep these quantities constant. The fundamental postulate that all microstates are equally probable applies to this restricted set of states, and hence these are the states we would like to count. Let Q generically denote all conserved charges other than the energy. In addition to E (and V), the macroscopic specification of the system now includes another set of parameters Q, which define the total charge of each kind in the box. The microcanonical distribution function which counts the states at energy E and total charge Q is given by (taking Q to be discrete)

$$\Omega(E, Q) \equiv \delta(E - E_\alpha)\delta_{Q,Q_\alpha} . \tag{2.109}$$

The entropy and the temperature, pressure, C_V, etc. are defined as before and now depend upon Q.

In the case of toroidal compactifications that we have been considering, the conserved quantities are the total winding number and the discrete momenta in each compact spatial dimension.

By introducing a chemical potential for each conserved charge the density of states is given by

$$\Omega(E,Q) = \int_{-i\pi}^{i\pi} \frac{d\mu}{2\pi i} \int_{L-i\infty}^{L+i\infty} \frac{d\beta}{2\pi i} e^{\beta E + 2i\pi\mu Q}\, Z(\beta,\nu)\,, \tag{2.110}$$

where the contour in β is taken to lie to the right of any singularity in Z just as before and the canonical partition function is defined as

$$Z(\beta,\mu) \equiv \sum_\alpha e^{-\beta E_\alpha - \mu q_\alpha}\,. \tag{2.111}$$

The main difference compared to the case where no conservation laws are taking into account is that the Hagedorn singularity will depend on the chemical potentials and in the radius of the compact dimensions.

2.8.1 Hagedorn singularity

Let us consider for simplicity chemical potentials in one dimension. The partition function is again given by (2.83) by replacing

$$W(q,\bar q,\{R\}) \to W(q,\bar q,\{R\},\{\lambda\},\{\nu\}) \tag{2.112}$$

where

$$\begin{aligned} W(q,\bar q,\{R\},\{\lambda\},\{\nu\}) &= \prod_{i=1}^{\bar d} W_0(q,\bar q,R_i,\lambda_i,\nu_i)\,, \\ W_0(q,\bar q,R_i,\lambda_i,\nu_i) &= \frac{\sqrt{\alpha'\tilde\tau_2}}{R_i} \sum_{p_i,w_i=-\infty}^{\infty} q^{\frac{\alpha'}{4}\left(\frac{p_i}{R_i}-\frac{w_iR_i}{\alpha'}\right)^2} \bar q^{\frac{\alpha'}{4}\left(\frac{p_i}{R_i}+\frac{w_iR_i}{\alpha'}\right)^2} e^{-(\nu_i p_i + \lambda_i w_i)} \end{aligned} \tag{2.113}$$

and (λ_i,ν_i) correspond to the chemical potentials associated to the momenta and winding conserved numbers.

In the limit $\tilde\tau_2 \to 0$, the sum over p and w in the factor $W_0(q,\bar q,R,\lambda,\nu)$ can be approximated by integrals. By performing the change of variables $x = \tilde\tau_1/\tilde\tau_2$, $y = \sqrt{\alpha'}\tilde\tau_2(p/R + wR/\alpha')$ and $z = \sqrt{\alpha'}\tilde\tau_2(p/R - wR/\alpha')$, eq (2.83) becomes

$$\begin{aligned} Z_1 \;\sim\; & \int_0^\Lambda \frac{d\tilde\tau_2}{\tilde\tau_2}\tilde\tau_2^{(d-3)/2} \int dx\, dy\, dz \exp\left(-\frac{\beta^2}{4\pi\alpha'\tilde\tau_2} + \right. \\ & \left. \frac{1}{2\tilde\tau_2}\left[\frac{4\pi}{1+x^2} - \pi(y^2+z^2) - i\pi x(y^2-z^2) - \gamma_+ y - \gamma_- z\right]\right) \end{aligned} \tag{2.114}$$

where

$$\gamma_\pm = \frac{1}{\sqrt{\alpha'}}\left(\nu R \pm \frac{\alpha'\lambda}{R}\right) . \tag{2.115}$$

By using the saddle point approximation, we can integrate over x, y and z [11] leading to

$$Z_1 \sim \int_0^\Lambda \frac{d\tilde{\tau}_2}{\tilde{\tau}_2}\tilde{\tau}_2^{d/2}\exp\left(-\frac{\beta^2}{4\pi\alpha'\tilde{\tau}_2} + \frac{1}{16\pi\tilde{\tau}_2}\left(\sqrt{8\pi^2+\gamma_+^2}+\sqrt{8\pi^2+\gamma_-^2}\right)^2\right) . \tag{2.116}$$

As we see the partition function still preserves the form of (2.86) and thus the nature of the Hagedorn singularity is unchanged. However the inverse Hagedorn temperature depends on the chemical potentials and on the radius of the compact dimensions and is given by

$$\beta_H = \left(\sqrt{\beta_0^2+\alpha'\gamma_+^2}+\sqrt{\beta_0^2+\alpha'\gamma_-^2}\right) , \tag{2.117}$$

where $\beta_0 = \pi\sqrt{8\alpha'}$. As the chemical potential is increased, the Hagedorn temperature decreases [131]. The general expression for several chemical potentials replaces $\gamma_\pm^2$ by $\gamma_\pm^2 = \sum_i(\nu_i R_i \pm \alpha'\lambda_i/R_i)^2/\alpha'$. The free energy is still given by (2.87) but now with $\beta_H = \beta_H(\lambda,\nu)$ given by (2.117).

Finally just as we did in section 2.7 the total density of states, and thus the thermodynamical quantities, can be computed from (2.110) by integrating over the complex β plane.

[11] The saddle point is at the real values of $y = -(\gamma_+/2\pi)(1+ix)^{-1}$ and $z = -(\gamma_-/2\pi)(1-ix)^{-1}$ where x lies on the imaginary axis at $ix = (\sqrt{\beta_0^2+\alpha'\gamma_+^2}-\sqrt{\beta_0^2+\alpha'\gamma_-^2})/(2\beta_H)$.

Chapter 3

String gas cosmology at finite temperature

In this chapter we present the work made in [57], in which inspired by the BV scenario, the dilaton-gravity equations of motion [79] were numerically solved with some types of "stringy matter". Adiabatic evolution (which implies constant entropy), weak string coupling and thermodynamical equilibrium are always assumed in our analysis. For simplicity, we analyze Type IIA or IIB closed string theory on a T^9 torus, with no branes[1]. In particular, we consider the following two regimes:

- (i) Hagedorn matter at high energy densities in a very small homogeneous and isotropic universe with a common compactification radius $\sim \sqrt{\alpha'}$, and
- (ii) an almost-radiation dominated regime with two independent scale factors, associated with the large and small dimensions.

In the latter case, the lightest Kaluza Klein (KK) and winding mode contributions are also taken into account.

In both of these regimes, our matter is manifestly T-duality invariant. This symmetry is broken in our set-up only by the (arbitrary) choice of initial values. The main relevant questions in the two cases are respectively: which is the evolution of the universe at early times for a free Hagedorn string gas in thermal equilibrium? Assuming large and small dimensions as initial conditions, how do they evolve? In particular, do the small dimensions remain small?

[1] In [80] it has been shown that fundamental strings, even in the presence of D-branes, are still the dominant degrees of freedom for the realization of the BV scenario.

String matter in the Hagedorn phase was already briefly discussed in [79] where it was realized that, to a first approximation in which the energy is constant, it leads to a very slow evolution of the universe, as in eq. (3.34). We complete the analysis of [79] by relaxing this approximation and imposing the conservation laws for KK and winding modes. We find that in any practical sense there is no departure from the behavior dictated by eq. (3.34), and thus no relevant dynamics emerges in this set-up.

On the other hand, the evolution of the small and large dimensions in case (ii) is much more interesting. First of all, we will see that when there is only pure radiation, the small dimensions can be stabilized and kept small relative to the large dimensions. When we introduce matter in the form of KK and winding modes, the winding modes will be able to distinguish large and small dimensions, leading respectively to a positive/negative contribution to the pressure along the large/small dimensions, as shown in eqs. (3.56) and (3.57). As we will see it turns out that there exists a wide range of parameters for which the small dimensions actually remain small (see fig. 3.4), while the large ones expand as required in the presence of radiation and string matter (see fig. 3.5).

3.1 Dilaton gravity

We shall study the dilaton gravity equations of motion with a massless dilaton field Φ corresponding to the low-energy effective action of string theory in $D+1$ space-time dimensions, described by [6, 7, 79]:

$$S = \int d^{D+1}x\sqrt{-g}\left[e^{-2\Phi}\left\{R + 4(\nabla\Phi)^2\right\} + \mathcal{L}_M\right], \tag{3.1}$$

where g is the determinant of the background metric $g_{\mu\nu}$, and $\mathcal{L}_M$ corresponds to the Lagrangian of some matter. The coupling of Φ with gravity is the standard one arising in string theory. Hereafter we shall consider the case with $D = 9$. We are interested in the case where the whole universe is small and compact, corresponding to a flat T^9-torus. In this case, if one considers field configurations that are spatially homogeneous, assuming only slow time-dependence and adopting an adiabatic approximation it has been shown in [6, 79] that the action (3.1) exhibits a duality symmetry, a low energy manifestation of the string T-duality $R \to 1/R$ symmetry. The ansatz for the metric and dilaton we use is

$$ds^2 = -dt^2 + \sum_{i=1}^{9} R_i^2(t)dx_i^2\,, \quad R_i = e^{\lambda_i(t)}\,, \quad \Phi = \Phi(t)\,. \tag{3.2}$$

Here the R_i denotes the i-th scale-factor of the torus. The equations of motion simplify if one introduces a shifted dilaton, ψ, via

$$\psi \equiv 2\Phi - \sum_{i=1}^{9} \lambda_i \,. \tag{3.3}$$

Given the metric (3.2) the equations of motions of the dilaton-gravity system are then [79]

$$\begin{aligned} -\sum_{i=1}^{9} \dot{\lambda}_i^2 + \dot{\psi}^2 &= e^{\psi} E, & (3.4)\\ \ddot{\lambda}_i - \dot{\psi}\dot{\lambda}_i &= \frac{1}{2} e^{\psi} P_i, & (3.5)\\ \ddot{\psi} - \sum_{i=1}^{9} \dot{\lambda}_i^2 &= \frac{1}{2} e^{\psi} E, & (3.6) \end{aligned}$$

with E the total energy and P_i the total pressure along the i-th direction found by multiplying the total spatial volume of the space by the energy density and pressure appearing in $\mathcal{L}_M$ of (3.1). Here a dot denotes derivative with respect to cosmic time, t. These equations are manifestly invariant under the duality symmetry [6, 79]

$$\lambda_i \to -\lambda_i, \quad \Phi \to \Phi - \sum_i \lambda_i, \tag{3.7}$$

under which ψ defined in eq. (3.3) is left invariant. It is typically assumed that the scale factors R_i are the same in all directions, *i.e.* $R_i = R$. In contrast, we also consider in section 3.4 a scenario where the background is homogeneous and isotropic in d-spatial large dimensions and $(9-d)$-spatial small dimensions. We denote the large and small dimensions with their corresponding scale factors, as

$$R = e^{\mu} \,, \quad r = e^{\nu} \,. \tag{3.8}$$

In this case eqs. (3.4)-(3.6) take the form

$$\begin{aligned} -d\dot{\mu}^2 - (9-d)\dot{\nu}^2 + \dot{\psi}^2 &= e^{\psi} E, & (3.9)\\ \ddot{\mu} - \dot{\psi}\dot{\mu} &= \frac{1}{2} e^{\psi} P_d, & (3.10)\\ \ddot{\nu} - \dot{\psi}\dot{\nu} &= \frac{1}{2} e^{\psi} P_{9-d}, & (3.11)\\ \ddot{\psi} - d\dot{\mu}^2 - (9-d)\dot{\nu}^2 &= \frac{1}{2} e^{\psi} E, & (3.12) \end{aligned}$$

where

$$\begin{aligned} P_d &= -\frac{\partial F}{\partial \mu_i}, \qquad \forall i = 1, \ldots, d \,, & (3.13)\\ P_{9-d} &= -\frac{\partial F}{\partial \nu_i}, \qquad \forall i = d+1, \ldots, 9 \,, & (3.14) \end{aligned}$$

in terms of the free energy F.

3.2 The one loop string partition function with compact dimensions

Consider type IIA/B string theory with all dimensions compactified in a rectangular tori (*i.e.* simple products of circles), where $(9-d)$ spatial dimensions are compactified on small radii all equal to a common value denoted r, whereas the remaining d directions are taken very large, and all equal to R. As we saw section 2.7 the free energy can be written as

$$F^{(d)}(\beta) = -\frac{V_d}{2\pi\sqrt{\alpha'}}\int_{-\frac{1}{2}}^{\frac{1}{2}} d\tilde{\tau}_1 \int_0^{\infty} \frac{d\tilde{\tau}_2}{\tilde{\tau}_2^{(3+d)/2}}\,[\Lambda(r;\tilde{\tau})]^{9-d} \sum_{p=1}^{\infty} e^{-\frac{\beta^2 p^2}{4\pi\tilde{\tau}_2}}\,|M_2|^2(\tilde{\tau})\,, \qquad (3.15)$$

where

$$\Lambda(r;\tilde{\tau}) = \sum_{m,n=-\infty}^{+\infty} q^{\frac{1}{4}(\frac{m}{r}+nr)^2}\,\bar{q}^{\frac{\alpha'}{4}(\frac{m}{r}-nr)^2}\,, \qquad (3.16)$$

represents the contributions of the whole KK and winding modes along the small dimensions. In deriving eq.(3.15) the winding modes along the large dimensions have been completely neglected and the sum over the KK modes has been approximated by an integral over continuous momenta. Note that p in eq.(3.15) runs only over positive *odd* numbers and correspond to taking the correct quantum statistic for bosons and fermions. Taking only the term $p=1$ in the above sum corresponds to replacing the quantum bosonic/fermionic distribution with the classical Maxwell-Boltzmann distribution. V_d is the volume of the large dimensions in $(4\pi^2)$ units[2]:

$$V_d \equiv \frac{1}{(4\pi^2)^{d/2}}(2\pi R)^d = R^d\,, \qquad (3.17)$$

The M_2 factor in eq. (3.15) encodes the contribution to the free energy of the whole tower of massive strings, and can be expanded in powers of q:

$$M_2(\tilde{\tau}) = \frac{\theta_2(\tilde{\tau})^4}{\eta(\tilde{\tau})^{12}} = \sum_{N=0}^{\infty} D(N)\,q^N\,. \qquad (3.18)$$

Here θ_2 and η are modular functions on the torus (see *e.g.* the appendix A for an explicit expression) and $D(N)$ is the degeneracy factor at level N ($D(0)=16$, for example). The value of N corresponds to each string mass level. The $\tilde{\tau}_1$ and $\tilde{\tau}_2$ integrals in eq. (3.15)

[2]Recall that we are setting $\alpha'=1$.

can be easily performed. It is convenient to consider the term with $N = \bar{N} = m_i = n_i = 0$ in eq. (3.15) separately from the remaining ones. This is the contribution of the purely massless states, which we will henceforth denote as F_{rad} (where "rad" stands for radiation). We get

$$F^{(d)}_{rad} = -\frac{R^d}{2\pi} D(0)^2 \Gamma\left(\frac{d+1}{2}\right)(4\pi)^{(d+1)/2}\zeta(d+1)(1-2^{-(d+1)})\beta^{-d-1}\,, \tag{3.19}$$

where $\xi(x)$ is the Riemann zeta-function. The remaining "matter" terms give

$$F^{(d)}_{mat} = -\frac{V_d}{\pi}\sum_{m_i,n_i}\sum_{p=1}^{\infty}\sum_{N,\bar{N}}\left(\frac{2\pi M}{\beta p}\right)^{(d+1)/2}\delta_{m_i n_i+N-\bar{N},0}\, D(N)D(\bar{N})\, K_{(d+1)/2}(\beta p M)\,, \tag{3.20}$$

where $K_{(d+1)/2}$ are modified Bessel functions, and

$$M = M(m_i, n_i, r_i, N, \bar{N}) \equiv \sqrt{\sum_{i=1}^{9-d}\left(\frac{m_i^2}{r^2}+n_i^2 r^2\right)+2(N+\bar{N})}\,. \tag{3.21}$$

The total free energy is given by

$$F^{(d)} = F^{(d)}_{rad} + F^{(d)}_{mat}\,. \tag{3.22}$$

The infinite sum over N and $\bar{N}$ is not always convergent. In fact, the degeneracy factors $D(N)$, for large values of N, have a leading exponential behavior $D(N) \sim \exp(2\pi\sqrt{2N})$. On the other hand, for large values of its argument, the modified Bessel function $K_n(z)$ admits an asymptotic expansion whose leading term is $\sim \exp(-z)$. Hence, the sum over N and $\bar{N}$ in eq. (3.20) converges only for $\beta > \beta_H = 2\pi\sqrt{2}$. The temperature $T_H = 1/\beta_H$ is the Hagedorn temperature.

As long as we deal with a range of energies where eq. (3.15) converges and no large energy fluctuations are present, we can work with the canonical ensemble. On the other hand, for high energy densities a microcanonical description has to be used. In this case the energy density of states is governed by the analytic structure of the canonical partition function $Z = \exp(-\beta F)$, in the complex β-plane. Taking into account the leading singularities of (3.15), we can parametrize the partition function as

$$Z(\beta, R) \simeq \frac{e^{\Lambda(\beta,R)}}{\beta-\beta_H}\left(\frac{\eta_K}{\beta-\beta_K}\right)^{18}\left(\frac{\eta_W}{\beta-\beta_W}\right)^{18}, \tag{3.23}$$

where η_K and η_W are defined by eq. (3.26) with $\beta_K = \beta_H - \eta_K$ and $\beta_W = \beta_H - \eta_W$. $\Lambda(\beta, R)$ is an entire function in β. The microcanonical energy distribution function $\Omega(E)$, is then given by

$$\Omega(E) = \int_{\beta_H-i\infty}^{\beta_H+i\infty}\frac{d\beta}{2\pi i}\,Z(\beta,R)\,e^{\beta E} \simeq \sum_{i=H,K,W}\oint_{C_i}\frac{d\beta}{2\pi i}\,Z(\beta,R)\,e^{\beta E}, \tag{3.24}$$

where C_H, C_K and C_W are the three contours encircling respectively the poles in β_H, β_K and β_W in the complex β-plane. The entropy and the rest of the thermodynamical quantities easily follow from eq. (3.24).

3.3 Hagedorn regime

According to the original BV proposal [21], the very early universe was compact over all nine spatial dimensions with radii $r \sim 1$ in string units. In this section we study the dilaton-gravity equations (3.4)-(3.6), with E and P the total energy and pressure of a free string gas in thermal equilibrium in such a compact, small universe. For simplicity, we consider the string gas associated with type IIA/IIB string theory compactified on a square T^9-torus, simple product of nine circles, with radii all equal to a common value $r = e^\lambda$. Although the strict thermodynamical limit $V \to \infty$ cannot be taken for this system, thermodynamics is still trustable as long as the system contains many degrees of freedom. In our case, this implies having an energy density $\rho \gg 1$. Our first step is then to derive the equation of state of the string matter in this regime, or equivalently the energy and pressure entering in eqs. (3.4)-(3.6).

3.3.1 Microcanonical ensemble

As we already saw in section 2.7, for a totally compact space at high energy, the leading singularity (a simple pole) of the partition function at $\beta = \beta_H$ is not sufficient to establish the thermodynamical properties of the system [21, 64]. The first next-to-leading singularities are poles of order 18 with a dependence on the compactification radius. In this case a useful parametrization of the one-loop partition function Z is given in eq. (3.23), from which one computes the density of states $\Omega(E)$ by means of eq.(3.24) and thus the associated entropy $S = \log \Omega(E)$ [3]. It reads

$$S(E,r) \simeq \beta_H E + \log \left\{ 1 - \frac{1}{\Gamma(18)\eta_{KW}^{18}} \left[(\eta_K E)^{17} \eta_W^{18} e^{-\eta_K E} + (\eta_W E)^{17} \eta_K^{18} e^{-\eta_W E} \right] \right\}, \quad (3.25)$$

[3] We have numerically checked that the term $\Lambda(\beta, R)$ in eq. (3.23) is negligible and thus is not reported in the following.

where $\beta_H = 2\sqrt{2}\pi$, and

$$\begin{aligned}
\eta_K &= \sqrt{2}\pi\left[2-\sqrt{4-\frac{2}{r^2}}\right], \\
\eta_W &= \sqrt{2}\pi\left[2-\sqrt{4-2r^2}\right], \\
\eta_{KW} &= \sqrt{2}\pi\left\{\left[2-\sqrt{4-\frac{2}{r^2}}\right]-\left[2-\sqrt{4-2r^2}\right]\right\}.
\end{aligned} \tag{3.26}$$

The energy as a function of r is given directly by (3.25), since S = constant, by the assumption of adiabatic evolution. On the other hand, the temperature and pressure, defined as

$$\frac{1}{T}=\frac{\partial S}{\partial E}, \qquad P=\frac{T}{9}\frac{\partial S}{\partial(\log r)}, \tag{3.27}$$

yield

$$\frac{1}{T} = \frac{1}{T_H}-\frac{1}{xE\,\Gamma(18)}\left[\left(\frac{z}{w}\right)^{18}y^{17}(17-y)e^{-y}+\left(\frac{y}{w}\right)^{18}z^{17}(17-z)e^{-z}\right], \tag{3.28}$$

$$P = \frac{T}{9}\frac{\dot{x}}{x}\frac{1}{\dot{\lambda}}, \tag{3.29}$$

where

$$x\equiv 1-\frac{1}{\Gamma(18)}\left(\frac{yz}{w}\right)^{18}\left(\frac{e^{-y}}{y}+\frac{e^{-z}}{z}\right), \tag{3.30}$$

and $y\equiv\eta_K E$, $z\equiv\eta_W E$, $w\equiv\eta_{KW}E$. As usual in the microcanonical ensemble, the temperature is a derived quantity (from S and E) and its explicit form is needed only to compute the pressure P. When the radius r is close to unity, T and P are approximately given by

$$\frac{1}{T}\sim\frac{1}{T_H}+C_1E^{17}e^{-\tilde{\eta}E}, \tag{3.31}$$

$$P\sim C_2E^{17}e^{-\tilde{\eta}E}, \tag{3.32}$$

where $\tilde{\eta}\simeq\eta_K\simeq\eta_W$ for $r\sim 1$, and C_1 and C_2 are certain polynomial functions of $\tilde{\eta}$ and η_{KW}.

We numerically solved the dilaton-gravity equations (3.4)-(3.6) using a standard Runge-Kutta routine. We adopted initial conditions around $E_0\sim 1000$, $r_0\sim 1$, which comes from the requirement of T-duality. The shifted dilaton is chosen to satisfy the condition $e^{\Phi}\ll 1$ to ensure that the string coupling constant is initially small and hence that perturbation theory and the ideal gas approximation are trustable. The initial condition for $\dot{\lambda}$ is somewhat arbitrary and we have carried out simulations for a wide variety of different initial values of $\dot{\lambda}$. Notice that $\dot{\psi}_0^2$ is fixed by the constraint equation (3.4) and that the negative solution is taken to remain in the perturbative regime of small string coupling constant.

For initial conditions $r_0 \in [0.8, 1.2]$ and $E_0 \sim 1000$, the temperature is very close to the Hagedorn temperature with a nearly constant value. This is clear from eq. (3.31), since the last term in (3.31) is vanishingly small relative to the first term, due to the exponential suppression given by $e^{-\tilde{\eta}E}$. Similarly we have $P \simeq 0$ for the above initial conditions from eq. (3.32). Therefore the system is effectively described by a pressureless dust as shown in fig. 3.1. In this case one has $\dot{\lambda} \simeq Ae^{\psi}$ from eq. (3.5), with A an integration constant. Subtracting eq. (3.6) from eq. (3.4), we find a simple relation, $\ddot{(e^{-\psi})} = E/2$. Taking note that E is nearly constant ($E \simeq E_0$), the analytic solutions of eq. (3.4)-(3.6) in the Hagedorn regime may be written as

$$e^{-\psi} \simeq \frac{E_0}{4}t^2 + Bt + \frac{B^2 - dA^2}{E_0}, \tag{3.33}$$

$$\lambda \simeq \lambda_0 + \frac{1}{\sqrt{d}}\log\left|\frac{(E_0 t + 2B - 2\sqrt{d}A)(B + \sqrt{d}A)}{(E_0 t + 2B + 2\sqrt{d}A)(B - \sqrt{d}A)}\right|. \tag{3.34}$$

A and B are integration constants depending on the initial values for $\dot{\lambda}$, ψ and $\dot{\psi}$. In particular

$$A = \dot{\lambda}_0 e^{-\psi_0}, \qquad B = -\dot{\psi}_0 e^{-\psi_0}, \tag{3.35}$$

and d is the number of dimensions (we are now considering the case with $d = D = 9$). Notice that due to eq. (3.4), $\dot{\psi}_0$, and thus B, can not be taken to be vanishing.

In fig. 3.1 we plot the evolution of r that corresponds to the analytic solution (3.34), together with the full numerical results. They show very good agreement, which implies that the Hagedorn regime is well described by a state with a constant energy and negligible pressure. This actually ensures the validity of the analytic estimation in ref. [79] discussed briefly in its appendix.

As long as $\dot{\lambda}_0$ is positive (negative), the radius grows (decreases) towards the asymptotic value

$$r_\infty = e^{\lambda_0}\left|\frac{B + \sqrt{d}A}{B - \sqrt{d}A}\right|^{1/\sqrt{d}}, \tag{3.36}$$

with $\dot{r}$ getting smaller with time (see fig. 3.1). We have checked this for values of r_0 very close to 1, up to $r_0 = (1 + 1 \times 10^{-15})$, and found no substantial changes in behavior. If one chooses exactly $r_0 = 1$, eq. (3.25) should be replaced by another similar relation, since now the two poles of order 18 in the β-plane approach each other to a single pole

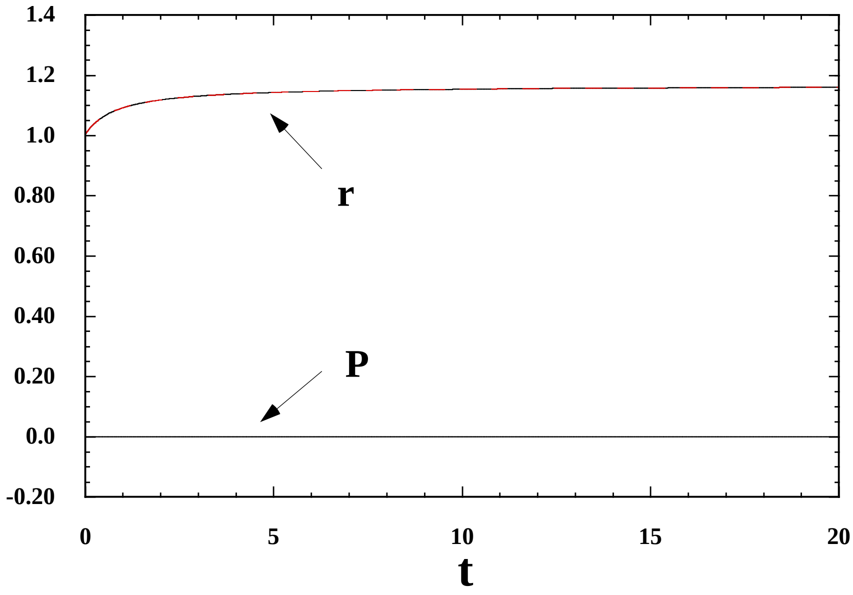

Figure 3.1: The evolution of r and P for the initial conditions $\lambda_0 = 10^{-3}$, $\dot{\lambda}_0 = 0.2$, $\psi_0 = -5$ and $E_0 = 10^3$ in the Hagedorn regime. $\dot{\psi}_0$ is determined by the constraint equation (3.4). We plot the evolution of r both using the analytic approximation (3.34) and the full numerical result, which show very good agreement each other.

of order 36. The above results apply also in this case: the pressure is almost zero and the evolution of the system is very slow in time.

For initial values of E in the range $E_0 \in [500, 5000]$, the dynamics of the system is practically the same as explained above. For initial energies $E_0 \geq 1000$ the scale factor is essentially constant in time. On the other hand the system is typically unstable for $E_0 \ll 1000$ and not thermodynamically meaningful for such low values of E_0.

3.3.2 Conservation laws

We can also address the question of what happens when conservation laws are taken into account. In the case of the toroidal compactification we consider, the conserved quantities are taken to be the total winding number N_i and the KK momenta M_i in each compact dimension ($i = 1, \ldots, 9$). As we saw in section 2.8, this is performed by introducing a chemical potential for each conserved charge [131].

A crucial difference with respect to the previous case is that the leading singularity of the partition function, namely the Hagedorn temperature, is enough to study the thermal behavior of the system. In particular, the leading singularity now depends on the compactification radius induced by the conservation laws. The entropy of such an ideal gas can be easily computed, yielding

$$S(E,r) \simeq \beta_H E - \frac{\pi}{4\sqrt{2}\,E}\sum_{i=1}^{9}\left(\frac{M_i^2}{r^2}+N_i^2 r^2\right) - 9\,\ln E\,. \tag{3.37}$$

We see that in eq. (3.37) there are two suppression terms with respect to the leading term $\beta_H E$ in eq. (3.25). This is expected, because eq. (3.25) counts states with all values of charges whereas (3.37) counts a smaller set of states in which the value of the charge is fixed. Moreover, the number of states decreases as M_i or N_i are increased, since less energy is available for the oscillators. The temperature and pressure obtained from the entropy (3.37) are:

$$\frac{1}{T} = \frac{1}{T_H} - \frac{9}{E} + \frac{1}{2\sqrt{2}\pi}\frac{1}{E^2}\sum_{i=1}^{9}\left(\frac{M_i^2}{r^2}+N_i^2 r^2\right)\,, \tag{3.38}$$

$$P = \frac{T}{9}\frac{\pi}{2\sqrt{2}}\frac{1}{E}\sum_{i=1}^{9}\left(\frac{M_i^2}{r^2}-N_i^2 r^2\right)\,. \tag{3.39}$$

The pressure vanishes if one imposes vanishing winding and KK charge, $M_i = N_i = 0$. The energy E evolves very slowly for E_0 of order 1000 in which case the radius r asymptotically approaches a constant value after some growth from $r_0 \simeq 1$, thereby showing similar behavior to fig. 3.1. When $M_i, N_i \neq 0$, due to presence of the E factor in the denominator of eq. (3.39), the system evolves similarly to the case $M_i = N_i = 0$, as long as the summation terms in eqs. (3.5) and (3.6) are unimportant. The dynamics changes if M_i and N_i are of order 10^5, since the scale factor can have respectively a significant expansion or decreasing rate. As expected, winding modes prevent expansion, whereas KK modes, as standard matter, favorite it.

As a last remark, notice that the string gas defined by eqs. (3.38) and (3.39) has a negative specific heat. Systems with negative specific heat are unstable in non-compact spaces, but actually can be in thermal equilibrium in a finite space. Along the lines of [63, 65, 134], we have evaluated the critical volume V_c under which the system is in equilibrium and found that this is actually the case for $V_c \gg 1$, implying that the system can be actually in equilibrium with radiation. It should also be emphasized that this system is trustable as long as string corrections are completely neglected. When string interactions are included, the system is most likely to undergo a phase transition [61] whose details are so far unknown.

3.4 Almost-radiation regime

Assuming that some dimensions (d) start to expand while the remaining $(9-d)$ dimensions remain small by means of some mechanism, the system will eventually reach a temperature below the Hagedorn regime where the dynamics is mostly governed by massless states, *i.e.* radiation[4]. In this case, it is important to see the conditions under which the large dimensions continue to expand. At the same time, it is important to study whether the small dimensions remain small or if they also enter an expanding phase. Different from the Hagedorn regime discussed in the section 3, the microcanonical ensemble agrees with the canonical one for the range of temperatures and energy densities involved now. In the following we shall use the latter ensemble, which is more convenient for practical purposes.

Assuming again thermodynamical equilibrium and adiabatic evolution, we shall solve the dilaton-gravity equations (3.9)-(3.12), with E and P obtained from the free energy of a string gas at temperatures below T_H. The entropy S is conserved with time under the assumption of adiabatic evolution:

$$\frac{d}{dt}S = \frac{d}{dt}\left(\beta^2\frac{\partial F}{\partial \beta}\right) = 0\,. \tag{3.40}$$

Eq. (3.40) is solved by letting β and the scale factors $\lambda_i = \log R_i$, be slowly varying functions of time $[\beta \to \beta(t),\ \lambda_i \to \lambda_i(t)]$. In this way one can derive a differential equation whose solution gives $\beta = \beta(\lambda_i)$ with S =constant. We denote the radii of the large d dimensions, taken all equal, by $R = e^{\mu}$, whereas the radii of the $(9-d)$ small dimensions, again all equal, by $r = e^{\nu}$.

3.4.1 Pure radiation

As a first step, let us consider the case of pure radiation. The energy and pressure are easily evaluated from eq. (3.19) :

$$E^{(d)}_{rad} = F^{(d)}_{rad} + \beta\frac{\partial F^{(d)}_{rad}}{\partial\beta} = \frac{dR^d}{2\pi}D(0)^2\Gamma\left(\frac{d+1}{2}\right)(4\pi)^{\frac{d+1}{2}}\zeta(d+1)(1-2^{-(d+1)})\beta^{-d-1}\,, \tag{3.41}$$

whereas the pressure $P^{(d)}_{rad}$ for the d spatial dimensions is given by

$$P^{(d)}_{rad} = -\frac{1}{d}\frac{\partial F^{(d)}_{rad}}{\partial(\ln R)} = -F^{(d)}_{rad} = \frac{E^{(d)}_{rad}}{d}\,, \tag{3.42}$$

[4]Notice that the system is already in an almost radiation regime for $\beta \geq 11$ and it is essentially governed by pure radiation only for $\beta \geq 14$, in string units.

which corresponds to the equation of state for radiation in d spatial dimensions. Eq. (3.41) is nothing but the d-dimensional generalization of the Stefan-Boltzmann law in presence of $D(0)^2/2$ bosonic and fermionic degrees of freedom. Since $F_{rad}^{(d)}$ does not depend on r, the pressure along the small dimensions vanish:

$$P_{rad}^{(9-d)} = -\frac{1}{9-d}\frac{\partial F_{rad}^{(d)}}{\partial(\ln r)} = 0\,. \tag{3.43}$$

From the adiabatic equation (3.40), we easily get the following relation

$$\beta = \beta_0 \frac{R}{R_0}\,, \tag{3.44}$$

relating the temperature and scale factor in a radiation-dominated universe, with β_0 and R_0 being initial conditions satisfying $\beta(R_0) = \beta_0$. In this case the dilaton-gravity equations (3.9)-(3.12) read

$$\ddot{\psi} = \frac{1}{2}d\dot{\mu}^2 + \frac{1}{2}(9-d)\dot{\nu}^2 + \frac{1}{2}\dot{\psi}^2\,, \tag{3.45}$$

$$\ddot{\mu} = \dot{\psi}\dot{\mu} + \frac{1}{2}e^{\psi}P_{rad}^{(d)}\,, \tag{3.46}$$

$$\ddot{\nu} = \dot{\psi}\dot{\nu}\,, \tag{3.47}$$

together with the constraint equation

$$\dot{\psi}^2 = e^{\psi}E_{rad}^{(d)} + d\dot{\mu}^2 + (9-d)\dot{\nu}^2\,. \tag{3.48}$$

Eq. (3.47) is integrated to give

$$\dot{\nu} = \dot{\nu}_0 e^{\psi-\psi_0}\,, \tag{3.49}$$

where $\dot{\nu}_0$ are ψ_0 are the initial values of $\dot{\nu}$ are ψ. We see from eq. (3.49) that when $\dot{\nu}_0$ is positive (negative) the expansion rate for the small dimensions is always positive (negative). In order to avoid unbounded growth of the dilaton towards the strongly coupled regime ($e^{\Phi} \geq 1$), it is natural to consider the case with negative $\dot{\psi}$. In this case the absolute value of $\dot{\nu}$ decreases with time.

In the absence of the pressure $P_{rad}^{(d)}$ in eq. (3.46), the evolution of the large dimensions is similar to that of the small ones. In the case $\dot{\psi}_0 < 0$ and $\dot{\mu}_0 > 0$, we have $\ddot{\mu} < 0$ for $P_{rad}^{(d)} = 0$ from eq. (3.46). This corresponds to the universe with expanding large dimensions with a decreasing Hubble rate. Numerically we found that the evolution of the system in this case is trivial, namely the large dimensions soon approach a nearly constant value with very small $\dot{\mu}$.

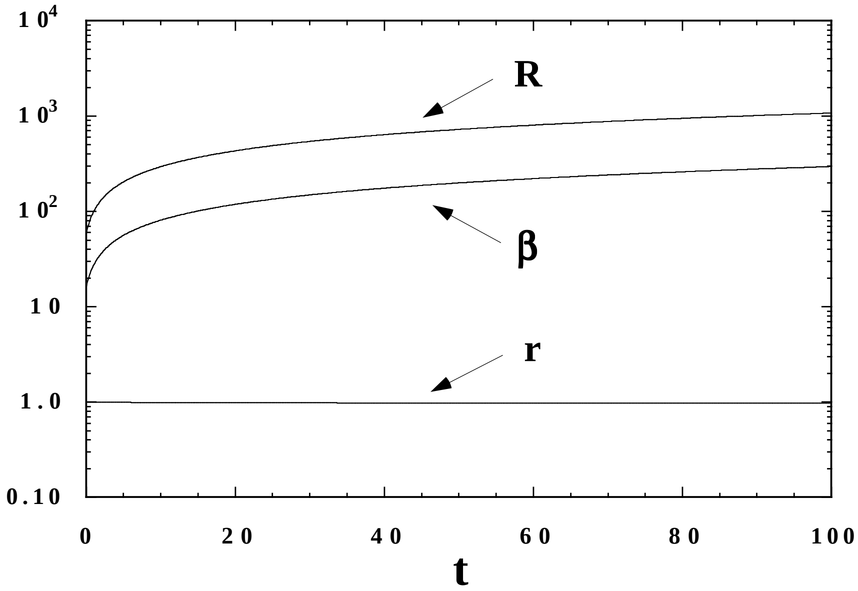

Figure 3.2: The evolution of R, r and β for the pure radiation case with $d = 3$. We choose the initial conditions $\dot{\mu}_0 = 1.0$, $\mu_0 = 4.0$, $\dot{\nu}_0 = -0.01$, $\nu_0 = 0.0$, $\psi_0 = -16$ and $\beta_0 = 15$.

When the pressure $P_{rad}^{(d)}$ is taken into account, this works as a positive source term in eq. (3.46). Therefore it is possible to make the r.h.s. of eq. (3.46) positive even when $\dot{\psi} < 0$ and $\dot{\mu} > 0$. We have made numerical simulations with initial conditions $R_0 \gg r_0 \sim 1$, $\dot{\psi}_0 < 0$, and several different values of $\dot{\mu}_0$ and $\dot{\nu}_0$. As long as $\dot{\mu}_0$ is positive, the large dimensions expand in the presence of the pressure due to radiation. The contribution of the pressure term in eq. (3.46) inhibits the rapid decrease of $\dot{\mu}$, thereby leading to different evolution of R compared to the case with $P_{rad}^{(d)} = 0$. The expansion rate $\dot{\nu}$ for the small dimensions is exponentially suppressed with the decrease of ψ [see eq. (3.49)]. Therefore unless the initial value of $|\dot{\nu}|$ is much larger than unity, the radius r can stay small around $r \sim 1$.

We have numerically succeeded to obtain ideal solutions with growing R and small, roughly constant, r satisfying $r \ll R$. One typical evolution is plotted in fig. 3.2. These solutions can be achieved by choosing initial values with $\dot{\mu}_0 \geq |\dot{\nu}_0|$, $R_0 \gg r_0 \sim 1$ and $\dot{\psi}_0 < 0$. When $\dot{\mu}_0 \ll |\dot{\nu}_0|$ holds initially, it is difficult to keep the small dimensions small relative to the large ones. If $\dot{\mu}_0 < 0$, we have $\ddot{\mu}_0 > 0$ from eq. (3.46). This leads to the growth of the expansion rate $\dot{\mu}$. Since $\dot{\mu}$ continues to be negative by the time it crosses zero, the large dimensions contract during this stage. After $\dot{\mu}$ changes sign, R begins to grow. This implies that bouncing solutions may be obtained if $\dot{\mu}_0 < 0$. We have

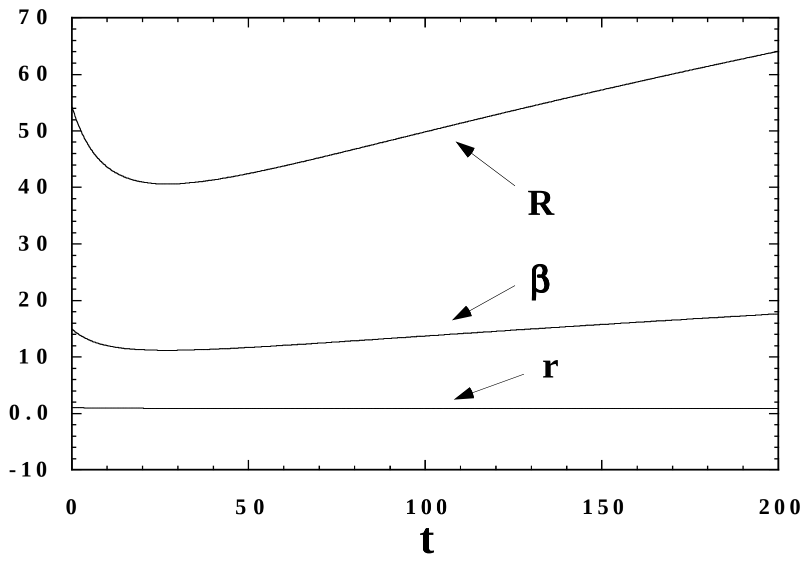

Figure 3.3: The evolution of R, r and β for the pure radiation case with $d=3$. We choose the initial conditions $\dot{\mu}_0=-0.04$, $\mu_0=4.0$, $\dot{\nu}_0=-0.01$, $\nu_0=0.0$, $\psi_0=-16$ and $\beta_0=15$.

numerically found that this is actually the case, see fig. 3.3. During the contracting phase, the temperature increases according to eq. (3.44). The temperature is maximum at the bounce where R is minimum. In the context of Pre-Big-Bang [7] or Ekpyrotic Cosmologies [11, 12], nonsingular bouncing solutions are difficult to construct unless loop or derivative corrections are added to the tree-level action [135–137]. It is quite interesting to be able to obtain bouncing solutions only by including radiation in dilaton-gravity equations.

It is worth investigating the asymptotic behavior of the dynamical system of eqs. (3.45)-(3.47), along the lines of [79]. Introducing new parameters, $\dot{\mu}\equiv\xi$, $\dot{\nu}\equiv\eta$, $\dot{\psi}\equiv f$, and using the fact that the pressure in the large dimensions is connected to the energy $E_{rad}^{(d)}$ through eq. (3.42), eqs. (3.45)-(3.48) become:

$$\dot{f} = \frac{d}{2}\left[\xi^2+\left(\frac{9}{d}-1\right)\eta^2\right]+\frac{1}{2}f^2\,, \qquad (3.50)$$

$$\dot{\xi} = -\frac{1}{2}\left[\xi^2+\left(\frac{9}{d}-1\right)\eta^2\right]+f\xi+\frac{f^2}{2d}\,, \qquad (3.51)$$

$$\dot{\eta} = f\eta\,. \qquad (3.52)$$

Since we are considering the case with decreasing $\dot{\psi}$, η asymptotically approaches zero from eq. (3.49), *i.e.* $\eta=0$ is an attractive solution. In this case, the analysis is closely related with the one outlined in the appendix of [79]. In particular one finds that the line described by

$$f/\xi = -d\,, \quad \eta = 0\,, \tag{3.53}$$

is an attractor. From eq. (3.3) the time-derivative of the dilaton is given as $2\dot{\Phi} = f + d\xi + (9-d)\eta$. Therefore we have $\dot{\Phi} = 0$ for the attractor (3.53), again in complete analogy to the case of the single scale factor [79]. Substituting eq. (3.53) for eq. (3.51) and integrating this relation, one finds

$$\xi \propto \frac{2}{(d+1)t}\,, \quad R \propto t^{\frac{2}{d+1}}\,. \tag{3.54}$$

This indicates that the late time evolution for the large dimensions can be described by that of the standard radiation dominant phase in FRW cosmology for $d = 3$, even in the presence of the small dimensions. The key point is that the pressure in the small dimensions vanishes for the massless case, thereby leading to $\eta = 0$ as an attractor. Notice that a cosmological solution of this kind has been also obtained in standard General Relativity and in a purely Kaluza-Klein extra-dimensional scenario by [138].

We also analyzed the evolution of the system by varying the value of d, and found that the situation is not basically changed compared to the $d = 3$ case discussed above. As long as the initial conditions satisfy $\dot{\mu}_0 \geq |\dot{\nu}_0|$, $R_0 \gg r_0 \sim 1$ and $\dot{\psi}_0 < 0$, the large dimensions continue to expand due to the presence of radiation while the small dimensions are kept to be small ($r \ll R$). In this case the large dimensions asymptotically approach the radiation-dominant FRW solution (3.54). When the initial value of $\dot{\mu}$ is negative, we found that it is possible to have a bouncing cosmological solution that approaches the expanding FRW universe given by (3.54).

3.4.2 Inclusion of matter

Although massless states, pure radiation, dominate the thermodynamical ensemble in this phase, this dominant contribution has a trivial dependence on the small dimensions r. In particular, $F^{(d)}_{rad}$ does not depend on r and the pressure along the small dimensions trivially vanishes. It is then important to see if and how matter terms can alter this behavior. For this purpose, we study the leading terms that have an explicit dependence on r in the infinite sums appearing in eq. (3.20). We have numerically estimated that it is enough to consider the first KK and winding modes along a small direction, *i.e.* the terms with $\{N = \bar{N} = 0, m_i = (1,0,\ldots,0), n_i = 0\}$ (as well as m_i and n_i exchanged) in eq. (3.20), plus the remaining $8-d$ inequivalent permutations. The energy $E^{(d)}_{mat}$ and pressures $P^{(d)}_{mat}$ and $P^{(9-d)}_{mat}$ along the large and small dimensions associated with these states are simply evaluated starting from the general expression (3.20). The equation of state for these leading order terms are:

$$E_{mat}^{(d)} = -V_d\,C(\beta)^{(d)}\Bigg\{\left[\frac{1}{r^{(d+1)/2}}\frac{1-d}{2}K_{(d+1)/2}\left(\frac{\beta}{r}\right)+\frac{1}{r^{(d+1)/2}}\frac{\beta}{r}K'_{(d+1)/2}\left(\frac{\beta}{r}\right)\right]+ \left[r^{(d+1)/2}\frac{1-d}{2}K_{(d+1)/2}\left(\beta r\right)+r^{(d+1)/2}\beta r K'_{(d+1)/2}\left(\beta r\right)\right]\Bigg\}, \quad (3.55)$$

$$P_{mat}^{(d)} = V_d\,C(\beta)^{(d)}\left[\frac{1}{r^{(d+1)/2}}K_{(d+1)/2}\left(\frac{\beta}{r}\right)+r^{(d+1)/2}K_{(d+1)/2}(\beta r)\right], \quad (3.56)$$

$$P_{mat}^{(9-d)} = V_d\,C(\beta)^{(d)}\beta\left[\frac{1}{r^{(d+3)/2}}K_{(d-1)/2}\left(\frac{\beta}{r}\right)-r^{(d+3)/2}K_{(d-1)/2}(\beta r)\right], \quad (3.57)$$

where K_n are modified Bessel functions, the prime denotes derivative with respect to r, and

$$C(\beta)^{(d)} = \left(\frac{2\pi}{\beta}\right)^{(d+1)/2}\frac{(18-2d)}{\pi}D(0)^2\,. \quad (3.58)$$

Here $D(0)^2 = 256$ is a string degeneracy factor. Note that we only consider the $p=1$ term in eq. (3.20), implying the approximation of the bosonic/fermionic statistics with the Maxwell-Boltzmann distribution. The pressure $P_{mat}^{(d)}$ along the large dimensions is always positive, which aids expansion of the universe in addition to the pressure $P_{rad}^{(d)}$ from the massless states. The first and second terms in square brackets in eqs. (3.56) and (3.57) come from the KK and winding mode, respectively. The above equations (3.55)-(3.57) are all manifestly invariant under the duality symmetry (3.7) acting on the small dimensions, $r \to 1/r$. Notice that the winding modes give rise to a standard positive pressure along the large dimensions (second term in (3.56)) but negative along the small ones (second term in (3.57)).

We numerically solved the dilaton-gravity equations (3.9)-(3.12), with $E = E_{rad}^{(d)} + E_{mat}^{(d)}$, $P^{(d)} = P_{rad}^{(d)} + P_{mat}^{(d)}$ and $P^{(9-d)} = P_{rad}^{(9-d)} + P_{mat}^{(9-d)}$, by carefully taking into account the adiabaticity condition (3.40). The pressure $P_{mat}^{(9-d)}$ for the small dimensions vanishes at the self-dual critical radius $r=1$. Therefore it is expected that the effect of the massive states for the small dimensions is weak around $r \sim 1$. In fact we have numerically found that this is the case. As seen from the case (b) in fig. 3.4, the evolution of the small dimensions is hardly altered by including the massive mode for the initial value of r very close to unity. From eq. (3.57) one notes that $P_{mat}^{(9-d)} < 0$ for $0 < r < 1$ and $P_{mat}^{(9-d)} > 0$ for $r > 1$ (the asymptotic values are $P_{mat}^{(9-d)} \to 0$ for $r \to 0$ and $r \to \infty$). This indicates that the pressure of the massive state makes the small dimensions contract for $0 < r < 1$ while its effect tends to expand the small dimensions for $r > 1$.

The effect of the massive states emerges by choosing the initial values of r_0 that are

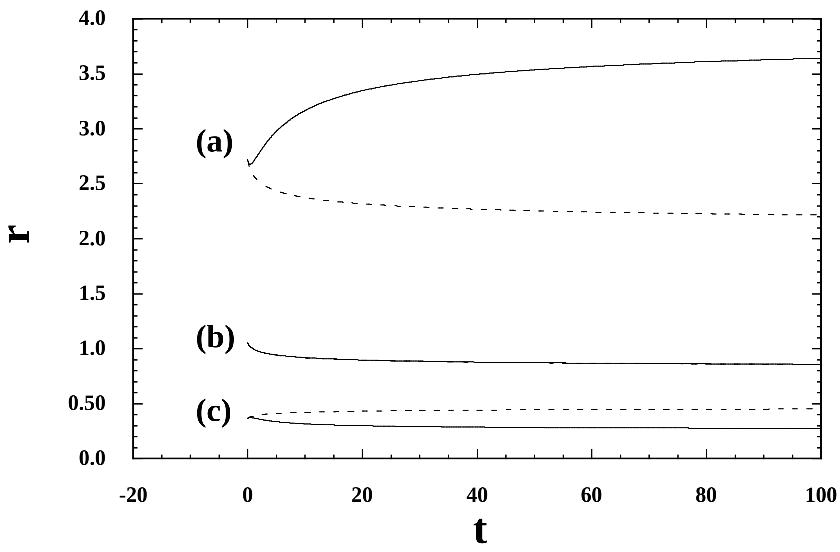

Figure 3.4: The evolution of the small dimensions for $d = 4$ when the massive states are taken into account (solid curve). The dotted curves correspond to the case where the massive states are neglected (only massless states). We choose the initial conditions $\dot{\mu}_0 = 1.0$, $\mu_0 = 4.0$, $\psi_0 = -16$, $\beta_0 = 12$ with (a) $\dot{\nu}_0 = -0.1$, $\nu_0 = 1.0$, (b) $\dot{\nu}_0 = -0.1$, $\nu_0 = 0.05$, (c) $\dot{\nu}_0 = 0.1$, $\nu_0 = -1.0$.

slightly smaller or larger than unity. For $0 < r_0 < 1$ with $\dot{\nu}_0 > 0$, the small dimensions can be larger than $r = 1$ for large initial values of $\dot{\nu}$. In this case the small dimensions continue to grow after they cross $r = 1$. When $\dot{\nu}_0$ is not large ($\dot{\nu}_0 \ll 1$), the massive effect can lead to the contraction of the small dimensions due to the negative pressure for $r < 1$. As found from the case (c) in fig. 3.4, the small dimensions always increase in the massless case, whereas the small dimensions begin to contract if the massive effect is included. Therefore we can keep the $(9-d)$ dimensions small ($0 < r < 1$) for these initial conditions.

We have also made numerical simulations for $r_0 > 1$. When $\dot{\mu}_0 > 0$ and $\dot{\nu}_0 > 0$, both large and small dimensions expand in the presence of positive pressures. If $\dot{\nu}_0$ is largely negative, the small dimensions contract by passing through $r = 1$. Meanwhile, if $|\dot{\nu}_0| \ll 1$, the small dimensions can exhibit bouncing with $r > 1$, instead of crossing $r = 1$ [see the case (a) in fig. 3.4). This means that the radius r can grow in the presence of the massive states. Since the small dimensions continue to expand after the bounce, this is not an ideal case where the small dimensions stay small. Nevertheless the small dimensions can be made small compared to the large dimensions as long as $\dot{\mu}_0 \geq \dot{\nu}_0$.

When the massive states are taken into account, this gives rise to an extra source

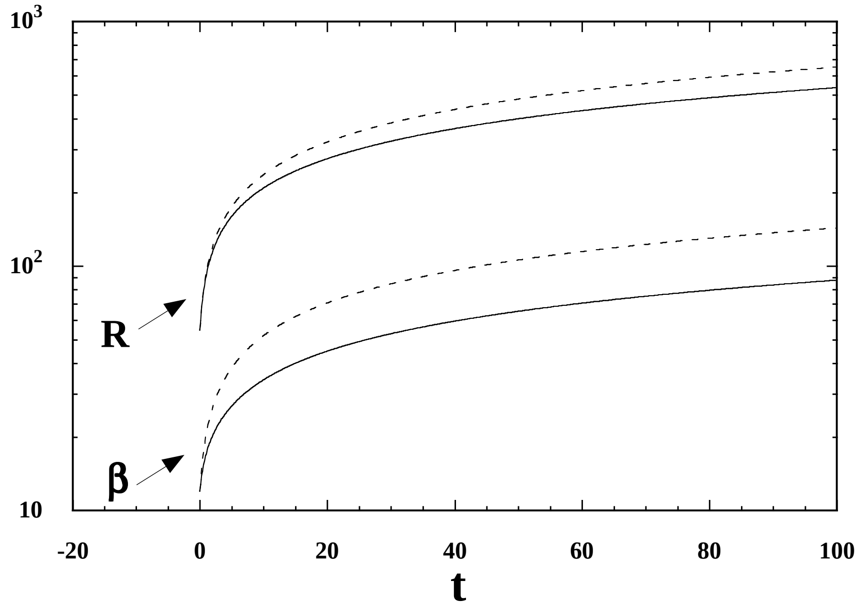

Figure 3.5: The evolution of the large dimensions and β that corresponds to the case (c) in fig. 3.4. The dotted curves correspond to the case where the massive states are neglected (massless states only).

term for the energy E in eq. (3.9). Then $|\dot{\psi}_0|$ gets larger compared to the pure massless case. Typically this leads to the suppression of the r.h.s. of eq. (3.10) via the e^{ψ} term, thereby yielding the smaller expansion rate $\dot{\mu}$ for the large dimensions. The large dimensions in the massive case grow slower relative to the massless case, as seen in fig. 3.5. In addition, the massive effect suppresses the growth of β, *i.e.* the temperature decreases faster in the massless case.

As expected, the massive terms get smaller as the initial value of β is increased. For example, in the cases shown in fig. 3.4 and 3.5, the system is effectively described by the massless states for $\beta_0 > 15$. We also analyzed the behavior of the system by varying the value d with $1 \leq d \leq 8$. We found that the numerical results are quite similar to the case explained above ($d = 4$). As long as the conditions, $\dot{\mu}_0 \geq |\dot{\nu}_0|$ and $R_0 \geq r_0 \sim 1$, are satisfied, the small dimensions are kept small, while the large ones expand as required in the presence of radiation and string matter.

We have also considered the case with the conservation of KK and/or winding modes. Under this circumstance, the partition function must be taken summing only over the configurations that respect the conservation laws. As before, this is done by introducing a chemical potential corresponding to each conserved quantity in the partition function. It turns out that no significant changes are found compared to the case where no conservation laws are imposed.

We then analyzed the case where the massive string states are taken into account in addition to the pure radiation. The presence of the massive states typically leads to a slower expansion of the "large" dimensions relative to the massless case (see fig. 3.5). Meanwhile the behavior of the small dimensions strongly depends on the initial conditions for r and $\dot{r}$, resulting in either expansion or contraction of the small dimensions (see fig. 3.4). The radius r can be kept small as long as r is initially close to unity, since the pressure vanishes at the duality symmetric radius ($r = 1$). The vanishing of the pressure at $r = 1$ is a purely stringy effect, since it is due to winding modes, whose negative contribution compensates that of KK states. The important point is that, even in the presence of the massive state, there exist a wide range of the initial condition space for which the small dimensions are stabilized around the self-dual radius and are kept small relative to the large ones. These behaviors are found to be insensitive to the number of large dimensions, d. We also considered the case for the conservation of KK and winding modes and found no substantial change compared to the case without imposing the conservation laws.

Chapter 4

Plane waves and Penrose limits

In 1976 Penrose [93] showed that any space-time (any solution of the Einstein field equations) has a limiting space-time which is a plane wave, which physical interpretation was described as follows:

There is a "physical" interpretation of the above mathematical procedure,which is the following. We envisage a succession of observers traveling in the space-time M whose world lines approach the null geodesic γ *more and more closely; so we picture these observers as traveling with greater and greater speeds, approaching that of light. As their speeds increase they must correspondingly recalibrate their clocks to run faster and faster (assuming that all space-time measurements are referred to clock measurements in the standard way) so that in the limit the clocks measure the affine parameter* x^0 *along* γ*. (Without clock recalibration a degenerate space-time metric would result.) In the limit the observers measure the space-time to have the plane wave structure* W_γ*.*

Then in 2000 Gueven [139] showed that any solution of a supergravity theory has plane wave limits which are also solutions.

Moreover, as we will see in the end of this chapter, the Penrose limit metric, encodes diffeomorphism invariant information about the original space-time metric; in particular it gives us information about the growth of curvature and geodesic deviation along a null geodesic.

On the other hand the relevance of dealing with a pp-wave background in string theory is that in this background there is a natural light cone gauge and in many cases string theory can be quantized exactly.

By taking into consideration these two facts, the study of the Penrose limit of cosmological singularities and further on the study of string propagation in these backgrounds, is one possibility to make a first step toward the study of string propagation in singular and time dependent space-time backgrounds.

In this chapter we will introduce the basics of Penrose limits and plane wave metrics in order to be prepared for the explicit computation of the Penrose limits of space-time singularities.

4.1 Plane waves

4.1.1 Plane waves in Rosen coordinates

If we consider a metric $g_{\mu\nu}$ with a small perturbation $h_{\mu\nu}$ of the Minkowski background metric

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}\,, \tag{4.1}$$

the Einstein equations to linear order in $h_{\mu\nu}$ reduce to a wave equation, i.e. a gravitational plane wave solution of general relativity. Gravitational waves are transversally polarized. For example, a wave traveling in the (t,z) direction would be

$$ds^2 = -dt^2 + dz^2 + (\delta_{ij} + h_{ij}(z-t))dy^i dy^j\,. \tag{4.2}$$

Using light cone coordinates $U = z - t$, $V = (z+t)/2$ it takes the form:

$$ds^2 = 2dUdV + (\delta_{ij} + h_{ij}(U))dy^i dy^j\,, \tag{4.3}$$

and so if we drop the assumption that h_{ij} is “small” a plane wave metric $\bar{g}_{\mu\nu}$ in *Rosen coordinates* is

$$d\bar{s}^2 = 2dUdV + C_{ij}(U)dY^i dY^j\,. \tag{4.4}$$

Plane wave metrics are characterized by a single matrix-valued function of U, but two metrics with different C_{ij} could be isometric.

4.1.2 Plane waves in Brinkmann coordinates

Since one of the characteristics of the plane wave metrics is the existence of a covariantly constant null vector ∂_V, let us derive a metric for a space-time admitting such a covariantly constant null vector.

Let Z be a parallel (covariantly constant) null vector of the $(d+2)$-dimensional Lorentzian metric $g_{\mu\nu}$, $\nabla_\mu Z^\nu = 0$ which is equivalent to:

$$\nabla_\mu Z_\nu + \nabla_\nu Z_\mu = 0\,, \tag{4.5}$$

$$\nabla_\mu Z_\nu - \nabla_\nu Z_\mu = 0\,, \tag{4.6}$$

in other words, Z is a Killing vector field and also a gradient vector field. We can assume that $Z = \partial_v$ for some coordinate v. Then, since Z is null $g_{vv} = 0$. On the other hand, the Killing equation implies that all the components of the metric are v-independent, $\partial_v g_{\mu\nu} = 0$. From this, it follows that eq (4.5) is satisfied for $\mu = v$ or $\nu = v$ and therefore it remains

$$\nabla_\alpha Z_\beta = \nabla_\beta Z_\alpha \,, \tag{4.7}$$

where $\{x^\mu\} = \{v, x^\alpha\}$, which implies that $Z_\alpha = g_{\alpha v} = \partial_\alpha u$.

Therefore the general form of a metric admitting a parallel null vector with coordinates $\{u, v, x^a\}$, $a = 1, \ldots d$, is

$$\begin{aligned} ds^2 &= 2g_{\alpha v}dx^\alpha dv + g_{\alpha\beta}dx^\alpha dx^\beta \\ &= 2dudv + g_{uu}(u,x^c)du^2 + 2g_{au}(u,x^c)dx^a du + g_{ab}(u,x^c)dx^a dx^b \\ &= 2dudv + K(u,x^c)du^2 + 2A_a(u,x^c)dx^a du + g_{ab}(u,x^c)dx^a dx^b \end{aligned} \tag{4.8}$$

Metrics with $g_{ab} = \delta_{ab}$,

$$d\bar{s}^2 = 2dudv + K(u,x^c)du^2 + 2A_a(u,x^c)dx^a du + d\bar{x}^2 \,, \tag{4.9}$$

are called *plane-fronted waves with parallel rays* or *pp-waves*. "Plane-fronted" refers to the fact that the wave fronts $u = const$ are planar and "parallel rays" refers to the existence of a parallel null vector. There are residual coordinate transformations that leave this form invariant.

In particular, when $A_a = 0$ and $K(u, x^a)$ is quadratic in the x^a we have a *plane wave* in *Brinkmann coordinates*:

$$d\bar{s}^2 = 2dudv + A_{ab}(u)x^a x^b du^2 + d\bar{x}^2 \,, \tag{4.10}$$

Thus in Brinkmann coordinates a plane wave metric is almost uniquely characterized by a single symmetric matrix-valued function $A_{ab}(u)$ since there are very few residual coordinate transformations that leave the metric invariant.

Moreover the only non-vanishing component of the Riemann curvature tensor of the plane wave metric is given by

$$\bar{R}_{uaub} = -A_{ab} \,, \tag{4.11}$$

the only one non-trivial component of the Ricci tensor is

$$\bar{R}_{uu} = -\delta^{ab}A_{ab} \equiv -\mathrm{tr}A \tag{4.12}$$

and the Ricci scalar is

$$\bar{R} = 0 \,. \tag{4.13}$$

Therefore the metric is flat if and only if $A_{ab} = 0$ and the vacuum Einstein equations imply that A_{ab} is traceless.

Let us now look at the traceless part of the Riemann tensor, the Weyl tensor,

$$\bar{C}_{uaub} = -(A_{ab} - \frac{1}{d}\delta_{ab}\mathrm{tr}A)\,. \tag{4.14}$$

We see that it vanishes if and only if A_{ab} is pure trace,

$$A_{ab}(u) = A(u)\delta_{ab} \tag{4.15}$$

and therefore for $d > 1$ the plane wave metric is conformally flat. By looking at the plane wave metric in Rosen coordinates we see that for $d = 1$, every plane wave is conformally flat.

When the Ricci tensor is non-zero, A_{ab} has non-vanishing trace, then plane waves solve the Einstein equations with null matter or null fluxes, i.e. with an energy-momentum tensor $T_{\mu\nu}$ whose only non-vanishing component is T_{uu},

$$T_{\mu\nu} = \rho(u)\delta_{\mu u}\delta_{\nu u}\,, \tag{4.16}$$

for example, null Maxwell fields $A(u)$ with field strength $F = du \wedge A'(u)$, and their higher-rank generalizations. Physical matter (with positive energy density) corresponds to $R_{uu} > 0$ or $\mathrm{tr}A < 0$.

4.1.3 From Rosen to Brinkmann coordinates

By making the change of variables

$$\begin{aligned} U &= u \\ V &= v + \frac{1}{2}\dot{\bar{E}}_{ai}\bar{E}^i_b x^a x^b \\ Y^i &= \bar{E}^i_a x^a\,, \end{aligned} \tag{4.17}$$

the plane wave metric in Rosen coordinates, eq. (4.4), is transformed to the one in Brinkmann coordinates, eq. (4.10), with

$$A_{ab} = \ddot{\bar{E}}_{ai}\bar{E}^i_b\,, \tag{4.18}$$

which can also be written as the harmonic oscillator equation $\ddot{\bar{E}}_{ai} = A_{ab}\bar{E}_{bi}$. Here $\bar{E}^a_i$ is a vielbein for C_{ij}

$$C_{ij} = \bar{E}^a_i \bar{E}^b_j \delta_{ab}\,, \tag{4.19}$$

and satisfies the symmetry condition

$$\dot{\bar{E}}_{ai}\bar{E}^i_b = \dot{\bar{E}}_{bi}\bar{E}^i_a \tag{4.20}$$

(such an $\bar{E}$ can always be found and is unique up to U-independent orthogonal transformations [99]).

Therefore the relation between $C_{ij}(U)$ and $A_{ab}(u)$ is that A_{ab} is the curvature of C_{ij},

$$A_{ab} = -\bar{E}^i_a\bar{E}^j_b\bar{R}_{UiUj}\,. \tag{4.21}$$

If the metric $C_{ij}(u)$ is diagonal, $C_{ij}(u) = \bar{e}_i(u)^2\delta_{ij}$, one can choose $\bar{E}^a_i = \bar{e}_i\delta^a_i$. Then the symmetry condition is automatically satisfied because a diagonal matrix is symmetric, and one finds that A_{ab} is also diagonal

$$A_{ab} = \frac{\ddot{\bar{e}}_a}{\bar{e}_a}\delta_{ab}\,. \tag{4.22}$$

Conversely, given a diagonal plane wave in Brinkmann coordinates, to obtain the metric in Rosen coordinates one needs to solve the harmonic oscillator equations

$$\ddot{\bar{e}}_i(u) = A_{ii}(u)\bar{e}_i(u)\,. \tag{4.23}$$

In particular, the Rosen metric determined by $C_{ij}(u)$ is flat if and only if $\bar{e}_i(u) = a_iu + b_i$ for some constants a_i, b_i.

4.2 Geodesics and Harmonic oscillators

The geodesic equations can be obtained from the Lagrangian,

$$L = \dot{u}\dot{v} + \frac{1}{2}A_{ab}(u)x^ax^b\dot{u}^2 + \frac{1}{2}\dot{\vec{x}}^2 \tag{4.24}$$

together with the constraint

$$L = \epsilon\,, \tag{4.25}$$

where the overdot denotes a derivate with respect to the affine parameter τ and $\epsilon = 0/-1$ for massless/massive particles. The *light cone momentum*

$$p_v = \frac{\partial L}{\partial \dot{v}} = \dot{u} \tag{4.26}$$

is conserved and the *light cone gauge* is

$$u = p_v\tau \tag{4.27}$$

so by setting $p_v = 1$ we have $u = \tau$. The equations of motion for the transverse coordinates are

$$\ddot{x}^a = A_{ab}(u)x^b\,, \tag{4.28}$$

which correspond to a non-relativistic harmonic oscillator with frequency matrix $w^2_{ab} = -A_{ab}$.

On the other hand, the harmonic oscillator equation arises as well in the geodesic deviation equation for the transverse coordinates. One way to detect the presence of a curvature singularity is to study the tidal forces acting on extended objects or families of freely falling particles, i.e. by means of the geodesic deviation equation:

$$\frac{D^2}{D\tau^2}\delta x^\mu = \bar{R}^\mu_{\ \nu\lambda\rho}\dot{x}^\nu\dot{x}^\lambda\delta x^\rho\,, \tag{4.29}$$

where δx^μ is the separation vector between nearby geodesics. Let us consider the family of geodesics ∂_u of plane waves parametrized by the transverse coordinates (v, x^a), and choose δx^μ to connect points on nearby geodesics with the same value of $\tau = u$. Thus $\delta u = 0$, and the geodesic deviation equation for the transverse separations δx^a reduces to

$$\frac{d^2}{du^2}\,\delta x^a = -\bar{R}^a_{ubu}\,\delta x^b = A_{ab}\,\delta x^b\,. \tag{4.30}$$

We then see that A_{ab} contains direct physical information. For negative eingenvalues of A_{ab}, physical matter, this tidal force is attractive, leading to a focusing of the geodesics. On the other hand, for vacuum plane waves the tidal force is attractive in some directions and repulsive in the other. Finally if $A_{ab}(u)$ diverges then the tidal forces become infinite and therefore the plane wave space-time is singular at such points. If such a singularity occurs at $u = u_0$, since u is the affine parameter along the null geodesic, this shows that any geodesic starting off at a finite value of u will reach the singularity in finite proper time. Thus the space-time is geodesically incomplete and ends at $u = u_0$.

Therefore a plane wave metric is singular if and only if A_{ab} diverges somewhere.

4.3 Homogeneous plane waves

In Brinkmann coordinates, because of the explicit dependence of the metric on u and the transverse coordinates, only the isometry generated by the parallel null vector $Z = \partial_v$ is manifest. On the other hand, in Rosen coordinates the metric depends neither on V nor on the transverse coordinates Y^k and therefore the Killing vectors are $Z = \partial_V$ and $d\ \partial_{Y^k}$.

All Killing vectors V can be found by solving the Killing equations

$$L_V \bar{g}_{\mu\nu} = \nabla_\mu V_\nu + \nabla_\nu V_\mu = 0\,, \tag{4.31}$$

in Brinkmann coordinates [103], which turn out to be $Z = \partial_v$ and the $2d$

$$X(f_{(K)}) \equiv X_{(K)} = f_{(K)a}\partial_a - \dot{f}_{(K)a}x^a\partial_v\,, \tag{4.32}$$

where $f_{(K)a}$, $K = 1, \cdots, 2d$ are the $2d$ linearly independent solutions of the harmonic oscillator equation

$$\ddot{f}_a(u) = A_{ab}(u) f_b(u)\,. \tag{4.33}$$

Then, the corresponding $2d$ Killing vectors

$$Q_a = X(q_{(a)})\,, \qquad P_a = X(p_{(a)})\,, \tag{4.34}$$

and Z satisfy the canonically normalized Heisenberg algebra

$$\begin{aligned} [Q_a, Z] &= [P_a, Z] = 0 \\ [Q_a, Q_b] &= [P_a, P_b] = 0 \\ [Q_a, P_b] &= \delta_{ab} Z\,, \end{aligned} \tag{4.35}$$

where $p_{(a)}$ and $q_{(b)}$ arise by splitting the solutions $f_{(J)}$ into two sets of solutions

$$\{f_{(J)}\} \to \{p_a, q_{(a)}\} \tag{4.36}$$

characterized by the initial conditions

$$\begin{aligned} p_{(a)b}(u_0) &= \delta_{ab} \qquad & \dot{p}_{(a)b}(u_0) &= 0 \\ q_{(a)b}(u_0) &= 0 \qquad & \dot{q}_{(a)b}(u_0) &= \delta_{ab}\,. \end{aligned} \tag{4.37}$$

This algebra acts transitively on the null hyperplanes $u = const$. However for special choices of $A_{ab}(u)$, there may be more Killing vectors, which could arise from internal symmetries of A_{ab}.

4.3.1 Cahen-Wallach plane waves

The Cahen-Wallach metric [140, 141] is such that A_{ab} is u-independent

$$d\bar{s}^2 = 2dudv + A_{ab}x^a x^b du^2 + d\bar{x}^2 \tag{4.38}$$

and has an extra Killing vector $X = \partial_u$. Since A_{ab} is u-independent, it can be diagonalized by a u-independent orthogonal transformation acting on the x^a. Moreover A_{ab} can be rescaled, $A_{ab} \to \mu^2 A_{ab}$ by the coordinate transformation

$$(u, v, x^a) \to (\mu u, \mu^{-1} v, x^a)\,. \tag{4.39}$$

The appearence of the extra Killing vector renders the metric homogeneous and since the Riemann curvature tensor is covariantly constant,

$$\nabla_\mu \bar{R}_{\lambda\nu\rho\sigma} = 0 \Leftrightarrow \partial_u A_{ab} = 0 \tag{4.40}$$

the metric is locally symmetric. The additional Killing vector $X = \partial_u$ extends the Heisenberg algebra to the harmonic oscillator algebra, with X playing the role of the number operator or harmonic oscillator Hamiltonian. X and Z conmute and the rest of the conmutators are

$$\begin{aligned} [X, Q_a] &= P_a \\ [X, P_a] &= A_{ab} Q_b \end{aligned} \tag{4.41}$$

which corresponds to the harmonic oscillator algebra.

The Cahen-Wallach metric eq. (4.38) can be generalized with the profile

$$A_{ab}(u) = (e^{uf} B e^{-uf})_{ab}\,, \tag{4.42}$$

which reduces to eq. (4.38) for $f_{ab} = 0$, and presents the extra Killing vector

$$X = \partial_u + f_{ab} z^b \partial_a\,. \tag{4.43}$$

These homogeneous plane waves are completly non-singular and geodesically complete, and they are solutions to the vacuum Einstein equations if and only if B_{ab} is traceless.

By making the change of coordinates

$$x^a \to (e^{-uf})_{ab} x^b\,, \tag{4.44}$$

the metric becomes manifestly independent of u,

$$d\bar{s}^2 = 2dudv + k_{ab} x^a x^b du^2 + 2 f_{ab} x^a dx^b du + d\bar{x}^2\,, \tag{4.45}$$

with

$$k_{ab} = B_{ab} - f^2_{ab}\,. \tag{4.46}$$

In this coordinates the additional Killing vector is just $X = \partial_u$ and the corresponding conserved charge is the lightcone Hamiltonian which is that of an harmonic oscillator coupled to the constant magnetic field f_{ab}.

4.3.2 Singular homogeneous plane waves

The plane waves with u-dependent A_{ab} such that

$$A_{ab}(u) = u^{-2} c_{ab}\,, \tag{4.47}$$

for some constant matrix c_{ab} are still homogeneous but not symmetric. In this case, the plane wave metric

$$d\bar{s}^2 = 2dudv + c_{ab}x^a x^b \frac{du^2}{u^2} + d\bar{x}^2 \tag{4.48}$$

is invariant under the boost eq (4.39), corresponding to the extra Killing vector

$$X = u\partial_u - v\partial_v\,. \tag{4.49}$$

The generalization of these metrics is done by considering the profile

$$A_{ab} = u^{-2}(e^{(\log u)f}\, c\, e^{-(\log u)f})_{ab}\,. \tag{4.50}$$

They have null singularities at $u = 0$.

4.4 The Penrose limit

As it was already mention at the beginning of the chapter, the Penrose limit construction associates to every space-time metric $g_{\mu\nu}$ and choice of null geodesic γ in that spacetime a plane wave metric $\bar{g}_{\mu\nu}$. The "traditional" receipt to take the Penrose limit is roughly the following:

- Rewrite the original metric $g_{\mu\nu}$ in "adapted" or "Penrose" coordinates

$$ds^2 = 2dUdV + C(U,V,Y^k)dV^2 + 2C_i(U,V,Y^k)dY^i dV + C_{ij}(U,V,Y^k)dY^i dY^j\,. \tag{4.51}$$

- Perform a change of coordinates ($\Omega \in \mathbb{R}$)

$$(U,V,Y^k) \quad \to \quad (u, \Omega^2\bar{v}, \Omega y_k) : g_\Omega\,. \tag{4.52}$$

- Take the Penrose limit to obtain the Penrose metric $\bar{g}_{\mu\nu}$

$$d\bar{s}^2 = \lim_{\Omega\to 0} \Omega^{-2} ds^2_\Omega \tag{4.53}$$

  which corresponds to the metric of a plane wave in Rosen coordinates.

In what follows the procedure will be explain carefully.

4.4.1 Adapted coordinates

Consider metrics of the form

$$ds^2 = 2dUdV + C(U,V,Y^k)dV^2 + 2C_i(U,V,Y^k)dY^i dV + C_{ij}(U,V,Y^k)dY^i dY^j\,, \quad (4.54)$$

where C, C_i and C_{ij} can depend on all the coordinates (U,V,Y^k). This class of metrics is characterized by the fact that $g_{UV}=1$ and $g_{UU}=g_{Ui}=0$. In particular, ∂_U is null and is also a geodesic vector field, with U playing the role of an affine parameter along the null geodesic integral curves of ∂_U. Indeed, ∂_U is a geodesic if

$$\nabla_{\partial_U}\partial_U = 0 \quad \Leftrightarrow \quad \Gamma_{\mu UU} = 0\,. \quad (4.55)$$

Since $g_{UU}=0$, it follows that $g_{\mu U}$ is U independent, as it certainly is.

Thus the above metric defines a special kind of null geodesic congruence: in the region of validity of the above coordinate system there is a unique null geodesic passing through any point, and we can therefore parametrize the points by the value of the affine parameter U on that geodesic and the transverse coordinates (V,Y^i) labelling the geodesics.

Now, given any particular null geodesic $x^\mu(\tau)$ of a space-time with metric $g_{\mu\nu}$, (U,V,Y^i) are "Penrose coordinates" or "adapted coordinates" if the metric takes the above form, (4.54), with $x^\mu(\tau)$ corresponding to the geodesic $V=Y^i=0$ with $U=\tau$.

Such a coordinate system always exists and can be shown by means of the Hamilton-Jacobi (HJ) formalism [58, 100] which is as follows: The momenta,

$$p_\mu = g_{\mu\nu}\frac{dx^\nu}{d\tau}\,, \quad (4.56)$$

associated with the above null congruence ($\dot U = 1, \dot V = \dot Y^k = 0$) are

$$p_V = 1\,, \quad p_U = p_{Y^k} = 0\,, \quad (4.57)$$

so that, in arbitrary coordinates x^μ, one has

$$p_\mu = \partial_\mu V\,. \quad (4.58)$$

Thus, since the geodesic congruence is null, $g^{\mu\nu}\partial_\mu V\partial_\nu V = 0$, one can identify

$$V(x^\mu) = S(x^\mu) \quad (4.59)$$

with the solution of the Hamilton-Jacobi equation

$$g^{\mu\nu}\,\partial_\mu S\,\partial_\nu S = 0\,, \quad (4.60)$$

corresponding to the null congruence (4.57)

$$\dot{x}^\mu = g^{\mu\nu}\partial_\nu S\,. \tag{4.61}$$

Likewise, any solution S of the equations (4.60,4.61) gives rise to a null geodesic congruence,

$$\dot{x}^\rho\nabla_\rho\dot{x}^\mu = g^{\rho\sigma}g^{\mu\nu}\nabla_\rho\partial_\nu S\partial_\sigma S = \frac{1}{2}g^{\mu\nu}\partial_\nu(g^{\rho\sigma}\partial_\rho S\partial_\sigma S) = 0\,, \tag{4.62}$$

and $V = S$ is the corresponding null adapted coordinate.

There usually exists a "complete" solution to the HJ equation (4.60), complete in the sense that it depends on $d+2$ (the space-time dimension) integration constants [142][1]. For a complete solution to the HJ equation with integration constants α_μ, the associated geodesic congruence is $x^\mu = x^\mu(\tau, \alpha_\mu, x_0^\mu)$, where x_0^μ are the positions of the geodesics at $\tau = 0$, $x_0^\mu = x^\mu(0, \alpha_\mu, x_0^\mu)$. The initial value surface parameterized by the x_0^μ can be represented by the equation $F(x^\mu) = 0$.

Given a particular null geodesic γ, the integration constants α_k can be uniquely fixed. We can let $p_\mu^0 = g_{\mu\nu}\dot{x}^\nu|_{\tau=0}$ to be the momentum of the geodesic γ at $\tau = 0$. The mass-shell condition $g^{\mu\nu}p_\mu p_\nu = 0$ is scale invariant and therefore there are d independent momenta. These can be used to determine the integration constants of the HJ function S via the equation

$$p_\mu^0 = \partial_\mu S|_{\tau=0}\,. \tag{4.63}$$

Therefore we can use the HJ equation to embed a given null geodesic into a null geodesic congruence determined by the solution S [105].

We can then parameterize the null geodesic congruence as described above (the integration constants are specified by the momentum of the null geodesic γ)

$$x^\mu = x^\mu(\tau, x_0^\nu) \tag{4.64}$$

$$F(x_0^\mu) = 0\,. \tag{4.65}$$

Then we set

$$\begin{aligned} U &= \tau \\ V &= S(x_0^\mu)\,. \end{aligned} \tag{4.66}$$

Since $\dot{S} = 0$ then $S(x^\mu) = S(x_0^\mu)$. The coordinates Y^k are found by solving the equations $F(x_0^\mu) = 0$ and $S(x_0^\mu) = V$. Therefore the first equation in (4.65),

$$x^\mu = x^\mu(U, x_0^\nu(V, Y^k)) = x^\mu(U, V, Y^k)\,, \tag{4.67}$$

[1]It is not always guaranteed that such a complete solution exists, though in all the cases that we consider here it does. The most general solution can be constructed from a complete solution [143].

is the transformation which relates a coordinate system on a space-time to the Penrose coordinates.

Since there is no natural choice for the hypersurface $F = 0$, it must be thought of as a gauge fixing condition, the Penrose limit metric is independent of it, corresponding to different ways of labelling the geodesics of the congruence on which the adapted coordinates are based.

In these coordinates the metric takes the form (4.54):

$$g_{UU} = g_{\mu\nu}\frac{\partial x^\mu}{\partial\tau}\frac{\partial x^\nu}{\partial\tau} = 0 \tag{4.68}$$

because the geodesics $x^\mu(\tau, x_0^\nu)$ are null. Moreover

$$g_{UV} = g_{\mu\nu}\frac{\partial x^\mu}{\partial\tau}\frac{\partial x^\nu}{\partial V} = g_{\mu\nu}g^{\mu\rho}\partial_\rho S\frac{\partial x^\mu}{\partial V} = \frac{\partial x^\mu}{\partial V}\partial_\mu S = \frac{\partial V}{\partial V} = 1 \tag{4.69}$$

and

$$g_{Ui} = g_{\mu\nu}\frac{\partial x^\mu}{\partial\tau}\frac{\partial x^\nu}{\partial Y^i} = g_{\mu\nu}g^{\mu\rho}\frac{\partial S}{\partial x^\rho}\frac{\partial x^\nu}{\partial Y^i} = \frac{\partial V}{\partial Y^i} = 0\,. \tag{4.70}$$

4.4.2 Taking the limit

By inspection we see that by dropping the second and third terms in eq. (4.54) and keeping only the U dependence of the coefficient C_{ij}, $C_{ij}(U) = C_{ij}(U,0,0)$, the metric, eq. (4.54), is the plane wave metric in Rosen coordinates. Being more precise, the Penrose limit can be understood as a boost accompanied by a commensurate uniform rescaling of the coordinates in such a way that the affine parameter along the null geodesic remains invariant.

Therefore consider the original metric in adapted coordinates, eq. (4.54), and perform the boost

$$(U,\, V,\, Y^k) \quad\to\quad (\Omega^{-1}U,\, \Omega V,\, Y^k)\,. \tag{4.71}$$

Since taking the limit $\Omega \to 0$, infinite boost, results in a singular metric, one must as well rescale the metric as

$$(U,\, V,\, Y^k) \quad\to\quad (\Omega U,\, \Omega V,\, \Omega Y^k) \tag{4.72}$$

$$ds^2 \quad\to\quad \Omega^{-2}ds^2\,. \tag{4.73}$$

The net effect is thus the asymmetric scaling

$$(U,\, V,\, Y^k) \quad\to\quad (U,\, \Omega^2 V,\, \Omega Y^k) : g_\Omega \tag{4.74}$$

of the coordinates, leaving the affine parameter U invariant, plus an overall rescaling of the metric.

Finally the Penrose limit metric, $\bar{g}_{\mu\nu}$ arises by taking the infinite boost and large volume limit $\Omega \to 0$

$$Penrose\,Limit \quad \Rightarrow \quad d\bar{s}^2 = \lim_{\Omega\to 0} \Omega^{-2} ds^2_\Omega = 2dUdV + C_{ij}(U)dY^i dY^j\,, \tag{4.75}$$

and results in the plane wave metric in Rosen coordinates.

A coordinate transformation $(U, V, Y^k) \to (u, v, x^a)$ takes the metric into the standard Brinkmann form eq. (4.10), with A_{ab} related to the only non-vanishing component of the Riemann curvature tensor , eq. (4.21),

$$A_{ab}(u) = -\bar{R}_{aUbU}(u) = -\bar{R}_{iuju}(u)\bar{E}^i_a(u)\bar{E}^j_b(u) \tag{4.76}$$

with $\bar{E}^i_a$ an orthonormal coframe for the transverse metric $\bar{g}_{ij}$ satisfying the symmetry condition eq.(4.20).

One may think that the plane wave limit space-time sees only an infinitesimal neighbourhood of a small segment of the geodesic blown-up to cover all of space-time, but this is not the case. By transforming to Brinkmann coordinates one recovers the entire original null geodesic, with the affine parameter U running from $-\infty$ to $+\infty$ unless the geodesic runs into a curvature singularity.

4.4.3 Covariance of the Penrose limit

It was just shown that the Penrose limit construction associates to any choice of Lorentzian metric and null geodesic a plane wave metric. But one can ask to what extent this plane wave metric depends on the choice of null geodesic. For small values of the affine parameter, a null geodesic γ is characterized by specifying the initial position $\gamma(0)$ and the initial velocity $\dot{\gamma}(0)$. So, the Penrose limit only cares on the initial direction of the geodesic. Consider γ_1 and γ_2, two null geodesics starting at the same point but such that $\dot{\gamma}_1(0) = \lambda\dot{\gamma}_2(0)$ for some nonzero constant λ, then the geodesics are related by a rescaling of the affine parameter. The resulting Penrose limits are related by a rescaling of u, which can be absorbed by a rescaling of the v coordinate. Thus the Penrose limit depends on the actual curved traced by the geodesic and not on how it is parametrized.

Furthermore Penrose limits have the *covariance property*: if two null geodesic are related by an isometry their Penrose limits are themselves isometric.

4.5 Some elementary hereditary properties of Penrose limits

Geroch has called a property of space-times *hereditarity* if, whenever a family of space-times have that property, all the limits of this family also have this property [144]. In this context one considers a one-parameter family of space-times (M_λ, g_λ) for $\lambda > 0$ and tries to study the properties of the limit space-time as $\lambda \to 0$.

In the context of Penrose limits a property of space-time is hereditary if, whenever a spacetime has this property, all its Penrose limits also have this property [99].

Therefore the first step to check if a property of a space-time is hereditary is to check if it is preserved under the coordinate transformation eq (4.74) and the accompanying scaling of the metric. Looking at generally covariant (coordinate independent) properties of a space-time metric amounts to checking if the property is invariant under a finite scaling of the metric before looking at that $\lambda \to 0$ limit.

One evident property is the following [144]: if there is some tensor field constructed from the Riemann tensor and its derivatives which vanishes for all $\lambda > 0$, then it also vanishes for $\lambda = 0$. In particular,

- the Penrose limit of a Ricci-flat metric is Ricci-flat,
- the Penrose limit of a conformally flat metric, vanishing Weyl tensor, is conformally flat,
- the Penrose limit of a locally symmetric metric, vanishing covariant derivative of the Riemann tensor, is locally symmetric.

On the other hand, if we have an Einstein metric with cosmological constant or scalar curvature, the Penrose limit is not of the same type since the Ricci tensor,

$$R_{\mu\nu}(g) = \Lambda g_{\mu\nu} \tag{4.77}$$

is only invariant under simultaneous scaling of the metric g and Λ,

$$R_{\mu\nu}(\lambda^{-2}g) = R_{\mu\nu}(g) = (\lambda^2\Lambda)(\lambda^{-2}g_{\mu\nu}) \tag{4.78}$$

and the Ricci scalar is not scale-invariant. Thus

- the Penrose limit of an Einstein metric is Ricci-flat.

Geroch [144] argued that the number of linearly independent Killing vectors can never decrease in the limit. Moreover his argument is also applicable to Killing spinors and supersymmetris and so the number of supersymmetries preserved by a supergravity configuration can never decrease in the Penrose limit [99].

4.6 Penrose limits of some "cosmological" backgrounds

In order to illustrate the Penrose procedure in this section we are going to calculate in detail the Penrose limit of the Schwarzschild metric, and mention the Penrose limit metrics of some cosmological backgrounds such as AdS black hole, the FRW and the Kasner metrics.

4.6.1 Black Hole

We start with the Schwarzschild metric

$$ds^2 = -f(r)dt^2 + f(r)^{-1}dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2)\,, \tag{4.79}$$

where

$$f(r) = 1 - \frac{2m}{r}\,, \tag{4.80}$$

and m is the ADM mass of the black hole. Since we are interested in a family of geodesics parametrized by the transverse coordinates we will consider the motion to take place in the polar plane, $\phi = const$. Such curves correspond to geodesics on S^2.

Therefore for $\phi = const$ the Lagrangian,

$$L = \frac{1}{2}(-f(r)\dot{t}^2 + f(r)^{-1}\dot{r}^2 + r^2\dot{\theta}^2)\,, \tag{4.81}$$

gives us the equations of motion

$$\begin{aligned} \dot{\theta} &= lr^{-2}\,, \\ \dot{t} &= f(r)^{-1}E\,, \end{aligned} \tag{4.82}$$

where E and l are the conserved energy and angular momentum respectively. For null geodesics $L = 0$, which becomes

$$\dot{r}^2 = E^2 - l^2 f(r)r^{-2} \equiv E^2 - 2V_{eff}(r)\,. \tag{4.83}$$

Here $V_{eff}(r)$ is the usual effective potential, with respect to which $r(u)$ satisfies the Newtonian equation of motion

$$\ddot{r} = -V'_{eff}(r)\,. \tag{4.84}$$

The Hamilton-Jacobi function $S(x^\mu)$, is constructed by solving the conditions

$$p_\mu = \partial_\mu S\,, \quad g^{\mu\nu}p_\mu p_\nu = 0\,. \tag{4.85}$$

Plugging the ansatz

$$S = -Et + l\theta + \rho(r)\,, \tag{4.86}$$

into the null constraint, we find a equation for $\rho(r)$,

$$\rho'^2 = f(r)^{-2}E^2 - l^2 f(r)^{-1} r^{-2} = f(r)^{-2}\dot{r}^2\,, \tag{4.87}$$

which is solved by

$$\rho(r) = \int f(r)^{-1}\dot{r}dr = \int f(r)^{-1}\dot{r}^2 d\tau\,. \tag{4.88}$$

Choosing U to be the affine parameter τ, and $V = S(x^\mu)$ as a new coordinate, we can change variables to an adapted coordinate system $(U, V, \tilde{\theta}, \tilde{\phi})$:

$$\begin{aligned}
d\phi &= d\tilde{\phi} \\
d\theta &= \dot{\theta}(U)dU + d\tilde{\theta} \\
&= lr(U)^{-2}dU + d\tilde{\theta} \\
dr &= \dot{r}(U)dU \\
dt &= -E^{-1}dV + E^{-1}ld\theta + E^{-1}d\rho(r(U)) \\
&= -E^{-1}dV + E^{-1}l^2 r(U)^{-2}dU + E^{-1}ld\tilde{\theta} + (Ef(r(U))^{-1} - E^{-1}l^2 r(U)^{-2})dU \\
&= -E^{-1}dV + E^{-1}ld\tilde{\theta} + Ef(r(U))^{-1}dU\,.
\end{aligned} \tag{4.89}$$

By plugging this into the metric one finds

$$\begin{aligned}
ds^2 &= 2dUdV + E^{-2}r(U)^2\dot{r}(U)^2 d\tilde{\theta}^2 + r(U)^2\sin^2(\tilde{\theta} + l\int r(U)^{-2})d\tilde{\phi}^2 \\
&\quad + \cdots\,,
\end{aligned} \tag{4.90}$$

where the dots refer to terms involving dV. Once we are in adapted coordinates (dU only appears in the term $2dUdV$), the Penrose limit is obtained by dropping the other dV pieces and the explicit dependence on coordinates other than U. Thus the Penrose limit of the Schwarzschild metric for non zero angular momentum l,

$$d\bar{s}^2 = 2dudv + E^{-2}r(u)^2\dot{r}(u)^2 d\theta^2 + r(u)^2\sin^2(l\int r(u)^{-2})d\phi^2\,. \tag{4.91}$$

For zero angular momentum one must replace $\tilde{\theta} \to \theta_0 + \tilde{\theta}$, with $\theta_0 \neq 0$, since one can not zoom in on the geodesic passing through $\tilde{\theta} = 0$ where there is a coordinate singularity. The Penrose limit turns out to be

$$d\bar{s}^2 = 2dudv + E^{-2}r(u)^2\dot{r}(u)^2 d\theta^2 + r(u)^2 d\phi^2\,. \tag{4.92}$$

We can write the metric in Brinkmann coordinates by using eq (4.22),

$$\begin{aligned}
A_{11}(u) &= (r(u)\dot{r}(u))^{-1}\frac{d^2}{du^2}(r(u)\dot{r}(u))\,, \\
A_{22}(u) &= (r(u)\sin l\int r(u)^{-2})^{-1}\frac{d^2}{du^2}(r(u)\sin l\int r(u)^{-2})\,.
\end{aligned} \tag{4.93}$$

Using eq (4.83) and eq (4.84) we can write the Penrose limit of the Schwarzschild metric in Brinkmann coordinates as

$$d\bar{s}^2 = 2dudv + \frac{3ml^2}{r(u)^5}(x_1^2 - x_2^2)du^2 + dx_1^2 + dx_2^2 \,. \tag{4.94}$$

This metric is singular if and only if $r(u_0) = 0$ for some u_0, in other words, there will be a singularity precisely when the original geodesic runs into the singularity.

By consider the dominant term in eq (4.83) one can calculate the limit $r \to 0$ which simplifies to the homegenous plane wave metric

$$d\bar{s}^2 = 2dudv + \frac{6}{25}\frac{1}{u^2}(x_1^2 - x_2^2)du^2 + dx_1^2 + dx_2^2 \,. \tag{4.95}$$

This is a universal behavior since there is no l^2 or m dependence and as we will see it generalizes to arbitrary dimension. As we will see in the next chapter this $1/u^2$ behavior of the metric in Brinkmann coordinates is a universal behavior of the Penrose limits of singular metrics.

For the $l = 0$ case the equation of motion for $r(u)$ is

$$\dot{r}^2 = E^2 \,, \tag{4.96}$$

and then eq (4.92) can be written as

$$d\bar{s}^2 = 2dUdV + E^2U^2(d\tilde{\theta}^2 + \sin^2\theta_0 d\tilde{\phi}^2) \,. \tag{4.97}$$

This corresponds to the flat metric in Rosen coordinates and one may wonder that it is not singular anymore. However in the original spacetime the geodesic $r(U) = -EU$ only exists for $U < 0$ and ends at $U = 0$ and in this sense there is a singularity, even though it seems to disappear after one transforms to Brinkmann coordinates and extends the range of U to $U > 0$.

4.6.2 AdS Black Hole

In this case the metric takes the form

$$ds^2 = -f(r)dt^2 + f(r)^{-1}dr^2 + r^2 d\Omega_3^2 + R^2[d\phi^2 + sin^2\phi d\Omega_4^2], \tag{4.98}$$

with

$$f(r) = \left(\frac{r^2}{R^2} + 1 - \frac{M}{r^2}\right) \,. \tag{4.99}$$

By following the same procedure it turns out that its Penrose limit is [145]

$$d\bar{s}^2 = -4dudv - \frac{\mu^2}{R^4}\left[\sum_{a=1}^{4} x_a^2 + (1+\frac{MR^2}{r^4})\sum_{b=5}^{7} x_b^2 + (1-3\frac{MR^2}{r^4})x_8^2\right] du^2 + \sum_{i=1}^{8} dx_i^2 \tag{4.100}$$

where $\mu = \dot{\phi}R^2$ is the angular momentum and E is the energy. However in the limit $r \to 0$ one recovers the $1/u^2$ behaviour.

4.6.3 Kasner

A cosmological solution to the vacuum Einstein equations that is homogeneous but not isotropic is the Kasner metric,

$$\begin{aligned} ds^2 &= dt^2 - t^{2P_1}dx_1^2 - t^{2P_2}dx_2^2 - t^{2P_3}dx_3^2 \\ &\sum_i P_i^2 = \sum_i P_i = 1\,. \end{aligned} \tag{4.101}$$

The set of exponents $\{P_i\}$ as constrained above have the properties that they are all smaller than one, and they can not all have the same sign. If n of the exponents are positive so that the Universe expands as time increases in those n directions then the remaining exponents are negative, and the Universe shrinks in those directions as time increases.

The metric can also be rewritten as

$$ds^2 = t^{(p^2-1)/2}(dt^2 - dz^2) - t^{1+p}dx^2 - t^{1-p}dy^2. \tag{4.102}$$

Its Penrose limit along the (t, z) plane in Rosen coordinates reads as

$$d\bar{s}^2 = 2dudv + u^{\frac{2(1+p)}{1+p^2}}dx^2 + u^{\frac{2(1-p)}{1+p^2}}dy^2, \tag{4.103}$$

which written in Brinkman coordinates[2] is

$$d\bar{s}^2 = 2dudv + \left[\frac{(1+p)(p-p^2)}{(1+p^2)^2}x^2 + \frac{(p-1)(p+p^2)}{(1+p^2)^2}y^2\right]\frac{1}{u^2}du^2 + dx^2 + dy^2\,, \tag{4.104}$$

showing once again this $1/u^2$ behaviour.

4.7 Penrose limit without the limit

Before [59], the precise nature of the Penrose limit and the extent to which it encodes generally covariant properties of the original space-time was somewhat elusive, also

[2] See [101] for related matters.

because the usual definition and practical implementations of the Penrose limit, which require changing of coordinates and taking a limit, look rather non-covariant.

In what follows it will be provided a *completely covariant characterization and definition of the Penrose limit* wave profile matrix $A_{ab}(u)$ which does not require taking any limit and which shows that $A_{ab}(u)$ directly encodes diffeomorphism invariant information about the original space-time metric.

In [59] (see also [58]) it was shown that the wave profile $A_{ab}(u)$ of the Penrose limit plane wave metric associated to a null geodesic γ in a space-time with metric $g_{\mu\nu}$ can be obtained directly from the curvature tensor of the original metric,

$$A_{ab}(u) = -R_{a+b+}|_{\gamma}\,. \tag{4.105}$$

Here the components refer to a parallel pseudo-orthonormal frame along γ,

$$ds^2|_{\gamma} = 2E^+E^- + \delta_{ab}E^aE^b \tag{4.106}$$

with $E_+ \equiv \partial_u$ the tangential direction. This relation is reminiscent of, but should not be confused with, the already known relation (4.11), expressing the sole non-vanishing curvature component of the plane wave metric (4.10) in terms of $A_{ab}(u)$.

$A_{ab}(u)$ can be characterized as the transverse null geodesic deviation matrix [98] of the original metric,

$$\frac{d^2}{du^2}Z^a = A_{ab}(u)Z^b \tag{4.107}$$

with Z the transverse geodesic deviation vector. And as we will see in the next subsection, the equivalence of (4.107) and the characterization (4.105) of $A_{ab}(u)$ is a standard result in the theory of null congruences. It turns out that in practice this is not only a geometrically transparent but frequently also a calculationally efficient way of determining the wave profile $A_{ab}(u)$.

Figure 4.1 schematically summarizes the two procedures to obtain the Penrose limit metric.

4.7.1 The Penrose Limit and the Null Geodesic Deviation Equation

To establish the relation between (4.105) and (4.107), we embed the null geodesic γ into some (arbitrary) null geodesic congruence. Via parallel transport one can construct a parallel pseudo-orthonormal frame E^A, $A = +, -, a$, along the null geodesic congruence,

$$ds^2 = 2E^+E^- + \delta_{ab}E^aE^b \;, \qquad \nabla_u E^A = 0 \tag{4.108}$$

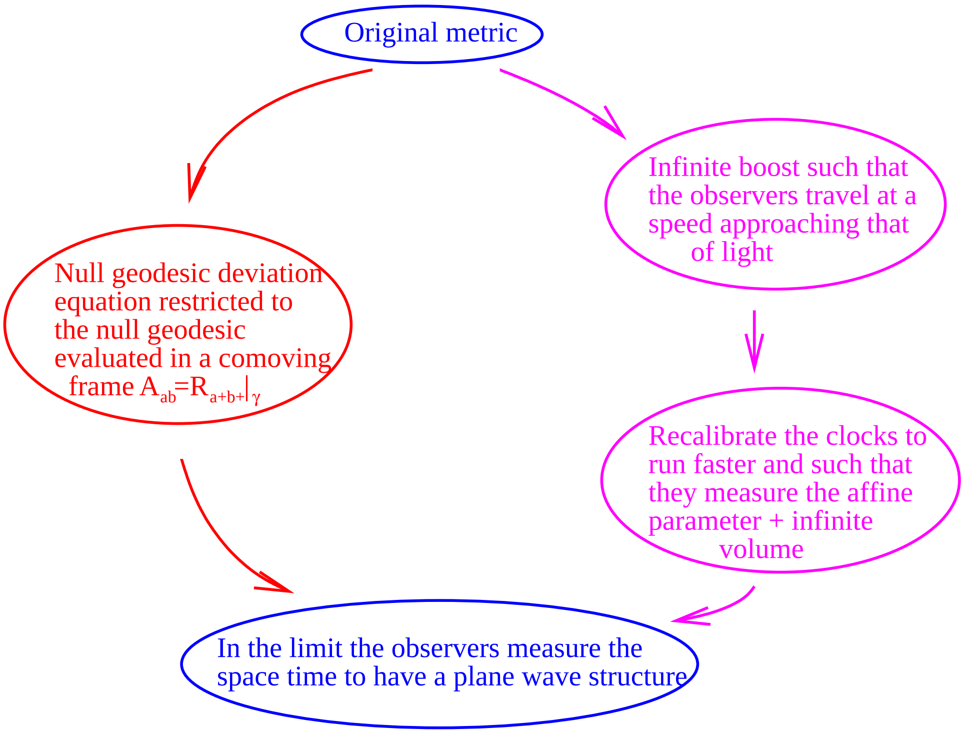

Figure 4.1: Two different ways of calculating the Penrose limit metric.

such that the component E_+ of the co-frame E_A is

$$E_+ = \dot{x}^\mu \partial_\mu \ , \qquad E_+|_\gamma = \partial_u \ \ , \tag{4.109}$$

i.e. the restriction of E_+ to every null geodesic is the tangent vector of the null geodesic.

Infinitesimally the congruence is characterized by the *connecting vectors* Z representing the separation of corresponding points on neighbouring curves and satisfying the equation

$$L_{E_+} Z = [E_+, Z] = \nabla_{E_+} Z - \nabla_Z E_+ = 0 \ \ . \tag{4.110}$$

In a parallel frame, covariant derivatives along the congruence become partial derivatives,

$$E_+^\mu \nabla_\mu (Z^A E_A) = \nabla_u (Z^A E_A) = (\partial_u Z^A) E_A \ \ , \tag{4.111}$$

and since E_+ is null one has

$$g(E_+, E_+) = 0 \Rightarrow (\nabla_A E_+)^- = 0 \ \ . \tag{4.112}$$

Hence (4.110) implies that $(d/du)Z^- = 0$, and we can set $Z^- = 0$ without loss of generality. Then, using the geodesic equation $\nabla_u E_+ = 0$, one finds that

$$\nabla_Z E_+ = Z^b (\nabla_b E_+)^a E_a + Z^b (\nabla_b E_+)^+ E_+ \tag{4.113}$$

and the connecting vector equation (4.110) becomes

$$\frac{d}{du}Z^a = B^a_{\ b}Z^b \tag{4.114}$$

with

$$B^a_{\ b} = (\nabla_b E_+)^a \equiv E^a_\nu E^\mu_b \nabla_\mu E^\nu_+ \quad , \tag{4.115}$$

and Z^+ determined by the Z^a via

$$\frac{d}{du}Z^+ = Z^b(\nabla_b E_+)^+ \quad . \tag{4.116}$$

For later use we note that (4.112) implies that the trace of B is

$$\mathrm{tr}\, B \equiv B^a_{\ a} = \nabla_\mu E^\mu_+ = \frac{1}{\sqrt{-g}}\partial_\mu(\sqrt{-g}\dot{x}^\mu) \quad , \tag{4.117}$$

explaining the ubiquity of logarithmic derivatives in the examples to be discussed below.

It follows from (4.114) that the transverse components Z^a satisfy the null geodesic deviation equation

$$\frac{d^2}{du^2}Z^a = A_{ab}(u)Z^b \quad . \tag{4.118}$$

where

$$A^a_{\ b} = \frac{d}{du}B^a_{\ b} + B^a_{\ c}B^c_{\ b} \quad . \tag{4.119}$$

Note that (4.118) is just a (time-dependent) harmonic oscillator equation with $(-A_{ab}(u))$ the matrix of frequency squares.

A routine calculation now shows that

$$A^a_{\ b} = E^a_\nu E^\mu_b R^\nu_{\ \lambda\rho\mu}\dot{x}^\lambda \dot{x}^\rho = -R^a_{\ +b+} \quad , \tag{4.120}$$

with R the Riemann curvature tensor of the metric g, establishing the equivalence of (4.105) and (4.107). In the following chapter we will use this geodesic deviation approach to calculate the Penrose limits of a variety of cosmological backgrounds.

4.8 Strings in plane wave backgrounds

As we said in the begining of this chapter the relevance of dealing with plane wave backgrounds is that in many cases string theory can be exactly quantized and solved. In particular, string theory in plane wave backgrounds simplifies dramatically due to the existence of a natural light-cone gauge.

Before the discovery of the new maximally supersymmetric solution of IIB supergravity was found, two maximally supersymmetric solutions of the IIB supergravity were known. The first is the flat ten dimensional Minkowski space and the second is the $AdS_5 \times S^5$ supported by Ramond-Ramond charges [146–148]. The recent maximally supersymmetric solution of IIB supergravity is a ten dimensional plane wave space supported by the RR 5-form flux [92]. The plane wave RR background is an exact solution of the equations of motion of IIB supergravity which are the equations of motion for the massless modes of the type IIB superstring. Moreover this new solution is an exact solution of type IIB string theory, there are no α' corrections since the possible corrections to the equations of motion consistent with supersymmetry, involve combinations of the curvature and derivatives of the curvature which are equal to zero[92, 149].

As an example, let us consider the simplest gravitational plane wave backgrounds (4.10) with $A_{ab}(u) = -f^2\delta_{ab}$ supported by a Ramond-Ramond 5 form $F_{u1234} = F_{u5678} = 2f$. This background preserves the maximal number of 32 supersymmetries [92] and it is related by a Penrose limit (boost along a circle of S^5 combined with a rescaling of the coordinates and of the radius) to the $AdS_5 \times S^5$ background [94]. In the light cone gauge

$$u = p_v \tau \,, \quad \Gamma^u \theta^I = 0 \,, \tag{4.121}$$

where $\theta^I (I = 1, 2)$ are the two real positive chirality $10-d$ MW spinors, Γ^μ is the chiral representation for the 32×32 Dirac matrices in terms of the 16×16 matrices γ^μ. The resulting quadratic light cone action [95] can be written like the flat space GS action, in a $2-d$ spinor form and describes 8 free massive $2-d$ scalars and 8 free massive Majorana $2-d$ fermionic fields $\psi = (\theta^1, \theta^2)$ propagating in flat $2-d$ world-sheet

$$L = L_B + L_F \,, \quad L_B = \frac{1}{2}(\partial_+ X^I \partial_- X^I - m^2 X_I^2) \,, \quad m \equiv p^u f \tag{4.122}$$

$$L_F = i(\theta^1 \bar{\gamma}^v \partial_+ \theta^1 + \theta^2 \bar{\gamma}^v \partial_- \theta^2 - 2m\theta^1 \bar{\gamma}^v \Pi \theta^2) \,, \quad \bar{\gamma}^u \theta^I = 0 \,, \tag{4.123}$$

where $\partial_\pm = \partial_\sigma \pm \partial_\tau$. Thus, the equations of motion are

$$\partial_+ \partial_- X^I + m^2 X^I = 0 \,, \tag{4.124}$$

$$\partial_+ \theta^1 - m\Pi\theta^2 = 0 \,, \quad \partial_- \theta^2 + m\Pi\theta^1 = 0 \,. \tag{4.125}$$

The general solutions satisfying the closed string boundary conditions,

$$X^I(\sigma + 1, \tau) = X^I(\sigma, \tau) \,, \quad \theta(\sigma + 1, \tau) = \theta(\sigma, \tau) \,, 0 \leq \sigma \leq 1 \,, \tag{4.126}$$

are [96]:

$$\begin{aligned} X^I(\sigma,\tau) &= \cos m\tau X_0^I + m^{-1}\sin m\tau P_0^I + i\sum_{n\neq 0}\frac{1}{w_n}\left(\phi_n^1(\sigma,\tau)\alpha_n^{1\,I} + \phi_n^2(\sigma,\tau)\alpha_n^{2\,I}\right) \\ \theta^1(\sigma,\tau) &= \cos m\tau\theta_0^1 + \sin m\tau\Pi\theta_0^2 + \sum_{n\neq 0} c_n\left(\phi_n^1(\sigma,\tau)\theta_n^1 + i\frac{w_n-k_n}{m}\phi_n^2(\sigma,\tau)\Pi\theta_n^2\right) \\ \theta^2(\sigma,\tau) &= \cos m\tau\theta_0^2 - \sin m\tau\Pi\theta_0^1 + \sum_{n\neq 0} c_n\left(\phi_n^2(\sigma,\tau)\theta_n^2 - i\frac{w_n-k_n}{m}\phi_n^1(\sigma,\tau)\Pi\theta_n^1\right)\,, \end{aligned}$$

where the basis functions $\phi_n^{1,2}(\sigma,\tau)$ are

$$\phi_n^1(\sigma,\tau) = \exp(-i(w_n\tau - k_n\sigma))\,, \quad \phi_n^2(\sigma,\tau) = \exp(-i(w_n\tau + k_n\sigma)) \tag{4.127}$$

and

$$w_n = \sqrt{k_n^2+m^2}\,,\ \ n>0\,; \qquad w_n = -\sqrt{k_n^2+m^2}\,,\ \ n<0\,; \tag{4.128}$$

$$k_n \equiv 2\pi n\,, \qquad c_n = \frac{1}{\sqrt{1+(w_n-k_n)^2/m^2}}\,, \qquad n=\pm1,\pm2,\cdots \tag{4.129}$$

Given these solutions, one can now quantize the $2-d$ fields X^I and θ^I by promoting as usual the coordinates and momenta to operators and replacing the classical Poisson brackets by equal-time commutators of quantum coordinates and momenta. As usual one can calculate the light-cone Hamiltonian, which in terms of the normalized oscillators

$$\begin{aligned} a_0^I &= \frac{1}{\sqrt{2m}}(P_0^I + imX_0^I)\,, & \bar{a}_0^I &= \frac{1}{\sqrt{2m}}(P_0^I - imX_0^I)\,, \\ \alpha_{-n}^{\mathcal{I}I} &= \sqrt{\frac{w_n}{2}}a_n^{\mathcal{I}I}\,, & \alpha_n^{\mathcal{I}I} &= \sqrt{\frac{w_n}{2}}\bar{a}_n^{\mathcal{I}I}\,, \quad n=1,2,\cdots \\ \theta_0 &= \frac{1}{\sqrt{2}}(\theta_0^1 + i\theta_0^2)\,, & \bar{\theta}_0 &= \frac{1}{\sqrt{2}}(\theta_0^1 - i\theta_0^2)\,, \\ \theta_{-n}^{\mathcal{I}} &= \frac{1}{\sqrt{2}}\eta_n^{\mathcal{I}}\,, & \theta_n^{\mathcal{I}} &= \frac{1}{\sqrt{2}}\bar{\eta}_n^{\mathcal{I}}\,, \quad n=1,2,\cdots \end{aligned}$$

takes the explicit form

$$H = f(a_0^I\bar{a}_0^I + 2f\bar{\theta}_0\bar{\gamma}^v\Pi\theta_0 + 4) + \frac{1}{\alpha' p^u}\sum_{\mathcal{I}=1,2}\sum_{n=1}^{\infty}\sqrt{n^2+(\alpha' p^u f)^2}(a_n^{\mathcal{I}I}\bar{a}_n^{\mathcal{I}I} + \eta_n^{\mathcal{I}}\bar{\gamma}^v\bar{\eta}_n^{\mathcal{I}})\,. \tag{4.130}$$

Thus as we just saw in this plane wave background is easy to exactly solve the superstring theory.

In a similar manner one can solve string models in general non-singular homogeneous plane wave backgrounds[108], and in a class of singular homogeneous plane wave backgrounds[104].

Finally as a remark we must keep on mind that since the Penrose limit relates in a precise way a general supergravity background to a pp-wave, it allow us to understand string theory on many backgrounds at least in the Penrose limit. This opens the possibility to probe string theory in backgrounds that were hitherto inaccessible, for example backgrounds with RR fields.

Chapter 5

Penrose limits near space-time singularities

Since singularities of $A_{ab}(u)$ result from curvature singularities of the original space-time, it is of interest to analyze the nature of Penrose limits of space-time singularities, as they encode information about the rate of growth of curvature and geodesic deviation as one approaches the singularity of the original space-time along a null geodesic. In particular as was shown in the previous chapter, the geometric significance of the Penrose limit, is that A_{ab} is the standard transverse null geodesic deviation matrix of the original metric along the null geodesic $\gamma(u)$.

Since the study of string propagation in singular, and perhaps time-dependent, space-time backgrounds is one of our main interests, then the first step toward this goal is to determine the Penrose limits of space-time singularities in general. In this chapter we are mainly interested in answering the following questions: *What are the Penrose limits of space-time singularities in general? Is there any universal behavior?* As we will see and as it was found in [58, 59], the Penrose limits of space-time singularities show a remarkably universal behavior.

In [99–102], it was found that for a variety of particular brane and cosmological backgrounds the exact Penrose limit is characterized by a wave profile of the special form

$$A_{ab} \sim u^{-2} \,. \tag{5.1}$$

Moreover, in [59, 105] it was observed that the Penrose limit of space-time singularities of cosmological FRW and Schwarzschild-like metrics, i.e. the leading behavior of the profile $A_{ab}(u)$ as one approaches the singularity, is also of the above form.

In the following we will calculate the Penrose limit of some cosmological backgrounds by means of the geodesic deviation approach. Afterwards, it will be shown that in this case the Penrose limits exhibit a remarkably universal behavior in the sense that for a large class of black hole, cosmological and null singularities, indeed all the Szekeres-Iyer metrics [106, 107] with singularities of "power-law type", one obtains plane wave metrics of the form

$$d\tilde{s}^2 = 2dudv + c_{ab}x^a x^b \frac{du^2}{u^2} + d\bar{x}^2 \,, \tag{5.2}$$

with c_{ab} constant and with eigenvalues bounded by $1/4$. Due to the existence of the scale invariance $(u,v) \to (\lambda u, \lambda^{-1} v)$, these singular plane waves are *homogeneous* reflecting the scaling behavior of the original power-law singularities.

Moreover, as we will see chapter 6, string theory in singular homogeneous plane wave backgrounds is exactly solvable [104, 108, 109, 150]. On the other hand, it has also been shown that in a class of such models the string oscillator modes can be analytically continued across the singularity [104]. Since the Penrose limit can be considered as the origin of a string expansion around the original background [99], the above observations about the relation of these backgrounds to interesting space-time singularities provide additional impetus for understanding string theory in an expansion around such metrics.

5.1 The Penrose Limits of a Static Spherically Symmetric Metric

Even though in section 4.6 we already showed explicit examples in the computation of the Penrose limit in this section we are going to illustrate the modern way to take the limit, namely, the geodesic deviation approach. We will determine all the Penrose limits of a static spherically symmetric metric. We start with the metric in Schwarzschild-like coordinates (the extension to isotropic coordinates, brane-like metrics with extended world volumes, or null metrics is straightforward and is discussed in section 5.2),

$$\begin{aligned} ds^2 &= -f(r)dt^2 + g(r)dr^2 + r^2 d\Omega_d^2 \\ d\Omega_d^2 &= d\theta^2 + \sin^2\theta d\Omega_{d-1}^2 \ . \end{aligned} \tag{5.3}$$

Taking the Penrose limit entails first choosing a null geodesic. Because of the rotational symmetry in the transverse direction, without loss of generality we can choose the null geodesic to lie in the (t,r,θ)-plane. The symmetries reduce the geodesic equations to the first integrals

$$\begin{aligned} \dot{t} &= E/f(r) \\ \dot{\theta} &= L/r^2 \\ \dot{r}^2 &= E^2/f(r)g(r) - L^2/g(r)r^2 \ , \end{aligned} \tag{5.4}$$

where E and L are the conserved energy and angular momentum respectively. This defines a natural geodesic congruence, corresponding to the Hamilton-Jacobi function

$$S = -Et + L\theta + R(r) \tag{5.5}$$

with

$$(\frac{d}{dr}R)^2 = gf^{-1}E^2 - r^{-2}gL^2 \tag{5.6}$$

and allows us to calculate B_{ab}.

We first construct the parallel frame. We have

$$E_+ = \dot{r}\partial_r + \dot{t}\partial_t + \dot{\theta}\partial_\theta \ , \qquad E_+|_\gamma = \partial_u \ , \tag{5.7}$$

and we will not need to be more specific about E_-. The transverse components are $E_a = (E_1, E_{\hat{a}})$, with $\hat{a} = 2, \ldots, d$ referring to the transverse $(d-1)$-sphere. Since there is no evolution in these directions, the $E_{\hat{a}}$ are the obvious parallel frame components

$$E_{\hat{a}} = \frac{1}{r\sin\theta} e_{\hat{a}} \tag{5.8}$$

with $e_{\hat{a}}$ an orthonormal coframe for $d\Omega^2_{d-1}$. The transverse $SO(d)$-symmetry implies

$$\begin{aligned} B_{1\hat{a}} &= A_{1\hat{a}} = 0 \\ B_{\hat{a}\hat{b}}(u) &= B(u)\delta_{\hat{a}\hat{b}} \\ A_{\hat{a}\hat{b}}(u) &= A(u)\delta_{\hat{a}\hat{b}} \ . \end{aligned} \tag{5.9}$$

Moreover, because of (4.117) we have

$$B_{11}(u) = \nabla_\mu \dot{x}^\mu(u) - (d-1)B(u) \tag{5.10}$$

so that we only have to calculate $B_{22}(u) = B(u)$, for which one finds (with, say, $e_2 = \partial_\phi$)

$$B_{22} = \Gamma^\phi_{\phi r}\dot{r} + \Gamma^\phi_{\phi\theta}\dot{\theta} = \partial_u \log(r(u)\sin\theta(u)) \ , \tag{5.11}$$

or

$$B_{\hat{a}\hat{b}}(u) = \delta_{\hat{a}\hat{b}}\partial_u \log(r(u)\sin\theta(u)) \ . \tag{5.12}$$

Since

$$\mathrm{tr}\, B = \partial_u \log\left(\dot{r} r^d \sin^{d-1}\theta \sqrt{f(r)g(r)}\right) \tag{5.13}$$

one finds

$$B_{11}(u) = \partial_u \log\left(r(u)\dot{r}(u)\sqrt{f(r(u))g(r(u))}\right) \ . \tag{5.14}$$

Now, in general, for $B_{ab}(u)$ of the logarithmic derivative form

$$B_{ab}(u) = \delta_{ab}\partial_u \log K_a(u) \tag{5.15}$$

one has

$$A_{ab}(u) = \delta_{ab} K_a(u)^{-1}\partial_u^2 K_a(u) \tag{5.16}$$

and therefore

$$\begin{aligned} A_{11} &= (r\dot{r}\sqrt{fg})^{-1}\partial_u^2(r\dot{r}\sqrt{fg}) \\ A_{\hat{a}\hat{b}} &= \delta_{\hat{a}\hat{b}}(r\sin\theta)^{-1}\partial_u^2(r\sin\theta) \ . \end{aligned} \tag{5.17}$$

In particular, for the transverse components one has the universal result

$$A_{\hat{a}\hat{b}}(u) = \delta_{\hat{a}\hat{b}}\left(\frac{\ddot{r}(u)}{r(u)} - \frac{L^2}{r(u)^4}\right) \ . \tag{5.18}$$

5.1.1 Schwarzschild metric

In section 4.6.1 we explicitly computed the Penrose limit of the Schwarzschild metric, however in this section we we will consider it again as a concrete example of the calculations above. The $D = (d+2)$-dimensional Schwarzschild metric is[1]

$$ds^2 = -f(r)dt^2 + f(r)^{-1}dr^2 + r^2 d\Omega_d^2 \tag{5.19}$$

where

$$f(r) = 1 - \frac{2m}{r^{d-1}} \ . \tag{5.20}$$

We are assuming that $D \geq 4$ and note that evidently only for $D = 4$ is m the ADM mass of the black hole.

In this case we have

$$\dot{r}^2 = E^2 - L^2 f(r) r^{-2} \equiv E^2 - 2V_{eff}(r) \ , \tag{5.21}$$

[1] The Penrose limit for certain special (radial $L = 0$) null geodesics in the AdS-Schwarzschild metric has been discussed before e.g. in [145, 151]. The general case, using the Hamilton-Jacobi method to construct adapted coordinates, was presented in [105]. For a general discussion of limits of the Schwarzschild metric not depending on additional parameters like L see [152].

where $V_{eff}(r)$ is the usual effective potential, with respect to which $r(u)$ satisfies the Newtonian equation of motion

$$\ddot{r} = -V'_{eff}(r) \ . \tag{5.22}$$

It follows that

$$\begin{aligned} A_{22}(u) = \ldots = A_{dd}(u) &= \frac{\ddot{r}(u)}{r(u)} - \frac{L^2}{r(u)^4} \\ &= -\frac{(d+1)mL^2}{r(u)^{d+3}} \ , \end{aligned} \tag{5.23}$$

where $r(u)$ is the solution to the geodesic (effective potential) equation (5.21). Moreover, since the Schwarzschild metric is a vacuum solution, this is a vacuum plane wave with $\mathrm{tr}\, A(u) = \delta^{ab}A_{ab}(u) = 0$, so that

$$A_{11}(u) = \frac{(d+1)(d-1)mL^2}{r(u)^{d+3}} \ . \tag{5.24}$$

There are a number of facts that can be readily deduced from this result:

- First of all, we see that the Penrose limit of the Schwarzschild metric is flat for radial null geodesics, $L = 0$. We could have anticipated this on general grounds because in this case the setting is $SO(d+1)$-invariant, implying $A_{ab}(u) \sim \delta_{ab}$, which is incompatible with $\mathrm{tr}\, A = 0$ unless $A_{ab}(u) = 0$. This should, however, not be interpreted as saying that the radial Penrose limit of the Schwarzschild metric is Minkowski space. Rather, the space-time "ends" at the value of u at which $r(u) = 0$, say at $u = 0$. Perhaps the best way of thinking of this metric is as a time-dependent orbifold of the kind studied recently in the context of string cosmology (see e.g. [153] and references therein).

- We also learn that the Penrose limit is a symmetric plane wave (u-independent wave profile) if $r(u) = r_*$ is a null geodesic at constant r. Setting $\ddot{r} = \dot{r} = 0$, one finds that

$$r_*^{d-1} = (d+1)m \tag{5.25}$$

(the familiar $r = 3m$ photon orbit for $D = 4$), with the constraint

$$r_*^2 = \frac{d-1}{d+1}\frac{L^2}{E^2} \tag{5.26}$$

on the ratio L/E. Precisely because they lead to symmetric plane waves, with a well-understood string theory quantization, such constant r Penrose limits have attracted some interest in the literature.

- Moreover we see that the resulting plane wave metric for $L \neq 0$ is singular if and only if the original null geodesic runs into the singularity, which will happen for sufficiently small values of L/E.

We will now take a closer look at the u-dependence of the wave profile near the singularity $r(u) = 0$. We thus consider sufficiently small values of L/E in order to avoid the angular momentum barrier.

For small values of r, the dominant term in the differential equation (5.21) for r is (unless $L = 0$, a case we already dealt with above)

$$\dot{r} = \sqrt{2m} L r^{-(d+1)/2} \quad . \tag{5.27}$$

This implies that

$$r(u)^{d+3} = \frac{mL^2(d+3)^2}{2} u^2 \quad . \tag{5.28}$$

Thus the behavior of the Penrose limit of the Schwarzschild metric as $r \to 0$ is

$$A_{11}(u) = -\omega_{SS}'^2(d) u^{-2} \tag{5.29}$$

and

$$A_{22}(u) = \ldots = A_{dd}(u) = -\omega_{SS}^2(d) u^{-2}, \tag{5.30}$$

with frequencies

$$\omega_{SS}'^2(d) = -\frac{2(d^2-1)}{(d+3)^2} \quad . \tag{5.31}$$

and

$$\omega_{SS}^2(d) = \frac{2(d+1)}{(d+3)^2} \quad . \tag{5.32}$$

We note the following:

- First of all, in this limit one finds a singular homogeneous plane wave of the type (4.47). As we will see later, this scale invariance of the near-singularity Penrose limit can be attributed to the power-law scaling behavior of the near-singularity Schwarzschild metric.

- Moreover, the dependence on L and m has dropped out. The metric thus exhibits a universal behavior near the singularity which depends only on the space-time dimension $D = d+2$, but neither on the mass of the black hole nor on the angular momentum of the null geodesic used to approach the singularity. For example, for $D = 4$ one has

$$\omega_{SS}^2(d=2) = \frac{6}{25} \quad . \tag{5.33}$$

- The frequencies are bounded by
$$\omega'^2_{SS}(d) < 0 < \omega^2_{SS}(d) < \frac{1}{4} \ . \tag{5.34}$$
- Finally, we note that the above result is also valid for (A)dS black holes since the presence of a cosmological constant is irrelevant close to the singularity.

5.1.2 FRW metric

As another example we consider the Penrose limit of the $D = (n+1)$-dimensional FRW metric
$$ds^2 = -dt^2 + a(t)^2(dr^2 + f_k(r)^2 d\Omega^2_{n-1}) \ , \tag{5.35}$$
where $f_k(r) = r, \sin r, \sinh r$ for $k = 0, +1, -1$ respectively.

Since the spatial slices are maximally symmetric, up to isometries there is a unique null geodesic and hence a unique Penrose limit. So without loss of generality we shall consider null geodesics which have vanishing angular momentum on the transverse sphere.

Then, with a suitable scaling of the affine parameter, the null geodesic equations can be written as
$$\frac{d}{du}t(u) = \pm a(t(u))^{-1} \ , \qquad \frac{d}{du}r(u) = a(t(u))^{-2} \tag{5.36}$$
(and in what follows, we choose the upper sign in the first equation). Thus
$$E_+ = \partial_u = a^{-1}\partial_t + a^{-2}\partial_r \ , \tag{5.37}$$
and this can be extended to a parallel pseudo-orthonormal frame by
$$\begin{aligned} E_- &= \frac{1}{2}(-a\partial_t + \partial_r) \\ E_a &= (af_k)^{-1}\hat{e}_a \ , \end{aligned} \tag{5.38}$$
where $\hat{e}_a$ is an orthonormal frame for $d\Omega^2_d$, $d = n-1$.

The transverse rotational symmetry implies that $B_{ab}(u) = B(u)\delta_{ab}$ and $A_{ab}(u) = A(u)\delta_{ab}$. Therefore, to determine $B(u)$ it suffices to compute the trace of $B_{ab}(u)$,
$$\mathrm{tr}\, B = \partial_u \log\left(a^{n-1} f_k^{n-1}\right) \ , \tag{5.39}$$
implying
$$B(u) = \partial_u \log(af_k) \ . \tag{5.40}$$
Using $\frac{d^2}{dr^2} f_k = -kf_k$, one finds
$$A_{ab}(u) = \delta_{ab}A(u) = \delta_{ab}\left(\frac{\ddot{a}(u)}{a(u)} - \frac{k}{a(u)^4}\right) \ . \tag{5.41}$$

This is the precise analogue of the expression (5.18) obtained in the static spherically symmetric case, the spatial curvature k now playing the role of the angular momentum L^2.

This can now be rewritten in a variety of ways to obtain insight into the properties of this FRW plane wave. For example, writing this in terms of t-derivatives (in order to make use of the Friedmann equations), we find

$$A(u(t)) = \frac{1}{a(t)^2}(\frac{a''(t)}{a(t)} - \frac{k + a'(t)^2}{a(t)^2}) \ , \tag{5.42}$$

where $a(t)$ is determined by the Einstein (Friedmann) equations, $u(t)$ by $du = a(t)dt$, and $a' = \frac{d}{dt}a$. The Friedmann equations

$$\begin{aligned} \frac{a'(t)^2 + k}{a(t)^2} &= \frac{16\pi G}{n(n-1)}\rho(t) \\ \frac{a''(t)}{a(t)} &= -\frac{8\pi G}{n(n-1)}[(n-2)\rho(t) + nP(t)] \ , \end{aligned} \tag{5.43}$$

imply

$$\frac{a'(t)^2 + k}{a(t)^2} - \frac{a''(t)}{a(t)} = \frac{8\pi G}{(n-1)}[\rho(t) + P(t)] \ , \tag{5.44}$$

so that one finds that the wave profile of the FRW plane wave can be written compactly as

$$A(u) = -\frac{8\pi G}{n-1}\frac{\rho(u) + P(u)}{a(u)^2} \ . \tag{5.45}$$

One immediate consequence is that the Penrose limit is flat if and only if $\rho + P = 0$, corresponding to having as the only matter content a cosmological constant. This is in agreement with the result [99] that every Penrose limit of a maximally symmetric space-time is flat.

We will now study the behavior of $A(u)$ near a singularity, and to be specific we choose the usual equation of state

$$P(t) = w\rho(t) \ . \tag{5.46}$$

We consider $w > -1$ ($w = -1$ would correspond to the case $\rho + P = 0$ already dealt with above) and introduce the positive parameter

$$h(n, w) = \frac{2}{n(1+w)} \tag{5.47}$$

and the positive constant (constant by the continuity equation for ρ)

$$C_h = \frac{16\pi G}{n(n-1)}\rho(t)a(t)^{2/h} \ , \tag{5.48}$$

in terms of which the Friedmann equations read

$$a'(t)^2 = C_h a(t)^{(2h-2)/h} - k \tag{5.49}$$
$$a''(t) = \frac{h-1}{h} C_h a(t)^{(h-2)/h} \ . \tag{5.50}$$

Thus the universe is decelerating for $0 < h < 1$ and accelerating for $h > 1$, the critical case $h = 1$ corresponding to $w_c = -1 + 2/n$ (the familiar dark energy threshold $w_c = -1/3$ for $n = 3$).

We first consider the case $k = 0$. In that case one has

$$a(t) \sim t^h \ , \tag{5.51}$$

and therefore

$$a(u) \sim u^{h/h+1} \ . \tag{5.52}$$

It then follows immediately from (5.41) that, more explicitly, the u-dependence of $A(u)$ is[2]

$$A(u) = -\omega_{FRW}^2(h, k=0) u^{-2} \ , \tag{5.53}$$

where

$$\omega_{FRW}^2(h, k=0) = \frac{h}{(1+h)^2} \ . \tag{5.54}$$

We see that the Penrose limit of a spatially flat FRW universe with equation of state $P = w\rho$ is exactly a singular homogeneous plane wave of the type (4.47).

The frequency square $\omega_{FRW}^2(h, k=0)$ has the following properties:

- Since

$$\omega_{FRW}^2(h, k=0) = \omega_{FRW}^2(1/h, k=0) \ , \tag{5.55}$$

  for every accelerating (inflating) solution of the $k = 0$ Friedmann equations there is precisely one decelerating solution with the same Penrose limit. The self-dual point $h = 1$ corresponds to the linear time-evolution $a(t) \sim t$.

- The frequency squares are again bounded by

$$\omega_{FRW}^2(h, k=0) \leq \frac{1}{4} \ , \tag{5.56}$$

  with equality attained for $h = 1$.

[2]This generalizes the result reported in [99].

- Curiously, the frequencies obtained in the Penrose limit of the Schwarzschild metric (in all but one of the directions) are precisely those of a dust-filled FRW universe, $P = w = 0$, of the same dimension $n = d + 1$,

$$\omega_{SS}^2(d) = \omega_{FRW}^2(h, k = 0) \quad , \tag{5.57}$$

  e.g. $6/25$ for $n = 3$.

It is clear that for $k = 0$, when only the first term in (5.41) is present, this homogeneous u^{-2}-behavior is a consequence of the exact power-law behavior of $a(t)$ and hence $a(u)$. Let us now consider what happens for $k \neq 0$, when there is a competition between the two terms in (5.41) as one approaches the singularity.

One might like to argue that, even for $k \neq 0$, one finds the same behavior provided that the matter term dominates over the curvature term in the Friedmann equation (5.49) as $a \to 0$. This happens for $0 < h < 1$, and this argument is correct as one can also see that in this range the first term in (5.41), proportional to u^{-2}, indeed dominates over the second (curvature) term which (cf. (5.52)) is proportional to $u^{-4h/(h+1)}$. Thus for $0 < h < 1$ the near-singularity limit of the FRW plane wave is a homogeneous plane wave with k-independent frequencies (5.54),

$$0 < h < 1: \quad \omega_{FRW}^2(h, k) = \omega_{FRW}^2(h, k = 0) = \frac{h}{(1 + h)^2} \quad . \tag{5.58}$$

Now let us look at what happens as one passes from a decelerating to a critical ($h = 1$) and then accelerating ($h > 1$) universe. First of all, for $h = 1$, both terms on the right hand side of the Friedmann equation (5.49) contribute equally (they are constant), and correspondingly both terms in (5.41) are proportional to u^{-2}. Thus one finds a homogeneous plane wave, but with a curvature-induced shift of the frequency,

$$h = 1: \quad \omega_{FRW}^2(h = 1, k) = \omega_{FRW}^2(h = 1, k = 0) + kc^2 = \frac{1}{4} + kc^2 \tag{5.59}$$

for some constant c. In particular, in the spatially closed case $k = +1$ (this requires $C_h > 1$), one now finds frequency squares that are larger than $1/4$. This is a borderline behavior in the sense that, as can easily be seen from (5.49), the initial singularity for $k = +1$ ceases to exist for $h > 1$.

It thus remains to discuss the case $k = -1$ and $h > 1$. Given the previous discussion, one might be tempted to think that now the second term in (5.41) will dominate over the first, leading to a non-homogeneous and more singular $u^{-4h/(h+1)}$-behavior. This is, however, not the case, as (5.52) now represents the leading behavior at large $a(u)$. At small $a(u)$, the leading behavior is, exactly as for $h = 1$, determined by the constant curvature term in (5.49). Thus even in this case one finds a singular homogeneous plane wave, with frequency

$$h > 1 : \quad \omega^2_{FRW}(h, k = -1) = \frac{1}{4} - c^2 \tag{5.60}$$

once again bounded from above by 1/4.

5.2 Generalizations

5.2.1 Brane metrics

It is straightforward to generalize the analysis of section 2.1 to include longitudinal world volume directions,

$$f(r)(-dt^2) \to f(r)(-dt^2 + d\vec{y}^2) \quad . \tag{5.61}$$

A parallel frame in the brane world volume directions is $E_i = f^{-1/2}\partial_{y^i}$, and

$$B_{ij} = \delta_{ij}\partial_u \log f(r(u))^{1/2} \tag{5.62}$$

which in turn leads to

$$A_{ij} = \delta_{ij} f(r(u))^{-1/2}\partial_u^2 \log f(r(u))^{1/2} \quad . \tag{5.63}$$

The remaining of the components of A are as in section 2.1.

5.2.2 Isotropic coordinates

Likewise, for isotropic coordinates,

$$ds^2 = -f(r)dt^2 + h(r)(dr^2 + r^2 d\Omega_d^2) \quad , \tag{5.64}$$

a straightforward calculation reveals that

$$\operatorname{tr} B = \partial_u \log\left(\dot{r} r^d f^{\frac{1}{2}} h^{\frac{d+1}{2}} \sin^{d-1}(\theta)\right) \tag{5.65}$$

and

$$B_{22} = \partial_u \log\bigl(h^{\frac{1}{2}} r \sin(\theta)\bigr) \ . \tag{5.66}$$

These lead to

$$\begin{aligned} B_{11}(u) &= \partial_u \log\bigl(r\dot{r} h f^{1/2}\bigr) \\ B_{\hat{a}\hat{b}}(u) &= \delta_{\hat{a}\hat{b}}\partial_u \log(r h^{1/2} \sin\theta) \end{aligned} \tag{5.67}$$

with the corresponding second-derivative expressions for $A_{ab}(u)$. Again it is easy to include longitudinal directions.

5.2.3 Null singularities

Finally, we consider spherically symmetric null metrics of the form

$$ds^2 = 2g(x)dxdy + f(x)d\Omega_2^2 \ . \tag{5.68}$$

The geodesic equations are

$$\dot{x} = Pg^{-1} \quad , \quad \dot{y} = -\frac{L^2}{P}f^{-1} \quad , \quad \dot{\theta} = f^{-1}L \ , \tag{5.69}$$

where P and L are constants of motion. In this case, one finds

$$\mathrm{tr}\, B = \partial_u \log\bigl(\dot{x} g f \sin(\theta)\bigr) \tag{5.70}$$

and

$$B_{22} = \partial_u \log\bigl(f^{\frac{1}{2}} \sin(\theta)\bigr) \ . \tag{5.71}$$

Therefore, we have

$$\begin{aligned} K_1(u) &= Pf(u)^{1/2} \\ K_2(u) &= f(u)^{1/2} \sin\theta(u) \ . \end{aligned} \tag{5.72}$$

In particular, in terms of $r(x) = f(x)^{1/2}$, A_{22} once again takes the standard form (5.18)

$$A_{22}(u) = \frac{\ddot{r}(u)}{r(u)} - \frac{L^2}{r(u)^4} \ . \tag{5.73}$$

5.3 The Universality of Penrose Limits of Power-Law Type Singularities

In the previous section we have presented some evidence for a remarkable

> **Conjecture:** Penrose limits of physically reasonable space-time singularities are singular homogeneous plane waves with wave profile $A_{ab}(u) \sim u^{-2}$.

In this section we will show how to prove this conjecture for a very large class of physical singularities of spherically symmetric type. We will in the process also see some examples of "extreme" stress-energy tensors that give rise to a different behavior.

5.3.1 Szekeres-Iyer metrics

The homogeneity of the Penrose limit (geodesic deviation) that we have found in the above examples appears to reflect a power-law scaling behavior of the metric near the singularity. Thus to assess the generality of this kind of result, one needs to inquire about the generality of space-time singularities exhibiting such a power-law behavior.

In [106] (see also [107]), in the context of investigations of the Cosmic Censorship Hypothesis, Szekeres and Iyer studied a large class of four-dimensional spherically symmetric metrics they dubbed "metrics with power-law type singularities". Such metrics encompass practically all known singular spherically symmetric solutions of the Einstein equations, in particular all the FRW metrics, Lemaître-Tolman-Bondi dust solutions, cosmological singularities of the Lifshitz-Khalatnikov type, as well as other types of metrics with null singularities.

In "double-null form", these metrics (in $d+2$ dimensions) take the form

$$ds^2 = -e^{A(U,V)}dUdV + e^{B(U,V)}d\Omega_d^2 \ , \tag{5.74}$$

where $A(U,V)$ and $B(U,V)$ have expansions

$$\begin{aligned} A(U,V) &= p\ln x(U,V) + \text{regular terms} \\ B(U,V) &= q\ln x(U,V) + \text{regular terms} \end{aligned} \tag{5.75}$$

near the singularity surface $x(U,V)=0$.

Generically, the residual coordinate transformations $U \to U'(U)$, $V \to V'(V)$ preserving the form of the metric (5.74) can be used to make $x(U,V)$ linear in U and V,

$$x(U,V) = kU + lV \ , \qquad k,l = \pm 1, 0 \ , \tag{5.76}$$

with $\eta = kl = 1, 0, -1$ corresponding to space-like, null and time-like singularities respectively. This choice of gauge essentially fixes the coordinates uniquely, and thus the "critical exponents" p and q contain diffeomorphism invariant information.

The Schwarzschild metric, for example, has $p = (1-d)/d$ and $q = 2/d$, as is readily seen by starting with the metric in Eddington-Finkelstein or Kruskal-Szekeres coordinates and transforming to the Szekeres-Iyer gauge. Alternatively, if one is just interested in the leading behavior of the metric, one can simply expand the metric near $r=0$ and then go to tortoise-coordinates.

We will focus on the behavior of these geometries near the singularity at $x=0$, where the metric is

$$ds^2 = -x^p dUdV + x^q d\Omega_d^2 \ . \tag{5.77}$$

For generic situations this leading behavior is sufficient to discuss the physics near the singularity. In certain special cases, for particular values of p, q or for null singularities, this leading behavior cancels in certain components of the Einstein tensor and the subleading terms in the above metric become important for a full analysis of the singularities [106, 107]. The analysis then becomes more subtle and we will not discuss these cases here. In the following we will consider exclusively the metric (5.77) which, for $\eta \neq 0$ and generic values of p and q, captures the dominant behavior of the physics near the singularity.

For $\eta \neq 0$ we define $y = kU - lV$ and choose $k = \eta l = 1$. Then the metric takes the form

$$ds^2 = \eta x^p dy^2 - \eta x^p dx^2 + x^q d\Omega_d^2 \ . \tag{5.78}$$

With the further definition $r = x^{q/2}$ (for $q \neq 0$), this has the standard form of a spherically symmetric metric. We will come back to this below in order to be able to make direct use of the analysis of section 5.1.

For $\eta = 0$, on the other hand, we choose $x = U$, $y = -V$, so that the metric is

$$ds^2 = x^p dx dy + x^q d\Omega_d^2 \ , \tag{5.79}$$

which has the form of the spherically symmetric null metrics analyzed in section 5.2.3.

5.3.2 Null Geodesics of Szekeres-Iyer Metrics

In terms of the conserved momenta P and L associated with y and, say, the colatitude θ of the d-sphere,

$$d\Omega_d^2 = d\theta^2 + \sin^2\theta d\Omega_{d-1}^2 \ , \tag{5.80}$$

in particular

$$x^q \dot{\theta} = L \ , \tag{5.81}$$

the null geodesic condition (for any η) is equivalent to

$$\dot{x}^2 = P^2 x^{-2p} + \eta L^2 x^{-p-q} \ , \tag{5.82}$$

To understand the null geodesics near $x = 0$, we begin by extracting as much information as possible from this equation, recalling that due to the expansion around $x = 0$ we can only trust the leading behavior of this equation as $x \to 0$.

Unless $p = q$, one of the two terms on the right-hand-side of (5.82) will dominate as $x \to 0$, and thus the generic behavior of a null geodesic near $x = 0$ is identical to that of a geodesic with either $L = 0$ or $P = 0$. In the former case, one finds

$$\text{Behavior 1:} \qquad x(u) \sim u^{1/(p+1)} \tag{5.83}$$

unless $p = -1$ when $x(u) \sim \exp u$. We are only interested in those geodesics which run into the singularity at $x = 0$ at finite u. This happens only for $p > -1$. In the latter case, corresponding to null geodesics which asymptotically, as $x \to 0$, behave like geodesics with $P = 0$, we evidently need $\eta = +1$ (a space-like singularity), which leads to

$$\text{Behavior 2:} \qquad x(u) \sim u^{2/(p+q+2)} \tag{5.84}$$

unless $p + q = -2$ which again leads to an exponential behavior. These null geodesics run into the singularity at finite u for $p + q > -2$.

For $\eta = +1$, the situation regarding null geodesics that reach the singularity at finite u is summarized in the following table.

Conditions on (P, L)	Constraints on (p, q)	Behavior
$P \neq 0, L = 0$	$p > -1$	1
$P = 0, L \neq 0$	$p + q > -2$	2
$P \neq 0, L \neq 0$	$p > q, p > -1$	1
$P \neq 0, L \neq 0$	$p < q, p + q > -2$	2
$P \neq 0, L \neq 0$	$p = q > -1$	$1 = 2$

(5.85)

For $\eta = -1$, the situation is largely analogous, the main difference being that now the second term in (5.82) acts as an angular momentum barrier preventing e.g. geodesics with $L \neq 0$ for $q > p$ from reaching the singularity at $x = 0$. These cases are indicated by a '$-$' in the table below. For the same reason, for $p = q$ one finds the constraint $|P| > |L|$.

Conditions on (P, L)	Constraints on (p, q)	Behavior
$P \neq 0, L = 0$	$p > -1$	1
$P = 0, L \neq 0$		$-$
$P \neq 0, L \neq 0$	$p > q, p > -1$	1
$P \neq 0, L \neq 0$	$p < q$	$-$
$\|P\| > \|L\|$	$p = q > -1$	$1 = 2$

(5.86)

Finally, for $\eta = 0$ we find Behavior 1 for all values of p and q, with the corresponding constraint $p > -1$.

5.3.3 Penrose Limits of Power-Law Type Singularities

We will now determine the Penrose limits of the Szekeres-Iyer metrics along the null geodesics reaching the singularity $x = 0$ at finite u.

For $\eta \neq 0$ we notice that the metric is simply a special case of a spherically symmetric metric and thus can be treated using the analysis of section 5.1. Indeed, with $t = y$ and $r = x^{q/2}$ $(q \neq 0)$, the metric (5.78) takes the form

$$\begin{aligned} ds^2 &= \eta x^p dy^2 - \eta x^p dx^2 + x^q d\Omega^2 && (5.87)\\ &= \eta r^{2p/q} dt^2 - \frac{4\eta}{q^2} r^{2(p-q+2)/q} dr^2 + r^2 d\Omega^2 && (5.88) \end{aligned}$$

where in the second line the notation of t and r is adapted to the case of $\eta = -1$ where the singularity is time-like and t is time. We will continue to use this notation even for space-like singularities where t is actually space like.

The case $q = 0$ is special, but actually corresponds to a shell crossing singularity [106] which is usually not considered to be a true singularity as the transverse sphere is of constant radius $x^q = 1$. Such singularities arise for instance for certain collisions of spherical dust shells. From here on we will only discuss $q \neq 0$.

Referring to section 5.1 where such a spherically symmetric metric was treated, we can identify

$$\begin{aligned} f(r) &= -\eta r^{2p/q} && (5.89)\\ g(r) &= -\frac{4\eta}{q^2} r^{2(p-q+2)/q}. && (5.90) \end{aligned}$$

We can now appeal to (5.14, 5.12) to deduce that

$$B_{ab} = \delta_{ab} \partial_u \log(K_a(u)) \qquad (5.91)$$

with

$$\begin{aligned} K_1(u) &= \dot{r}(u) r(u)^{2(p+1)/q}\\ K_2(u) &= r(u) \sin(\theta(u)) && (5.92) \end{aligned}$$

As shown in section 5.2.3, it is straightforward to extend this analysis to the case of null singularities, $\eta = 0$, with the result that the expressions for $K_a(u)$ are identical to those given for $\eta = \pm 1$ in (5.92). We can thus treat all three cases simultaneously.

It follows from the analysis of the previous section that the only possibility of interest for $r(u) = x(u)^{q/2}$ is the power-law behavior

$$r(u) = u^a \ , \qquad (5.93)$$

with

$$\begin{aligned} &\text{Behavior 1:} \quad p > -1 \quad a = q/2(p+1) && (5.94)\\ &\text{Behavior 2:} \quad p + q > -2 \quad a = q/(p+q+2) \ . && (5.95) \end{aligned}$$

Clearly, then, $K_1(u)$ is also a simple power of u. Specifically one has (since we are interested in the logarithmic derivatives of $K_1(u)$, proportionality factors are irrelevant)

$$\begin{aligned} &\text{Behavior 1:} \quad K_1(u) \sim r(u) \\ &\text{Behavior 2:} \quad K_1(u) \sim r(u)^{p/q} \ . \end{aligned} \tag{5.96}$$

Thus the corresponding component of $A_{ab}(u)$ is

$$\begin{aligned} &\text{Behavior 1:} \quad A_{11}(u) = \frac{\ddot{K}_1(u)}{K_1(u)} = a(a-1)u^{-2} \\ &\text{Behavior 2:} \quad A_{11}(u) = \frac{\ddot{K}_1(u)}{K_1(u)} = pa/q(pa/q-1)u^{-2} \ . \end{aligned} \tag{5.97}$$

and the Penrose limit behaves as a singular homogeneous plane wave in this direction. Since $b(b-1)$ has a minimum $-1/4$ at $b = 1/2$, this leads to the bound

$$\omega_1^2 \leq \frac{1}{4} \ . \tag{5.98}$$

This is the same range that we found empirically for both the Schwarzschild and FRW plane waves near the singularity.

The behavior of A_{22} is more subtle due to the dependence of $K_2(u)$ on $\sin\theta(u)$. The general behavior is as in (5.18), namely

$$A_{22}(u) = \frac{\ddot{r}(u)}{r(u)} - \frac{L^2}{r(u)^4} \ . \tag{5.99}$$

With the power-law behavior $r(u) = u^a$, the first term is always proportional to u^{-2}. This term is dominant as $u \to 0$ when $a < 1/2$, while it is the second term that dominates for $a > 1/2$ (and leads to a strongly singular plane wave with profile $\sim u^{-4a}$). In the special case $a = 1/2$, both terms are proportional to u^{-2}. Thus one has, for $L \neq 0$,

$$r(u) = u^a \quad a < \tfrac{1}{2}: \quad A_{22}(u) \to -\omega_2^2 u^{-2} \ , \qquad \omega_2^2 = a(1-a) < \frac{1}{4} \tag{5.100}$$

$$a = \tfrac{1}{2}: \quad A_{22}(u) \to -\omega_2^2 u^{-2} \ , \qquad \omega_2^2 = \frac{1}{4} + L^2 \geq \frac{1}{4} \tag{5.101}$$

$$a > \tfrac{1}{2}: \quad A_{22}(u) \to -L^2 u^{-4a} \ . \tag{5.102}$$

When $\eta = 0$, Behavior 1 arises in the entire p-q plane. Thus $\omega_1^2 \leq 1/4$ and a can take any of the three values above with the special value $a = 1/2$ corresponding to the line $q = p + 1$. When $p \geq q$ and $\eta \neq 0$, $a = q/2(p+1)$ and thus always $a < 1/2$. On the other hand, when $p < q$ and $\eta = 1$ we see that $a = q/(p+q+2)$ can take on any value, with $a = 1/2$ along the line $q = p + 2$ and $a > 1/2$ for $q > p + 2$. When $p < q$ and $\eta = -1$ we cannot reach the singularity along a geodesic with $L \neq 0$.

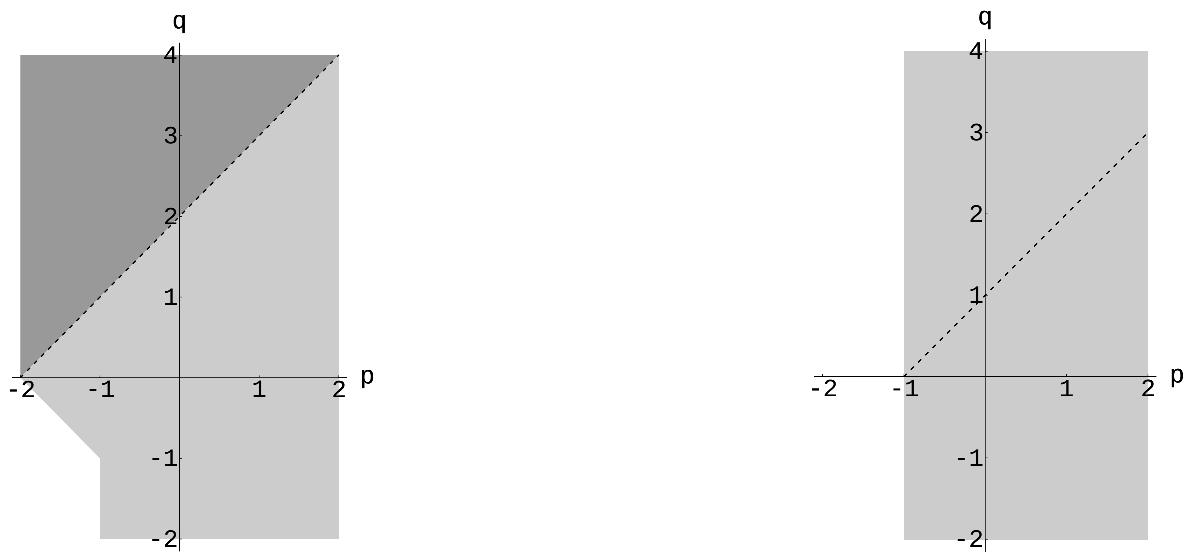

Figure 5.1: The Penrose Limit Phase Diagram in the $p-q$ plane for (a) space-like ($\eta = +1$) and (b) time-like ($\eta = -1$) singularities. Singular HPWs arise in the light-shaded regions whereas in the dark-shaded region there are Penrose limits leading to strongly singular (and non-homogeneous) plane waves. (a) The diagram is bounded on the left by the lines $p = -1$ and $p + q = -2$. The dashed line $a = 1/2 \Leftrightarrow q = p + 2$ separates the two regions, and only along that line one finds singular HPWs with $\omega_2^2 > 1/4$. (b) For $\eta = -1$, one finds singular HPWs with $\omega_2^2 \leq 1/4$ for all (p, q) with $p > -1$, $\omega_2^2 = 1/4$ arising only along the dashed line $a = 1/2$ for zero angular momentum, $L = 0$.

When $L = 0$, only the first term in (5.99) is present, and one thus finds (5.100) for all values of a. Since $L = 0$ implies Behavior 1, this means $a = q/2(p+1)$. Along the special line $q = 2(p+1)$ one has $a = 1$ and one finds the "flat" Penrose limit $A_{11}(u) = A_{22}(u) = 0$. In particular, this happens for radial null geodesics in the Schwarzschild metric ($p = (1-d)/d$ and $q = 2/d$), as already noticed in section 5.1.1.

These results are summarized in figure 5.1(a) for $\eta = 1$ and in 5.1(b) for $\eta = -1$.

5.4 The Role of the Dominant Energy Condition

We thus see that while we frequently obtain a singular HPW with $\omega_a^2 \leq 1/4$ in the Penrose limit, other possibilities do arise. For time-like singularities, the situation is clear:

> Penrose Limits of time-like spherically symmetric singularities of power-law type are singular HPWs with frequency squares bounded from above by 1/4.

We will now show that for space-like singularities a different behavior can occur only when the strict Dominant Energy Condition (DEC) is violated, in particular, that the strongly singular region (the dark-shaded region in figure 5.1(a)) is excluded by the requirement that the DEC be satisfied but not saturated.

We begin by recalling the definition of the *Dominant Energy Condition* on the stress-energy tensor T^μ_ν (or Einstein tensor G^μ_ν) [98]: for every time-like vector v^μ, $T_{\mu\nu}v^\mu v^\nu \geq 0$, and $T^\mu_\nu v^\nu$ is a non-space-like vector. This may be interpreted as saying that for any observer the local energy density is non-negative and the energy flux causal.

Next we recall that a stress-energy tensor is said to be of type I [98] if T^μ_ν has one time-like and three (more generally, $d+1$) space-like eigenvectors. The corresponding eigenvalues are $-\rho$ (ρ the energy density) and the principal pressures P_α, $\alpha = 1, \ldots, d{+}1$. For a stress-energy tensor of type I, the DEC is equivalent to

$$\rho \geq |P_\alpha| \quad . \tag{5.103}$$

We say that the *strict* DEC is satisfied if these are strict inequalities and we will see that the "extremal" matter content (or equation of state) for which at least one of the inequalities is saturated will play a special role in the following.

The Einstein tensor of the metric (5.78) is diagonal (appendix D),

$$\begin{aligned} G_x^x &= -\frac{1}{2}d(d-1)x^{-q} - \frac{1}{8}\eta dq((d-1)q+2p)x^{-(p+2)} \\ G_y^y &= -\frac{1}{2}d(d-1)x^{-q} + \frac{1}{8}\eta dq(2p+4-(d+1)q)x^{-(p+2)} \\ G_j^i &= -\frac{1}{2}(d-1)(d-2)\delta_j^i x^{-q} + \frac{1}{8}\eta(4p-4q+4qd-d(d-1)q^2)\delta_j^i x^{-(p+2)} \end{aligned} \tag{5.104}$$

and hence clearly of type I. For space-like singularities, $\eta = +1$, we have energy density $\rho = -G_x^x$, radial pressure $P_r = G_y^y$ and transverse pressures $P_i = G_i^i$, while for $\eta = -1$ the roles of G_x^x and G_y^y are interchanged.

Since for $q > p+2$ the first term in G_x^x and G_y^y dominates over the second term as $x \to 0$, it is obvious that for $q > p+2$ the relation between ρ and P_r becomes extremal as $x \to 0$,

$$G_x^x - G_y^y \to 0 \quad \Leftrightarrow \quad \rho + P_r \to 0 \ . \tag{5.105}$$

Put differently, $q \leq p+2$ is a necessary condition for the strict DEC to hold. Since strongly singular plane waves (the dark-shaded region in figure 5.1(a)) arise only for $q > p+2$, we have thus established that

> Penrose Limits of space-like spherically symmetric singularities of power-law type satisfying the strict Dominant Energy Condition are singular HPWs.

Since frequency squares exceeding $1/4$ can only occur along the line $q = p+2$ itself, we can also conclude that

> the resulting frequency squares ω_a^2 are bounded from above by $1/4$ unless one is on the border to an extremal equation of state.

A more detailed analysis of the DEC (as performed for $d = 2$ in [106]), shows that the actual region in which the strict DEC is satisfied (taking into account also the conditions involving the transverse pressures P_i), is more constrained. For space-like singularities, this is the (infinite) region bounded by the lines

$$q = 2/d, \quad q = p+2, \quad q = 2(p+1) \ , \tag{5.106}$$

displayed as the highlighted region A of figure 5.2(a) (drawn here for $d = 2$). A look at this figure confirms the results we have obtained above.

For time-like singularities, the region where the strict DEC is satisfied is considerably smaller - it is a finite subset of the strip bounded by the lines $q = 0$ and $q = 2/d$,

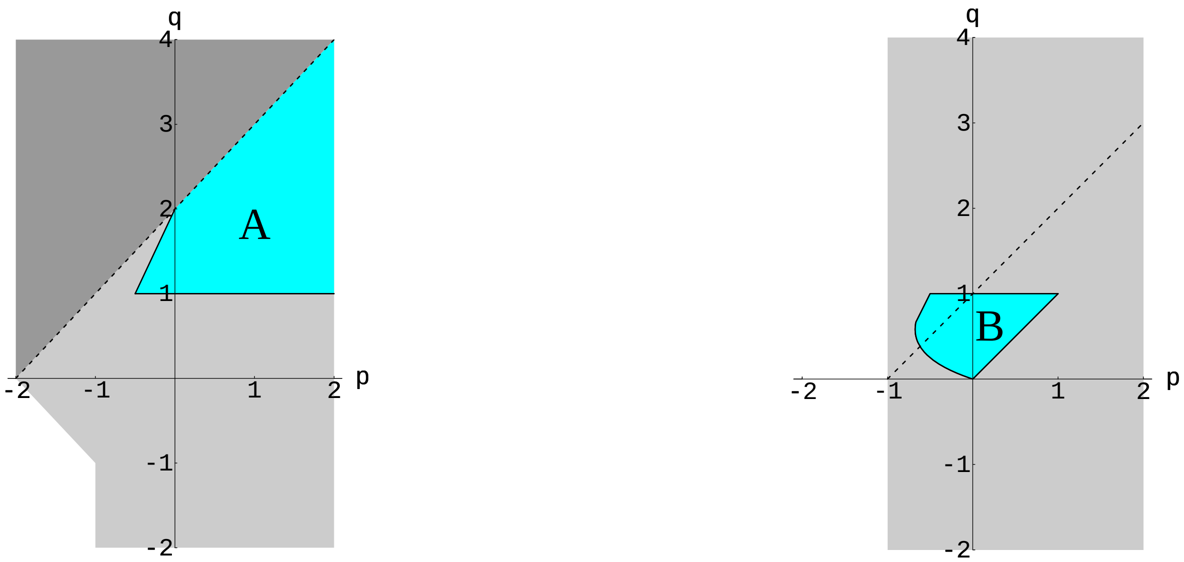

Figure 5.2: The Penrose Limit + DEC Phase Diagram in the $p-q$ plane for (a) space-like ($\eta = +1$) and (b) time-like ($\eta = -1$) singularities. In the highlighted regions A and B the DEC is satisfied (but not saturated). (a) the strongly singular (and non-homogeneous) plane waves of the dark-shaded region with extremal equation of state are excluded, and singular HPWs with $\omega_2^2 \geq 1/4$ arise only along the boundary $q = p + 2$ to the extremal equation of state. (b) the Penrose limits are singular HPWs with $\omega_2^2 \leq 1/4$, with $\omega_2^2 = 1/4$ only along the dashed line $q = p + 1$.

indicated (for $d = 2$) as the highlighted region B of figure 5.2(b). While of no consequence for the present discussion, the fact that in region B the pressures P_r and P_i cannot simultaneously be positive plays an important role in the discussion of Cosmic Censorship in [106].

The fact that the Penrose limits of time-like singularities always behave as u^{-2}, while in the space-like case strongly singular Penrose limits can arise (even though only for metrics violating the strict DEC), might give the impression that time-like (naked) singularities are in some sense better behaved than space-like (censored) singularities. We believe that this should rather be viewed as an indication that massless particles are inadequate for probing the geometry of time-like singularities since, for large regions in the (p, q)-parameter space, the angular momentum barrier prevents non-radial null geodesics from reaching (and hence probing) the singularity. From this point of view, it is much more significant that for space-like singularities massless particle probes with arbitrary angular momentum all detect homogeneous singular plane waves provided that the strict DEC is satisfied.

We conclude this chapter with a comment on the case of null singularities of power-law type ($\eta = 0$) which are analyzed in [107]. As mentioned in section 4.1, in this case some of the leading components of the Einstein tensor vanish and hence one (somewhat trivially) ends up with an extremal equation of state. Thus no interesting constraints arise from imposing the DEC, and using only the leading form (5.79) of the metric cannot be the basis for a full analysis which is more subtle and will be left for future work.

Chapter 6

Strings & thermodynamics in homogeneous plane waves

As we saw in the previous chapter, singular homogeneous plane waves are the Penrose limit of a large class of cosmological backgrounds. Moreover the plane wave metric is singular if and only if $A_{ab}(u)$ is singular at some finite value of u, i.e. if and only if the tidal forces along the null geodesic diverge in the original space-time. Since this infinite tidal force along time-like or null geodesics enters in the usual general relativistic definitions of singularities, we see that the Penrose limit retains precisely the information that characterizes space-time singularities.

On the other hand, if one would like to consider string theory as the unified theory of all fundamental forces of nature, it then becomes of interest to understand string theory in a generic cosmological background. However string theory can not be solved in generic cosmological backgrounds and therefore as a first step one would like to solve it in singular homogeneous plane wave backgrounds.

One of the most important applications of string theory is the study of energy regimes where the quantum effects of gravity become relevant. In many cases, for example the dynamics of the initial state of the universe, thermodynamical properties of the system must be taken into account. To deal with these issues, it is therefore necessary to have an understanding of the thermodynamical aspects of string theory. As we saw in chapter 1, the free energy of a gas of strings propagating in a flat background converges only for temperatures less than a limiting temperature, the Hagedorn temperature. This blow-up of the free energy is due to the exponential growth in the density of single particle states as a function of the mass, and is known to lead to a first order phase transition. Therefore studying the thermodynamics in the singular homogeneous background will shed light on the thermodynamics of singular cosmological backgrounds. As we will see in the following, strings propagating in a singular homogeneous plane wave background present a similar behavior as strings propagating in a flat background do.

However since string theory in singular homogeneous plane waves lead to a time dependent system the usual notions of thermodynamical equilibrium does not apply and an alternative procedure to study the non-equilibrium thermodynamics via a density matrix must be used.

6.1 Solution of string theory model

6.1.1 Classical equations and quantization

In this chapter we will show the bosonic solution of string theory in singular homogeneous plane waves as done in [104]. First we will consider a background where all the frequencies are the same

$$A_{ab} = -\omega_a^2 \delta_{ab} u^{-2} = -ku^{-2}\,, \tag{6.1}$$

i.e.

$$ds^2 = 2dudv - \frac{k}{u^2} X^{i2} du^2 + dX^{i2} \qquad i = 1, \ldots, d\,, \tag{6.2}$$

where d is the number of transverse dimensions. If we consider the case of a string propagating in a curved background with metric $G_{\mu\nu}(X)$ the Polyakov action takes the form of a non-linear sigma-model

$$S_P = \frac{1}{4\pi\alpha'} \int d^2\sigma \sqrt{h} h^{\alpha\beta} \partial_\alpha X^\mu \partial_\beta X^\nu G_{\mu\nu}(X)\,. \tag{6.3}$$

So, for strings propagating in the singular scale invariant homogeneous plane wave background, the bosonic part of the string lagrangian in the light cone gauge

$$u = p_v \tau\,, \tag{6.4}$$

where $p_v = p^u$ is the light-cone momentum, is

$$L_B = \frac{1}{2} \left(\partial_+ X^i \partial_- X^i - \frac{k}{\tau^2} X^{i2} \right)\,, \tag{6.5}$$

where $\partial_\pm = \partial_\tau \pm \partial_\sigma$ giving rise to the equation of motion:

$$\partial_+ \partial_- X^i + \frac{k}{\tau^2} X^i = 0\,. \tag{6.6}$$

The key property about this lagrangian, compared to the one corresponding to plane waves, is that there is no dependence on p_v due to the extra isometry of HPWs[1]. As

[1] In that case, the metric is $ds^2 = 2dudv - kx^{i2}du^2 + dX^{i2}$ and the lagrangian in the light cone gauge has an explicit p_v dependence, $L_B = \frac{1}{2}(\partial_+ X^i \partial_- X^i - p^{u2} k X^{i2})$

we will see this independence of p_v will imply that the density matrix associated to the system diverges at a parameter which does not depend on the metric and which in the flat limit corresponds to the Hagedorn temperature of strings in flat space.

The solution to eq (6.6) expanded in Fourier modes corresponds to an infinite collection of oscillators with time dependent frequencies

$$\begin{aligned} X^i(\sigma,\tau) &= x_0^i(\tau) + \frac{i}{2}\sqrt{2\alpha'}\sum_{n=1}^{\infty}\frac{1}{n}\Big[Z(2n\tau)\big(\alpha_n^i \mathrm{e}^{2in\sigma} + \tilde{\alpha}_n^i \mathrm{e}^{-2in\sigma}\big) \\ &- Z^*(2n\tau)\big(\alpha_{-n}^i \mathrm{e}^{-2in\sigma} + \tilde{\alpha}_{-n}^i \mathrm{e}^{2in\sigma}\big)\Big] , \end{aligned} \tag{6.7}$$

where

$$Z(2n\tau) \equiv \mathrm{e}^{-i\frac{\pi}{2}\nu}\sqrt{\pi n\tau}\,\left[J_{\nu-\frac{1}{2}}(2n\tau) - iY_{\nu-\frac{1}{2}}(2n\tau)\right] , \qquad \nu \equiv \frac{1}{2}(1+\sqrt{1-4k}) , \tag{6.8}$$

$$x_0^i(\tau) = \frac{1}{\sqrt{2\nu-1}}\big(\tilde{x}^i\ \tau^{1-\nu} + 2\alpha'\tilde{p}^i\tau^{\nu}\big) , \quad k \neq \frac{1}{4} , \tag{6.9}$$

$$x_0^i(\tau) = \sqrt{t}(\tilde{x}^i + 2\alpha'\tilde{p}^i\ \log\tau) , \quad k = \frac{1}{4} , \tag{6.10}$$

$$\tilde{x}^i = \frac{\sqrt{\alpha'}}{\sqrt{2}}(a_0^i + a_0^{i\dagger}) , \quad \tilde{p}^i = \frac{1}{i\sqrt{2\alpha'}}(a_0^i - a_0^{i\dagger}) . \tag{6.11}$$

Here $J_{\nu-\frac{1}{2}}(z)$ and $Y_{\nu-\frac{1}{2}}(z)$ are the usual Bessel functions. Asymptotically,

$$Z(2n\tau) \cong \mathrm{e}^{-2in\tau}\big[1 + O(\tau^{-1})\big] , \tag{6.12}$$

and satisfies the equation of motion

$$\ddot{Z}(2n\tau) + (1 + \frac{k}{4\tau^2 n^2})Z(2n\tau) = 0 . \tag{6.13}$$

For large τ the oscillator part of (6.7) reduces to that of the flat-space theory. However the zero mode part does not present this asymptotic flatness behavior since it does not reduce to $x^i_{0\,flat}(\tau) = \tilde{x}^i + 2\alpha'\tilde{p}^i\tau$. This is a consequence of the scale invariance, $\tau \to a\tau$, $a =$ constant, of the equation of motion, eq (6.6), restricted to the zero mode. Therefore the Fock-space vacuum for the zero mode part is different from the flat space zero mode vacuum at all scales.

When $k = 0$, the functions Z and $Z^\star$ reduce to plane waves $\mathrm{e}^{\mp 2in\tau}$ and therefore one recovers the flat space solution. On the other hand, the case $k = \frac{1}{4}$ corresponds to a limiting value and the solution depends on J_0 and Y_0 Bessel functions. For higher values of k, ν becomes imaginary and the Bessel functions have a singular behavior at $\tau = 0$. Therefore from now on k would be consider to be in the range $0 < k < \frac{1}{4}$.

By imposing that X^i are real functions it follows that the oscillators satisfy $\alpha^i_{-n} = (\alpha^i_n)^\dagger$ and $\tilde{\alpha}^i_{-n} = (\tilde{\alpha}^i_n)^\dagger$.

By considering the canonical momentum,

$$\Pi^i(\tau,\sigma) = \frac{1}{2\pi\alpha'}\partial_\tau X^i\,, \tag{6.14}$$

the total momentum carried by the string,

$$p_0^i(\tau) = \int_0^\pi d\sigma \Pi^i = \frac{1}{2\alpha'}\dot{x}_o^i(\tau)\,, \tag{6.15}$$

and imposing the canonical commutation relations,

$$[\Pi^i(\sigma,\tau), X^j(\sigma',\tau)] = -i\delta^{ij}\delta(\sigma-\sigma') \qquad [X^i(\sigma,\tau), X^j(\sigma',\tau)] = 0\,, \tag{6.16}$$

follows the standard commutation relations,

$$[\alpha^i_n, \alpha^j_m] = [\tilde{\alpha}^i_n, \tilde{\alpha}^j_m] = n\delta^{ij}\delta_{n+m}\,, \quad [\alpha^i_n, \tilde{\alpha}^i_m] = 0\,, \tag{6.17}$$

where we used the Wronskian:

$$ZW^\star - Z^\star W = -2i\,, \tag{6.18}$$

where

$$W(2n\tau) = \frac{1}{2n}\left[\frac{\nu}{\tau}Z(2n\tau) - \dot{Z}(2n\tau)\right]\,. \tag{6.19}$$

The zero mode part oscillators satisfy

$$[\alpha^i_0, \alpha^{j\,\dagger}_0] = \delta^{ij}\,, \tag{6.20}$$

which implies

$$[\tilde{x}^i, \tilde{p}^j] = i\delta^{ij}\,, \qquad [x_0^i(\tau), p_0^j(\tau)] = i\delta^{ij}\,. \tag{6.21}$$

6.1.2 Light-cone Hamiltonian

The light-cone Hamiltonian,

$$H = -p_u = \frac{1}{\alpha' p_v}\mathcal{H} = \frac{1}{\alpha' p_v}\frac{1}{8\pi\alpha'}\int_0^\pi d\sigma\,(\dot{X}^{i2} - X'^{i2} + \frac{k}{\tau^2}X^{i2}) \tag{6.22}$$

corresponds to a time-dependent harmonic oscillator which written in terms of the solutions to the equations of motion and invariant oscillators reads

$$\mathcal{H} = \mathcal{H}_0(\tau) + \frac{1}{2}\sum_{i=1}^{8}\sum_{n=1}^{\infty}[\Omega_n(\tau)(\alpha^i_{-n}\alpha^i_n + \tilde{\alpha}^i_{-n}\tilde{\alpha}^i_n + dn) - B_n(\tau)\alpha^i_n\tilde{\alpha}^i_n - B^\star_n\alpha^i_{-n}\tilde{\alpha}^i_{-n}]\,, \tag{6.23}$$

where $\mathcal{H}_0(\tau)$ is the Hamiltonian of the zero modes and

$$\begin{aligned}
\Omega_n(\tau) &= (1+\frac{\nu}{4\tau^2 n^2})|Z|^2+|W|^2-\frac{\nu}{2n\tau}(ZW^\star+Z^\star W)\,, \\
B_n(\tau) &= (1+\frac{\nu}{4\tau^2 n^2})Z^2+W^2-\frac{\nu}{\tau n}ZW\,.
\end{aligned} \tag{6.24}$$

By using

$$\tilde{x}^i=\sqrt{\frac{\alpha'}{2}}(a_0^i+a_0^{i\dagger})\,,\qquad \tilde{p}^i=\frac{1}{i\sqrt{2\alpha'}}(a_0^i-a_0^{i\dagger})\,, \tag{6.25}$$

we can write the zero mode part as

$$\mathcal{H}_0(\tau)=c_0(\tau)a_0^{i\,2}+c_0(\tau)^\star(a_0^{i\,\dagger})^2+d_0(\tau)(a_0^{i\,\dagger}a_0^i+1/2)\,, \tag{6.26}$$

where

$$c_0(\tau)\ :\ \frac{1}{8(2\nu-1)}\left\{\frac{1-\nu}{2\tau^{2\nu}}-\frac{2\nu}{\tau^{2-2\nu}}+\frac{4\omega^2}{i\tau}\right\}\,,\quad \omega^2\neq 1/4 \tag{6.27}$$

$$\frac{1}{8\tau}\left\{(\frac{1}{8}+\frac{\omega^2}{2})-[\frac{\log^2(\tau+1)}{2}+2\omega^2\log^2\tau] -i[\frac{\log(\tau+1)}{2}+2\omega^2\log\tau]\right\}\qquad \omega^2=1/4\,, \tag{6.28}$$

and

$$d_0(\tau)\ :\ \frac{1}{4(2\nu-1)}\left\{\frac{1-\nu}{2\tau^{2\nu}}+\frac{2\nu}{\tau^{2-2\nu}}\right\}\,,\quad \omega^2\neq 1/4 \tag{6.29}$$

$$\frac{1}{4\tau}\left\{(\frac{1}{8}+\frac{\omega^2}{2})+\frac{\log^2(\tau+1)}{2}+2\omega^2\log^2\tau\right\}\,,\quad \omega^2=1/4\,. \tag{6.30}$$

The asymptotic behavior of the functions for large $n\tau$ is

$$\begin{aligned}
\Omega_n(\tau) &= 2+\frac{\omega^2}{4n^2\tau^2}+\dots\,, \\
B_n(\tau) &= \omega^2 \mathrm{e}^{-4in\tau}\left(-\frac{i}{8n^3\tau^3}+\dots\right)\,.
\end{aligned} \tag{6.31}$$

The light cone hamiltonian eq. (6.22) is not diagonal, but it can be diagonalized by introducing new operators,

$$\mathcal{A}_n^i(\tau)=\alpha_n^i\,f_n(\tau)+\tilde{\alpha}_{-n}^i\,g_n^\star(\tau),\quad \mathcal{A}_n^{i\,\dagger}(\tau)=\alpha_{-n}^i\,f_n^\star(\tau)+\tilde{\alpha}_n^i\,g_n(\tau), \tag{6.32}$$

$$\tilde{\mathcal{A}}_n^i(\tau)=\alpha_{-n}^i\,g_n^\star(\tau)+\tilde{\alpha}_n^i\,f_n(\tau),\quad \tilde{\mathcal{A}}_n^{i\,\dagger}(\tau)=\alpha_n^i\,g_n(\tau)+\tilde{\alpha}_{-n}^i\,f_n^\star(\tau)\,, \tag{6.33}$$

where $\mathcal{A}_n^\dagger=\mathcal{A}_{-n}$, $\tilde{\mathcal{A}}_n^\dagger=\tilde{\mathcal{A}}_{-n}$, satisfying the commutation relations

$$[\mathcal{A}_n^i(\tau),\mathcal{A}_m^{j\,\dagger}(\tau)]=\delta_{nm}\delta^{ij}\,,\quad [\tilde{\mathcal{A}}_n^i(\tau),\tilde{\mathcal{A}}_m^{j\,\dagger}(\tau)]=\delta_{nm}\delta^{ij}\,, \tag{6.34}$$

$$[\mathcal{A}_n^i(\tau), \tilde{\mathcal{A}}_m^{j\,\dagger}(\tau)] = 0. \tag{6.35}$$

The functions $f_n(\tau)$ and $g_n(\tau)$ are

$$\begin{aligned} f_n(\tau) &= \frac{\sqrt{\omega_n}}{2n} \mathrm{e}^{2i\omega_n\tau}[Z(2n\tau) + \frac{i}{2\omega_n}\dot{Z}(2n\tau)]\,, \\ g_n(\tau) &= \frac{\sqrt{\omega_n}}{2n} \mathrm{e}^{-2i\omega_n\tau}[-Z(2n\tau) + \frac{i}{2\omega_n}\dot{Z}(2n\tau)]\,, \end{aligned} \tag{6.36}$$

and

$$\omega_n \equiv \omega_n^i(\tau) = \sqrt{n^2 + \frac{\omega_i^2}{4\tau^2}}. \tag{6.37}$$

Plugging in these new operators the diagonalized light cone hamiltonian is

$$\mathcal{H} = \mathcal{H}_0(\tau) + \sum_{i=1}^{8}\sum_{n=1}^{\infty} \omega_n \, [\mathcal{A}_n^{i\,\dagger}(\tau)\mathcal{A}_n^i(\tau) + \tilde{\mathcal{A}}_n^{i\,\dagger}(\tau)\tilde{\mathcal{A}}_n^i(\tau)] + h(\tau)\,, \tag{6.38}$$

where $h(\tau)$ is a normal ordering c-function,

$$h(\tau) = \sum_{i=1}^{8}\left(\sqrt{n^2 + \frac{\omega_i^2}{4\tau^2}} + \frac{\omega_i}{4\tau}\right)\,. \tag{6.39}$$

This corresponds to an infinite collection of oscillators with time dependent frequencies given by eq. (6.37).

6.2 Implications of Scale Invariance

As we have seen, the string mode frequencies in the scale invariant plane wave are p_v-independent but τ-dependent

$$\omega_n^i(\tau) = \sqrt{n^2 + \frac{\omega_i^2}{4\tau^2}} \ . \tag{6.40}$$

This highlights the special feature of the scale invariant plane waves, namely that for fixed τ their large n behavior is exactly that of flat space,

$$\omega_n^i \stackrel{n\to\infty}{\longrightarrow} n \ , \tag{6.41}$$

The above behavior of strings in scale invariant homogeneous plane waves should be contrasted with the behavior of the string modes in the symmetric plane waves, e.g. the BFHP plane wave [92, 95–97], with constant wave profile,

$$A_{ab}(u) = -\mu_a^2\delta_{ab} \ . \tag{6.42}$$

In this case one finds (up to the same overall factor as in (6.37)) the now τ-independent but p_v-dependent frequencies

$$\tilde{\omega}_n^i = \sqrt{n^2 + \alpha'^2 p_v^2 \mu_a^2} \ . \tag{6.43}$$

While these also behave as $\sim n$ for fixed p_v, the integration over the light-cone momentum implies that for the symmetric plane waves (or any u-dependence of the frequencies other than u^{-2} times a bounded function of u) there can be significant departures from the flat space behavior even for $n \to \infty$.

This seems to indicate that the scale invariant singular homogeneous plane waves have properties more in common with strings in flat space than either have with the symmetric plane waves. Another manifestation of this is the fact that for $u > 0$ the $\omega_a \to 0$ limit of the metric with profile (6.1) is the flat metric, while rescaling μ_a in (6.42) is an isometry of the metric (so that the flat space emerging at $\mu_a = 0$ is a non-Hausdorff limit).

In particular, this has direct implications for the issue we want to study in this paper, namely the Hagedorn behavior of strings in scale invariant plane waves, which depends on the properties of the string modes and the exponential growth of the number of states at large n. On the basis of the above reasoning, one might expect this behavior to be identical to that in flat space, and we will confirm this by an explicit calculation in the next section. For the symmetric plane waves, on the other hand, a different behavior may be expected and has indeed been found, see e.g. [110–114].

6.3 Thermodynamics of time-dependent systems

We wish to initiate a study of the non-equilibrium thermodynamics of string theory on the time-dependent background corresponding to the singular family of homogeneous plane wave space-times. This is complicated by the fact that we have an explicitly time-dependent Hamiltonian.

For present purposes we will adopt the point of view (see e.g. [116–118]) that suitable analogues of the Boltzmann thermal state for a time-independent Hamiltonian system,

$$\hat{\rho}_H = \frac{\mathrm{e}^{-\beta H}}{\mathrm{tr}\,\mathrm{e}^{-\beta H}} \tag{6.44}$$

can be constructed as density operators of the form

$$\hat{\rho}_I = \frac{\mathrm{e}^{-\beta I}}{\mathrm{tr}\,\mathrm{e}^{-\beta I}} \ , \tag{6.45}$$

where I is an *invariant* of the system, i.e. a possibly explicitly time-dependent operator in the Heisenberg picture satisfying the equation

$$\frac{d}{dt}I = \frac{\partial}{\partial t}I + i[H, I] = 0 \tag{6.46}$$

(note that for a time-independent system evidently the Hamiltonian H itself satisfies this equation). In particular, therefore, the density operator satisfies the quantum counterpart of the classical Liouville theorem for the phase space density, namely the Liouville-von Neumann equation

$$\mathrm{tr}\,\hat{\rho}_I = 1 \ , \qquad \frac{d}{dt}\hat{\rho}_I = \frac{\partial}{\partial t}\hat{\rho}_I + i[H, \hat{\rho}_I] = 0 \ . \tag{6.47}$$

For the harmonic oscillator systems under considerations, such invariants are easy to come by (as we will briefly recall below), and different choices correspond to different initial "thermal" ensembles with partition functions

$$Z_I(\beta) = \mathrm{tr}\,\mathrm{e}^{-\beta I} \ . \tag{6.48}$$

A convenient, but by no means the unique acceptable, choice is the "instantaneous" density operator based on the choice

$$I = H(t_0) \tag{6.49}$$

(which will be explicitly time-dependent when written in terms of Heisenberg operators). The corresponding density operator reduces to the standard choice in the case of a time-independent system, and it provides an "adiabatic" approximation to the system provided that $H(t)$ various sufficiently slowly with time near $t = t_0$.

6.4 Calculation of the "thermal" partition function

In this section we will present the calculation of the density operator corresponding to the invariant $I = H_{lc}(\tau_0)$. We first do the calculation for a plane wave solution with non-trivial dilaton and no other non-trivial fields in the bosonic string theory [104] and will discuss later the generalization to type II string theories.

We will add to the invariant the light-cone momentum p_v in such a way that in the flat space limit the density matrix becomes the density matrix for evolution by the time-like Hamiltonian $1/\sqrt{2}(p_u + p_v)$. Since we are interested in the density operator corresponding to the exponential of our invariant,

$$\hat{\rho}_I = \frac{\mathrm{e}^{-\frac{\beta}{\sqrt{2}}(I+\hat{p}_v)}}{\mathrm{tr}\,\mathrm{e}^{-\frac{\beta}{\sqrt{2}}(I+\hat{p}_v)}} \ . \tag{6.50}$$

Analogously to what happens to strings in flat space we are interested in what happens to $\hat{\rho}_I$ as the parameter $\beta \to 0$. The only possible source of a singular behavior is the trace in the denominator, the "thermal" partition function, and thus we will concentrate on its evaluation:

$$Z_I(\beta) = \mathrm{tr}\, \mathrm{e}^{-\frac{\beta}{\sqrt{2}}(\hat{p}_v + I)} . \tag{6.51}$$

We of course need to implement the level matching condition to evaluate this trace. In terms of either of the oscillator bases the level matching condition is simply the difference between the left and right number operators.

$$N - \tilde{N} = \sum_{i=1}^{d}\sum_{n=1}^{\infty}(\alpha^i_{-n}\alpha^i_n - \tilde{\alpha}^i_{-n}\tilde{\alpha}^i_n) = \sum_{i=1}^{d}\sum_{n=1}^{\infty}(\mathcal{A}^{i\dagger}_n\mathcal{A}^i_n - \tilde{\mathcal{A}}^{i\dagger}_n\tilde{\mathcal{A}}^i_n) \tag{6.52}$$

We thus need to evaluate

$$Z_I(\beta) = \int dp_v \int d\lambda \mathrm{e}^{-\frac{\beta p_v}{\sqrt{2}}}\ \mathrm{tr}\, \mathrm{e}^{-\frac{\beta}{\sqrt{2}}I + 2\pi i\lambda(N-\tilde{N})} . \tag{6.53}$$

A convenient way to deal with this integral is by introducing the complex variable $\tilde{\tau}$,

$$\tilde{\tau} = \tilde{\tau}_1 + i\tilde{\tau}_2 = \lambda + i\frac{\beta}{2\sqrt{2}\alpha'\pi p_v} . \tag{6.54}$$

The trace inside the integral is

$$\begin{aligned} \mathrm{tr}\, \mathrm{e}^{-2\pi\tilde{\tau}_2 I + 2\pi i\tilde{\tau}_1(N-\tilde{N})} &= \prod_{n=1}^{\infty} |\sum_{M=0}^{\infty} \mathrm{e}^{(-2\pi\tilde{\tau}_2\omega_n + 2\pi i\tilde{\tau}_1 n)M}|^{2d} \\ &\quad \times \sum_{M=0}^{\infty} \mathrm{e}^{-2\pi\tilde{\tau}_2\omega_0 dM} \mathrm{e}^{-2\pi\tilde{\tau}_2 h(\tau_0)} \\ &= (\prod_{n=-\infty}^{\infty} \frac{1}{1 - \mathrm{e}^{-2\pi\tilde{\tau}_2\omega_n + 2\pi i\tilde{\tau}_1 n}})^d \mathrm{e}^{-2\pi\tilde{\tau}_2 h(\tau_0)} \end{aligned} \tag{6.55}$$

For concreteness, we will from now on consider the case where all the frequencies are equal, $\omega^i = \omega$ (such plane waves arise for instance as the Penrose limits of FRW metrics [58, 59, 99]). The normal ordering term is,

$$h(\tau_0) = d\left(\sum_{n=1}^{\infty}\sqrt{n^2 + \frac{\omega^2}{4\tau_0^2}} + \frac{\omega}{4\tau_0}\right) , \tag{6.56}$$

and is divergent. Actually $h(\tau_0)$ has two divergences, the first one arising from a sum over n of n - a quadratic divergence, the second one arising from a sum over n of $1/n$ - a logarithmic divergence. In the present case (generalizing bosonic string theory in Minkowski background) this quadratic divergence is canceled by a generalized zeta function regularization which removes the divergence coming from the sum over n. The details of the resummation are given in the appendix A with the result that ($q = \omega/2\tau_0$),

$$\frac{h(\tau_0)}{d} = \frac{1}{2}\sum_{n\in Z}\sqrt{n^2+q^2} \tag{6.57}$$

$$= -\frac{q}{\pi}\sum_{k=1}^{\infty}\frac{K_1(2\pi kq)}{k} - \frac{q^2}{4}\Gamma(-1) \tag{6.58}$$

$$= \Delta_0(q) - \frac{q^2}{4}\Gamma(-1), \tag{6.59}$$

where K_1 is a modified Bessel function of the second kind. Note that the term proportional to q^2 needs interpreting as it is infinite. The source of this infinity is a subleading divergence of the sum over n in $h(\tau_0)$ mentioned above. At $q = 0$ it is easy to see that we get the usual result of zeta function regularization that $\zeta = -1/12$ as is shown in appendix A.

We can also expand and re-sum this series for the case that q is small by utilizing the binomial expansion of the square root. Again the calculation is presented in the appendix A and the result is,

$$h(\tau_0) = \frac{q}{2} + \zeta(-1) + \frac{q^2}{2}\zeta(1) + \sum_{k=2}^{\infty}\frac{\Gamma(3/2)\zeta(2k-1)}{\Gamma(3/2-k)k!}q^{2k}. \tag{6.60}$$

ζ function regularization gives $\zeta(-1) = -1/12$ but there remains the complex infinity of $\zeta(1)$.

This remaining divergence is more subtle, but before we consider it in more detail we first make the following observation. This divergence is in the term in $h(\tau_0)$ that is quadratic in q. Recall that the massive modular functions of the appendix A for the case that $b_1 = b_2 = 0$, transform as

$$D_{0,0}(\tilde{\tau}_1, \tilde{\tau}_2; q) = D_{0,0}(-\tilde{\tau}_1/|\tilde{\tau}|^2, \tilde{\tau}_2/|\tilde{\tau}|^2; q|\tilde{\tau}|) \tag{6.61}$$

A term in $h(\tau_0)$ that is proportional to q^2 satisfies trivially the identity,

$$e^{\alpha\tilde{\tau}_2 q^2} = e^{\alpha(\tilde{\tau}_2/|\tilde{\tau}|^2)(q|\tilde{\tau}|)^2}, \tag{6.62}$$

and thus any term in h that is proportional to q^2 is also modular covariant.

In principle there can be a logarithmic divergence arising in a non-linear sigma model though it generally vanishes on shell or can be removed by a singular field redefinition as such models are formally (1-loop) scale invariant. In string theory one requires that the quantization preserves conformal invariance and divergences are again regularized via singular field redefinitions and appropriate subtractions [104]. For us at the moment it is important simply to note that any subtraction that removes the divergence will be proportional to q^2 and thus will not destroy the modular covariance of the state sum.

Referring to the definition of the generalized theta function in appendix A, we see that we can now write Z_I in the form (recalling that for the moment we are considering bosonic string theory with $d = 24$ transverse dimensions),

$$Z_I(\beta) = \int_0^\infty \frac{d\tilde{\tau}_2}{\tilde{\tau}_2^2} \int_{-1/2}^{1/2} d\tilde{\tau}_1 \mathrm{e}^{-\frac{\beta^2}{4\pi\alpha'\tilde{\tau}_2}} D_{0,0}(\tilde{\tau}_1, \tilde{\tau}_2; \frac{\omega}{2\tau_0})^{-24} . \tag{6.63}$$

The divergence in this integral comes from the region where $p_v \to \infty$ corresponding to $\tilde{\tau}_2 \to 0$ and we can set $\tilde{\tau}_1 = 0$ to determine the leading behavior of the integral. To easily determine the behavior of the integrand we first use the modular transformation property of $D_{0,0}$,

$$D_{0,0}(\tilde{\tau}_1, \tilde{\tau}_2; q) = D_{0,0}(-\frac{\tilde{\tau}_1}{|\tilde{\tau}|^2}, \frac{\tilde{\tau}_2}{|\tilde{\tau}|^2}; q|\tilde{\tau}|) \tag{6.64}$$

In this limit $\tilde{\tau}_1 = 0$, $\tilde{\tau}_2 \to 0$ the transverse partition function behaves as

$$D_{0,0}(\tilde{\tau}_1, \tilde{\tau}_2; q)^{-24} \to \exp[-\frac{2\pi}{\tilde{\tau}_2}\, 24\Delta_0(q\tilde{\tau}_2)] , \tag{6.65}$$

and so,

$$Z_I \sim \int_0^\infty \frac{d\tilde{\tau}_2}{\tilde{\tau}_2^2} \int_{-1/2}^{1/2} d\tilde{\tau}_1 \mathrm{e}^{-\frac{1}{4\pi\alpha'\tilde{\tau}_2}[\beta^2 + 8.24\pi^2\alpha'\Delta_0(q\tilde{\tau}_2)]} . \tag{6.66}$$

By expanding $\Delta_0(q)$ for small q at first order, see eq. (A.20)

$$\Delta_0(q\tilde{\tau}_2) = -\frac{1}{12} + \frac{1}{2}\frac{w}{2\tau_0}\tilde{\tau}_2 \tag{6.67}$$

we can evaluate the dominant behavior of Z_I by taking the limit $\tau_2 \to 0$, and we find that it diverges at a value of $\beta = \beta_c$ independent of k/τ_0:

$$\beta_c^2 = 16\pi^2\alpha'. \tag{6.68}$$

This is precisely the value of β (as inverse temperature) that corresponds to the Hagedorn temperature for strings in flat space, $\beta_c = \beta_H$. Since Z_I diverges as $\beta \to \beta_c$ we see that the density matrix $\hat{\rho}_I$, becomes ill defined at $\beta = \beta_c$ signaling a breakdown in the non-equilibrium thermodynamic description.

For the type II superstring theories, we have a similar background with dilaton field. Apart from the change in dimension, the calculation of the bosonic part of the density matrix carries through as above. To include the fermionic contribution we notice that in the light cone gauge the fermionic Lagrangian is given by

$$L_F = \psi^i \partial_{\bar{z}} \psi^i + \bar{\psi}^i \partial_z \bar{\psi}^i , \tag{6.69}$$

so the fermions do not couple to the parameters of the metric, and therefore its contribution to the "thermal" partition function will be the same as the one in a flat background. Calculating the "thermal" partition function as above we discover that the density matrix of type II strings in singular homogeneous plane waves is singular at a value $\beta = \beta_c$ where $\beta_c = \beta_H = 2\pi\sqrt{2\alpha'}$.

Chapter 7

Conclusions

This thesis has been devoted to the study of some aspects of string cosmology. In the first part we have studied some aspects of string-gas cosmology at finite temperature mainly based on [57]. In the second part, our objective was the study of strings in time-dependent backgrounds. In particular, we studied the universal behavior of the Penrose limit metric close to space-time singularities, based on [58, 59]. Then we looked at the solvable string model and studied its thermodynamics based on [60].

In chapters 2 and 3 we studied string-gas cosmology at finite temperature in a toroidal universe. We made use of the dilation-gravity equations of motion, satisfying the $R \to 1/R$ duality symmetry, to study the evolution of the system, which was assumed adiabatic. Our set up was as follows: the universe, initially homogeneous, isotropic and in thermal equilibrium, is filled with an ideal gas of closed strings. The 9 spatial dimensions, compactified on a 9-torus, evolve adiabatically starting from a Hagedorn regime.

In order to find an acceptable equation of state that describes such a system in string theory at finite temperature, we evaluated the energy and pressure in a microcanonical approach. Close to the Hagedorn regime, the scale factor R exhibited a slow time evolution around $R \sim 1$, as shown in figure 3.1. In this case the dynamics of the system is effectively described by a nearly constant energy and negligible pressure. We found that the analytic solution in the Hagedorn regime shows very good agreement with the full numerical result. No substantial changes were observed even when the conservation of Kaluza-Klein and winding modes was imposed, as long as the conserved charges are of order one.

We have also investigated a "low" temperature regime, in which the equation of state was derived in a canonical context. We first considered the dynamics of 3 "large" and 6 "small" compact dimensions in the presence of a pure gas of radiation (given by the massless states). It turned out, as expected, that there exist interesting cosmological solutions where the large dimensions continue to expand while the small dimensions remain nearly constant and small relative to the large ones (see figure 3.2). The attractor solutions for the large dimensions can be described by the evolution of the standard radiation dominant phase in FRW cosmology whereas the small dimensions always asymptotically approach a constant value. We also found bouncing solutions for the large dimensions if their Hubble rates are initially negative (see fig. 3.3).

We then analyzed the case where the massive string states are taken into account in addition to the pure radiation. The presence of the massive states typically leads to a slower expansion of the "large" dimensions relative to the massless case (see fig. 3.5). Meanwhile the behavior of the small dimensions strongly depends on the initial conditions for r and $\dot{r}$, resulting in either expansion or contraction of the small dimensions (see fig. 3.4). The radius r can be kept small as long as r is initially close to unity, since the pressure vanishes at the duality symmetric radius ($r = 1$). The vanishing of the pressure at $r = 1$ is a purely stringy effect since it is due to winding modes, whose negative contribution compensates that of KK states. The important point is that, even in the presence of massive states there exist a wide range of the initial condition space for which the small dimensions are stabilized around the self-dual radius and are kept small relative to the large ones. These behaviors were found to be insensitive to the number of large dimensions d. We also considered the case for the conservation of KK and winding modes and found no substantial change compared to the case without imposing the conservation laws.

Concerning the second part of this thesis we showed that space-time singularities exhibit a remarkably universal homogeneous u^{-2}-behavior in the Penrose limit. We established this in complete generality for time-like singularities of power-law type and also showed that for space-like singularities of power-law type, for which more singular Penrose limits are possible, this u^{-2}-behavior is implied by demanding the strict DEC.

Perhaps the main implications of this result are for the study of string theory in singular and/or time-dependent backgrounds. In general, because of the simplifications brought about by the existence of a natural light cone gauge [154, 155], plane wave (and more generally pp-wave) backgrounds provide an ideal setting for studying such problems. Now, as we have seen, the Penrose limits of a large class of singularities are always at least as singular as u^{-2}. Thus "weakly singular" plane waves with profile $\sim u^{-\alpha}$, $\alpha < 2$, while perhaps interesting as toy-models of time-dependent backgrounds in string theory [109, 150, 156], do not actually arise as Penrose limits of standard cosmological or other singularities. Moreover, a strongly singular behavior with $\alpha > 2$ can only arise for metrics violating the strict DEC. This singles out the singular HPWs with profile $\sim u^{-2}$ as the backgrounds to consider in order to obtain insight into the properties of string theory near physically reasonable space-time singularities.

String theory in precisely this class of backgrounds, and in precisely the frequency range $\omega_a^2 < 1/4$ that we generically found in chapter 5, has already been studied in some detail and shown to be exactly solvable in chapter 6. The significance of this bound on the frequencies for string theory (also noted in the earlier studies [109, 150]) lies in the fact that the properties of Bessel functions allow for an analytic continuation of string modes through the singularity in this case (due to the standard shift by 1/4 in the Bessel equation). Other aspects of string theory in this class of backgrounds still remain (and deserve) to be explored. The above observations about the relation of these backgrounds to interesting space-time singularities provide additional impetus for understanding string theory in an expansion around such metrics.

This analysis can be generalized in various ways. In particular, even though we have only considered spherically symmetric metrics, the results we have obtained are certainly not restricted to spherical symmetry. A prototypical anisotropic example is the Kasner metric whose Penrose limit (together with those of other anisotropic or inhomogeneous cosmological models) was studied by Kunze [101] and shown in section 4.6.3. In many of these examples one obtains either an exact u^{-2}-behavior (for particular Penrose limits of the Kasner metric) or as the dominant contribution in the near-singularity limit. It would be interesting to establish a more general result along these lines, in particular also in view of the role played by the Kasner metrics (and Bianchi IX cosmologies) in the BKL discussion of the general solution of Einstein's equations near space-like singularities [158, 159].

Going beyond spherical symmetry may also shed light on a peculiar feature of the spherically symmetric metrics of power-law type studied in [106, 107] and here. Namely, we have seen that a rather special role is played by the "extremal" equations of state for which at least one of the inequalities in the DEC is saturated. As noted in [106, 107] this type of equation of state is not ruled out by physical considerations alone, and arises "rather too easily" near the spherically symmetric singularities of power law type that we have been considering. Indeed, as we have seen in section 5.4, whenever $q > p + 2$ one obtains an extremal equation of state close to the singularity because then the "extremal" contribution to the curvature from the transverse sphere dominates the longitudinal (radial/time) contribution. Thus this may well be an artifact of spherical symmetry.

Finally we concluded our analysis by calculating, in a suitable approach for time-dependent systems, the "thermal" partition function of strings propagating in these singular homogeneous plane wave backgrounds. Contrary to the case of symmetric plane wave backgrounds we found that the partition function diverges when the parameter β takes on a value that is the same as the inverse of the Hagedorn temperature for strings propagating in flat space. This suggests that strings propagating in singular homogeneous plane wave backgrounds behave in a similar way to strings propagating in flat space.

This thesis contributes in a small way to the progress in string cosmology. Despite all the work done over many years, string cosmology is a broad young field and much work has yet to be done. Hopefully one day as a result of continuing work in the field, the question *where does the universe come from?* will be closer to finding an answer.

Appendix A

Jacobi Theta functions

A.1 Theta functions

The Dedekind eta function is defined as

$$\eta(\tilde{\tau}) = q^{1/24} \prod_{n=1}^{\infty} (1 - q^n) \,, \tag{A.1}$$

where $q = \exp(2\pi i \tilde{\tau})$, and under modular transformations it transforms as

$$\begin{aligned} \eta(\tilde{\tau} + 1) &= e^{i\pi/12} \eta(\tilde{\tau}) \\ \eta(-1/\tilde{\tau}) &= \sqrt{-i\tilde{\tau}}\, \eta(\tilde{\tau}) \end{aligned} \tag{A.2}$$

The Jacobi theta functions are defined as

$$\begin{aligned} \theta_{00}(\nu, \tilde{\tau}) &= \theta_3(\nu|\tilde{\tau}) = \theta \begin{bmatrix} 0 \\ 0 \end{bmatrix} (\nu, \tilde{\tau}) = \sum_{n=-\infty}^{\infty} q^{n^2/2} z^n \,, \\ \theta_{01}(\nu, \tilde{\tau}) &= \theta_4(\nu|\tilde{\tau}) = \theta \begin{bmatrix} 0 \\ \frac{1}{2} \end{bmatrix} (\nu, \tilde{\tau}) = \sum_{n=-\infty}^{\infty} (-1)^n q^{n^2/2} z^n \,, \\ \theta_{10}(\nu, \tilde{\tau}) &= \theta_2(\nu|\tilde{\tau}) = \theta \begin{bmatrix} \frac{1}{2} \\ 0 \end{bmatrix} (\nu, \tilde{\tau}) = \sum_{n=-\infty}^{\infty} q^{(n-1/2)^2/2} z^{n-1/2} \,, \\ \theta_{11}(\nu, \tilde{\tau}) &= -\theta_1(\nu|\tilde{\tau}) = \theta \begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \end{bmatrix} (\nu, \tilde{\tau}) = -i \sum_{n=-\infty}^{\infty} (-1)^n q^{(n-1/2)^2/2} z^{n-1/2} \,. \end{aligned} \tag{A.3}$$

Their product representations are

$$\begin{aligned}
\theta_{00}(\nu,\tilde{\tau}) &= \prod_{m=1}^{\infty}(1-q^m)(1+zq^{m-1/2}(1+z^{-1}q^{m-1/2})\,,\\
\theta_{01}(\nu,\tilde{\tau}) &= \prod_{m=1}^{\infty}(1-q^m)(1-zq^{m-1/2}(1-z^{-1}q^{m-1/2})\,,\\
\theta_{10}(\nu,\tilde{\tau}) &= 2e^{\pi i\tilde{\tau}/4}\cos\pi\nu\prod_{m=1}^{\infty}(1-q^m)(1+zq^{m-1/2}(1+z^{-1}q^{m})\,,\\
\theta_{11}(\nu,\tilde{\tau}) &= -2e^{\pi i\tilde{\tau}/4}\sin\pi\nu\prod_{m=1}^{\infty}(1-q^m)(1-zq^{m-1/2}(1-z^{-1}q^{m})\,.
\end{aligned} \tag{A.4}$$

Their modular transformations are

$$\begin{aligned}
\theta_{00}(\nu,\tilde{\tau}+1) &= \theta_{01}(\nu,\tilde{\tau})\,,\\
\theta_{01}(\nu,\tilde{\tau}+1) &= \theta_{00}(\nu,\tilde{\tau})\,,\\
\theta_{10}(\nu,\tilde{\tau}+1) &= \exp(\pi i/4)\theta_{10}(\nu,\tilde{\tau})\,,\\
\theta_{11}(\nu,\tilde{\tau}+1) &= \exp(\pi i/4)\theta_{11}(\nu,\tilde{\tau})\,,
\end{aligned} \tag{A.5}$$

and

$$\begin{aligned}
\theta_{00}(\nu/\tilde{\tau},-1/\tilde{\tau}) &= (-i\tilde{\tau})^{1/2}\exp(\pi i\nu^2/\tilde{\tau})\theta_{00}(\nu,\tilde{\tau})\,,\\
\theta_{01}(\nu/\tilde{\tau},-1/\tilde{\tau}) &= (-i\tilde{\tau})^{1/2}\exp(\pi i\nu^2/\tilde{\tau})\theta_{10}(\nu,\tilde{\tau})\,,\\
\theta_{10}(\nu/\tilde{\tau},-1/\tilde{\tau}) &= (-i\tilde{\tau})^{1/2}\exp(\pi i\nu^2/\tilde{\tau})\theta_{01}(\nu,\tilde{\tau})\,,\\
\theta_{11}(\nu/\tilde{\tau},-1/\tilde{\tau}) &= -i(-i\tilde{\tau})^{1/2}\exp(\pi i\nu^2/\tilde{\tau})\theta_{11}(\nu,\tilde{\tau})\,.
\end{aligned} \tag{A.6}$$

A.2 Generalized Theta functions

The generalized Theta function is given by

$$D_{b_1,b_2}(\tilde{\tau}_1,\tilde{\tau}_2;q) \equiv e^{2\pi\tilde{\tau}_2\Delta_{b_1}(q)}\prod_{n=-\infty}^{\infty}(1-e^{-2\pi\tilde{\tau}_2\sqrt{(n+b_1)^2+q^2}+2\pi i\tilde{\tau}_1(n+b_1)-2\pi i b_2})\,, \tag{A.7}$$

where

$$\Delta_b(q) \equiv -\frac{q}{\pi}\sum_{p=1}^{\infty}\frac{\cos(2\pi bp)}{p}K_1(2\pi qp)\,, \tag{A.8}$$

and K_1 is a modified Bessel function of the second kind. Its modular properties are[1]

$$D_{b_1,b_2}(\tilde{\tau}_1,\tilde{\tau}_2;q) = D_{b_2,-b_1}(-\frac{\tilde{\tau}_1}{|\tilde{\tau}|^2},\frac{\tilde{\tau}_2}{|\tilde{\tau}|^2};q|\tilde{\tau}|) = D_{b_1,b_2+b_1}(\tilde{\tau}_1+1,\tilde{\tau}_2;q)\,. \tag{A.9}$$

[1] This modular transformation is valid since q is not related to the torus modular parameter, otherwise the modular transformation would be $D_{b_1,b_2}(\tilde{\tau},q) = D_{b_1,b_2}(-1/\tilde{\tau},q/|\tilde{\tau}|)$ [111].

In the detailed calculations of this thesis we only need $D_{0,0}(\tilde{\tau}_1, \tilde{\tau}_2; q)$ and therefore just $\Delta_0(q)$. Lets see how it arises.

Consider

$$F = \sum_{n \in Z} \sqrt{n^2 + q^2} \tag{A.10}$$

where we will put $q = \omega/2\tau_0$ at the end of the calculation and the normal ordering term $h(\tau_0) = F/2$ and we want find out how $h(\tau_0)$ is related to $\Delta_0(q)$. In zeta function regularization the first step we take is to use Poisson resummation. Then,

$$F = \sum_{k \in Z} \int_{-\infty}^{\infty} dy e^{-2\pi i k y} \sqrt{y^2 + q^2}. \tag{A.11}$$

Rewriting the square root term as an integral,

$$\sqrt{y^2 + q^2} = \frac{1}{\Gamma(-1/2)} \int_0^{\infty} \frac{dt}{t^{3/2}} e^{-t(y^2+q^2)} \tag{A.12}$$

we can then write

$$\begin{aligned} F &= \frac{1}{\Gamma(-1/2)} \sum_{k \in Z} \int_{-\infty}^{\infty} dy e^{-2\pi i k y} \int_0^{\infty} \frac{dt}{t^{3/2}} e^{-t(y^2+q^2)} && \text{(A.13)} \\ &= -\frac{2q}{\pi} \sum_{k=1}^{\infty} \frac{K_1(2\pi k q)}{k} - \frac{q^2}{2}\Gamma(-1), && \text{(A.14)} \end{aligned}$$

Alternatively the binomial expansion of F for small q^2 is as follows.

$$\begin{aligned} F &= q + 2\sum_{n=1}^{\infty} \sqrt{n^2 + q^2} && \text{(A.15)} \\ &= q + 2(\sum_{n=1}^{\infty}\sum_{k=0}^{\infty} \frac{\Gamma(3/2)}{\Gamma(3/2-k)} \frac{q^{2k}}{n^{2k-1}k!} && \text{(A.16)} \\ &= q + 2\sum_{k=0}^{\infty} \frac{\Gamma(3/2)\zeta(2k-1)}{\Gamma(3/2-k)k!} q^{2k} && \text{(A.17)} \\ &= q + 2\zeta(-1) + q^2\zeta(1) + 2\sum_{k=2}^{\infty} \frac{\Gamma(3/2)\zeta(2k-1)}{\Gamma(3/2-k)k!} q^{2k} && \text{(A.18)} \end{aligned}$$

So we find that

$$\begin{aligned} h(\tau_0) &= \Delta_0(q) - \frac{q^2}{4}\Gamma(-1) && \text{(A.19)} \\ &= -\frac{1}{12} + \frac{q}{2} + \frac{q^2}{2}\zeta(1) + \sum_{k=2}^{\infty} \frac{(-)^k\Gamma(k-\frac{1}{2})}{\Gamma(-\frac{1}{2})\Gamma(k+1)} \zeta(2k-1) q^{2k} && \text{(A.20)} \end{aligned}$$

Appendix B

Saddle point approximation

The saddle point approximation is a way to approximately calculate the integral

$$F(\lambda) = \int_C \phi(z) e^{\lambda f(z)} dz\,, \tag{B.1}$$

where C is a given contour in the complex plane and both $\phi(z)$ and $f(z)$ are analytic functions in a region $\mathcal{G}$ containing the curve C. This method works as long as $f(z)$ has a sharp maximum along C, such that one can expand around it. Since $f(z)$ is an analytic function, it will have saddle-points rather than minima or maxima.

Suppose $f(z)$ has a saddle point at $z = z_0$ where $f'(z) = 0$, thus

$$f(z) = f(z_0) + c_0(z - z_0)^2 + \cdots \tag{B.2}$$

around z_0. Assume, for simplicity only, that $c_0 \neq 0$. Introduce radial variable $a - a_0 = \rho e^{i\phi}$, c_0 and denote u and v the real and imaginary parts of f. Then

$$\begin{aligned} u(\rho,\phi) - u(z_0) &= \rho^2 r_0 \cos(2\phi + \theta_0) + \cdots \\ v(\rho,\phi) - v(z_0) &= \rho^2 r_0 \sin(2\phi + \theta_0) + \cdots\,. \end{aligned} \tag{B.3}$$

One should approach z_0 such that the imaginary part vanishes, and there is no cancellation in the integral. Moreover, one should get a maximum for u along such direction and hence $u(\rho,\phi) - u(z_0) < 0$. We then get

$$\phi_m = \frac{\pi - \theta_0}{2} + m\pi\,, \qquad m = 0, 1\,. \tag{B.4}$$

By plugging in the expansion (B.2) into (B.1):

$$F(\lambda) \sim e^{\lambda f(z_0)} \phi(z_0) \int_C e^{\frac{\lambda}{2} f''(z_0)(z-z_0)^2 + \cdots} dz\,. \tag{B.5}$$

Recalling that $f'' = 2c_0 = 2r_0 e^{i\theta_0}$ and making the change of variable $z - z_0 = \rho e^{i[(2m+1)\pi - \theta_0]/2}$ one obtains

$$F(\lambda) \sim e^{\lambda f(z_0)} \left[\phi(z_0) \sqrt{\frac{2\pi}{\lambda |f''(z_0)|}} e^{i\phi_m} + O(\lambda^{-3/2}) \right] . \tag{B.6}$$

The choice $m = 0$ or $m = 1$ depends on the orientation of the contour integral on C. See [160] for a more rigorous statement and proof of the saddle point approximation.

Appendix C

Orbifold

C.1 Compactification

Let us consider a periodic scalar field such that

$$X \sim X + 2\pi R \,. \tag{C.1}$$

Since the world-sheet action is the same as in the noncompact case, the equations of motion are unchanged. However the periodicity has two effects. First, string states must be single-valued under the identification (C.1), i.e. the operator $\exp(2\pi i R p)$ which translates strings once around the periodic dimension must leave states invariant, so the center of mass momentum is quantized

$$k = n/R \,, \qquad n \in \mathbb{Z} \,. \tag{C.2}$$

The second effect is that a closed string may wind around the compact dimension,

$$X(\sigma + 2\pi) = X(\sigma) + 2\pi R w \,, \qquad w \in \mathbb{Z} \,, \tag{C.3}$$

where the integer w is the winding number. Then, the most general configuration for $X(\sigma, \tau)$ satisfying the two dimensional wave equation and the closed string boundary conditions is

$$X(\sigma,\tau) = x + \frac{n}{R}\tau + 2Rw\sigma + \frac{i}{2}\sum_{n\neq 0}\frac{1}{n}[\alpha_n e^{-2in(\tau-\sigma)} + \tilde{\alpha}_n e^{-2in(\tau+\sigma)}] \,. \tag{C.4}$$

The partition function for X is now

$$\begin{aligned} Z_X &= (q\bar{q})^{-1/24}\,\mathrm{Tr}(q^{L_0}\bar{q}^{\bar{L}_0}) \\ &= |\eta(\tilde{\tau})|^{-2} \sum_{n,w=-\infty}^{\infty} \Lambda_{n,w} \quad , \end{aligned} \tag{C.5}$$

where $\Lambda_{n,w}$ is the lattice contribution

$$\Lambda_{n,w} = q^{\frac{\alpha'}{4}\left(\frac{n}{R}+\frac{wR}{\alpha'}\right)^2} \bar{q}^{\frac{\alpha'}{4}\left(\frac{n}{R}-\frac{wR}{\alpha'}\right)^2} . \tag{C.6}$$

The oscillator sum is the same as in the noncompact case but the momentum integration is replaced by a sum over n and w. By using the Poisson resummation formula,

$$\sum_{n=-\infty}^{\infty} \exp(-\pi a n^2 + 2\pi i b n) = a^{-1/2} \sum_{m=-\infty}^{\infty} \exp\left[-\frac{\pi(\mathrm{m}-\mathrm{b})^2}{\mathrm{a}}\right] \tag{C.7}$$

the partition function can be written as

$$\begin{aligned} Z_X &= |\eta(\tilde{\tau})|^{-2} \sum_{n,w=-\infty}^{\infty} \exp\left[-\pi\tilde{\tau}_2\left(\frac{\alpha' n^2}{R^2}+\frac{w^2R^2}{\alpha'}\right)+2\pi i\tilde{\tau}_1 n w\right] \\ &= \frac{R}{\sqrt{\alpha'\tilde{\tau}_2}}|\eta(\tilde{\tau})|^{-2} \sum_{n,w=-\infty}^{\infty} \exp\left(-\frac{\pi R^2|m-w\tilde{\tau}|^2}{\alpha'\tilde{\tau}_2}\right) . \end{aligned} \tag{C.8}$$

C.2 Orbifold

The orbifold arises when we consider a manifold M that has a discrete symmetry group G. The new manifold $\tilde{M} \equiv M/G$ is obtained from the old one by modding out the symmetry group G. If G is freely acting (M has no fixed-points under the G action) then $\tilde{M}$ is a smooth manifold. On the other hand, if G has fixed points, then $\tilde{M}$ has conical singularities at the fixed points known as orbifold singularities.

In the orbifold construction we should keep only the operators invariant under the orbifold transformation. Therefore we have to insert a projector in the trace: $(1+g)/2$, where g is the non-trivial orbifold group element acting on states, and include the twisted states. Thus for a scalar field propagating in the orbifold the partition function is

$$\begin{aligned} Z^{\text{orb}} &= Z^{\text{untwisted}} + Z^{\text{twisted}} \\ &= \frac{1}{2}\sum_{h,g=0}^{1} Z[^h_g] \end{aligned} \tag{C.9}$$

where $h=0$ labels the untwisted sector, $h=1$ the twisted sector, $g=0$ implies no projection and $g=1$ implies projection.

Let us now for the purposes of this thesis consider a scalar on a circle of radius R, under the action $X \to X+\pi R$ (freely acting orbifold). $Z[^0_0]$ is simply given by (C.5). The lattice shift $X \to X+\pi R$ acts in the states as $|n,w\rangle \to (-1)^n|n,w\rangle$ and leaves the oscillators invariant. Therefore $Z[^0_1]$ is of the form

$$Z[^0_1] = \sum_{n,w\in\mathbb{Z}} (-1)^n \frac{\exp\left[\frac{i\pi\tilde{\tau}\alpha'}{2}\left(\frac{n}{R}+\frac{wR}{\alpha'}\right)^2 - \frac{i\pi\bar{\tilde{\tau}}\alpha'}{2}\left(\frac{n}{R}-\frac{wR}{\alpha'}\right)^2\right]}{\eta\bar{\eta}} . \tag{C.10}$$

The computation of $Z\left[{1 \atop 0}\right]$ can be made by noting that the twisted boundary condition is similar to that of a circle of half the radius, so that $w \to w + 1/2$, or by performing a $\tilde\tau \to -1/\tilde\tau$ transformation on $Z\left[{0 \atop 1}\right]$. Both methods give

$$Z\left[{1 \atop 0}\right] = \sum_{n,w\in\mathbb{Z}} \frac{\exp\left[\frac{i\pi\tilde\tau\alpha'}{2}\left(\frac{n}{R} + \left(w+\frac{1}{2}\right)\frac{R}{\alpha'}\right)^2 - \frac{i\pi\bar{\tilde\tau}\alpha'}{2}\left(\frac{n}{R} - \left(w+\frac{1}{2}\right)\frac{R}{\alpha'}\right)^2\right]}{\eta\bar\eta} . \tag{C.11}$$

Finally $Z\left[{1 \atop 1}\right]$ can be obtained from $Z\left[{1 \atop 0}\right]$ by a $\tilde\tau \to \tilde\tau + 1$ transformation or by inserting the group element:

$$Z\left[{1 \atop 1}\right] = \sum_{n,w\in\mathbb{Z}} (-1)^n \frac{\exp\left[\frac{i\pi\tilde\tau\alpha'}{2}\left(\frac{n}{R} + \left(w+\frac{1}{2}\right)\frac{R}{\alpha'}\right)^2 - \frac{i\pi\bar{\tilde\tau}}{2}\left(\frac{n}{R} - \left(w+\frac{1}{2}\right)\frac{R}{\alpha'}\right)^2\right]}{\eta\bar\eta} . \tag{C.12}$$

Now let us consider type IIB string theory compactified on S_1/Z_2, where Z_2 is generated by g, the product of a translation $\pi\beta$ along the circle and $(-1)^F$, with F the spacetime fermion operator. Since we already know what is the effect of the translation along the circle let us focus on the action of the $(-)^F$ operator.

Let us introduce the level one $SO(2n)$ characters

$$I_{2n} = \frac{1}{2\eta^n}(\theta_3^n + \theta_4^n) , \qquad V_{2n} = \frac{1}{2\eta^n}(\theta_3^n - \theta_4^n) , \tag{C.13}$$

$$S_{2n} = \frac{1}{2\eta^n}(\theta_2^n + i^n\theta_1^n) , \qquad C_{2n} = \frac{1}{2\eta^n}(\theta_2^n - i^n\theta_1^n) . \tag{C.14}$$

Then, leaving aside all contributions from transverse bosons and from the measure over the moduli, the partition function of the type IIB superstring in $10d$ is simply

$$Z = |V_8 - S_8|^2 . \tag{C.15}$$

In this notation, the massless spectrum is manifest; in particular, $|V_8|^2$ describes the universal bosonic modes of the NS-NS sector (graviton, dilaton and antisymmetric tensor), while $|S_8|^2$ describes the additional scalar, the 2-form and the self dual 4-form of the RR sector. Now the action of the $(-1)^F$ is clear: it acts as $+1$ to the bosonic states and -1 to the fermionic states. In other words, for $F = F_L + F_R$, the total space-time fermion number, $(-1)^F = R_{SC}\bar R_{SC}$, where R_{SC} contributes with a $+1$ to I_8 and V_8 and with a -1 to S_8 and C_8. Thus by acting on the partition function with $(-1)^F$ we can straightforwardly calculate $Z\left[{0 \atop 1}\right]$.

Now, modular invariance is restored by introducing the twisted sector, which can be calculated by $Z\left[{1 \atop 0}\right](\tilde\tau) \equiv Z\left[{0 \atop 1}\right](-1/\tilde\tau)$. Finally, $Z\left[{1 \atop 1}\right]$ is calculated by taking into consideration the $(-1)^F$ action on $Z\left[{1 \atop 0}\right]$ [161, 162].

It then follows that the partition function, or likewise the free energy, of type IIB string theory compactified on S_1/Z_2, where Z_2 is generated by g, the product of a translation $\pi\beta$ along the circle and $(-1)^F$, is given by, eq(2.63),

$$\begin{aligned} F(\beta) &= -\frac{1}{\beta}\frac{V_{25}}{(4\pi\alpha')^{25/2}}\int_{\mathcal{F}}\frac{d^2\tilde{\tau}}{2\tilde{\tau}_2^{11/2}}\frac{1}{2|\eta|^{24}} \\ &\times \sum_{n,w\in\mathbb{Z}}\left\{|\theta_2^4|^2\Lambda_{n,w}(-1)^n + |\theta_4^4|^2\Lambda_{n,w+\frac{1}{2}} + |\theta_3^4|^2\Lambda_{n,w+\frac{1}{2}}(-)^n\right\}. \end{aligned} \quad (C.16)$$

C.3 Unfolding technique

In the following appendix we briefly discuss the unfolding technique (UT) used in this paper to calculate the torus contribution to the vacuum energy density. This technique was first introduced in string theory in the context of strings at finite temperature [128, 129]. It has also been efficiently used in the context of threshold corrections to gauge couplings in heterotic string theories (see *e.g.* [163, 164], and more recently [165]). It has also been used in [166] to study the quantum stability of type IIB orbifolds and orientifolds with Scherk-Schwarz SUSY breaking.

One loop closed string amplitudes on the torus always involve an integral over the fundamental region $\mathcal{F}$ which is given by the complex upper half plane $\tilde{\tau}_2 > 0$ modded out by the modular group $PSL(2,\mathbb{Z})$. Analytic integration over this region is difficult to perform even for zero-point amplitudes, *i.e.* in computing the partition function of a model. The UT gives a systematic procedure that allows one to unfold $\mathcal{F}$ into the strip S: $\tilde{\tau}_2 > 0$, $-1/2 \leq \tilde{\tau}_1 \leq 1/2$. In many instances, such as in the models studied in this paper, such technique is essential to be able to compute the torus partition function.

The integrand Z_T of a torus partition function is generically given by a sum of terms. Although Z_T is always modular invariant in any consistent string theory, each term in the sum generically is not, being mapped to another term under the modular group. It is always possible, however, to choose a set of representative terms Z_T^i having as an orbit under the action of the modular group the whole Z_T. We can thus integrate a given set of representatives of the orbit over the unfolding of $\mathcal{F}$, namely the strip S, instead of integrating the whole Z_T over $\mathcal{F}$:

$$\int_{\mathcal{F}} Z_T = \int_{S/G}\sum_{g\in G}\sum_i g\left[Z_T^i\right] = \int_S \sum_i Z_T^i\,. \quad (C.17)$$

In particular we will show how the UT allows to rewrite eq.(2.63) as (2.64). As can be easily shown, each of the three terms in curly brackets in (2.63) is not modular invariant.

The first term is invariant under $\Gamma_0[2]/\mathbf{Z}_2$ which is given by the set of unimodular transformations

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

with c even and d odd, where $\mathbf{Z}_2$ acts as $a, b, c, d \to -a, -b, -c, -d$. The two remaining terms can be reached from the second one by the action of S and ST, where $S(\tilde{\tau} \to -1/\tilde{\tau})$ and $T(\tilde{\tau} \to \tilde{\tau}+1)$ are the usual $SL(2,\mathbb{Z})$ generators. Moreover, $PSL(2,\mathbb{Z}) = \{I, S, ST\} \times \Gamma_0[2]/\mathbf{Z}_2$. The lattice contribution of the first term in (2.63) can be written as:

$$\begin{aligned} \frac{\sqrt{\alpha'\tilde{\tau}_2}}{R}\Lambda_{n,w}(-)^n &= \sum_{n,w\,\in\,\mathbb{Z}} e^{-\frac{\pi R^2}{\alpha'\tilde{\tau}_2}|n+1/2+w\tilde{\tau}|^2} = \sum_{n\,\in\,\mathbb{Z}\,\text{odd},\; w\in\,\mathbb{Z}\,\text{even}} e^{-\frac{\pi R^2}{4\alpha'\tilde{\tau}_2}|n+w\tilde{\tau}|^2} \\ &= \sum_{p\,\in\,\mathbb{Z}\,\text{odd}} \sum_{\substack{c\,\in\,\mathbb{N}\,\text{odd}, \\ d\,\in\,\mathbb{Z}\,\text{even}}} e^{-\frac{\pi R^2}{4\tilde{\tau}_2\alpha'}p^2\,|c+d\tilde{\tau}|^2}, \end{aligned} \tag{C.18}$$

where we have performed a Poisson resummation and then the change of variable $n = pc$, $w = pd$ with $c \geq 0$ and p the minimum common divisor between n and w.

By applying $\tilde{\Gamma}_0[2] = \Gamma_0[2]/(\mathbf{Z}_2 \times T)$ to

$$\sum \exp\left[-\frac{R^2}{4\alpha'\tilde{\tau}_2}p^2\right], \tag{C.19}$$

we obtain

$$\sum \exp\left[-\frac{R^2}{4\alpha'\tilde{\tau}_2}p^2\;|c+d\tilde{\tau}|^2\right]. \tag{C.20}$$

Therefore (2.63) can finally be written as (2.64):

$$\begin{aligned} F(\beta) &= -\frac{V_{25}}{(2\pi\sqrt{\alpha'})^{26}}\int_{\mathcal{F}}\frac{d^2\tilde{\tau}}{4\tilde{\tau}_2^6}(1+S+ST)\sum_{g\in\tilde{\Gamma}_0[2]} g\left[\frac{|\theta_2^4|^2}{|\eta|^{24}}\sum_{p\in\mathbb{Z}}\frac{1-(-)^p}{2}e^{-p^2\frac{\pi R^2}{4\alpha'\tilde{\tau}_2}}\right] \\ &= -\frac{V_{25}}{(2\pi\sqrt{\alpha'})^{26}}\int_{S}\frac{d^2\tilde{\tau}}{4\tilde{\tau}_2^6}\frac{|\theta_2^4|^2}{|\eta|^{24}}\sum_{p\in\mathbb{Z}}\frac{1-(-)^p}{2}e^{-p^2\frac{\pi R^2}{4\alpha'\tilde{\tau}_2}}, \end{aligned} \tag{C.21}$$

where S is the strip $\tilde{\tau}_2 > 0$, $-1/2 \leq \tilde{\tau}_1 \leq 1/2$, the unfolded region of $\mathcal{F}$.

Appendix D

Curvature of Szekeres-Iyer metrics

For reference purposes we give here the non-vanishing components of the Ricci and Einstein tensors of the metric (5.78),

$$ds^2 = \eta x^p dy^2 - \eta x^p dx^2 + x^q d\Omega_d^2 \tag{D.1}$$

(for $d = 2$, these results can be inferred from [106]). Indices i, j refer to the metric $\hat{g}_{ij}$ of the transverse sphere (or some other transverse space), with $\hat{R}_{ij}$ and $\hat{R}$ the corresponding Ricci tensor and Ricci scalar.

Ricci Tensor

$$\begin{aligned} R_{xx} &= \frac{1}{4}(2p + 2qd + pqd - q^2 d)x^{-2} \\ R_{yy} &= \frac{1}{4}p(qd-2)x^{-2} \\ R_{ij} &= \hat{R}_{ij} + \frac{1}{4}\eta q(qd-2)\hat{g}_{ij}x^{q-p-2} \\ &= (d-1)\hat{g}_{ij} + \frac{1}{4}\eta q(qd-2)\hat{g}_{ij}x^{q-p-2} \end{aligned} \tag{D.2}$$

Ricci Scalar

$$\begin{aligned} R &= \hat{R}x^{-q} - \frac{1}{4}\eta(4p + 4qd - d(d+1)q^2)x^{-(p+2)} \\ &= d(d-1)x^{-q} - \frac{1}{4}\eta(4p + 4qd - d(d+1)q^2)x^{-(p+2)} \end{aligned} \tag{D.3}$$

Einstein Tensor

$$
\begin{aligned}
G^x_x &= -\frac{1}{2}\hat{R}x^{-q} - \frac{1}{8}\eta dq((d-1)q+2p)x^{-(p+2)} \\
&= -\frac{1}{2}d(d-1)x^{-q} - \frac{1}{8}\eta dq((d-1)q+2p)x^{-(p+2)} \\
G^y_y &= -\frac{1}{2}\hat{R}x^{-q} + \frac{1}{8}\eta dq(2p+4-(d+1)q)x^{-(p+2)} \\
&= -\frac{1}{2}d(d-1)x^{-q} + \frac{1}{8}\eta dq(2p+4-(d+1)q)x^{-(p+2)} \\
G^i_j &= \hat{G}^i_j x^{-q} + \frac{1}{8}\eta(4p-4q+4qd-d(d-1)q^2)\delta^i_j x^{-(p+2)} \\
&= -\frac{1}{2}(d-1)(d-2)\delta^i_j x^{-q} + \frac{1}{8}\eta(4p-4q+4qd-d(d-1)q^2)\delta^i_j x^{-(p+2)} \quad \text{(D.4)}
\end{aligned}
$$

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