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De Sitter Entropy in Higher Derivative Theories of Gravity

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Abstract: A theorem on higer-order derivative theories of gravity is proved. We find that the de Sitter/anti-de Sitter metric is always a solution of any generally covariant theory of gravity. With this theorem and a general form of entropy function for de Sitter spacetimes, we show how to calculate the entropy of de Sitter spacetime in a generally covariant theory of gravity without knowing the details of the modified metric. As an example, a general formula of dS entropy in Lovelock gravity is obtained.

Keywords: de Sitter entropy; higher-order derivative gravity; entropy function; Lovelock gravity

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1. Introduction

One of the main challenges of General Relativity (GR) is the ultraviolet (UV) divergent problem. This problem emerges in cosmological or black-hole type singularities at the classical level. Over the last decades, a large number of attempts have been made to resolve these divergences. Among them, the following attempts are widely accepted: (i) construct a UV-complete theory of gravity by the replacement of spacetime metric with more fundamental approaches such as string theory or loop quantum gravity; (ii) keep the framework of GR and instead modify the matter stress energy tensor sourcing the gravity; (iii) a case somewhere in between. The metric theory is still adopted, but modifications of Einstein's equations at the UV scale are suggested. This leads to a general approach called higher derivative theories, which have been of great interest in the last decades since these theories are renormalizable, and their study might include vital clues as to how to quantize gravity.

On the other hand, recently, there has been increasing interest in the study of de Sitter (dS) spacetime, partially because recent cosmological observations, particularly those of type Ia supernovae [1,2], suggest that our universe is expanding at an accelerating rate. This means that our universe is asymptotically de Sitter spacetime. The other motivation comes from the well-known (A)dS/CFT correspondence [3,4], which argues that physics in the bulk is equivalent to the CFT on the boundary. This is closely related to an issue of particular interest, which is the finite entropy of the de Sitter horizon, which was first explored by Gibbons and Hawking in [5], following the pioneering works by Bekenstein and Hawking [6–10]. It is useful to know what we mean by entropy, or equivalently, how to understand the de Sitter entropy from the view point of statistics (or microstates). This issue has been explored by Maldacena and Strominger [11] for $2+1$ dimensions. A clue may be found by studying what would happen if we consider dS entropy at very high energies close to the Planck scale, where Einstein's theory breaks down and the higher-order curvature corrections to the Einstein–Hilbert action become important. It is often believed that the corrections of higher-order terms might provide hints for the understanding of the microscopic origin of dS entropy.

By now, it is well-known that for a special class of black holes, known as BPS black holes, the microscopic origin of black hole entropy can be understood in the framework of string theory [12]. Recently, an elegant method has been proposed by Sen [13] for calculating the BPS entropy even in the presence of higher-order curvature terms in the gravity action. This method is generally called the entropy function method (see [14] and its recent progress [15–29]), which can give the BPS entropy without knowing the details of the space-time geometry. An important question that immediately comes to our mind is why the entropy function method can know the BPS entropy without knowing the modified geometry.

Our work probes along this line by exploring the entropy of dS space in higher derivative theories of gravity and showing a remarkable feature of dS space with which one can obtain the dS entropy without knowing the details of the modified dS geometry. This property is based on an important theorem that states that the dS (and the AdS) space is always a solution of any generally covariant theories of gravity. We start by giving a full proof of this theorem.

Let us consider generally covariant higher derivative gravity theories with cosmological constant Λ:

$$I = \int d^D x \sqrt{-det\ g}\mathcal{L}(\Lambda, g_{\mu\nu}, R_{\mu\nu\rho\sigma}, \nabla_{\alpha_1} R_{\mu\nu\rho\sigma}, \cdots \nabla_{(\alpha_1 \cdots} \nabla_{\alpha_m)} R_{\mu\nu\rho\sigma}). \tag{1}$$

Variation over the metric $g_{\mu\nu}$ yields the equations of motion [30]:

$$\frac{\partial \mathcal{L}}{\partial g^{\mu\nu}} + E_\mu{}^{\rho\sigma\kappa} R_{\nu\rho\sigma\kappa} - 2\nabla^\rho \nabla^\sigma E_{\mu\rho\sigma\nu} - \frac{1}{2} g_{\mu\nu}\mathcal{L} + \{\nabla \tilde{\mathcal{L}}\}_{\mu\nu} = 0, \tag{2}$$

where $E_\mu{}^{\rho\sigma\kappa}$ is defined as

$$E^{\mu\nu\rho\sigma} \equiv \frac{\partial \mathcal{L}}{\partial R_{\mu\nu\rho\sigma}} - \nabla_{\alpha_1} \frac{\partial \mathcal{L}}{\partial \nabla_{\alpha_1} R_{\mu\nu\rho\sigma}} + \cdots + (-1)^m \nabla_{(\alpha_1 \cdots \alpha_m)} \frac{\partial \mathcal{L}}{\partial \nabla_{(\alpha_1 \cdots} \nabla_{\alpha_m)} R_{\mu\nu\rho\sigma}} \tag{3}$$

and $\tilde{\mathcal{L}}$ stands for a sum of terms constructed with $\Lambda, g_{\mu\nu}, R_{\mu\nu\rho\sigma}$ and its covariant derivatives. Without loss of generality, we can divide the terms in $\mathcal{L}$ into three types: $\mathcal{L}_i$ ($i = 1,\ 2,\ 3$). $\mathcal{L}_1$ is a sum of terms containing contractions of $R_{\mu\nu\rho\sigma}$ only, without any of its covariant derivatives; $\mathcal{L}_2$ are those of $R_{\mu\nu\rho\sigma}$ and its covariant derivatives; and $\mathcal{L}_3$ are those of only covariant derivatives of $R_{\mu\nu\rho\sigma}$. By doing so, the equations of motion (2) can be rewritten as

$$\frac{\partial \mathcal{L}}{\partial g^{\mu\nu}} + g_{\mu\lambda} \frac{\partial \mathcal{L}_1}{\partial R_{\lambda\rho\sigma\kappa}} R_{\nu\rho\sigma\kappa} + g_{\mu\lambda} \frac{\partial \mathcal{L}_2}{\partial R_{\lambda\rho\sigma\kappa}} R_{\nu\rho\sigma\kappa} - \frac{1}{2} g_{\mu\nu}\mathcal{L} + \{\nabla \hat{\mathcal{L}}\}_{\mu\nu} = 0, \tag{4}$$

where $\hat{\mathcal{L}}$ stand for terms constructed with $\Lambda, g_{\mu\nu}, R_{\mu\nu\rho\sigma}$ and the associated covariant derivatives.

We now proceed to prove that the following metric is always a solution of the field Equation (4):

$$ds^2 = -(1 - \lambda r^2)dt^2 + (1 - \lambda r^2)^{-1} dr^2 + r^2 d\Omega^2_{D-2}, \tag{5}$$

where λ is a constant to be determined by Λ and coefficients of all higher curvature terms in the action. A concrete example is the dS metric in Einstein's theory, with $\lambda = \Lambda/3$. For the background (5), the nonvanishing components of the Riemann curvature tensor are

$$R_{\alpha\beta\gamma\delta} = \lambda(g_{\alpha\gamma} g_{\beta\delta} - g_{\alpha\delta} g_{\beta\gamma}), \quad R_{ijk\ell} = \lambda(g_{ik} g_{j\ell} - g_{i\ell} g_{jk}), \quad R_{\alpha i\beta j} = \beta g_{\alpha\beta} g_{ij}, \tag{6}$$

where $\alpha, \beta, \gamma, \delta = r, t$ and $i, j, k, \ell = \theta, \phi, \ldots$. Using these properties it is not difficult to show that the Lagrangian density $\mathcal{L}$ is a function of λ and Λ only, i.e., $\mathcal{L}(\lambda, \Lambda)$. Since the curvature tensor can be expressed simply by the metric tensor, whose covariant derivatives are all zero, the covariant derivatives of the curvature tensor therefore all vanish. As a consequence, the last term in the left hand side of (4) vanishes identically. For the same reason, the third term in (4) disappears since every term in $\mathcal{L}_2$ contains the covariant derivatives of the Riemann tensor. Moreover, the above properties also imply that the first and second terms in the left-hand side of (4) take the form $f(\lambda,\Lambda)g_{\mu\nu}$ and $g(\lambda,\Lambda)g_{\mu\nu}$, respectively, with both f and g functions of λ and Λ only. To prove this, we note that the metric (5) is a solution of the equation of motion (4), as long as $f(\lambda,\Lambda)+g(\lambda,\Lambda)-\frac{1}{2}\mathcal{L}(\lambda,\Lambda)=0$ is satisfied. This leads to the following theorem:

Theorem 1. *For any generally covariant, purely metrical theory of gravity, the dS (or AdS) metric is always a solution, with the higher-order curvature terms in the action changing only the space-time curvature.*

2. Entropy for De Sitter Space in Higher Derivative Theories of Gravity

In this section, we calculate the entropy of dS space in higher derivative theories of gravity. In Einstein gravity, the entropy is proportional to one quarter the event horizon area, A_H. This, however, has been proved to be no longer true in higher derivative theories of gravity. In [31], it is shown that the entropy should take the form of a geometric expression evaluated at the event horizon. Several methods were developed to calculate the entropy of the higher derivative gravity: One was proposed by Wald [30,32], leading to the so-called Wald's formula [33]. Another is related to the Euclidean entropy of the space as shown in [34]. The dS entropy can be also obtained by assuming that space-time satisfies the first law of thermodynamics as shown in [35]. One of the authors of [36] has suggested an essentially thermodynamic argument that generalizes the entropy function method to dS space, and thus one can in principle calculate the entropy of dS space with higher order curvature corrections without knowing the details of the modified metric. This is due to the observation that the contribution from the modified temperature due to higher curvature corrections is the same in magnitude as that from the change in the entropy function due to the modified horizon, and the two are exactly canceled [36]. In this section, we firstly give a brief review of these related works. Then, we apply it to the calculation of dS entropy in higher derivative theories of gravity.

2.1. Brief Review of Noetherian Entropy and Entropy Function

In this subsection, we briefly review some important results made in the previous works [13,30,32], following the framework of Lagrangian field theories developed by Wald and viewing the Lagrangian as an n-form $\mathbf{L}(\psi)$, where $\psi=\{g_{ab},R_{abcd},\Phi_s,F^I_{ab},\cdots\}$ denotes the dynamical fields considered. Under this definition, the variation of $\mathbf{L}$ is

$$\delta\mathbf{L}=\mathbf{E}_\psi\delta\psi+d\mathbf{\Theta}, \tag{7}$$

where $\mathbf{\Theta}$ is an $(n-1)$-form, which is called the symplectic potential form, and $\mathbf{E}_\psi$ denotes equations of motion for the dynamical fields. Now, suppose ξ be any smooth vector field on the space-time manifold; one can then define a Noether current

$$\mathbf{J}[\xi]=\mathbf{\Theta}(\psi,\mathcal{L}_\xi\psi)-\xi\cdot\mathbf{L}. \tag{8}$$

It was shown that $d\mathbf{J}[\xi]=0$ when the equations of motion are satisfied; therefore, an $(n-2)$-form $\mathbf{Q}[\xi]$ can be introduced and an "on shell" formula can be obtained:

$$\mathbf{J}[\xi]=d\mathbf{Q}[\xi]. \tag{9}$$

Based on the first law of black hole thermodynamics, Wald [30,32] showed that for general stationary black holes, the black hole entropy is a kind of Noether charge at horizon,

$$S_{BH}=2\pi\int_{\mathcal{H}}\mathbf{Q}[\xi]\,, \tag{10}$$

where ξ denotes the Kklling field on the horizon, and $\mathcal{H}$ is the bifurcation surface of the horizon.

The Noether charge $\mathbf{Q}[\xi]$ can be also extended to the "off shell" form, as done in [37]

$$\mathbf{J}[\xi] = d\mathbf{Q}[\xi] + \xi^a \mathbf{C}_a \,, \tag{11}$$

where $\mathbf{C}_a$ is locally constructed out of the dynamical fields in a covariant manner and $\mathbf{C}_a = 0$ reduces to the previous definition (9) "on shell". The Noether charge defined in (11) can be written as

$$\mathbf{Q} = \mathbf{Q}^F + \mathbf{Q}^g + \cdots \tag{12}$$

with

$$\mathbf{Q}^F_{a_1\cdots a_{n-2}} = \frac{\partial \mathcal{L}}{\partial F^I_{ab}} \xi^c A^I_c \boldsymbol{\epsilon}_{aba_1\cdots a_{n-2}} \,, \tag{13}$$

$$\mathbf{Q}^g_{a_1\cdots a_{n-2}} = -\frac{\partial \mathcal{L}}{\partial R_{abcd}} \nabla_{[c}\xi_{d]} \boldsymbol{\epsilon}_{aba_1\cdots a_{n-2}} \,. \tag{14}$$

The "$\cdots$" terms are not important for our following discussion, and $\mathcal{L}$ is the Lagrangian density.

On the other hand, it was shown that the entropy of a kind of extremal black holes that have the near horizon geometry $AdS_2 \times S^{n-2}$ can be obtained by extremizing the so-called "entropy function" F with respect to the moduli on the horizon [13]:

$$S_{BH} = 2\pi F = 2\pi(e_i q_i - f(\vec{u}, \vec{v}, \vec{e}, \vec{p})), \tag{15}$$

where f is defined as

$$f(\vec{u}, \vec{v}, \vec{e}, \vec{p}) = \int dx^2 \cdots dx^{n-1} \sqrt{-\det g}\, \mathcal{L}. \tag{16}$$

Now, let $\mathcal{L}_{\hat{\lambda}}$ be a deformation of $\mathcal{L}$ in which we rescale all factors of Riemann tensor $R_{\alpha\beta\gamma\delta}$ by $\hat{\lambda} R_{\alpha\beta\gamma\delta}$ and define $f_{\hat{\lambda}} = \sqrt{-\det g}\, \mathcal{L}_{\hat{\lambda}}$; then, the following relation holds:

$$\left.\frac{\partial f_{\hat{\lambda}}}{\partial \hat{\lambda}}\right|_{\hat{\lambda}=1} = \int_{\mathcal{H}} \sqrt{-\det g}\, R_{\alpha\beta\gamma\delta} \frac{\partial \mathcal{L}}{\partial R_{\alpha\beta\gamma\delta}} dx^1 \cdots dx^{n-2} = f - e_i \frac{\partial f}{\partial e_i}, \tag{17}$$

where $\alpha, \beta, \gamma, \delta$ are summed over the coordinates r and t.

Let us consider a specific example: an n–dimensional spherically symmetric black hole in asymptotically flat space-time. Since $\boldsymbol{\Theta} = 0$, if ξ is a killing vector, we find by integrating over a Cauchy surface $\mathcal{C}$ on Equation (9)

$$\int_{\mathcal{C}} \mathbf{J} = -\int_{\mathcal{C}} \xi \cdot \mathbf{L} = \int_{\mathcal{C}} d\mathbf{Q}[\xi] = \int_{\infty} \mathbf{Q} - \int_{\mathcal{H}} \mathbf{Q}, \tag{18}$$

where $\mathcal{H}$ denotes the interior boundary, and we have used the Stokes theorem. For an asymptotically flat, static spherically symmetric black hole, one can simply choose $\xi = \partial_t = \frac{\partial}{\partial t}$; then, the free energy of the system is shown to be [36]

$$\mathcal{F} = \mathcal{E} - \int_{\mathcal{H}} \mathbf{Q}[\xi^a], \tag{19}$$

where $\mathcal{E}$ is the "canonical energy", and $\mathcal{F} = TI_E$, with T and I_E the temperature and Euclidean action of the system, respectively. Variation of Equation (19) leads to

$$\delta\mathcal{F} = \delta\mathcal{E} - \delta \int_{\mathcal{H}} \mathbf{Q}[\xi^a]. \tag{20}$$

Let us consider a stretched region near the horizon ranged from r_H to $r_H + \delta r$, combining (12); then,

$$\delta \int_{\mathcal{H}} \mathbf{Q}[\xi] = \int_{r_H}^{r_H+\delta r} \Big(\mathbf{Q}^{\mathrm{F}}[\xi] + \mathbf{Q}^g[\xi] \Big). \tag{21}$$

Together with the killing equation $\nabla_{[a}\xi_{b]} = 2\kappa\epsilon_{ab}$ (where κ is the surface gravity of the hole), one gets [38]

$$\int_{r_H}^{r_H+\delta r} \mathbf{Q}^g[\partial_t] = \delta r\left[\kappa' E + \kappa E'\right]_{r_H} + \mathcal{O}(\delta r^2)\,, \tag{22}$$

$$\int_{r_H}^{r_H+\delta r} \mathbf{Q}^F[\partial_t] = q_I e_I \delta r + e_I q_I' \delta r + \mathcal{O}(\delta r^2). \tag{23}$$

where

$$E(r) \equiv -\int_{\mathcal{H}} \frac{\partial \mathcal{L}}{\partial R_{abcd}} \epsilon_{ab}\epsilon_{cd} dx^1 \cdots dx^{n-2}. \tag{24}$$

The above formula is exactly the Wald formula for entropy up to a factor of 2π [30]. According to the definition of entropy function in (4.4) of [38], we find $E(r)$ is related to the entropy by $2\pi E(r_H) = S$.

On the other hand, for the free energy, we also find that

$$\delta\mathcal{F} = -\int_{r_H}^{r_H+\delta r} f dr = -f(r_H)\delta r + \mathcal{O}(\triangle r^2). \tag{25}$$

Substituting Equations (22), (23) and (25) into Equation (20), we obtain

$$F\delta r = -S\delta T, \tag{26}$$

where $F = (-f(r_H) + q_I e_I)$ is the entropy function for extremal black hole as shown in (15), and we have used the relation $\delta\mathcal{E} = T\delta S - e_I q_I' \delta r$. In the limit $\delta r \to 0$, we obtain an equation which governs the entropy function for non-extremal black holes

$$ST' = -F, \tag{27}$$

where prime denotes derivative with respect to r.

The asymptotically dS or AdS cases, however, are slightly different due to the definition of the Hamiltonian. Our previous work [36] shows that the following relation holds, given by

$$(TS)' - \mathcal{E}' + e_I q_I' = -F_{BH} + \left[\left(\frac{g_{tt}^{BH}}{g_{tt}^{AdS}}\right)^{1/2}\right]_{r=R} F_{dS}, \tag{28}$$

where $F_{BH} = e_I q_I - f_{BH}$ and $F_{dS} = -f_{dS}$ denote the entropy functions of the black hole and AdS, respectively, and R is a cutoff of the dS (AdS) space, which is usually chosen as dS (AdS) radius. The integration form of Equation (28) gives the free energy

$$\mathcal{F} = \int_{r_H}^{R} F_{BH} - \left[\left(\frac{g_{tt}^{BH}}{g_{tt}^{AdS}}\right)^{1/2}\right]_{r=R} \int_0^R F_{dS}. \tag{29}$$

The entropy then can be evaluated by using the formula $S = -\frac{\partial\mathcal{F}}{\partial T}$. Special attention has been paid to pure dS (AdS) case with higher curvature corrections in our previous work [39], where we have shown that the entropy of dS space in any higher derivative gravity theories can be computed by

$$S = S_0 + \gamma S_1 = \frac{1}{T_{(0)}} \int_0^{r_H^{(0)}} (F_0 + \gamma F_1) dr, \tag{30}$$

where script (0) denotes the variables computed by using the non-perturbative metric, and F_0 and F_1 represent entropy functions with and without higher derivative corrections, respectively. Both F_0 and F_1 are calculated by using the non-perturbative metric. γ is a small quanta showing the coupling strength. This implies that one can obtain the entropy of dS spacetime in any higher derivative gravity theories without knowing the corrected metric. Equation (30) can be also rewritten as

$$S = \frac{1}{T_{(0)}} \int_0^l F dr, \tag{31}$$

where the cosmological radius l and the temperature $T_{(0)}$ are calculated by using the non-perturbative metric, namely, the dS metric in Einstein gravity. All higher derivative corrections are contained in the entropy function F. In what follows, we calculate the explicit form of dS entropy for higher derivative theories of gravity.

2.2. Ds Entropy Function for Higher Derivative Gravity

Combining Equation (31) with the theorem in the first section, we show that the metric of the form (5) can be used to quickly derive the dS entropy, without knowing the exact form of λ. To demonstrate the validity of this statement, in this subsection, we show a concrete example. In the framework of string theory, the higher-order curvature terms are generated by the slope expansion. Although such terms may introduce graviton ghosts and violate unitarity, it was shown in [40,41] that ghosts can be avoided if the modified field equations induced by the stringy corrections remain in second order. One of the important second-order gravity theories in higher dimensional spaceötimes is known as the Lovelock gravity [42]. The Lagrangian density for the Lovelock gravity in D dimensions is $\mathcal{L} = \sum_{m=0}^{[D/2]} c_m \mathcal{L}_m$, where $\mathcal{L}_m$ is given by

$$\mathcal{L}_m = \frac{1}{2^m} \delta^{a_1 b_1 \ldots a_m b_m}_{c_1 d_1 \ldots c_m d_m} R_{a_1 b_1}{}^{c_1 d_1} \ldots R_{a_m b_m}{}^{c_m d_m} , \tag{32}$$

α_m is the m'th order coupling constant, $[D/2]$ denotes the integer part of $D/2$ and the Latin indices a, b, c and d run from 0 to $D-1$. The δ symbol is a totally antisymmetric product of $2m$ Kronecker deltas normalized to take the values of ± 1. The term $\mathcal{L}_0 = 1$ is the cosmological term, while $\mathcal{L}_1 = \delta^{a_1 b_1}_{c_1 d_1} R_{a_1 b_1}{}^{c_1 d_1}/2$ is the Einstein term. In general $\mathcal{L}_m$ is the Euler class of a $2m$ dimensional manifold. Let us first consider a static, spherically symmetric space-time. The metric of interest is of the form

$$ds^2 = -e^{2\varepsilon(r)} dt^2 + e^{-2\varepsilon(r)} dr^2 + r^2 \sum_{i,j}^{D-2} h_{ij} dx^i dx^j, \tag{33}$$

where $\varepsilon(r)$ is a function of r only, and $h_{ij} dx^i dx^j$ stands for the line element of a $(D-2)$-dimensional Einstein space, with $V_{D-2} = \int d^{D-2}\sqrt{h}$ the volume of this $(D-2)$-dimensional hypersurface at the horizon. With this metric ansatz, we have $\mathcal{R}_{ijkl} = k(h_{ik}h_{jl} - h_{il}h_{jk})$, $\mathcal{R}_{ij} = k(D-3)h_{ij}$ and $\mathcal{R} = k(D-2)(D-3)$, with k the curvature constant, whose value determines the geometry of the horizon. Without loss of generality, one may take the constant curvature $k = 1$, 0, and -1, with the $(D-2)$-dimensional hypersurface being spherical, flat and hyperbolic space, respectively. For simplicity, we set $k = 1$ in the following.

Following [43], for general static geometries, the action is expected to be of the form

$$I \to I_{(s)} = \frac{1}{16\pi G_D} \int d^{D-1}x N \sqrt{g_{(s)}} \sum_{m=0}^{[D/2]} c_m \mathcal{L}_m^{(s)}, \tag{34}$$

where $N = \sqrt{-g_{00}}$, and the subscript (s) denotes the spatial section and G_D is the Newton's constant in D dimension. The action principle yields the equation of motion of the form [44]

$$\sum_{m=0}^{n} \tilde{c}_m \psi^m = \frac{2G_D M}{r^{D-1}}, \tag{35}$$

where $\psi = r^{-2}(1 - e^{2\varepsilon})$. Here $0 < n \leq [D/2]$, M is an integration constant, and $\tilde{c}_m \equiv \frac{(D-3)!}{(D-2m-1)!} c_m$. Equation (35) incorporates all information concerning the static spherically symmetric spacetimes in the Lovelock gravity. One special solution to the equation of motion (35), which is of great interest in this letter, is the pure dS/AdS solution (with no black holes in it), corresponding to $M = 0$ in Equation (35). In this case, Equation (35) becomes

$$\sum_{m=0}^{n} \tilde{c}_m \psi^m = 0, \tag{36}$$

indicating that ψ is a function of the constants c_i and is independent of r:

$$\psi = \lambda(c_0, c_1, \cdots, c_n). \tag{37}$$

From the definition of ψ, one finds that $e^{2\varepsilon} = 1 - \lambda r^2$. In other words, the modified metric of dS/AdS space is of the following form:

$$ds^2 = -(1 - \lambda r^2)dt^2 + \frac{dr^2}{1 - \lambda r^2} + r^2 \sum_{i,j}^{D-2} h_{ij} dx^i dx^j. \tag{38}$$

This confirms our statement that higher-order curvature terms only change the coefficient of r^2 in dS/AdS space. In particular, if we consider the case with $n = 2$, which corresponds to the Gauss–Bonnet term, the equation of motion gives the value of λ:

$$\lambda = \frac{-\tilde{c}_1 \pm \sqrt{\tilde{c}_1^2 - 4\tilde{c}_0 \tilde{c}_2}}{2\tilde{c}_2}. \tag{39}$$

Without loss of generality, we can choose $\tilde{c}_0 = \mp 1/l^2$ (l denotes the cosmological radius) and $\tilde{c}_1 = 1$ for dS/AdS, respectively. By doing so, we find that (39) is exactly the solution given in [43] when there is no black hole inside the dS/AdS space. In [38], it is also shown that only the solution taking "+" sign is reliable, since the solution with "−" is unstable.

In the following, we focus on the dS spacetime. The temperature associated with a cosmological horizon $r_c = \sqrt{1/\lambda}$ is given by

$$T = \frac{\sqrt{\alpha k}}{2\pi}. \tag{40}$$

Since λ is a constant determined by $\tilde{c}_m (m = 0, 1, \ldots, n)$ through (36) and (37). Analytically, it becomes difficult to find λ once we consider the Lovelock gravity with $n > 4$.

To go around this difficulty, we apply Equation (31), which allows us to obtain the dS entropy without knowing many details of the modified geometry. The key is to find the entropy function. For simplicity, we choose $\tilde{c}_0 = -1/l^2$ and $\tilde{c}_1 = 1$ (hence $c_0 = -(D-1)(D-2)/l^2$ and $c_1 = 1$), reproducing the standard action without higher-order terms. With this choice, the dS line-element is the standard one $e^{2\varepsilon} = \left(1 - r^2/l^2\right)$. Generalizing Sen's definition, the entropy function of dS space is given by

$$F = \frac{1}{16\pi G_D} \int d^{D-2}x \sqrt{-g} \sum_{m=0}^{n} c_m \mathcal{L}_m, \tag{41}$$

where $0 < n \leq [(D-1)/2]$ and $\mathcal{L}_m$ computed by the line element (38) in its non-perturbative form with $\lambda = l^{-2}$ is given by

$$\mathcal{L}_m = \frac{D!}{(D-2m)!}\frac{1}{l^{2m}}. \tag{42}$$

It is straightforward to give the entropy function of dS space in Lovelock gravity:

$$F = \frac{V_{D-2}r^{D-2}}{16\pi G_D}\sum_{m=0}^{n}\frac{c_m}{l^{2m}}\frac{D!}{(D-2m)!}. \tag{43}$$

On the other hand, (40) tells us that the temperature of dS space in Einstein gravity reads $T_{(0)} = \frac{1}{2\pi l}$. Combining (31) and (43) one directly obtain

$$S = \frac{V_{D-2}l^{D}}{8G_D(D-1)}\sum_{m=0}^{n}\frac{c_m}{l^{2m}}\frac{D!}{(D-2m)!}. \tag{44}$$

It is not difficult to check that this formula reproduces the correct entropy of dS space; for instance, the $n = 2$ case, which corresponds to the Gauss–Bonnet gravity. Using Equation (44), we obtain

$$S = \frac{V_{D-2}l^{D-2}}{4G_D}\left(1 + \frac{D(D-2)(D-3)c_2}{2l^2}\right),$$

exactly the entropy of Guass-Bonnet gravity [36].

What we should emphasize here is that entropy formula (44) is valid for any Lovelock gravity. During our calculation, we do not need the modified metric at all. In other words, one can get the dS entropy in any Lovelock gravity without knowing the modified metric. Moreover, a remarkable feature of (44) is that we express the dS entropy in arbitrary order Lovelock gravity in terms of l (or $r_c^{(0)} = l$)—a cosmological radius calculated in the framework of Einstein's gravity. Under this consideration, our result is different from a previous work made by Cai in [45], where the author derived the entropy of a spherically symmetric black hole in Lovelock gravity by assuming that space-time satisfies the first law of thermodynamics. When it is applied to dS space, [45] shows that the entropy is given by

$$S = \frac{V_{D-2}r_c^{D-2}}{4G_D}\sum_{m=1}^{n}\frac{m(D-2)}{(D-2m)}\tilde{c}_m(r_c^{-2})^{m-1}. \tag{45}$$

where $r_c = \sqrt{1/\lambda}$ is the cosmological radius of Lovelock gravity and λ is given by (37) (here, we have re-expressed the formula in terms of our language). Obviously, $\lambda \neq 1/l^2$ in most cases. To express the entropy of dS space in terms of l, one has to solve the equation of motion (36). This, however, becomes impossible once $n \geq 5$. In this sense, we say that one cannot have an explicit expression for dS entropy in terms of (45). Therefore, one of the advantages of our result is that we can express the dS entropy in an explicit way. Although (45) fails to give an explicit expression for dS entropy in some cases, it can be applied to check the validity of our result. To do this, we have to find a perturbative solution to the equation of motion (36) by assuming $\lambda = (1+\lambda_1)/l^2$ with λ_1 a small term compared to 1. Then, we insert it into the equation of motion (36) and expand the formula to the first order, i.e., neglecting all the terms higher than $\tilde{c}_i$. By doing so, the expression of λ_1 turns out to be

$$\lambda_1 = -l^2\sum_{m=2}^{n}\frac{\tilde{c}_m}{l^{2m}} + \mathcal{O}(\tilde{c}_m).$$

The expression of λ is then obtained. Substituting the formula of λ obtained in this way into (45), one quickly recovers the formula (44).

3. Conclusions

We conclude with some remarks about Sen's entropy function method. We wish that the approach addressed here concerning the computation of dS entropy may shed a light on understanding the nature of the entropy function. From Sen's remarkable paper [13], we learn that the most distinguished feature of the entropy function method is the powerful ability to obtain the entropy of a specific class of extremal black holes in higher derivative gravity with little knowledge about the space-time geometry. To ensure the method works well, we requires a hypothesis, which is addressed by Sen in [14]:

> In any generally covariant theory of gravity coupled to matter fields, the near horizon geometry of a spherically symmetric extremal black hole in D dimensions has $SO(2,1) \times SO(D-1)$ isometry.

This is to assume that in any generally covariant theory of gravity, a spherically symmetric extremal black hole has the near horizon geometry $\mathrm{AdS}_2 \times \mathrm{S}^{D-2}$. The validity of this postulate has been proved in the 4 and 5-dimensional cases, as shown in [46]. However, with this postulate, we are still lacking knowledge to understand the entropy function method. Our results obtained in this letter may provide a possible understanding of the entropy function method as follows:

(i) De Sitter space has a geometry structure that is similar to the near horizon geometry $\mathrm{AdS}_2 \times \mathrm{S}^{D-2}$ of a spherically symmetric extremal black hole and has an analogue symmetry $SO(1,2) \times SO(D-1)$.

(ii) De Sitter geometry and the entropy function method share the same property, namely, the entropy can be calculated without knowing the exact metric form.

With these common grounds, we may expect that the crucial point for the entropy function method to be valid is the near horizon geometry structure. However, a full understanding of the relationship between our analysis and the entropy function method remains an outstanding challenge.

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