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The Kerr–Bumblebee exact massive and massless scalar quasibound states and Hawking radiation

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Abstract In this letter, we will focus on the Klein–Gordon equation with rotating axially symmetric black hole solution of the Einstein–Bumblebee theory, so called the Kerr–Bumblebee black hole, as its 3 + 1 background space-time. We start with constructing the covariant Klein–Gordon equation component by component and with the help of the ansatz of separation of variables, we successfully separate the polar part and found the exact solution in terms of Spheroidal Harmonics while the radial exact solution is discovered in terms of the Confluent Heun function. The quantization of the quasibound state is done by applying the polynomial condition of the Confluent Heun function that is resulted in a complex-valued energy levels expression for a massive scalar field, where the real part is the scalar particle's energy while the imaginary part represents the quasibound stats's decay. And for a massless scalar, a pure imaginary energy levels is obtained. The quasibound states, thus, describe the decaying nature of the relativistic scalar field bound in the curved Kerr–Bumblebee space-time. We also investigate the Hawking radiation of the Kerr–Bumblebee black hole's apparent horizon via the Damour–Ruffini method by making use the obtained exact scalar's wave functions. The radiation distribution function and the Hawking temperature are successfully obtained.

1 Introduction

The first black hole solution of the Einstein's general theory of relativity (1915) became available shortly after the work by Schwarzschild in 1916. Shortly afterwards, the first slowly rotating metric was developed in 1918 by [1,2]. The development of the rotating black hole solution was slow moving since then. However, almost all astrophysically significant bodies in nature are rotating and as the rotating body collapses, the conservation of the constant angular momentum will cause the rotation to speed up. It took 45 years for Roy Kerr, [3] to find the rigorous solution of the only possible stationary, axially symmetric and asymptotically flat solution of the Einstein theory of gravity, In recent years, the effect of the frame dragging caused by a rotating gravitational central body has received special interest and importance again as after more than 85 years of the theoretical predictions, it has become possible to directly measure this particular effect [4–6].

Despite of the Einstein theory of gravity is the most successful theory of gravitation in the level of solar system, one of the major challenges in theoretical gravitational physics is to integrate the Einstein theory with the standard model of particle physics, which accommodate many other interactions. A possible guide to this problem is the spontaneous symmetry breaking, which plays a key role in the elementary particle physics. The symmetry breaking in the early universe is very likely to happen since the temperature may have been high enough to set it off. One of the possible symmetry breaking that could happen in the quest of quantization of general relativity is the of Lorentz symmetry breaking [7]. The primary Lorentz-violating coupling in the gravitational sector of the Standard Model Extension takes the form $s^{\mu\nu}R_{\mu\nu}$, where $s^{\mu\nu}$ is a tensor coupling with a non-zero background configuration while the $R_{\mu\nu}$ is the Ricci tensor. The preferred frames of the background configuration is defined by $s^{\mu\nu}$ that makes, this coupling violates the Lorentz symmetry.

One of the modified theories of gravity that incorporate such effect is the Bumblebee theory of gravity where theory starts with this following integral of action [7],

$$S = \int \left[\frac{c^4}{16\pi G} \left(R + \zeta B^\mu B^\nu R_{\mu\nu} \right) - \frac{1}{4} B^{\mu\nu} B_{\mu\nu} - V \right] \sqrt{-g} d^4 x, \quad (1)$$

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where B^{μ} is a dynamical vector field called the Bumblebee field. It induces a spontaneous breaking of the Lorentz symmetry by causing a non-zero vacuum expectation value [8]. The ζ is a real valued coupling constant controlling the non-minimal curvature (gravity)-coupling and the Bumblebee vector fields. The $B_{\mu\nu}$ corresponds to the Bumblebee vector field, which is a generalization of the Maxwell tensor as follows,

$$B_{\mu\nu} = \partial_{\mu} B_{\nu} - \partial_{\nu} B_{\mu}, \tag{2}$$

and the potential, V, is driving the Lorentz violation. It is chosen to have a minimum at,

$$B^{\mu} B_{\mu} \pm b^2 = 0. \tag{3}$$

Three years ago, [7] has successfully found the novel rotating black hole solution, called the Kerr–Bumblebee black hole, of the Bumblebee theory gravity and becomes the interest of our scalar quasibound states and apparent horizon's Hawking radiation investigation.

In 14 September 2015, the gravitational wave signal of a binary black hole merger was directly detected for the first time [9]. Together with the quite recent Hawking radiation of the optical black hole analog [10], make the investigation of black hole spectroscopy a new emerging interest. The quasibound states, quasinormal modes, and shadows of black holes are among the most interesting characteristics of such an astrophysical objects in the observational measurable spectra that is generated as particles crossing into the black hole [11]. The quasibound states, that is also known as quasistationary resonance, are relativistic bound states outside the black hole's event horizon. The states are localized in the black hole's finite gravitational potential well. Thus, the quasibound states is leaking, crossing into the black hole causing the spectrum to have complex valued frequencies where the real part is associated as the scalar's energy while the imaginary part determines the stability of the system. It is possible, in principle, to extract some information about the physics of black holes as well as to validate some alternative/modified theories of gravity from these quasibound states [11]. Analogously to atomic transitions emitting photons, level transitions of axions around black holes emit gravitons [12].

The spectroscopy of black hole is also a new emerging interest in condensed matter physics as many different kinds of analogue models has been proposed after Unruh's prediction, for example Bose–Einstein condensates, electromagnetic wave guides, graphene, optical black hole, sonic black hole and ion rings [13–17]. In optical domain, [18] proposed the idea of an optical black hole for the first time in the year of 2000. The idea is that propagation of light in a moving medium resembles many features of a motion in a curved space-time background.

However, due to the complexity of the equations involved, especially the radial equation, analytical methods were used less often and only for certain problems. The vast majority of these studies made use of numerical techniques such as the asymptotical analysis, WKB, and continued fraction to investigate the specific task at hand. In recent years, thanks to the development of a special functions, so-called the Confluent Heun functions, several authors have successfully worked out and presented novel exact scalar quasibound states solutions respectively in the analog Schwarzschild black hole, the charged and chargeless Lense–Thirring black hole, the Reissner–Nordström black hole, the magnetic Ernst black hole and for the case of $f(R)$ theory's static spherically symmetric black hole background [19–25]. The importance of this particular special function in black hole physics was mentioned in [26]. Despite of the complexity, the radial equations of the governing relativistic Klein–Gordon equations are successfully solved in terms of the Confluent Heun and the General Heun functions. The polynomial condition of the Confluent Heun and the General Heun functions are found to be directly related to the quasibound states' energy quantization.

In this present work, we are going to show in detail the analytical derivation of both massive and massless scalar quasibound states' exact solutions in a rotating Kerr–Bumblebee black hole background. The exact wave function comprises of a harmonic temporal part, the Spheroidal Harmonics as the angular part and the Confluent Heun function as the radial part. The energy levels expression is obtained from the polynomial condition of the Confluent Heun function (see Appendix B) and further investigation of the wave function shows us that a quasibound states behave like an ingoing wave close to the black hole horizon and vanishing far away of the horizon. In the last section, using exact wave solution, the Hawking radiation of the apparent black hole's horizon is investigated and the Hawking temperature is obtained.

2 The Kerr–Bumblebee metric

2.1 The metric

In the Boyer–Lindquist coordinate, the line element of the Kerr–Bumblebee space-time that is generated by a rotating massive object in Bumblebee gravity, with mass M, real positive parameter b, Bumblebee parameter $L = 1 + l_B$ and angular momentum per unit mass a is given by [7],

$$ds^2 = -\left(1 - \frac{r_s r}{\rho^2}\right) c^2 dt^2 - \frac{2 r_s r L a \sin\theta}{\rho^2} c dt d\phi + \frac{\rho^2}{\Delta} dr^2 + \rho^2 d\theta^2 + \frac{A \sin^2\theta}{\rho^2} d\phi^2, \tag{4}$$

where,

$$\rho^2 = r\,(r + b) + L a^2 \cos^2\theta\ , \tag{5}$$

$$L\Delta = r\,(r+b) - r_s r + La^2, \tag{6}$$

$$\frac{A\sin^2\theta}{\rho^2} = \left[r\,(r+b) + La^2 + \frac{2r_s r La^2}{\rho^2}\sin^2\theta\right]\sin^2\theta\ . \tag{7}$$

It is possible to express the metric in the Cartesian coordinate by this following relation,

$$x = \sqrt{r\,(r+b) + La^2}\sin\theta\cos\phi\ , \tag{8}$$

$$y = \sqrt{r\,(r+b) + La^2}\sin\theta\sin\phi\ , \tag{9}$$

$$z = \sqrt{r\,(r+b)}\cos\theta\ . \tag{10}$$

The Kerr–Bumblebee space-time is singular when $\rho^2 = 0$ and $\Delta = 0$. The first condition, $\rho^2 = 0 = r\,(r+b) + La^2\cos^2\theta$ happens at $\theta = \frac{\pi}{2}$ and as $x^2 + y^2 = r\,(r+b) + La^2$, we get,

$$r\,(r+b) = 0 \leftrightarrow x^2 + y^2 = La^2. \tag{11}$$

The singularity forms a ring singularity lying on $x - y$ plane with radius $a\sqrt{L}$ from the Cartesian coordinate's origin.

The black hole horizons are found by solving the second condition, i.e. $\Delta = 0 = r\,(r+b) - r_s r + La^2$. The solution of the quadratic equation represent the outer and inner horizons respectively r_+, r_- as follows,

$$r_\pm = \frac{r_s}{2} - \frac{b}{2} \pm \sqrt{\frac{(b-r_s)^2}{4} - La^2}. \tag{12}$$

And in the case of $a = 0$, the static spherically symmetric Schwarzschild–Bumblebee black hole is recovered and setting $b = 0$ and $L = 1$, the regular Schwarzschild black hole is recovered. Moreover, without nulling the spin parameter but setting $b = 0$ and $L = 1$, we obtain the regular Kerr black hole.

Before working with the Klein–Gordon equation in the Kerr–Bumblebee black hole background, we are going to find the Kerr–Bumblebee metric inverse as the first step. Let us rewrite line element (4) we can write the charged Lense–Thirring metric as follows,

$$[g_{\mu\nu} = \begin{pmatrix} -\left(1 - \frac{r_s r}{\rho^2}\right) & 0 & 0 & -\frac{r_s r\sqrt{L}a\sin^2\theta}{\rho^2} \\ 0 & \frac{\rho^2}{\Delta} & 0 & 0 \\ 0 & 0 & \rho^2 & 0 \\ -\frac{r_s r\sqrt{L}a\sin^2\theta}{\rho^2} & 0 & 0 & \frac{A\sin^2\theta}{\rho^2} \end{pmatrix} \tag{13}$$

and as we need to calculate the metric tensor determinant, the metric must be modified and written as a block matrix as follows,

$$g_{\mu\nu} = \begin{pmatrix} \frac{\rho^2}{\Delta} & 0 & 0 & 0 \\ 0 & \rho^2 & 0 & 0 \\ 0 & 0 & -\left(1 - \frac{r_s r}{\rho^2}\right) & -\frac{r_s r\sqrt{L}a\sin^2\theta}{\rho^2} \\ 0 & 0 & -\frac{r_s r\sqrt{L}a\sin^2\theta}{\rho^2} & \frac{A\sin^2\theta}{\rho^2} \end{pmatrix} \tag{14}$$

$$= \begin{pmatrix} \tilde{A} & 0 \\ 0 & \tilde{B} \end{pmatrix}. \tag{15}$$

The determinant of the block matrix can be calculated using this following formula,

$$\det\left(g_{\mu\nu}\right) = g = \left|\tilde{A}\right|\left|\tilde{B}\right|, \tag{16}$$

and after some algebra, we obtain,

$$g = L\rho^4\sin^2\theta\ . \tag{17}$$

2.2 The metric inverse

The metric inverse is then calculated by making use of the special property of the block matrix as follows,

$$g_{\mu\nu} = \begin{pmatrix} \tilde{A} & 0 \\ 0 & \tilde{B} \end{pmatrix} \longrightarrow g^{\mu\nu} = \begin{pmatrix} \tilde{A}^{-1} & 0 \\ 0 & \tilde{B}^{-1} \end{pmatrix}, \tag{18}$$

$$\tilde{A}^{-1} = \begin{pmatrix} \frac{\Delta}{\rho^2} & 0 \\ 0 & \frac{1}{\rho^2} \end{pmatrix}, \tag{19}$$

$$\tilde{B}^{-1} = \frac{1}{\left|\tilde{B}\right|}\begin{pmatrix} \frac{A\sin^2\theta}{\rho^2} & \frac{r_s r\sqrt{L}a\sin^2\theta}{\rho^2} \\ \frac{r_s r\sqrt{L}a\sin^2\theta}{\rho^2} & -\left(1 - \frac{r_s r}{\rho^2}\right) \end{pmatrix}, \tag{20}$$

$$\left|\tilde{B}\right| = \frac{g}{\left(\frac{\rho^4}{\Delta}\right)} = -L\Delta\sin^2\theta\ . \tag{21}$$

The inverse of the metric tensor is as follows,

$$g^{\mu\nu} = \begin{pmatrix} -\frac{1}{L\Delta}f_{\phi\phi} & 0 & 0 & -\frac{1}{L\Delta}\frac{r_s r\sqrt{L}}{\rho^2}a \\ 0 & \frac{\Delta}{\rho^2} & 0 & 0 \\ 0 & 0 & \frac{1}{\rho^2} & 0 \\ -\frac{1}{L\Delta}\frac{r_s r\sqrt{L}}{\rho^2}a & 0 & 0 & \frac{1}{L\Delta\sin^2\theta}\left(1 - \frac{r_s r}{\rho^2}\right) \end{pmatrix}, \tag{22}$$

where,

$$f_{\phi\phi} = r\,(r+b) + La^2 + \frac{r_s r La^2}{\rho^2}\sin^2\theta\ . \tag{23}$$

3 The Klein–Gordon equation

3.1 Construction

A relativistic scalar field in a curved space-time, regardless massive or massless, is represented by the covariant Klein–Gordon equation. The quantum relativistic matter wave equation reads as follows,

$$-\hbar^2\nabla_\mu\nabla^\mu\psi + k^2c^2 = 0, \tag{24}$$

where ∇_μ is the covariant derivative.

Operating the covariant derivatives, we proceed as follows,

$$\nabla_\mu \nabla^\mu \psi = \nabla_\mu \partial^\mu \psi = \partial_\mu \partial^\mu \psi + \Gamma^\mu_{\mu\nu} \partial^\nu \psi, \tag{25}$$

where, we also can derive this following identity,

$$\begin{aligned}\Gamma^\alpha_{\alpha\beta} &= \frac{g^{\alpha\gamma}}{2}\left(\partial_\alpha g_{\beta\gamma} + \partial_\beta g_{\alpha\gamma} - \partial_\gamma g_{\alpha\beta}\right) = \frac{1}{2} g^{\alpha\gamma} \partial_\beta g_{\alpha\gamma} \\ &= \frac{1}{2} g^{\alpha\gamma} \frac{\partial g_{\alpha\gamma}}{\partial x^\beta} = \frac{1}{2} g^{\alpha\gamma} \left(\frac{2}{\sqrt{-g} g^{\alpha\gamma}} \frac{\partial \sqrt{-g}}{\partial x^\beta}\right) \\ &= \frac{1}{\sqrt{-g}} \frac{\partial \sqrt{-g}}{\partial x^\beta},\end{aligned} \tag{26}$$

so, with the help of this property of the Christoffel symbol, we get,

$$\begin{aligned}\nabla_\mu \nabla^\mu \psi &= \partial_\mu \partial^\mu \psi + \left(\frac{1}{\sqrt{-g}} \partial_\nu \sqrt{-g}\right) \partial^\nu \psi \\ &= \frac{1}{\sqrt{-g}} \partial_\mu \left(\sqrt{-g} \partial^\mu \psi\right) \\ &= \frac{1}{\sqrt{-g}} \partial_\mu \left(\sqrt{-g} g^{\mu\nu} \partial_\nu \psi\right),\end{aligned} \tag{27}$$

and finally, we can express the Klein–Gordon equation in the terms of partial derivatives and the metric tensor components as follows,

$$\left\{-\hbar^2 \left[\frac{1}{\sqrt{-g}} \partial_\mu \sqrt{-g} g^{\mu\nu} \partial_\nu\right] + k^2 c^2\right\} \psi = 0, \tag{28}$$

where $E_0 = kc^2$ is the scalar's rest energy per unit mass. So, we have $k = 1$ for massive scalars and $k = 0$ for massless scalars.

As we have obtained the metric determinant and the metric inverse, the Laplace–Beltrami operator of the Klein–Gordon equation can be found component per component as follows,

$$\begin{aligned}&\frac{1}{\sqrt{-g}} \partial_0 \sqrt{-g} g^{00} \partial_0 \\ &= -\left\{\left(r\,(r+b) + La^2\right)^2 - L^2 \Delta a^2 \sin^2\theta\right\} \frac{\partial^2_{ct}}{\Delta \rho^2},\end{aligned} \tag{29}$$

$$\frac{1}{\sqrt{-g}} \partial_0 \sqrt{-g} g^{03} \partial_3 = -\frac{1}{L\Delta} \frac{r_s r}{\rho^2} \sqrt{L} a \partial_{ct} \partial_\phi, \tag{30}$$

$$\frac{1}{\sqrt{-g}} \partial_1 \sqrt{-g} g^{11} \partial_1 = \frac{1}{r^2} \partial_r \left(\Delta \partial_r\right), \tag{31}$$

$$\frac{1}{\sqrt{-g}} \partial_2 \sqrt{-g} g^{22} \partial_2 = \frac{1}{r^2 \sin\theta} \partial_\theta \left(\sin\theta \partial_\theta\right), \tag{32}$$

$$\frac{1}{\sqrt{-g}} \partial_3 \sqrt{-g} g^{30} \partial_0 = -\frac{1}{L\Delta} \frac{r_s r}{\rho^2} \sqrt{L} a \partial_{ct} \partial_\phi, \tag{33}$$

$$\frac{1}{\sqrt{-g}} \partial_3 \sqrt{-g} g^{33} \partial_3 = \frac{L\Delta - a^2 \sin^2\theta}{L\Delta \sin^2\theta\ \rho^2} \partial^2_\phi. \tag{34}$$

Combining all of the components, we obtain the full Klein–Gordon equation in Kerr–Bumblebee black hole background as follows,

$$\begin{aligned}&\left[\left\{-\frac{1}{\Delta\rho^2}\left\{\left(r\,(r+b) + La^2\right)^2 - L^2 \Delta a^2 \sin^2\theta\right\} \partial^2_{ct}\right.\right. \\ &-2 \frac{1}{L\Delta} \frac{r_s r}{\rho^2} \sqrt{L} a \partial_{ct} \partial_\phi + \frac{1}{\rho^2} \partial_r \left(\Delta \partial_r\right) \\ &+ \frac{1}{\rho^2 \sin\theta} \partial_\theta \left(\sin\theta\ \partial_\theta\right) \\ &\left.\left.+ \frac{L\Delta - a^2 \sin^2\theta}{L\Delta \sin^2\theta\ \rho^2} \partial^2_\phi\right\} - \frac{k^2 c^2}{\hbar^2}\right] \psi = 0.\end{aligned} \tag{35}$$

Due to the present temporal and azimuthal symmetry, we can use this following separation ansatz [27],

$$\psi\,(t, r, \theta, \phi) = e^{i \frac{E}{c} ct - i m_l \phi} R\,(r)\, T\,(\theta)\,. \tag{36}$$

3.2 The polar equation

Substituting the separation ansatz into the Eq. (35) and multiplying the whole by $\frac{r^2}{\psi(t,r,\theta,\phi)}$, also defining these following dimensionless variables, i.e. $\Omega^2 = \frac{\omega^2 r_s^2}{c^2}$ and $\Omega_0^2 = \frac{E_0^2 r_s^2}{\hbar^2 c^2}$ where $E_0 = kc^2$, to get,

$$\begin{aligned}&\left\{-\frac{1}{\Delta\rho^2}\left\{\left(r\,(r+b) + La^2\right)^2 - L^2 \Delta a^2 \sin^2\theta\right\}\left(-\frac{\omega^2}{c^2}\right)\right. \\ &-2 \frac{1}{L\Delta} \frac{r_s r}{\rho^2} \sqrt{L} a \left(\frac{\omega m_l}{c}\right) + \frac{1}{R\rho^2} \partial_r \left(\Delta \partial_r R\right) \\ &+ \frac{1}{\rho^2 \sin\theta} \partial_\theta \left(\sin\theta\ \partial_\theta\right) \\ &\left.+ \frac{L\Delta - a^2 \sin^2\theta}{L\Delta \sin^2\theta\ \rho^2} \left(-m_l^2\right)\right\} - \frac{k^2 c^2}{\hbar^2} = 0.\end{aligned} \tag{37}$$

Let us define these two dimensionless energy parameters,

$$\Omega = \frac{E r_s}{\hbar c},\ \Omega_0 = \frac{E_0 r_s}{\hbar c}. \tag{38}$$

Multiplying the whole wave equation by ρ^2 and using the identity $\sin^2\theta\ = 1 - \cos^2\theta$, we obtain a full radial-polar equation as follows,

$$\begin{aligned}&\left[\frac{1}{\sin\theta} \partial_\theta \left(\sin\theta\ \partial_\theta\right) - \frac{m_l^2}{\sin^2\theta}\right. \\ &\left. - \left(\frac{\Omega_0^2 La^2}{r_s^2} - \frac{\Omega^2 L^2 a^2}{r_s^2}\right) \cos^2\theta\ \right] \\ &+ \left[\frac{1}{R} \partial_r \left(\Delta \partial_r R\right) + \frac{\Omega^2}{r_s^2} \frac{\left(r\,(r+b) + La^2\right)^2}{\Delta}\right. \\ &+ \frac{\Omega^2 L^2 a^2}{r_s^2} + \frac{m_l^2 a^2}{L\Delta}\end{aligned}$$

$$-2\frac{\left(r\,(r+b)+L(a^2-\Delta)\right)a}{\sqrt{L}\Delta}\left(\frac{\Omega m_l}{r_s}\right)$$
$$-\frac{\Omega_0^2}{r_s^2}r\,(r+b)\Bigg]=0, \tag{39}$$

and the polar part can be separated as follows,

$$\frac{1}{\sin\theta}\partial_\theta\left(\sin\theta\;\partial_\theta\right)-\frac{m_l^2}{\sin^2\theta}$$
$$-\left(\frac{\Omega_0^2La^2}{r_s^2}-\frac{\Omega^2L^2a^2}{r_s^2}\right)\cos^2\theta\;+\lambda_l^{m_l}=0. \tag{40}$$

In the case of $L=1$ and $a=0$, it's clear that the separation constant $\lambda_l^{m_l}=l\,(l+1)$ and the polar wave solution is the Legendre polynomial, $P_l^{m_l}(\cos\theta)$. But, for in general case with non zero L and a, the polar solution is the Spheroidal Function, $S_l^{m_l}$, as follows [28],

$$T(\theta)=S_l^{m_l}\left(c,\cos\theta\;\right)=\sum_{r=-\infty}^{\infty}d_r^{lm_l}\left(c\right)P_{l+r}^{m_l}\left(\cos\theta\;\right), \tag{41}$$

where,

$$c=\frac{\Omega_0^2La^2}{r_s^2}-\frac{\Omega^2L^2a^2}{r_s^2}. \tag{42}$$

3.3 The radial equation

Having found the exact solution of the polar part, we are left with this following lengthy radial equation,

$$\partial_r\left(\Delta\partial_rR\right)+\left[\frac{\Omega^2}{r_s^2}\frac{\left(r\,(r+b)+La^2\right)^2}{\Delta}+\frac{\Omega^2L^2a^2}{r_s^2}\right.$$
$$-2\frac{\left(r\,(r+b)+La^2-L\Delta\right)a}{\sqrt{L}\Delta}\left(\frac{\Omega m_l}{r_s}\right)+\frac{m_l^2a^2}{L\Delta}$$
$$\left.-\frac{\Omega_0^2}{r_s^2}r\,(r+b)-\lambda_l^{m_l}\right]R=0. \tag{43}$$

Due to the issue of complexity, the radial equation must be treated carefully. As the condition of $\Delta=0$ leads to two solutions, it can be expressed in this following form,

$$\Delta=(r-r_-)(r-r_+), \tag{44}$$
$$\partial_r(\Delta\partial_rR)=(r-r_-)\partial_rR+(r-r_+)\partial_rR$$
$$+(r-r_-)(r-r_+)\partial_r^2R, \tag{45}$$
$$\frac{r_+-r_-}{\Delta}=\frac{\delta_r}{\Delta}=\frac{1}{r-r_+}-\frac{1}{r-r_-}. \tag{46}$$

We can now rearrange the radial equation (43) as follows,

$$\partial_r\left(\Delta\partial_rR\right)+\left[\frac{1}{\Delta}\left\{\frac{O}{r_s}\left(r\,(r+b)+La^2\right)-\frac{m_la}{\sqrt{L}}\right\}^2\right.$$
$$\left.-\left\{\frac{\Omega_0^2}{r_s^2}\left(r\,(r+b)+La^2\right)+K_l^{m_l}\right\}\right]R=0, \tag{47}$$

where we have defined a constant, $K_l^{m_l}$, to shorten the equation as follows,

$$K_l^{m_l}=-\frac{\Omega^2}{r_s^2}L^2a^2-\frac{\Omega_0^2}{r_s^2}La^2-2\sqrt{L}\frac{\Omega m_la}{r_s}+\lambda_l^{m_l}. \tag{48}$$

Now operating the differential respect to the r in the first term of the radial equation (47) followed by multiplying the whole by δ_r^2, we obtain this following expression,

$$\delta_r^2\partial_r^2R+\left\{\frac{1}{r-r_+}+\frac{1}{r-r_-}\right\}\delta_r^2\partial_rR$$
$$+\left[\left(\frac{1}{r-r_+}-\frac{1}{r-r_-}\right)^2\left\{\frac{\Omega}{r_s}\left(r\,(r+b)+La^2\right)-\frac{m_la}{\sqrt{L}}\right\}^2\right.$$
$$-\delta_r\left(\frac{1}{r-r_+}-\frac{1}{r-r_-}\right)$$
$$\left.\times\left\{\frac{\Omega_0^2}{r_s^2}\left(r\,(r+b)+La^2\right)+K_l^{m_l}\right\}\right]R=0. \tag{49}$$

As the region of interest is outside the outer horizon, i.e. $r_+\le r<\infty$, let us use these following new variables,

$$x=r-r_+=\delta_ry\rightarrow dx=dr=\delta_rdy, \tag{50}$$
$$r-r_-=x+r_+-r_-=\delta_ry+\delta_r=\delta_r\,(y+1\,), \tag{51}$$

that shift the region of interest to be $0\le y<\infty$. In the terms of y, the radial equation becomes as the following,

$$\partial_y^2R+\left\{\frac{1}{y}+\frac{1}{y+1}\right\}\partial_yR+[F.T.+S.T.]\,R=0. \tag{52}$$

where,

$$F.T.=\left[\frac{1}{\delta_r}\left(\frac{1}{y}-\frac{1}{y+1}\right)\right.$$
$$\left.\times\left\{\frac{\Omega}{r_s}\left((\delta_ry+r_+)^2+b\,(\delta_ry+r_+)+La^2\right)-\frac{m_la}{\sqrt{L}}\right\}\right]^2, \tag{53}$$

and,

$$S.T.=\left(\frac{1}{y}-\frac{1}{y+1}\right)$$
$$\times\left\{\frac{\Omega_0^2}{r_s^2}\left((\delta_ry+r_+)^2+b\,(\delta_ry+r_+)+La^2\right)+K_l^{m_l}\right\} \tag{54}$$

Let us consider the $F.T.$,

$$F.T.=\frac{1}{\delta_r}\frac{1}{y\,(y+1)}\left\{\frac{\Omega}{r_s}\left(\delta_r^2y^2+2y\delta_r\left(r_++\frac{b}{2}\right)\right)\right.$$
$$\left.+\underbrace{\frac{\Omega}{r_s}\left(r_+\,(r_++b)+La^2\right)-\frac{m_la}{\sqrt{L}}}_{K_1}\right\}. \tag{55}$$

Using the fractional decomposition, $\frac{y}{y+1} = 1 - \frac{1}{y+1}$, we can proceed as follows,

$$F.T. = \left[\frac{\Omega}{r_s}\delta_r + \frac{1}{y}\frac{K_1}{\delta_r} + \frac{1}{y+1}\underbrace{\left\{(r_+ + r_- + b)\frac{\Omega}{r_s} - \frac{K_1}{\delta_r}\right\}}_{K_3}\right]^2, \tag{56}$$

or, in the expanded form, we have,

$$first\ term = \frac{\Omega^2}{r_s^2}\delta_r^2 + \frac{K_1^2}{\delta_r^2 y^2} + \frac{K_3^2}{(y+1)^2} + \frac{2\Omega K_1}{r_s y} + \frac{2\Omega\delta_r K_3}{r_s(y+1)} + \frac{2K_1K_3}{\delta_r}\left(\frac{1}{y} - \frac{1}{y+1}\right). \tag{57}$$

Now let us proceed the $S.T.$,

$$S.T. = -\frac{1}{y(y+1)}\left\{\frac{\Omega_0^2}{r_s^2}\left(\delta_r^2 y^2 + 2\delta_r y\left(r_+ + \frac{b}{2}\right)\right) + \underbrace{\frac{\Omega_0^2}{r_s^2}\left(r_+(r_+ + b) + La^2\right) + K_l^{m_l}}_{K_2}\right\}, \tag{58}$$

and again, making use the fractional decomposition, $\frac{y}{y+1} = 1 - \frac{1}{y+1}$ again, we obtain,

$$S.T. = -\frac{\Omega_0^2}{r_s^2}\delta_r^2 + \frac{1}{y+1}\left(\underbrace{\frac{\Omega_0^2}{r_s^2}\delta_r^2 - 2\delta_r\left(r_+ + \frac{b}{2}\right)\frac{\Omega_0^2}{r_s^2} + K_2}_{K_4}\right) - \frac{K_2}{y}. \tag{59}$$

The $F.T.$ and $S.T.$ have successfully been expressed in the terms of $\frac{1}{y+1}$ and $\frac{1}{y}$ and now, we are going to convert the radial equation into its normal form following the Appendix A as follows,

$$\partial_y^2 R + \left\{\frac{1}{y} + \frac{1}{y+1}\right\}\partial_y R + \left[\frac{\Omega^2}{r_s^2}\delta_r^2 + \frac{K_1^2}{\delta_r^2 y^2} + \frac{K_3^2}{(y+1)^2} + \frac{2\Omega K_1}{r_s y} + \frac{2\Omega\delta_r K_3}{r_s(y+1)} + \frac{2K_1K_3}{\delta_r}\left(\frac{1}{y} - \frac{1}{y+1}\right) - \frac{\Omega_0^2}{r_s^2}\delta_r^2 + \frac{K_4}{y+1} - \frac{K_2}{y}\right]R = 0. \tag{60}$$

Here we identify,

$$p = \frac{1}{y} + \frac{1}{y+1}, \tag{61}$$

$$q = \frac{\Omega^2}{r_s^2}\delta_r^2 + \frac{K_1^2}{\delta_r^2 y^2} + \frac{K_3^2}{(y+1)^2} + \frac{2\Omega K_1}{r_s y} + \frac{2\Omega\delta_r K_3}{r_s(y+1)} + \frac{2K_1K_3}{\delta_r}\left(\frac{1}{y} - \frac{1}{y+1}\right) - \frac{\Omega_0^2}{r_s^2}\delta_r^2 + \frac{K_4}{y+1} - \frac{K_2}{y}. \tag{62}$$

Doing the algebra carefully and grouping the coefficients of $\frac{1}{y}$ and $\frac{1}{y+1}$, we finally obtain this following normal form of the radial equation,

$$\frac{d^2Y(y)}{dy^2} + K(y)Y(y) = 0, \tag{63}$$

where,

$$K(y) = -\frac{1}{2}\frac{dp}{dx} - \frac{1}{4}p^2 + q = \left(\frac{\Omega^2}{r_s^2}\delta_r^2 - \frac{\Omega_0^2}{r_s^2}\delta_r^2\right) + \frac{1}{y}\left(-\frac{1}{2} + \frac{2\Omega K_1}{r_s} + \frac{2K_1K_3}{\delta_r} - K_2\right) + \frac{1}{y^2}\left(\frac{1}{4} + \frac{K_1^2}{\delta_r^2}\right) + \frac{1}{y+1}\left(\frac{1}{2} + \frac{2\Omega\delta_r K_3}{r_s} - \frac{2K_1K_3}{\delta_r} + K_4\right) + \frac{1}{(y+1)^2}\left(\frac{1}{4} + K_3^2\right), \tag{64}$$

and the $Y(y)$ is interconnected with the $R(y)$ by,

$$Y(y) = \sqrt{(y(y+1)}R(y). \tag{65}$$

And comparing with the $K(x) = -\frac{1}{2}\frac{dp}{dx} - \frac{1}{4}p^2 + q$ of Confluent Heun's differential equation (see Appendix B), the Confluent Heun's parameters we can be solved as follows,

$$y = -x, \tag{66}$$

$$\alpha = 2\frac{\delta_r}{r_s}\sqrt{\Omega_0^2 - \Omega^2}, \tag{67}$$

$$\beta = \frac{2i}{\delta_r}\left[\frac{\Omega}{r_s}\left(r_+(r_+ + b) + La^2\right) - \frac{m_l a}{\sqrt{L}}\right], \tag{68}$$

$$\gamma = \frac{2i}{\delta_r}\left[\frac{\Omega}{r_s}\left(r_-(r_- + b) + La^2\right) - \frac{m_l a}{\sqrt{L}}\right], \tag{69}$$

$$\delta = \frac{(r_+ - r_-)}{r_s^2}\left[\Omega_0^2(r_+ + r_- + 2b) - 2\Omega^2(r_+ + r_- + b)\right], \tag{70}$$

$$\eta = \left(r_+(r_+ + b) + La^2\right)\left(\frac{2\Omega^2 - \Omega_0^2}{r_s^2}\right) - \frac{2\Omega a m_l}{r_s\sqrt{L}} - K_l^{m_l} - \frac{2}{\delta_r^2}\left(\frac{\Omega}{r_s}\left(r_+(r_+ + b) + La^2\right) - \frac{m_l a}{\sqrt{L}}\right) \times \left(\frac{\Omega}{r_s}\left(r_-(r_- + b) + La^2\right) - \frac{m_l a}{\sqrt{L}}\right). \tag{71}$$

After finding all of the Confluent Heun's parameters, we can present the complete exact radial solution of the Klein–

Gordon equation in the Kerr–Bumblebee black hole background as follows,

$$R = e^{-\frac{1}{2}\alpha y}\left[Ay^{\frac{1}{2}\beta}(y+1)^{\frac{1}{2}\gamma}\,\mathrm{HeunC}(-y)\right], \tag{72}$$

$$y = \frac{r-r_+}{\delta_r}. \tag{73}$$

And the complete exact wave function is as the following,

$$\psi = e^{i\frac{E}{\hbar c}ct}Y_\ell^{m_l}(\theta,\phi)\,e^{-\frac{1}{2}\alpha\left(\frac{r-r_+}{\delta_r}\right)}\left(\frac{r-r_-}{\delta_r}\right)^{\frac{1}{2}\gamma} \times\left[A\left(\frac{r-r_+}{\delta_r}\right)^{\frac{1}{2}\beta}\mathrm{HeunC}\left(\frac{r-r_+}{\delta_r}\right) +B\left(\frac{r-r_+}{\delta_r}\right)^{-\frac{1}{2}\beta}\mathrm{HeunC}'\left(\frac{r-r_+}{\delta_r}\right)\right]. \tag{74}$$

3.4 Energy quantization

The radial quantization condition is related to the number of zeros of the radial wave. The Confluent Heun function will have n zeros if it this following polynomial condition to make it a function with degree of n (see Appendix B) is fulfilled,

$$\frac{\delta}{\alpha}+\frac{\beta+\gamma}{2} = -n, \tag{75}$$

or explicitly,

$$\frac{\frac{(r_+^2-r_-^2)}{r_s^2}\left[\Omega_0^2-2\Omega^2-\frac{2b}{r_++r_-}\left(\Omega^2-\Omega_0^2\right)\right]}{2\frac{\delta_r}{r_s}\sqrt{\Omega_0^2-\Omega^2}} +\frac{i}{\delta_r}\left\{\frac{\Omega}{r_s}\left(r_+(r_++b)+r_-(r_-+b)+2La^2\right)-\frac{2m_l a}{\sqrt{L}}\right\} = -n. \tag{76}$$

and by cancelling out some pairs of r_s, we obtain,

$$\frac{(r_+^2-r_-^2)}{\delta_r}\frac{\frac{E_0 r_s}{\hbar c}\left\{1+\frac{2b}{r_++r_-}-\left(2+\frac{2b}{r_++r_-}\right)\frac{E^2}{E_0^2}\right\}}{2\left(1-\frac{E^2}{E_0^2}\right)^{\frac{1}{2}}} +\frac{i}{\delta_r}\left\{\frac{\Omega}{r_s}\left(r_+(r_++b)+r_-(r_-+b)+2La^2\right)-\frac{2m_l a}{\sqrt{L}}\right\} = -n. \tag{77}$$

Notice that the scalar energy levels have a dependence on the azimuthal quantum number m_l that is directly coupled to the black hole's spin parameter a. This is similar with the Zeeman effect when a Hydrogenic atom is immersed in a magnetized space. The existence of the term can be understood as an interaction between the orbiting scalar field with magnetic state m_l, with the black hole's angular momentum a.

It is also interesting to investigate the energy levels in the limit $Er_s \to 0$. Here we obtain,

$$E = E_0\sqrt{1-\left[\frac{\frac{E_0 r_s}{\hbar c}\left(r_+^2-r_-^2\right)}{2\delta_r\left(i\frac{2m_l a}{\sqrt{L}\delta_r}+n\right)}\right]^2}, \tag{78}$$

$$E - E_0 \approx -\frac{E_0}{2}\left[\frac{\frac{E_0 r_s}{\hbar c}\left(r_+^2-r_-^2\right)}{2\delta_r\left(i\frac{2m_l a}{\sqrt{L}d_r}+n\right)}\right]^2. \tag{79}$$

Contrary to the case of static black holes' weak field limit, where the energy levels are purely real [21–23], which can easily be reproduce by setting $a=0$. For a rotating black holes, the energy levels are complex valued anyway. This is expected as, classically, the calculation show that an astrophysical rotating black hole with $a = 0.998\frac{r_s}{2}$ has 32% effectivity in converting the mass of its accretion disc to be radiation. The value is $5\times$ greater than of the non rotating Schwarzschild black hole [29].

For the case of the massless scalar particle, by setting $E_0 = 0$, purely imaginary energy levels expression is found as follows,

$$E_n = \frac{\hbar c\left(in\delta_r+\frac{2m_l a}{\sqrt{L}}\right)}{r_+(r_++b)+r_-(r_-+b)+2La^2}. \tag{80}$$

So, any massless particle can quickly be absorbed by the Kerr–Bumblebee black hole.

It is important to mention that when we take $r_Q = 0$, this makes $r_+ \to r_s$ and $r_- \to 0$ which recovers the energy levels (79) in chargeless Lense–Thirring black hole,

$$E - E_0 \approx \frac{E_0}{2}\left[\frac{\frac{E_0 r_s}{\hbar c}r_s^2}{2\left(i\frac{2m_l a}{\delta_r}+n\right)}\right]^2. \tag{81}$$

In [30], there is mention the so-called "second polynomial condition" of the Confluent Heun function, i.e. the Δ_n condition. One can work out the algebra and find out that the second polynomial condition does not change the energy equation. Instead, it limits the value of l for states with main quantum number n, i.e. $l \le n-1$.

3.5 Quasibound states in extreme regions

Now, we are going to investigate the behaviour of the exact quasibound states solution in two extreme regions, i.e. very near to the black hole's outer horizon, $r \to r_+$ and asymptotic behaviour far away from the black hole's horizon $r \to \infty$. Remember that the quasibound states are quantized states. Thus, the Confluent Heun functions are always polynomial functions.

Let us first consider how the quasibound states behave very near to the apparent horizon by taking the limit $r \to r_+$. We investigate as follows, as $x = \frac{r-r_s}{\delta_r}$ is approaching $x = 0$, the Confluent Heun functions, $\mathrm{HeunC}(0) = \mathrm{HeunC}'(0) \approx 1$. Also the exponential $e^{-\frac{1}{2}\alpha\left(\frac{r-r_s}{\delta_r}\right)} \approx 1$. Thus, we get this following expression very near to the horizon,

$$\psi_{\to r_s} = e^{i\frac{E}{\hbar c}ct} Y_\ell^{m_l}(\theta,\phi)\left[A\left(\frac{r-r_s}{\delta_r}\right)^{\frac{1}{2}\beta} + B\left(\frac{r-r_s}{\delta_r}\right)^{-\frac{1}{2}\beta}\right]. \tag{82}$$

Now, let us define a new radial variable $\frac{r-r_s}{\delta_r} = \zeta r - \zeta_0$ and expressing the Heun's β parameter following (68) or equivalently, $\beta = i|\beta|$ to get this following expression,

$$\psi_{\to r_s} = e^{i\frac{E}{\hbar c}ct} Y_\ell^{m_l}(\theta,\phi)\left[A(\zeta r-\zeta_0)^{\frac{i|\beta|}{2}} + B(\zeta r-\zeta_0)^{-\frac{i|\beta|}{2}}\right], \tag{83}$$

and using the complex identity,

$$z^i = e^{1\ln(z)}, \tag{84}$$

together with,

$$\cos z = \frac{1}{2}\left(e^{iz} + e^{-iz}\right), \tag{85}$$

we get,

$$\psi_{\to r_s} = e^{i\frac{E}{\hbar c}ct} Y_\ell^{m_l}(\theta,\phi)\left[C\cos(\zeta r-\zeta_0)\right], \tag{86}$$

which represent a purely ingoing cosine wave. So, the quasi-bound states are purely ingoing waves very close to black hole's horizon and as $r \to \infty$, the exponential function $e^{-\frac{1}{2}\alpha\left(\frac{r-r_s}{r_s}\right)} \approx 1$ is definitely suppressing the Heun polynomials quenching the whole wave at the asymptotic infinity.

4 Hawking radiation

In the previous section, the complete expression of the polar and radial waves in terms of the Spheroidal Harmonics and the Confluent Heun functions have been presented in detail. In this section, we are going to investigate the Hawking radiation of the Kerr–Bumblebee black hole's apparent horizon. We start with this following exact radial solution of the wave function,

$$R = e^{-\frac{1}{2}\alpha\left(\frac{r-r_+}{\delta_r}\right)}\left(\frac{r-r_-}{\delta_r}\right)^{\frac{1}{2}\gamma}\left[A\left(\frac{r-r_+}{\delta_r}\right)^{\frac{1}{2}\beta}\mathrm{HeunC}\left(\frac{r-r_+}{\delta_r}\right) + B\left(\frac{r-r_+}{\delta_r}\right)^{-\frac{1}{2}\beta}\mathrm{HeunC}'\left(\frac{r-r_+}{\delta_r}\right)\right]. \tag{87}$$

Very close to the event horizon r_+, the radial wave is to be expanded in the lowest order of r and the Heun functions can be approximated as,

$$\mathrm{HeunC}(0) = \mathrm{HeunC}'(0) = 1, \tag{88}$$

also the exponential, $e^{-\frac{1}{2}\alpha\left(\frac{r-r_+}{\delta_r}\right)} = 1$,

$$R = \left(\frac{r_+-r_-}{\delta_r}\right)^{\frac{1}{2}\gamma}\left[B\left(\frac{r-r_+}{\delta_r}\right)^{-\frac{1}{2}\beta} + A\left(\frac{r-r_+}{\delta_r}\right)^{\frac{1}{2}\beta}\right], \tag{89}$$

$$\beta = \frac{2i}{\delta_r}\left[\frac{\Omega}{r_s}\left(r_+(r_++b)+La^2\right) - \frac{m_l a}{\sqrt{L}}\right]. \tag{90}$$

The radial wave consists of two independent parts as follows,

$$R = \begin{cases} \psi_{+in} = A\left(\frac{r_+-r_-}{\delta_r}\right)^{\frac{1}{2}\gamma}\left(\frac{r-r_+}{\delta_r}\right)^{\frac{1}{2}\beta} & ingoing \\ \psi_{+out} = B\left(\frac{r_+-r_-}{\delta_r}\right)^{\frac{1}{2}\gamma}\left(\frac{r-r_+}{\delta_r}\right)^{-\frac{1}{2}\beta} & outgoing. \end{cases} \tag{91}$$

Suppose there is an ingoing wave hitting the apparent horizon r_+. This will induce a particle-antiparticle pair where the particle part will be reflected and the antiparticle will be transmitted, going through the horizon, reaching $r = 0$. The analytical continuation of the wave function $\psi\left(\frac{r-r_+}{\delta_r}\right)$ can be calculated as follows,

$$\left(\frac{r-r_+}{\delta_r}\right)^\lambda = \left(\frac{r_+}{\delta_r}\right)^\lambda\left(\frac{r}{r_+}-1\right)^\lambda \to \left(\frac{r_+}{\delta_r}\right)^\lambda\left[\left(\frac{r}{r_+}-1\right)+i\epsilon\right]^\lambda$$
$$= \begin{cases} \left(\frac{r_+}{\delta_r}\right)^\lambda\left(\frac{r-r_+}{\delta_r}\right)^\lambda & , r > r_+ \\ \left(\frac{r_+}{\delta_r}\right)^\lambda\left|\frac{r-r_+}{\delta_r}\right|^\lambda e^{i\lambda\pi} & , r < r_+. \end{cases} \tag{92}$$

This analytical continuation enable us to obtain the $\psi_{-out} = \psi_{+out}\left(\left(\frac{r-r_+}{\delta_r}\right) \to \left(\frac{r-r_+}{\delta_r}\right)e^{i\pi}\right)$ simply by replacing $\left(\frac{r-r_+}{\delta_r}\right) \to -\left(\frac{r-r_+}{\delta_r}\right) = \left(\frac{r-r_+}{\delta_r}\right)e^{i\pi}$ as follows,

$$\psi_{-out} = B\left(\frac{r_+-r_-}{\delta_r}\right)^{\frac{1}{2}\gamma}\left(\left(\frac{r-r_+}{\delta_r}\right)e^{i\pi}\right)^{-\frac{1}{2}\beta},$$
$$= \psi_{+out}e^{-\frac{1}{2}i\pi\beta} \tag{93}$$

$$\left|\frac{\psi_{-out}}{\psi_{+in}}\right|^2 = \left|\frac{\psi_{+out}}{\psi_{+in}}\right|^2 e^{-i2\pi\beta} \tag{94}$$

$$= \left|\frac{\psi_{+out}}{\psi_{+in}}\right|^2 e^{\frac{4\pi}{\delta_r}\left[\frac{E}{\hbar c}\left(r_+(r_++b)+La^2\right)-\frac{m_l a}{\sqrt{L}}\right]} \tag{95}$$

$$= \left|\frac{\psi_{+out}}{\psi_{+in}}\right|^2 e^{-4\pi\frac{Er_s}{\hbar c}\left(r_+(r_++b)+La^2\right)-\frac{m_l a}{\sqrt{L}}} = e^\zeta. \tag{96}$$

The exponent e^ζ is called the radiation relative probability. The amplitude of the pair productions to occur is described by that function. As the observer stays outside the black hole horizon, the absolute probability of the process to occur outside the horizon needs to be found by exploiting the summing of all probabilities to create no pair, 1 pair, 2 pairs and so on as follows,

$$C_\omega\left(1+e^\zeta+\left(e^\zeta\right)^2+\cdots\right) = 1 \to C_\omega = 1-e^\zeta. \tag{97}$$

The probability to create j pairs of particle-anti-particle is given by,

$$C_\omega \left(e^\zeta\right)^j = \left(1 - e^\zeta\right) e^{-j\zeta}. \tag{98}$$

So, the normalized distribution function of the all the possible pair production that may occur is obtained as follows,

$$n\left(\omega\right) = \sum_{n=0}^{\infty} n\left(1 - e^\zeta\right) e^{-n\zeta} = \frac{1}{e^\zeta - 1}. \tag{99}$$

From the analysis above, the number of particles emitted out of the Kerr–Bumblebee even horizon is represented by this following distribution function,

$$\left\langle \frac{\psi_{out}}{\psi_{in}} \middle| \frac{\psi_{out}}{\psi_{in}} \right\rangle = 1 = \left|\frac{B}{A}\right|^2 \left|1 - e^\zeta\right|, \tag{100}$$

$$\left|\frac{B}{A}\right|^2 = \frac{1}{e^\zeta - 1}. \tag{101}$$

The Hawking temperature, T_H, is then obtained from this following modification,

$$\begin{aligned} &\frac{4\pi}{\delta_r}\left[\frac{E}{\hbar c}\left(r_+\left(r_+ + b\right) + La^2\right) - \frac{m_l a}{\sqrt{L}}\right] \\ &= \frac{4\pi}{\delta_r}\left[\frac{\omega}{c}\left(r_+\left(r_+ + b\right) + La^2\right) - \frac{m_l a}{\sqrt{L}}\right] \\ &= \frac{\hbar\left(\omega - \omega_J\right)}{\left[\frac{\delta_r c\hbar}{4\pi\left(r_+\left(r_+ + b\right) + La^2\right)}\right]}, \end{aligned} \tag{102}$$

$$\omega_J = \frac{m_l a}{\sqrt{L}c\left(r_+\left(r_+ + b\right) + La^2\right)}. \tag{103}$$

Finally, we obtain the apparent horizon's temperature as follows,

$$T_H = \frac{\delta_r c\hbar}{4\pi k_B\left(r_+\left(r_+ + b\right) + La^2\right)}. \tag{104}$$

By setting $a = b = 0$ and $L = 1$, $\delta_r \to r_s$ and $r_+ \to r_s$ and we obtain the Schwarzschild's Hawking radiation,

$$T_H = \frac{c\hbar}{4\pi k_B r_s}. \tag{105}$$

5 Conclusions

In this work, we successfully solve the Klein–Gordon wave equation in the Kerr–Bumblebee black hole space-time background. The exact analytical massive and massless scalar quasibound states' quantized energy levels (77), (80) and their wave functions (74) are obtained. It is important to mention that since the obtained solutions are exact, the are valid for all region of interest, i.e. $r_+ \leq r < \infty$. This is a remarkable improvement of the asymptotical method whose solutions solve only for either very close to the horizon region of very far away from the horizon.

By nulling the spin a and taking $L = 1$, in the small black hole limit, we reproduce the real valued energy levels of the Schwarzschild massive quasibound state's as,

$$\frac{E_n}{E_0} \approx 1 - \frac{\kappa^2}{2n^2}, \kappa = \left(\frac{E_0 r_s}{\hbar c}\right)^2. \tag{106}$$

The result that resembles the Hydrogenic atom energy expression $\frac{1}{n^2}$ is also sound in many previously published works [31–36].

Moreover, we also investigate the behaviour of the exact wave solutions in the two extreme regions, i.e. the near horizon and at infinity. Near the Kerr–Bumblebee black hole's horizon, the quasibound states behave like a purely ingoing wave (82) and become vanishing states at infinity.

With exact relativistic Klein–Gordon solutions in hand, the [37] method is applied to investigate the Hawking temperature of the black hole's apparent horizon. The method uses the Klein pair production scenario where the pair production occurring at the horizon is induced by an incoming particle. The induced particle goes to infinity while the induced anti-particle goes towards the black hole. From there, we make a summation of all possible pair productions and obtain the radiation distribution function (99). Comparing it with the bosonic distribution function, the Hawking temperature of the Kerr–Bumblebee black hole's apparent horizon is obtained (104).

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Code availability My manuscript has no associated code/software. [Author's comment: Code/Software sharing not applicable to this article as no code/software was generated or analysed during the current study.]

Appendix A: Normal form

An ordinary differential equation is said to be in the normal form if is solved explicitly for the highest derivative [28]. One may start with a general form of the linear second order ordinary differential equation as follows,

$$\frac{d^2y}{dx^2} + p(x)\frac{dy}{dx} + q(x)y = 0. \tag{107}$$

In order to bring the linear second order ordinary differential equation above to its normal form, we express the $y(x)$ in this particular form, which is specially designed to remove the first order derivative terms [38],

$$y = Y(x)e^{-\frac{1}{2}\int p(x)dx}, \tag{108}$$

$$\frac{dy}{dx} = \frac{dY}{dx}e^{-\frac{1}{2}\int p(x)dx} - \frac{1}{2}Ype^{-\int p(x)dx}, \tag{109}$$

$$\frac{d^2y}{dx^2} = \frac{d^2Y}{dx^2}e^{-\frac{1}{2}\int p(x)dx} - \frac{1}{2}\frac{dY}{dx}pe^{-\frac{1}{2}\int p(x)dx} - \frac{1}{2}Y\frac{dp}{dx}e^{-\frac{1}{2}\int p(x)dx} + \frac{1}{4}Yp^2e^{-\frac{1}{2}\int p(x)dx}. \tag{110}$$

Substituting the expressions to (107), a lot of things cancel out and it resulted in this following equation where the first order derivative term has been removed,

$$\frac{d^2Y}{dx^2} + \left(-\frac{1}{2}\frac{dp}{dx} - \frac{1}{4}p^2 + q\right)Y = 0, \tag{111}$$

$$Y = ye^{\frac{1}{2}\int p(x)dx}. \tag{112}$$

It is important to mention that if $Q(x) = -\frac{1}{2}\frac{dp}{dx} - \frac{1}{4}p^2 + q < 0$ and $Y(x)$ is a nontrivial solution of (112), then $Y(x)$ does not oscillate at all and has at most one zero. And if $Q(x) = -\frac{1}{2}\frac{dp}{dx} - \frac{1}{4}p^2 + q > 0$ and $\int_1^\infty Q(x)dx = \infty$, then $Y(x)$ has infinitely many zeros on the positive x-axis [38].

Appendix B: Normal form of confluent Heun equation

Let us consider the Confluent Heun differential equation [30],

$$\frac{d^2y}{dx^2} + \left(\alpha + \frac{\beta+1}{x} + \frac{\gamma+1}{x-1}\right)\frac{dy}{dx} + \left(\frac{\mu}{x} + \frac{\nu}{x-1}\right)y = 0, \tag{113}$$

$$\nu = \frac{1}{2}(\alpha + \beta + \gamma + \alpha\beta + \beta\gamma) + \delta + \eta, \tag{114}$$

$$y = A\,\text{HeunC}(\alpha, \beta, \gamma, \delta, \eta, x) + Bx^{-\beta}\,\text{HeunC}(\alpha, -\beta, \gamma, \delta, \eta, x), \tag{115}$$

$$y = A\,\text{HeunC}(x) + Bx^{-\beta}\,\text{HeunC}'(x), \tag{116}$$

$$\frac{\delta}{\alpha} + \frac{\beta+\gamma}{2} + 1 = -n_r, \quad n_r \in \mathbb{Z}. \tag{117}$$

Now, let us express Confluent Heun's differential equation in its the normal form by recognizing p and q function (see Appendix A). First, we recognize,

$$p = \alpha + \frac{\beta+1}{x} + \frac{\gamma+1}{x-1}, q = \frac{\mu}{x} + \frac{\nu}{x-1}, \tag{118}$$

$$y = \text{HeunC} = Y(x)e^{-\frac{1}{2}\alpha x}x^{-\frac{1}{2}(\beta+1)}(x-1)^{-\frac{1}{2}(\gamma+1)}, \tag{119}$$

and this leads to,

$$-\frac{1}{2}\frac{dp}{dx} = \frac{1}{x^2}\left(\frac{\beta+1}{2}\right) + \frac{1}{(x-1)^2}\left(\frac{\gamma+1}{2}\right), \tag{120}$$

$$-\frac{1}{4}p^2 = -\frac{\alpha^2}{4} - \frac{1}{x^2}\left(\frac{\beta^2+1+2\beta}{4}\right) - \frac{1}{(x-1)^2}\left(\frac{\gamma^2+1+2\gamma}{4}\right) - \frac{2}{x}\left(\frac{\alpha\beta+\alpha}{4}\right) - \frac{2}{x-1}\left(\frac{\alpha\gamma+\alpha}{4}\right) - \frac{2}{x(x-1)}\left(\frac{\beta\gamma+1+\beta+\gamma}{4}\right), \tag{121}$$

$$-\frac{1}{2}\frac{dp}{dx} - \frac{1}{4}p^2 + q = -\frac{\alpha^2}{4} + \frac{\frac{1}{2}-\eta}{x} + \frac{\frac{1}{4}-\frac{\beta^2}{4}}{x^2} + \frac{-\frac{1}{2}+\delta+\eta}{x-1} + \frac{\frac{1}{4}-\frac{\gamma^2}{4}}{(x-1)^2}. \tag{122}$$

Combining everything, we get the Confluent Heun equation's normal form,

$$\frac{d^2Y}{dx^2} + \left(-\frac{\alpha^2}{4} + \frac{\frac{1}{2}-\eta}{x} + \frac{\frac{1}{4}-\frac{\beta^2}{4}}{x^2} + \frac{-\frac{1}{2}+\delta+\eta}{x-1} + \frac{\frac{1}{4}-\frac{\gamma^2}{4}}{(x-1)^2}\right)Y = 0, \tag{123}$$

$$Y = e^{\frac{1}{2}\alpha x}x^{\frac{1}{2}(\beta+1)}(x-1)^{\frac{1}{2}(\gamma+1)}\,\text{HeunC}\,. \tag{124}$$

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