GLOBAL ANOMALIES IN STRING THEORY

Edward Witten*
Joseph Henry Laboratories
Princeton University
Princeton, New Jersey 08544

Abstract

Various questions involving global anomalies in particle theory and string theory are addressed. It is shown that the question of whether a manifold is a spin manifold is equivalent to a question about global anomalies in the propagation of a point particle. In the superstring case, it is shown that the measure of the heterotic theory has no global anomalies on any Riemann surface. This generalizes known one loop results. Also, a topological condition is derived which restricts the possible choices of Wilson lines for grand unified symmetry breaking. It is argued that the long-term development of global anomalies in string theory will involve eventual study of global anomalies in the determinant of the Dirac-Ramond operator.

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The gravitational interactions of a spin 1/2 particle propagating on a space-time manifold M are consistent only if several conditions are obeyed.* First of all, M must be a spin manifold on which there is no topological obstruction to defining spinors.[1] And if (in 4k+2 dimensions) we wish to discuss spinors of definite chirality, then M must be orientable. These conditions are needed just in order to make sense of the single particle Dirac equation $i\not{D}\psi=0$ (or $i\not{D}(1-\bar{\Gamma})\psi = 0$, in the chiral case; here $\bar{\Gamma} = \Gamma_1\Gamma_2\ldots\Gamma_{4k+2}$ is the chirality operator, the product of the gamma matrices Γ_i).

If we wish to formulate not just the one particle equation but the quantum field theory of spin one half particles on M, some additional conditions arise. It is necessary to be able to define the Dirac determinant $\det i\not{D}$--or, in the chiral case, $\det i\not{D}(\frac{1-\bar{\Gamma}}{2})$. In the chiral case, one encounters an anomaly in perturbation theory in trying to define the determinant.[2] To cancel this anomaly, it is necessary to introduce fields of various spin, perhaps including antisymmetric tensor fields.[3] Even when perturbative anomalies are cancelled the consistency of the theory is still not guaranteed. It is necessary to consider global anomalies. Thus let $f:M\to M$ be a diffeomorphism, not continuously connected to the identity, which leaves fixed the spin structure of M. One must ask whether the Dirac determinant is invariant under f. To answer this question, it is convenient to define the "mapping cylinder" $(M\times S^1)_f$. It is defined as follows: in the Cartesian product $M\times I$ ($I=[0,1]$ is the unit interval) one identifies $(x,0)$ with $(f(x),1)$ for any $x\in M$. The η invariant of the Dirac operator on $(M\times S^1)_f$ is defined as follows.[5] If λ_i are the Dirac eigenvalues on $(M\times S^1)_f$ then

$$\eta = \lim_{s\to 0} \sum_i \text{sign}\, \lambda_i \exp -s|\lambda_i| \tag{1}$$

Then it can be shown[4] that the change Δf of the chiral Dirac determinant on M is related to the η invariant on $(M\times S^1)_f$:

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*Most of the original lecture was devoted to reviewing the derivation of equation (2). As this material has appeared elsewhere,[4] I have chosen in the written version to expand on the concluding portion of the lecture.

$$\Delta_f \ln \det i\not{D}\left(\frac{1-\bar{\Gamma}}{2}\right) = \frac{i\pi n}{2}((M\times S^1)_f) \bmod 2\pi i \qquad (2)$$

Of course, (2) must be summed over all fermion multiplets, and if perturbative anomaly cancellation involved antisymmetric tensor fields, their contribution must be included too, to get the proper formula for the global anomaly. In [4], equation (2) was used to show that ten dimensional supergravity theories (which are of interest because they are the low energy limits of superstring theories) have no global anomalies when formulated on S^{10}. For the future it will be extremely interesting to learn whether these theories have global anomalies when formulated on $S^4\times K$ for various K. Although equation (2) is an appropriate starting point for addressing this question, practicable methods to evaluate this formula for a large class of ten manifolds (and diffeomorphisms) are currently unknown.

Of course, if one finds that the change in the determinant under a diffeomorphism f is not zero, one must be careful in drawing conclusions. Before concluding that the theory is inconsistent, one must make sure that there is a sound physical reason that lack of invariance under f would correspond to inconsistency. In essence, lack of invariance under f leads to inconsistency if upon decompactification from $S^4\times K$ to $R^4\times K$, f has compact support. Lack of invariance under f will then cause the physically relevant Feynmann path integral to vanish.[6] On $S^4\times K$, a "dangerous" f (under which the determinant must be invariant if the theory is to make sense) is one that leaves fixed a copy of K ($p\times K$, where p is the "point at infinity" on S^4). Later, we will discuss the physical meaning of global anomalies in certain cases in which f does not obey this condition.

To recapitulate what I have said so far, in assessing the consistency of the quantum field theory of spin 1/2 particles on a manifold M, there are several steps:

(a) One must make sure that M has a spin structure; and, if chiral fermions are to be considered, one must ask whether M is orientable.

(b) One must ensure the absence of dangerous local and global anomalies in the Dirac determinant.

Of course, merely determining whether a theory is consistent should not satisfy us. We also wish to extract its physical content. I will simply single out one aspect of this:

(c) One would like to determine the symmetries of the quantum field theory, allowing for possible effects of anomalies and other subtleties that might spoil an apparent symmetry.

In these notes I will discuss certain aspects of problems (a), (b), and (c) in field theory, and then discuss the generalization to string theory

While (b) and (c) are manifestly questions that involve anomalies, this is not so for (a). However, as we will now see, the question of whether a manifold admits spinors can be interpreted as a question about global anomalies on the world line of a point particle.

There is an action[7] for a point particle that possesses world line supersymmetry:

$$I = \int d\tau \; [\frac{1}{2} g_{ij}(x(\tau)) \frac{dx^i}{d\tau} \frac{dx^j}{d\tau} + \frac{i}{2} \psi^i(\tau) \; (g_{ij} \frac{d}{d\tau} + \frac{dx^k}{d\tau} \omega_{kij}(x(\tau))) \; \psi^j(\tau)] \tag{3}$$

Here x^i are coordinates on M, ω_{kij} is the spin connection of M, and the ψ^i are real anticommuting variables, i being a tangent vector index of M. (In the mathematical sense, the ψ^i take values in the pull-back of the tangent space of M to the line or circle parametrized by τ.) Now, (3) possesses the world-line supersymmetry

$$\delta x^i = i\psi^i \varepsilon$$

$$\delta\psi^i = \frac{dx^i}{d\tau} \varepsilon - i\omega_k{}^i{}_\ell \; \psi^k \psi^\ell \varepsilon \tag{4}$$

where ε is an anticommuting constant. (ε is a scalar under tranformations of M. There is a very different point particle action that possesses space-time supersymmetry.[8]) The conserved quantity corresponding to (4) is $Q = \psi_i \frac{dx^i}{d\tau}$. Upon quantization, the ψ_i obey $\{\psi_i(\tau), \psi_j(\tau)\} = g_{ij}(x(\tau))$, so they are gamma matrices, in effect. The wave functions--on which gamma matrices act--must then be spinor fields on M. As for $\frac{dx^i}{d\tau}$, we note that the canonical momentum is

$$p_i = \frac{\delta I}{\delta(\frac{dx^i}{d\tau})} = g_{ij}\frac{dx^j}{d\tau} + \frac{i}{4}\omega_{ijk}[\psi^j,\psi^k] \qquad (5)$$

As is usual in quantum mechanics, canonical quantization corresponds to $p_i = -i\frac{\partial}{\partial x^i}$. (5) can therefore be inverted to give $g_{ij}\frac{dx^j}{d\tau} = -iD_i$, where

$$D_i = \frac{\partial}{\partial x^i} + \frac{1}{4}\omega_{ijk}[\psi^j,\psi^k] \qquad (6)$$

is the usual covariant derivative acting on spinors. The conserved charge $Q = \psi_i\frac{dx^i}{d\tau} = -i\,\psi^i D_i$ is thus the Dirac operator.

This construction seems to make sense for arbitrary M. But if M is not a spin manifold, spinors and Dirac operators cannot be defined on M. We must run into trouble somewhere. How does this happen? One approach to quantization of (4) is path integrals. In this approach, one must define the fermion effective action $\sqrt{\det Y}$ where Y is the "world line Dirac operator"

$$Y = i\left(\frac{d}{d\tau}\delta^i{}_j + \frac{dx^k}{d\tau}\omega_k{}^i{}_j\right) \qquad (7)$$

We need the square root of det Y since the ψ^i are real. What we will see is that $\sqrt{\det Y}$ is afflicted with a global anomaly precisely when M is not a spin manifold.

It is convenient to take the world line to be a circle S^1.* Thus, we will study the eigenvalue problem $Y\phi=\lambda\phi$, where Y is defined on a circle, $0<\tau<2\pi$. In essence, $A^i{}_j = \frac{dx^k}{d\tau}\omega_k{}^i{}_j$ is an O(n) gauge field on the circle, n being the dimension of M. The Dirac equation on the circle should require a single gamma matrix Γ obeying $\Gamma^2=1$; for Γ we can pick the 1x1 matrix $\Gamma = 1$. So $Y = i(\frac{d}{d\tau} + A)$ is indeed a one dimensional Dirac operator.

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*We will use periodic boundary conditions for the fermions, corresponding to calculating Tr $e^{-\beta H}(-1)^F$. Antiperiodic ones would give Tr $e^{-\beta H}$.

In one dimension, the only gauge covariant quantity characterizing the gauge field A is the rotation matrix

$$R = P \exp \int_0^{2\pi} d\tau \, A(\tau) \tag{8}$$

If, say, n=2k, R can always be brought to canonical form, with "rotation angles" $\theta_1 \ldots \theta_k$:

$$R = \begin{pmatrix} \cos\theta_1 & \sin\theta_1 & & & & & \\ -\sin\theta_1 & \cos\theta_1 & & & & & \\ & & \cos\theta_2 & \sin\theta_2 & & & \\ & & -\sin\theta_2 & \cos\theta_2 & & & \\ & & & & \ddots & & \\ & & & & & \cos\theta_k & \sin\theta_k \\ & & & & & -\sin\theta_k & \cos\theta_k \end{pmatrix} \tag{9}$$

For given R, A can be gauge transformed into any form that obeys (8). A convenient choice is

$$A = \frac{1}{2\pi} \begin{pmatrix} 0 & \theta_1 & & & & & \\ -\theta_1 & 0 & & & & & \\ & & 0 & \theta_2 & & & \\ & & -\theta_2 & 0 & & & \\ & & & & \ddots & & \\ & & & & & 0 & \theta_k \\ & & & & & -\theta_k & 0 \end{pmatrix} \tag{10}$$

With this choice, the eigenvalues of Y are easily determined. They are $n \pm \frac{\theta_i}{2\pi}$, for arbitrary integer n. Hence formally

$$(\det Y)_0 = \prod_n \prod_i \left(n + \frac{\theta_i}{2\pi}\right)\left(n - \frac{\theta_i}{2\pi}\right) = \prod_{n,i} \left(n^2 - \frac{\theta_i^2}{(2\pi)^2}\right) \tag{11}$$

where the subscript "o" means that the divergent infinite product in (11) needs to be regularized. Noting that the right-hand side of (11) is periodic in each θ_i with a double zero whenever any θ_i is an integer multiple of 2π, we may guess that the proper formula is

$$\det Y = \prod_{i=1}^{k} \sin^2 \theta_i/2 \tag{12}$$

To derive (12), one may divide (11) by an infinite constant, giving

$$\det Y = \prod_{i=1}^{k} \left(\frac{\theta_i^2}{4} \prod_{n\neq 0} \left(1 - \frac{\theta_i^2}{4\pi^2 n^2}\right)\right) \tag{13}$$

The convergent product in (13) can be evaluated to give (12).

For the square root of the determinant we have then

$$\sqrt{\det Y} = \prod_{i=1}^{k} \sin \theta_i/2 \tag{14}$$

However, the sign of (14) is ill-defined, because the θ_i are well-defined only modulo 2π. This is the potential source of a global anomaly.

For a given world line γ, we can just define $\sqrt{\det Y}$ to be positive. For some other world line γ', the sign of $\sqrt{\det Y}$ must be determined by smoothly interpolating from γ to γ' and requiring that $\sqrt{\det Y}$ should vary smoothly. Thus, we find a mapping $\phi: S^1 \times I \to M$, where S^1 is parametrized by the time τ, and I by an auxiliary variable u, $0 < u \leq 1$. ϕ is chosen so that $\phi(\tau,0)$ is the curve γ, while $\phi(\tau,1)$ is γ'; thus, as u varies from 0 to 1, $\phi(\tau,u)$ is a one parameter family of curves varying from γ to γ'. Requiring $\sqrt{\det Y}$ to vary smoothly then determines its sign at γ' in terms of the sign chosen at γ.

Thus, at least for curves within a single homotopy class of curves in $\pi_1(M)$, we can determine the sign of $\sqrt{\det Y}$ for all curves in terms of a single overall sign choice. Is there an inconsistency in this procedure? An inconsistency would arise if we find an interpolation $\phi(\tau,u)$ that starts and ends at the <u>same</u> curve γ with the property that (requiring it to vary smoothly with u) $\sqrt{\det Y(u=1)} = - \sqrt{\det Y(u=0)}$.

Mathematically, an interpolation from γ to itself via a one parameter family of curves corresponds to a map

$$\phi: S^1 \times S^1 \to M \tag{15}$$

Here the first S^1 is parametrized by time τ, and the second by u, $0<u<1$, but now we identify $u=1$ with $u=0$. Thus, the first S^1 is mapped into M the same way at $u=1$ as at $u=0$. Given such a map, we can define for each u a rotation matrix

$$R(u) = P \exp \int_0^{2\pi} d\tau \ A(\tau;u) \tag{16}$$

As u varies from 0 to 1, R(u) sweeps out a closed curve in the O(n) manifold ($R(1) = R(0)$). This closed curve defines an element of $\pi_1(O(n)) \cong Z_2$. It is shown in the literature on spin manifolds (for instance, by Hawking and Pope [1]) that M does not admit a spin structure if for some $\phi: S^1 \times S^1 \to M$, R(u) is non-zero in $\pi_1(O(n))$. We can now easily see that there is a global anomaly in precisely this situation. The non-zero element of $\pi_1(O(n))$ is related to a 2π rotation, so R(u) is non-trivial in $\pi_1(O(n))$ if (say) one θ_i increases by 2π as u is increased from 0 to 1 and the other θ_j do not change. But (14) is odd under a 2π increase in any of the θ_i. So $\sqrt{\det Y(u=1)} = -\sqrt{\det Y(u=0)}$ precisely when R(u) is non-zero in $\pi_1(O(n))$.

At this point we have shown that studying global anomalies on the world line amounts to asking whether M has a spin structure. Actually, our treatment is complete only when $\pi_1(M) = 0$. We will not attempt here to unravel some further subtleties that arise when $\pi_1(M) \neq 0$.

Concerning the other problems on our list above, we have already discussed those aspects of (b) that we will need for our later discussion of string theory. Therefore, we move on to discuss certain aspects of (c). Readers who are so inclined can skip the following section and jump directly to the remarks on strings.

If we wish to discuss quantum gravity, we are interested in a situation in which the metric g of M is not fixed but is one of the dynamical variables. When we try to define $\sqrt{\det Y}$ for a curve in M, we may regard it as a functional not just of the curve γ but also of the metric along the curve. We ask whether $\sqrt{\det Y}$ is single-valued as a functional of the curve and metric.

In the above discussion, we considered a map $\phi: S^1 \times S^1 \to M$, which can be viewed as a one parameter family (u, labeling the second S^1, is the parameter) of maps of S^1 into M--with a fixed metric on M. Instead of taking a fixed metric on M, we could let the metric on M be u dependent--and so consider a one parameter family of maps of S^1 into a one parameter family of space-times. (I will take the liberty of referring to a one parameter family of metrics on M as defining a one parameter family of space-times.) Non-trivial one parameter families of metrics on M are in one to one correspondence with topological classes of diffeomorphisms $f:M \to M$. Thus, let f be such a diffeomorphism and g a metric on M, and suppose g transforms into g^f under f. Then $g(u) = (1-u)g + ug^f$ is a one parameter family of metrics on M. It is intimately connected with the mapping cylinder $E = (M\times S^1)_f$ discussed earlier; one can take the metric of E to be $ds^2 = du^2 + g_{ij}(u)\ dx^i dx^j$.

A one parameter family of maps of S^1 into a one parameter family of space times can be defined as a map $\phi:(S^1\times S^1) \to E$ which is of the special form $\phi(\tau,u) = (x(\tau,u),u)$. A fancier way to consider this is as follows. The mapping cylinder E is a fiber bundle over S^1, the fibration $\beta:E \to S^1$ being defined by $\beta(x,u)=u$. Likewise, $S^1\times S^1$ is a fiber bundle over the second S^1, the fibration $\alpha:S^1\times S^1 \to S^1$ being simply $\alpha(\tau,u)=u$. By a one parameter family of maps of S^1 into a one parameter family of space-times we mean simply a map $\phi:S^1\times S^1 \to E$ such that the diagram

$$\begin{array}{ccc} S^1\times S^1 & \xrightarrow{\ \phi\ } & E \\ & {\scriptstyle\alpha}\searrow \quad \swarrow{\scriptstyle\beta} & \\ & S^1 & \end{array} \qquad (17)$$

commutes, in the sense that $\beta\phi = \alpha$.

Physically, this means the following. The map $\phi(\tau,u)$ defines (for each u) a curve in M. As u varies, both the curve and the metric of M vary. But at u=1, the curve and metric are the same as at u=0, up to a diffeomorphism of M. As before, we can calculate $H(u) = \sqrt{\det Y(u)}$ as a function of u. If all is well on heaven and earth, we may expect H(1) = H(0).

If instead H(1) = -H(0), what conclusion should we draw? One might be tempted to conclude that the theory is inconsistent if $H(1) \neq H(0)$ for some choice of E and ϕ. This is far from being the right interpretation. To understand this requires a brief digression.

M is a spin manifold (if the theory under discussion makes sense), so it has one or more spin structures. In studying fermions on M, we choose a spin structure μ.* A diffeomorphism $f:M \to M$ maps μ either into itself or into another spin structure $\tilde{\mu}$. If $f(\mu) = \mu$, μ can be extended to a spin structure on $E = (M\times S^1)_f$. Such an extension is not possible if $f(\mu) \neq \tilde{\mu}$.

What diffeomorphisms $f:M \to M$ are symmetries of our theory? One necessary condition is that $f(\mu) = \mu$. If $f(\mu) \neq \mu$, that means that f changes the boundary conditions obeyed by the fermions, and so is not a symmetry of the theory formulated with spin structure μ.

Returning to the map ϕ in (17), we want to know what it means if $H(1) = -H(0)$ for some ϕ and E. Our previous discussion of global anomalies shows that this means that E is not a spin manifold. This implies that $f(\mu) \neq \mu$ (since, as we just discussed, if $f(\mu) = \mu$, then E inherits a spin structure from the spin structures of M and S^1). But--as we discussed a moment ago--if $f(\mu) \neq \mu$, then f is not a symmetry of the theory with spin structure μ. So this is the interpretation of finding that $H(1) = -H(0)$: it means that the diffeomorphism f is not a symmetry of the theory. There is no paradox in this; a diffeomorphism f that behaves as described here never has compact support in space-time (once time is decompactified). What we have discovered is that global anomalies in a one parameter family of maps into a one parameter family of space-times are one symptom of how a classical symmetry may fail to be a quantum symmetry. What has been described here is a complex way of looking at a relatively simple restriction on diffeomorphisms (a true symmetry f must leave the spin structure fixed), but it will stand us in

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*As discussed later, choosing a spin structure μ means deciding whether fermions propagate around non-contractible loops with periodic or anti-periodic boundary conditions. The spectrum of elementary particle masses depends on μ. We do not sum over μ. The situation will be completely different later when we consider spin structures on the string world sheet, since the physical role of the world sheet is completely different from that of space-time.

good stead later when we discuss the analogous phenomenon in string theory.

Although somewhat outside our line of development, I would like to pause at this point to discuss a purely mathematical application of the global anomaly in $\sqrt{\det Y}$. Let M be a manifold, and let $\Omega(M)$ be the space of all (oriented, unbased) maps $S^1 \to M$. We wish to ask whether the infinite dimensional manifold $\Omega(M)$ is orientable. To answer this we must find a definition of orientability in the finite dimensional case which makes sense in the infinite dimensional situation.

One notion of orientability for finite dimensional manifolds is that an n dimensional manifold Q is orientable if it admits a real, everywhere non-zero n form (volume element) ε. This notion does not seem to generalize to the infinite dimensional case. For even n, another criterion for orientability is that a manifold Q is orientable if it admits a (real) two form ω which is everywhere nondegenerate. (Nondegeneracy of ω means that at any point $p \varepsilon Q$, for any non-zero tangent vector V^i, $V^i \omega_{ij} \neq 0$. If ω is everywhere nondegenerate, the n form $\varepsilon = \omega^{n/2}$ is everywhere non-zero, so our first criterion for orientability is obeyed.) This notion is rather narrow since even an orientable, even dimensional manifold does not necessarily admit an everywhere nondegenerate two form ω. However, any n dimensional manifold Q admits (if n is even) a two form ω that is nondegenerate except on a subspace of dimension n-1. Introducing a Riemannian metric and raising an index gives a matrix $\omega^i{}_j = g^{ik}\, \omega_{kj}$. It is easy to see that, for any closed loop σ in Q, $\sqrt{\det \omega}$ changes sign in traversing σ if and only if the orientation of Q changes sign in traversing σ. So a finite dimensional manifold Q (of even dimension) is orientable if, picking any two form ω that is smooth and generically nondegenerate, $\sqrt{\det \omega}$ can be defined smoothly throughout Q.

This notion is readily implemented for $\Omega(M)$. Indeed, $Z = iY$ is a real, skew hermitian operator that can be interpreted as a two form on $\Omega(M)$. To see this, we must ask what is a tangent vector at a point γ in $\Omega(M)$. A point γ in $\Omega(M)$ is a loop $x^i(\tau)$ in M. A tangent vector to γ is an infinitesimal displacement $\delta x^i(\tau)$ or $\delta \tilde{x}^i(\tau)$ of this loop. Then the quantity

$$Z(\delta x, \delta\tilde{x}) = \int_0^{2\pi} d\tau\ \delta x^i(\tau)\ \left(g_{ij} \frac{d}{d\tau} + \frac{dx^k}{d\tau}\ \omega_{kij}\right)\ \delta\tilde{x}^j(\tau)$$

is bilinear, and $Z(\delta x, \delta\tilde{x}) = -Z(\delta\tilde{x}, \delta x)$, so Z defines a two form on $\Omega(M)$. We thus can consider $\Omega(M)$ to be orientable if $\sqrt{\det Z}$ is globally definable. We know the criterion for this--M must be a spin manifold. So we conclude that it is reasonable to say that $\Omega(M)$ is orientable if and only if M is a spin manifold.

Returning now to our main theme, we want to discuss--and to implement as far as possible--the steps analogous to (a), (b), and (c) above in the context of string theory. At least some of the steps should be evident:

(a') Analogous to anomalies in the propagation of a single point particle, we must discuss anomalies in the propagation of a single string. While world-line anomalies probe whether M has a spin structure, world-sheet anomalies will probe certain analogous obstacles to consistency of string theory.

(b') Analogous to anomalies in the Dirac determinant which enters the second quantized Dirac field theory, we will have to study anomalies in some generalization of the Dirac determinant that will enter the second quantized string field theory.

(c') After settling questions of consistency, we will want to study more general anomalies that affect the question of which apparent symmetries are actually valid.

As one might expect, the richness of string theory makes all of these questions much more difficult than their field theory analogues. It is possible to give a fairly thorough discussion of (a') and to say something about the other subjects. We will discuss these matters in turn.

First we will discuss (a')--anomalies in the propagation of a single string. This question is most interesting in the heterotic case,[9] since in that case the measure associated with the Euclidean world-sheet integrals is complex. We will discuss global world-sheet anomalies using the covariant, conformal gauge choice discussed in the latter part of the second paper in [9], since the expression given there for the measure is quite convenient for our purposes. The fact that supersymmetry is not manifest in this formalism is irrelevant for our purposes.

Two of the basic ingredients in world-sheet integrals are a Riemann surface Σ (which possesses a conformal structure over which we must integrate) and a map $\phi:\Sigma \to M$ of Σ into the space-time manifold M. Global anomalies always involve a one parameter family of objects of some kind. Global anomalies arise when the function space over which one is integrating is not simply connected, and the effective action does not return to its original value (modulo $2\pi i$) in traversing a non-contractible loop in function space. In string theory, there are several possibilities for which one parameter family of objects one may consider:

(i) One can investigate a one parameter family of Riemann surfaces. This arises if one is given a topologically non-trivial diffeomorphism $h:\Sigma\to\Sigma$. Generically, the metric g of Σ will be transformed into some other metric g^h by h. This leads in the usual way to a one parameter family of metrics on Σ, $g^u = (1-u)g + ug^h$, $0<u<1$, which induce a one parameter family of conformal structures on Σ. This family is conveniently studied in terms of the mapping cylinder $(\Sigma\times S^1)_h$.

(ii) Keeping Σ fixed, one can study a one parameter family of maps of Σ into space-time. This amounts to consideration of a map $\phi:\Sigma\times S^1\to M$.

(iii) One can let both Σ and the map into space-time vary. One is then dealing with a one parameter family of maps of a one parameter family of Riemann surfaces into space-time. This amounts to the consideration of a map $\phi:(\Sigma\times S^1)_h\to M$.

In principle, (i) and (ii) are special cases of (iii). ((i) corresponds to the case in which ϕ maps $(\Sigma\times S^1)_h$ to a point, and (ii) to the case h=1.) However, (i) and (ii) are such natural special cases that it is reasonable to single them out. It is (ii) that corresponds most directly to the question which arises--and was discussed above--for a point particle. Neither (i) nor (ii) nor (iii) is vacuous. Anomalies of type (i) were shown in [9] to lead, at the one loop level, to the requirement that the Yang-Mills gauge group be O(32) or $E_8\times E_8$--the same conclusion that follows from consideration of perturbative anomalies in space-time.[3] We will show below that no new information comes from anomalies of type (i) at the multiloop level. As for anomalies of type (ii), they lead to the quantization of the Wess-Zumino interaction, a phenomenon that will be explored elsewhere,[10] and to a restriction on the allowed values of magnetic charge.[11] Anomalies of type (iii) have not been considered before (except for the special cases of type (i) and (ii)); we will see later that they lead to new constraints on allowed compactification.

Considering first (i), let us recall the formula of [9] for the effective world-sheet measure. For heterotic superstrings, there are ten right-moving fermions ψ_i, i=1...10. They come with a spin structure α (one must sum over α*), and contribute to the measure a factor $(\det_\alpha i\not{\partial}\,(\frac{1+\bar{\rho}}{2}))^{10}$. Here $\frac{1+\bar{\rho}}{2}$ is the chirality projection operator, and the notation $\det_\alpha$ is meant to emphasize that the determinant depends on the choice of spin structure α. There is also a Rarita-Schwinger ghost, of the same chirality. It has the same spin structure α, and contributes a factor $(\det_\alpha R\,(\frac{1+\bar{\rho}}{2}))^{-1}$, where R is the Rarita-Schwinger operator.** Finally, if one realizes the gauge group by fermions, there are two groups of sixteen left-moving fermions with (in general) two different spin structures β and γ. They contribute $(\det_\beta i\not{\partial}\,(\frac{1-\bar{\rho}}{2}))^{16}$ $\cdot(\det_\gamma i\not{\partial}\,(\frac{1-\bar{\rho}}{2}))^{16}$. For $E_8\times E_8$ gauge group, one sums independently over β and γ; for O(32), they are restricted to obey $\beta=\gamma$. The effective measure is hence

$$S = (\det_\alpha i\not{\partial}(\tfrac{1+\bar{\rho}}{2}))^{10}\,(\det_\beta i\not{\partial}(\tfrac{1-\bar{\rho}}{2}))^{16}\,(\det_\gamma i\not{\partial}(\tfrac{1-\bar{\rho}}{2}))^{16}\,(\det_\alpha R(\tfrac{1+\bar{\rho}}{2}))^{-1} \tag{18}$$

This expression is free of perturbative anomalies, as was noted in [9] (using formulas in [2] to compare the Rarita-Schwinger and Dirac contributions).

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* Summing over α is a way to project onto the supersymmetric sector of states of definite G parity.

**R is the operator $\psi_\mu+i\not{\partial}\psi_\mu$, ψ_μ being a vector-spinor.

Now, we will certainly not try to prove here that the multiloop diagrams are physically acceptable in all respects. We will limit ourselves to the narrower problem of proving that S, in (18), is invariant under any diffeomorphism $h:\Sigma\to\Sigma$. This has already been proved (using explicit formulas for the determinants and standard theorems about theta functions) at the one loop level,[9] so the only novelty is the generalization to higher loops.

S depends on α, β, and γ and on the metric g of Σ. We will prove that $S(\alpha,\beta,\gamma,g) = S(\alpha,\beta,\gamma,g^h)$ if h is any diffeomorphism that leaves fixed α, β, and γ (g^h is the conjugate of g by h). Whether this is so or not is a well-defined question, since one can interpolate continuously from g to g^h. Once it is established that $S(\alpha,\beta,\gamma,g)$ is invariant under any diffeomorphism that leaves fixed α, β, and γ, no new information comes from considering a diffeomorphism h that maps α, β, and γ into other spin structures α^h, β^h, and γ^h. One can just define $S(\alpha^h, \beta^h,\gamma^h,g^h) = S(\alpha,\beta,\gamma,g)$. There is no way to test this statement, since as spin structures are discrete, there is no way to interpolate smoothly from α, β, and γ to α^h, β^h, and γ^h. So what really must be done is to prove that $S(\alpha,\beta,\gamma,g) = S(\alpha,\beta,\gamma,g^h)$ if h is such that α, β, and γ are invariant under it.

We first consider the special case $\alpha=\beta=\gamma$. As $\det_\alpha i\not{D}\left(\frac{1+\bar{\rho}}{2}\right) \cdot \det_\alpha i\not{D}\left(\frac{1-\bar{\rho}}{2}\right)$ is real, positive, and anomaly free, S simplifies to

$$\tilde{S} = \left(\det_\alpha i\not{D}\left(\frac{1-\bar{\rho}}{2}\right)\right)^{22} \left(\det_\alpha R\left(\frac{1+\bar{\rho}}{2}\right)\right)^{-1} \qquad (19)$$

In view of the basic equation (2) for global anomalies, the change in $\tilde{S}$ under h is

$$\Delta \ln \tilde{S} = \frac{i\pi}{2}\left(22\eta(D) + (\eta(R) - \eta(D))\right) \qquad (20)$$

where $\eta(D)$ and $\eta(R)$ are the eta invariants of the Dirac and Rarita-Schwinger operators on the mapping cylinder $Q = (\Sigma\times S^1)_h$.* Now we use

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* The appearance of $(\eta_R-\eta_D)$ as the Rarita-Schwinger contribution (rather than η_R as one might have guessed) follows from the fact that a three dimensional vector is a two dimensional vector plus scalar. A similar combination appears in (21) below for the same reason. See [4] for further discussion.

the fact* that the spin cobordism group is trivial in three dimensions, so Q is the boundary of a four dimensional spin manifold B. The Atiyah-Patodi-Singer theorem[5] then asserts that

$$\frac{1}{2}\eta(D) = \text{index}(D) - \int_B \hat{A}(R)$$

$$\frac{1}{2}\eta(R) = (\text{index}(R) - \text{index}(D)) - \int_B (K(R) - \hat{A}(R)) \qquad (21)$$

Here index(D) and index(R) are the Dirac and Rarita-Schwinger index on B (with boundary conditions explained in [5]); $\hat{A}$ and K are the curvature polynomials whose integral over a four manifold <u>without</u> boundary would equal index(R) and index(D). Substituting (21) in (20), we may modulo $2\pi i$ drop index(D) and index(R), since they are even in four dimensions (as explained for instance in [4]), so we get

$$\Delta \ell n \tilde{S} = i\pi \int_B (-20\,\hat{A} - K) \bmod 2\pi i \qquad (22)$$

This vanishes because $K = -20\hat{A}$ in four dimensions.**

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* An oriented manifold has $w_1=0$; if it has a spin structure, then $w_2=0$ in addition. (Here w_k are Stiefel-Whitney classes.) If $w_1=w_2=0$, the tangent bundle is trivial when restricted to the two skeleton, and hence (since $\pi_2(O(N))=0$) it is also trivial when restricted to the three skeleton. Hence $w_3=0$ if $w_1=w_2=0$. Consequently, a three dimensional (oriented) spin manifold has all Stiefel-Whitney classes and all Stiefel-Whitney (or Z_2) characteristic numbers zero. Hence ([12], p. 42) such a manifold (if it has no boundary) bounds an oriented manifold. Except in 8k+1 or 8k+2 dimensions, every spin manifold that bounds an oriented manifold bounds a spin manifold ([12], pp. 46-7). So every three dimensional (closed, oriented) spin manifold is the boundary of a spin manifold.

** $\hat{A}$ corresponds to the term of order x_i^2 in $\prod_{i=1}^{2} \frac{x_i/2}{\sinh x_i/2}$, and K to the term of order x_i^2 in $\prod_{i=1}^{2} \frac{x_i/2}{\sinh x_i/2} \cdot \sum_{j=1}^{2} (2\cosh x_j)$.

We still must consider the possibility that α, β, and γ are not all equal. The correction factor between S and $\tilde{S}$ is

$$U_0 = \left(\frac{\det_\beta\ i\not{D}\ (\frac{1-\bar{\rho}}{2})}{\det_\alpha\ i\not{D}\ (\frac{1-\bar{\rho}}{2})}\right)^{16} \left(\frac{\det_\gamma\ i\not{D}\ (\frac{1-\bar{\rho}}{2})}{\det_\alpha\ i\not{D}\ (\frac{1-\bar{\rho}}{2})}\right)^{16} \tag{23}$$

and we must prove that this is invariant under any diffeomorphism $h:\Sigma\to\Sigma$ that fixes α, β, and γ. Actually, we will prove the stronger statement that

$$U = \left(\frac{\det_\beta\ i\not{D}\ (\frac{1-\bar{\rho}}{2})}{\det_\alpha\ i\not{D}\ (\frac{1-\bar{\rho}}{2})}\right)^{8} \tag{24}$$

is invariant under any diffeomorphism h that fixes α and β. Eight is incidentally the lowest exponent for which this is true, as one loop calculations show.

Since α and β are invariant under h, they both extend to spin structures (which we will also call α and β) on the mapping cylinder $T = (\Sigma\times S^1)_h$. We have two different Dirac operators on T (with spin structures α and β); the two η invariants associated with these two Dirac operators will be called $\eta_\alpha(D)$ and $\eta_\beta(D)$. In view of equation (2), the change in U under a diffeomorphism is

$$\Delta \ln U = 8\,\frac{i\pi}{2}\left(\eta_\alpha(D) - \eta_\beta(D)\right) \bmod 2\pi i \tag{25}$$

The general theory for calculating differences such as $\eta_\alpha(D)-\eta_\beta(D)$ has been developed by Atiyah, Patodi, and Singer in the last paper in [5]. As we do not wish to calculate $\eta_\alpha(D)-\eta_\beta(D)$, but only to prove that it is an integer multiple of 1/2 (so that (25) vanishes modulo $2\pi i$), a short cut is available. Actually, for <u>any</u> orientable three manifold T (not necessarily a mapping cylinder) and any two spin structures α and β, we will prove

$$8\,\frac{\pi}{2}\left(\eta_\alpha(T) - \eta_\beta(T)\right) = 0 \bmod 2\pi \tag{26}$$

A key to proving this is to understand what is the nature of the difference between two spin structures α and β. When a vector is parallel transported around a closed curve γ in T, it is rotated by a rotation matrix $R_\gamma \epsilon$ O(3). When a spinor (with spin structure α or β) is parallel transported around γ, it returns rotated by a matrix $R_\gamma(\alpha)$ or $R_\gamma(\beta)$ which is a matrix that represents the same R_γ in the spinor representation. Since the spinor representation is double-valued, it is not necessarily so that $R_\gamma(\beta)$ equals $R_\gamma(\alpha)$. In general $R_\gamma(\beta) = (-1)^{n(\gamma)} \cdot R_\gamma(\alpha)$, where $n(\gamma)=\pm 1$ for each γ. The mapping $\gamma \to n(\gamma)$ is a homomorphism $\pi_1(T)\to Z_2$. Given one spin structure α, possible choices of another spin structure β are in one to one correspondence with homomorphisms $\mu:\pi_1 \to Z_2$.

Our goal will be to represent the difference between α and β spin structures as an interaction with an auxiliary SU(2) or O(3) gauge field that will have $F_{ij}{}^a=0$, so it will just enter in modifying the law of parallel transport around closed loops. To this end, think of a mapping $\nu:T\to O(3)$. Any such mapping ν maps a closed curve γ in T into a closed curve $\nu(\gamma)$ in O(3), so it induces a homomorphism $\pi_1(T)\to\pi_1(O(3)) \approx Z_2$. We want to show that <u>every</u> homomorphism $\mu:\pi_1(T)\to Z_2$ is induced in this way from a mapping $\nu:T\to O(3)$. To this end, pick a triangulation of T.* This means roughly that we realize T as a collection of tetrahedra glued together on their faces. The vertices of the tetrahedra are called 0-simplices S_0, the edges are called 1-simplices S_1, the faces are called 2-simplices S_2, and the interiors are called 3-simplices S_3. The boundary ∂S_q of a q-simplex S_q is topologically a q-1 sphere S^{q-1}. If for some q simplex S_q a mapping $\nu_0:\partial S_q \to O(3)$ has been defined, the obstruction to extending ν_0 over S_q is an element of $\pi_{q-1}(O(3))$.

Now, given a homomorphism $\mu:\pi_1(T)\to\pi_1(O(3)) \approx Z_2$, we wish to find a mapping $R:T\to O(3)$ which "induces" μ in the manner described in the last paragraph. The strategy is to first define R on 0-simplices, then on 1-simplices, then on 2-simplicies, and finally on 3-simplices. On 0-simplices, one may just take $\nu(x)=1$ for any 0-simplex x. On 1-simplices, the definition of R is determined up to homotopy by μ. (Pick

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*This is a standard concept in topology; see for instance [13]. A brief exposition was given in [11].

any ordering of the one simplices. Define R first on the "first" one simplex, then on the "second," and finally on the last. At each stage the choice of R is arbitrary unless the 1-simplex which is being added completes a closed loop, in which case μ determines a topological restriction on R which must be obeyed. There never is an inconsistency in obeying these restrictions since μ is a homomorphism of $\pi_1(T)$ into $\pi_1(O(3)) \approx Z_2$.) Now we try to extend R over two-simplices. The boundary of a two-simplex y is a curve γ. γ is trivial in $\pi_1(T)$ (since it bounds y) so $\mu(\gamma)=0$ in $\pi_1(O(3))$. Hence, having defined R on one simplices to induce the homomorphism μ of fundamental groups, the mapping $R:\gamma\to O(3)$ is topologically trivial for any γ which bounds y and can be extended to $R:y\to O(3)$. Having thus defined R over two simplices, it can always be extended over three simplices, since the obstruction would lie in $\pi_2(O(3))=0$. This completes the proof that every homomorphism $\mu:\pi_1(T)\to Z_2 \approx \pi_1(O(3))$ is induced by some mapping $R:T\to O(3)$.

Now, we will wish to study the SU(2) gauge field $A_i = R^{-1}\partial_i R$. In integer spin representations of SU(2), this is completely trivial, a pure gauge. In half integer spin representations, R (being double-valued) is not well defined, but $A_i = R^{-1}\partial_i R$ is well-defined. However, in half integer spin representations of SU(2), A_i is trivial only locally, not globally. In such a representation, for any closed curve γ,

$$P \exp \int_\gamma A\cdot dx = (-1)^{n(\gamma)} \tag{27}$$

where $\gamma\to n(\gamma)$ is our homomorphism μ.

Now we wish to pick two representations P and Q of SU(2) with the following properties:

(x) They are real and of the same dimension.

(y) They have the same quadratic Casimir operator.

(z) In P the center of SU(2) is represented by +1, and in Q by -1.

The lowest dimension for which such representations exist is 8. This is why U in (24) will turn out to be single-valued with the exponent 8. The minimal choice is that P should be the $3 \oplus 1 \oplus 1 \oplus 1 \oplus 1 \oplus 1$ of SU(2), while Q is $2 \oplus 2 \oplus 2 \oplus 2$.

To evaluate (25), we must understand the quantities $8\eta_\alpha$ and $8\eta_\beta$. As regards $8\eta_\alpha$, it is the eta invariant for eight spinors with spin structure α. There is no harm in saying they lie in the P representation of SU(2) and interact with the gauge field $A_i = R^{-1}\partial_i R$--since that gauge field is trivial in that representation. So we say

$$8\eta_\alpha = \eta'_{\alpha P} \tag{28}$$

where $\eta_{\alpha P}$ is the eta invariant for a fermion in the P representation of SU(2). The eight fermions interacting with spin structure β, on the other hand, will be treated in a slightly less trivial way. Eight fermions with spin structure β are exactly equivalent to eight fermions in the Q representation of SU(2) with spin structure α--since the not-quite-trivial SU(2) gauge field $A_i = R^{-1}\partial_i R$ has, in the Q representation, the sole effect of flipping the boundary condition, turning spin structure α into β. So

$$8\eta_\beta = \eta_{\alpha Q} \tag{29}$$

Now we will use the Atiyah-Patodi-Singer theorem to evaluate $\eta_{\alpha Q}$ and $\eta_{\alpha P}$ --or at least their difference. Let $W = T\times I$; I is a unit interval with parameter u, $0<u<1$. In contrast to the rather trivial gauge fields we have been discussing, we now consider a highly non-trivial instanton gauge field on W. Let $B_i(x_k,u) = uA_i(x_k)$, $B_u = 0$ (x_k being coordinates for T). This B_i interpolates from 0 at u=0 to A_i at u=1. The Atiyah-Patodi-Singer applied to the manifold W with gauge field B gives

$$\frac{1}{2}\left(\eta_{\alpha P} - 8\eta_\alpha\right) = \text{index}_P(D) - \int_W \hat{A}_P$$

$$\frac{1}{2}\left(\eta_{\alpha Q} - 8\eta_\alpha\right) = \text{index}_Q(D) - \int_W \hat{A}_Q \tag{30}$$

Here $\text{index}_P(D)$ and $\text{index}_Q(D)$ are the index of the Dirac operator on W (with spin structure α) in the P or Q representation. These are even, since P and Q are real. $\hat{A}_P$ and $\hat{A}_Q$ are the curvature polynomials related to the Dirac index in four dimensions; they are equal, since we chose P and Q to have the same quadratic Casimir operator. The left-hand side of (30) involves $\eta_{\alpha P}-8\eta_\alpha$ and $\eta_{\alpha Q}-8\eta_\alpha$ since the boundary of W has two components; the u=1 component contributes $\eta_{\alpha P}$ or $\eta_{\alpha Q}$, while the u=0 component contributes $8\eta_\alpha$. Subtracting the two equations in (30) and using the facts just noted, we learn

$$\eta_{\alpha P} - \eta_{\alpha Q} = 0 \text{ modulo } 4 \tag{31}$$

Equations (28), (29), and (31) imply that $\Delta \ell n U$ in (25) is zero. This completes the proof that the measure $S(\alpha,\beta,\gamma;g)$ is invariant under diffeomorphisms tht leave α, β, and γ fixed.

This proof required only very general considerations and few detailed facts about Riemann surfaces. Of course, it leaves open many other questions tied to the consistency of the theory whose resolution may require deeper knowledge of Riemann surfaces. For instance, it is expected from supersymmetry that $\sum_{\alpha,\beta,\gamma} S(\alpha,\beta,\gamma;g) = 0$ but I do not think that this can be proved using only the methods above.

Returning to our list of problems, this completes what we will say here about anomalies involving purely a one parameter family of conformal structures on Σ. We now turn very briefly to discuss anomalies involving mappings into space-time of a fixed Riemann surface Σ. I will not describe any new examples of anomalies (beyond those described in [10] and [11]), but I will comment briefly on the setting for this problem.

As soon as we consider non-trivial maps ϕ of Σ into the space-time M, the spin connection and gauge field in space-time become highly relevant. The spin connection $\omega_i{}^k{}_\ell$ is an O(10) connection on the tangent bundle T of M. Its pull-back to Σ is an O(10) gauge field $a_\alpha{}^k{}_\ell = \frac{\partial x_i}{\partial \sigma^\alpha}\omega_i{}^k{}_\ell$. The factor $\left(\det_\alpha i\not{\partial}\left(\frac{1+\bar\rho}{2}\right)\right)^{10}$ in (18) becomes replaced by $\det_{\alpha T} i\not{D}\left(\frac{1+\bar\rho}{2}\right)$ where $\det_{\alpha T}$ is the determinant for ten fermions (with spin structure α) interacting with a_α. Henceforth we will abbreviate positive or negative chirality determinants as $\det_\pm$. Likewise we have in spacetime an $E_8 \times E_8$ bundle $V = V_1 \oplus V_2$ with the two E_8 gauge fields

$A_{(1)i}$, $A_{(2)i}$. There is no convenient way for E_8 gauge fields to act on 16 fermions. But pragmatically, on the four skeleton of M (all that enters in discussing world-sheet anomalies) the structure group of any E_8 bundle can be reduced to O(16). So for our limited purposes we probably lose little in assuming $A_{(1)i}$ and $A_{(2)i}$ are O(16) gauge fields. Anyway, at present it is the best we can do. Then $A_{(1)}$ and $A_{(2)}$ can be pulled back to gauge fields $B_{(1)\alpha} = \frac{\partial x^i}{\partial \sigma^\alpha} A_{(1)i}$, $B_{(2)\alpha} = \frac{\partial x^i}{\partial \sigma^\alpha} A_{(2)i}$ on Σ. The factors in (18) involving $(\det_\beta)^{16}$ and $(\det_\gamma)^{16}$ become $\det^{(-)}_{\beta V_1}$ and $\det^{(-)}_{\gamma V_2}$. In addition, one more potentially anomalous term must be included in the measure. In space-time there is a two form B_{ij}. Its interaction with the string is $i\int d^2\sigma\, \epsilon^{\alpha\beta}\, \partial_\alpha x^i\, \partial_\beta x^j\, B_{ij}(x(\sigma))$, which we will abbreviate as $i\int_\Sigma \phi^* B$. ($\phi^* B$ is the pullback of B from M to Σ via ϕ.) So the measure is

$$S = (\det{}^+_{\alpha T})(\det{}^-_{\beta V_1})(\det{}^-_{\gamma V_2})(\det{}^+_\alpha R)^{-1} \exp i \int_\Sigma \phi^* B \tag{32}$$

Now, one difference between the string and the point particle is that in the particle case we only had to consider global anomalies on the world line, but in the case of the string both perturbative and global anomalies on the world sheet must be considered. Indeed, the perturbative anomalies cancel [9] if one considers a trivial map of Σ into space-time. But as soon as one considers a non-trivial map it appears, at first sight, that the anomaly cancellation would be spoiled by so-called sigma model anomalies[14] involving the spin connection and gauge field of space-time. This would indeed be so if we considered only the product of determinants in (32). But the last factor involving B saves the day.[15] Indeed, the sigma model anomaly can be canceled in (32) if one accompanies a gauge and local Lorentz transformation in space-time with

$$\delta B = \mathrm{tr}\,(A d\Lambda) - \mathrm{tr}\,(\omega d\theta) \tag{33}$$

Here Λ and θ are the parameters of infinitessimal gauge and local Lorentz transformations. This is the same gauge transformation law that cancels anomalies in space-time--a satisfying relation between space-time and world-sheet anomalies.

(33) implies that the gauge invariant field strength H of B obeys not dH=0 but

$$dH = \mathrm{Tr}\, F^2 - \mathrm{tr}\, R^2 \tag{34}$$

This, in turn, has a certain topological interpretation [16]. In de Rham cohomology, dH=0 while $\mathrm{Tr}F^2$ and $\mathrm{tr}R^2$ represent the first Pontryagin classes (with real coefficients) of the $E_8 \times E_8$ bundle V and the tangent bundle T. So (34) implies that the theory is only consistent if T and V are such that

$$P_1(V;R) = P_1(T;R) \tag{35}$$

Here $P_1(\ ;R)$ denotes Pontryagin classes with real coefficients. It is satisfying to see that the world-sheet theory is "aware" of this relation, which also follows from space-time considerations.

Now, Pontryagin classes with real coefficients have a natural generalization--Pontryagin classes $P_1(\ ;Z)$ with integer coefficients. These entered and were explained in [11]. (In that paper, P_1 was sometimes called the second Chern class C_2--which is equivalent for real bundles.) It was found there that magnetic monopoles are not permitted in string theory unless their magnetic charge is such as to not contribute to $P_1(V;Z)$. This result can be unified with (35) if we assume that the actual requirement for consistency of the theory is*

$$P_1(V;Z) = P_1(T;Z) \tag{36}$$

Although I believe that (36) is probably needed in generality for consistency of the theory, it would be beyond the scope of the present notes to try to prove this in full. Many technicalities arise, some of which will be mentioned later. I will limit myself to providing new evidence for (36) by showing that it is required for consistency in a

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*By $P_1(F;Z)$ for an E_8 bundle F I mean the fourth cohomology class that is the first obstruction to triviality of F. If F is induced from an O(16) bundle $\tilde{F}$ by the embedding $O(16) \subset E_8$, $P_1(F;Z)$ should be taken to mean $P_1(\tilde{F};Z)$.

new physical situation. In the situation I have in mind, little can be learned by considering a one parameter family of maps of a fixed Riemann surface into space-time (corresponding, as was described earlier, to a map $\phi:\Sigma\times S^1\to M$). It will be necessary to consider a one parameter family of maps of a one parameter family of Riemann surfaces into space-time. This corresponds to a map $\phi:T\to M$ where $T = (\Sigma\times S^1)_h$ is a mapping cylinder. Physically, the rationale for considering anomalies in this situation is as follows. If u is a parameter for S^1, then as u is varied, both Σ and its map into M vary. But on returning these to their original values, we require that the amplitude associated with the string propagation should return to its original value.

The situation we wish to consider is that of compactification of the ten dimensional theory on $M^4\times K$, M^4 being four dimensional Minkowski space and K some compact six manifold. We assume that the spin connection of K is embedded in one E_8 factor of the gauge group, breaking $E_8\times E_8$ to $E_6\times E_8$, $O(10)\times U(1)\times E_8$, or $O(10)\times E_8$ depending on whether K has SU(3), U(3), or O(6) holonomy.[17] (Actually, if K has U(3) holonomy, there are several other possibilities for the unbroken group[18].) By itself, this does not introduce any global anomalies, since embedding the spin connection in one E_8 factor gives a vector-like non-linear sigma model on the string world sheet.

If now K is not simply connected, it is possible to introduce grand unified symmetry breaking via Wilson lines. This means that one picks a homomorphism $\pi_1(K)\to G$ of the fundamental group of K into the unbroken group G. This breaks G to the subgroup that commutes with the image of $\pi_1(K)$ in G. The Wilson lines define a flat vector bundle V. What we wish to investigate is whether there are global anomalies associated with the choice of Wilson lines. Global anomalies, being a topological notion, can only detect topological invariants of V. By far the simplest topological invariant of V is its first Pontryagin class, so it is natural to look for global anomalies associated with the Wilson lines via $P_1(V;Z)$.

Rather than trying to be general, we will consider a simple example. Let $K = (S^5/Z_n)\times S^1$. Here S^5/Z_n is the "lens space" consisting of three complex variables (Z_1,Z_2,Z_3) with $|Z_1|^2 + |Z_2|^2 + |Z_3|^2 = 1$

and with (Z_1,Z_2,Z_3) considered equivalent to $(Z_1\alpha,Z_2\alpha,Z_3\alpha)$ where $\alpha =$ exp $22\pi i/n$. Ignoring the S^1 factor, we will consider Wilson lines associated with S^5/Z_n only. As $\pi_1(S^5/Z_n) = Z_n$, we must pick a single Wilson line U obeying $U^n=1$. It always fits into a subgroup $U(8) \subset O(16) \subset E_8$, and in that subgroup it can be written

$$U = \begin{pmatrix} \exp 2\pi i k_1/n & & & \\ & \exp 2\pi i k_2/n & & \\ & & \ddots & \\ & & & \exp 2\pi i k_8/n \end{pmatrix} \qquad (37)$$

Actually, we have two such matrices, one for each E_8, and one of them is restricted to commute with the embedded spin connection (requiring $k_6=k_7=k_8$ if K has SU(3) holonomy). We want to calculate global anomalies associated with the choice of U.

A map $\phi:T\to K$ can (since $T=(\Sigma\times S^1)_h$ is three dimensional) only "cover" a three dimensional subspace of K. We will pick this subspace to be the subspace of S^5/Z_n defined by $Z_3=0$. This itself is a lens space $L=S^3/Z_n$.

Now, we want to choose a Riemann surface Σ and a map $h:\Sigma\to\Sigma$ such that we can find a degree one map $\phi:(\Sigma\times S^1)_h\to L$. We will pick Σ to be a torus with periodic coordinates (τ,σ), $0<\sigma,\tau<2\pi$. As for h, it will be the map $h(\tau,\sigma) = (\tau+n\sigma,\sigma)$. Thus, $(\Sigma\times S^1)_h$ will consist of triples (τ,σ,u) with $(\tau,\sigma,0)$ identified with $(\tau+n\sigma,\sigma,1)$. As h has been chosen to leave σ invariant, $T = (\Sigma\times S^1)_h$ is actually in this case a fiber bundle over a torus $T_0 = S^1\times S^1$ spanned by σ and u. The fiber S^1 is spanned by τ.

L is also a fiber bundle over a two manifold, in this case a two sphere S^2 with coordinates $\vec{w} = Z^*\vec{\sigma}Z$ (as we set $Z_3=0$ to define L, there are two Z_i; $\vec{\sigma}$ are the standard Pauli matrices). From $\sum_i|Z_i|^2 = 1$ it follows that $\vec{w}^2=1$; the map $\{Z_i\}\to\{w_k\}$ is a fibration $L\to S^2$.

To pick a degree one map $\phi:T\to L$, we begin by picking a degree one map $\phi_0:T_0\to S^2$. There is a simple analytic formula for a degree two map $T_0\to S^2$ given by polar coordinates ($w_1 =$ sin$2\pi u$ cosσ, $w_2 =$ sin$2\pi a$ sinσ, $w_3 =$ cos$2\pi u$). Although there seems to be no simple analytic formula for a degree one map, such a map exists. (Pick a disc D in T_0 and map its boundary and exterior to a single point in S^2. D is thus effectively compactified to a sphere, which has a degree one map onto S^2. This completes the specification of the map $T_0 \to S^2$.) Now, L is an S^1 fiber bundle over S^2, so pulling back this fiber bundle via the map $\phi_0 : T_0 \to S^2$ gives an S^1 fiber bundle over T_0. The total space of this fiber bundle is a three manifold T fibered over T_0. ϕ_0 "lifts" automatically to a degree one map $\phi : T \to L$. Actually T was chosen earlier to be isomorphic to the manifold T just encountered, and ϕ is the desired degree one map $\phi : T \to L$.

Now, we want to calculate the global anomaly associated with the Wilson line U of equation (37). While E_8 can be realized on 16 real fermions, it is convenient, in view of the diagonal form in (37), to think of these as eight complex fermions ψ_a, a = 1...8 and their complex conjugates. A given fermion ψ_a interacts with an Abelian gauge field $A^{(a)}$ in space-time. $A^{(a)}$ is a pure gauge locally but has global holonomy $U = \exp 2\pi i k_a/n$. Now $A^{(a)}$ "pulls-back" to an abelian gauge field $A^{(a)}_\alpha = \frac{\partial x^i}{\partial \sigma^\alpha} A^{(a)}_i$ (here σ^α is τ, σ, or u). $A^{(a)}_\alpha$ is again a pure gauge locally. Its global holonomy can be inferred from that of $A^{(a)}$:

$$\exp i \int_0^{2\pi} d\tau \; A^{(a)}_\tau = \exp 2\pi i k_a/n$$

$$\exp i \int_0^{2\pi} d\sigma \; A^{(a)}_\sigma = \exp i \int_0^1 du \; A^{(a)}_u = 1 \qquad (38)$$

Now we wish to calculate the change in the effective action of the fermion ψ_a between u=0 and u=1. This is the variation under h of

$$X = \frac{\det^-_\beta \left(A^{(a)}\right)}{\det^-_\beta (0)} \qquad (39)$$

Here $\det^-_\beta (A_a)$ is a negative chirality fermion determinant with spin structure β and abelian gauge field $A^{(a)}$; $\det^-_\beta(0)$ is a negative chirality determinant without $A^{(a)}$. X is the factor by which the world sheet measure is corrected by the interaction of ψ_a with $A^{(a)}$. Since the theory was invariant under h at $A^{(a)}=0$, it is the correction factor X that we must study.

By our usual formula (2) (multiplied by a factor of 2 since ψ^a is complex while (2) refers to Majorana-Weyl fermions), the change in X under h is

$$\Delta \ln X = i\pi \left(\eta_\beta(A^{(a)}) - \eta_\beta(0)\right)_{\mathrm{mod} 2\pi i} \qquad (40)$$

where η_β $(A^{(a)}(0))$ and η_β are the η invariants on T with and without $A^{(a)}$.

Now, to evaluate (40), we will pick a manifold B whose boundary is T and use the Atiyah-Patodi-Singer theorem. B is conveniently chosen as follows. The circle S^1 with coordinate τ is the boundary of a disc D with polar coordinates (ρ,τ), $0 < \rho < 1$, $0 < \tau < 2\pi$. The S^1 bundle $T \to T_0$ can be extended to a disc bundle $B \to T_0$. (The metric for B may be, for instance, $ds^2 = d\rho^2 + \rho^2 (d\tau - nu d\sigma)^2 + d\sigma^2 + du^2$.) B has boundary T. Actually, B is a spin manifold only if n is even, and we will content ourselves with this case. (The more general case can be studied by using a spin_c structure on B.) Also, the hitherto arbitrary h invariant spin structure β on Σ must be chosen to be extendable over B.

The next step is to extend the U(1) gauge field $A^{(a)}$ on T to a U(1) gauge field (which we will also call $A^{(a)}$) on B. Since $A^{(a)}$ is topologically non-trivial, it is awkward to write an explicit formula for it. But a suitable formula for $F = dA^{(a)}$ is

$$F = \frac{k^{(a)}}{n} \left[d\rho \; (d\tau - nu \; d\sigma) - (1-\rho) \; ndu \; d\sigma\right] \qquad (41)$$

This formula has the following key properties:

(1) It is invariant under $u \to u+1$, $\tau \to \tau + n\sigma$, so it actually is defined on B, not just some covering space of B.

(2) $dF = 0$, so locally $F = dA$ for some A.

(3) Restricted to T ($\rho = 1$, $d\rho = 0$), $F = 0$. So the gauge field A associated with F is on T locally a pure gauge. It also has the global holonomy desired for $A^{(a)}$. For instance, $\exp i \int_0^{2\pi} d\tau \, A_\tau^{(a)} = \exp i \int_D F = \exp \frac{2\pi i \; k^{(a)}}{n}$, as desired.

Having found the right gauge field, we can evaluate (40) via the Atiyah-Patodi-Singer theorem. In fact

$$\frac{1}{2}\eta_\beta\,(A^{(a)}) = \text{index}_\beta\,(A^{(a)}) - \int_B \hat{A}\,(A^{(a)})$$

$$\frac{1}{2}\eta_\beta(0) = \text{index}_\beta(0) - \int_B \hat{A}\,(0) \qquad (42)$$

Here $\text{index}_\beta\,(A^{(a)})$ and $\text{index}_\beta(0)$ are the Dirac index with and without the gauge field $A^{(a)}$. Also, $\hat{A}(A^a)$ and $\hat{A}(0)$ are the polynomials related to the index theorem, with and without $A^{(a)}$. In fact $A(A^{(a)}) - A(0) = \frac{1}{2(2\pi)^2} F\wedge F$. So we get

$$\Delta \ell n X = -2\pi i \int_B (\hat{A}(A^a) - \hat{A}(0)) = -\frac{2\pi i}{2\cdot(2\pi)^2} \int_B F\wedge F$$

$$= \frac{2\pi i}{2\cdot(2\pi)^2} \int_0^1 d\rho \int_0^{2\pi} d\tau \int_0^{2\pi} d\sigma \int_0^1 du \left(\frac{k^{(a)}}{n}\right)^2 n(1-\rho)$$

$$= 2\pi i\,\frac{(k^{(a)})^2}{2n} \bmod 2\pi i \qquad (43)$$

The total global anomaly is obtained by summing (43) overall fermion species (a), and vanishes if*

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*At first sight, it is not clear why Wilson lines are such that $\frac{1}{2}\Sigma k^{(a)2}$ is always an integer. This is paradoxical, since with $\pi_1(K) = Z_n$, we must at most get mod n, not mod 2n, global anomalies. The resolution of the paradox is as follows. With $\pi_1(K) = Z_n$, the only Wilson lines that can be defined are those for which $U^n = 1$ not just in U(8) but also in E_8 when U(8) is embedded in E_8 via $U(8) \subset O(16) \subset E_8$. The key point is that the adjoint of E_8 contains the double-valued spinor of O(16). As a result of this, when n is even, all Wilson lines that can be defined in E_8 have $\Sigma_a k^{(a)}$ even--which always makes $\frac{1}{2}\Sigma k^{(a)2}$ an integer. When n is odd, so that B is not a spin manifold, there are further intricacies which we will not try to unravel.

$$\sum_a \frac{1}{2} k^{(a)2} = 0 \bmod n \tag{44}$$

But as was explained in [11], the contribution of the Wilson lines to $P_1(V;Z)$ is $\frac{1}{2} \sum_a k^{(a)^2} y^2$, where y is the generator of $H^2(K;Z)$. As $\ell y^2=0$ in $H^4(K;Z)$ if and only if $\ell=0 \bmod n$, equation (44) can be regarded as further evidence for the general validity of the requirement (36) that $P_1(V;Z) = P_1(T;Z)$ for consistency.

This calculation raises many questions. First of all, how would one generalize it to establish a universal requirement $P_1(V;Z) = P_1(T;Z)$? As this equation holds rationally because of perturbative anomalies, it is only the torsion piece that we have to worry about. A torsion part of P_1 is necessarily a torsion element of $H^4(K;Z)$, the fourth cohomology group of K with integer coefficients. By the universal coefficients theorem, this is related to a torsion class in $H_3(K;Z)$, the third homology group of K with integer coefficients. Such a class can be represented by an immersed three manifold L. To try to find a global anomaly that forces $P_1(V;Z) = P_1(T;Z)$, one must then look for a Riemann surface Σ, a map $h:\Sigma\to\Sigma$, and a degree one map $\phi:(\Sigma\times S^1)_h\to L$. If this can be found, one can try to repeat the above calculation and find a global anomaly. While I expect that this last step can be pushed through, I will not attempt it here. A more difficult problem arises because, in general, L is such that for any Σ and h there is no degree one map $\phi:(\Sigma\times S^1)_h\to L$. How then can a requirement on P_1 emerge? While I do not know the answer to this question, it is encouraging to note that a similar problem arises in the point particle case. A manifold M fails to be a spin manifold if there is an immersed two manifold N with $w_2(N)=0$. (w_2 is the second Stiefel-Whitney class of M.) Our discussion of global anomalies detected the failure of M to be a spin manifold only if there is some $N\varepsilon M$ with a degree one map $S^1\times S^1\to N$ and $w_2(N)\neq 0$. In practice, this means N must be S^2 or $S^1\times S^1$. A more refined treatment of the particle case, exhibiting the fact that the point particle theory is ill defined if there is <u>any</u> $N \varepsilon M$ with $w_2(N)\neq 0$, would hopefully generalize to a string theory argument establishing that (irrespective of the topology of L) consistency requires $P_1(V;Z) = P_1(T;Z)$.

This leads to another question. A three manifold L in space-time must be topologically non-trivial (and so non-zero in $H_3(K;Z)$) if a global anomaly is to arise from a degree one map $\phi:(\Sigma\times S^1)_h\to L$. But why must L be a torsion class in $H_3(K;Z)$? If not, L has nothing to do with $H^4(K;Z)$ or P_1. For instance, suppose we compactify not on $K = (S^5/Z_n)\times S^1$ but on $\tilde{K} = (S^3/Z_n) \times S^3$. We could still do the calculation leading to (43), using the subspace $L = S^3/Z_n$ of $\tilde{K}$. But L now has nothing to do with H^4 --and in fact $H^4(K;Z) = 0$. So is not this evidence that the global world sheet anomaly in general can not be interpreted in terms of Pontryagin classes? The answer to this question is rather surprising. If L is not a torsion class in $H_3(K;Z)$, but an element of infinite order, it is always possible to add a Wess-Zumino interaction whose integral over L is non-zero. Adding this term with suitable (fractional) coefficient, it cancels any anomaly associated with fermion determinants on L. This is a curious situation in which a Wess-Zumino interaction is needed for consistency.

Now let us briefly discuss the physical applications of these results. It must be stressed that the anomaly cancellation in (44) can involve a cancellation of a problem in one E_8 against an equal and opposite problem in the second E_8. In fact, it is intriguing to speculate that nature may operate in that way. In that case, Wilson lines carrying out grand unified symmetry breaking in "our" E_8 would define a topologically non-trivial bundle, and grand unified symmetry breaking would be unavoidable (in the sector of the theory based on this bundle) for topological reasons.

This completes what I will say about consistency conditions coming from anomalies on the string world sheet. Although much more remains to be done, it is plausible that a fairly complete picture will emerge in the not too distant future. When these questions are all resolved, we will have merely settled the preliminaries, analogous in the point particle case to making sure that M is a spin manifold. It is necessary then to move on and explore global anomalies in the full-fledged second quantized theory.

In the field theory case, this involves considering anomalies in the determinant of the operator $i\not{D}$. That operator has the following properties. In Minkowski space, its zero modes are the physical, on mass shell states. It flips the chirality, so that in chiral theories it maps physical modes of the right chirality into unphysical modes of the wrong chirality, and vice-versa. Here chirality is measured by an operator $\bar{\Gamma}$ that anticommutes with all the gamma matrices in $\not{D} = \Sigma_i \Gamma^i D_i$.

In string theory, then, we want an operator which (a) generalizes the Dirac operator; (b) vanishes for physical states; (c) anticommutes with some analogue of the chirality operator $\bar{\Gamma}$.

The operator we want is just the operator introduced by Ramond in his original attempt[19] to generalize the Dirac equation to string theory! It is often called the Dirac-Ramond operator. In a modern language, it is the supersymmetry operator on the string world sheet. In the original version, there was a single Dirac-Ramond operator for open strings and two for closed strings. In the heterotic case, there is a single right-moving supercharge on a closed string. Temporarily suppressing the degrees of freedom that carry gauge quantum numbers, the supercharge can be derived from the action with N = 1/2 supersymmetry[20,15]

$$I = \int d^2\sigma \, [g_{ij}(x(\sigma)) \, \partial_+ x^i \, \partial_- x^j + i\psi^i \, (g_{ij} \, \partial_- + \partial_- x^k \, \omega_{kij}) \, \psi^j \tag{45}$$

Here g_{ij} is the metric of space-time, $\partial_\pm = \frac{\partial}{\partial\sigma^\pm}$ where $\sigma^\pm = \frac{1}{\sqrt{2}}(\tau\pm\sigma)$, ψ^j are the fermions of the Ramond-Neveu-Schwarz model, and a possible Wess-Zumino interaction (plus an infinity of possible couplings to massive modes of the string) is being suppressed. The conserved charge is

$$Q = \int_0^{2\pi} d\sigma \; g_{ij} \, \psi^i \, \left(\frac{\partial x^j}{\partial\tau} + \frac{\partial x^j}{\partial\sigma}\right) = \int_0^{2\pi} d\sigma \; \psi^i \, \left(-i \, \frac{D}{Dx^i(\sigma)} + g_{ij} \, \frac{\partial x^j}{\partial\sigma}\right) \tag{46}$$

The first form is the classical expression, and the second arises from it by canonical quantization, since--much as in our discussion of the point particle--canonical quantization gives $g_{ij} \frac{dx^j}{d\sigma} = -i \frac{D}{Dx^i(\sigma)}$ where $\frac{D}{Dx^i(\sigma)} = \frac{\delta}{\delta x^i(\sigma)} + \omega_{ijk}(x(\sigma)) \, \psi^j(\sigma) \, \psi^k(\sigma)$. Q, which was invented as a generalization of the usual Dirac operator, indeed has many similarities to it. For instance (in a suitable formalism) on mass shell states $|\Lambda\rangle$

are those that obey $Q|\Lambda\rangle = 0$. Also, the analogue of the chirality operator $\bar{\Gamma}$ is the operator $(-1)^F$ (or, in an older language, G parity) which anticommutes with all the $\psi^i(\sigma)$. It certainly obeys $(-1)^F Q = -Q(-1)^F$. Moreover, in superstring theory physical states $|\Lambda\rangle$ obey $(-1)^F|\Lambda\rangle = +|\Lambda\rangle$, while Q maps these to states obeying $(-1)^F|\Lambda\rangle = -|\Lambda\rangle$. Thus the operator Q relevant in superstring theory is a chiral Dirac-Ramond operator, which like the chiral Dirac operator $i\not{D}$ on a finite dimensional manifold maps physical states of positive chirality to unphysical states of negative chirality (and on mass shell states are zero modes of Q or $i\not{D}$).

I would like to make a few remarks aimed at clarifying in what sense Q is a generalization of the finite dimensional Dirac operator. The following remarks may also clarify the last section of [21]. Let M be the space time manifold, and let $\Omega(M)$ be the corresponding loop space. A Riemannian metric $ds^2 = g_{ij}\, dx^i dx^j$ induces a Riemannian metric on $\Omega(M)$ as follows. As we discussed before in our discussion of whether $\Omega(M)$ is orientable, a tangent vector at a point $\gamma\in\Omega(M)$--γ corresponding to a loop $x^i(\sigma)$--is a tangent field δx^i along the loop. The metric on $\Omega(M)$ is defined by saying $\langle\delta x, \delta x\rangle = \int_0^{2\pi} d\sigma\; g_{ij}\,(x(\sigma))\;\delta x^i(\sigma)\;\delta x^j(\sigma)$.

Now, viewing $\Omega(M)$ as an infinite dimensional Riemannian manifold, it is natural to try to define a Dirac operator on $\Omega(M)$ as on a finite dimensional Riemannian manifold. To do so, we need gamma matrices Γ^I, I being a tangent vector index on $\Omega(M)$. But a tangent vector index I on $\Omega(M)$ is really a pair of indices (i,σ)--since a loop $x^i(\sigma)$ in M can be varied at any point σ and in any direction i tangent to M. Gamma matrices Γ^I are thus really fields $\Gamma^i(\sigma)$--and they obey a Clifford algebra $\{\Gamma^i(\sigma), \Gamma^j(\sigma')\} = 2g^{ij}\,\delta(\sigma-\sigma')$. These are the canonical commutation relations for fermions! The Dirac operator on $\Omega(M)$ would then be

$$Q_0 = -i\sum_I \Gamma^I \frac{D}{Dx^I} \tag{47}$$

But in fact $\sum_I$ is a short hand for $\sum_i \int d\sigma$, so

$$Q_0 = -i\int_0^{2\pi} d\sigma\; \psi^i(\sigma)\,\frac{D}{Dx^i(\sigma)} \tag{48}$$

where henceforth we call the gamma matrices by their more familiar name $\psi^i(\sigma)$.

Two things may be noted here. First, Q_0 is almost, but not quite, the Dirac-Ramond operator. The second term in (46) is missing in (48). Second, Q_0 is a formal construction which--even when M is flat and everything can be diagonalized by Fourier transforms--doesn't make sense. As is explained in every book on quantum field theory, the second term in Q is needed to make an operator that makes sense, with a well-defined spectrum, on the infinite dimensional manifold $\Omega(M)$. Adding the second piece to Q may not be sufficient to make a meaningful operator, but it is certainly necessary!

Still, armed with our success in interpreting the first term in Q, we try to find a reasonable interpretation of the second one. Consider then a finite dimensional spin manifold W. On W we have a Dirac operator $i\not{D}$. If in addition a continuous isometry of W is given, generated by a Killing vector field K^i, then a simple generalization of the Dirac operator is $i\not{D}_K = i\not{D} + \Gamma^i K_i$. In fact, $i\not{D}_K$ is not just a simple generalization of the Dirac operator. It is in its own right an operator associated with a rich mathematical theory. It can be used[22] to prove the localization theorem associated with the Atiyah-Singer index theorem. Its analogue for the de Rham complex[21] is related[23] to equivariant cohomology, and its analogue for the $\bar{\partial}$ operator of a complex manifold is related to a system of holomorphic Morse inequalities[24] which so far have not attracted much mathematical attention.

Now, on the loop space of maps $S^1 \to M$, there is a continuous symmetry induced from rotations of the circle. It maps a loop $x^i(\sigma)$ to $x^i(\sigma+\varepsilon)$, for any ε. The infinitesimal form of the transformation is $\delta x^i(\sigma) = \frac{\partial x^i}{\partial\sigma}$. This formula shows that the $(i\sigma)$ component of the associated Killing vector field K^I is just $\delta x^i(\sigma) = \partial x^i/\partial\sigma$. The analogue of $\gamma^i K_i$ on a finite dimensional manifold is thus

$$Q_1 = \sum \Gamma^I K_I = \int_0^{2\pi} d\sigma\ \psi_i(\sigma)\,\frac{\partial x^i}{\partial\sigma} \tag{49}$$

Q_1 is just the second piece of the Dirac-Ramond operator! The Dirac-

Ramond operator $Q = Q_0+Q_1$ is thus an analogue of the operator $i\not{\partial}_K$ on a finite dimensional manifold. It is remarkable that unlike $i\not{\partial}$, $i\not{\partial}_K$ has a meaningful infinite dimensional analogue.

We now see that $(-1)^F$, which anticommutes with all the gamma matrices $\psi^i(\sigma)$, is really the chirality operator of $\Omega(M)$. As in the finite dimensional case, the chiral Dirac-Ramond operator should make sense only if $\Omega(M)$ is orientable. Is this so? We concluded earlier that $\Omega(M)$ is orientable precisely if M is a spin manifold. Thus, superstring theory should make sense only if space-time is a spin manifold--a satisfactory result. Moving on in the same vein, to define a Dirac-like operator on $\Omega(M)$ should be possible only if $\Omega(M)$ is itself a spin manifold. Is this so? The same sort of reasoning we used to decide if $\Omega(M)$ is orientable would indicate that $\Omega(M)$ is a spin manifold precisely if $w_3(M)$--the third Stiefel-Whitney class of space-time--is zero. Happily, an orientable spin manifold M always has $w_3(M) = 0$.

Now we want to consider a more realistic situation with left-movers as well as right movers along the string. Thus we add to (45) additional terms described in [15] involving left-moving fermions λ^A and preserving N = 1/2 supersymmetry:

$$\Delta I = \int d^2\sigma \; [i\lambda^A(g_{AB}\,\partial_+ + A_{iAB}(x(\sigma))\,\partial_+x^i)\,\lambda^B - \frac{1}{2}F_{ijAB}(x(\sigma))\,\psi^i\psi^j\lambda^A\lambda^B] \tag{50}$$

For any fixed loop $\gamma\epsilon\Omega(M)$, quantization of the λ^A gives an infinite dimensional Hilbert space H_γ. As γ varies, H_γ varies smoothly, giving an infinite dimensional vector bundle X over $\Omega(M)$. In the presence of the λ^A, the Dirac-Ramond operator thus acts not on "ordinary" spinor fields on $\Omega(M)$ but on spinor fields with values in X. The supercharge or Dirac-Ramond operator is still

$$Q = \int_0^{2\pi} d\sigma\; \psi^i(x(\sigma))\;\left(-i\,\frac{D}{Dx^i(\sigma)} + g_{ij}\,\frac{dx^j}{d\sigma}\right) \tag{51}$$

but now the connection in $\frac{D}{Dx^i}$ is modified to include a connection on X. Indeed, by analogy with our previous discussion, canonical quantization of (50) reveals that the covariant derivative is now

$$\frac{D}{Dx^i} = \frac{\delta}{\delta x^i(\sigma)} + \omega_{ijk}\, \psi^j(\sigma)\, \psi^k(\sigma) + A_{iAB}\, \lambda^A(\sigma)\, \lambda^B(\sigma) \qquad (52)$$

Thus, Q is in this case an analogue of $i\not{D}_A + \Gamma\cdot K$ on a finite dimensional manifold, where $i\not{D}_A$ is the Dirac equation acting on spinors with values in some auxiliary vector bundle with connection A.

At this point, one may be tempted to believe that the analogue of $\det i\not{D}$ in field theory is $\det Q$ in string theory. For at least one reason, this is not quite right in the case of closed strings. One is required to project onto states that are invariant under translations of σ, $\sigma\to\sigma+c$. Translations of σ are generated by a "momentum" operator P. One must project onto states of P=0. Since [P,Q]=0, Q maps the P=0 subspace of Hilbert space into itself. Let $\tilde{Q}$ be the restriction of Q to the P=0 subspace. I believe that $\det\tilde{Q}$ is the proper string theoretical generalization of $\det i\not{D}$. I will refer to it as the "equivariant determinant" of Q.

An analogous concept can be considered, but does not seem too natural, in field theory. Given a manifold M and a continuous symmetry generated by a Killing vector field K, we could restrict the Dirac operator to act on the K invariant spinor fields and calculate the corresponding equivariant determinant $\det i\tilde{\not{D}}$. Like the ordinary Dirac determinant, the equivariant determinant may be afflicted with anomalies. We may call these equivariant anomalies. Equivariant anomalies may not be a natural concept in field theory, but in the case of perturbative anomalies there is a natural way to calculate them. In fact, the same family index theorem that gives a topological interpretation of ordinary perturbative Dirac anomalies[25] also gives a topological interpretation of equivariant perturbative Dirac anomalies. When coupled with the localization theorem mentioned earlier, it predicts the equivariant anomaly in terms of ordinary anomalies of a suitable Dirac operator defined on the fixed point set of K--the subspace M_0 of M consisting of points invariant under K. By contrast, there is no localization theorem for global equivariant anomalies.[26]

Now, what can we expect to learn from Dirac-Ramond anomalies? I do not believe that perturbative Dirac-Ramond anomalies will teach us much we do not already know. In fact, the localization theorem would relate the perturbative equivariant anomalies on $\Omega(M)$ to anomalies of a suitable operator on $\Omega_0(M)$--the subspace of loops that are invariant under translations of σ. But such loops, obeying $x^i(\sigma) = x^i(\sigma+c)$ for any c, are constant maps $S^1 \to M$. So $\Omega_0(M) \approx M$, and perturbative equivariant anomalies are related to calculations on M. Thus--while many details of this argument must be worked out--perturbative equivariant anomalies can be (and presumably already have been[2,3]) understood from suitable calculations on the finite dimensional manifold M.

Global equivariant anomalies are another matter, since there is no localization theorem for global equivariant anomalies. There is no telling what secrets may be locked in global equivariant Dirac-Ramond anomalies--or how long it will be before we know enough about string theory to unlock them. Although general arguments[4] show that field theoretic global anomalies could not explain why we do not live in uncompactified ten dimensional space, it is not obvious that this is impossible for Dirac-Ramond global anomalies. It is also possible that some presently known superstring theories may be rendered wholly inconsistent by global Dirac-Ramond anomalies.

A localization argument similar to the one just sketched was used elsewhere[21] to argue that in the supersymmetric nonlinear sigma model in 1+1 dimensions, formulated on S^1, any state with zero energy in lowest order actually has precisely zero energy. (Under special conditions, but not generically, this follows more simply from an index theorem.) This has applications to string theory in proving that states massless in the field theoretic limit are actually massless in string theory. The argument in [21] was carried out for N=1 supersymmetry with periodic fermion boundary conditions in both directions. Superstring theory requires both this and other cases. It is not clear if the argument generalizes to the other cases.

In the field theory case, we discussed one other question about global anomalies on the particle world line. Let $g: M \to M$ be a gauge transformation or diffeomorphism of space-time, and $E = (M \times S^1)_g$ the associated cylinder. We repeat here equation (17) for convenience:

$$\begin{array}{ccc} S^1\times S^1 & \xrightarrow{\phi} & E \\ & {\scriptstyle\alpha}\searrow \quad \swarrow{\scriptstyle\beta} & \\ & S^1 & \end{array} \tag{53}$$

Here ϕ is a map that makes the above diagram commute. We found that a global anomaly associated with such a picture means that g is not a symmetry of the quantum theory.

In string theory the analogous picture is

$$\begin{array}{ccc} (\Sigma\times S^1)_h & \xrightarrow{\phi} & E \\ & {\scriptstyle\alpha}\searrow \quad \swarrow{\scriptstyle\beta} & \\ & S^1 & \end{array} \tag{54}$$

Here Σ is a Riemann surface, h is a diffeomorphism of Σ, E is again $(M\times S^1)_g$, and we again require $\beta\phi=\alpha$. An anomaly associated with such a picture presumably means, as in field theory, that g is not a valid symmetry.

I will make no effort to investigate this situation systematically, but I cannot resist mentioning one example. Let Λ be the "big" gauge transformation associated with QCD instantons. In QCD, Λ is conserved, and this is a nuisance. It means that the physical world has a quantum number called θ_{QCD} defined by saying $\Lambda|\Omega\rangle = e^{i\theta_{QCD}}|\Omega\rangle$. (Since Λ commutes with all local operators, θ_{QCD} is the same for all states in a given "world".) The appearance of θ_{QCD} makes trouble. It is the origin of the strong CP problem. How much happier we would be if the QCD Hamiltonian did not commute with Λ! There would be no θ_{QCD}, and no strong CP problem.

Let us probe for conservation of Λ in superstring theory. To this end, we take space-time to be $M=S^1\times S^3\times K$, where S^1 is "time," S^3 is "space" on which Λ acts, and K is the Kaluza-Klein space. We then consider the mapping cylinder $E_\Lambda = (M\times S^1)_\Lambda$. (As Λ is a gauge transformation rather than a diffeomorphism, $(M\times S^1)_\Lambda$ is just $M\times S^1$ with a modified $E_8\times E_8$ vector bundle.) Because of the relation of Λ with instantons, and the fact that the instanton field has a non-zero P_1, the equation $P_1(V) = P_1(T)$ needed to avoid anomalies is violated on E_Λ. So there are anomalies (even perturbative ones) in (54), and Λ is not a symmetry. We are thus led to hope that superstring theory will solve the strong CP problem. In fact, the arguments just described seem to be a rather baroque way to understand the appearance of axions in superstring theory.[16] While that can be understood more straightforwardly, the example reassures us that we are interpreting correctly (54), which may have other applications.

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