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Divergence equations and uniqueness theorem of static spacetimes with conformal scalar hair

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We reexamine the Israel-type proof of the uniqueness theorem of the static spacetime outside the photon surface in the Einstein-conformal scalar system. We derive in a systematic fashion a new divergence identity which plays a key role in the proof. Our divergence identity includes three parameters, allowing us to give a new proof of the uniqueness.

Subject Index E01

1. Introduction

The Bocharova–Bronnikov–Melnikov–Bekenstein (BBMB) solution [1,2] is a static black hole solution to the Einstein-conformal scalar system in four dimensions.[1] A natural question to be asked is whether this solution exhausts all the static black holes in this theory. The original uniqueness proof [6,7] of static black holes in vacuum general relativity demonstrates the uniqueness of the boundary value problem of the elliptic system between the event horizon and spatial infinity. In the BBMB solution, the photon surface composed of (unstable) closed circular orbits of photons appears at the points where the coefficient of the Ricci tensor vanishes in the Einstein equation. This feature prevents us applying the global boundary value problem outside the event horizon. Nevertheless, the uniqueness property of the static region outside the photon surface has been properly addressed in Refs. [8,9], where it has been shown to be isometric to the BBMB solution.[2]

To prove the uniqueness theorem, two technically and conceptually distinct methods are currently available. The BBMB uniqueness has been demonstrated in Ref. [8] by a way similar to Refs. [6, 7], relying on certain divergence identities. The other proof, in Ref. [9], follows the argument in Ref. [13] based on the conformal transformation and positive mass theorem [14]. Meanwhile, for the uniqueness of black holes in vacuum Einstein, Einstein–Maxwell, and Einstein–Maxwell–dilaton systems, the argument by Robinson [7], which is regarded as a simplification of Israel's proof, has been reexamined in Ref. [15]. A significant achievement in Ref. [15] is to provide a systematic way to derive the divergence identities exploiting the proper deviation from the Schwarzschild metric. The obstruction tensors are of great use in finding a series of divergence identities even in

[1] Assuming analyticity at the photon surface, it has been shown that a higher-dimensional counterpart fails to admit a regular event horizon [3–5].

[2] See Refs. [10,11] for similar arguments under an additional strong assumption on the photon surface, and Ref. [12] for perturbative analysis in the vacuum Einstein system.

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stationary metrics [16]. Then, it is natural to ask if the procedure developed in Ref. [15] works in the Einstein-conformal scalar system and also for the uniqueness proof of photon surfaces.

In this paper we apply the procedure of Ref. [15] to the Einstein-conformal scalar system. We shall see that it indeed works, and find a new divergence identity with three parameters. Since the derivation for the divergence identities found in Ref. [8] was rather non-trivial, the systematic way to derive the identity will be of some help in similar situations for other systems. Finally, we shall prove the uniqueness of the static photon surface again.

The rest of the paper is organized as follows. In Sect. 2 we describe the Einstein-conformal scalar system and the setup of the current paper. In Sect. 3, we develop the procedure of Ref. [15] to the Einstein-conformal scalar system. Finally, we will give a summary in Sect. 4. In the Appendix, we present the relation to Ref. [8] in detail.

2. The BBMB black hole and setup

In this section we describe the Einstein-conformal scalar system and the basic equations for static spacetimes. The action for the Einstein-conformal scalar system is represented by

$$S = \int d^4x\sqrt{-g}\left(\frac{1}{2\kappa}R - \frac{1}{2}(\nabla\phi)^2 - \frac{1}{12}R\phi^2\right), \tag{1}$$

where $\kappa = 8\pi G$ is a gravitational constant. The field equations are given by

$$\left(1 - \frac{\kappa}{6}\phi^2\right)\left(R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu}\right) = \kappa\left(\nabla_\mu\phi\nabla_\nu\phi - \frac{1}{2}g_{\mu\nu}(\nabla\phi)^2 + \frac{1}{6}(g_{\mu\nu}\nabla^2 - \nabla_\mu\nabla_\nu)\phi^2\right) \tag{2}$$

and

$$\nabla^2\phi - \frac{1}{6}R\phi = 0. \tag{3}$$

Taking the trace of Eq. (2), one finds $R = 0$, so that the field equations are simplified to

$$\left(1 - \frac{\kappa}{6}\phi^2\right)R_{\mu\nu} = \kappa\left(\frac{2}{3}\nabla_\mu\phi\nabla_\nu\phi - \frac{1}{6}g_{\mu\nu}(\nabla\phi)^2 - \frac{1}{3}\phi\nabla_\mu\nabla_\nu\phi\right) \tag{4}$$

and

$$\nabla^2\phi = 0. \tag{5}$$

This theory admits a static black hole solution (the BBMB solution) [1,2]. Its metric and scalar field are

$$ds^2 = -\left(1 - \frac{m}{r}\right)^2 dt^2 + \left(1 - \frac{m}{r}\right)^{-2} dr^2 + r^2 d\Omega_2^2 \tag{6}$$

and

$$\phi = \pm\sqrt{\frac{6}{\kappa}}\frac{m}{r - m}, \tag{7}$$

where $d\Omega_2^2$ is the metric of the unit two-sphere and m is the mass, which is supposed to be positive. The event horizon is located at $r = m$.

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One important feature of the Einstein-conformal scalar system is that it may admit points satisfying $\phi = \pm\sqrt{6/\kappa}$, where the prefactor of the Ricci tensor in Eq. (4) vanishes. This means that the effective gravitational constant diverges. For the BBMB solution, this occurs precisely at the photon surface $r = 2m$. As far as the outside region of the photon surface is concerned, the uniqueness property has been settled to be affirmative [8,9]. As stated in Sect. 1, we will reexamine the proof of Ref. [8] and then present an elegant way to find the divergence identities used in the proof.

The generic form of a static metric is written as

$$ds^2 = -V^2(x^k)dt^2 + g_{ij}(x^k)dx^i dx^j, \tag{8}$$

where V is the norm of the static Killing vector. The event horizon is located at $V = 0$. We also assume that the conformal scalar field is also static, $\phi = \phi(x^i)$. The Einstein equation becomes

$$\left(1 - \frac{\kappa}{6}\phi^2\right) V D^2 V = \frac{\kappa}{6}\left[V^2 (D\phi)^2 + 2\phi V D^i V D_i \phi\right] \tag{9}$$

and

$$\left(1 - \frac{\kappa}{6}\phi^2\right)\left({}^{(3)}R_{ij} - V^{-1} D_i D_j V\right) = \kappa\left(\frac{2}{3} D_i\phi D_j\phi - \frac{1}{6} g_{ij}(D\phi)^2 - \frac{1}{3}\phi D_i D_j \phi\right), \tag{10}$$

where D_i and ${}^{(3)}R_{ij}$ are the covariant derivative and the Ricci tensor with respect to the three-dimensional metric g_{ij}, respectively. Note here that the front factors on the left-hand side of Eqs. (9) and (10) vanish at the surface S_p determined by $\phi = \pm\sqrt{6/\kappa} =: \phi_p$.[3] The equation for the scalar field is written as

$$D_i(V D^i \phi) = 0. \tag{11}$$

The asymptotic conditions at infinity are given as

$$V = 1 - \frac{m}{r} + \mathcal{O}\left(\frac{1}{r^2}\right), \tag{12}$$

$$g_{ij} = \left(1 + \frac{2m}{r}\right)\delta_{ij} + \mathcal{O}\left(\frac{1}{r^2}\right), \tag{13}$$

$$\phi = \mathcal{O}\left(\frac{1}{r}\right). \tag{14}$$

Equations (9) and (11) give us

$$D_i[(1-\varphi)D^i\Phi] = 0, \tag{15}$$

where $\Phi := (1+\varphi)V$ and $\varphi := \pm\sqrt{\kappa/6}\,\phi$. Then, one considers Ω, the region bounded by S_p and the two-sphere S_∞ at spatial infinity. The volume integration of Eq. (15) over Σ shows the relation between V and ϕ as [8]

$$\phi = \pm\sqrt{\frac{6}{\kappa}}(V^{-1} - 1). \tag{16}$$

[3] After our proof is completed, one realizes that S_p coincides with the photon surface due to the uniqueness.

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Through Eq. (16), we see that $V = 1/2$ at S_p.

Using the relation in Eq. (16), the Einstein equation implies

$$D^2 v = 0, \tag{17}$$

where $v := \ln V$. Henceforth, we can regard v as a kind of radial coordinate. It allows us to decompose the $t =$ constant hypersurface Σ into the radial direction and the foliation of the $v =$ constant surfaces S_v. As a consequence, the (i, j)-component of the Einstein equation becomes

$$(2V - 1)^{(3)}R_{ij} = D_i D_j v + (4V + 1) D_i v D_j v - g_{ij} (Dv)^2. \tag{18}$$

The curvature invariant is expressed in terms of geometrical quantities associated with S_v as

$$R_{\mu\nu}R^{\mu\nu} = \frac{1}{(2V-1)^2\rho^2}\left[\left(2(1-V)k_{ij} - \frac{1}{\rho}h_{ij}\right)^2 + \left(-2(1-V)k + \frac{1+2V}{\rho}\right)^2 + \frac{8(1-V)^2}{\rho^2}(\mathcal{D}\rho)^2\right] + \frac{1}{\rho^4}, \tag{19}$$

where h_{ij} is the induced metric of S_v, $\mathcal{D}_i$ is the covariant derivative with respect to h_{ij}, and ρ is the lapse function, $\rho := (D_i v D^i v)^{-1/2}$. Moreover, k_{ij} is the extrinsic curvature of S_v defined by $k_{ij} := h_i^k D_k n_j$, where n_i is the unit normal vector to S_v on Σ, and k is the trace part of k_{ij}. Using the lapse function, n_i is expressed by $n_i = \rho D_i v$. From Eq. (19), at $V = 1/2$ we have to impose

$$\mathcal{D}_i \rho|_{S_p} = 0, \qquad k_{ij}|_{S_p} = \frac{1}{\rho_p} h_{ij}|_{S_p}, \tag{20}$$

otherwise the curvature invariant diverges. The first equation of Eq. (20) shows that ρ is constant on S_p. We write the constant as $\rho_p := \rho|_{S_p}$. The second condition of Eq. (20) implies that the surface S_p is totally umbilic. We can see that, under the conditions of Eq. (20), the Kretschmann invariant, $R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma}$, is also finite at S_p [8].

In the proof of uniqueness in Ref. [8] the following divergence identities are presented without any explanation:

$$D_i\left(\frac{n^i}{\rho}\right) = 0, \tag{21}$$

$$D_i\left[\frac{(\rho k - 2)n^i}{(2V-1)\rho^{\frac{3}{2}}}\right] = -\frac{1}{2V-1}\rho^{-\frac{1}{2}}(\tilde{k}_{ij}\tilde{k}^{ij} + \rho^{-1}\mathcal{D}^2\rho), \tag{22}$$

$$D_i((k\xi + \eta)n^i) = -(\tilde{k}_{ij}\tilde{k}^{ij} + \rho^{-1}\mathcal{D}^2\rho)\xi, \tag{23}$$

where $\xi := (2V-1)\rho^{-\frac{1}{2}}$, $\eta := 2(2V+1)\rho^{-\frac{3}{2}}$, and $\tilde{k}_{ij}$ is the traceless part of k_{ij}. By the volume integration of these equations over Ω and the use of Stokes' theorem, one gets one equality and two inequalities. These inequalities are consistent with each other only when the equalities hold in both. This gives the result that Ω is spherically symmetric. Finally, Ref. [17] shows that Ω is unique as the BBMB solution.

Compared to Eq. (21), the derivation of Eqs. (22) and (23) is far from trivial. In the following we will discuss a systematic way to derive them by applying the argument of Ref. [15].

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3. Generalization of the divergence equations and uniqueness

In this section we develop the systematic derivation of the divergence equations following Ref. [15]. The obtained divergence equation allows us to show the uniqueness of the photon surface of the BBMB solution.

First, we wish to find a current J^i satisfying

$$D_i J^i = (\text{terms with a definite sign}). \tag{24}$$

The right-hand side is required to have a definite sign and to consist of a sum of tensors which vanish if and only if the spacetime is the BBMB solution. A candidate for such a set of tensors is

$$H_{ij} = D_i D_j v + \frac{3V}{1-V} D_i v D_j v - \frac{V}{\rho^2(1-V)} g_{ij}. \tag{25}$$

Note that this is a symmetric and traceless tensor. A simple calculation shows that H_{ij} vanishes for the BBMB solution. The expression for H_{ij} in terms of geometric quantities on S_v is useful for later discussions:

$$H_{ij} = \frac{1}{\rho}\tilde{k}_{ij} - \frac{2}{\rho^2} n_{(i}\mathcal{D}_{j)}\rho + \frac{1}{2\rho}(h_{ij} - 2n_i n_j)\left(k - \frac{2V}{\rho(1-V)}\right), \tag{26}$$

where we used the Einstein equation. Using H_{ij}, we can also construct a vector H_i which vanishes for the BBMB solution as

$$H_i = -\rho^2 H_{ij} D^j v. \tag{27}$$

Here we suppose that J_i has the form

$$J_i = f_1(v) g_1(\rho) D_i \rho + f_2(v) g_2(\rho) D_i v. \tag{28}$$

The divergence of Eq. (28) is written by

$$\begin{aligned} D_i J^i &= (f_1' g_1 + f_2 g_2') D_i v D^i \rho + f_1 g_1' (D\rho)^2 + f_2' g_2 (Dv)^2 + f_1 g_1 D^2 \rho \\ &= -\rho^3 f_1 g_1 \left[|H_{ij}|^2 - \left(\frac{g_1'}{\rho g_1} + \frac{3}{\rho^2} \right) |H_i|^2 \right] + \rho f_1 g_1 D^i v H_i S_1 + \frac{1}{\rho^2} f_1 g_2 S_2, \end{aligned} \tag{29}$$

where

$$S_1 := \frac{f_1'}{f_1} + \frac{f_2}{f_1}\frac{g_2'}{g_1} + \frac{(4V-1)(3V-1)}{(2V-1)(1-V)} + \frac{4\rho V}{1-V}\frac{g_1'}{g_1}, \tag{30}$$

$$S_2 := \frac{2\rho V}{1-V}\frac{g_1}{g_2} S_1 - \frac{4\rho V^2}{(1-V)^2}\frac{g_1}{g_2}\left[\rho\frac{g_1'}{g_1} + \frac{8V^2 - 7V + 2}{2V(2V-1)}\right] + \frac{f_2'}{f_1}. \tag{31}$$

The prime denotes differentiation with respect to each argument of the functions. In the second equality of Eq. (29) we used[4]

$$D^2\rho = -\rho^3 |D_i D_j v|^2 + \frac{3}{\rho}(D\rho)^2 + \frac{1}{2V-1}\left(D_i \rho D^i v - \frac{4V}{\rho}\right). \tag{32}$$

[4] With the aid of Eq. (18), direct calculation from the definition of ρ gives this.

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To control the sign of the right-hand side of Eq. (29) we require $S_1 = S_2 = 0$. Following Ref. [15], to have decoupled equations we suppose that g_1 and g_2 have the form

$$g_1 = -c\rho^{-(c+1)}, \qquad g_2 = \rho^{-c}, \tag{33}$$

where c is an integration constant. Then, we have two ordinary differential equations for f_1 and f_2:

$$f_2 + f_1' + \left[\frac{(4V-1)(3V-1)}{(2V-1)(1-V)} - \frac{4V(1+c)}{1-V}\right] f_1 = 0, \tag{34}$$

$$f_2' + \frac{4cV^2}{(1-V)^2}\left[\frac{8V^2-7V+2}{2V(2V-1)} - (c+1)\right] f_1 = 0. \tag{35}$$

The solutions are given by

$$f_1 = \frac{1}{4}(2V-1)^{-1}(1-V)^{1-2c}(a + b(2V-1)^2), \tag{36}$$

$$f_2 = \frac{1}{4}(2V-1)^{-1}(1-V)^{-2c}\left[(a+b)(2cV-2V+1) - 8bcV^2(1-V)\right], \tag{37}$$

where a and b are integration constants. Using the fact that

$$\frac{1}{2}\left|2\rho^2 H_{i[j}D_{k]}v - g_{i[j}H_{k]}\right|^2 = \rho^2|H_{ij}|^2 - \frac{3}{2}|H_i|^2, \tag{38}$$

the divergence equation is rearranged as

$$D_iJ^i = \frac{cf_1}{2\rho^c}\left[\left|2\rho^2 H_{i[j}D_{k]}v - g_{i[j}H_{k]}\right|^2 + (2c-1)|H_i|^2\right]. \tag{39}$$

To fix the sign of the right-hand side in Eq. (39), we require

$$f_1 \geq 0, \qquad c \geq \frac{1}{2}. \tag{40}$$

With $\frac{1}{2} \leq V < 1$, it is easy to see that the former is guaranteed by

$$a \geq 0, \qquad a + b \geq 0. \tag{41}$$

Now, let us integrate Eq. (39) over Ω. Using Stokes' theorem, we have

$$\int_\Omega D_iJ^i d\Sigma = \int_{S_\infty} J_in^i dS - \int_{S_p} J_in^i dS \geq 0. \tag{42}$$

Using the asymptotic behaviors near the spatial infinity ($\rho \simeq |\partial_r V|^{-1} \simeq r^2/m$, $k \simeq 2/r$), the first term of the right-hand side is estimated as

$$\int_{S_\infty} J_in^i dS = -\pi(a+b)m^{1-c}. \tag{43}$$

The second term has to be carefully estimated. First, we have

$$\int_{S_p} J_in^i dS = -\frac{ac}{4}\left(\frac{1}{2}\right)^{1-2c}\frac{1}{\rho_p^{c+1}}\int_{S_p}\frac{k\rho_p - 2}{2V-1}dS - \frac{1}{4}\left(\frac{1}{2}\right)^{-2c}(a+b-2ac)\frac{1}{\rho_p^{c+1}}A_p, \tag{44}$$

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where A_p is the area of the surface S_p. Here, note that the Gauss equation with the Einstein equation gives

$$ {}^{(2)}R = \frac{2}{\rho^2} + k^2 - k_{ij}k^{ij} + \frac{2(\rho k - 4V)}{(2V-1)\rho^2}, \tag{45} $$

and then

$$ \lim_{V\to 1/2} {}^{(2)}R = \lim_{V\to 1/2} \frac{2(\rho k - 2)}{(2V-1)\rho^2} \tag{46} $$

holds, where we used Eq. (20). Using this and the Gauss–Bonnet theorem for the first term on the right-hand side of Eq. (44), we arrive at

$$ \int_{S_p} J_i n^i dS = -\frac{\pi ac}{2}\left(\frac{1}{2}\right)^{1-2c}\frac{1}{\rho_p^{c-1}}\chi - \frac{1}{4}\left(\frac{1}{2}\right)^{-2c}(a+b-2ac)\frac{1}{\rho_p^{c+1}}A_p, \tag{47} $$

where χ is the Euler characteristic. As a consequence, Eq. (42) implies

$$ (a+b)\left(A_p - \pi\rho_p^2\left(\frac{4m}{\rho_p}\right)^{1-c}\right) + ac\left(\pi\rho_p^2\chi - 2A_p\right) \geq 0. \tag{48} $$

Under the parameter range in Eqs. (40) and (41), we get the pair of inequalities

$$ \pi\rho_p^2\left(\frac{4m}{\rho_p}\right)^{1-c} \leq A_p \leq \frac{1}{2}\pi\rho_p^2\chi. \tag{49} $$

Setting $c = 1$ gives $\chi \geq 2$, meaning that the only allowed topology of S_p is spherical ($\chi = 2$). Setting $\chi = 2$ implies that the equality holds, and it occurs if and only if H_{ij} vanishes. This is the case that the spacetime is spherically symmetric. According to Ref. [17], the spacetime is unique as the BBMB solution.

Before closing this section, we comment on the relation to the divergence identities in Ref. [8]. For $b = 0$ and $c = 1/2$, Eq. (39) coincides with Eq. (22), and for $a = 0$ and $c = 1/2$, Eq. (23). See the Appendix for the details.

4. Summary

In this paper we reexmined the Israel-type proof for the uniqueness of the photon surface in the Einstein-conformal scalar system. Following Ref. [15], we derived a new divergence identity with three parameters and gave a new proof of the uniqueness. In Ref. [15], vacuum Einstein, Einstein–Maxwell, and Einstein–Maxwell–dilaton systems were addressed. Therefore, the current study indicates the power of the systematic procedure presented there. The deep physical/mathematical reason is expected to be hidden behind the presence of such a procedure.

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Appendix A. Relation of Eq. (39) to Eqs. (22) and (23)

Using Eq. (21), or the equivalent equality

$$k\rho = n^i D_i \rho, \tag{A.50}$$

we have

$$J_i = \frac{-cf_1\rho k + f_2}{\rho^{c+1}} n_i - cf_1 \frac{\mathcal{D}_i\rho}{\rho^{c+1}}. \tag{A.51}$$

Using

$$D^i\left(\frac{\mathcal{D}_i\rho}{\rho^{c+1}}\right) = \frac{\mathcal{D}^2\rho}{\rho^{c+1}} - c\frac{(\mathcal{D}\rho)^2}{\rho^{c+2}}, \tag{A.52}$$

the left-hand side of Eq. (39) becomes

$$D^iJ_i = D^i\left(\frac{-cf_1\rho k + f_2}{\rho^{c+1}} n_i\right) - cf_1\frac{\mathcal{D}^2\rho}{\rho^{c+1}} + c^2 f_1\frac{(\mathcal{D}\rho)^2}{\rho^{c+2}}. \tag{A.53}$$

The right-hand side of Eq. (39) is expressed as

$$c\frac{f_1}{\rho^c}\left[\tilde{k}_{ij}\tilde{k}^{ij} + \frac{c}{\rho^2}(\mathcal{D}\rho)^2 + \frac{2c-1}{2}\left(k - \frac{2V}{\rho(1-V)}\right)^2\right]. \tag{A.54}$$

Thus, we have

$$D^i\left(\frac{-cf_1\rho k + f_2}{\rho^{c+1}} n_i\right) = c\frac{f_1}{\rho^c}\left[\tilde{k}_{ij}\tilde{k}^{ij} + \frac{2c-1}{2}\left(k - \frac{2V}{\rho(1-V)}\right)^2\right] + cf_1\frac{\mathcal{D}^2\rho}{\rho^{c+1}}. \tag{A.55}$$

Now we consider the $c = 1/2$ case; then, $f_1 = (1/4)(2V-1)^{-1}[a + b(2V-1)^2]$ and $f_2 = (1/4)(2V-1)^{-1}[a + b(1-2V)(1+2V)]$. Setting $b = 0$ ($a = 0$), we can see that Eq. (A.55) becomes Eq. (22) (Eq. (23)).

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