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Marginally trapped surfaces in Minkowski 4-space invariant under a rotation subgroup of the Lorentz group

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Abstract A local classification of spacelike surfaces in Minkowski 4-space, which are invariant under spacelike rotations, and with mean curvature vector either vanishing or lightlike, is obtained. Furthermore, the existence of such surfaces with prescribed Gaussian curvature is shown. A procedure is presented to glue several of these surfaces with intermediate parts where the mean curvature vector field vanishes. In particular, a local description of marginally trapped surfaces invariant under spacelike rotations is exhibited.

Keywords Lightlike mean curvature vector, Marginally trapped surfaces, Lorentz group

1 Introduction

In General Relativity, a spacelike surface in a four-dimensional Lorentzian manifold is called *marginally trapped* if its mean curvature vector is proportional to one of the null normals, by an either positive or negative function. When such function is arbitrary, the surface is called marginally outer trapped, or MOTS, for short. The study of these families of surfaces has been quite active in recent years (see for instance [1; 7; 11]).

In general, it is customary to ask these surfaces to be closed, i. e., compact and without boundary. However, some results concerning the non-existence of closed

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MOTS can be found in the literature. Among the classical ones, a result due to R. Penrose, [9] (see also [8]), implies the non-existence of closed MOTS in the Minkowski spacetime when it bounds a compact domain. Moreover, in [7], it is proved that in any strictly static spacetime, no marginally trapped surface which is not globally extremal (i. e., its mean curvature vector is not zero at least in a point) can exist. Also, A. Carrasco and M. Mars, [2], have shown the non-existence of MOTS bounding a domain and entering a region of a static spacetime where the Killing vector field is timelike, and with the additional assumptions of dominant energy condition and an outer untrapped barrier. Thus, some authors are beginning to relax the definition, letting the surface to be non-compact.

In order to gain some idea of the properties of marginally trapped surfaces in particular spacetimes, classification results were obtained for marginally trapped surfaces with positive relative nullity in Lorentzian space forms [3] and in Robertson-Walker spaces [4]. In [6] marginally trapped surfaces which are invariant under a boost transformation in four-dimensional Minkowski space were studied.

We consider the Minkowski 4-space $\mathbb{L}^4$ endowed with its standard metric $-\mathrm{d}x_1^2 + \mathrm{d}x_2^2 + \mathrm{d}x_3^2 + \mathrm{d}x_4^2$. In this paper, we are interested in studying marginally trapped surfaces in Minkowski 4-space which are invariant under the following group of isometries:

$$\mathbf{G}_s = \left\{ B_\theta = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \cos(\theta) & \sin(\theta) \\ 0 & 0 & -\sin(\theta) & \cos(\theta) \end{pmatrix} : \theta \in \mathbb{R} \right\}.$$

Due to the previous non-existence results, we relax the definition of these surfaces, in the sense that we consider non-closed marginally trapped surfaces (i. e. either non-compact or with boundary). However, with little more efforts, it is possible to study a more general family of surfaces, namely, those whose mean curvature vector is either lightlike or zero, and invariant under $\mathbf{G}_s$. Thus, the main result of this paper is Theorem 1, where a local classification of such surfaces is obtained. In particular, this classification includes marginally trapped surfaces and those surfaces with vanishing mean curvature vector, which are invariant by $\mathbf{G}_s$. For the sake of simplicity, we say that a surface is extremal at a point p if its mean curvature vector field is zero at p. Needless to say, an extremal surface has everywhere vanishing mean curvature vector field.

Further, a gluing procedure is presented to construct $\mathbf{G}_s$-invariant spacelike surfaces for which the mean curvature vector is lightlike or zero on certain parts. This allows to obtain examples of various surfaces occurring in the classification given in [12]. The point is that the examples constructed using this method have up to infinitely many regions where the mean curvature vector of each such region can be chosen to be future or past-pointing *as desired*, and among two consecutive regions, there is an extremal subset.

In the final section, it is shown that it is possible to construct surfaces which are invariant by $\mathbf{G}_s$, whose mean curvature vector is lightlike or zero, and with prescribed Gaussian curvature. In particular, those with constant Gaussian curvature are given explicitly.

The main mathematical tool consist of the local theory of surfaces. Its origins go back almost two centuries ago, when C. F. Gauss [5] and other authors started its development for surfaces in the Euclidean 3-space. Since then, this powerful theory has been used successfully in an overwhelming number of situations. Nowadays, this is the standard technique to study surfaces in Mathematics. At the end of the day, it can be summarized in a small collection of formulae, bringing to light interesting geometric properties of surfaces in Physics.

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2 Preliminaries

Let $(\mathbb{L}^4, \widetilde{g})$ be the four-dimensional Lorentz-Minkowski space with the flat metric given in local coordinates by

$$\widetilde{g} = -\mathrm{d}x_1^2 + \mathrm{d}x_2^2 + \mathrm{d}x_3^2 + \mathrm{d}x_4^2.$$

For a connected immersed surface S in $\mathbb{L}^4$, we denote by g the induced metric on S. We will assume that this metric g is positive-definite, i.e., the surface is *spacelike*. Let $\widetilde{\nabla}$ and ∇ denote the Levi-Civita connections on $(\mathbb{L}^4, \widetilde{g})$ and (S, g), respectively. Then, if X and Y are two smooth vector fields tangent to S, the Gauss formula gives the decomposition of the vector $\widetilde{\nabla}_X Y$ into its tangential and normal parts, i.e.,

$$\widetilde{\nabla}_X Y = \nabla_X Y + \mathbf{K}(X,Y),$$

where $\mathbf{K} : \mathfrak{X}(S) \times \mathfrak{X}(S) \to \mathfrak{X}^{\perp}(S)$ is called the *shape tensor* or *second fundamental form* of S in $\mathbb{L}^4$. If η is a normal vector to the surface, the Weingarten formula gives the decomposition of the vector $\widetilde{\nabla}_X \eta$ into its tangential and normal parts, i.e.,

$$\widetilde{\nabla}_X \eta = -A_\eta(X) + \nabla_X^{\perp} \eta,$$

where $\nabla^{\perp}$ is the normal connection in the normal bundle of S and the endomorphism A_η on $\mathfrak{X}(S)$ is called the *shape operator* associated with η. The shape tensor and shape operator are related by $\widetilde{g}(\mathbf{K}(X,Y),\eta) = g(A_\eta(X),Y)$. The *mean curvature vector* $\mathbf{H}$ is defined as the normalized trace of the shape tensor,

$$\mathbf{H} = \frac{1}{2}\mathrm{tr}_g \mathbf{K} \in \mathfrak{X}^{\perp}(S).$$

The component of $\mathbf{H}$ along a given normal direction η, denoted by h_η, is called the *expansion along* η, i.e., $h_\eta = \widetilde{g}(\mathbf{H},\eta) = \mathrm{tr}_g(A_\eta)/2$.

Let us consider a local orthonormal basis $\{\eta_1, \eta_2\}$ of the normal bundle of the spacelike surface S in $\mathbb{L}^4$, where η_1 is future-pointing timelike and η_2 is spacelike. If we denote by A_i the shape operator associated with η_i, $i = 1,2$, the shape tensor can be written as

$$\mathbf{K}(X,Y) = -g(A_1(X),Y)\eta_1 + g(A_2(X),Y)\eta_2,$$

for any tangent vector fields X, Y to S. Assume that $X(u,v)$ is a local parametrization on the surface S. Then, from the classical theory of surfaces (see, e.g. [13]), with the notation $2h_i = \mathrm{tr}_g(A_i)$ and

$$E = \widetilde{g}(X_u, X_u), F = \widetilde{g}(X_u, X_v), G = \widetilde{g}(X_v, X_v),$$
$$e_i = \widetilde{g}(X_{uu}, \eta_i), f_i = \widetilde{g}(X_{uv}, \eta_i), g_i = \widetilde{g}(X_{vv}, \eta_i),$$

we obtain

$$2h_i = \frac{e_i G - 2 f_i F + g_i E}{EG - F^2}, \quad i = 1, 2.$$

Another useful local basis $\{\mathbf{k}, \mathbf{l}\}$ of the normal bundle of S can be chosen such that both vectors are null, future-pointing and satisfy the normalization condition $\widetilde{g}(\mathbf{k}, \mathbf{l}) = -1$. In the following, we choose

$$\mathbf{k} = \frac{1}{\sqrt{2}}(\eta_1 - \eta_2) \text{ and } \mathbf{l} = \frac{1}{\sqrt{2}}(\eta_1 + \eta_2).$$

With respect to this normal basis the mean curvature vector field $\mathbf{H}$ becomes

$$\mathbf{H} = -\frac{\sqrt{2}}{2}(h_1 + h_2)\mathbf{k} - \frac{\sqrt{2}}{2}(h_1 - h_2)\mathbf{l}.$$

In particular, the *expansions along* $\mathbf{k}$ *and* $\mathbf{l}$ are given by

$$\Theta_k = \frac{\sqrt{2}}{2}(h_1 - h_2) \text{ and } \Theta_l = \frac{\sqrt{2}}{2}(h_1 + h_2).$$

Besides the extrinsic mean curvature, also the intrinsic Gaussian curvature K of the surface can be expressed in terms of the coefficients of the first and second fundamental forms as (see, e.g. [13]),

$$K = \frac{-\det(A_1) + \det(A_2)}{\det(g)} = \frac{-e_1 g_1 + e_2 g_2 + f_1^2 - f_2^2}{EG - F^2}.$$

A spacelike surface S in $\mathbb{L}^4$ is called *invariant under spacelike rotations* if it is invariant under the following group $\mathbf{G}_s$ of linear isometries of $\mathbb{L}^4$:

$$\mathbf{G}_s = \left\{ B_\theta = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \cos(\theta) & \sin(\theta) \\ 0 & 0 & -\sin(\theta) & \cos(\theta) \end{pmatrix} \ : \ \theta \in \mathbb{R} \right\},$$

i.e., if $B_\theta S = S$, for any $\theta \in \mathbb{R}$.

Since we regard these surfaces as geometric objects, the main tool to study them consist of introducing natural (local) parameterizations, which can be achieved by making use of the action of the group and finding a suitable profile curve. It is worth pointing out that when we let a surface be *only* of class C^∞ and not analytical, we might get a very complicated curve. More problems arise when the surface is immersed, but not imbedded. Even worse, since the codimension is two, the surface does not need to be orientable. As a consequence, we will restrict our study to a local setting.

Note that the set of fixed points of $\mathbf{G}_s$ is $\{(x_1,x_2,x_3,x_4)\in\mathbb{L}^4 : x_3 = x_4 = 0\}$, so we need the following subset

$$\mathscr{P} = \{(x_1,x_2,x_3,x_4)\in\mathbb{L}^4 : x_4 = 0, x_3 > 0\}.$$

With the help of $\mathscr{P}$, we can introduce a parametrization $X(t,\theta)$ of S as follows. Given a smooth curve $\alpha : I\subset\mathbb{R}\to\mathscr{P}$, $t\mapsto\alpha(t) = (\alpha_1(t),\alpha_2(t),\alpha_3(t),0)$, the parametrization can be written as

$$X(t,\theta) = (\alpha_1(t),\alpha_2(t),\alpha_3(t)\cos(\theta),\alpha_3(t)\sin(\theta)), \quad t\in I,\theta\in\mathbb{R}.$$

We denote by Σ_α the parameterized surface associated with α, as a subset of $\mathbb{L}^4$,

$$\Sigma_\alpha = \{X(t,\theta) = \alpha(t)\cdot B_\theta : t\in I,\theta\in\mathbb{R}\}\subset S.$$

We recall that Σ_α might not cover the whole original surface S, but it would be a big enough open subset. Next, without loss of generality we can assume that the spacelike curve α is arc-length parameterized, i.e., $\widetilde{g}(\alpha'(t),\alpha'(t)) = 1$. The derivatives of $X(t,\theta)$ are

$$X_t = \left(\alpha_1'(t),\alpha_2'(t),\alpha_3'(t)\cos(\theta),\alpha_3'(t)\sin(\theta)\right) \text{ and}$$
$$X_\theta = (0,0,-\alpha_3(t)\sin(\theta),\alpha_3(t)\cos(\theta)).$$

The Riemannian metric of the surface reads

$$g = \mathrm{d}t^2 + \alpha_3^2\mathrm{d}\theta^2.$$

A globally defined orthonormal tangent frame on Σ_α is $u_1 = X_t$ and $u_2 = X_\theta/\alpha_3$, and a globally defined orthonormal basis of the normal bundle of Σ_α is given by

$$\eta_1 = \frac{1}{\sqrt{1+(\alpha_1')^2}}\left(1+(\alpha_1')^2,\alpha_1'\alpha_2',\alpha_1'\alpha_3'\cos(\theta),\alpha_1'\alpha_3'\sin(\theta)\right),$$
$$\eta_2 = \frac{1}{\sqrt{1+(\alpha_1')^2}}\left(0,-\alpha_3',\alpha_2'\cos(\theta),\alpha_2'\sin(\theta)\right),$$

with η_1 future-pointing timelike and η_2 spacelike. A straightforward computation shows that the components of the second fundamental form are given by

$$e_1 = -\frac{\alpha_1''}{\sqrt{1+(\alpha_1')^2}},\ f_1 = 0,\ g_1 = -\frac{\alpha_1'\,\alpha_3\,\alpha_3'}{\sqrt{1+(\alpha_1')^2}}, \tag{1}$$
$$e_2 = \frac{\alpha_2'\,\alpha_3'' - \alpha_2''\,sf\alpha_3'}{\sqrt{1+(\alpha_1')^2}},\ f_2 = 0,\ g_2 = -\frac{\alpha_2'\,\alpha_3}{\sqrt{1+(\alpha_1')^2}}.$$

Hence the shape operators associated with η_1 and η_2 are simultaneously diagonalizable, i.e., the normal curvature $R^\perp$ of the normal bundle vanishes identically. The expansions along η_1 and η_2 are

$$2h_1 = -\frac{\alpha_1'\,\alpha_3' + \alpha_3\,\alpha_1''}{\alpha_3\sqrt{1+(\alpha_1')^2}} \text{ and } 2h_2 = -\frac{\alpha_2' + \alpha_3(\alpha_2''\,\alpha_3' - \alpha_2'\,\alpha_3'')}{\alpha_3\sqrt{1+(\alpha_1')^2}}. \tag{2}$$

The Gaussian curvature of a spacelike surface which is invariant under a spacelike rotation is

$$K = -\frac{\alpha_3''}{\alpha_3}. \tag{3}$$

3 Classification theorem and a gluing procedure

The following classification is local, i. e., a surface S which is invariant by $\mathbf{G}_s$ will be locally congruent to the surfaces in the next theorem.

Theorem 1 *Let Σ_α be a surface in $\mathbb{L}^4$ which is invariant under spacelike rotations. Assume that its mean curvature vector satisfies $\|\mathbf{H}\| = 0$. Then, the generating curve $\alpha(t) = (\alpha_1(t), \alpha_2(t), \alpha_3(t), 0)$ is locally described by one of the following cases:*

(A) *Given a smooth function $\tau : I \subset (0,\infty) \longrightarrow \mathbb{R}$, choose a function $\varepsilon : I \longrightarrow \{1,-1\}$ such that $\varepsilon\tau$ is also smooth. Define the coordinate functions $\alpha_i : I \longrightarrow \mathbb{R}$, $i = 1,2,3$, as follows*

$$\alpha_1(t) = \int \varepsilon(t)\tau(t)\mathrm{d}t, \quad \alpha_2(t) = \int \tau(t)\mathrm{d}t, \quad \alpha_3(t) = t. \tag{4}$$

Moreover, the mean curvature vector of Σ_α is

$$\mathbf{H} = \frac{\tau + t\tau'}{2t\sqrt{1+\tau^2}}\,(\varepsilon\eta_1 - \eta_2).$$

(B) *Given a smooth positive function $\alpha_3 : I \subset \mathbb{R} \to \mathbb{R}$, and two constants $\varepsilon_1, \varepsilon_2 = \pm 1$, define the functions*

$$\xi(t) = \int \frac{\mathrm{d}t}{\alpha_3(t)}, \quad \alpha_1(t) = \varepsilon_1 \int \left\{\sinh(\xi(t)) - \alpha_3'(t)\cosh(\xi(t))\right\} \mathrm{d}t, \tag{5}$$

and

$$\alpha_2(t) = \varepsilon_2 \int \left\{\cosh(\xi(t)) - \alpha_3'(t)\sinh(\xi(t))\right\} \mathrm{d}t. \tag{6}$$

Moreover, the mean curvature vector of Σ_α is

$$\mathbf{H} = \frac{\cosh(\xi(t))\left(1 - \alpha_3'(t)^2 - \alpha_3(t)\,\alpha_3''(t)\right)}{2\alpha_3(t)\sqrt{1+\alpha_1'(t)^2}}\,(\varepsilon_1\eta_1 - \varepsilon_2\eta_2).$$

In addition, in Case B, given two unit curves $\alpha(t) = (\alpha_1(t), \alpha_2(t), \alpha_3(t), 0)$ and $\beta(t) = (\beta_1(t), \beta_2(t), \beta_3(t), 0)$, such that $\alpha_3(t) = \beta_3(t)$, there exists an affine isometry F of $\mathbb{L}^4$ satisfying $F(\Sigma_\alpha) = \Sigma_\beta$.

Proof We recall the generating spacelike unit curve $\alpha\colon J \longrightarrow \mathscr{P}$, $\alpha(t) = (\alpha_1(t), \alpha_2(t), \alpha_3(t), 0)$ of Σ_α, with $\alpha_3(t) > 0$. Now, we consider two subsets $J_0 = \{t \in J : \alpha_3'(t) = \pm 1\}$ and $J_1 = \{t \in J : \alpha_3'(t) \neq \pm 1\}$, which are not intervals in general. Since α_3' is continuous, J_1 is an open subset of J, i. e., it is either the empty set or made of countable many open intervals. From a topological point of view, J_1 might be empty or a mixture of intervals, accumulation points and isolated points. To make some progress, we need to work on open intervals included in J_0 and J_1.
Case A. We assume that there exists an open interval $I \subset J_0$ where $\alpha_3'(t)^2 = 1$. By a change of parameter, we can assume without loss of generality that $I \subset (0,\infty)$ and $\alpha_3(t) = t$ on I. From now on, we work on I. Since α is unit, we know that

$\alpha_1'(t)^2 = \alpha_2'(t)^2$. Thus, there exists a function $\varepsilon : I \longrightarrow \{-1,1\}$ such that $\varepsilon\,\alpha_1'(t) = \alpha_2'(t)$. Now, (4) are trivially satisfied.
Case B. We assume that there exists an open interval $I \subset J_1$, so we work on I. Since α is arc-length parameterized, we have $(-\alpha_1' + \alpha_2')(\alpha_1' + \alpha_2') = 1 - (\alpha_3')^2$. By shrinking I if necessary, we can introduce an *angle* function $\xi(t)$ and a constant $\varepsilon = \pm 1$, such that

$$-\alpha_1'(t)+\alpha_2'(t) = \varepsilon\,(1+\alpha_3'(t))\,\exp(\xi(t)), \quad \alpha_1'(t)+\alpha_2'(t) = \varepsilon\,(1-\alpha_3'(t))\,\exp(-\xi(t)).$$

In this way, we obtain the following expressions:

$$\alpha_1'(t) = \frac{1}{2}\left\{\varepsilon\,(1-\alpha_3'(t))\,\exp(\xi(t)) - \varepsilon\,(1+\alpha_3'(t))\,\exp(-\xi(t))\right\}, \tag{7}$$

$$\alpha_2'(t) = \frac{1}{2}\left\{\varepsilon\,(1-\alpha_3'(t))\,\exp(\xi(t)) + \varepsilon\,(1+\alpha_3'(t))\,\exp(-\xi(t))\right\}. \tag{8}$$

Since we are assuming $\|\mathbf{H}\| = 0$, there exists a function $\delta : I \longrightarrow \{1,-1\}$ such that $h_1 = \delta\, h_2$. Bearing in mind (7) and (8), we substitute this in (2), obtaining

$$\begin{aligned}&(1+\delta(t))\,\alpha_3'(t)\,\left(1-\alpha_3(t)\,\xi'(t)\right)\,\sinh(\xi(t))\\ &\quad + \left\{\alpha_3'(t)^2\,\left(\delta(t)\alpha_3(t)\xi'(t)-1\right) + \alpha_3(t)\,\left(\xi'(t) + (\delta(t)-1)\,\alpha_3''(t)\right) - \delta(t)\right\}\\ &\quad \times \cosh(\xi(t)) = 0. \end{aligned} \tag{9}$$

Now, two cases arise naturally.

1. We suppose that there exists an open interval I^+ such that $\delta(t) = 1$ for any $t \in I^+$. We work in this interval. Equation (9) becomes

   $$0 = \left(\cosh(\xi(t))\,\alpha_3'(t)^2 - 2\sinh(\xi(t))\,\alpha_3'(t) + \cosh(\xi(t))\right)\,\left(\alpha_3(t)\xi'(t) - 1\right).$$

   Now, we suppose that there exists a $t_0 \in I^+$ such that $0 = \cosh(\xi(t_0))\,\alpha_3'(t_0)^2 - 2\sinh(\xi(t_0))\,\alpha_3'(t_0) + \cosh(\xi(t_0))$. However, from this equation, we obtain $\alpha_3'(t_0) = \tanh(\xi(t_0)) \pm \sqrt{-1}\,\mathrm{sech}(\xi(t_0))$, which is impossible. Thus, on the whole I^+ (at least), we obtain

   $$\xi(t) = \int \frac{\mathrm{d}t}{\alpha_3(t)}.$$

   Inserting this in (7) and (8) gives the expressions (5) and (6) for the case $\varepsilon_1 = \varepsilon_2 = \varepsilon$.
2. We suppose that there exists an open interval I^- such that $\delta(t) = -1$ for any $t \in I^-$. We work in this interval. Equation (9) becomes

   $$-\alpha_3'(t)^2\,\left(\alpha_3(t)\xi'(t)+1\right) + \alpha_3(t)\,\left(\xi'(t) - 2\alpha_3''(t)\right) + 1 = 0.$$

   From here, we compute $\xi'(t) = \frac{-1}{\alpha_3(t)} - \frac{2\alpha_3''(t)}{\alpha_3'(t)^2-1}$. Now, we obtain

   $$\xi(t) = -\int \frac{\mathrm{d}t}{\alpha_3(t)} - \ln\left|\frac{\alpha_3'(t)-1}{\alpha_3'(t)+1}\right|.$$

When inserting this in (7) and (8), one cannot forget the signs, i.e., $(1+\alpha_3')$ $\exp(\ln|\frac{\alpha_3'-1}{1+\alpha_3'}|)=\pm(\alpha_3'-1)$. Bearing this in mind, two cases arise. However, it is possible to deal with both at the same time by choosing suitable constants $\varepsilon_1,\varepsilon_2=\pm1$, obtaining again expressions (5) and (6). Thus, there is no loss of generality if we redefine the angle function as $\xi(t)=\int(1/\alpha_3(t))\mathrm{d}t$.

Let $\beta(t)$ be another arc-length parameterized spacelike curve, with $\beta_3(t)=\alpha_3(t)$. Then, we have

$$\int\frac{\mathrm{d}t}{\beta_3(t)}=\int\frac{\mathrm{d}t}{\alpha_3(t)}+c_0,$$

with c_0 an integration constant. A straightforward computation shows

$$(\beta_1',\beta_2')=(\alpha_1',\alpha_2')\begin{pmatrix}\widetilde{\varepsilon}_1 & 0\\ 0 & \widetilde{\varepsilon}_2\end{pmatrix}\begin{pmatrix}\cosh(c_0) & \sinh(c_0)\\ \sinh(c_0) & \cosh(c_0)\end{pmatrix},$$

with $\widetilde{\varepsilon}_1,\widetilde{\varepsilon}_2=\pm1$. If the integration constants of (5) and (6) are denoted by α_1^0 and α_2^0, we call $v=(\alpha_1^0,\alpha_2^0,0,0)$. The affine isometry $F:\mathbb{L}^4\to\mathbb{L}^4$,

$$F(x_1,x_2,x_3,x_4)=(x_1,x_2,x_3,x_4)\begin{pmatrix}\widetilde{\varepsilon}_1 & 0 & 0 & 0\\ 0 & \widetilde{\varepsilon}_2 & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1\end{pmatrix}\begin{pmatrix}\cosh(c_0) & \sinh(c_0) & 0 & 0\\ \sinh(c_0) & \cosh(c_0) & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1\end{pmatrix}+v,$$

satisfies $F\circ\alpha=\beta$ and thus $F(\Sigma_\alpha)=\Sigma_\beta$. □

In Cases A and B of the previous theorem, the domain of the curve α might not be connected. If we ask the domain I of α to be an interval, we will say that the surface Σ_α is of *type A* or *type B*, according to Cases A or B, respectively. In particular, surfaces of type A and B have to be connected, orientable, and any normal lightlike vector can be globally defined.

Corollary 1 1. *A surface of type A is a MOTS if, and only if, the function ε is a global constant. In addition, a surface of type A is marginally trapped if, and only if, the function ε is a global constant and $\tau(t)+t\tau'(t)$ is globally positive or negative.*

2. *Any surface of type B is a MOTS. In addition, a surface of type B is marginally trapped if, and only if, the function $1-(\alpha_3')^2-\alpha_3\alpha_3''$ is globally positive or negative.*

By a result in [9] (see also [8]), a closed surface of type A or B bounding a domain cannot exist. Thus, a good second alternative is completeness.

Corollary 2 *Let Σ_α be a surface of type B in $\mathbb{L}^4$. If $\alpha_3:\mathbb{R}\longrightarrow\mathbb{R}$ is a smooth function such that $\alpha_3(t)\geq a_0>0$ for some real constant a_0, then the surface Σ_α is complete.*

Proof The metric of the surface Σ_α of type B satisfies

$$g \geq \mathrm{d}t^2 + a_0^2 \mathrm{d}\theta^2,$$

and it is defined for any $t, \theta \in \mathbb{R}$. This means that Σ_α is complete. □

From the proof of Theorem 1, we find a characterization of the extremal spacelike surfaces which are invariant under a spacelike rotation.

Corollary 3 *A spacelike surface in $\mathbb{L}^4$ is extremal and invariant under a spacelike rotation if, and only if, it is locally congruent to a surface Σ_α whose profile curve $\alpha : I \subset (0,\infty) \to \mathscr{P}$, $\alpha(t) = (\alpha_1(t), \alpha_2(t), \alpha_3(t), 0)$, is given by one of the following cases:*

1. $\alpha_1(t) = a\varepsilon_1 \ln(t) + b$, $\alpha_2(t) = a\varepsilon_2 \ln(t) + c$, $\alpha_3(t) = t$, *with* $a > 0$, $b, c \in \mathbb{R}$ *and* $\varepsilon_1, \varepsilon_2 = \pm 1$.
2. $\alpha_1(t) = \frac{\varepsilon_1}{2}(a^2 + 1 - b) \ln \left| a + t + \sqrt{t^2 + 2at + b} \right|$, $\alpha_2(t) = \frac{\varepsilon_2}{2}(a^2 - 1 - b) \ln \left| a + t + \sqrt{t^2 + 2at + b} \right|$, $\alpha_3(t) = \sqrt{t^2 + 2at + b}$, *where* $\varepsilon_1, \varepsilon_2 = \pm 1$, *and* $a, b \in \mathbb{R}$.

Remark 1 Surfaces of type A and B are not excluding. Indeed, by considering $\alpha_3(t) = t$ in Theorem 1, we obtain Case 1 of Corollary 3, which is a description of all curves generating surfaces which are simultaneously of type A and B.

Remark 2 All surfaces of type A are flat, i.e., their Gaussian curvature $K = 0$.

Remark 3 Given a surface of type A, if the mean curvature vector is future-pointing, by considering the function $-\varepsilon$, we obtain a surface with past-pointing mean curvature vector, and viceversa. A similar situation holds for surfaces of type B by changing ε_1 by $-\varepsilon_1$.

Remark 4 Given a surface of type A, with the very same function τ (and same function ε) it is possible to construct infinitely many curves, and thus surfaces of type A. However, two of them are related by the translation defined by considering different integration constants in the expressions of functions α_1 and α_2. In addition, if we change ε by $-\varepsilon$, the reflection by a suitable hyperplane links both surfaces.

Remark 5 For a surface of type A, with the function $\tau(t) = c \in \mathbb{R}$, both shape operators A_1 and A_2 are of rank 1. Such a surface is called pseudo-isotropic. See, e.g. [10] for properties of such surfaces.

Remark 6 Surfaces of type A as graphs, locally. Given a surface S of type A, with the function ε locally constant. We restrict this remark to an interval J where ε is constant. Then, the surface is included in the null hyperplane $\mathscr{H} = \{(x_1, x_2, x_3, x_4) \in \mathbb{L}^4 : x_1 = \varepsilon x_2\}$. From a Set Theory point of view, one can identify $\mathscr{H}$ with $\mathbb{R}^3$, where the surface is parameterized as $Y(t,\theta) = (\int \tau(t)\mathrm{d}t, t\cos\theta, t\sin\theta)$. We just call $T(t) = \int \tau(t)\mathrm{d}t$, so we can identify a region of S with the set $\{(T(\sqrt{y^2+z^2}), y, z) : \sqrt{y^2+z^2} \in J\}$. Conversely, any surface of type A can be locally seen as a graph over an annulus centered at the origin of $\mathbb{R}^2$. Furthermore, given a real constant $a > 0$, a disk $D(a) = \{(y,z) \in \mathbb{R}^2 : y^2 + z^2 < a^2\}$ and a smooth function $T : D(a) \longrightarrow \mathbb{R}$, such that T is invariant by transformations of the form $(y,z) \mapsto (y\cos\theta - z\sin\theta, y\sin\theta + z\cos\theta)$, $\theta \in \mathbb{R}$, then the graph $S = \{(T(y,z),y,z) : (y,z) \in D(a)\}$ can be imbedded in $\mathbb{L}^4$ as a surface whose mean curvature vector field satisfies $\|\mathbf{H}\| = 0$, and admitting a parametrization of a surface of type A except in the point touching the plane of fixed points of $\mathbf{G}_s$.

We consider two bounded spacelike surfaces of type A and B, and suppose that the mean curvature vector of a surface of type A is constantly either past or future-pointing near one of its boundaries. In such case, we describe a method to glue them in one new spacelike surface which is invariant by a spacelike rotation with an intermediate region satisfying $\mathbf{H} = 0$.

Proposition 1 *Let Σ_α and Σ_β be two surfaces of type A and B as in Theorem 1 with generating curves $\alpha : (a,b) \longrightarrow \mathscr{P}$, $\beta : (c,d) \longrightarrow \mathscr{P}$, with $0 \le a < b < c < d \le \infty$. Assume that there is a constant $\omega > 0$ such that the function ε is constant on the interval $(b-\omega, b)$. Then, there exists an affine isometry $F : \mathbb{L}^4 \longrightarrow \mathbb{L}^4$, a real number $\nu > 0$ and a unit spacelike curve $\gamma : (a,d) \longrightarrow \mathscr{P}$, satisfying that the surface Σ_γ is invariant by a spacelike rotation, $\gamma|_{(a,b-\nu)} = \alpha|_{(a,b-\nu)}$, $F(\Sigma_{\gamma|_{(c+\nu,d)}}) = \Sigma_{\beta|_{(c+\nu,d)}}$ and the mean curvature of the region $\Sigma_{\gamma|_{(b,c)}}$ vanishes identically.*

Proof Because the function ε is constant on the interval $(b-\omega, b)$, there is no loss of generality if we assume $\varepsilon(t) = \varepsilon_1\varepsilon_2$ for any $b - \omega < t < b$ (by changing ε_1). In such case, the surface Σ_β is unique up to an affine isometry as in Theorem 1.

We choose $\nu \in \mathbb{R}$ such that $0 < \nu < \min\{(d-c)/4, (b-a)/4, \omega\}$ and consider two smooth functions $f_i : (a,d) \longrightarrow \mathbb{R}$, $i = 1,2$, satisfying

1. $0 \le f_i \le 1$, $i = 1,2$;
2. $f_1(t) = 0$ and $f_2(t) = 1$ for any $t \in (a,c)$;
3. $f_1(t) = 1$ and $f_2(t) = 0$ for any $t \in (c+\nu, d)$.

Note that $f_1' = f_2' = 0$ on the intervals (a,c) and $(c+\nu,d)$. We define the smooth function $\gamma_3 : (a,d) \longrightarrow (0,\infty)$, given by $\gamma_3(t) = t f_2(t) + \beta_3(t) f_1(t)$. It is straightforward to check that $\gamma_3(t) = t$ for any $t \in (a,c)$ and $\gamma_3(t) = \beta_3(t)$ for any $t \in (c+\nu,d)$. In particular, $\gamma_3(t) = \alpha_3(t)$ on the interval (a,b).

We define $\tilde{\xi}(t) = \displaystyle\int \frac{\mathrm{d}t}{\gamma_3(t)}$, with the additional condition $\tilde{\xi}(t) = \xi(t)$ for any $t \in (c+\nu,d)$, which can be achieved by choosing a suitable integration constant. Bearing in mind Case B in Theorem 1, we define $\tilde{\beta}_1, \tilde{\beta}_2 : (a,d) \longrightarrow \mathscr{P}$, satisfying $\tilde{\beta}_1(t) = \beta_1(t)$ and $\tilde{\beta}_2(t) = \beta_2(t)$ for any $t \in (c+\nu,d)$.

Next, we consider two smooth functions $f_3, f_4 : (a,d) \longrightarrow \mathbb{R}$ such that

1. $0 \le f_i \le 1$, $i = 3,4$;
2. $f_3(t) = 1$ and $f_4(t) = 0$ for any $t \in (a, b-\nu)$;
3. $f_3(t) = 0$ and $f_4(t) = 1$ for any $t \in (b,d)$.

Let $\tau(t)$ be the function in the definition of the curve α. Next, we define the smooth functions $\gamma_i : (a,d) \longrightarrow \mathbb{R}$, $i = 1,2$, given by

$$\gamma_1(t) = \int \Big(\varepsilon(t)\,\tau(t)\,f_3(t) + \tilde{\beta}_1'(t)\,f_4(t)\Big)\,\mathrm{d}t, \quad \gamma_2(t) = \int \Big(\tau(t)\,f_3(t) + \tilde{\beta}_2'(t)\,f_4(t)\Big)\,\mathrm{d}t,$$

but satisfying $\gamma_1(t) = \alpha_1(t)$ and $\gamma_2(t) = \alpha_2(t)$ for any $t \in (a, b-\nu)$. As above, it is only necessary to choose suitable integration constants. Indeed, given $t \in (a, b-\nu)$, $\gamma_1'(t) = \varepsilon(t)\, f_3(t)\, \tau(t) + f_4(t)\, \tilde{\beta}_1'(t) = \varepsilon(t)\, \tau(t)$, and $\gamma_2'(t) = f_3(t)\, \tau(t) + f_4(t)\, \tilde{\beta}_2'(t) = \tau(t)$. Next, given $t \in (b, d)$, $\gamma_i'(t) = \tilde{\beta}_i'(t)$, for $i = 1, 2$, and $\gamma_3(t) = t$. Note that by Corollary 3, the surface $\Sigma_{\gamma|_{(b,c)}}$ satisfies $\mathbf{H} = 0$. Now, bearing in mind Remark 1, given $t \in (b-\nu, b)$, we see $\tilde{\beta}_i'(t) = 2\varepsilon_i \exp(\xi_0)\, t$, $i = 1, 2$. Thus, since $0 < \nu < \omega$, $\gamma_1'(t) = \varepsilon(t)\, (f_3(t)\, \tau(t) + f_4(t)\, \frac{\varepsilon_1}{\varepsilon(t)} \exp(\xi_0)\, t) = \varepsilon(t)\, \big(f_3(t)\, \tau(t) + f_4(t)\, \varepsilon_2$
$\exp(\xi_0)\, t\big) = \varepsilon(t)\, \gamma_2'(t)$. Finally, we define the curve

$$\gamma : (a, d) \longrightarrow \mathscr{P}, \quad \gamma(t) = (\gamma_1(t), \gamma_2(t), \gamma_3(t), 0),$$

and its associated surface Σ_γ. From the above computations, it is easy to check that $\|\gamma'\| = 1$. Needless to say, Σ_α is an open subset of Σ_γ. It only remains to point out that the open subset $\Sigma_{\gamma|_{(c+\nu,d)}}$ of Σ_γ might not be the original $\Sigma_{\beta|_{(c+\nu,d)}}$, but they will be congruent by an affine isometry, as in Theorem 1. $\square$

Corollary 4 *Let Σ_α and Σ_β two surfaces of type A, whose generating curves $\tau_\alpha : (a, b) \longrightarrow \mathbb{R}$ and $\tau_\beta : (c, d) \longrightarrow \mathbb{R}$ satisfy $0 < a < b < c < d \leq \infty$. Then, there exists a unit spacelike curve $\gamma : (a, d) \longrightarrow \mathscr{P}$, a real number $\nu > 0$ and two translations $F_\alpha, F_\beta : \mathbb{L}^4 \longrightarrow \mathbb{L}^4$ such that Σ_γ is a surface of type A, $F_\alpha(\Sigma_{\gamma|_{(a,b-\nu)}}) = \Sigma_{\alpha|_{(a,b-\nu)}}$, $F_\beta(\Sigma_{\gamma|_{(c+\nu,d)}}) = \Sigma_{\beta|_{(c+\nu,d)}}$ and the mean curvature of the region $\Sigma_{\gamma|_{(b,c)}}$ vanishes identically.*

Corollary 5 *Let Σ_α and Σ_β two surfaces of type B, with profile curves $\alpha : (a, b) \longrightarrow \mathscr{P}$ and $\beta : (c, d) \longrightarrow \mathscr{P}$, $-\infty \leq a < b < c < d \leq \infty$. Then, there exists a unit spacelike curve $\gamma : (a, d) \longrightarrow \mathscr{P}$, a real number $\nu > 0$ and two affine isometries $F_\alpha, F_\beta : \mathbb{L}^4 \longrightarrow \mathbb{L}^4$ such that Σ_γ is a surface of type B, $F_\alpha(\Sigma_{\gamma|_{(a,b-\nu)}}) = \Sigma_{\alpha|_{(a,b-\nu)}}$, $F_\beta(\Sigma_{\gamma|_{(c+\nu,d)}}) = \Sigma_{\beta|_{(c+\nu,d)}}$ and the mean curvature of the region $\Sigma_{\gamma|_{(b,c)}}$ vanishes identically..*

All necessary ideas to prove these two corollaries are contained in the proof of Proposition 1 and in Remark 4.

Remark 7 The methods explained in Proposition 1 and its two corollaries give the possibility to construct surfaces S satisfying the following conditions:

1. S is invariant under a spacelike rotation group.
2. The mean curvature vector of S satisfies $\|\mathbf{H}\| = 0$, with (infinitely many countable) regions $\{S_n : n \in N \subset \mathbb{N}\}$ where its mean curvature vector $\mathbf{H} \neq 0$.
3. Each region S_n can be either of type A or B.
4. The mean curvature vector of each region S_n can be set either future or past-pointing, as *desired*.
5. Among two *adjacent* regions S_n and S_{n+1}, there is an open subset which is extremal, i.e., $\mathbf{H} = 0$.

In particular, it is possible to construct examples of several of the types given in the classification introduced in [12].

4 The Gaussian curvature

We show that there exist surfaces in $\mathbb{L}^4$ invariant by $\mathbf{G}_s$, whose mean curvature vector field satisfies $\|\mathbf{H}\| = 0$ and with prescribed Gaussian curvature. As an application, we compute all such surfaces which have constant Gaussian curvature.

Corollary 6 *Let $\kappa : I \subset \mathbb{R} \to \mathbb{R}$ be a smooth function and $t_0 \in I$. There exist $\delta > 0$ and a unit curve $\alpha : (t_0 - \delta, t_0 + \delta) \subset \mathbb{R} \to \mathscr{P}$, such that α is a profile curve of a spacelike surface $\Sigma_\alpha(t,\theta)$ of type B and whose Gaussian curvature at every point (t,θ) is $\kappa(t)$. Moreover, if $\kappa(t)\,\alpha_3(t)^2 - \alpha_3'(t)^2 + 1$ never vanishes on $(t_0 - \delta, t_0 + \delta)$, the surface Σ_α is marginally trapped.*

Proof Given a smooth function $\kappa : I \subset \mathbb{R} \to \mathbb{R}$ and let $\alpha_3 : (t_0 - \delta, t_0 + \delta) \to \mathscr{P}$ be a positive solution of the differential equation $\alpha_3''(t) = -\kappa(t)\,\alpha_3(t)$ [see (3)]. The result then follows from Theorem 1. □

Corollary 7 *There do not exist extremal spacelike surfaces of type B with constant Gaussian curvature in $\mathbb{L}^4$.*

Proof If we take κ constant in the previous Corollary, the solution of the differential equation $\kappa\alpha_3(t)^2 + \alpha_3'(t)^2 - 1 = 0$ is either $\alpha_3(t) = \varepsilon/\sqrt{-\kappa}$ if $\kappa < 0$ or $\alpha_3(t) = \varepsilon \sinh((t-c)\sqrt{\kappa})/\sqrt{\kappa}$ if $\kappa > 0$, with $\varepsilon = \pm 1$ and $c \in \mathbb{R}$. By (3), using these expressions in the differential equation $\alpha_3''(t) = -\kappa\,\alpha_3(t)$ gives a contradiction in both cases. □

Example 1 A surface of type B is flat if and only if a profile curve $\alpha : (-b/a, +\infty) \subset \mathbb{R} \to \mathscr{P}$, with $a, b \in \mathbb{R}$, $|\,a\,| \neq 0, 1$, is given by

$$\begin{aligned}
\alpha_1(t) &= \frac{\varepsilon}{2}\left\{\frac{1-a}{1+a}(at+b)^{\frac{a+1}{a}} + \frac{1+a}{1-a}(at+b)^{\frac{a-1}{a}}\right\},\\
\alpha_2(t) &= \frac{\varepsilon}{2}\left\{\frac{1-a}{1+a}(at+b)^{\frac{a+1}{a}} - \frac{1+a}{1-a}(at+b)^{\frac{a-1}{a}}\right\},\\
\alpha_3(t) &= at+b,
\end{aligned}$$

or a profile curve $\alpha : I \subset \mathbb{R} \to \mathscr{P}$ is given by

$$\alpha(t) = \left(\varepsilon_1\, b \cosh\left(\frac{t}{b}\right), \varepsilon_2\, b \sinh\left(\frac{t}{b}\right), b, 0\right),$$

with $b \in \mathbb{R}_0^+$.

Example 2 Given $K > 0$, we compute the profile curve α of a surface of type B with constant Gaussian curvature K^2. By (3), we need to solve the differential equation $\alpha_3''(t) = -K^2\alpha_3(t)$, whose general solution is

$$\alpha_3(t) = c_1 \cos(Kt + c_2), \quad \text{with } c_1, c_2 \in \mathbb{R},\ c_1 \neq 0.$$

As $\alpha_3(t)$ has to be positive, we can choose $I = (-\frac{\pi+2c_2}{2K}, \frac{\pi-2c_2}{2K})$ if $c_1 > 0$ or $I = (\frac{\pi-2c_2}{2K}, \frac{3\pi-2c_2}{2K})$ if $c_1 < 0$, as the domain of $\alpha_3(t)$. According to Theorem 1, we need to compute a primitive of $1/\alpha_3(t)$, which is

$$\xi(t) = \frac{1}{c_1} \ln\left|\frac{1+\sin(Kt+c_2)}{1-\sin(Kt+c_2)}\right| + \xi_0,$$

being $\xi_0 \in \mathbb{R}$. This way, by taking ε_1, $\varepsilon_2 = \pm 1$, the coordinate functions of $\alpha(t) = (\alpha_1(t), \alpha_2(t), \alpha_3(t), 0)$ are

$$\begin{aligned}\alpha_1(t) &= \varepsilon_1 \left(\sinh(\xi(t)) + c_1 \sin(Kt + c_2)\cosh(\xi(t))\right),\\ \alpha_2(t) &= \varepsilon_2 \left(\cosh(\xi(t)) + c_1 \sin(Kt + c_2)\sinh(\xi(t))\right),\\ \alpha_3(t) &= c_1 \cos(Kt + c_2).\end{aligned}$$

Finally, the mean curvature vector of Σ_α is

$$\mathbf{H} = \frac{\cosh(\xi(t))\left(1 + c_1^2 K^2 \cos(2(Kt + c_2))\right)}{2c_1 \cos(Kt + c_2)\sqrt{1 + (\alpha_1'(t))^2}}(\varepsilon_1\eta_1 - \varepsilon_2\eta_2).$$

Example 3 Given $K > 0$, we compute the profile curve α of a surface of type B with constant Gaussian curvature $-K^2$. By (3), we need to solve the differential equation $\alpha_3''(t) = K^2\alpha_3(t)$, whose general solution is

$$\alpha_3(t) = c_1 \exp(Kt) + c_2 \exp(-Kt), \quad \text{with } c_1, c_2 \in \mathbb{R},\ c_1^2 + c_2^2 > 0.$$

We choose an interval I where $\alpha_3(t)$ is positive. We discuss some cases.

Case $c_1c_2 > 0$. Given $\xi_0 \in \mathbb{R}$, the angle function is

$$\xi(t) \;=\; \frac{1}{K\sqrt{c_1c_2}} \arctan\left(\frac{c_1 \exp(Kt)}{\sqrt{c_1c_2}}\right) + \xi_0.$$

Case $c_1c_2 < 0$. Given $\xi_0 \in \mathbb{R}$, the angle function is

$$\xi(t) \;=\; \frac{1}{2K\sqrt{-c_1c_2}} \ln\left|\frac{2c_1 \exp(Kt) - 2\sqrt{-c_1c_2}}{2c_1 \exp(Kt) + 2\sqrt{-c_1c_2}}\right| \xi_0.$$

Case $c_2 = 0$. Given $\xi_0 \in \mathbb{R}$, the angle function is

$$\xi(t) \;=\; -\frac{1}{Kc_1 \exp(Kt)} + \xi_0.$$

Case $c_1 = 0$. Given $\xi_0 \in \mathbb{R}$, the angle function is

$$\xi(t) \;=\; \frac{\exp(Kt)}{Kc_2} + \xi_0.$$

It only remains to compute $\alpha_1(t)$ and $\alpha_2(t)$. To do so, we choose ε_1, $\varepsilon_2 = \pm 1$, and then

$$\begin{aligned}\alpha_1(t) &= \varepsilon_1 \left(\sinh(\xi(t)) - K\left(c_1 \exp(\xi(t)) - c_2 \exp(\xi(t))\right)\cosh(\xi(t))\right),\\ \alpha_2(t) &= \varepsilon_2 \left(\cosh(\xi(t)) - K\left(c_1 \exp(\xi(t)) - c_2 \exp(\xi(t))\right)\sinh(\xi(t))\right).\end{aligned}$$

Finally, the mean curvature vector of Σ_α is

$$\mathbf{H} \;=\; \frac{\cosh(\xi(t))\left(1 - 2K^2(c_1^2 \exp(2Kt) + c_2^2 \exp(-2Kt))\right)}{2\alpha_3(t)\sqrt{1 + \alpha_1'(t)^2)}}(\varepsilon_1\eta_1 - \varepsilon_2\eta_2).$$

Corollary 8 *Let S be a spatial surface in* $\mathbb{L}^4$ *invariant by* $\mathbf{G}_s$ *satisfying* $\|\mathbf{H}\| = 0$ *with constant Gaussian curvature K. Then, S is locally congruent to either a surface of type A or one among Examples* 1, 2 *and* 3.

5 Conclusions

In this paper, we have studied spacelike surfaces in Minkowski 4-space which are invariant by a rotation group of isometries and whose mean curvature vector field is lightlike or zero. Our main result is the classification of such surfaces in Theorem 1, from which it follows that there are two types of surfaces, that we call of type A and B, which are not excluding. As a consequence, a long list of corollaries is exhibited. Among them, we locally describe MOTS and marginally trapped surfaces. Furthermore, given up to countable infinitely many surfaces of either type A or B whose mean curvature vector might be either future or past-pointing, (and some reasonable conditions), we describe a method to glue them in just one surface whose mean curvature vector is null, which are invariant by a spacelike rotation group, and having intermediate extremal regions among two regions of type A or B. Also, we pay attention to the Gaussian curvature, showing the possibility to construct surfaces of type B with prescribed Gaussian curvature (at least, theoretically). Among them, the list of surfaces with constant Gaussian curvature is exhibited.

These constructions may lead to the study of generalized horizons in Minkowski 4-space as well as in other spacetimes, since they are foliated by marginally trapped surfaces.

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