# The quest for Quantum Gravity:

## from Weyl invariance to transverse theories

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# The Quest For Quantum Gravity:

## From Weyl Invariance To Transverse Theories

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*A la niña que creyó entender la Relatividad General*
*desde la cocina de su casa...*

The following papers, some of them unrelated to the content of this document, where published by the candidate while this thesis was being developed

1) *Hawking Radiation from Universal Horizons,*
   M. Herrero-Valea, Stefano Liberati, **R. Santos-Garcia**
   JHEP 04 (2021) 255

2) *Massless Positivity in Graviton Exchange*
   M. Herrero-Valea, **R. Santos-Garcia**, A. Tokareva
   arXiv:2011.11652

3) *Weighing the Vacuum Energy*
   E. Alvarez, J. Anero, **R. Santos-Garcia**
   Phys.Rev.D 103 (2021) 8, 084032

4) *Non-minimal Tinges of Unimodular Gravity*
   M. Herrero-Valea, **R. Santos-Garcia**
   JHEP 09 (2020) 041

5) *Structural stability of spherical horizons*
   E. Alvarez, J. Anero, **R. Santos-Garcia**
   arXiv:2006.02463

6) *CFT in Conformally Flat Spacetimes*
   E. Alvarez, **R. Santos-Garcia**
   Phys.Rev.D 101 (2020) 12, 125009

7) *A note on unimodular* $N = 1$, $d = 4$ *AdS supergravity*
   C. P. Martin, J. Anero, **R. Santos-Garcia**
   JCAP 03 (2020) 006

8) *Off-shell unimodular* $N = 1$, $d = 4$ *supergravity*
   C. P. Martin, J. Anero, **R. Santos-Garcia**
   JHEP 01 (2020) 145

9) *Weyl anomalies and the nature of the gravitational field*
   E. Alvarez, J. Anero, **R. Santos-Garcia**
   arXiv:1907.03781

10) *Conformal invariance versus Weyl invariance*
    E. Alvarez, J. Anero, **R. Santos-Garcia**
    Eur.Phys.J.C 80 (2020) 2, 95

11) *Massive unimodular gravity*
    E. Alvarez, J. Anero, G. Milans del Bosch, **R. Santos-Garcia**
    Class.Quant.Grav. 37 (2020) 13, 135001

12) *Quasilocal energy and compactification*
E. Alvarez, J. Anero, G. Milans del Bosch, **R. Santos-Garcia**
JHEP 1806 (2018) 069

13) *Physical content of quadratic gravity*
E. Alvarez, J. Anero, S. Gonzalez-Martin, **R. Santos-Garcia**
Eur.Phys.J. C78 (2018) no.10, 794

14) *One-loop counterterms in first order quantum gravity*
J. Anero, **R. Santos-Garcia**
arXiv:1706.02622

## ABSTRACT

This thesis, written as a compendium of articles, addresses some of the fundamental problems encountered when trying to build a theory of Quantum Gravity. Taking General Relativity (GR) as the starting point, its well-known non-renormalizable character leads to the need for an ultraviolet completion. Besides, the Cosmological Constant (CC) problem is still one of the cornerstones of Theoretical Physics. The following articles explore possible insights into these problems within the context of gravitational Effective Field Theories.

The first article is devoted to the study of quadratic (in curvature) theories of gravity when treated in the *First Order formalism*, where the metric and the connection are considered as independent fields. These renormalizable theories are quadratic in the derivatives of the connection and do not contain quartic propagators, leaving *a priori* some room for unitarity. Nevertheless, it is not clear whether these theories include a graviton or whether they are free of *ghosts*, as all the dynamics is now encoded in the connection field. A complete study of the propagating degrees of freedom is then needed. In this work, we analyze the spin content of a generic torsion-free connection by constructing a complete basis of 22 six index spin projectors. We find that these theories generically propagate a spin three piece together with several lower spin components.

One of the classical solutions to the CC problem is to consider Weyl invariant theories, as they forbid a CC term in the action. This symmetry is a generalization of the usual conformal invariance to cases where gravity is present. In the second article of the thesis, we carry out an analysis of the (in)equivalence of conformal and Weyl invariant theories for the gravitational field. The most general Lagrangian for spin two particles up to dimension six operators is explored, corresponding to the low-energy expansion of linear and quadratic (in curvature) theories of gravity. We carry out a full classification of the theories invariant under linearized (transverse) diffeomorphism, linearized Weyl transformations, and the usual conformal and scale symmetries.

In the last part of the thesis, the theory of Unimodular Gravity (UG) is examined. This theory is an alternative low-energy description of gravity defined as the truncation of GR to unit determinant metrics. In UG the CC does not couple directly to gravity due to the unimodular constraint, and thus, it possesses a completely different nature. In particular, it does not receive radiative corrections, partially solving the CC problem. Apart from the character of the CC, UG is found to be classically equivalent to GR, and the question of the full (in)equivalence of both theories is still an open debate when quantum corrections are considered. The potential differences arising when studying the coupling to matter are investigated, via the introduction of a non-minimally coupled scalar field. We compute all the one-loop divergences in both theories and find a *physical* combination of couplings whose *running* differs for intermediate values of the non-minimal coupling.

## RESUMEN

Esta tesis, escrita como un compendio de artículos, aborda algunos de los problemas fundamentales que se plantean al intentar construir una teoría de la gravedad cuántica. Tomando como punto de partida la Relatividad General (RG), su conocido carácter no normalizable conduce a la necesidad de una compleción ultravioleta. Además, el problema de la Constante Cosmológica (CC) sigue siendo una de las piedras angulares de la Física Teórica. Los siguientes artículos exploran posibles soluciones a estos problemas en el contexto de las Teorías de Campos Efectivas de la gravedad.

El primer artículo está dedicado al estudio de las teorías gravitatorias cuadráticas (en curvatura) cuando se tratan en el *formalismo de Primer Orden*, donde la métrica y la conexión se consideran campos independientes. Estas teorías renormalizables son cuadráticas en las derivadas de la conexión y no contienen propagadores cuárticos, dejando *a priori* cierto espacio para la unitaridad. Sin embargo, no está claro si estas teorías incluyen un gravitón o si están libres de *fantasmas*, ya que toda la dinámica está ahora codificada en el campo de la conexión. Se necesita entonces un estudio completo de los grados de libertad que se propagan. En este trabajo, analizamos el contenido de espín de una conexión genérica libre de torsión construyendo una base completa de 22 proyectores de espín de seis índices. Encontramos que estas teorías propagan genéricamente una pieza de espín tres junto con varias componentes de espín inferiores.

Una de las soluciones clásicas al problema de la CC es considerar teorías invariantes Weyl, ya que prohíben un término de CC en la acción. Esta simetría es una generalización de la invariancia conforme habitual a los casos en los que la gravedad está presente. En el segundo artículo de la tesis, realizamos un análisis de la (in)equivalencia de las teorías conformes e invariantes Weyl para el propio campo gravitatorio. Se explora el Lagrangiano más general para partículas de espín dos hasta operadores de dimensión seis correspondientes a la expansión de baja energía de las teorías lineales y cuadráticas (en curvatura). Llevamos a cabo una clasificación completa de las teorías invariantes bajo difeomorfismos (transversos) linealizados, transformaciones de Weyl linealizadas, y las simetrías conformes y de escala.

En la última parte de la tesis se examina la teoría de la Gravedad Unimodular (GU). Esta teoría es una descripción alternativa de baja energía de la gravedad definida como el truncamiento de la RG a métricas con determinante unidad. En la GU la CC no se acopla directamente a la gravedad debido a la restricción unimodular, y por tanto, posee una naturaleza completamente distinta. En particular, no recibe correcciones radiativas, lo que resuelve parcialmente el problema de la CC. Aparte del carácter de la CC, la GU resulta ser clásicamente equivalente a la RG, y la cuestión de la (in)equivalencia completa de ambas teorías sigue siendo un debate abierto cuando se consideran las correcciones cuánticas. Se investigan las posibles diferencias que surgen al estudiar el acoplamiento a la materia, mediante la introducción de un campo escalar no mínimamente acoplado. Calculamos todas las divergencias a un bucle en ambas teorías y encontramos una combinación de constantes de acoplo *física* cuya variación con la energía difiere para valores intermedios del acoplo no mínimo.

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*All these people think love's for show*
*but I will die for you in secret.*

# CONTENTS

# INTRODUCTION

Theoretical Physics is facing stimulating times. Although the discovery of the Higgs Boson at the LHC [1, 2] completed the Standard Model of particle physics – the theory describing fundamental particles and their interactions –, many new questions came along regarding the precise value of the Higgs mass. Together with other long-standing issues such as the puzzle of neutrino masses, the matter-antimatter asymmetry of the universe, or the nature of dark matter, all these issues require the introduction of *new physics*. Furthermore, a full understanding of the gravitational interaction is still the *Pot of Gold* of Theoretical Physics. Despite the success of General Relativity (GR) [3] as a low-energy theory describing a wide range of gravitational phenomena, its quantum completion is still an open issue. It seems, however, that we could be seeing some light at the end of the tunnel (or rather *hearing* another type of *waves*). The successful current generation of cosmological experiments, and more importantly the direct detection of gravitational waves by the LIGO-VIRGO collaboration [4], have allowed us to probe new regimes in gravitational physics which were not accessible before. More so, next-generation experiments will push this program further towards the edge of our current understanding and support the exploration of possible modifications of GR and their impact on observable physics.

## THE QUEST FOR A COMPLETE THEORY OF GRAVITATION

This thesis, presented as a compilation of three articles, aims to shed some light on some of the problems arising when trying to get to a quantum theory of Gravitation. The philosophy followed here has been to break down this problem into various subproblems and explore possible solutions to some of them. With this in mind, we have tried to get some insight into how a Quantum Gravity theory might look like. But before we get into the heart of the matter, let us take a step back and remember where we are coming from. In the following, we will take a general overview of the discoveries and breakthroughs that have shaped our current understanding of the universe.

### *Physics as a cube: a tale of three fundamental constants*

Most of the physics of the last centuries can be summarized and understood via three fundamental constants of nature: Newton's constant, $G_N$, the speed of light in vacuum, $c$, and Planck's constant, $\hbar$. Each of these constants embodies a basic prinpicle of Theoretical Physics. Let us briefly discuss the new features that came along with each of them

- The first of the three, $G_N$, was introduced by Newton in his universal law of gravitation back in 1687, as a proportionality factor between the gravitational force felt by two masses and the quotient between the product of the masses and the distance between them. This dimensionful constant gives us an idea of the strength of the gravitational interaction, and in particular, its small value makes gravity the weakest force compared to the other three fundamental interactions (taking for example the gravitational force felt by two protons in a nucleus, it is 38 orders of magnitude weaker than the strong force between them).

- The constancy and finiteness of the speed of light in all inertial reference frames was introduced by Einstein in his theory of Special Relativity (SR) in 1905 [5]. This idea completely revolutionized the way we treated space and time, unifying those into a single entity. Being a universal dimensionful constant, $c$ can be used to transmute dimensional quantities like length and time (e.g. $x = ct$). The same can be applied to mass and energy, as encoded in Einstein's famous formula

$$E = \sqrt{(pc)^2 + (mc^2)^2}.$$

  Let us highlight that this formula associates an energy to massives particles but also to massless ones, proportional to their momentum. Before SR, the notion of massless particles could not even be addressed using Newtonian mechanics.

- The last constant belongs to the realm of Quantum Mechanics (QM). The introduction of the quantum world implied a complete change of paradigm in physics. In this context, the constant $\hbar$ is the minimum value that the physical action can have. This translates into the quantization of physical quantities, such as linear momentum or energy, meaning that they now come in tiny packets called *quanta*. This theory governs the interaction of the fundamental and smallest constituents of matter and departures from the deterministic viewpoint of classical mechanics, due to properties like the uncertainty principle, which implies that we cannot measure conjugate quantities such as momentum and position with infinite precision. It also gives one of the most incredible features of nature, namely that energy fluctuations violating energy conservation are allowed as long as they happen in a very small period of time. Moreover, when combined with SR, this has game-changing consequences.

After introducing the basic building blocks, we can turn to the construction of the cube shown in figure 1. Simple at first sight, it summarizes the foundation of modern Theoretical Physics.[1] Each of the axes corresponds to one of the three aforementioned fundamental constants (note that we take the inverse of the speed of light for simplicity, as will be explained shortly). Different vertices are reached when each of the constants is either switched on or off, depending on whether they are relevant for the phenomena we want to study. For instance, processes involving small velocities in comparison with the maximum velocity $c$ are dubbed as non-relativistic as they correspond to taking the limit $c \to \infty$, and thus, $c^{-1} \to 0$. For speeds comparable to $c$, special-relativistic effects become important and are desbribed by moving one unit along the $c^{-1}$ axis.

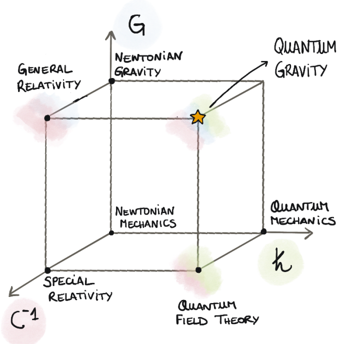


Figure 1: Pictorial representation of the theories governed by each of the fundamental constants. The axes correspond to Newton's constant, the inverse of the speed of light, and Planck's constant.

Once the stage has been settled, we can explore the cube starting from the origin. It represents the long-known Newtonian mechanics, a classical (as opposed to quantum) theory without gravity, valid as long as the speeds considered remain well bellow the speed of light. Even if we take it as the starting point, it took many great minds to get there. From that point, Newton already took a step up and conquered the vertex of the so-called Newtonian Gravity, and the first part of the 20th century was devoted to exploring the other two directions, arriving at SR and QM.

As a next step, various of these movements can be combined, switching on two constants simultaneously. Looking at the lower face of the cube, we can climb to the first hybrid point, where both $G_N$ and $c$ enter the game. This point corresponds to the General Theory of Relativity, also due to Einstein in 1915 [3]. This theory generalizes the ideas of SR to spacetimes where gravity is present. In this theory, gravity is treated as a field, fully characterized by a symmetric tensor called the metric tensor, $g_{\mu\nu}$. Einstein's equations provide the equations

[1] As a remark, this idea was already discussed by G. Gamow, D. Ivanenko and L. Landau in *Zh. Russ. Fiz. Khim. Obstva. Chast' Fiz. 60 (1928) 13* (in Russian).

of motion for the gravitational field in the presence of matter or energy, and take the form

$$R_{\mu\nu} - \frac{1}{2}(R + 2\Lambda)g_{\mu\nu} = \frac{8\pi G_N}{c^4} T_{\mu\nu} \tag{1}$$

where $R_{\mu\nu}$ and $R$ stand for the Ricci tensor and the Ricci scalar respectively and are constructed from the metric and its second derivatives, and $\Lambda$ represents the Cosmological Constant (CC) term.[2] The right-hand side corresponds to the energy-momentum tensor that contains all the information about any kind of energy density in spacetime. These equations entail a very profound and geometrical interpretation of gravity: the energy density affects the geometry of spacetime, and at the same time, gravity tells matter and energy how to move on this geometry. The action from where these equations of motion are obtained is called the Einstein-Hilbert action,[3]

$$S_{EH} = -\frac{1}{2\kappa^2} \int d^4x \sqrt{|g|}\,(R + 2\Lambda), \tag{2}$$

where $\kappa^2 = (8\pi G_N)/c^4$ contains the two relevant constants, as advertised.

The other interesting interplay, starring $c$ and $\hbar$, takes us to Quantum Field Theory (QFT). This is a framework that allows for the unification of SR and QM, where brand new phenomena arise. We know from QM that quantum energy fluctuations happen in small intervals of time, and now, when SR kicks in, that energy can be converted into mass, and therefore, into particles. It also provided the introduction of antiparticles, opening a new world of possibilities. As the standard lore says, anything that is allowed in QFT will indeed happen. For example, the vacuum can produce a particle-antiparticle pair that then annihilates back into the vacuum. Therefore, we immediately depart from a static picture of the vacuum and replace it with a boiling sea of quantum fluctuations. New tools are then needed to treat all these processes and their contribution to physical observables. After the path integral formulation of QFT in terms of the action, Feynman introduced a simple technique to compute these contributions to physical quantities in perturbation theory. Each of the relevant processes can be depicted as a diagram, and some rules (which can be extracted from the action) can then be applied to compute each of their contributions. Moreover, these diagrams can be ordered in a series expansion depending on the number of internal loops (closed paths),[4] which corresponds to an expansion in powers of $\hbar$. Higher loops are thus more suppressed and a hierarchy between each of the orders exists. The computation of these contributions was not straightforward though, and the pioneers of QFT faced a lot of apparent

[2]Incidentaly, talking about famous constants, we will see that the CC embodies one of the biggest open problems of Theoretical Physics itself.

[3]We take the mostly minus convention for the metric.

[4]Processes with more loops can be generated for example if the created particle or antiparticle further splits into another pair.

problems coming from divergent integrals whose interpretation was not clear. After years of work, a technical procedure called renormalization was introduced to absorb these infinities and render the physical observable quantities finite. In a nutshell, it consists of introducing a finite[5] number of terms, dubbed counterterms, so that they can subtract the infinite part of the non-observable quantities that were appearing in the initial description of the theory. However, this procedure only works for some theories, which are therefore called renormalizable. In this context, the Standard Model arose as a renormalizable QFT describing the strong, weak, and electromagnetic interactions, and has turned out to be arguably one of the most tested scientific theories of all time.

Before going into the theory where all three constants are important, let us comment on a concept that also involves the use of the three. Combining these dimensionful quantities, we can build up some universal units of time, length, and mass (or energy), called Planck's units.[6] The existence of these units gives us a universal way of measuring these quantities which has nothing to do with the particularities of our world or galaxy. The Planck mass can be constructed by equating two different lengths built by using GR, namely $MG_N/c^2$, or QFT, $\hbar/Mc$, so that we get another constant given by $M_p = \sqrt{\hbar c/G_N}$. This mass is huge, $M_p \sim 10^{19} m_{\text{proton}}$, and turns out to be very important in the discussion that will follow, as the combination of GR and QM becomes necessary at this energy.

But what happens then when the three fundamental constants are switched on? This is indeed the *Holy Grail* of Theoretical physics: a theory of Quantum Gravity (QG), and the hope of a unified treatment of the four fundamental interactions of nature. To reach this star point in the cube, we see that we would need to blend GR and QFT together in a consistent way. Unfortunately, this is proven not to be straightforward and is in turn one of the greatest open problems in Theoretical Physics, and certainly, one that has caused rivers of ink to flow in the last decades. At this point, one could ask whether a QG theory is actually necessary.[7] This seems to be the case when looking at one of the most amazing avatars appearing in GR: Black Holes (BH). These objects are a direct consequence of the existence of a maximum velocity, namely $c$. If the gravitational field is strong enough, the escape velocity of an object can exceed the speed of light, and thus, nothing can escape from such a gravitational field. This is the mechanism at work in a BH. Once an object passes through a certain distance from the center of the BH, called the event horizon, nothing can escape from it – not even light. But in a turn of events, it happens that this is not

[5] The fact that the number of counterterms needed is finite, is key for renormalization to work. We will shortly see that this is not possible in the case of GR.

[6] Although they play a main role in most areas of theoretical physics, they are most of the time fixed to unity for simplicity. We work with $c$ and $\hbar$ equal to unity during most of this thesis.

[7] The idea that gravity is not a *fundamental* interaction but rather an emergent phenomenum that has to be treated in a thermodynamical framework is explored in Emergent Gravity theories [6, 7].

*strictly* true. When quantum mechanical effects are considered close to the event horizon, a black-body type of radiation called Hawking radiation [8] is predicted. One of the consequences of this phenomenon is a decrease in the area of the BH given by a formula containing the three fundamental constants $\hbar$, $c$, and $G_N$. This fact, together with the existence of a singularity in a tiny region of the center of the BH which GR cannot resolve, makes BHs the perfect arena for *testing* QG effects.

At this point, there are two possible ways to address the construction of a quantum theory of gravity. The first one modifies the high-energy degrees of freedom and looks for a complete theory first, and tries to make contact with the physics we know within that framework afterwards. This top-bottom approach has culminated in String Theory, the strongest candidate so far for a complete theory of QG. Nevertheless, there are still open issues regarding how to precisely recover the low-energy theory that describes our universe. The bottom-up approach on the other hand, takes GR as a starting point and looks for minimal extensions to find a high-energy completion of it. The latter is the viewpoint taken throughout this thesis. Let us then go through the problems found when trying to quantize GR, which will be adressed in the three articles contained in the thesis.

### *General Relativity and the issues to go quantum*

The failed marriage attempt between QFT and GR has been a very announced and suffered one. From the beginning, some conceptual problems could already be foreseen when looking at one of the foundational principles of QFT. In QFT the spacetime is a fixed entity corresponding to flat Minkowski spacetime, which is then used to define key aspects like causality. When gravity becomes quantum, however, this spacetime can fluctuate and even the notion of causality is not well defined anymore. To surpass this issue, we can study the quantum fluctuations of the metric around a fixed background expanding it as

$$g_{\mu\nu} = \bar{g}_{\mu\nu} + \kappa h_{\mu\nu}, \tag{3}$$

where the first term corresponds to the fixed background field and $h_{\mu\nu}$ represents the graviton fluctuations (and $\kappa$ is defined right below (2)). This procedure is called the Background Field method [9]. In this way, the usual quantization methods of QFT can be applied to the graviton fluctuations. Unlike the QFT describing the other three fundamental interactions, we find that GR is non-renormalizable. This is tied to the fact that Newton's constant has dimensions of inverse energy squared. This has far-reaching consequences when looking at the contributions to the scattering amplitude coming from different loop orders. As higher loops are considered, the number of internal fields grows, and thus, more propagators or higher interaction vertices are needed. In particular, the interaction vertices of the gravitons, as computed from the expansion of the action (2), contain different powers of Newton's constant depending on the number of fields involved. From (3) we can see that each graviton contributes with one power of $\kappa$. As the action has an extra $\kappa^{-2}$, we can see that the propagators do not carry any power of $\kappa$, whereas the three and four-vertices pick up one and two extra powers respectively. Taking for instance the tree-level and one-loop processes in the graviton-graviton scattering diplayed in figure 2, we see that they are of different order in $\kappa$. By dimensional analysis,

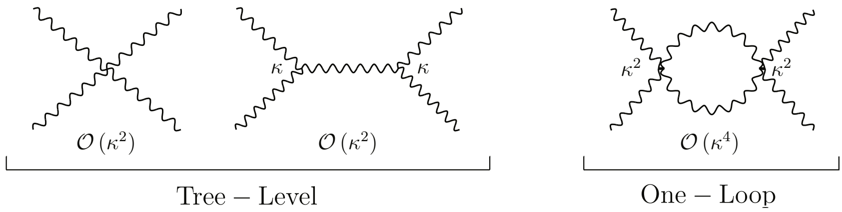


Figure 2: Tree-level and one-loop diagrams for a graviton-graviton scattering.

higher-order contributions will contain higher powers of the external momenta to compensate for the extra powers of $\kappa$. This translates into a series of terms with an increasing number of derivatives. One-loop counterterms correspond to quadratic invariants of the curvature (terms with four derivatives such as the Ricci scalar squared), and consequently, higher-dimensional curvature invariants arise at higher loops. Subsequently, an infinite number of counterterms would be needed to absorb the infinities contained in the different terms at all orders of perturbation theory. The only possible way out then is the vanishing of all these invariants when the equations of motion are imposed. This is true at one-loop for pure gravity [10], but already fails at the two-loop level [11], and thus, GR is non-renormalizable.

Although we know that this is an unavoidable problem, the hierarchical structure of the scattering amplitudes in terms of loop contributions allows for the theory to be treated as an EFT valid in a finite range of energies. Recalling the equivalence between energy and momentum, the scattering amplitude at an energy $E$ can be symbolically written as the series

$$\mathcal{M} \sim G_N[1 + G_N E^2 + (G_N E^2)^2 + \dots], \tag{4}$$

where the factors of $\hbar$ and $c$ are fixed to one. Therefore, as long as $E \ll (1/G_N)^{1/2} \equiv M_p$, we can truncate the perturbative series up to some loop order and work with it as an approximate theory, given that the non-renormalizable pieces are highly suppressed by powers of $G_N$. This is true up to energies close to the Planck mass $M_p$, where the perturbative expansion breaks down and higher terms in the expansion cannot be ignored. This signals the need for *new* *physics* at least at energies of the order of the Planck mass – where the star is reached in the cube.

But this is not the only problem of GR and its unification with the rest of interactions. The so-called Cosmological Constant (CC) problem, namely the explanation of why the universe is expanding in an accelerated way due to the presence of a tiny but positive CC constant, given by $(8\pi G_N)^{-1}\Lambda \sim 10^{-47} GeV^4$, is one of the greatest challenges in Theoretical Physics. This constant, introduced by Einstein himself so that the solution for a static universe could be accommodated in the theory, was later acknowledged by him as his *biggest failure* when the true expanding nature of the universe was shown by Hubble. After that, solutions for an expanding universe where found, which where independent of the value of the CC. This lead the comunity to assume that the CC was zero for a long time, till the accelerated expansion of the universe was discovered [12, 13]. Nonetheless, finding an explanation for a zero CC was not an easy task either. Lorentz invariance implies that the contribution to the vacuum energy from quantum fluctuations takes the same form as the contribution from a CC-like term given by

$$\left\langle T_{\mu\nu} \right\rangle = \left\langle \rho_V \right\rangle g_{\mu\nu}, \tag{5}$$

where $\langle \rho_V \rangle$ corresponds to the vacuum energy density. These terms add up to the bare CC appearing in equations (1) and we can thus define an effective CC constant as

$$\Lambda_{\text{eff.}} = \Lambda + 8\pi G_N \left\langle \rho_V \right\rangle. \tag{6}$$

Looking at the form of the effective CC, it is easy to explore the two aspects of the problem. The first one is related to the lack of an explanation for the particular tiny value of the CC, and the need for a mechanism or principle that sets it to this value. The second one is related to its fine-tuning. Using the QFT framework to compute the zero-point energy density contributions, and trusting GR up to the Planck mass, we obtain that the vacuum density goes as $\langle \rho_V \rangle \sim M_p^4$. To match the experimental data, the sum of the bare cosmological constant plus the vacuum energy contributions, as computed in the QFT framework, must *unnaturaly* cancel up to 118 decimal points. Many attempts to solve this problem have been made, but its solution is still *the big elephant in the room*.

## EFT APPROACH TO SOME OF THE PROBLEMS OF GR

During the completion of this thesis, we have mainly focused on some possible ways out to the two problems of GR that were introduced in the previous section, that is, its non-renormalizability and the CC problem. The idea behind the articles presented here (as well as some of the other publications carried out during these years), is to modify GR in a minimal way to explore alternatives for its UV completion. In the diagram of figure 3, a summary of the issues analyzed in these works can be found, as well as the insights and drawbacks that each of them presents. In the following, we will briefly introduce the main points included in each of the three articles.[8]

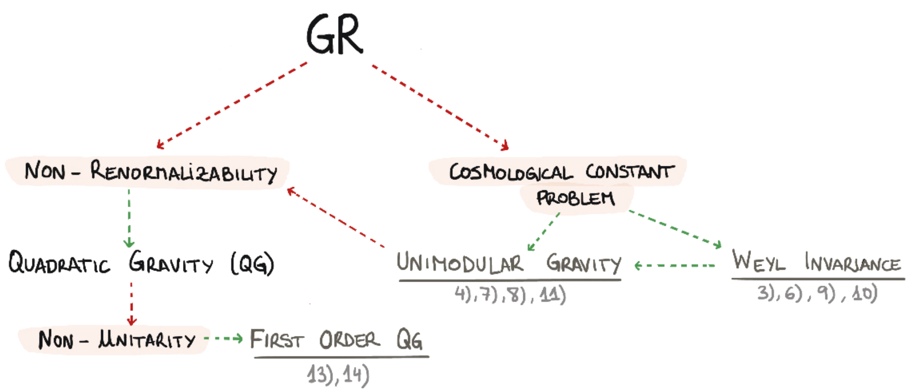


Figure 3: Summary of the research directions and their connections. Red lines indicate the drawbacks found at each step, whereas the green lines correspond to the insights followed. The numbers in gray correspond to the precise articles where these ideas have been followed, matching the number on the list included at the beginning of the thesis.

*Non-renormalizability of GR and First Order quadratic gravity*

One of the first solutions to try to bypass the non-renormalizability of GR consists on considering theories that are quadratic in the curvature, instead of the linear one given by (2). In principle, one can write three of such quadratic gravitational invariants, but only two of them are independent when using the Gauss-Bonnet identity. To this end, we consider the action

$$S_{quad} = \int d^4x \sqrt{|g|} \, (\alpha R^2 + \beta R_{\mu\nu} R^{\mu\nu}). \tag{7}$$

The appealing feature of these theories is the fact that the coupling constants appearing in front of each of the quadratic invariants are dimensionless. As opposed to the GR case, the loop expansion contains a finite number of terms

[8] Please refer to the full introduction of each of the papers for specific and more technical details.

and therefore, the divergences can be absorbed by a finite number of counterterms. Thus, it can be proven that these theories are renormalizable for some regions of the parameter space of $\alpha$ and $\beta$ [14]. The second important difference with GR is that quadratic theories are quartic in the derivatives of the metric. As a consequence, the graviton propagator is quartic in the momentum. This momentum dependence implies that these theories dominate at high energies and, therefore, can work as a UV completion of GR. Nevertheless, a mechanism is needed to recover GR at low-energies and describe the observable universe. This can be achieved for instance by directly appending the EH term to the action, which will gobern the theory at low energies.[9] Other mechanisms include minimally coupling a scalar field with a potential, which after renormalization produces a counterterm of the form $\Delta\mathcal{L} = \sqrt{|g|}C\phi^2 R$. An EH-like term can be then generated from the spontaneous breaking of scale invariance via a vacuum expectation value $< \phi >= v$ for the scalar field, at an energy scale given by $M^2 = Cv^2$. This term will then dominate in the infrared and we can recover the GR phenomenology.

Unfortunately, these theories suffer from another fatal problem coming from the quartic momentum dependence. We know from the Källen-Lehmann spectral representation [15, 16] that any propagator can be written as the integral sum of some positive density multiplied by quadratic propagators. Taking a quartic propagator and decomposing it in such a way, it is straightforward to see that one of the two pieces carries the wrong sign, thus signaling the presence of a *ghost* [17] and the corresponding non-unitarity of the theories. The first of the three articles presented in this thesis tries to give a possible way out to the non-unitarity problem of quadratic theories (that is, the left branch of figure 3). To do so, we treat these theories in the *First Order (FO) formalism*. In the FO approach, the metric and the connection are represented by completely independent fields, as opposed to their relation in the usual *Second Order formalism* where the Levi Civita connection is constructed from the metric tensor and its derivatives. Once we take this into account, the first consequence is straightforward: the action is now quadratic in the derivatives of the connection so that no quartic propagators appear. Nonetheless, FO quadratic theories come with new features that need to be carefully analyzed. In particular, the solution space of these theories is bigger than that of usual quadratic gravity, and more degrees of freedom may be present coming from the three-index connection field. Therefore, the behavior of these new components needs to be investigated to fully assess the unitarity issue. Moreover, when expanded around flat spacetime, the graviton propagator vanishes. Hence, the gravitational degrees of freedom must come from a spin two-piece hidden in the connection field. This is studied in detail in the first article included in the thesis.

[9] This is indeed the usual action considered in the literature when talking about quadratic gravity.

*Insights from Weyl invariance*

Changing gears, let us focus on another concept that has been constantly popping up along the thesis: Weyl invariance. This symmetry is nothing but the generalization of scale or conformal invariance to the case in which gravity is present. All these symmetries embody the old and intuitive idea that at very high-energies, masses should be unimportant, and hence, the theories describing our universe could enter a conformal or scale-invariant phase. But why is this symmetry appealing in the first place? Weyl invariance manifests itself as the invariance under local rescalings of the metric, namely, $g_{\mu\nu} \to \Omega^2(x) g_{\mu\nu}$. Therefore, if the universe were Weyl invariant, no CC term would be allowed, as the volume term in the Lagrangian would not be invariant under such transformations.[10] Moreover, as we will see in the second paper of the thesis, Weyl invariance turns out to be quite restrictive. It could then be the case that if a fully Weyl invariant quantum theory existed, no counterterm respecting all the symmetries of the theory would exist, and thus, the theory could be *finite*. Although these utopic ideas would help with the CC problem and the possible finiteness of such theories, it is a fact that our universe is not Weyl invariant, at least up to the energies avalaible to us in the LHC. Moreover, most of these theories suffer from quantum anomalies already at the one-loop level (see [18] for a review on conformal anomalies and [19] for a comparison when gravity is dynamical), so it is not even clear whether fully consistent Weyl invariant theories exist at all (see [20] for a related discussion). Contributions to the CC would then arise at the symmetry-breaking scale.

In any case, although necessarily broken at low energies, Weyl invariance is a recurring symmetry in Theoretical Physics. Accordingly, aspects like the differences among scale, conformal and Weyl invariance, together with the subtleties arising for dynamical gravity, have been thoroughly analyzed in the literature, as well as in this thesis. The first of these points, for instance, has already been broadly investigated in several works for different types of couplings of matter to gravity. In the second paper of this thesis, however, an analysis of this (in)equivalence is carried out for the gravitational field itself. To do that, we study the most general low-energy action describing spin two particles as the linear combination of operators quadratic in the field, with two and four derivatives. These type of operators arise as the low-energy limit of theories linear and quadratic in the curvature. We compute the conditions on the coefficients of the operators for the theories to be scale, conformal, or Weyl invariant, combining them with the linearized version of the symmetries of gravitational theories, namely, (transverse) diffeomorphism invariance.

---

[10] The volume measure is not invariant under Weyl transformations and it transforms as $\sqrt{g}\, d^4x \to \Omega^4 \sqrt{g}\, d^4x$.

*Unimodular Gravity: invisible to the CC*

Following the path of possible insights on the CC problem, Unimodular Gravity (UG) poses a very interesting alternative to GR, in which the CC has a different character. UG is defined as a truncation of GR to metrics with unit determinant, $\hat{g} = 1$, and its action can be written as

$$S_{UG} = -\frac{1}{2\kappa^2} \int d^4x \, (\hat{R} + 2\Lambda), \tag{8}$$

where a hat[11] is used to denote that the Ricci scalar is constructed with the unimodular metric $\hat{g}_{\mu\nu}$. We have explicitly written a CC-like term to highlight the fact that these types of constant contributions in the action are irrelevant in UG, as they do not couple to gravity. Therefore, the CC does not gravitate, that is, it does not couple to the gravitational field. Moreover, the unit determinant constraint has further implications. First of all, the variation of the determinant must vanish, and thus, the metric variations are found to be traceless, $\delta\hat{g} = \hat{g}^{\mu\nu}\delta\hat{g}_{\mu\nu} = 0$. Taking into account the infinitesimal transformation of the metric under general diffeomorphisms, $\delta\hat{g}_{\mu\nu} = \mathcal{L}_{\xi}\hat{g}_{\mu\nu}$, the tracelessness condition implies that UG is only invariant under transverse diffeomorphisms. The symmetry group is then reduced to volume-preserving diffeomorphisms, whose algebra dubbed as *TDiff* corresponds to the infinitesimal diffeomorphisms generated by transverse vectors.

Differences aside, both theories share many properties that may not be obvious at first sight. On the one hand, the three gauge conditions from *TDiff* are enough to reduce the 5 massive graviton degrees of freedom down to the 2 massless ones [21]. On the other hand, they share classically equivalent equations of motion. Due to the tracelessness condition of the variations, the unimodular equations of motion are just the traceless part of Einstein's equations, namely,

$$R_{\mu\nu} - \frac{1}{4}Rg_{\mu\nu} = \kappa^2 \left( T_{\mu\nu} - \frac{1}{4}Tg_{\mu\nu} \right). \tag{9}$$

Nevertheless, we can reintroduce the trace into the game by using the Bianchi identities, $\nabla^{\mu}R_{\mu\nu} = \frac{1}{2}\nabla_{\nu}R$, and the conservations of the energy-momentum tensor, implying

$$\nabla_{\nu}(\hat{R} + \kappa^2 T) = 0 \quad \rightarrow \quad \hat{R}_{\mu\nu} - \frac{1}{2}\left(\hat{R} + 2\Lambda\right)\hat{g}_{\mu\nu} = \kappa^2 T_{\mu\nu}, \tag{10}$$

where $\Lambda \equiv -\frac{1}{4}(\hat{R} + \kappa^2 T)$. Here we can appreciate the main difference between GR and UG. While in GR the CC appears like a parameter in the Lagrangian,

[11] A note of caution here. In the third paper of the thesis, the notation is changed and $g_{\mu\nu}$ is used for unimodular metric and $\tilde{g}_{\mu\nu}$ for the unconstrained ones. We have chosen to change the notation as $g_{\mu\nu}$ has been used for GR metric throughout the introduction.

here it corresponds to an integration constant which is in principle fixed by boundary conditions of our universe. This would then mitigate one of the aspects of the CC problem as we saw it, as the CC appears as a fixed integration constant in the Lagrangian and is not modified by radiative corrections. It does not help, however, with the other facet of the problem, namely, the precise tiny value of $\Lambda$. Classically then, the equations of motion are equivalent to the ones found in GR with a family of fixed cosmological constants. In particular, it can be shown that we can obtain the usual BH solutions[12] as well as the FRW cosmological solutions, from UG [22].

The big question is then whether both theories are also equivalent at the quantum level. There has been a long debate on trying to understand whether UG is actually a different theory than GR in any additional aspect apart from the fact that the vacuum energy does not gravitate. In the third paper of this thesis, we explore some possible differences between these two theories at the quantum level. The basic idea is that whereas on-shell gravitons are transverse and traceless in both UG and GR, their trace propagates off-shell in GR and contributes to the renormalization of the CC. In UG, as a consequence of fixing the condition $\hat{g} = 1$, the trace of the graviton is always absent, even off the mass-shell. With this in mind, we compute one-loop corrections for these theories when non-minimally coupled to a massive scalar field.

Let us comment on another important aspect of UG and its interplay with Weyl invariance. When taking into account quantum corrections in UG, or more precisely, when the path integral formulation is used, technical difficulties arise due to working with constrained metrics. This is a subtle issue and computations of purely gravitational corrections turned out to give contradictory results when carried out with unconstrained or constrained metrics [22, 23]. The first approach uses an alternative description of UG, obtained by writing the constrained metric in terms of an unconstrained one via the transformation, $\hat{g}_{\mu\nu} = \hat{g}^{-1/4} g_{\mu\nu}$, where the unit determinant condition is automatically fulfilled for any $g$. With this construction, an extra Weyl invariance is introduced, and we can always gauge fix the new theory to go back to UG. In terms of the unconstrained metric, the action can be written as

$$S_{\text{WTDiff}} \equiv -\frac{1}{2\kappa^2} \int d^4x \, |g|^{1/4} \left\{ R + \frac{3}{32} \frac{\nabla_\mu g \nabla_\nu g}{g^2} g^{\mu\nu} \right\}, \tag{11}$$

and is usually called *WTDiff*, after this extra Weyl symmetry on top of the transverse one. This is the formulation that will be used in the third paper of the thesis. With it, we can work with unrestricted fields in the path integral formulation, but the price to pay is the complicated gauge sector needed to fix both symmetries.

[12] Actually Schwarzschild himself used a unimodular metric in its original derivation of his BH solution.

$$
\begin{array}{ccc}
S_{TDiff}[\hat{g}] & \xrightarrow{(\hat{g}_{\mu\nu}=g^{-1/4}g_{\mu\nu})} & S_{WTDiff}[g^{-1/4}g_{\mu\nu}] \\
\Big\downarrow{\scriptstyle \delta\hat{g}_{\mu\nu}} & & \Big\downarrow{\scriptstyle \delta g_{\mu\nu}} \\
\boxed{\hat{R}_{\mu\nu}-\frac{1}{4}\hat{R}\hat{g}_{\mu\nu}=0} & \xrightarrow{(\hat{g}_{\mu\nu}=g^{-1/4}g_{\mu\nu})} & \boxed{R_{\mu\nu}-\frac{1}{4}Rg_{\mu\nu}=f(g)} \\
{\scriptstyle \nabla_\nu\hat{R}=\nabla^\mu\hat{R}_{\mu\nu}}\Big\downarrow & & \Big\downarrow{\scriptstyle \nabla_\nu R=\nabla^\mu R_{\mu\nu}} \\
\boxed{\hat{R}_{\mu\nu}-\frac{1}{2}(\hat{R}+2\Lambda)\hat{g}_{\mu\nu}=0} & \xrightarrow{(\hat{g}_{\mu\nu}=g^{-1/4}g_{\mu\nu})} & \boxed{R_{\mu\nu}-\frac{1}{2}(R+2\Lambda)g_{\mu\nu}=h(g)}
\end{array}
$$

As a final remark, one can close the loop and show that the CC can also be reintroduced in this *WTDiff* formulation via the Bianchi identity. This is shown schematically for UG and *WTDiff* theories in vacuum (but can be easily generalized to the case of a non-vanishing energy-momentum tensor) in the diagram bellow. There, all the transformations between the restricted and unrestricted metric are indicated, where $f(g)$ and $h(g)$ are functions of $g$ and its derivatives.[13] As expected, both theories are trivially equivalent when the Weyl gauge in *WTDiff* is fixed to $g = 1$.

*Plan of the thesis*

After this general introduction, the second part of the thesis contains the collection of three articles in the published version. Each of them is presented in a separated chapter. We leave the conclusions for the final section, where the main results of these works are summarized.

[13]More precisely,

$$
\begin{aligned}
f(g) &= \frac{7}{32}\left(\frac{\nabla_\mu g\nabla_\nu g}{g^2}-\frac{1}{4}\frac{\nabla_\alpha g\nabla^\alpha g}{g^2}g_{\mu\nu}\right)-\frac{1}{4}\left(\frac{\nabla_\mu\nabla_\nu g}{g}-\frac{1}{4}\frac{\Box g}{g}g_{\mu\nu}\right)\\
h(g) &= \frac{7}{32}\frac{\nabla_\mu g\nabla_\nu g}{g^2}-\frac{17}{64}\frac{\nabla_\alpha g\nabla^\alpha g}{g^2}g_{\mu\nu}-\frac{1}{4}\frac{\nabla_\mu\nabla_\nu g}{g}+\frac{1}{4}\frac{\Box g}{g}g_{\mu\nu}
\end{aligned}
\tag{12}
$$

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# Collection of articles

# 1

## PHYSICAL CONTENT OF QUADRATIC GRAVITY

This chapter contains the article

- Enrique Alvarez, Jesus Anero, Sergio Gonzalez-Martin and Raquel Santos-Garcia,

  "Physical content of Quadratic Gravity",

  *Eur.Phys.J.C* **78** (2018) 10, 794, arXiv: 1802.05922 [hep-th].

Eur. Phys. J. C (2018) 78:794
https://doi.org/10.1140/epjc/s10052-018-6250-x

THE EUROPEAN PHYSICAL JOURNAL C

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# Physical content of quadratic gravity

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**Abstract** We have recently undergone an analysis of gravitational theories as defined in first order formalism, where the metric and the connection are treated as independent fields. The physical meaning of the connection field has historically been somewhat elusive. In this paper, a complete spin analysis of the torsionless connection field is performed, and its consequences are explored. The main properties of a hypothetical consistent truncation of the theory are discussed as well.

**Contents**



## 1 Introduction

Theories of gravity where the lagrangian is quadratic in the Riemann tensor [1,2] are known to be well behaved in the ultraviolet (they are often asymptotically free) but suffer from the fatal drawback of not being unitary (cf. [3] for a general review, and [4] for a recent analysis similar in spirit to ours). The distinctive flavor of our approach, as compared with previous literature on the subject (confer in particular the work of Biswas et al. [5–9] and also [10]), is that we work in the first order formalism.

It has been recently pointed out [11–13] that when considering quadratic theories of gravity in first order formalism (which is not equivalent to the usual, second order one[1]) where the metric and the connection are considered as independent physical fields, no quartic propagators appear and the theory is not obviously inconsistent. This framework is a good candidate for a unitary and renormalizable theory of the gravitational field, leading to a posible ultraviolet (UV) completion of General Relativity (GR). Recent work, following related lines, has been done regarding a possible UV completion of GR by modifying the usual second order quadratic gravity [14–16].

Those theories depend on a number of independent coupling constants, which can be grouped into three big classes, corresponding to the Riemann tensor squared, the Ricci tensor squared, and the scalar curvature squared. Althoug there are many similarities with the second order approach used in the above references, there are also crucial differences. The most important of which is that we do not have explicit violation of the positivity in the spectral function (that is, we do not have propagators falling off at infinity faster that $\frac{1}{p^2}$). This

[1] Even for the Einstein–Hilbert first order lagrangian the equivalence is lost as soon as fermionic matter is considered.

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is the reason why we have endeavored a systematic approach following our ideas from basic principles, even at the risk of rederiving some results already known in the second order approach. Of course not all of them hold true in our case. We shall point out the main differences in the main body of the paper.

In particular, there is one worrisome fact. When considering the theory around a flat background there is no propagator for the graviton. This means that either the theory is not a theory of gravity at all, or else all the dynamics of the gravitational field is determined by the three index connection field.

Of course the idea that the true dynamics of gravitation is better conveyed by the connection field than by the metric has a long history (cf. for example to the classic paper [17]). It is the closest analogue to the usual gauge theories, and can be easily related to physical experiments and observations. In fact in [11–13] we have shown that there are possible physical static connection sources that produce a $V(r) = \frac{C}{r}$ potential between them. This is at variance with what happens in the usual quadratic theories as formulated in second order, in which the natural potential is a scale invariant one $V(r) = Cr$. This forces many authors to include an Einstein–Hilbert (linear in the scalar curvature) piece in the action from the very beginning if one wants to reproduce solar-system observational constraints (cf [1,2] for a lucid discussion). Another possibility is a spontaneous symmetry breaking of the scale invariance of quadratic theories, so that the EH term is generated and dominates in the infrared (see e.g [18–22] regarding this issue).

The static connection sources in [11–13] were of the form $J_{\mu\nu\lambda} \sim j_\mu T_{\nu\lambda} + \cdots$, where $j_\mu$ was a conserved current and $T_{\mu\nu}$ was the energy-momentum tensor. The physical meaning of those sources is not clear, to say the least. In order to get a better grasp on the workings of the theory, it would be helpful to disentangle the different physical spins contained in the connection.

Our aim in this paper is precisely to perform a complete analysis of the physical content of the connection field. There are a priori 40 independent components in this field. We shall analyze them by generalizing the usual spin projectors [23–25] to the three-index case, and expanding the action in terms of these projectors. We shall find that generically there is a spin 3 component, which disappears only when the coefficient of the Riemann square term vanishes. This property is however not stable with respect to quantum corrections, that will make this term reappear even if the classical coefficient is fine tuned to zero. Kinematically, there is also a set of three spin 2 components, five spin 1 components and three spin 0 components.

Let us now summarize the contents of our paper. First we quickly review, mostly to establish our conventions, the spin content of the usual lagrangian linear in curvature (Einstein–Hilbert) both in second and in first order formalism. Then we tackle the spin analysis of theories quadratic in curvature, again both in second order and first order formalism. Extensive use is made of a new set of spin projectors, which are explained in the appendices.

Throughout this work we follow the Landau–Lifshitz spacelike conventions, in particular

$$R^{\mu}_{\ \nu\rho\sigma} = \partial_\rho \Gamma^{\mu}_{\nu\sigma} - \partial_\sigma \Gamma^{\mu}_{\nu\rho} + \Gamma^{\mu}_{\lambda\rho}\Gamma^{\lambda}_{\nu\sigma} - \Gamma^{\mu}_{\lambda\sigma}\Gamma^{\lambda}_{\nu\rho} \tag{1.1}$$

and we define the Ricci tensor as

$$R_{\mu\nu} \equiv R^{\lambda}_{\ \mu\lambda\nu} \tag{1.2}$$

The commutator with our conventions is

$$\begin{aligned} [\nabla_\mu, \nabla_\nu] V^\lambda &= R^{\lambda}_{\ \rho\mu\nu} V^\rho \\ [\nabla_\mu, \nabla_\nu] h^{\alpha\beta} &= h^{\beta\lambda} R^{\alpha}_{\ \lambda\mu\nu} + h^{\alpha\lambda} R^{\beta}_{\ \lambda\mu\nu} \end{aligned} \tag{1.3}$$

## 2 Lagrangians linear in curvature (Einstein–Hilbert) in second order formalism

Let us begin by quickly reviewing some well-known results on the quadratic (one loop) approximation of General Relativity (GR), as derived from the Einstein–Hilbert (EH) lagrangian. We do that mainly to establish our notation and methodology.

We expand the EH action around flat space by taking

$$g_{\mu\nu} = \eta_{\mu\nu} + \kappa h_{\mu\nu} \tag{2.1}$$

We are interested in the quadratic order of the expansion. The operator mediating the interaction between the metric perturbation reads

$$S = \frac{1}{2} \int d^4x\, h^{\mu\nu} K^{\text{EH}}_{\mu\nu\rho\sigma} h^{\rho\sigma} \tag{2.2}$$

where the operator reads

$$\begin{aligned} K^{\text{EH}}_{\mu\nu\rho\sigma} \equiv& -\frac{1}{8}\left(\eta_{\mu\rho}\eta_{\nu\sigma} + \eta_{\mu\sigma}\eta_{\nu\rho}\right)\Box \\ &+ \frac{1}{8}\left(\partial_\mu\partial_\rho\eta_{\nu\sigma} + \partial_\mu\partial_\sigma\eta_{\nu\rho} - \partial_\nu\partial_\rho\eta_{\mu\sigma} + \partial_\nu\partial_\sigma\eta_{\mu\rho}\right) \\ &+ -\frac{1}{4}\left(\partial_\rho\partial_\sigma\eta_{\mu\nu} + \eta_{\rho\sigma}\partial_\mu\partial_\nu\right) + \frac{1}{4}\eta_{\mu\nu}\eta_{\rho\sigma}\Box \end{aligned} \tag{2.3}$$

In order to better understand the physical content of this action, we can decompose the symmetric tensor $h_{\mu\nu}$ as

$$\begin{aligned} h_{\mu\nu} =& h^2_{\mu\nu} + \Box^{-1}\left(\partial_\mu A_\nu + \partial_\nu A_\mu\right) - \frac{\partial_\mu\partial_\nu}{\Box}\Phi \\ &+ \frac{1}{3}\left(\eta_{\mu\nu} - \frac{\partial_\mu\partial_\nu}{\Box}\right)\psi \end{aligned} \tag{2.4}$$

where as we shall see $h^2_{\mu\nu}$ corresponds to the spin 2 part of the field. The other fields are defined as follows

$$\phi \equiv \partial^\rho \partial^\sigma h_{\rho\sigma} \equiv \Box \Phi$$
$$h \equiv \eta^{\mu\nu} h_{\mu\nu}$$
$$A_\mu \equiv \partial^\sigma h_{\mu\sigma}; \quad \partial_\mu A^\mu = \Box\Phi \tag{2.5}$$

Under linearized diffeomorphisms

$$\delta h_{\mu\nu} = \partial_\mu \xi_\nu + \partial_\nu \xi_\mu \tag{2.6}$$

these transform as

$$\delta\phi = 2\Box^2 \xi$$
$$\delta h = 2\Box\xi$$
$$\delta A_\mu = \Box \xi^T_\mu + 2\Box\partial_\mu \xi \tag{2.7}$$

where we have split $\xi_\mu$ in its transverse ($\xi^T_\mu$) and longitudinal ($\partial_\mu \xi$) parts.

From the transformation properties, it is clear that there is a scalar gauge invariant combination

$$\delta\psi \equiv \delta\,(h - \Phi) = 0 \tag{2.8}$$

As stated before, we want to carry out an analysis of the spin content of the fields in the theory using the spin projectors defined in "Appendix A". The action of these spin projectors[2] over $h_{\mu\nu}$ gives

$$h^{0w}_{\mu\nu} \equiv (P^w_0 h)_{\mu\nu} = \Box^{-2}\partial_\mu\partial_\nu\phi = \frac{\partial_\mu\partial_\nu\Phi}{\Box};$$
$$\delta h^{0w}_{\mu\nu} = 2\partial_\mu\partial_\nu\xi$$
$$h^{0s}_{\mu\nu} \equiv (P^s_0 h)_{\mu\nu} = \frac{1}{3}\left\{\eta_{\mu\nu} - \frac{\partial_\mu\partial_\nu}{\Box}\right\}\psi; \qquad \delta h^{0s}_{\mu\nu} = 0$$
$$h^1_{\mu\nu} \equiv (P_1 h)_{\mu\nu} = \Box^{-1}\left(\partial_\mu A_\nu + \partial_\nu A_\mu\right) - 2\frac{\partial_\mu\partial_\nu\Phi}{\Box};$$
$$\delta h^1_{\mu\nu} = \partial_\mu \xi^T_\nu + \partial_\nu \xi^T_\mu$$
$$\begin{aligned} h^2_{\mu\nu} \equiv (P_2 h)_{\mu\nu} &= h_{\mu\nu} - \Box^{-1}\left(\partial_\mu A_\nu + \partial_\nu A_\mu\right) + \Box^{-2}\partial_\mu\partial_\nu\phi \\ &\quad - \frac{1}{3}\{h\eta_{\mu\nu} - \Box^{-1}(\partial_\mu\partial_\nu h + \phi\eta_{\mu\nu}) + \Box^{-2}\partial_\mu\partial_\nu\phi\} \\ &= h_{\mu\nu} - \Box^{-1}(\partial_\mu A_\nu + \partial_\nu A_\mu) + \frac{\partial_\mu\partial_\nu}{\Box}\Phi \\ &\quad - \frac{1}{3}\left(\eta_{\mu\nu} - \frac{\partial_\mu\partial_\nu}{\Box}\right)\psi; \qquad \delta h^2_{\mu\nu} = 0 \end{aligned} \tag{2.9}$$

and integrating by parts we get

$$\int d(vol)\, h^{0s}_{\mu\nu}\Box h^{\mu\nu}_{0s} = \int d(vol)\,\frac{1}{3}\psi\Box\psi$$
$$\int d(vol)\,(h^{0s}_{\mu\nu} + h^{0w}_{\mu\nu})\Box\,(h^{\mu\nu}_{0s} + h^{\mu\nu}_{0w}) = \int d(vol)\left(\Phi\Box\Phi + \frac{1}{3}\psi\Box\psi\right)$$
$$\int d(vol)\, h^1_{\mu\nu}\Box h^{\mu\nu}_1 = \int d(vol)\,\left(-2A_\mu A^\mu - 2\Phi\Box\Phi\right)$$
$$\int d(vol)\, h^2_{\mu\nu}\Box h^{\mu\nu}_2 = \int d(vol)\left(h_{\mu\nu}\Box h^{\mu\nu} - \frac{1}{3}\psi\Box\psi + \Phi\Box\Phi + 2A_\mu A^\mu\right) \tag{2.10}$$

Then the Einstein–Hilbert action can be rewritten in terms of the projectors as

$$S^{\text{EH}} = -\frac{1}{8}\int d^4x\, h^{\mu\nu}(P_2 - 2P^s_0)_{\mu\nu\rho\sigma}\Box h^{\rho\sigma} \tag{2.11}$$

At this point, one can ask the question of whether it is possible to write a local lagrangian that contains only the spin 2 part of $h_{\mu\nu}$. Indeed the spin two part can be written as

$$\begin{aligned} h^2_{\mu\nu} &= h_{\mu\nu} - \frac{\partial_\mu\partial^\rho h_{\rho\nu} + h_{\mu\rho}\partial^\rho\partial_\nu}{\Box} + \frac{\partial_\mu\partial_\nu\partial_\rho\partial_\sigma h^{\rho\sigma}}{\Box^2} \\ &\quad - \frac{1}{3}\left\{h\,\eta_{\mu\nu} - \frac{\partial_\mu\partial_\nu}{\Box}h - \eta_{\mu\nu}\frac{\partial^\rho\partial^\sigma h_{\rho\sigma}}{\Box} + \frac{\partial_\mu\partial_\nu\partial^\rho\partial^\sigma h_{\rho\sigma}}{\Box^2}\right\} \end{aligned} \tag{2.12}$$

where we can see that we have a term which goes as $\frac{1}{\Box^2}$. This means that if we do not want to get non-local inverse powers of the d'Alembert operator, the simplest monomial that contains spin 2 only is going to be given by

$$S_2 \equiv \frac{1}{\kappa^6}\int d^4x\, h^2_{\mu\nu}\Box^4 h^{\mu\nu}_2 \tag{2.13}$$

which as is well-known suffers from several unitarity and causality problems associated to higher derivative lagrangians.[3] It would seem that the (harmless as we shall see) spin 0 addition is a necessary ingredient in a unitary Lorentz invariant spin 2 theory. We will come back to this point at the end of this work.

Let us go back to the EH action (2.11). With the help of (2.9), we can further decompose it in terms of the different fields contained in $h_{\mu\nu}$

$$S^{\text{EH}} = -\frac{1}{8}\int d^4x\,\left[h^{\mu\nu}\Box h_{\mu\nu} + 2A_\mu A^\mu + \Phi\Box\Phi - \psi\Box\psi\right] \tag{2.14}$$

[2] It has to be understood that when writting the action of the projectors in terms of derivatives and box operators, it is implicit that these correspond to the ones of flat space.

[3] Note that this action has a larger gauge symmetry, namely

$$\delta h_{\mu\nu} = (P_1)_{\mu\nu\rho\sigma}\,\Lambda^{\rho\sigma}_1 + \left(P^s_0\right)_{\mu\nu\rho\sigma}\,\Lambda^{\rho\sigma}_2 + \left(P^w_0\right)_{\mu\nu\rho\sigma}\,\Lambda^{\rho\sigma}_3$$

where $\Lambda^{\mu\nu}_i$ are arbitrary fields.

The equations of motion read

$$\frac{\delta S}{\delta h^{\mu\nu}} = \Box h_{\mu\nu} = 0$$
$$\frac{\delta S}{\delta \psi} = \Box \psi = 0$$
$$\frac{\delta S}{\delta \Phi} = \Box \Phi = \phi = 0$$
$$\frac{\delta S}{\delta A_\mu} = A_\mu = 0 \tag{2.15}$$

so that $A_\mu = \phi = 0$, leaving just 5 free components in $h_{\mu\nu}$ on shell.

In order to find the propagator, we need to introduce a gauge fixing term to make (2.3) invertible. Let us choose the harmonic (de Donder) gauge condition given by the operator

$$\begin{aligned} K^{\text{gf}}_{\mu\nu\rho\sigma} = &-\frac{1}{8}\left(\partial_\mu\partial_\rho\eta_{\nu\sigma} + \partial_\mu\partial_\sigma\eta_{\nu\rho} + \partial_\nu\partial_\rho\eta_{\mu\sigma} + \partial_\nu\partial_\sigma\eta_{\mu\rho}\right) \\ &-\frac{1}{4}\left(\eta_{\rho\sigma}\partial_\mu\partial_\nu + \eta_{\mu\nu}\partial_\rho\partial_\sigma\right) \\ &-\frac{1}{8}\eta_{\mu\nu}\eta_{\rho\sigma}\Box \\ = &-\frac{1}{4}\left(P_1 + \frac{3}{2}P_0^s + \frac{1}{2}P_0^w - \frac{\sqrt{3}}{2}P^\times\right)_{\mu\nu\rho\sigma}\Box \end{aligned} \tag{2.16}$$

in such a way that

$$\begin{aligned} K^{\text{EH+gf}}_{\mu\nu\rho\sigma} &= -\frac{1}{8}\left(\eta_{\mu\rho}\eta_{\nu\sigma} + \eta_{\mu\sigma}\eta_{\nu\rho} - \eta_{\mu\nu}\eta_{\rho\sigma}\right)\Box \\ &= -\frac{1}{4}\left(P_2 + P_1 - \frac{1}{2}P_0^s + \frac{1}{2}P_0^w - \frac{\sqrt{3}}{2}P^\times\right)_{\mu\nu\rho\sigma}\Box \end{aligned} \tag{2.17}$$

The propagator is easily found to be

$$\begin{aligned} \Delta_{\mu\nu\rho\sigma} &= -\frac{1}{4}\left(\eta_{\mu\rho}\eta_{\nu\sigma} + \eta_{\mu\sigma}\eta_{\nu\rho} - \eta_{\mu\nu}\eta_{\rho\sigma}\right)\Box^{-1} \\ &= -4\left(P_2 + P_1 - \frac{1}{2}P_0^s + \frac{1}{2}P_0^w - \frac{\sqrt{3}}{2}P^\times\right)_{\mu\nu\rho\sigma}\Box^{-1} \end{aligned} \tag{2.18}$$

We are also interested in computing the interaction energy between two external, conserved currents $T^{\mu\nu}_{(1)}$ and $T^{\mu\nu}_{(2)}$

$$\begin{aligned} W\left[T_{(1)}, T_{(2)}\right] &= \int d^4x\, T^{\mu\nu}_{(1)}\Delta_{\mu\nu\rho\sigma}T^{\rho\sigma}_{(2)} \\ &= \int d^4x \left(T^{\mu\nu}_{(1)}\Box^{-1}T_{(2)\mu\nu} - \frac{1}{2}T_{(1)}\Box^{-1}T_{(2)}\right) \end{aligned} \tag{2.19}$$

One may reasonably feel a little nervous about the negative sign of the spin 0 component in (2.11) as well as in (2.18). Let us demonstrate in a very explicit way that in spite of what it seems, the Einstein–Hilbert propagator is positive definite when saturated with physical sources.

First we assume that massless gravitons are the carriers of the interaction. In momentum space we choose

$$k^\mu = (\kappa, 0, 0, \kappa) \tag{2.20}$$

and the conservation of energy-momentum implies

$$\begin{aligned} T^{00}(k) &= T^{33}(k) \\ T^{0i}(k) &= T^{3i}(k) \end{aligned} \tag{2.21}$$

Then, an easy computation leads to the expression for the free energy in terms of the components of the two external conserved sources $T^{\mu\nu}_{(1)}$ and $T^{\mu\nu}_{(2)}$ as

$$\begin{aligned} &W\left[T_{(1)}, T_{(2)}\right] \\ &\quad = \int \frac{d^4k}{k^2}\left\{\frac{1}{2}\left(T^{11}_{(1)} - T^{22}_{(1)}\right)\left(T^{11}_{(2)} - T^{22}_{(2)}\right) + 2T^{12}_{(1)}T^{12}_{(2)}\right\} \end{aligned} \tag{2.22}$$

which is positive semi-definite in case of identical sources $T^{\mu\nu}_{(1)} = T^{\mu\nu}_{(2)}$.

Moreover, for static sources the energy-momentum tensor reads (all other components vanish)

$$T^{00}_{(1,2)} \equiv M_{(1,2)}\delta^{(3)}\left(\mathbf{x} - \mathbf{x}_{(1,2)}\right) \tag{2.23}$$

and in momentum space

$$T^{00}_{(1,2)}(k) \equiv M_{(1,2)}\,\delta(k^0)\,e^{i\mathbf{k}\mathbf{x}_{(1,2)}} \tag{2.24}$$

it follows that

$$\begin{aligned} W\left[T_{(1)}, T_{(2)}\right] &= \frac{1}{2C}M_1M_2\int\frac{d^3k}{\mathbf{k}^2}e^{i\mathbf{k}(\mathbf{x}_1 - \mathbf{x}_2)} \\ &= \frac{\pi}{2C}\frac{M_1M_2}{|\mathbf{x}_1 - \mathbf{x}_2|} \end{aligned} \tag{2.25}$$

where we have represented

$$\int dk_0 \equiv \frac{1}{C} \tag{2.26}$$

Therefore, the free energy is definite positive, as it should.

## 3 Lagrangians linear in curvature in first order formalism

Let us now make the exercise of reanalyzing this same theory in first order formalism, in which the metric and the connection are independent. We shall find after some roundabout that the physical content of the theory is the same as we previously found in the last paragraph.

We start with the Einstein–Hilbert action

$$S^{\text{EH}} \equiv -\frac{1}{2\kappa^2}\int d^n x\sqrt{|g|}g^{\mu\nu}R_{\mu\nu}\left[\Gamma\right] \tag{3.1}$$

and we expand it around Minkowski spacetime as

$$\begin{aligned} g_{\mu\nu} &\equiv \eta_{\mu\nu}+\kappa h_{\mu\nu}\\ \Gamma^{\alpha}_{\beta\gamma} &\equiv A^{\alpha}_{\beta\gamma} \end{aligned} \tag{3.2}$$

where $A^{\alpha}_{\beta\gamma}$ is the quantum field for the connection, which is symmetric in the last two indices as we are restricting ourselves to the torsionless case.

After this expansion the action can be written as

$$S^{\text{EH}} = -\int d^n x\left\{h^{\gamma\epsilon}N_{\gamma\epsilon\;\lambda}^{\;\;\alpha\beta}A^{\lambda}_{\alpha\beta}+\tfrac{1}{2}A^{\tau}_{\gamma\epsilon}K^{\gamma\epsilon\;\alpha\beta}_{\;\tau\;\;\lambda}A^{\lambda}_{\alpha\beta}\right\} \tag{3.3}$$

where the operators mediating the interactions have the form

$$\begin{aligned} N_{\gamma\epsilon\;\lambda}^{\;\;\alpha\beta} &= \frac{1}{2\kappa}\left\{\frac{1}{2}\left(\eta_{\gamma\epsilon}\eta^{\alpha\beta}-\delta^{\alpha}_{\gamma}\delta^{\beta}_{\epsilon}-\delta^{\alpha}_{\epsilon}\delta^{\beta}_{\gamma}\right)\partial_\lambda\right.\\ &\quad -\frac{1}{4}\left(\eta_{\gamma\epsilon}\delta^{\beta}_{\lambda}\partial^{\alpha}+\eta_{\gamma\epsilon}\delta^{\alpha}_{\lambda}\partial^{\beta}-\delta^{\alpha}_{\gamma}\delta^{\beta}_{\lambda}\partial_\epsilon\right.\\ &\quad\left.\left.-\delta^{\beta}_{\gamma}\delta^{\alpha}_{\lambda}\partial_\epsilon-\delta^{\alpha}_{\epsilon}\delta^{\beta}_{\lambda}\partial_\gamma-\delta^{\beta}_{\epsilon}\delta^{\alpha}_{\lambda}\partial_\gamma\right)\right\}\\ K^{\gamma\epsilon\;\alpha\beta}_{\;\tau\;\;\lambda} &= \frac{1}{\kappa^2}\left\{\frac{1}{4}[\delta^{\epsilon}_{\tau}\delta^{\gamma}_{\lambda}\eta^{\alpha\beta}+\delta^{\gamma}_{\tau}\delta^{\epsilon}_{\lambda}\eta^{\alpha\beta}+\delta^{\beta}_{\lambda}\delta^{\alpha}_{\tau}\eta^{\gamma\epsilon}+\delta^{\alpha}_{\lambda}\delta^{\beta}_{\tau}\eta^{\gamma\epsilon}\right.\\ &\quad\left.-\delta^{\beta}_{\tau}\delta^{\gamma}_{\lambda}\eta^{\alpha\epsilon}-\delta^{\beta}_{\tau}\delta^{\epsilon}_{\lambda}\eta^{\alpha\gamma}-\delta^{\alpha}_{\tau}\delta^{\epsilon}_{\lambda}\eta^{\beta\gamma}-\delta^{\alpha}_{\tau}\delta^{\gamma}_{\lambda}\eta^{\beta\epsilon}]\right\} \end{aligned} \tag{3.4}$$

From the path integral, the contribution to the effective action reads

$$e^{iW[\eta_{\mu\nu}]} = \int \mathcal{D}h\mathcal{D}A\; e^{iS_{\text{FOEH}}[h,A]} \tag{3.5}$$

and using the background expansion (3.3) we can integrate over $\mathcal{D}A$ yielding

$$e^{iW} = \int \mathcal{D}h e^{\left\{-\frac{i}{2}\int d^n x\;\sqrt{|g|}\;\frac{1}{2}h^{\mu\nu}D_{\mu\nu\rho\sigma}h^{\rho\sigma}\right\}} \tag{3.6}$$

where

$$\begin{aligned} D_{\mu\nu\rho\sigma} &= \frac{1}{4}(\eta_{\mu\rho}\eta_{\nu\sigma}+\eta_{\mu\sigma}\eta_{\nu\rho}-2\eta_{\mu\nu}\eta_{\rho\sigma})\Box\\ &\quad+\frac{1}{2}(\eta_{\mu\nu}\partial_\rho\partial_\sigma+\eta_{\rho\sigma}\partial_\mu\partial_\nu)\\ &\quad-\frac{1}{8}(\eta_{\mu\rho}\partial_\nu\partial_\sigma+\eta_{\mu\sigma}\partial_\nu\partial_\rho+\eta_{\nu\rho}\partial_\mu\partial_\sigma+\eta_{\nu\sigma}\partial_\mu\partial_\rho)\\ &\quad-\frac{1}{8}(\eta_{\mu\rho}\partial_\sigma\partial_\nu+\eta_{\mu\sigma}\partial_\rho\partial_\nu+\eta_{\nu\rho}\partial_\sigma\partial_\mu+\eta_{\nu\sigma}\partial_\rho\partial_\mu) \end{aligned} \tag{3.7}$$

We now expand this operator in the basis of projectors (see "Appendix A") so that

$$D_{\mu\nu\rho\sigma} = \frac{1}{2}\left(P_2-(n-2)P^s_0\right)_{\mu\nu\rho\sigma}\;\Box \tag{3.8}$$

and in this way the action can be rewritten (for $n=4$) as

$$S^{\text{EH}} = -\frac{1}{8}\int d^4x\; h^{\mu\nu}(P_2-2P^s_0)_{\mu\nu\rho\sigma}\Box h^{\rho\sigma} \tag{3.9}$$

In conclusion, we obtain the same result when we treat the theory in second order formalism (2.11) and in first order formalism, for the particular case of the Einstein–Hilbert action.

## 4 Lagrangians quadratic in curvature in second order formalism

Let us now begin the study of Lagrangians quadratic in the spacetime curvature, first in the usual second order formalism.

The most general action in this set (the connection is assumed in this section to be the metric one) is

$$S^{\text{SOQ}} \equiv \int d^n x\sqrt{|g|}(\alpha R^2+\beta R_{\mu\nu}R^{\mu\nu}+\gamma R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma}) \tag{4.1}$$

When we expand around flat space $g_{\mu\nu}=\eta_{\mu\nu}+\kappa h_{\mu\nu}$ it follows that

$$\begin{aligned} S^{\text{SOQ}} &= \kappa^2\int d^n x h^{\mu\nu}\left\{\alpha\left[\partial_\mu\partial_\nu\partial_\rho\partial_\sigma\right.\right.\\ &\quad\left.-\left(\eta_{\rho\sigma}\partial_\mu\partial_\nu+\eta_{\mu\nu}\partial_\rho\partial_\sigma\right)\Box+\eta_{\mu\nu}\eta_{\rho\sigma}\Box^2\right]\\ &\quad+\frac{\beta}{4}\left[2\partial_\mu\partial_\nu\partial_\rho\partial_\sigma-\frac{1}{2}\left(\eta_{\mu\rho}\partial_\nu\partial_\sigma+\eta_{\mu\sigma}\partial_\nu\partial_\rho\right.\right.\\ &\quad\left.+\eta_{\nu\rho}\partial_\mu\partial_\sigma+\eta_{\nu\sigma}\partial_\mu\partial_\rho\right)\Box-\left(\eta_{\rho\sigma}\partial_\mu\partial_\nu+\eta_{\mu\nu}\partial_\rho\partial_\sigma\right)\Box\\ &\quad\left.+\frac{1}{2}\left(\eta_{\mu\rho}\eta_{\nu\sigma}+\eta_{\mu\sigma}\eta_{\nu\rho}\right)\Box^2+\eta_{\mu\nu}\eta_{\rho\sigma}\Box^2\right]\\ &\quad+\frac{\gamma}{4}\left[4\partial_\mu\partial_\nu\partial_\rho\partial_\sigma+2\left(\eta_{\mu\rho}\eta_{\nu\sigma}+\eta_{\mu\sigma}\eta_{\nu\rho}\right)\Box^2\right.\\ &\quad\left.\left.-2\left(\eta_{\mu\rho}\partial_\nu\partial_\sigma+\eta_{\mu\sigma}\partial_\nu\partial_\rho+\eta_{\nu\rho}\partial_\mu\partial_\sigma+\eta_{\nu\sigma}\partial_\mu\partial_\rho\right)\Box\right]\right\}\\ &\quad\times h^{\rho\sigma} \end{aligned} \tag{4.2}$$

We can write the operator in terms of spin projectors as

$$\begin{aligned} &K^{\text{SOQ}}_{\mu\nu\rho\sigma}\\ &\quad=\kappa^2\left(\alpha(n-1)P^s_0+\frac{\beta}{4}(P_2+nP^s_0)+\gamma(P_2+P^s_0)\right)_{\mu\nu\rho\sigma}\\ &\times\Box^2=\frac{\kappa^2}{4}\left(c_1P_2+c_2P^s_0\right)_{\mu\nu\rho\sigma}\;\Box^2 \end{aligned} \tag{4.3}$$

where $c_1=\beta+4\gamma$ and $c_2=4(n-1)\alpha+n\beta+4\gamma$.

If we use the action of spin projectors over the graviton decomposition (2.9), the action can be rewritten as

$$S^{\text{SOQ}} = \frac{\kappa^2}{4}\int d^n x \times \left[c_1\left(h^{\mu\nu}\Box^2 h_{\mu\nu} + 2A_\mu\Box A^\mu + \phi^2 - \frac{1}{3}\psi\Box^2\psi\right) + \frac{c_2}{3}\psi\Box^2\psi\right] \tag{4.4}$$

Let us at this point make a short aside on the higher derivative scalar terms. Consider the lagrangian [26]

$$L = \frac{1}{2}\left(\partial_\mu\psi\right)^2 + \frac{1}{2}C\psi\Box^2\psi \tag{4.5}$$

and introduce an auxiliary field, $\chi$, so that

$$L = \frac{1}{2}\left(\partial_\mu\psi\right)^2 + C\,\partial_\mu\psi\partial^\mu\chi - \frac{1}{2}C\,\chi^2 \tag{4.6}$$

The EM for the auxiliary field just yields

$$\chi = -\Box\psi \tag{4.7}$$

which just reproduces the original action. Now we can define

$$\Psi \equiv \psi + C\chi \tag{4.8}$$

The mixing term disappears and the action diagonalizes to

$$L = \frac{1}{2}\left(\partial_\mu\Psi\right)^2 - \frac{1}{2}C^2\left(\partial_\mu\chi\right)^2 - \frac{1}{2}C\,\chi^2 \tag{4.9}$$

It follows that the auxiliary field becomes a ghost no matter the value of the constant $C$. When there is no canonical kinetic term for the field $\psi$ this mechanism is not at work. However, such a term is always generated by the Einstein–Hilbert (linear in the space-time curvature) piece of the gravitational lagrangian. This linear piece is physically unavoidable, even if it is not present in the classical lagrangian, it will be generated by radiative corrections.[4]

Going back to our analysis, we can obtain the equations of motion for the quadratic action (4.4)

$$\frac{\delta S}{\delta h^{\mu\nu}} = c_1\Box^2 h_{\mu\nu} = 0$$
$$\frac{\delta S}{\delta\psi} = (c_2 - c_1)\Box^2\psi = 0$$
$$\frac{\delta S}{\delta\phi} = c_1\phi = c_1\Box\Phi = 0$$

$$\frac{\delta S}{\delta A^\mu} = c_1\Box A_\mu = 0 \tag{4.10}$$

Please note that the equations of motion have four derivatives so that the only way in which we can fix this problem is by taking $c_1 = c_2 = 0$. This implies

$$\beta + 4\gamma = \beta + 4\alpha = 0 \tag{4.11}$$

In this case the lagrangian is proportional to the Gauss-Bonnet density, i.e. $\alpha = 1, \beta = -4, \gamma = 1$ and $n = 4$, and the operator (4.3) reduces to

$$K^{\text{GB}}_{\mu\nu\rho\sigma} = 0 \tag{4.12}$$

This fact follows from the identity

$$R^2 - 4R^{\mu\nu}R_{\mu\nu} + R^{\mu\nu\rho\sigma}R_{\mu\nu\rho\sigma} = \text{total derivative} \tag{4.13}$$

Let us now obtain the propagator for the general quadratic action (4.4), again in the harmonic gauge (2.16) with a gauge parameter $-\frac{1}{2\xi}$. The operator reads

$$K^{\text{SOQ+gf}}_{\mu\nu\rho\sigma} = \frac{1}{8}\left\{\frac{1}{\xi}P_1 + 2\kappa^2 c_1\Box P_2 + \left(2\kappa^2 c_2\Box + \frac{n-1}{2\xi}\right)P_0^s + \frac{1}{2\xi}P_0^w - \frac{\sqrt{n-1}}{2\xi}P_0^\times\right\}_{\mu\nu\rho\sigma}\Box \tag{4.14}$$

and inverting it we get

$$\Delta_{\mu\nu\rho\sigma} \equiv (K^{-1})^{\text{SOQ+gf}}_{\mu\nu\rho\sigma} = \frac{8}{k^2}\left\{\xi P_1 + \frac{1}{2\kappa^2 c_1 k^2}P_2 + \frac{\xi}{\kappa^2 c_2 k^2} \times\left[\left(2\kappa^2 c_2 k^2 + \frac{n-1}{2\xi}\right)P_0^w + \frac{1}{2\xi}P_0^s + \frac{\sqrt{n-1}}{2\xi}P_0^\times\right]\right\}_{\mu\nu\rho\sigma} \tag{4.15}$$

provided $c_1 \neq 0$ and $c_2 \neq 0$.

Now the interaction energy between external static sources, for $n = 4$, is proportional to

$$W^{\text{SOQ+gf}} \propto T^{\mu\nu}\Delta^{\text{SOQ+gf}}_{\mu\nu\rho\sigma}T^{\rho\sigma} = \frac{4}{\kappa^2 k^4}\left[\frac{1}{c_1}\left(T_{\mu\nu}T^{\mu\nu} - \frac{1}{3}T^2\right) + \frac{1}{3c_2}T^2\right] \tag{4.16}$$

This result is independent of the gauge fixing, and for the particular case $2c_1 = -c_2$, the dependence on the sources is proportional to the Einstein–Hilbert one

$$W^{\text{SOQ+gf}}\Big|_{c_2=-2c_1} \propto \frac{4}{\kappa^2 k^4}\frac{1}{c_1}\left(T_{\mu\nu}T^{\mu\nu} - \frac{1}{2}T^2\right) \tag{4.17}$$

However, the factor $\frac{1}{k^4}$ in momentum space leads to a confining (linear) potential in position space.

[4] If we restrict ourselves only to the $R^2$ terms, i.e. $\beta = \gamma = 0$, we get

$$S_{R^2} = \kappa^2\alpha\int d^n x\;\psi\Box^2\psi$$

so that the equation of motion reads

$$\Box^2\psi = 0$$

From this we can see that there is a gauge invariant ghostly state.

### 4.1 Adding a term linear in the scalar curvature

It has been argued in [11–13] that a term linear in the spacetime curvature will be generated by quantum corrections, even if it is not initially present in the classical lagrangian. It is then of interest to consider the quadratic action plus the Einstein–Hilbert action

$$S^{\text{Q+EH}} \equiv \int d^n x \sqrt{|g|} \times \left(-\frac{\lambda}{2\kappa^2} R + \alpha R^2 + \beta R_{\mu\nu} R^{\mu\nu} + \gamma R_{\mu\nu\rho\sigma} R^{\mu\nu\rho\sigma}\right) \tag{4.18}$$

We can use the same harmonic gauge fixing (2.16) with parameter $\xi$, so that the total operator can be written in terms of projectors as

$$K^{\text{Q+EH+gf}}_{\mu\nu\rho\sigma} = \frac{1}{8}\left\{\frac{1}{\xi} P_1 + (2\kappa^2 c_1 \Box + \lambda) P_2 + \left(2\kappa^2 c_2 \Box + \frac{n-1}{2\xi} - \lambda(n-2)\right) P_0^s + \frac{1}{2\xi} P_0^w - \frac{\sqrt{n-1}}{2\xi} P_0^\times\right\}_{\mu\nu\rho\sigma} \Box \tag{4.19}$$

Inverting the operator the propagator reads

$$\Delta^{\text{Q+EH+gf}}_{\mu\nu\rho\sigma} = \frac{8}{k^2}\Bigg\{\xi P_1 + \frac{1}{2\kappa^2 c_1 k^2 + \lambda} P_2 + \frac{\xi}{\kappa^2 c_2 k^2 - \frac{\lambda(n-2)}{2}} \times \left[\left(2\kappa^2 c_2 k^2 + \frac{n-1}{2\xi} - \lambda(n-2)\right) P_0^w + \frac{1}{2\xi} P_0^s + \frac{\sqrt{n-1}}{2\xi} P_0^\times\right]\Bigg\}_{\mu\nu\rho\sigma} \tag{4.20}$$

Once we have the propagator, it is easy to check that the interaction energy between two external, static sources, for $n = 4$, is proportional to

$$\begin{aligned} W &\propto T^{\mu\nu} (K^{-1})^{\text{Q+EH+gf}}_{\mu\nu\rho\sigma} T^{\rho\sigma} \\ &= \frac{8}{\lambda}\left[\left(\frac{1}{k^2} - \frac{1}{(k^2 + \frac{\lambda}{2\kappa^2 c_1})}\right)\left(T_{\mu\nu}T^{\mu\nu} - \frac{1}{3}T^2\right) + \frac{2}{n-2}\left(\frac{1}{2(k^2 - \frac{\lambda(n-2)}{2\kappa^2 c_2})} - \frac{1}{2k^2}\right)\frac{T^2}{3}\right] \\ &= \frac{8}{\lambda k^2}\left(T_{\mu\nu}T^{\mu\nu} - \frac{n-1}{3(n-2)}T^2\right) \\ &\quad - \frac{8}{\lambda}\left[\frac{1}{(k^2 + \frac{\lambda}{2\kappa^2 c_1})}\left(T_{\mu\nu}T^{\mu\nu} - \frac{1}{3}T^2\right) - \frac{1}{2(k^2 - \frac{\lambda(n-2)}{2\kappa^2 c_2})}\frac{T^2}{3}\right] \end{aligned} \tag{4.21}$$

Notice that the only contributions to the free energy come from $P_2$ and $P_0^s$ as the rest of spin operators do not contribute when saturated with the sources. The spin 2 piece can be rewritten as

$$\frac{8}{k^2(2\kappa^2 c_1 k^2 + \lambda)} P_2 = \frac{8}{\lambda}\left[\frac{1}{k^2} - \frac{1}{(k^2 + \frac{\lambda}{2\kappa^2 c_1})}\right] P_2 \tag{4.22}$$

The first term comes from the Einstein–Hilbert action, giving the well-known massless pole, whereas the second term corresponds to a massive $k^2 = -\frac{\lambda}{2\kappa^2 c_1}$ spin 2 pole with negative residue, coming from the quadratic action.

The spin 0 piece has the form

$$\frac{8}{2k^2(\kappa^2 c_2 k^2 - \frac{\lambda(n-2)}{2})} P_0^s = \frac{16}{\lambda(n-2)} \times \left[\frac{1}{2(k^2 - \frac{\lambda(n-2)}{2\kappa^2 c_2})} - \frac{1}{2k^2}\right] P_0^s \tag{4.23}$$

In this case, the first term is a massive $k^2 = \frac{\lambda(n-2)}{2\kappa^2 c_2}$ spin 0 pole with positive residue, coming from the quadratic piece of the action. The second term is again the massless spin 0 pole with negative residue that we already encountered when studying the EH action.

## 5 Lagrangians quadratic in curvature in first order formalism

Let us now enter into the main topic of this paper, namely the general situation in which the physics is conveyed by the graviton as well as by the connection field. Actually, as was pointed out in [3], when considering a metric fluctuating around flat space there is no kinetic term for the graviton, so that all the physics is encoded in the connection field. This is the main reason why we underwent a systematic analysis of the spin content of the said connection field. We consider the general action

$$S_{\text{FOQ}} \equiv \int d^n x \sqrt{|g|} \times (\alpha R[\Gamma]^2 + \beta R[\Gamma]_{\mu\nu} R[\Gamma]^{\mu\nu} + \gamma R[\Gamma]_{\mu\nu\rho\sigma} R[\Gamma]^{\mu\nu\rho\sigma}) \tag{5.1}$$

and we again use the expansion around Minkowski spacetime given by

$$\begin{aligned} g_{\mu\nu} &\equiv \eta_{\mu\nu} + \kappa h_{\mu\nu} \\ \Gamma^\alpha_{\beta\gamma} &\equiv A^\alpha_{\beta\gamma} \end{aligned} \tag{5.2}$$

where $A^{\alpha}_{\beta\gamma}$ is the quantum field for the connection, which is symmetric in the last two indices as we are restricting ourselves to the torsionless case.

The action reduces to a kinetic term for the connection field

$$S_{\text{FOQ}} = \int d^n x A^{\tau}_{\mu\nu} K^{\mu\nu}{}_{\tau}{}^{\rho\sigma}{}_{\lambda} A^{\lambda}_{\rho\sigma} \tag{5.3}$$

where the operator reads

$$\begin{aligned} K^{\mu\nu}{}_{\tau}{}^{\rho\sigma}{}_{\lambda} &= \alpha\Big\{\frac{1}{2}\left(\eta^{\mu\nu}\delta^{\sigma}_{\tau}\partial_{\lambda}\partial^{\rho} + \eta^{\mu\nu}\delta^{\rho}_{\tau}\partial_{\lambda}\partial^{\sigma} + \eta^{\rho\sigma}\delta^{\nu}_{\lambda}\partial_{\tau}\partial^{\mu}\right.\\ &\left.+\eta^{\rho\sigma}\delta^{\mu}_{\lambda}\partial_{\tau}\partial^{\nu}\right) - \eta^{\mu\nu}\eta^{\rho\sigma}\partial_{\lambda}\partial_{\tau}\\ &- \frac{1}{4}\left(\delta^{\nu}_{\lambda}\delta^{\sigma}_{\tau}\partial^{\mu}\partial^{\rho} + \delta^{\mu}_{\lambda}\delta^{\sigma}_{\tau}\partial^{\nu}\partial^{\rho} + \delta^{\nu}_{\lambda}\delta^{\rho}_{\tau}\partial^{\mu}\partial^{\sigma}\right.\\ &\left.+\delta^{\mu}_{\lambda}\delta^{\rho}_{\tau}\partial^{\nu}\partial^{\sigma}\right)\Big\}\\ &+ \beta\Big\{\frac{1}{4}\left(\eta^{\mu\rho}\delta^{\sigma}_{\tau}\partial_{\lambda}\partial^{\nu} + \eta^{\nu\rho}\delta^{\sigma}_{\tau}\partial_{\lambda}\partial^{\mu} + \eta^{\mu\sigma}\delta^{\rho}_{\tau}\partial_{\lambda}\partial^{\nu}\right.\\ &\left.+\eta^{\nu\sigma}\delta^{\rho}_{\tau}\partial_{\lambda}\partial^{\mu}\right)\\ &+ \frac{1}{4}\left(\eta^{\mu\rho}\delta^{\nu}_{\lambda}\partial_{\tau}\partial^{\sigma} + \eta^{\nu\rho}\delta^{\mu}_{\lambda}\partial_{\tau}\partial^{\sigma} + \eta^{\mu\sigma}\delta^{\nu}_{\lambda}\partial_{\tau}\partial^{\rho}\right.\\ &\left.+\eta^{\nu\sigma}\delta^{\mu}_{\lambda}\partial_{\tau}\partial^{\rho}\right) - \frac{1}{2}\left(\eta^{\mu\rho}\eta^{\nu\sigma} + \eta^{\nu\rho}\eta^{\mu\sigma}\right)\partial_{\lambda}\partial_{\tau}\\ &- \frac{1}{4}\left(\eta^{\mu\rho}\delta^{\nu}_{\lambda}\delta^{\sigma}_{\tau} + \eta^{\nu\rho}\delta^{\mu}_{\lambda}\delta^{\sigma}_{\tau} + \eta^{\mu\sigma}\delta^{\nu}_{\lambda}\delta^{\rho}_{\tau}\right.\\ &\left.+\eta^{\nu\sigma}\delta^{\mu}_{\lambda}\delta^{\rho}_{\tau}\right)\Box\Big\}\\ &+ \gamma\Big\{\eta_{\lambda\tau}\Big[\frac{1}{2}\left(\eta^{\mu\rho}\partial^{\sigma}\partial^{\nu} + \eta^{\nu\rho}\partial^{\sigma}\partial^{\mu}\right.\\ &\left.+\eta^{\mu\sigma}\partial^{\rho}\partial^{\nu} + \eta^{\nu\sigma}\partial^{\rho}\partial^{\mu}\right) - \left(\eta^{\mu\rho}\eta^{\nu\sigma} + \eta^{\nu\rho}\eta^{\mu\sigma}\right)\Box\Big]\Big\} \end{aligned} \tag{5.4}$$

In the "Appendix B" we have studied the spin projectors for connection fields $A \in \mathcal{A}$, where $\mathcal{A}$ is the space of torsionless connections (see "Appendix C" for metric, torsionful connections). There are two main sectors in this space: the one corresponding to connections symmetric in the three indices (B.1), $A_S$, and the one endowed with the hook symmetry (B.2), $\mathcal{A}_{\mathcal{H}}$, each one with 20 components. The spin content of the symmetric sector is

$$\underline{20}_S = (\underline{3}) \oplus (\underline{2}) \oplus 2\ (\underline{1}) \oplus 2\ (\underline{0}) \tag{5.5}$$

and the spin content of the hook one is given by

$$\underline{20}_H = 2\ (\underline{2}) \oplus 3\ (\underline{1}) \oplus (\underline{0}) \tag{5.6}$$

There are 12 mutually orthogonal projectors on these different sectors. Projectors on the symmetric sector are represented by roman letters and indexed by the spin, $\mathrm{P}_s$, whereas projectors in the hook sector are represented by calligraphic letters also indexed by the spin, $\mathcal{P}_s$. Nevertheless, this is not enough to expand the most general linear operator

$$K : \mathcal{A} \to \mathcal{A} \tag{5.7}$$

which has dimension 22. In order to find a basis for this space, we need to add 10 new operators to the above set, which are not mutually orthogonal anymore. These new operators will be denoted as $\mathfrak{P}_s$, where $s$ stands for the spin. Explicit expressions can be found in the "Appendix B.3".

Once we have obtained the complete basis for this space, we can expand the general operator in terms of these spin operators as

$$\begin{aligned} (K_{\text{FOQ}})^{\mu\nu}{}_{\tau}{}^{\rho\sigma}{}_{\lambda} &= (-2(2\gamma+\beta)\,\mathrm{P}^s_0 - (4\gamma+9\alpha+2\beta)\,\mathcal{P}^s_0\\ &+ (2\gamma-\beta)\,\mathfrak{P}^x_0 - \frac{4}{3}(3\gamma+5\beta)\,\mathrm{P}^s_1\\ &- 2\gamma\,\mathcal{P}^s_1 - \frac{4}{3}(3\gamma+\beta)\,\mathcal{P}^t_1 - (2\gamma+\beta)\,\mathfrak{P}^{wx}_1\\ &+ 4\beta\,\mathfrak{P}^{ss}_1 - 2(2\gamma+\beta)\,(\mathrm{P}_2+\mathcal{P}_2)\\ &- 4\gamma\,\mathcal{P}^s_2 + 2(\beta+\gamma)\,\mathfrak{P}^x_2 - 4\gamma\mathrm{P}_3)^{\mu\nu}{}_{\tau}{}^{\rho\sigma}{}_{\lambda}\,\Box \end{aligned} \tag{5.8}$$

We also need to choose a gauge fixing, in this case we take

$$S_{\text{gf}} = \frac{1}{\chi}\int d^n x\ \eta^{\mu\nu}\eta^{\rho\sigma}\eta_{\tau\lambda}A^{\tau}_{\mu\nu}\Box A^{\lambda}_{\rho\sigma} \tag{5.9}$$

from where we can extract the operator which in terms of the projectors reads

$$\begin{aligned} (K_{\text{gf}})^{\mu\nu}{}_{\tau}{}^{\rho\sigma}{}_{\lambda} &= \frac{1}{\chi}(\mathrm{P}^w_0 + 3\,\mathrm{P}^s_0 + 3\,\mathcal{P}^s_0 - 3\,\mathfrak{P}^x_0 + \mathfrak{P}^{sw}_0 + \mathfrak{P}^{ws}_0\\ &+ \mathrm{P}_1 - \frac{5}{3}\,\mathrm{P}^s_1 + \mathcal{P}^w_1 + \frac{2}{3}\,\mathcal{P}^t_1 - \mathfrak{P}^{wx}_1\\ &+ \mathfrak{P}^{ws}_1 + \mathfrak{P}^{sw}_1 + \mathfrak{P}^{sx}_1 + 4\,\mathfrak{P}^{ss}_1)^{\mu\nu}{}_{\tau}{}^{\rho\sigma}{}_{\lambda}\,\Box \end{aligned} \tag{5.10}$$

From the decomposition of the gauge fixing operator we see that the gauge fixing term does not posses any spin 2 or spin 3 piece. Looking at the operator (5.8) for the three quadratic terms, we are going to have problems when $\gamma$ equals zero due to the fact that $\mathrm{P}_3$, $\mathcal{P}^s_2$ and $\mathcal{P}^s_1$ disappear from the scene. As we have seen, we cannot recover the spin 2 and spin 3 ones from the gauge fixing, so this leads to a non invertible operator, and thus, to new zero modes.

To understand this fact, let us focus in the simplest case where $\beta = \gamma = 0$. The operator for $R^2$ collapses to

$$(K_{R^2})^{\mu\nu}{}_{\tau}{}^{\rho\sigma}{}_{\lambda} = -9\,(\mathcal{P}^s_0)^{\mu\nu}{}_{\tau}{}^{\rho\sigma}{}_{\lambda}\,\Box \tag{5.11}$$

so that

$$(K_{R^2+\text{gf}})^{\mu\nu}{}_{\tau}{}^{\rho\sigma}{}_{\lambda} = \frac{1}{\chi}(\mathrm{P}^w_0 + 3\,\mathrm{P}^s_0 + (3-9\chi)\,\mathcal{P}^s_0$$

$$- 3\,\mathcal{P}_0^x + \mathcal{P}_0^{sw} + \mathcal{P}_0^{ws} + \mathrm{P}_1^w - \frac{5}{3}\,\mathrm{P}_1^s + \mathcal{P}_1^w + \frac{2}{3}\,\mathcal{P}_1^t - \mathcal{P}_1^{wx} + \mathcal{P}_1^{ws} + \mathcal{P}_1^{sw} + \mathcal{P}_1^{sx} + 4\,\mathcal{P}_1^{ss})^{\mu\nu}{}_{\tau}{}^{\rho\sigma}{}_{\lambda}\,\Box \tag{5.12}$$

It follows that there are a grand total of 13 new zero modes. They are listed in the "Appendix D". Physically, this means that the theory has extra gauge symmetry when considered at one loop order, in addition to the one it has for the full theory, namely diffeomorphism and Weyl invariance. We are not aware of any other physical system where this happens. For what we can say, these extra gauge symmetries are accidental, and will disappear when computing higher loop orders.

It is plain that the first order theory has a sector in which the connection reduces to the metric one. It is physically obvious that in this sector the theory should reduce to the one obtained in second order formalism. Let us then check what happens when the connection reduces to the Levi-Civita connection. Around flat space we have

$$A^{\lambda\,(\mathrm{LC})}_{\mu\nu} = \partial_\mu h^\lambda_\nu + \partial_\nu h^\lambda_\mu - \partial^\lambda h_{\mu\nu} \tag{5.13}$$

With this change we can extract an operator mediating interactions between the $h_{\mu\nu}$ and expand it in terms of the four-index spin projectors. In this way we can see how the six-index projectors and the four-index projectors talk to each other. The full correspondence is as follows

| $A_{\lambda\mu\nu}\,P^{\lambda\mu\nu}_{\alpha\beta\gamma}\,A^{\alpha\beta\gamma}$ | $h_{\mu\nu}\,P^{\mu\nu}_{\alpha\beta}\,h^{\alpha\beta}$ |
|---|---|
| $\mathrm{P}_0^w$ | $\frac{k^2}{4}\,P_0^w$ |
| $\mathrm{P}_0^s$ | $\frac{k^2}{36}(n-1)\,P_0^s$ |
| $\mathcal{P}_0^s$ | $\frac{2k^2}{9}(n-1)\,P_0^s$ |
| $\mathcal{P}_0^x$ | $\frac{k^2}{6}(n-1)\,P_0^s$ |
| $\mathcal{P}_0^{sw}$ | $-\frac{k^2}{3}\sqrt{n-1}\,P_0^\times$ |
| $\mathcal{P}_0^{ws}$ | $\frac{k^2}{12}\sqrt{n-1}\,P_0^\times$ |
| $\mathrm{P}_1^w$ | $\frac{k^2}{6}\,P_1$ |
| $\mathcal{P}_1^w$ | $\frac{k^2}{3}\,P_1$ |
| $\mathrm{P}_2$ | $\frac{k^2}{12}P_2 - \frac{k^2}{36}(n-4)\,P_0^s$ |
| $\mathcal{P}_2$ | $\frac{2k^2}{3}P_2 - \frac{2k^2}{9}(n-4)\,P_0^s$ |
| $\mathcal{P}_2^x$ | $\frac{k^2}{2}P_2 - \frac{k^2}{6}(n-4)\,P_0^s$ |

where $\mathrm{P}_1^s, \mathcal{P}_1^s, \mathcal{P}_1^w, \mathcal{P}_1^t, \mathcal{P}_1^{wx}, \mathcal{P}_1^{ws}, \mathcal{P}_1^{sw}, \mathcal{P}_1^{sx}, \mathcal{P}_1^{ss}, \mathcal{P}_1^{wst}, \mathcal{P}_2^s, \mathrm{P}_3$, do not contribute when the connection reduces to the metric one.

The end result is that spin 3 collapses to zero, and the surviving different spin 2 sectors of the first order theory degenerate into the unique spin 2 of the second order one. Moreover, spin 1 reduces to spin 1 when going to second order formalism, as well as spin 0 goes to spin 0.

In the process however, a power of $k^2$ has been generated. This power is the responsible for the lack of (perturbative) unitarity of the theory in second order formalism. This problem then appears in this particular sector of the first order theory as well.

Then, unless a consistent method is found to isolate this sector from the full first order theory (*id est*, a consistent truncation), the latter will inherit the unitarity problems of the second order one.

## 6 Conclusions

When analyzing the connection field, one easily finds that there is generically a spin 3 component. This might be a problem in the sense that it is well-known (cf. for example [27]) that it is not possible to build an interacting theory for spin 3 with a finite number of fields. Although we see no particular type of inconsistency to the order we have worked, it is always possible to avoid the presence of this spin 3 field altogether by choosing a particular set of coupling constants, namely, putting to zero the coefficient of the Riemann squared term. This combination is not stable by renormalization, so that this choice implies a fine tuning of sorts. In addition there are several spin 0, spin 1 and spin 2 fields. This proliferation of spins occurs even for the Einstein–Hilbert action when in first order formalism.

When the connection collapses to the metric (Levi-Civita) form, the spin 3 component disappears, and all spin 2 components are identified, but this sector suffers from the well-known unitarity problems present in second order formalism.

In conclusion it is unclear whether it will be possible to define a truncation of the gravity lagrangian quadratic in curvature in first order formalism in which the problems of unitarity are absent. It seems that the healthy sectors do not describe gravity, and the sectors that do describe gravity fall into the known unitarity problems. To be specific, let us define a scalar product in $\mathcal{A}$

$$\langle A_1|A_2\rangle \equiv \int d(vol)\,A^1_{\mu\nu\lambda}A_2^{\mu\nu\lambda} \tag{6.1}$$

Then the subspace $\mathcal{A}^\perp$ orthogonal to the metric connections

$$A^{(\mathrm{LC})}_{\mu\nu\lambda} \equiv \partial_\mu h_{\nu\lambda} - \partial_\nu h_{\lambda\mu} - \partial_\lambda h_{\mu\nu} \tag{6.2}$$

is defined by

$$A^\perp \in \mathcal{A}^\perp \Leftrightarrow \partial^\mu\left(A_{\mu\nu\lambda} - A_{\nu\mu\lambda} + A_{\lambda\nu\mu}\right) = 0 \tag{6.3}$$

which in terms of projectors reads

$$\begin{aligned} A^{\perp}{}_{\mu\nu\lambda} &= (\mathcal{P}_0^x)_{\mu\nu\lambda}^{\rho\sigma\tau}\,\Omega^1_{\rho\sigma\tau} + (\mathrm{P}_1^s)_{\mu\nu\lambda}^{\rho\sigma\tau}\,\Omega^2_{\rho\sigma\tau} \\ &+ (\mathcal{P}_1^t)_{\mu\nu\lambda}^{\rho\sigma\tau}\,\Omega^3_{\rho\sigma\tau} + (\mathcal{P}_1^{ss})_{\mu\nu\lambda}^{\rho\sigma\tau}\,\Omega^4_{\rho\sigma\tau} \\ &+ (\mathcal{P}_2^s)_{\mu\nu\lambda}^{\rho\sigma\tau}\,\Omega^5_{\rho\sigma\tau} + (\mathcal{P}_2^x)_{\mu\nu\lambda}^{\rho\sigma\tau}\,\Omega^6_{\rho\sigma\tau} \\ &+ (\mathrm{P}_3)_{\mu\nu\lambda}^{\rho\sigma\tau}\,\Omega^7_{\rho\sigma\tau} \end{aligned} \tag{6.4}$$

where $\Omega^i_{\rho\sigma\tau} \in \mathcal{A}$.

Now, if we want to write a local lagrangian involving $A^{\perp}$ only, we encounter the same problems we faced early on when we intended to write a lagrangian in terms of $h^2_{\mu\nu}$ only (2.13). For example, taking just the spin 3 part, due to the fact that $(\mathrm{P}_3)_{\mu\nu\lambda}^{\rho\sigma\tau}\,\Omega_{\rho\sigma\tau}$ goes as $\Box^{-3}$, we will need to have an action of the type

$$S_3 = \frac{1}{\kappa^{10}} \int d(vol)\, A^{(3)}{}_{\mu\nu\lambda} \Box^6 A^{(3)\mu\nu\lambda} \tag{6.5}$$

if we want it to be formally *local* (in the sense that no negative powers of $\Box$ appear).

It is perhaps worth remarking that some of these problems are shared even by theories linear in curvature, as soon as fermionic matter is coupled to gravity. In this case the first order formalism and the second order one are not equivalent, and in fact when treating the theory in first order formalism, spacetime torsion is generated on shell. This fact seems worthy of some extra research.

More work is clearly needed however before a good understanding of the first order formalism is achieved.

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## Appendix A: Spin content and spin projectors

In order to get the spin projectors for a symmetric tensor $h_{\mu\nu}$, let us start with a simple vector field $u^\mu$. If we consider a timelike reference momentum $k^\mu$ (with $k^2 > 0$), physics is simpler in the adapted frame where

$$k^\mu = \delta^\mu_0 \tag{A.1}$$

Therefore, the spin content of a vector $u^\mu$ which we represent as $\Box$ is

$$\begin{aligned} &s = 1 : u^i \quad 3 \text{ components}, \\ &s = 0 : u^0 \quad 1 \text{ component}. \end{aligned} \tag{A.2}$$

And the corresponding projectors in momentum space read

$$\begin{aligned} P^{(0)\beta}_{\alpha} &= \frac{k_\alpha k^\beta}{k^2} \equiv \omega_\alpha{}^\beta = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \\ P^{(1)\beta}_{\alpha} &= \delta^\beta_\alpha - \frac{k_\alpha k^\beta}{k^2} \equiv \theta_\alpha{}^\beta = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \end{aligned} \tag{A.3}$$

It should be noted that these operators are non-local in position space where $\frac{1}{k^2}$ stands for $\Box^{-1}$. We shall use both momentum and position space as equivalent. That is, we could as well write

$$\begin{aligned} \omega_\alpha{}^\beta &= \frac{\partial_\alpha \partial^\beta}{\Box} \\ \theta_\alpha{}^\beta &= \delta^\beta_\alpha - \frac{\partial_\alpha \partial^\beta}{\Box} \end{aligned} \tag{A.4}$$

so the traces read as follows

$$\begin{aligned} \mathrm{Tr}\ P_0 &= 1 \\ \mathrm{Tr}\ P_1 &= 3 \end{aligned} \tag{A.5}$$

As it is well-known, the metric $h_{\mu\nu}$ (or equivalently, the frame field, $h^a{}_\mu$) transforms in the euclidean setting under the representation $\underline{10} \equiv \Box\Box$ of SO(4), so the spin content and corresponding projectors are given by

$$\begin{aligned} &s = 2: \quad h^T_{ij} \equiv h_{ij} - \frac{1}{3} h \delta_{ij} \\ &\quad (P_2)^{\rho\sigma}_{\mu\nu} \equiv \frac{1}{2}(\theta^\rho_\mu \theta^\sigma_\nu + \theta^\sigma_\mu \theta^\rho_\nu) - \frac{1}{3}\theta_{\mu\nu}\theta^{\rho\sigma} \\ &s = 1: \quad h_{0i} \\ &\quad (P_1)^{\rho\sigma}_{\mu\nu} \equiv \frac{1}{2}(\theta^\rho_\mu \omega^\sigma_\nu + \theta^\sigma_\mu \omega^\rho_\nu + \theta^\rho_\nu \omega^\sigma_\mu + \theta^\sigma_\nu \omega^\rho_\mu) \\ &s = 0: \quad h_{00} \\ &\quad (P^w_0)^{\rho\sigma}_{\mu\nu} \equiv \omega_{\mu\nu}\omega^{\rho\sigma} \\ &s = 0: \quad h \equiv \delta^{ij} h_{ij} \end{aligned}$$

$$\left(P_0^s\right)_{\mu\nu}^{\rho\sigma} \equiv \frac{1}{3}\theta_{\mu\nu}\theta^{\rho\sigma} \tag{A.6}$$

These particular projectors have been studied previously by Barnes and Rivers [23,24]. They are complete in the symmetrized direct product

$$Sym(T_x \otimes T_x) \tag{A.7}$$

where $T_x$ is the tangent space at the point $x \in M$ of the space-time manifold.

It is convenient to define another projector

$$P_0 \equiv P_0^w + P_0^s \tag{A.8}$$

and the non-differential projectors are

$$I_{\mu\nu}^{\rho\sigma} \equiv \frac{1}{2}\left(\delta_\mu^\rho\delta_\nu^\sigma + \delta_\mu^\sigma\delta_\nu^\rho\right)$$
$$T_{\mu\nu}^{\rho\sigma} \equiv \frac{1}{4}\eta_{\mu\nu}\eta^{\rho\sigma} \tag{A.9}$$

Then we can write a closure relation for these projectors, to be specific,

$$(P_2)_{\mu\nu}^{\rho\sigma} + (P_1)_{\mu\nu}^{\rho\sigma} + (P_0)_{\mu\nu}^{\rho\sigma} = I_{\mu\nu}^{\rho\sigma} \tag{A.10}$$

These projectors are not enough though, as they do not form a base of the space of four-index tensors of the type of interest. Such a base is formed by five independent monomials, namely (permutations are implicit)

$$M_1 \equiv k_\mu k_\nu k_\rho k_\sigma$$
$$M_2 \equiv k_\mu k_\nu \eta_{\rho\sigma}$$
$$M_3 \equiv k_\mu k_\sigma \eta_{\rho\nu}$$
$$M_4 \equiv \eta_{\mu\nu}\eta_{\rho\sigma}$$
$$M_5 \equiv \eta_{\mu\rho}\eta_{\nu\sigma} \tag{A.11}$$

Therefore, in order to get a basis, we then need to add a new independent operator

$$\left(P_0^\times\right)_{\mu\nu}^{\rho\sigma} = \frac{1}{\sqrt{3}}(\omega_{\mu\nu}\theta^{\rho\sigma} + \theta_{\mu\nu}\omega^{\rho\sigma}) \tag{A.12}$$

that can be identified with the mixing of the two spin 0 components, $h$ and $h_{00}$. It is clear that this new operator cannot be orthogonal to the other four, since closure implies that the only operator orthogonal to the set that closes is the null operator.

## Appendix B: Spin content of the symmetric connection field

In this appendix, we decompose the operators mediating between two connection fields $A_{\mu\beta\gamma} \equiv g_{\alpha\mu}\Gamma^\alpha_{\beta\gamma}$ – symmetric in the last two indices, because we are assuming vanishing torsion – in terms of the spin projectors of this field. The procedure is analogue to the one followed in "Appendix A".

Since $A_{\mu\nu\lambda} = A_{\mu\lambda\nu}$,

$$A_{\mu\nu\lambda} \in \mathcal{A} \equiv T_x \otimes Sym\,(T_x \otimes T_x) \tag{B.1}$$

The quadratic kinetic operator in this space is

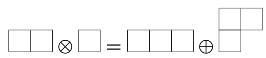
$$K \in \mathcal{A} \otimes \mathcal{A} \tag{B.2}$$

In order to disentangle the physical meaning of the gauge piece of the total action, we would like to expand $K$ as a sum of projectors with definite spin. There are 22 independent monomials to consider. Let un proceed by steps.

The projector into $\mathcal{A}$ – namely, the identity in this space – is

$$P_0 \equiv (P_0)_{\mu(\nu\lambda)}^{\alpha(\beta\gamma)} \equiv \frac{1}{2}\delta_\mu^\alpha\left(\delta_\nu^\beta\delta_\lambda^\gamma + \delta_\nu^\gamma\delta_\lambda^\beta\right) = \frac{1}{2}(1,0,0,1,0,0)$$
$$P_0^2 \equiv (P_0)_{\mu(\nu\lambda)}^{\alpha(\beta\gamma)}(P_0)_{\alpha(\beta\gamma)}^{a(bc)} = P_{\mu(\nu\lambda)}^{a(bc)} = \mathcal{P}_0$$
$$P_0\,\mathcal{A} = \mathcal{A} \tag{B.3}$$

(where the last equality in the first equation refers to the vector notation introduced in the "Appendix E"). The subspace $\mathcal{A}$ corresponds, in terms of representations of the tangent group SO(4), to the sum of a totally symmetric three-index tensor plus a tensor with the *hook* symmetry

$$\{2,0\} \otimes \{1\} = \{3,0\} \oplus \{2,1\}$$  (B.4)

In terms of dimensions this is $\underline{40} = \underline{20} + \underline{20}$. The Young projectors are

$$\mathrm{P}_S \equiv (\mathrm{P}_{\boxed{\alpha|\beta|\gamma}})_{\mu\nu\lambda}^{\alpha\beta\gamma} \equiv \frac{1}{6}\Big\{\delta_\mu^\alpha\delta_\nu^\beta\delta_\lambda^\gamma + \delta_\mu^\beta\delta_\nu^\gamma\delta_\lambda^\alpha + \delta_\mu^\gamma\delta_\nu^\alpha\delta_\lambda^\beta + \delta_\mu^\alpha\delta_\nu^\gamma\delta_\lambda^\beta + \delta_\mu^\beta\delta_\nu^\alpha\delta_\lambda^\gamma + \delta_\mu^\gamma\delta_\nu^\beta\delta_\lambda^\alpha\Big\}$$
$$= \frac{1}{6}(1,1,1,1,1,1) \tag{B.5}$$

and the hook representation

$$\mathcal{P}_H \equiv \left(P_{\boxed{\alpha|\beta},\boxed{\gamma}}\right)_{\mu\nu\lambda}^{\alpha\beta\gamma} \equiv \frac{1}{3}\Big\{\delta_\mu^\alpha\delta_\nu^\beta\delta_\lambda^\gamma + \delta_\mu^\alpha\delta_\nu^\gamma\delta_\lambda^\beta - \frac{1}{2}\delta_\nu^\alpha\delta_\mu^\beta\delta_\lambda^\gamma - \frac{1}{2}\delta_\nu^\alpha\delta_\lambda^\beta\delta_\mu^\gamma - \frac{1}{2}\delta_\lambda^\alpha\delta_\nu^\beta\delta_\mu^\gamma - \frac{1}{2}\delta_\lambda^\alpha\delta_\mu^\beta\delta_\nu^\gamma\Big\}$$
$$= \frac{1}{3}\left(1,-\frac{1}{2},-\frac{1}{2},1,-\frac{1}{2},-\frac{1}{2}\right) \tag{B.6}$$

It should be stressed that this projector is not symmetric in $(\alpha\beta)$, but rather in $(\beta,\gamma)$.

$$\left(\mathcal{P}_{\begin{smallmatrix}\alpha&\beta\\\gamma&\end{smallmatrix}}\right)^{\alpha\beta\gamma}_{\mu\nu\lambda}=\left(\mathcal{P}_{\begin{smallmatrix}\alpha&\gamma\\\beta&\end{smallmatrix}}\right)^{\alpha\beta\gamma}_{\mu\nu\lambda}$$
$$\left(\mathcal{P}_{\begin{smallmatrix}\alpha&\beta\\\gamma&\end{smallmatrix}}\right)^{\alpha\beta\gamma}_{\mu\nu\lambda}+\left(\mathcal{P}_{\begin{smallmatrix}\gamma&\alpha\\\beta&\end{smallmatrix}}\right)^{\alpha\beta\gamma}_{\mu\nu\lambda}+\left(\mathcal{P}_{\begin{smallmatrix}\beta&\gamma\\\alpha&\end{smallmatrix}}\right)^{\alpha\beta\gamma}_{\mu\nu\lambda}=0 \quad (B.7)$$

In the following, we will keep this notation: P for the projectors in the symmetric subspace and $\mathcal{P}$ for those in the hook subspace.

The Young projectors are symmetric, orthogonal and add to the identity in $\mathcal{A}$

$$\mathrm{P}_S^T=\mathrm{P}_S \quad \mathcal{P}_H^T=\mathcal{P}_H$$
$$\mathrm{P}_S\mathcal{P}_H=\mathcal{P}_H\mathrm{P}_S=0$$
$$\mathrm{P}_S+\mathcal{P}_H=P_0 \quad (B.8)$$

Then we can always write for any $A\in\mathcal{A}$

$$A=\mathcal{P}_0A=\mathrm{A}_S+\mathcal{A}_H \quad (B.9)$$

with

$$\mathrm{P}_S\mathrm{A}_S=\mathrm{A}_S$$
$$\mathcal{P}_H\mathcal{A}_H=\mathcal{A}_H \quad (B.10)$$

B.1 The totally symmetric tensor

Let us start by determining the spin content of the totally symmetric piece $\left(P_{\{3\}}A\right)_{\alpha\beta\gamma}\equiv A_{(\alpha\beta\gamma)}$.

We can decompose it in its spin components as

- First the spin 3 component, which is given in the rest frame by

$$A^T_{ijk}\equiv A_{ijk}-\frac{1}{5}\left(A_i\delta_{jk}+A_j\delta_{ik}+A_k\delta_{ij}\right) \quad (B.11)$$

where

$$A_i\equiv\sum_j A_{ijj} \quad (B.12)$$

There are of course 7 components in this set.
The spin 3 projector reads

$$(\mathrm{P}_3)^{\alpha\beta\gamma}_{\lambda\mu\nu}=\frac{1}{6}\Big(\theta^\alpha{}_\nu\theta^\beta{}_\mu\theta^\gamma{}_\lambda+\theta^\alpha{}_\mu\theta^\beta{}_\nu\theta^\gamma{}_\lambda$$
$$+\theta^\alpha{}_\nu\theta^\beta{}_\lambda\theta^\gamma{}_\mu+\theta^\alpha{}_\lambda\theta^\beta{}_\nu\theta^\gamma{}_\mu+\theta^\alpha{}_\mu\theta^\beta{}_\lambda\theta^\gamma{}_\nu$$
$$+\theta^\alpha{}_\lambda\theta^\beta{}_\mu\theta^\gamma{}_\nu\Big)$$
$$-\frac{1}{15}\Big(\theta^\alpha{}_\nu\theta^{\beta\gamma}\theta_{\mu\lambda}+\theta^{\alpha\gamma}\theta^\beta{}_\nu\theta_{\mu\lambda}$$
$$+\theta^{\alpha\beta}\theta^\gamma{}_\nu\theta_{\mu\lambda}+\theta^\alpha{}_\mu\theta^{\beta\gamma}\theta_{\nu\lambda}+\theta^{\alpha\gamma}\theta^\beta{}_\mu\theta_{\nu\lambda}$$
$$+\theta^{\alpha\beta}\theta^\gamma{}_\mu\theta_{\nu\lambda}+\theta^\alpha{}_\lambda\theta^{\beta\gamma}\theta_{\mu\nu}+\theta^{\alpha\gamma}\theta^\beta{}_\lambda\theta_{\mu\nu}$$
$$+\theta^{\alpha\beta}\theta^\gamma{}_\lambda\theta_{\mu\nu}\Big) \quad (B.13)$$

- The spin 2 component is given in the rest frame by

$$A^T_{0ij}\equiv A_{0ij}-\frac{1}{3}A_0\delta_{ij} \quad (B.14)$$

where

$$A_0\equiv\sum_i A_{0ii} \quad (B.15)$$

The projector reads

$$(\mathrm{P}_2)^{\alpha\beta\gamma}_{\lambda\mu\nu}=\tfrac{1}{6}\theta^\beta{}_\nu\theta^\gamma{}_\mu\omega^\alpha{}_\lambda+\tfrac{1}{6}\theta^\beta{}_\mu\theta^\gamma{}_\nu\omega^\alpha{}_\lambda$$
$$-\tfrac{1}{9}\theta^{\beta\gamma}\theta_{\mu\nu}\omega^\alpha{}_\lambda$$
$$+\tfrac{1}{6}\theta^\beta{}_\nu\theta^\gamma{}_\lambda\omega^\alpha{}_\mu+\tfrac{1}{6}\theta^\beta{}_\lambda\theta^\gamma{}_\nu\omega^\alpha{}_\mu$$
$$-\tfrac{1}{9}\theta^{\beta\gamma}\theta_{\lambda\nu}\omega^\alpha{}_\mu+\tfrac{1}{6}\theta^\beta{}_\mu\theta^\gamma{}_\lambda\omega^\alpha{}_\nu$$
$$+\tfrac{1}{6}\theta^\beta{}_\lambda\theta^\gamma{}_\mu\omega^\alpha{}_\nu-\tfrac{1}{9}\theta^{\beta\gamma}\theta_{\lambda\mu}\omega^\alpha{}_\nu$$
$$+\tfrac{1}{6}\theta^\alpha{}_\nu\theta^\gamma{}_\mu\omega^\beta{}_\lambda$$
$$+\tfrac{1}{6}\theta^\alpha{}_\mu\theta^\gamma{}_\nu\omega^\beta{}_\lambda-\tfrac{1}{9}\theta^{\alpha\gamma}\theta_{\mu\nu}\omega^\beta{}_\lambda$$
$$+\tfrac{1}{6}\theta^\alpha{}_\nu\theta^\gamma{}_\lambda\omega^\beta{}_\mu+\tfrac{1}{6}\theta^\alpha{}_\lambda\theta^\gamma{}_\nu$$
$$\times\,\omega^\beta{}_\mu-\tfrac{1}{9}\theta^{\alpha\gamma}\theta_{\lambda\nu}\omega^\beta{}_\mu$$
$$+\tfrac{1}{6}\theta^\alpha{}_\mu\theta^\gamma{}_\lambda\omega^\beta{}_\nu+\tfrac{1}{6}\theta^\alpha{}_\lambda\theta^\gamma{}_\mu\omega^\beta{}_\nu$$
$$-\tfrac{1}{9}\theta^{\alpha\gamma}\theta_{\lambda\mu}\omega^\beta{}_\nu$$
$$+\tfrac{1}{6}\theta^\alpha{}_\nu\theta^\beta{}_\mu\omega^\gamma{}_\lambda+\tfrac{1}{6}\theta^\alpha{}_\mu\theta^\beta{}_\nu\omega^\gamma{}_\lambda$$
$$-\tfrac{1}{9}\theta^{\alpha\beta}\theta_{\mu\nu}\omega^\gamma{}_\lambda+\tfrac{1}{6}\theta^\alpha{}_\nu\theta^\beta{}_\lambda\omega^\gamma{}_\mu$$
$$+\tfrac{1}{6}\theta^\alpha{}_\lambda\theta^\beta{}_\nu\omega^\gamma{}_\mu-\tfrac{1}{9}\theta^{\alpha\beta}\theta_{\lambda\nu}\omega^\gamma{}_\mu$$
$$+\tfrac{1}{6}\theta^\alpha{}_\mu\theta^\beta{}_\lambda\omega^\gamma{}_\nu$$
$$+\tfrac{1}{6}\theta^\alpha{}_\lambda\theta^\beta{}_\mu\omega^\gamma{}_\nu-\tfrac{1}{9}\theta^{\alpha\beta}\theta_{\lambda\mu}\omega^\gamma{}_\nu \quad (B.16)$$

- There are two spin 1 components. First the one that is given in the rest frame by

$$A_{ijk}\delta^{jk} \quad (B.17)$$

with projector

$$(\mathrm{P}^s_1)^{\alpha\beta\gamma}_{\lambda\mu\nu}=\frac{1}{15}(\theta^\alpha{}_\nu\theta^{\beta\gamma}\theta_{\mu\lambda}+\theta^{\alpha\gamma}\theta^\beta{}_\nu\theta_{\mu\lambda}+\theta^{\alpha\beta}\theta^\gamma{}_\nu\theta_{\mu\lambda}$$
$$+\theta^\alpha{}_\mu\theta^{\beta\gamma}\theta_{\lambda\nu}+\theta^{\alpha\gamma}\theta^\beta{}_\mu\theta_{\lambda\nu}+\theta^{\alpha\beta}\theta^\gamma{}_\mu\theta_{\lambda\nu}$$
$$+\theta^\alpha{}_\lambda\theta^{\beta\gamma}\theta_{\mu\nu}+\theta^{\alpha\gamma}\theta^\beta{}_\lambda\theta_{\mu\nu}+\theta^{\alpha\beta}\theta^\gamma{}_\lambda\theta_{\mu\nu}) \quad (B.18)$$

The other corresponds to

$$A_{00i} \quad (B.19)$$

and the projector is

$$(\mathrm{P}^w_1)^{\alpha\beta\gamma}_{\lambda\mu\nu} = \frac{1}{6}\Big(\theta^\gamma{}_\nu w^\alpha{}_\mu w^\beta{}_\lambda + \theta^\gamma{}_\mu w^\alpha{}_\nu w^\beta{}_\lambda + \theta^\gamma{}_\nu w^\alpha{}_\lambda w^\beta{}_\mu + \theta^\gamma{}_\lambda w^\alpha{}_\nu w^\beta{}_\mu + \theta^\gamma{}_\mu w^\alpha{}_\lambda w^\beta{}_\nu + \theta^\gamma{}_\lambda w^\alpha{}_\mu w^\beta{}_\nu + \theta^\beta{}_\nu w^\alpha{}_\mu w^\gamma{}_\lambda + \theta^\beta{}_\mu w^\alpha{}_\nu w^\gamma{}_\lambda + \theta^\alpha{}_\nu w^\beta{}_\mu w^\gamma{}_\lambda + \frac{1}{6}\theta^\alpha{}_\mu w^\beta{}_\nu w^\gamma{}_\lambda + \theta^\beta{}_\nu w^\alpha{}_\lambda w^\gamma{}_\mu + \theta^\beta{}_\lambda w^\alpha{}_\nu w^\gamma{}_\mu + \frac{1}{6}\theta^\alpha{}_\nu w^\beta{}_\lambda w^\gamma{}_\mu + \theta^\alpha{}_\lambda w^\beta{}_\nu w^\gamma{}_\mu + \theta^\beta{}_\mu w^\alpha{}_\lambda w^\gamma{}_\nu + \theta^\beta{}_\lambda w^\alpha{}_\mu w^\gamma{}_\nu + \frac{1}{6}\theta^\alpha{}_\mu w^\beta{}_\lambda w^\gamma{}_\nu + \theta^\alpha{}_\lambda w^\beta{}_\mu w^\gamma{}_\nu\Big) \tag{B.20}$$

- There are also two different spin zero components. The first one corresponds to

$$A_{000} \tag{B.21}$$

and its projector is

$$(\mathrm{P}^w_0)^{\alpha\beta\gamma}_{\lambda\mu\nu} = \frac{1}{6}\Big(\omega^\alpha{}_\nu\omega^\beta{}_\mu\omega^\gamma{}_\lambda + \omega^\alpha{}_\mu\omega^\beta{}_\nu\omega^\gamma{}_\lambda + \omega^\alpha{}_\nu\omega^\beta{}_\lambda\omega^\gamma{}_\mu + \omega^\alpha{}_\lambda\omega^\beta{}_\nu\omega^\gamma{}_\mu + \omega^\alpha{}_\mu\omega^\beta{}_\lambda\omega^\gamma{}_\nu + \omega^\alpha{}_\lambda\omega^\beta{}_\mu\omega^\gamma{}_\nu\Big) \tag{B.22}$$

while the second one corresponds to

$$A_{0ij}\delta^{ij} \tag{B.23}$$

with projector

$$(\mathrm{P}^s_0)^{\alpha\beta\gamma}_{\lambda\mu\nu} = \frac{1}{9}\Big(\theta^{\beta\gamma}\theta_{\mu\nu}w^\alpha{}_\lambda + \theta^{\beta\gamma}\theta_{ln}w^\alpha{}_\mu + \theta^{\beta\gamma}\theta_{\mu\lambda}w^\alpha{}_\nu + \theta^{\alpha\gamma}\theta_{\mu\nu}w^\beta{}_\lambda + \theta^{\alpha\gamma}\theta_{\lambda\nu}w^\beta{}_\mu + \theta^{\alpha\gamma}\theta_{\mu\lambda}w^\beta{}_\nu + \theta^{\alpha\beta}\theta_{\mu\nu}w^\gamma{}_\lambda + \theta^{\alpha\beta}\theta_{\lambda\nu}w^\gamma{}_\mu + \theta^{\alpha\beta}\theta_{\mu\lambda}w^\gamma{}_\nu\Big) \tag{B.24}$$

Altogether we have accounted for the 20 components in this set and the spin content is

$$\underline{20}_S = (\underline{3}) \oplus (\underline{2}) \oplus 2\ (\underline{1}) \oplus 2\ (\underline{0}) \tag{B.25}$$

Indeed, they satisfy the closure relation that symbolicall reads,

$$\mathrm{P}^s_0 + \mathrm{P}^w_0 + \mathrm{P}^s_1 + \mathrm{P}^w_1 + \mathrm{P}_2 + \mathrm{P}_3 = \mathrm{P}_S \tag{B.26}$$

B.2 The hook sector

Let us now work out the spin content of the 20 components of the diagram $P_{\{2,1\}}A$.

We will henceforth assume that connections are already projected into the corrresponding Young subspace, that is, when $A \in \mathcal{A}$,

$$\mathcal{A}^H_{\alpha\beta\gamma} \equiv (\mathcal{P}_H A)_{\alpha\beta\gamma} \equiv \frac{1}{3}\left(2A_{\alpha\beta\gamma} - A_{\beta\gamma\alpha} - A_{\gamma\alpha\beta}\right) = A_{\alpha\beta\gamma} \tag{B.27}$$

This implies cyclic symmetry

$$\mathcal{A}_{\alpha\beta\gamma} + \mathcal{A}_{\beta\gamma\alpha} + \mathcal{A}_{\gamma\alpha\beta} = 0 \tag{B.28}$$

Consider first components with one element in the direction of the momentum (that is the 0-th component in the rest frame). Remember that for the projectors acting in this subspace we are using the letter $\mathcal{P}$.

- There is only one spin zero, a trace that is given by

$$\sum_{i=1}^{3} A_{i0i} \tag{B.29}$$

that is

$$(\mathcal{P}^s_0)^{\alpha\beta\gamma}_{\lambda\mu\nu} = -\frac{1}{9}\theta^{\beta\gamma}\theta_{\mu\nu}w^\alpha{}_\lambda + \frac{2}{9}\theta^{\beta\gamma}\theta_{\nu\lambda}w^\alpha{}_\mu - \frac{1}{9}\theta^{\beta\gamma}\theta_{\mu\lambda}w^\alpha{}_\nu + \frac{1}{18}\theta^{\alpha\gamma}\theta_{\mu\nu}w^\beta{}_\lambda - \frac{1}{9}\theta^{\alpha\gamma}\theta_{\nu\lambda}w^\beta{}_\mu + \frac{1}{18}\theta^{\alpha\gamma}\theta_{\mu\lambda}w^\beta{}_\nu + \frac{1}{18}\theta^{\alpha\beta}\theta_{\mu\nu}w_\lambda{}^\gamma - \frac{1}{9}\theta^{\alpha\beta}\theta_{\nu\lambda}w_\mu{}^\gamma + \frac{1}{18}\theta^{\alpha\beta}\theta_{\mu\lambda}w_\nu{}^\gamma \tag{B.30}$$

- There are three spin 1 components. First

$$\frac{1}{2}\left(A_{j0i} - A_{i0j}\right) \tag{B.31}$$

corresponding to

$$(\mathcal{P}^s_1)^{\alpha\beta\gamma}_{\lambda\mu\nu} = -\frac{1}{4}\theta^\alpha{}_\nu\theta_\mu{}^\gamma w^\beta{}_\lambda + \frac{1}{4}\theta^\alpha{}_\mu\theta_\nu{}^\gamma w^\beta{}_\lambda + \frac{1}{4}\theta^\alpha{}_\mu\theta_\lambda{}^\gamma w^\beta{}_\nu - \frac{1}{4}\theta^\alpha{}_\lambda\theta_\mu{}^\gamma w^\beta{}_\nu - \frac{1}{4}\theta^\alpha{}_\nu\theta^\beta{}_\mu w_\lambda{}^\gamma + \frac{1}{4}\theta^\alpha{}_\mu\theta^\beta{}_\nu w_\lambda{}^\gamma + \frac{1}{4}\theta^\alpha{}_\mu\theta^\beta{}_\lambda w_\nu{}^\gamma - \frac{1}{4}\theta^\alpha{}_\lambda\theta^\beta{}_\mu w_\nu{}^\gamma \tag{B.32}$$

The second one is given by

$$A_{i00} \tag{B.33}$$

$$(\mathcal{P}_1^w)_{\lambda\mu\nu}^{\alpha\beta\gamma} = \frac{1}{12}\theta_\nu{}^\gamma w^\alpha{}_\mu w^\beta{}_\lambda - \frac{1}{6}\theta_\mu{}^\gamma w^\alpha{}_\nu w^\beta{}_\lambda + \frac{1}{12}\theta_\nu{}^\gamma w^\alpha{}_\lambda w^\beta{}_\mu + \frac{1}{12}\theta_\lambda{}^\gamma w^\alpha{}_\nu w^\beta{}_\mu - \frac{1}{6}\theta_\mu{}^\gamma w^\alpha{}_\lambda w^\beta{}_\nu + \frac{1}{12}\theta_\lambda{}^\gamma w^\alpha{}_\mu w^\beta{}_\nu + \frac{1}{12}\theta^\beta{}_\nu w^\alpha{}_\mu w_\lambda{}^\gamma - \frac{1}{6}\theta^\beta{}_\mu w^\alpha{}_\nu w_\lambda{}^\gamma - \frac{1}{6}\theta^\alpha{}_\nu w^\beta{}_\mu w_\lambda{}^\gamma + \frac{1}{3}\theta^\alpha{}_\mu w^\beta{}_\nu w_\lambda{}^\gamma + \frac{1}{12}\theta^\beta{}_\nu w^\alpha{}_\lambda w_\mu{}^\gamma + \frac{1}{12}\theta^\beta{}_\lambda w^\alpha{}_\nu w_\mu{}^\gamma - \frac{1}{6}\theta^\alpha{}_\nu w^\beta{}_\lambda w_\mu{}^\gamma - \frac{1}{6}\theta^\alpha{}_\lambda w^\beta{}_\nu w_\mu{}^\gamma - \frac{1}{6}\theta^\beta{}_\mu w^\alpha{}_\lambda w_\nu{}^\gamma + \frac{1}{12}\theta^\beta{}_\lambda w^\alpha{}_\mu w_\nu{}^\gamma + \frac{1}{3}\theta^\alpha{}_\mu w^\beta{}_\lambda w_\nu{}^\gamma - \frac{1}{6}\theta^\alpha{}_\lambda w^\beta{}_\mu w_\nu{}^\gamma \tag{B.34}$$

And there is also a spin 1 trace given by

$$(\mathcal{P}_1^t)_{\lambda\mu\nu}^{\alpha\beta\gamma} = -\frac{1}{6}\theta^a{}_\nu\theta^{\beta\gamma}\theta_{\lambda\mu} + \frac{1}{12}\theta^{ag}\theta^\beta{}_\nu\theta_{\lambda\mu} + \frac{1}{12}\theta^{\alpha\beta}\theta^\gamma{}_\nu\theta_{\lambda\mu} + \frac{1}{3}\theta^a{}_\mu\theta^{\beta\gamma}\theta_{ln} - \frac{1}{6}\theta^{ag}\theta^\beta{}_\mu\theta_{ln} - \frac{1}{6}\theta^{\alpha\beta}\theta^\gamma{}_\mu\theta_{ln} - \frac{1}{6}\theta^a{}_\lambda\theta^{\beta\gamma}\theta_{\mu\nu} + \frac{1}{12}\theta^{ag}\theta^\beta{}_\lambda\theta_{\mu\nu} + \frac{1}{12}\theta^{\alpha\beta}\theta^\gamma{}_\lambda\theta_{\mu\nu} \tag{B.35}$$

- Finally, there are two spin 2 projectors. The first one is the transverse traceless spin two component

$$\frac{1}{2}\left(A_{j0i} + A_{i0j}\right) - \frac{1}{3}\delta_{ij}\sum_{k=1}^{3} A_{k0k} \tag{B.36}$$

with projector

$$(\mathcal{P}_2)_{\lambda\mu\nu}^{\alpha\beta\gamma} = -\frac{1}{6}\theta^\beta{}_\nu\theta^\gamma{}_\mu w^\alpha{}_\lambda - \frac{1}{6}\theta^\beta{}_\mu\theta^\gamma{}_\nu w^\alpha{}_\lambda + \frac{1}{9}\theta^{\beta\gamma}\theta_{\mu\nu} w^\alpha{}_\lambda + \frac{1}{3}\theta^\beta{}_\nu\theta^\gamma{}_\lambda w^\alpha{}_\mu + \frac{1}{3}\theta^\beta{}_\lambda\theta^\gamma{}_\nu w^\alpha{}_\mu - \frac{2}{9}\theta^{\beta\gamma}\theta_{\lambda\nu} w^\alpha{}_\mu - \frac{1}{6}\theta^\beta{}_\mu\theta^\gamma{}_\lambda w^\alpha{}_\nu - \frac{1}{6}\theta^\beta{}_\lambda\theta^\gamma{}_\mu w^\alpha{}_\nu + \frac{1}{9}\theta^{\beta\gamma}\theta_{\lambda\mu} w^\alpha{}_\nu + \frac{1}{12}\theta^\alpha{}_\nu\theta^\gamma{}_\mu w^\beta{}_\lambda + \frac{1}{12}\theta^\alpha{}_\mu\theta^\gamma{}_\nu w^\beta{}_\lambda - \frac{1}{18}\theta^{\alpha\gamma}\theta_{\mu\nu} w^\beta{}_\lambda - \frac{1}{6}\theta^\alpha{}_\nu\theta^\gamma{}_\lambda w^\beta{}_\mu - \frac{1}{6}\theta^\alpha{}_\lambda\theta^\gamma{}_\nu w^\beta{}_\mu + \frac{1}{9}\theta^{\alpha\gamma}\theta_{\lambda\nu} w^\beta{}_\mu + \frac{1}{12}\theta^\alpha{}_\mu\theta^\gamma{}_\lambda w^\beta{}_\nu + \frac{1}{12}\theta^\alpha{}_\lambda\theta^\gamma{}_\mu w^\beta{}_\nu - \frac{1}{18}\theta^{\alpha\gamma}\theta_{\lambda\mu} w^\beta{}_\nu + \frac{1}{12}\theta^\alpha{}_\nu\theta^\beta{}_\mu w^\gamma{}_\lambda + \frac{1}{12}\theta^\alpha{}_\mu\theta^\beta{}_\nu w^\gamma{}_\lambda - \frac{1}{18}\theta^{\alpha\beta}\theta_{\mu\nu} w^\gamma{}_\lambda - \frac{1}{6}\theta^\alpha{}_\nu\theta^\beta{}_\lambda w^\gamma{}_\mu - \frac{1}{6}\theta^\alpha{}_\lambda\theta^\beta{}_\nu w^\gamma{}_\mu + \frac{1}{9}\theta^{\alpha\beta}\theta_{ln} w^\gamma{}_\mu + \frac{1}{12}\theta^\alpha{}_\mu\theta^\beta{}_\lambda w^\gamma{}_\nu + \frac{1}{12}\theta^\alpha{}_\lambda\theta^\beta{}_\mu w^\gamma{}_\nu - \frac{1}{18}\theta^{\alpha\beta}\theta_{\lambda\mu} w^\gamma{}_\nu \tag{B.37}$$

The second one corresponds to the spin 2 traceless connection field

$$A^T_{ijk} \equiv A_{ijk} - \frac{2t_i^1 - t_i^2}{5}\delta_{jk} - \frac{3t_j^2 - t_j^1}{10}\delta_{ik} - \frac{3t_k^2 - t_k^1}{10}\delta_{ij} \tag{B.38}$$

with projector

$$(\mathcal{P}_2^s)_{\lambda\mu\nu}^{\alpha\beta\gamma} = -\frac{1}{6}\theta^a{}_\nu\theta^\beta{}_\mu\theta^\gamma{}_\lambda + \frac{1}{3}\theta^a{}_\mu\theta^\beta{}_\nu\theta^\gamma{}_\lambda - \frac{1}{6}\theta^a{}_\nu\theta^\beta{}_\lambda\theta^\gamma{}_\mu - \frac{1}{6}\theta^a{}_\lambda\theta^\beta{}_\nu\theta^\gamma{}_\mu + \frac{1}{3}\theta^a{}_\mu\theta^\beta{}_\lambda\theta^\gamma{}_\nu - \frac{1}{6}\theta^a{}_\lambda\theta^\beta{}_\mu\theta^\gamma{}_\nu + \frac{1}{6}\theta^a{}_\nu\theta^{\beta\gamma}\theta_{\lambda\mu} - \frac{1}{12}\theta^{ag}\theta^\beta{}_\nu\theta_{\lambda\mu} - \frac{1}{12}\theta^{\alpha\beta}\theta^\gamma{}_\nu\theta_{\lambda\mu} - \frac{1}{3}\theta^a{}_\mu\theta^{\beta\gamma}\theta_{ln} + \frac{1}{6}\theta^{ag}\theta^\beta{}_\mu\theta_{ln} + \frac{1}{6}\theta^{\alpha\beta}\theta^\gamma{}_\mu\theta_{ln} + \frac{1}{6}\theta^a{}_\lambda\theta^{\beta\gamma}\theta_{\mu\nu} - \frac{1}{12}\theta^{ag}\theta^\beta{}_\lambda\theta_{\mu\nu} - \frac{1}{12}\theta^{\alpha\beta}\theta^\gamma{}_\lambda\theta_{\mu\nu} \tag{B.39}$$

Therefore, the spin content in this sector is

$$\underline{20}_H = 2\,(\underline{2}) \oplus 3\,(\underline{1}) \oplus (\underline{0}) \tag{B.40}$$

Finally, the closure relation in this space reads

$$\mathcal{P}_0^s + \mathcal{P}_1^s + \mathcal{P}_1^w + \mathcal{P}_1^t + \mathcal{P}_2 + \mathcal{P}_2^s = \mathcal{P}_H \tag{B.41}$$

### B.3 Mixed operators completing a basis of $\mathcal{L}(\mathcal{A}, \mathcal{A})$

Let us represent by $\mathcal{L}(\mathcal{A}, \mathcal{A})$ the space of linear mappings from $\mathcal{A}$ in $\mathcal{A}$. It is plain that a basis is given by (again, with implicit permutations)

$$M_1 \equiv k_\mu k_\nu k_\lambda k_\alpha k_\beta k_\gamma \quad M_2 \equiv \eta_{\nu\lambda} k_\mu k_\alpha k_\beta k_\gamma$$

$M_3 \equiv \eta_{\mu\nu}k_\lambda k_\alpha k_\beta k_\gamma \quad M_4 \equiv \eta_{\mu\alpha}k_\nu k_\gamma k_\beta k_\lambda$
$M_5 \equiv \eta_{\mu\beta}k_\nu k_\lambda k_\alpha k_\gamma \quad M_6 \equiv \eta_{\nu\beta}k_\mu k_\lambda k_\alpha k_\gamma$
$M_7 \equiv \eta_{\mu\alpha}\eta_{\beta\gamma}k_\nu k_\lambda \quad M_8 \equiv \eta_{\mu\beta}\eta_{\alpha\gamma}k_\nu k_\lambda$
$M_9 \equiv \eta_{\alpha\beta}\eta_{\lambda\gamma}k_\mu k_\nu \quad M_{10} \equiv \eta_{\alpha\lambda}\eta_{\beta\gamma}k_\mu k_\nu$
$M_{11} \equiv \eta_{\nu\lambda}\eta_{\beta\gamma}k_\mu k_\alpha \quad M_{12} \equiv \eta_{\nu\beta}\eta_{\lambda\gamma}k_\mu k_\alpha$
$M_{13} \equiv \eta_{\nu\lambda}\eta_{\alpha\gamma}k_\mu k_\beta \quad M_{14} \equiv \eta_{\nu\alpha}\eta_{\lambda\gamma}k_\mu k_\beta$
$M_{15} \equiv \eta_{\mu\alpha}\eta_{\nu\beta}\eta_{\lambda\gamma} \quad M_{16} \equiv \eta_{\mu\alpha}\eta_{\nu\lambda}\eta_{\beta\gamma}$
$M_{17} \equiv \eta_{\mu\beta}\eta_{\nu\alpha}\eta_{\lambda\gamma} \quad M_{18} \equiv \eta_{\mu\beta}\eta_{\nu\lambda}\eta_{\alpha\gamma}$
$M_{19} \equiv \eta_{\mu\nu}\eta_{\lambda\alpha}\eta_{\beta\gamma} \quad M_{20} \equiv \eta_{\mu\nu}\eta_{\lambda\beta}\eta_{\alpha\gamma}$
$M_{21} \equiv \eta_{\mu\lambda}\eta_{\nu\alpha}\eta_{\beta\gamma} \quad M_{22} \equiv \eta_{\mu\lambda}\eta_{\nu\beta}\eta_{\alpha\gamma}$

So far, we have obtained 12 different operators that satisfy the closure relation.

Given the fact that we have obtained up to now 12 projectors, which added to the identity in our space – see (B.26) and (B.41) –, it is plain that we are 10 operators short in order to get a complete basis on the space $\mathcal{L}(\mathcal{A}, \mathcal{A})$. The remaining operators (which are not, in general, projectors) correspond to the mixing of equal spin components of $A$. In the same sense that $P_0^\times$ in (A.12) corresponds to the mixing of the two spin 0 components of $h_{\mu\nu}$. Hence, we are going to classify them by their spin.

- There are three of them with spin 0

$$(\mathcal{P}_0^{sw})^{\alpha\beta\gamma\lambda\mu\nu} = \frac{4}{9}\theta^{\mu\nu}\omega^{\alpha\lambda}\omega^{\beta\gamma} + \frac{1}{9}\theta^{\lambda\nu}\omega^{\alpha\mu}\omega^{\beta\gamma} + \frac{1}{9}\theta^{\lambda\mu}\omega^{\alpha\nu}\omega^{\beta\gamma} + \frac{1}{9}\theta^{\mu\nu}\omega^{\alpha\gamma}\omega^{\beta\lambda} - \frac{2}{9}\theta^{\lambda\nu}\omega^{\alpha\gamma}\omega^{\beta\mu} - \frac{2}{9}\theta^{\lambda\mu}\omega^{\alpha\gamma}\omega^{\beta\nu} + \frac{1}{9}\theta^{\mu\nu}\omega^{\alpha\beta}\omega^{\gamma\lambda} - \frac{2}{9}\theta^{\lambda\nu}\omega^{\alpha\beta}\omega^{\gamma\mu} - \frac{2}{9}\theta^{\lambda\mu}\omega^{\alpha\beta}\omega^{\gamma\nu} + \frac{1}{9}\theta^{\beta\gamma}\omega^{\alpha\nu}\omega^{\lambda\mu} - \frac{2}{9}\theta^{\alpha\gamma}\omega^{\beta\nu}\omega^{\lambda\mu} - \frac{2}{9}\theta^{\alpha\beta}\omega^{\gamma\nu}\omega^{\lambda\mu} + \frac{1}{9}\theta^{\beta\gamma}\omega^{\alpha\mu}\omega^{\lambda\nu} - \frac{2}{9}\theta^{\alpha\gamma}\omega^{\beta\mu}\omega^{\lambda\nu} - \frac{2}{9}\theta^{\alpha\beta}\omega^{\gamma\mu}\omega^{\lambda\nu} + \frac{4}{9}\theta^{\beta\gamma}\omega^{\alpha\lambda}\omega^{\mu\nu} + \frac{1}{9}\theta^{\alpha\gamma}\omega^{\beta\lambda}\omega^{\mu\nu} + \frac{1}{9}\theta^{\alpha\beta}\omega^{\gamma\lambda}\omega^{\mu\nu} \tag{B.42}$$

$$(\mathcal{P}_0^{ws})^{\alpha\beta\gamma\lambda\mu\nu} = \frac{1}{9}\theta^{\mu\nu}\omega^{\alpha\lambda}\omega^{\beta\gamma} + \frac{1}{9}\theta^{\lambda\nu}\omega^{\alpha\mu}\omega^{\beta\gamma} + \frac{1}{9}\theta^{\lambda\mu}\omega^{\alpha\nu}\omega^{\beta\gamma} + \frac{1}{9}\theta^{\mu\nu}\omega^{\alpha\gamma}\omega^{\beta\lambda} + \frac{1}{9}\theta^{\lambda\nu}\omega^{\alpha\gamma}\omega^{\beta\mu} + \frac{1}{9}\theta^{\lambda\mu}\omega^{\alpha\gamma}\omega^{\beta\nu} + \frac{1}{9}\theta^{\mu\nu}\omega^{\alpha\beta}\omega^{\gamma\lambda} + \frac{1}{9}\theta^{\lambda\nu}\omega^{\alpha\beta}\omega^{\gamma\mu} + \frac{1}{9}\theta^{\lambda\mu}\omega^{\alpha\beta}\omega^{\gamma\nu} + \frac{1}{9}\theta^{\beta\gamma}\omega^{\alpha\nu}\omega^{\lambda\mu} + \frac{1}{9}\theta^{\alpha\gamma}\omega^{\beta\nu}\omega^{\lambda\mu} + \frac{1}{9}\theta^{\alpha\beta}\omega^{\gamma\nu}\omega^{\lambda\mu} + \frac{1}{9}\theta^{\beta\gamma}\omega^{\alpha\mu}\omega^{\lambda\nu} + \frac{1}{9}\theta^{\alpha\gamma}\omega^{\beta\mu}\omega^{\lambda\nu} + \frac{1}{9}\theta^{\alpha\beta}\omega^{\gamma\mu}\omega^{\lambda\nu} + \frac{1}{9}\theta^{\beta\gamma}\omega^{\alpha\lambda}\omega^{\mu\nu} + \frac{1}{9}\theta^{\alpha\gamma}\omega^{\beta\lambda}\omega^{\mu\nu} + \frac{1}{9}\theta^{\alpha\beta}\omega^{\gamma\lambda}\omega^{\mu\nu} \tag{B.43}$$

$$(\mathcal{P}_0^{x})^{\alpha\beta\gamma\lambda\mu\nu} = \frac{1}{6}\theta^{\alpha\gamma}\theta^{\lambda\nu}\omega^{\beta\mu} + \frac{1}{6}\theta^{\alpha\gamma}\theta^{\lambda\mu}\omega^{\beta\nu} + \frac{1}{6}\theta^{\alpha\beta}\theta^{\lambda\nu}\omega^{\gamma\mu} + \frac{1}{6}\theta^{\alpha\beta}\theta^{\lambda\mu}\omega^{\gamma\nu} \tag{B.44}$$

- There are six with spin 1

$$(\mathcal{P}_1^{wx})^{\alpha\beta\gamma\lambda\mu\nu} = \frac{1}{4}\theta^{\gamma\nu}\omega^{\alpha\mu}\omega^{\beta\lambda} + \frac{1}{4}\theta^{\gamma\mu}\omega^{\alpha\nu}\omega^{\beta\lambda} + \frac{1}{4}\theta^{\gamma\nu}\omega^{\alpha\lambda}\omega^{\beta\mu} + \frac{1}{4}\theta^{\gamma\mu}\omega^{\alpha\lambda}\omega^{\beta\nu} + \frac{1}{4}\theta^{\beta\nu}\omega^{\alpha\mu}\omega^{\gamma\lambda} + \frac{1}{4}\theta^{\beta\mu}\omega^{\alpha\nu}\omega^{\gamma\lambda} + \frac{1}{4}\theta^{\beta\nu}\omega^{\alpha\lambda}\omega^{\gamma\mu} + \frac{1}{4}\theta^{\beta\mu}\omega^{\alpha\lambda}\omega^{\gamma\nu} \tag{B.45}$$

$$(\mathcal{P}_1^{ws})^{\alpha\beta\gamma\lambda\mu\nu} = \frac{1}{9}\theta^{\gamma\nu}\theta^{\lambda\mu}\omega^{\alpha\beta} + \frac{1}{9}\theta^{\gamma\mu}\theta^{\lambda\nu}\omega^{\alpha\beta} + \frac{1}{9}\theta^{\gamma\lambda}\theta^{\mu\nu}\omega^{\alpha\beta} + \frac{1}{9}\theta^{\beta\nu}\theta^{\lambda\mu}\omega^{\alpha\gamma} + \frac{1}{9}\theta^{\beta\mu}\theta^{\lambda\nu}\omega^{\alpha\gamma} + \frac{1}{9}\theta^{\beta\lambda}\theta^{\mu\nu}\omega^{\alpha\gamma} + \frac{1}{9}\theta^{\alpha\nu}\theta^{\lambda\mu}\omega^{\beta\gamma} + \frac{1}{9}\theta^{\alpha\mu}\theta^{\lambda\nu}\omega^{\beta\gamma} + \frac{1}{9}\theta^{\alpha\lambda}\theta^{\mu\nu}\omega^{\beta\gamma} + \frac{1}{9}\theta^{\alpha\nu}\theta^{\beta\gamma}\omega^{\lambda\mu} + \frac{1}{9}\theta^{\alpha\gamma}\theta^{\beta\nu}\omega^{\lambda\mu} + \frac{1}{9}\theta^{\alpha\beta}\theta^{\gamma\nu}\omega^{\lambda\mu} + \frac{1}{9}\theta^{\alpha\mu}\theta^{\beta\gamma}\omega^{\lambda\nu} + \frac{1}{9}\theta^{\alpha\gamma}\theta^{\beta\mu}\omega^{\lambda\nu} + \frac{1}{9}\theta^{\alpha\beta}\theta^{\gamma\mu}\omega^{\lambda\nu} + \frac{1}{9}\theta^{\alpha\lambda}\theta^{\beta\gamma}\omega^{\mu\nu} + \frac{1}{9}\theta^{\alpha\gamma}\theta^{\beta\lambda}\omega^{\mu\nu} + \frac{1}{9}\theta^{\alpha\beta}\theta^{\gamma\lambda}\omega \tag{B.46}$$

$$(\mathcal{P}_1^{sw})^{\alpha\beta\gamma\lambda\mu\nu} = \frac{1}{9}\theta^{\gamma\nu}\theta^{\lambda\mu}\omega^{\alpha\beta} + \frac{1}{9}\theta^{\gamma\mu}\theta^{\lambda\nu}\omega^{\alpha\beta} - \frac{2}{9}\theta^{\gamma\lambda}\theta^{\mu\nu}\omega^{\alpha\beta} + \frac{1}{9}\theta^{\beta\nu}\theta^{\lambda\mu}\omega^{\alpha\gamma}$$

$$+\frac{1}{9}\theta^{\beta\mu}\theta^{\lambda\nu}\omega^{\alpha\gamma}-\frac{2}{9}\theta^{\beta\lambda}\theta^{\mu\nu}\omega^{\alpha\gamma}$$
$$-\frac{2}{9}\theta^{\alpha\nu}\theta^{\lambda\mu}\omega^{\beta\gamma}-\frac{2}{9}\theta^{\alpha\mu}\theta^{\lambda\nu}\omega^{\beta\gamma}$$
$$+\frac{4}{9}\theta^{\alpha\lambda}\theta^{\mu\nu}\omega^{\beta\gamma}-\frac{2}{9}\theta^{\alpha\nu}\theta^{\beta\gamma}\omega^{\lambda\mu}$$
$$+\frac{1}{9}\theta^{\alpha\gamma}\theta^{\beta\nu}\omega^{\lambda\mu}+\frac{1}{9}\theta^{\alpha\beta}\theta^{\gamma\nu}\omega^{\lambda\mu}$$
$$-\frac{2}{9}\theta^{\alpha\mu}\theta^{\beta\gamma}\omega^{\lambda\nu}+\frac{1}{9}\theta^{\alpha\gamma}\theta^{\beta\mu}\omega^{\lambda\nu}$$
$$+\frac{1}{9}\theta^{\alpha\beta}\theta^{\gamma\mu}\omega^{\lambda\nu}+\frac{4}{9}\theta^{\alpha\lambda}\theta^{\beta\gamma}\omega^{\mu\nu}$$
$$-\frac{2}{9}\theta^{\alpha\gamma}\theta^{\beta\lambda}\omega^{\mu\nu}-\frac{2}{9}\theta^{\alpha\beta}\theta^{\gamma\lambda}\omega^{\mu\nu} \tag{B.47}$$

$$(\mathcal{P}_1^{sx})^{\alpha\beta\gamma\lambda\mu\nu}=-\frac{2}{9}\theta^{\gamma\nu}\theta^{\lambda\mu}\omega^{\alpha\beta}-\frac{2}{9}\theta^{\gamma\mu}\theta^{\lambda\nu}\omega^{\alpha\beta}$$
$$+\frac{1}{9}\theta^{\gamma\lambda}\theta^{\mu\nu}\omega^{\alpha\beta}-\frac{2}{9}\theta^{\beta\nu}\theta^{\lambda\mu}\omega^{\alpha\gamma}$$
$$-\frac{2}{9}\theta^{\beta\mu}\theta^{\lambda\nu}\omega^{\alpha\gamma}+\frac{1}{9}\theta^{\beta\lambda}\theta^{\mu\nu}\omega^{\alpha\gamma}$$
$$+\frac{1}{9}\theta^{\alpha\nu}\theta^{\lambda\mu}\omega^{\beta\gamma}+\frac{1}{9}\theta^{\alpha\mu}\theta^{\lambda\nu}\omega^{\beta\gamma}$$
$$+\frac{4}{9}\theta^{\alpha\lambda}\theta^{\mu\nu}\omega^{\beta\gamma}+\frac{1}{9}\theta^{\alpha\nu}\theta^{\beta\gamma}\omega^{\lambda\mu}$$
$$-\frac{2}{9}\theta^{\alpha\gamma}\theta^{\beta\nu}\omega^{\lambda\mu}-\frac{2}{9}\theta^{\alpha\beta}\theta^{\gamma\nu}\omega^{\lambda\mu}$$
$$+\frac{1}{9}\theta^{\alpha\mu}\theta^{\beta\gamma}\omega^{\lambda\nu}-\frac{2}{9}\theta^{\alpha\gamma}\theta^{\beta\mu}\omega^{ln}$$
$$-\frac{2}{9}\theta^{\alpha\beta}\theta^{\gamma\mu}\omega^{\lambda\nu}+\frac{4}{9}\theta^{\alpha\lambda}\theta^{\beta\gamma}\omega^{\mu\nu}$$
$$+\frac{1}{9}\theta^{\alpha\gamma}\theta^{\beta\lambda}\omega^{\mu\nu}+\frac{1}{9}\theta^{\alpha\beta}\theta^{\gamma\lambda}\omega^{\mu\nu} \tag{B.48}$$

$$(\mathcal{P}_1^{ss})^{\alpha\beta\gamma\lambda\mu\nu}=\frac{1}{18}\theta^{\alpha\nu}\theta^{\beta\gamma}\theta^{\lambda\mu}$$
$$+\frac{1}{72}\theta^{\alpha\gamma}\theta^{\beta\nu}\theta^{\lambda\mu}$$
$$+\frac{1}{72}\theta^{\alpha\beta}\theta^{\gamma\nu}\theta^{\lambda\mu}+\frac{1}{18}\theta^{\alpha\mu}\theta^{\beta\gamma}\theta^{\lambda\nu}$$
$$+\frac{1}{72}\theta^{\alpha\gamma}\theta^{\beta\mu}\theta^{\lambda\nu}+\frac{1}{72}\theta^{\alpha\beta}\theta^{\gamma\mu}\theta^{\lambda\nu}$$
$$+\frac{2}{9}\theta^{\alpha\lambda}\theta^{\beta\gamma}\theta^{\mu\nu}+\frac{1}{18}\theta^{\alpha\gamma}\theta^{\beta\lambda}\theta^{\mu\nu}$$
$$+\frac{1}{18}\theta^{\alpha\beta}\theta^{\gamma\lambda}\theta^{\mu\nu} \tag{B.49}$$

$$(\mathcal{P}_1^{wst})^{\alpha\beta\gamma\lambda\mu\nu}=-\frac{1}{18}\theta^{\gamma\nu}\theta^{\lambda\mu}\omega^{\alpha\beta}$$
$$-\frac{1}{18}\theta^{\gamma\mu}\theta^{\lambda\nu}\omega^{\alpha\beta}-\frac{2}{9}\theta^{\gamma\lambda}\theta^{\mu\nu}\omega^{\alpha\beta}$$
$$-\frac{1}{18}\theta^{\beta\nu}\theta^{\lambda\mu}\omega^{\alpha\gamma}$$
$$-\frac{1}{18}\theta^{\beta\mu}\theta^{\lambda\nu}\omega^{\alpha\gamma}-\frac{2}{9}\theta^{\beta\lambda}\theta^{\mu\nu}\omega^{\alpha\gamma}$$
$$+\frac{5}{18}\theta^{\alpha\nu}\theta^{\lambda\mu}\omega^{\beta\gamma}+\frac{5}{18}\theta^{\alpha\mu}\theta^{\lambda\nu}\omega^{\beta\gamma}$$
$$+\frac{1}{9}\theta^{\alpha\lambda}\theta^{\mu\nu}\omega^{\beta\gamma}-\frac{2}{9}\theta^{\alpha\nu}\theta^{\beta\gamma}\omega^{\lambda\mu}$$
$$-\frac{1}{18}\theta^{\alpha\gamma}\theta^{\beta\nu}\omega^{\lambda\mu}-\frac{1}{18}\theta^{\alpha\beta}\theta^{\gamma\nu}\omega^{\lambda\mu}$$
$$-\frac{2}{9}\theta^{\alpha\mu}\theta^{\beta\gamma}\omega^{\lambda\nu}-\frac{1}{18}\theta^{\alpha\gamma}\theta^{\beta\mu}\omega^{\lambda\nu}$$
$$-\frac{1}{18}\theta^{\alpha\beta}\theta^{\gamma\mu}\omega^{\lambda\nu}+\frac{1}{9}\theta^{\alpha\lambda}\theta^{\beta\gamma}\omega^{\mu\nu}$$
$$+\frac{5}{18}\theta^{\alpha\gamma}\theta^{\beta\lambda}\omega^{\mu\nu}+\frac{5}{18}\theta^{\alpha\beta}\theta^{\gamma\lambda}\omega^{\mu\nu} \tag{B.50}$$

- Finally, there is one more with spin 2

$$(\mathcal{P}_2^{x})^{\alpha\beta\gamma\lambda\mu\nu}=\frac{1}{4}\theta^{\alpha\nu}\theta^{\gamma\lambda}\omega^{\beta\mu}+\frac{1}{4}\theta^{\alpha\lambda}\theta^{\gamma\nu}\omega^{\beta\mu}$$
$$-\frac{1}{6}\theta^{\alpha\gamma}\theta^{\lambda\nu}\omega^{\beta\mu}+\frac{1}{4}\theta^{\alpha\mu}\theta^{\gamma\lambda}\omega^{\beta\nu}$$
$$+\frac{1}{4}\theta^{\alpha\lambda}\theta^{\gamma\mu}\omega^{\beta\nu}-\frac{1}{6}\theta^{\alpha\gamma}\theta^{\lambda\mu}\omega^{\beta\nu}$$
$$+\frac{1}{4}\theta^{\alpha\nu}\theta^{\beta\lambda}\omega^{\gamma\mu}+\frac{1}{4}\theta^{\alpha\lambda}\theta^{\beta\nu}\omega^{\gamma\mu}$$
$$-\frac{1}{6}\theta^{\alpha\beta}\theta^{\lambda\nu}\omega^{\gamma\mu}+\frac{1}{4}\theta^{\alpha\mu}\theta^{\beta\lambda}\omega^{\gamma\nu}$$
$$+\frac{1}{4}\theta^{\alpha\lambda}\theta^{\beta\mu}\omega^{\gamma\nu}-\frac{1}{6}\theta^{\alpha\beta}\theta^{\lambda\mu}\omega^{\gamma\nu} \tag{B.51}$$

## Appendix C: Spin content of the antisymmetric connection field

In this appendix, we decompose the operators mediating between two connection fields $A_{\mu\beta\gamma}\equiv g_{\alpha\mu}\Gamma^{\alpha}_{\beta\gamma}$ – antisymmetric in the las two indices because we consider torsionful connections which fulfill the metricity condition– in terms of the spin projectors of this field. The procedure is analogue to the one followed in "Appendices A and B".

The subspace $\mathcal{A}$ corresponds, in terms of representations of the tangent group SO(4), to the sum of a totally antisymmetric three-index tensor plus a tensor with the *hook* symmetry

$$\{0,2\}\otimes\{1\}=\{0,3\}\oplus\{2,1\} \tag{C.1}$$

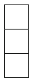
In terms of dimensions this is $\underline{24}=\underline{4}+\underline{20}$

### C.1 The totally antisymmetric tensor

We want to determine the spin content of the totally antisymmetric piece $A_{[\alpha\beta\gamma]}$, in this case there are only two monomials we can form

$$M_{23} = \delta^{[a}_{[\lambda}\delta^{\beta}_{\mu}\delta^{\gamma]}_{\nu]}$$
$$M_{24} = \delta^{[a}_{[\lambda}\delta^{\beta}_{\mu}k^{\gamma]}k_{\nu]} \tag{C.2}$$

The totally antisymmetric piece is represented as

$\{0, 3\}$  (C.3)

and the corresponding Young projectors are

$$\left(\bar{P}_{\begin{smallmatrix}\alpha\\ \beta\\ \gamma\end{smallmatrix}}\right)^{\alpha\beta\gamma}_{\mu\nu\lambda} \equiv \frac{1}{6}\Big\{\delta^{\alpha}_{\mu}\delta^{\beta}_{\nu}\delta^{\gamma}_{\lambda} + \delta^{\beta}_{\mu}\delta^{\gamma}_{\nu}\delta^{\alpha}_{\lambda} + \delta^{\gamma}_{\mu}\delta^{\alpha}_{\nu}\delta^{\beta}_{\lambda} - \delta^{\alpha}_{\mu}\delta^{\gamma}_{\nu}\delta^{\beta}_{\lambda} - \delta^{\beta}_{\mu}\delta^{\alpha}_{\nu}\delta^{\gamma}_{\lambda} - \delta^{\gamma}_{\mu}\delta^{\beta}_{\nu}\delta^{\alpha}_{\lambda}\Big\} = \frac{1}{6}\,(1, 1, 1, -1, -1, -1) \tag{C.4}$$

where the notation of the projectors in the same as in "Appendix B".

We can decompose it in its spin componets as

- First the spin 1 component

$$\frac{1}{2}\left(A_{j0i} - A_{i0j}\right) \tag{C.5}$$

with projector

$$(\bar{P}_1)^{\alpha\beta\gamma\lambda\mu\nu} = -\frac{1}{6}\theta^{\alpha\nu}\theta^{\beta\mu}\theta^{\gamma\lambda} + \frac{1}{6}\theta^{\alpha\mu}\theta^{\beta\nu}\theta^{\gamma\lambda} + \frac{1}{6}\theta^{\alpha\nu}\theta^{\beta\lambda}\theta^{\gamma\mu} - \frac{1}{6}\theta^{\alpha\lambda}\theta^{\beta\nu}\theta^{\gamma\mu} - \frac{1}{6}\theta^{\alpha\mu}\theta^{\beta\lambda}\theta^{\gamma\nu} + \frac{1}{6}\theta^{\alpha\lambda}\theta^{\beta\mu}\theta^{\gamma\nu} \tag{C.6}$$

- The spin 0 component is

$$A_{[ijk]} \tag{C.7}$$

with projector

$$(\bar{P}_0)^{\alpha\beta\gamma\lambda\mu\nu} = -\frac{1}{6}\omega^{\alpha\lambda}\theta^{\beta\nu}\theta^{\gamma\mu} + \frac{1}{6}\omega^{\alpha\lambda}\theta^{\beta\mu}\theta^{\gamma\nu} + \frac{1}{6}\omega^{\alpha\mu}\theta^{\beta\nu}\theta^{\gamma\lambda} - \frac{1}{6}\omega^{\alpha\mu}\theta^{\beta\lambda}\theta^{\gamma\nu} - \frac{1}{6}\omega^{\alpha\nu}\theta^{\beta\mu}\theta^{\gamma\lambda} + \frac{1}{6}\omega^{\alpha\nu}\theta(b,l)\theta^{\gamma\mu} + \frac{1}{6}\theta^{\alpha\nu}\omega^{\beta\lambda}\theta^{\gamma\mu} - \frac{1}{6}\theta^{\alpha\mu}\omega^{\beta\lambda}\theta^{\gamma\nu} - \frac{1}{6}\theta^{\alpha\nu}\omega^{\beta\mu}\theta^{\gamma\lambda} + \frac{1}{6}\theta^{\alpha\lambda}\omega^{\beta\mu}\theta^{\gamma\nu} + \frac{1}{6}\theta^{\alpha\mu}\omega^{\beta\nu}\theta^{\gamma\lambda} - \frac{1}{6}\theta^{\alpha\lambda}\omega^{\beta\nu}\theta^{\gamma\mu} - \frac{1}{6}\theta^{\alpha\nu}\theta^{\beta\mu}\omega^{\gamma\lambda} + \frac{1}{6}\theta^{\alpha\mu}\theta^{\beta\nu}\omega^{\gamma\lambda} + \frac{1}{6}\theta^{\alpha\nu}\theta^{\beta\lambda}\omega^{\gamma\mu} - \frac{1}{6}\theta^{\alpha\lambda}\theta^{\beta\nu}\omega^{\gamma\mu} - \frac{1}{6}\theta^{\alpha\mu}\theta^{\beta\lambda}\omega^{\gamma\nu} + \frac{1}{6}\theta^{\alpha\lambda}\theta^{\beta\mu}\omega^{\gamma\nu} \tag{C.8}$$

Finally it is easy to check that

$$\left(\bar{P}\right)^{\alpha\beta\gamma}_{\mu\nu\lambda} = (\bar{P}_1)^{\alpha\beta\gamma}_{\mu\nu\lambda} + (\bar{P}_0)^{\alpha\beta\gamma}_{\mu\nu\lambda} \tag{C.9}$$

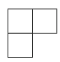
In terms of dimensions this is $\underline{4} = (\underline{1}) \oplus (\underline{0})$.

C.2 The antisymmetric hook sector

We determine the spin content of the antisymmetric hook piece $A_{\alpha[\beta\gamma]}$, in this case there are six monomials

$$M_{25} = \delta^{\alpha}_{\lambda}\delta^{[\beta}_{[\mu}\delta^{\gamma]}_{\nu]}$$
$$M_{26} = k^{\alpha}k_{\lambda}\,\delta^{[\beta}_{[\mu}\delta^{\gamma]}_{\nu]}$$
$$M_{27} = k^{\alpha}\delta^{[\beta}_{\lambda}\,k_{[\mu}\delta^{\gamma]}_{\nu]}$$
$$M_{28} = \delta^{\alpha}_{[\mu}k^{[\beta}k_{\nu]}\delta^{\gamma]}_{\lambda}$$
$$M_{29} = \delta^{\alpha}_{\lambda}\,k^{[\beta}k_{[\mu}\delta^{\gamma]}_{\nu]}$$
$$M_{30} = k^{\alpha}k_{\lambda}\,k^{[\beta}k_{[\mu}\delta^{\gamma]}_{\nu]} \tag{C.10}$$

The antisymmetric hook part corresponds to the piece

$\{2, 1\}$  (C.11)

The Young projectors reads

$$\bar{\mathcal{P}}_H \equiv \left(P_{\begin{smallmatrix}\alpha & \beta\\ \gamma & \end{smallmatrix}}\right)^{\alpha\beta\gamma}_{\mu\nu\lambda} \equiv \frac{1}{3}\Big\{\delta^{\alpha}_{\mu}\delta^{\beta}_{\nu}\delta^{\gamma}_{\lambda} - \delta^{\alpha}_{\mu}\delta^{\gamma}_{\nu}\delta^{\beta}_{\lambda} + \frac{1}{2}\delta^{\alpha}_{\nu}\delta^{\beta}_{\mu}\delta^{\gamma}_{\lambda} - \frac{1}{2}\delta^{\alpha}_{\nu}\delta^{\beta}_{\lambda}\delta^{\gamma}_{\mu} + \frac{1}{2}\delta^{\alpha}_{\lambda}\delta^{\beta}_{\nu}\delta^{\gamma}_{\mu} - \frac{1}{2}\delta^{\alpha}_{\lambda}\delta^{\beta}_{\mu}\delta^{\gamma}_{\nu}\Big\} = \frac{1}{3}\left(1, \frac{1}{2}, -\frac{1}{2}, -1, \frac{1}{2}, -\frac{1}{2}\right) \tag{C.12}$$

We can decompose it in its spin componets as

- There are two spin 2 component. The first one is the transverse traceless spin two component

$$\frac{1}{2}\left(A_{j0i} + A_{i0j}\right) - \frac{1}{3}\delta_{ij}\sum_{k=1}^{3}A_{k0k} \tag{C.13}$$

with projector

$$(\bar{\mathcal{P}}_2)^{\alpha\beta\gamma\lambda\mu\nu} = \frac{1}{4}\theta^{\alpha\nu}\omega^{\beta\mu}\theta^{\gamma\lambda} + \frac{1}{4}\theta^{\alpha\lambda}\omega^{\beta\mu}\theta^{\gamma\nu} - \frac{1}{6}\theta^{\alpha\gamma}\omega^{\beta\mu}\theta^{\nu\lambda} - \frac{1}{4}\theta^{\alpha\mu}\omega^{\beta\nu}\theta^{\gamma\lambda} - \frac{1}{4}\theta^{\alpha\lambda}\omega^{\beta\nu}\theta^{\gamma\mu} + \frac{1}{6}\theta^{\alpha\gamma}\omega^{\beta\nu}\theta^{\mu\lambda} - \frac{1}{4}\theta^{\alpha\nu}\theta^{\beta\lambda}\omega^{\gamma\mu} - \frac{1}{4}\theta^{\alpha\lambda}\theta^{\beta\nu}\omega^{\gamma\mu} + \frac{1}{6}\theta^{\alpha\beta}\omega^{\gamma\mu}\theta^{\nu\lambda} + \frac{1}{4}\theta^{\alpha\mu}\theta^{\beta\lambda}\omega^{\gamma\nu} + \frac{1}{4}\theta^{\alpha\lambda}\theta^{\beta\mu}\omega^{\gamma\nu} - \frac{1}{6}\theta^{\alpha\beta}\omega^{\gamma\nu}\theta^{\mu\lambda} \tag{C.14}$$

The second one corresponds to the spin 2 traceless connection field

$$A^T_{ijk} \equiv A_{ijk} - \frac{1}{2}t_j\delta_{ik} + \frac{1}{2}t_k\delta_{ij} \tag{C.15}$$

where $t_i = \sum_{j=1}^3 A_{jij}$, with projector

$$(\bar{\mathcal{P}}^s_2)^{\alpha\beta\gamma\lambda\mu\nu} = \frac{1}{6}\theta^{\alpha\nu}\theta^{\beta\mu}\theta^{\gamma\lambda} - \frac{1}{6}\theta^{\alpha\mu}\theta^{\beta\nu}\theta^{\gamma\lambda} - \frac{1}{6}\theta^{\alpha\nu}\theta^{\beta\lambda}\theta^{\gamma\mu} - \frac{1}{3}\theta^{\alpha\lambda}\theta^{\beta\nu}\theta^{\gamma\mu} + \frac{1}{6}\theta^{\alpha\mu}\theta^{\beta\lambda}\theta^{\gamma\nu} + \frac{1}{3}\theta^{\alpha\lambda}\theta^{\beta\mu}\theta^{\gamma\nu} + \frac{1}{4}\theta^{\alpha\gamma}\theta^{\beta\nu}\theta^{\lambda\mu} - \frac{1}{4}\theta^{\alpha\beta}\theta^{\gamma\nu}\theta^{\lambda\mu} - \frac{1}{4}\theta^{\alpha\gamma}\theta\beta\mu\theta^{\lambda\nu} + \frac{1}{4}\theta^{\alpha\beta}\theta^{\gamma\mu}\theta^{\lambda\nu} \tag{C.16}$$

- There are three spin 1 components. First

$$\frac{1}{2}\left(A_{j0i} - A_{i0j}\right) \tag{C.17}$$

with projector

$$(\bar{\mathcal{P}}^s_1)^{\alpha\beta\gamma\lambda\mu\nu} = -\frac{1}{3}\omega^{\alpha\lambda}\theta^{\beta\nu}\theta^{\gamma\mu} + \frac{1}{3}\omega^{\alpha\lambda}\theta^{\beta\mu}\theta^{\gamma\nu} - \frac{1}{6}\omega^{\alpha\mu}\theta^{\beta\nu}\theta^{\gamma\lambda} + \frac{1}{6}\omega^{\alpha\mu}\theta^{\beta\lambda}\theta^{\gamma\nu} + \frac{1}{6}\omega^{\alpha\nu}\theta^{\beta\mu}\theta^{\gamma\lambda} - \frac{1}{6}\omega^{\alpha\nu}\theta^{\beta\lambda}\theta^{\gamma\mu} - \frac{1}{6}\theta^{\alpha\nu}\omega^{\beta\lambda}\theta^{\gamma\mu} + \frac{1}{6}\theta^{\alpha\mu}\omega^{\beta\lambda}\theta^{\gamma\nu} - \frac{1}{12}\theta^{\alpha\nu}\omega^{\beta\mu}\theta^{\gamma\lambda} + \frac{1}{12}\theta^{\alpha\lambda}\omega^{\beta\mu}\theta^{\gamma\nu} + \frac{1}{12}\theta^{\alpha\mu}\omega^{\beta\nu}\theta^{\gamma\lambda} - \frac{1}{12}\theta^{\alpha\lambda}\omega^{\beta\nu}\theta^{\gamma\mu} + \frac{1}{6}\theta^{\alpha\nu}\theta^{\beta\mu}\omega^{\gamma\lambda} - \frac{1}{6}\theta^{\alpha\mu}\theta^{\beta\nu}\omega^{\gamma\lambda} + \frac{1}{12}\theta^{\alpha\nu}\theta^{\beta\lambda}\omega^{\gamma\mu} - \frac{1}{12}\theta^{\alpha\lambda}\theta^{\beta\nu}\omega^{\gamma\mu} - \frac{1}{12}\theta^{\alpha\mu}\theta^{\beta\lambda}\omega^{\gamma\nu} + \frac{1}{12}\theta^{\alpha\lambda}\theta^{\beta\mu}\omega^{\gamma\nu} \tag{C.18}$$

The second one is given by

$$A_{0i0} \tag{C.19}$$

corresponding to

$$(\bar{\mathcal{P}}^w_1)^{\alpha\beta\gamma\lambda\mu\nu} = \frac{1}{2}\omega^{\alpha\beta}\theta^{\gamma\nu}\omega^{\lambda\mu} - \frac{1}{2}w^{\alpha\gamma}\theta^{\beta\nu}\omega^{\lambda\mu} - \frac{1}{2}\omega^{\alpha\beta}\theta^{\gamma\mu}\omega^{\lambda\nu} + \frac{1}{2}\omega^{\alpha\gamma}\theta^{\beta\mu}\omega^{\lambda\nu} \tag{C.20}$$

And there is also a spin 1 trace

$$\sum_{j=1}^{3} A_{jij} \tag{C.21}$$

given by

$$(\bar{\mathcal{P}}^t_1)^{\alpha\beta\gamma\lambda\mu\nu} = -\frac{1}{4}\theta^{\alpha\gamma}\theta^{\beta\nu}\theta^{\lambda\mu} + \frac{1}{4}\theta^{\alpha\beta}\theta^{\gamma\nu}\theta^{\lambda\mu} + \frac{1}{4}\theta^{\alpha\gamma}\theta^{\beta\mu}\theta^{\lambda\nu} - \frac{1}{4}\theta^{\alpha\beta}\theta^{\gamma\mu}\theta^{\lambda\nu} \tag{C.22}$$

- There is only one spin zero, a trace that is given by

$$\sum_{i=1}^{3} A_{i0i} \tag{C.23}$$

that is

$$(\bar{\mathcal{P}}_0)^{\alpha\beta\gamma\lambda\mu\nu} = \frac{1}{6}\theta^{\alpha\gamma}\omega^{\beta\mu}\theta^{\lambda\nu} - \frac{1}{6}\theta^{\alpha\gamma}\omega^{\beta\nu}\theta^{\lambda\mu} - \frac{1}{6}\theta^{\alpha\beta}\omega^{\gamma\mu}\theta^{\lambda\nu} + \frac{1}{6}\theta^{\alpha\beta}\omega^{\gamma\nu}\theta^{\lambda\mu} \tag{C.24}$$

Finally it is easy to check that

$$\left(\bar{\mathcal{P}}_H\right)^{\alpha\beta\gamma}_{\mu\nu\lambda} = (\bar{\mathcal{P}}_2)^{\alpha\beta\gamma}_{\mu\nu\lambda} + (\bar{\mathcal{P}}^s_2)^{\alpha\beta\gamma}_{\mu\nu\lambda} + (\bar{\mathcal{P}}^s_1)^{\alpha\beta\gamma}_{\mu\nu\lambda} + (\bar{\mathcal{P}}^w_1)^{\alpha\beta\gamma}_{\mu\nu\lambda} + (\bar{\mathcal{P}}^t_1)^{\alpha\beta\gamma}_{\mu\nu\lambda} + (\bar{\mathcal{P}}_0)^{\alpha\beta\gamma}_{\mu\nu\lambda} \tag{C.25}$$

In terms of dimensions this is $\underline{20} = 2(\underline{2}) \oplus 3(\underline{1}) \oplus (\underline{0})$.

These projectors agree with the ones obtained by Sezgin and van Nieuwenhuizen [29].

## Appendix D: Zero modes for $R^2$

In Sect. 5 we had determined the quadratic one loop operator in the particular case where the lagrangian is proportional to $R^2$, the square of the scalar curvature.

$$\begin{aligned}(K_{R^2+\text{gf}})^{\mu\nu}_{\tau}{}^{\rho\sigma}_{\lambda} = \frac{1}{\chi}\Big(&\text{P}_0^w + 3\,\text{P}_0^s + (3-9\chi)\,\mathcal{P}_0^s - 3\,\mathcal{P}_0^x \\ &+ \mathcal{P}_0^{sw} + \mathcal{P}_0^{ws} + \text{P}_1^w - \frac{5}{3}\,\text{P}_1^s \\ &+ \mathcal{P}_1^w + \frac{2}{3}\,\mathcal{P}_1^t - \mathcal{P}_1^{wx} + \mathcal{P}_1^{ws} \\ &+ \mathcal{P}_1^{sw} + \mathcal{P}_1^{sx} + 4\,\mathcal{P}_1^{ss}\Big)^{\mu\nu}_{\tau}{}^{\rho\sigma}_{\lambda}\,\Box \end{aligned} \tag{D.1}$$

It can be checked that this operator has 13 independent zero modes, which are written in terms of the spin operators acting on an arbitrary field $\Omega_{\alpha\beta\gamma} \in \mathcal{A}$ as

$$\begin{aligned}
Z_1 &\equiv \left(\text{P}_0^w + \text{P}_0^s - \mathcal{P}_0^{ws}\right)_{\lambda\mu\nu}^{\alpha\beta\gamma}\ \Omega_{\alpha\beta\gamma}\\
Z_2 &\equiv \left(-\text{P}_1^w + \text{P}_1^s + 3\mathcal{P}_1^w - \tfrac{3}{8}\mathcal{P}_1^{sw} - \tfrac{3}{2}\mathcal{P}_1^{wst}\right)_{\lambda\mu\nu}^{\alpha\beta\gamma}\ \Omega_{\alpha\beta\gamma}\\
Z_3 &\equiv \left(2\mathcal{P}_1^w + \mathcal{P}_1^t - \tfrac{3}{2}\mathcal{P}_1^{sw}\right)_{\lambda\mu\nu}^{\alpha\beta\gamma}\ \Omega_{\alpha\beta\gamma}\\
Z_4 &\equiv \left(-2\text{P}_1^w + \mathcal{P}_1^w + \mathcal{P}_1^{ws} - \tfrac{1}{8}\mathcal{P}_1^{sw} - \tfrac{1}{2}\mathcal{P}_1^{wst}\right)_{\lambda\mu\nu}^{\alpha\beta\gamma}\ \Omega_{\alpha\beta\gamma}\\
Z_5 &\equiv \left(-2\text{P}_1^w + \mathcal{P}_1^w - \tfrac{3}{4}\mathcal{P}_1^{sw} + \mathcal{P}_1^{sx} - \mathcal{P}_1^{wst}\right)_{\lambda\mu\nu}^{\alpha\beta\gamma}\ \Omega_{\alpha\beta\gamma}\\
Z_6 &\equiv \left(-\tfrac{7}{6}\text{P}_1^w + \tfrac{14}{3}\mathcal{P}_1^w - \tfrac{21}{16}\mathcal{P}_1^{ws} + \mathcal{P}_1^{ss} - \tfrac{7}{4}\mathcal{P}_1^{wst}\right)_{\lambda\mu\nu}^{\alpha\beta\gamma}\ \Omega_{\alpha\beta\gamma}\\
Z_7 &\equiv (\mathcal{P}_1^s)_{\lambda\mu\nu}^{\alpha\beta\gamma}\ \Omega_{\alpha\beta\gamma}\\
Z_8 &\equiv (\mathcal{P}_1^{wx})_{\lambda\mu\nu}^{\alpha\beta\gamma}\ \Omega_{\alpha\beta\gamma}\\
Z_9 &\equiv (\text{P}_2)_{\lambda\mu\nu}^{\alpha\beta\gamma}\ \Omega_{\alpha\beta\gamma}\\
Z_{10} &\equiv (\mathcal{P}_2)_{\lambda\mu\nu}^{\alpha\beta\gamma}\ \Omega_{\alpha\beta\gamma}\\
Z_{11} &\equiv (\mathcal{P}_2^s)_{\lambda\mu\nu}^{\alpha\beta\gamma}\ \Omega_{\alpha\beta\gamma}\\
Z_{12} &\equiv (\mathcal{P}_2^x)_{\lambda\mu\nu}^{\alpha\beta\gamma}\ \Omega_{\alpha\beta\gamma}\\
Z_{13} &\equiv (\text{P}_3)_{\lambda\mu\nu}^{\alpha\beta\gamma}\ \Omega_{\alpha\beta\gamma}
\end{aligned} \tag{D.2}$$

It is quite remarkable that the system has extra gauge symmetries at one loop order that are not present in the exact lagrangian. The physical meaning of this is discussed in the main body of the paper.

## Appendix E: Fun with $S_3$

Let us highlight the procedure to get the spin projectors in a systematic way. Denoting the elements of permutation group of three elements $S_3$ acting on $T_{\alpha\beta\gamma} \in T \times T \times T$ as

$$\begin{aligned}
g_1 &\equiv \delta_\mu^\alpha \delta_\nu^\beta \delta_\lambda^\gamma\\
g_2 &\equiv \delta_\mu^\beta \delta_\nu^\gamma \delta_\lambda^\alpha\\
g_3 &\equiv \delta_\mu^\gamma \delta_\nu^\alpha \delta_\lambda^\beta\\
g_4 &\equiv \delta_\mu^\alpha \delta_\nu^\gamma \delta_\lambda^\beta\\
g_5 &\equiv \delta_\mu^\beta \delta_\nu^\alpha \delta_\lambda^\gamma\\
g_6 &\equiv \delta_\mu^\gamma \delta_\nu^\beta \delta_\lambda^\alpha
\end{aligned} \tag{E.1}$$

The most general projector in this space can be written as

$$P \equiv \sum_{i=1}^{i=6} C_i\, g_i \equiv \begin{pmatrix} U \\ V \end{pmatrix} \tag{E.2}$$

where we have defined the column vectors

$$U \equiv \begin{pmatrix} C_1 \\ C_2 \\ C_3 \end{pmatrix}\ V \equiv \begin{pmatrix} C_4 \\ C_5 \\ C_6 \end{pmatrix} \tag{E.3}$$

Those operators are not symmetric ones; rather the transpose operator is given by

$$(C_1, C_2, C_3, C_4, C_5, C_6)^T = (C_1, C_3, C_2, C_4, C_5, C_6) \tag{E.4}$$

It is important to keep this in mind when multiplying projectors.

On the other hand, it is not difficult to establish that

$$\begin{aligned}(P'')^{\mathbf{a}}_{\boldsymbol{\mu}} \equiv \sum_{\mathbf{c}} P^{\mathbf{a}}_{\mathbf{c}} \cdot (P')^{\mathbf{c}}_{\boldsymbol{\mu}} = M \begin{pmatrix} U' \\ V' \end{pmatrix} \equiv \begin{pmatrix} U'' \\ V'' \end{pmatrix}\\
= \begin{pmatrix} AU' + BV' \\ BU' + AV' \end{pmatrix}\end{aligned} \tag{E.5}$$

with

$$\begin{aligned}M \equiv \begin{pmatrix} A & B \\ B & A \end{pmatrix} A \equiv \begin{pmatrix} C_1 & C_3 & C_2 \\ C_2 & C_1 & C_3 \\ C_3 & C_2 & C_1 \end{pmatrix}\\
B \equiv \begin{pmatrix} C_4 & C_5 & C_6 \\ C_5 & C_6 & C_4 \\ C_6 & C_4 & C_5 \end{pmatrix}\end{aligned} \tag{E.6}$$

All this implies that

$$\left[P, P'\right] = \begin{pmatrix} 0 \\ C_{54} + C_{65} + C_{46} \\ C_{64} + C_{45} + C_{56} \\ C_{52} + C_{63} + C_{35} + C_{28} \\ C_{52} + C_{63} + C_{35} + C_{26} \\ C_{62} + C_{43} + C_{24} + C_{36} \\ C_{42} + C_{53} + C_{34} + C_{25} \end{pmatrix} \tag{E.7}$$

where

$$C_{ab} \equiv C_a C_b' - C_b C_a' \tag{E.8}$$

These formulas make it trivial to check all assertions about projectors, which have been nevertheless also verified with xAct [28].

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# 2

# CONFORMAL INVARIANCE VERSUS WEYL INVARIANCE

This chapter contains the article

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# Conformal invariance versus Weyl invariance

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**Abstract** The most general Lagrangian describing spin 2 particles in flat spacetime and containing operators up to (mass) dimension 6 is carefully analyzed, determining the precise conditions for it to be invariant under linearized (transverse) diffeomorphisms, linearized Weyl rescalings, and conformal transformations.

**Contents**



## 1 Introduction

Particle physics interactions, when considered at very high energy (probing then smaller and smaller distances), are expected to be independent of the individual masses of the particles themselves, which are negligible in comparison with the energy scale. Some sort of scale invariance is expected to be at work there. The same thing happens in second order phase transitions. The correlation length diverges, and again, scale invariance is at work. In fact in many known cases (in all unitary theories[1]) this symmetry is upgraded to full conformal symmetry [2]. There is however a caveat. In quantum field theory the well-known need to renormalize the bare quantities implies that an arbitary mass must be introduced. This is the origin of the dependence of coupling constants with the energy scale, encoded in the correspondiong beta-functions.

It is however only recently that the precise relationship between scale invariance, conformal invariance and Weyl invariance has been clarified (cf. [3,4] and references therein). This includes the precise conditions for scale invariant theories to become conformal invariant and also the existence of the so-called a-theorem for renormalizable theories (cf. [5] and [6] for a recent review). Most of the work done so far has been in flat spacetime, where the gravitational field is absent, or at most, non-dynamic.

When such a gravitational field is present [7] there are two possible generalizations of scale invariance. The most direct of those is the algebra of *conformal Killing vector fields* (CKV), that is, those that obey

$$\mathcal{L}(\xi)g_{\mu\nu} = \phi(x)g_{\mu\nu} \tag{1}$$

the fact that

$$\mathcal{L}([\xi,\eta]) = \mathcal{L}(\xi)\mathcal{L}(\eta) - \mathcal{L}(\eta)\mathcal{L}(\xi) \tag{2}$$

implies that the set of all CKV generate an algebra, which for Minkowski spacetime is the conformal algebra, $SO(2,n)$. In fact, the maximal possible dimension of the conformal algebra is precisely

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[1] There is a counterexample by Riva and Cardy [1] where scale invariance does not imply conformal invariance. The theory is the two-dimensional theory of elasticity which is not unitary.

$$d = \frac{(n+1)(n+2)}{2} \tag{3}$$

which is attained by conformally flat spacetimes (the ones with vanishing Weyl tensor [8,9]). Unfortunately, however, this property is not generic; that is, an arbitrary metric does not support any CKV, and the corresponding algebra has to be studied for each particular spacetime by itself.

The next most natural symmetry to study is Weyl invariance, the invariance of the action under local rescalings of the metric tensor.

$$g_{\mu\nu}(x) \to \Omega^2(x)\, g_{\mu\nu}(x) \tag{4}$$

this invariance, besides, can still be studied in the linear limit, when the gravitational field manifests itself as a perturbation of the Minkowski metric.

$$g_{\mu\nu}(x) = \eta_{\mu\nu} + \kappa h_{\mu\nu}(x) \tag{5}$$

Given that the previous expansion is exact, the linearized Weyl symmetry of the metric perturbation can be written as

$$\kappa\, \delta h_{\mu\nu} = 2\omega(x)\eta_{\mu\nu} + 2\kappa\, \omega\, \mathcal{L}_\xi h_{\mu\nu} \tag{6}$$

Let us remark the appearance of an order $h$ piece in the variation of the perturbation that will be relevant in our analysis.

In order for a flat spacetime theory to be scale invariant, the (Rosenfeld) energy-momentum tensor must be a total derivative (on-shell)

$$\eta^{\mu\nu} T_{\mu\nu} = \partial_\mu V^\mu \tag{7}$$

where $V^\mu$ is the *virial current* [10–12]. This is enough to guarantee the existence of a conserved scale current

$$j^\mu \equiv x^\lambda T^\mu_\lambda - V^\mu \tag{8}$$

In the particular case when the virial current is itself a divergence, that is, when

$$V^\mu = \partial_\nu \sigma^{\mu\nu} \tag{9}$$

then the theory is conformally invariant under the group $O(2, n)$, and the conserved current reads

$$K^{\mu\nu} \equiv \left(2x^\nu x_\rho - x^2 \delta^\nu_\rho\right) T^{\rho\mu} - 2x^\nu V^\mu + 2\sigma^{\mu\nu} \tag{10}$$

which also implies that the energy-momentum tensor can be improved.

In the present paper we want to clarify the precise relationship between Weyl invariant theories (WIFT) and conformal invariant theories (CFT) for systems where the gravitational field is dynamic, but still approachable as a fluctuation of flat spacetime. Our analysis then concerns mostly spin 2 theories in flat spacetime as described by a rank two symmetric field in Minkowski space. Our plan is to do it systematically, determining the conditions for scale invariance (which is still meaningful in flat space), conformal invariance and Weyl invariance.

We analyze first the most general Lagrangian containing dimension 4 operators, and then we do the same analysis for dimension 6 operators, containing two and four derivatives respectively (operators appearing in the weak field expansion of gravitational theories linear and quadratic in the curvature). We then study dimension 5 and dimension 6 operators with two derivatives. The analysis is, in some sense, the continuation of the one in [13,14] and also in [15]. Recent works regarding conformal invariance and Weyl invariance include [16–18]. We are always (with the only exception of our discussion of the improvement of the energy-momentum tensor) refering to actions defined as spacetime integrals, so that we allow for integration by parts, in other words, when we claim that an expression vanishes, we mean this only *up to total derivatives*.

## 2 Symmetries of the low energy spin 2 action

Let us revisit the possible symmetries of low dimension kinetic operators in spin 2 theories [13] in flat spacetime where the graviton is represented by a symmetric tensor $h_{\mu\nu}$. Lorentz invariance will be assumed throughout the paper. If we want the field equations to be given by differential operators of (at least) second order, then the Lagrangian has got to incorporate at least two derivatives. Let us begin our study with the operators of lowest possible dimension.

### 2.1 Dimension 4 operators

Our building blocks are the gravitational field, $h_{\alpha\beta}$ (assumed to be of mass dimension 1) and the spacetime derivatives, $\partial_\mu$. There are four different dimension four operators with two derivatives

$$\begin{aligned}
\mathcal{D}_1 &\equiv \frac{1}{4}\partial_\mu h_{\nu\rho}\partial^\mu h^{\nu\rho}\\
\mathcal{D}_2 &\equiv -\frac{1}{2}\partial^\lambda h_{\lambda\rho}\partial_\sigma h^{\rho\sigma}\\
\mathcal{D}_3 &\equiv \frac{1}{2}\partial^\lambda h \partial^\sigma h_{\lambda\sigma}\\
\mathcal{D}_4 &\equiv -\frac{1}{4}\partial_\mu h \partial^\mu h
\end{aligned} \tag{11}$$

There is another operator

$$\mathcal{D}_5 \equiv -\frac{1}{2}\partial_\sigma h_{\lambda\rho}\partial^\lambda h^{\rho\sigma} \tag{12}$$

which is equivalent to $\mathcal{D}_2$ modulo total derivatives:

$$\mathcal{D}_5 = \mathcal{D}_2 - \frac{1}{2}\partial_\lambda\left(\partial_\sigma h^\lambda_\rho h^{\rho\sigma}\right) + \frac{1}{2}\partial_\sigma\left(\partial_\lambda h^{\lambda\rho} h^\sigma_\rho\right) \tag{13}$$

Then, the most general action principle involving dimension 4 derivative operators reads

$$S = \int d^4x \sum_{i=1}^{4} \alpha_i\, \mathcal{D}_i \tag{14}$$

First, we are going to consider invariance under linearized diffeomorphisms, *LDiff* gauge symmetry. This is the one implemented in the pioneering work by Fierz–Pauli

$$\delta h_{\mu\nu} = \partial_\mu \xi_\nu + \partial_\nu \xi_\mu \tag{15}$$

The variation of the fragment of the action containing the $\mathcal{D}$ terms only (that is, with $m_i^2 = \lambda_\alpha = 0$) and upon integration by parts, yields

$$\begin{aligned}
\delta \int d^4x\, \mathcal{D}_1 &= \int d^4x\, \xi^\lambda\, \Box\, \partial^\nu\, h_{\nu\lambda}\\
\delta \int d^4x\, \mathcal{D}_2 &= -\int d^4x\, \xi^\lambda\left(\partial_\lambda\, \partial^\rho \partial^\sigma h_{\rho\sigma} + \Box\, \partial^\sigma h_{\lambda\sigma}\right)\\
\delta \int d^4x\, \mathcal{D}_3 &= \int d^4x\, \xi^\lambda\left(\partial_\lambda \partial^\rho \partial^\sigma h_{\rho\sigma} + \partial_\lambda \Box h\right)\\
\delta \int d^4x\, \mathcal{D}_4 &= -\int d^4x\, \xi^\lambda\, \partial_\lambda\, \Box h
\end{aligned} \tag{16}$$

so that linearized diffeomorphisms (*LDiff* henceforth) imposes some relations among the coupling constants

$$\alpha_1 = \alpha_2 = \alpha_3 = \alpha_4 \tag{17}$$

this is, $\alpha_i = 1\, \forall i$.

The second case we are analyzing is the invariance under transverse linearized diffeomorphisms (*LTDiff* henceforth), that is, diffeomorphisms such that their generating vector fields obey

$$\partial_\mu \xi^\mu = 0 \tag{18}$$

These conditions impose

$$\alpha_1 = \alpha_2 \tag{19}$$

but allow for arbitrary values of $\alpha_3$ and $\alpha_4$.

In third place, under linearized Weyl transformations, *LWeyl*, the variation of the metric reads

$$\delta h_{\alpha\beta} = \frac{2}{\kappa}\, \omega(x) \eta_{\alpha\beta} \tag{20}$$

and after integration by parts,[2]

$$\begin{aligned}
\delta \mathcal{D}_1 &= -\frac{1}{2}\, \omega\, \Box h\\
\delta \mathcal{D}_2 &= \omega\, \partial^\alpha \partial^\beta h_{\alpha\beta}\\
\delta \mathcal{D}_3 &= -\frac{\omega}{2}\left(4\partial^\alpha \partial^\beta h_{\alpha\beta} + \Box h\right)\\
\delta \mathcal{D}_4 &= 2\omega\, \Box h
\end{aligned} \tag{21}$$

where we have multiplied by $\frac{\kappa}{2}$ for simplicity. The invariance under LWeyl puts further constraints on the coupling constants, namely

$$\begin{aligned}
&\alpha_1 + \alpha_3 - 4\alpha_4 = 0\\
&4\alpha_3 - 2\alpha_2 = 0
\end{aligned} \tag{22}$$

In [13] we have dubbed *WTDiff* to the theory with *TDiff* invariance enhanced with linearized Weyl symmetry, *LWeyl*. This is the particular case of the above, corresponding to $\alpha_1 = \alpha_2 = 1$ and

$$\alpha_3 = \frac{1}{2}, \qquad \alpha_4 = \frac{3}{8} \tag{23}$$

A consistent non-linear completion of the actions which fullfill these requirements are the ones explained in the Appendix (110) namely actions proportional to

$$S_{WTDiff} = -\frac{1}{\kappa^2}\int d^4x\, g^{1/4}\left(R + \frac{3}{32}\frac{(\nabla g)^2}{g^2}\right) \tag{24}$$

Finally, we could consider only traceless graviton fields, $h_{\alpha\beta}$ such that $h \equiv \eta^{\alpha\beta} h_{\alpha\beta} = 0$. Obviously, in this case, $\mathcal{D}_3 = \mathcal{D}_4 = 0$, and for consistency, we can only implement *TDiff* with the coupling constants fixed to

$$\alpha_1 = \alpha_2 = 1 \tag{25}$$

### *2.1.1 Scale and conformal invariance*

The most general Lagrangian we are considering (without the mass terms) is obviously *scale invariant* under

$$\begin{aligned}
x^\mu &\to \lambda x^\mu\\
h_{\mu\nu} &\to \lambda^{-1} h_{\mu\nu}
\end{aligned} \tag{26}$$

with the assigned scaling dimensions. In order to make a full analysis of the scale and conformal invariance of the theory we have to compute the energy momentum tensor of these theories. In this case, and neglecting total derivatives, the metric (or Rosenfeld's) energy momentum tensor has the form

[2] Although we do not write the integrals explicitely, integration by parts is carried out in the analysis and total derivatives are not considered as stated in the introduction.

$$T_{\mu\nu} = \frac{1}{4}\alpha_1\left\{\partial_\mu h_{\alpha\beta}\partial_\nu h^{\alpha\beta} + 2\partial_\lambda h_{\mu\beta}\partial^\lambda h_\nu{}^\beta\right\} - \frac{1}{2}\alpha_2\left\{\partial_\mu h_{\nu\lambda}\partial_\sigma h^{\lambda\sigma} + \partial_\nu h_{\mu\lambda}\partial_\sigma h^{\lambda\sigma} + \partial^\lambda h_{\lambda\mu}\partial^\delta h_{\delta\nu}\right\} + \frac{1}{2}\alpha_3\left\{\partial^\alpha h_{\mu\nu}\partial^\lambda h_{\alpha\lambda} + \frac{1}{2}\left(\partial^\alpha h\partial_\mu h_{\alpha\nu} + \partial^\alpha h\partial_\nu h_{\mu\alpha} + \partial_\mu h\partial^\lambda h_{\lambda\nu} + \partial_\nu h\partial^\lambda h_{\lambda\mu}\right)\right\} - \frac{1}{4}\alpha_4\left\{2\partial^\alpha h\partial_\alpha h_{\mu\nu} + \partial_\mu h\partial_\nu h\right\} - \frac{1}{2}L\eta_{\mu\nu} \tag{27}$$

whose trace reads

$$T = \frac{3}{4}\alpha_1\,(\partial_\mu h_{\alpha\beta})^2 - \frac{3}{2}\alpha_2\,\partial_\lambda h^{\lambda\sigma}\partial^\rho h_{\rho\sigma} + \frac{3}{2}\alpha_3\partial^\alpha h\partial^\lambda h_{\lambda\alpha} - \frac{3}{4}\alpha_4\,(\partial_\mu h)^2 - 2\,L = L \tag{28}$$

The equations of motion (eom) read

$$\frac{\delta S}{\delta h^{\alpha\beta}} = -\frac{\alpha_1}{2}\Box h_{\alpha\beta} + \frac{\alpha_2}{2}\partial_\alpha\partial^\lambda h_{\lambda\beta} + \frac{\alpha_2}{2}\partial_\beta\partial^\lambda h_{\lambda\alpha} - \frac{\alpha_3}{2}\eta_{\alpha\beta}\partial^\lambda\partial^\sigma h_{\lambda\sigma} - \frac{\alpha_3}{2}\partial_\alpha\partial_\beta h + \frac{\alpha_4}{2}\eta_{\alpha\beta}\Box h \tag{29}$$

In fact we know that, as explained in the introduction, scale invariance implies that the trace of the energy momentum tensor can be written, on-shell, as a total derivative, that is, as the divergence of the virial (7). What happens is that whereas two Lagrangians that differ by a total derivative still generate the same eom, the specific form of the virial does depend on the particular form of the Lagrangian. This, in turn, also determines whether the virial itself can be written as a total derivative. Examining the contributions of the whole set of total derivative operators, leads to the convenient action

$$S = \int d^4x\left[\frac{\alpha_1}{4}\partial_\mu h_{\nu\rho}\partial^\mu h^{\nu\rho} - \frac{\alpha_2}{4}\left(\partial^\lambda h_{\lambda\rho}\partial_\sigma h^{\rho\sigma} + \partial_\sigma h_{\lambda\rho}\partial^\lambda h^{\rho\sigma}\right) + \frac{\alpha_3}{2}\partial^\lambda h\partial^\sigma h_{\lambda\sigma} - \frac{\alpha_4}{4}\partial_\mu h\partial^\mu h\right] \tag{30}$$

The trace of the energy momentum tensor can then be rewritten as

$$T = \left\{\alpha_1\left[\frac{1}{4}\partial^\mu[h^{\nu\rho}\partial_\mu h_{\nu\rho}] - \frac{1}{4}h_{\mu\nu}\Box h^{\mu\nu}\right] + \alpha_2\left[-\frac{1}{4}\partial_\sigma[h^{\rho\sigma}\partial^\lambda h_{\lambda\rho}] - \frac{1}{4}\partial_\sigma[h_{\lambda\rho}\partial^\lambda h^{\rho\sigma}] + \frac{1}{4}h^{\rho\sigma}\partial_\sigma\partial^\lambda h_{\lambda\rho} + \frac{1}{4}h_{\lambda\rho}\partial^\lambda\partial_\sigma h^{\rho\sigma}\right] + \alpha_3\left[\frac{1}{4}\partial^\sigma[h_{\lambda\sigma}\partial^\lambda h] + \frac{1}{4}\partial^\lambda[h\partial^\sigma h_{\lambda\sigma}] - \frac{1}{4}h_{\lambda\sigma}\partial^\sigma\partial^\lambda h - \frac{1}{4}h\partial^\lambda\partial^\sigma h_{\lambda\sigma}\right] + \alpha_4\left[-\frac{1}{4}\partial_\mu[h\partial^\mu h] + \frac{1}{4}h\Box h\right]\right\} \tag{31}$$

so that on-shell, we can express the trace as $T = \partial_\mu V^\mu$, where

$$V^\mu = \left\{\frac{\alpha_1}{4}h^{\nu\rho}\partial^\mu h_{\nu\rho} - \frac{\alpha_2}{4}h^{\rho\mu}\partial^\lambda h_{\lambda\rho} - \frac{\alpha_2}{4}h_{\lambda\rho}\partial^\lambda h^{\rho\mu} + \frac{\alpha_3}{4}h^\mu_\lambda\partial^\lambda h + \frac{\alpha_3}{4}h\partial^\sigma h^\mu_\sigma - \frac{\alpha_4}{4}h\partial^\mu h\right\} \tag{32}$$

It is also well-known that a conformal current can be formally constructed in case the virial can be so expressed as $V^\mu = \partial_\nu\sigma^{\mu\nu}$. In our case, and upon using the eom,

$$\sigma^{\mu\nu} = \left\{\frac{\alpha_1}{8}\eta^{\mu\nu}h^{\alpha\beta}h_{\alpha\beta} - \frac{\alpha_2}{4}h^{\rho\mu}h^\nu_\rho + \frac{\alpha_3}{4}h^{\mu\nu}h - \frac{\alpha_4}{8}\eta^{\mu\nu}h^2\right\} \tag{33}$$

In that way we get such a tensor (on-shell), for any value of the coupling constants.[3]

Once a particular form of an $\sigma^{\mu\nu}$ is found, there is a systematic way of improving the energy-momentum tensor [10]. The improvement consists on adding another piece to the initial energy momentum tensor, so that the trace of the new energy-momentum tensor is precisely cero, that is, we avoid the total derivative terms. The piece in [10] has the form

$$\Theta^{\mu\nu} = T^{\mu\nu} + \frac{1}{2}\partial_\lambda\partial_\rho X^{\lambda\rho\mu\nu} \tag{36}$$

where $X^{\lambda\rho\mu\nu}$ is symmetric $(\mu, \nu)$, and divergenceless. The precise for of the improvement reads

[3] We insist that this result (because this tensor is after all a total derivative) depends on the boundary terms that are neglected in order to write down the original Lagrangian. The monomials have to be written down as indicated; to be specific, in order to split $\mathcal{D}_2$ in the two pieces, there is an integration by parts

$$-\partial^\lambda h_{\lambda\rho}\partial_\sigma h^{\rho\sigma} - \partial^\lambda h_{\lambda\rho}\partial_\sigma h^{\rho\sigma} = -\partial^\lambda h_{\lambda\rho}\partial_\sigma h^{\rho\sigma} - \partial_\sigma h_{\lambda\rho}\partial^\lambda h^{\rho\sigma} - \partial^\lambda(h_{\lambda\rho}\partial_\sigma h^{\rho\sigma}) + \partial_\sigma(h_{\lambda\rho}\partial^\lambda h^{\rho\sigma}) \tag{34}$$

so we have two total derivatives appearing in order to interchange the two derivatives. Although this does not contribute to the equations of motion, it does contribute to the virial, $L = L' + \partial_\mu j^\mu \rightarrow V^\mu = V'^\mu + j^\mu$. It could well be the case that $V^\mu = \partial_\nu\sigma^{\mu\nu}$ but $V'^\mu + j^\mu \neq \partial_\nu\sigma'^{\mu\nu}$. If we take this contributions into account one of the total derivatives cancels one of the pieces of the virial proportional to $\alpha_2$ and the other one is summed with the other piece. We end up with

$$V^\mu = \left\{\frac{\alpha_1}{4}h^{\nu\rho}\partial^\mu h_{\nu\rho} - \frac{\alpha_2}{2}h^{\rho\mu}\partial^\lambda h_{\lambda\rho} + \frac{\alpha_3}{4}h^\mu_\lambda\partial^\lambda h + \frac{\alpha_3}{4}h\partial^\sigma h^\mu_\sigma - \frac{\alpha_4}{4}h\partial^\mu h\right\} \tag{35}$$

and we could not write this as the derivative of a two-index tensor.

$$X^{\lambda\rho\mu\nu} = g^{\lambda\rho}\sigma_+^{\mu\nu} - g^{\lambda\mu}\sigma_+^{\rho\nu} - g^{\lambda\nu}\sigma_+^{\mu\rho} + g^{\mu\nu}\sigma_+^{\lambda\rho} - \frac{1}{3}g^{\lambda\rho}g^{\mu\nu}\sigma_{+\alpha}^{\alpha} + \frac{1}{3}g^{\lambda\mu}g^{\rho\nu}\sigma_{+\alpha}^{\alpha} \qquad (37)$$

where $\sigma_+^{\mu\nu}$ stands for the symmetric part of $\sigma^{\mu\nu}$. The original analysis [11] was specific for $n = 4$ dimensions but it can be generalized to arbitrary dimension [19].

## 2.2 Dimension 6 operators

In this section, we want to study the dimension 6 operators, and among these, there are various possibilities. First we take operators with four derivatives and two $h_{\alpha\beta}$. After integration by parts, those are

$$\begin{aligned}
\mathcal{O}_1 &\equiv h_{\alpha\beta}\left(\partial^\alpha\partial^\beta\partial^\gamma\partial^\delta\right)h_{\gamma\delta}\\
\mathcal{O}_2 &\equiv h_{\alpha\beta}\left(\partial^\alpha\partial^\beta\eta^{\gamma\delta}\Box\right)h_{\gamma\delta}\\
\mathcal{O}_3 &\equiv h_{\alpha\beta}\left(\partial^\alpha\partial^\gamma\eta^{\beta\delta}\Box\right)h_{\gamma\delta}\\
\mathcal{O}_4 &\equiv h_{\alpha\beta}\left(\Box^2\eta^{\alpha\gamma}\eta^{\beta\delta}\right)h_{\gamma\delta}\\
\mathcal{O}_5 &\equiv h_{\alpha\beta}\left(\Box^2\eta^{\alpha\beta}\eta^{\gamma\delta}\right)h_{\gamma\delta}
\end{aligned} \qquad (38)$$

There is a small caveat here. There are also many operators with two derivatives and four $h_{\alpha\beta}$, that will be analyzed in the next section. It is the case that these operators do not appear as a limit of quadratic *Diff* or *TDiff* invariant theories; they can only appear as higher order contributions to Lagrangians linear in the curvature.

Let us consider the general theory involving dimension 6 operators which can come from quadratic theories of gravity, namely

$$S_{quad} = \kappa^2 \int d^4x \sum_{i=1}^{5} g_i\,\mathcal{O}_i \qquad (39)$$

Like in the previous section, we first study the *LDiff* symmetry, upon which the $\mathcal{O}$ transform as

$$\begin{aligned}
\delta\mathcal{O}_1 &= -4\xi_\lambda\partial^\lambda\partial^\alpha\partial^\beta\Box h_{\alpha\beta}\\
\delta\mathcal{O}_2 &= -2\xi_\lambda\left(\partial^\lambda\Box^2 h + \partial^\lambda\partial^\alpha\partial^\beta\Box h_{\alpha\beta}\right)\\
\delta\mathcal{O}_3 &= -2\xi^\lambda\left(\partial^\beta\Box^2 h_{\lambda\beta} + \partial_\lambda\partial^\alpha\partial^\beta\Box h_{\alpha\beta}\right)\\
\delta\mathcal{O}_4 &= -4\xi^\lambda\partial^\alpha\Box^2 h_{\alpha\lambda}\\
\delta\mathcal{O}_5 &= -4\xi^\lambda\partial_\lambda\Box^2 h
\end{aligned} \qquad (40)$$

after having integrated by parts. Then the symmetry under $LDiff$ imposes the following relations between the coupling constants

$$\begin{aligned}
&2g_1 + g_2 + g_3 = 0\\
&g_2 + 2g_5 = 0\\
&g_3 + 2g_4 = 0
\end{aligned} \qquad (41)$$

These still allow for arbitrary values of $g_1$ and $g_2$, and

$$\begin{aligned}
g_3 &= -(2g_1 + g_2)\\
g_4 &= \frac{2g_1 + g_2}{2}\\
g_5 &= -\frac{g_2}{2}
\end{aligned} \qquad (42)$$

In the second place we consider invariance under *LTDiff*, which imposes

$$g_3 + 2g_4 = 0 \qquad (43)$$

Finally for *LWeyl* symmetry, the variations read (multiplying by $\kappa/2$ again for simplicity)

$$\begin{aligned}
\delta\mathcal{O}_1 &= 2\omega\,\Box\partial^\alpha\partial^\beta h_{\alpha\beta}\\
\delta\mathcal{O}_2 &= \omega\,\Box^2 h + 4\omega\,\Box\partial^\alpha\partial^\beta h_{\alpha\beta}\\
\delta\mathcal{O}_3 &= 2\omega\,\Box\partial^\alpha\partial^\beta h_{\alpha\beta}\\
\delta\mathcal{O}_4 &= 2\omega\,\Box^2 h\\
\delta\mathcal{O}_5 &= 8\omega\,\Box^2 h
\end{aligned} \qquad (44)$$

so that the action is invariant under such tranformations whenever

$$\begin{aligned}
&2g_1 + 4g_2 + 2g_3 = 0\\
&g_2 + 2g_4 + 8g_5 = 0
\end{aligned} \qquad (45)$$

Now it is interesting to combine *LDiff* and *LWeyl*. In the case of dimension 4 operators, actions which are invariant under both symmetries do not exist. For dimension 6 operators, we can have *LWDiff* invariant theories as long as the coupling constants are constrained to have the following relations

$$g_1 = g_2,\quad g_3 = -3g_2,\quad g_4 = \frac{3}{2}g_2,\quad g_5 = -\frac{g_2}{2} \qquad (46)$$

These actions with *LWDiff* invariance are obtained as the weak field limit of the following quadratic theories

$$L = \sqrt{g}\left[\alpha\,R^2_{\alpha\beta\gamma\delta} + [-4\alpha + 6g_2]\,R^2_{\mu\nu} + (\alpha - 2g_2)\,R^2\right] \qquad (47)$$

Note that the term $\sqrt{g}$ is immaterial at the order we are working. The weak field expansion of the quadratic invariants is worked out in the Appendix A. For $n = 4$ spacetime dimension these theories can be rewritten as

$$L = \sqrt{g}\,(\alpha - 3g_2)\,E_4 + \sqrt{g}\,3g_2\,W_4 \qquad (48)$$

where $E_4$ is the four-dimensional Euler density

$$E_4 \equiv R^2_{\alpha\beta\gamma\delta} - 4\,R^2_{\alpha\beta} + R^2 \qquad (49)$$

It can be easily checked that the weak field expansion of $E_4$ around Minkowski spacetime vanishes. This is the origin of the arbitrary coefficient $\alpha$ in the above expression. The quantity $W_4$ is the square of the 4-dimensional Weyl tensor

$$W_4 \equiv R^2_{\alpha\beta\gamma\delta} - 2\, R^2_{\alpha\beta} + \frac{1}{3}\, R^2 \tag{50}$$

The actions which are precisely proportional to the Weyl squared tensor are the ones with $3g_2 = \alpha$. Nevertheless, at the linear level, the Euler density does not contribute, so that all the solutions will effectively correspond to actions proportional to $W_4$.

Also quite interesting are those actions that are *LWTDiff* invariant; that is *LWeyl* invariant, but *LDiff* invariant under transverse diffeomorphisms only. They are characterized by

$$\begin{aligned} g_1 &= -g_3 - 2g_2 = 6g_4 + 16g_5 \\ g_2 &= -2g_4 - 8g_5 \\ g_3 &= -2g_4 \end{aligned} \tag{51}$$

The most general quadratic *WTDiff* invariant Lagrangian is the one obtained by Weyl transforming the metric in the quadratic action with $\tilde{g}_{\mu\nu} = g^{-1/4} g_{\mu\nu}$ (this transformation ensures *TDiff* and automatically introduces a Weyl invariance). The expansion around flat spacetime reads

$$\begin{aligned} &\sqrt{\tilde{g}}\,(\alpha\,\tilde{R}_{\mu\nu\rho\sigma}\,\tilde{R}^{\mu\nu\rho\sigma} + \beta\,\tilde{R}_{\mu\nu}\,\tilde{R}^{\mu\nu} + \gamma\,\tilde{R}^2) \\ &= \left(\alpha + \frac{\beta}{2} + \gamma\right) \mathcal{O}_1 + \frac{1}{2}(\alpha - \gamma)\,\mathcal{O}_2 \\ &\quad - \left(2\alpha + \frac{\beta}{2}\right) \mathcal{O}_3 + \left(\alpha + \frac{\beta}{4}\right) \mathcal{O}_4 \\ &\quad - \left(\frac{20\alpha + 4\beta - 4\gamma}{64}\right) \mathcal{O}_5 + O(h^3) \end{aligned} \tag{52}$$

The weak field limit of these theories automatically satisfies the constraints needed for *LWTDiff* (51). The precise form of these theories after the Weyl transformation (120) is shown in the Appendix.

### 2.2.1 Scale and conformal invariance

*Scale invariance* is now lost with the assingment given to $h_{\mu\nu}$ (conformal weight one). If we have a theory incorporating dimension 6 operators only, it is possible to recover scale invariance, just by making the graviton inert (conformal weight 0). It is plain that this does not hold when we have both dimension 4 and dimension 6 operators in the theory.

On the other hand, the *conformal invariance* demands as usual, tracelessness of the (metric, or Rosenfeld) energy-momentum tensor. In this case, the energy-momentum tensor takes the form

$$\begin{aligned} T_{\mu\nu} &= 2g_1\{h_{\mu\lambda}\partial_\nu\partial^\lambda\partial^\alpha\partial^\beta h_{\alpha\beta} \\ &\quad + h_{\nu\lambda}\partial_\mu\partial^\lambda\partial^\alpha\partial^\beta h_{\alpha\beta}\} \\ &\quad + g_2\{h_{\lambda\mu}\partial_\nu\partial^\lambda\Box h + h_{\lambda\nu}\partial_\mu\partial^\lambda\Box h \\ &\quad + h_{\alpha\beta}\partial^\alpha\partial^\beta\partial_\mu\partial_\nu h + h_{\alpha\beta}\partial^\alpha\partial^\beta\Box h_{\mu\nu}\} \\ &\quad + g_3\{h_{\alpha\mu}\partial_\nu\partial_\beta\Box h^{\alpha\beta} + h_{\alpha\nu}\partial_\mu\partial_\beta\Box h^{\alpha\beta} \\ &\quad + h_{\alpha\mu}\partial^\alpha\partial^\beta\Box h_{\beta\nu} + h_{\alpha\beta}\partial^\alpha\partial_\lambda\partial_\mu\partial_\nu h^{\lambda\beta}\} \\ &\quad + 2g_4\{h_{\alpha\beta}\partial_\mu\partial_\nu\Box h^{\alpha\beta} + h_{\lambda\mu}\Box^2 h^\lambda_\nu\} + 2g_5\{h\partial_\mu\partial_\nu\Box h \\ &\quad + h_{\mu\nu}\Box^2 h\} - \frac{1}{2}\, L\eta_{\mu\nu} \end{aligned} \tag{53}$$

and the trace reads

$$T = 2\,L \tag{54}$$

The eom read

$$\begin{aligned} \frac{\delta S}{\delta h^{\alpha\beta}} &= 2g_1\partial_\alpha\partial_\beta\partial^\mu\partial^\nu h_{\mu\nu} + g_2\partial_\alpha\partial_\beta\Box h \\ &\quad + g_2\eta_{\alpha\beta}\partial^\mu\partial^\nu\Box h_{\mu\nu} + g_3\partial_\beta\partial^\lambda\Box h_{\alpha\lambda} + g_3\partial_\alpha\partial^\lambda\Box h_{\beta\lambda} \\ &\quad + 2g_4\Box^2 h_{\alpha\beta} + 2g_5\eta^{\alpha\beta}\Box^2 h \end{aligned} \tag{55}$$

and they imply that, on-shell, the Lagrangian indeed vanishes, $L = 0$ (up to total derivatives).

In order to study the virial in detail, let us start from the specific form of the Lagrangian

$$\begin{aligned} L &= 2\left\{g_1\left(\partial^\alpha\partial^\gamma h_{\alpha\beta}\partial^\beta\partial^\delta h_{\gamma\delta} + \partial^\alpha h_{\alpha\beta}\partial^\beta\partial^\gamma\partial^\delta h_{\gamma\delta}\right.\right. \\ &\quad \left. + \partial^\gamma h_{\alpha\beta}\partial^\beta\partial^\alpha\partial^\delta h_{\gamma\delta}\right) \\ &\quad + \frac{g_2}{2}\left(\partial^\alpha\partial^\beta h_{\alpha\beta}\Box h + \partial^\alpha\partial^\beta h\Box h_{\alpha\beta}\right. \\ &\quad \left. + 2\partial^\beta h_{\alpha\beta}\partial^\alpha\Box h + 2\partial^\beta h\Box\partial^\alpha h_{\alpha\beta}\right) \\ &\quad + g_3\left(\partial^\alpha\partial^\beta h_{\alpha\lambda}\Box h^\lambda_\beta + \partial^\alpha h_{\alpha\lambda}\partial^\beta\Box h^\lambda_\beta\right. \\ &\quad \left. + \partial^\beta h_{\alpha\lambda}\partial^\alpha\Box h^\lambda_\beta\right) \\ &\quad + g_4\left(\Box h_{\alpha\beta}\Box h^{\alpha\beta} + 2\partial^\mu h_{\alpha\beta}\partial_\mu\Box h^{\alpha\beta}\right) \\ &\quad \left. + g_5\left(\Box h\Box h + 2\partial^\mu h\partial_\mu\Box h\right)\right\} \end{aligned} \tag{56}$$

Now it is a simple matter to show that on-shell, $L = \partial_\mu\partial_\nu\sigma^{\mu\nu}$ with

$$\begin{aligned} \sigma^{\mu\nu} &= 2\left\{\frac{g_1}{2}\left(h^{\mu\alpha}\partial_\alpha\partial^\beta h^\nu_\beta + h^{\nu\alpha}\partial_\alpha\partial^\beta h^\mu_\beta\right)\right. \\ &\quad + \frac{g_2}{2}\left(h^{\mu\nu}\Box h + h\Box h^{\mu\nu}\right) \\ &\quad + \frac{g_3}{2}\left(h^{\mu\lambda}\Box h^\nu_\lambda + h^{\nu\lambda}\Box h^\mu_\lambda\right) + g_4\eta^{\mu\nu}h_{\alpha\beta}\Box h^{\alpha\beta} \\ &\quad \left. + g_5\eta^{\mu\nu}h\Box h\right\} \end{aligned} \tag{57}$$

This result is somewhat puzzling, because we have already indicated that this theory is *not* even scale invariant with the standard assignment of conformal weight for the graviton field (namely 1). The result is however logical if we remember that the low energy of the Weyl squared Lagrangian is of this form. The theory containing both dimension 4 as well as dimension 6 operators, should not be conformal however. This fact can be easily understood from a simpler example, namely a scalar Lagrangian, where all complications of indices can be avoided. Consider then the Lagrangian

$$L' = \alpha\phi\Box\phi + \frac{\beta}{M^2}\phi\Box^2\phi \tag{58}$$

which is equivalent to (up to total derivatives)

$$\begin{aligned} L &= -\alpha\, \partial_\mu\phi\partial^\mu\phi \\ &\quad - \frac{\beta}{M^2}\left(4\partial_\lambda\phi\Box\partial^\lambda\phi + 2\partial^\mu\partial_\lambda\phi\partial_\mu\partial^\lambda\phi + \Box\phi\Box\phi\right) \\ &= -\alpha L_1 - \frac{\beta}{M^2}L_2 \end{aligned} \tag{59}$$

This is our starting point. The eom read

$$\frac{\delta S}{\delta\phi} = \alpha\Box\phi + \frac{\beta}{M^2}\Box^2\phi = 0 \tag{60}$$

and the energy-momentum tensor

$$\begin{aligned} T_{\mu\nu} &= -\alpha\Big(\partial_\mu\phi\partial_\nu\phi - \frac{1}{2}\partial_\lambda\phi\partial^\lambda\phi\,\eta_{\mu\nu}\Big) \\ &\quad -\frac{\beta}{M^2}\Big(4\partial_\lambda\phi\partial_\mu\partial_\nu\partial^\lambda\phi + 4\partial_\mu\phi\Box\partial_\nu\phi + 4\partial_\mu\partial_\lambda\phi\partial_\nu\partial^\lambda\phi \\ &\quad +2\partial_\mu\partial_\nu\phi\Box\phi - \frac{1}{2}\left(4\partial_\lambda\phi\Box\partial^\lambda\phi + 2\partial^\sigma\partial_\lambda\phi\partial_\sigma\partial^\lambda\phi \right. \\ &\quad \left. +\Box\phi\Box\phi\right)\eta_{\mu\nu}\Big) \end{aligned} \tag{61}$$

The trace of the above reads

$$T = -\alpha\left(1-\frac{n}{2}\right)L_1 - \frac{\beta}{M^2}\left(2-\frac{n}{2}\right)L_2 = \alpha L_1 \tag{62}$$

Even if we are working in $n = 4$, we leave $n$ arbitrary to maintain the second piece and illustrate the point we want to make. Note that this is not proportional to the total Lagrangian, because the trace counts the number of derivatives. We can rewrite the trace as

$$\begin{aligned} T &= -\frac{\alpha}{2}\left(1-\frac{n}{2}\right)\left(\Box\phi^2 - 2\phi\Box\phi\right) \\ &\quad -\frac{\beta}{2M^2}\left(2-\frac{n}{2}\right)\left(\Box^2\phi^2 - 2\phi\Box^2\phi\right) \end{aligned} \tag{63}$$

which fails to be a total derivative when both $\alpha$ and $\beta$ are nonvanishing, because

$$1-\frac{n}{2} \neq 2-\frac{n}{2} \tag{64}$$

Note that this is true even if there are *WTDiff* (that is *TDiff* and *Weyl* invariant) theories linear as well as quadratic in the Riemann tensor. The weak field limit of those Weyl invariant theories fails to be conformal invariant.

### 2.3 Dimension 5 and dimension 6 operators (with 2 derivatives)

Next, we study dimension 5 and dimension 6 operators containing just two derivatives, so that they come from the weak field limit of theories linear in the curvature, when expanded to higher orders in the perturbation. In the previous sections, operators coming from the lowest (non-trivial) order of gravitational actions were analyzed. In that cases, the lowest order of *(T)Diff* and Weyl variations was enough to obtain the conditions for those actions to be invariant under such symmetries. In this case, however, different orders of the expansion are needed because of the two orders involved in the field variations

$$\begin{aligned} \delta_D(\kappa\, h_{\mu\nu}) &= \partial_\mu\xi_\nu + \partial_\nu\xi_\mu + \kappa\,\mathcal{L}_\xi h_{\mu\nu} \\ \delta_W(\kappa\, h_{\mu\nu}) &= 2\omega\eta_{\mu\nu} + 2\kappa\,\omega h_{\mu\nu} \end{aligned} \tag{65}$$

This translates into dimension 4 operators mixing with dimension 5 ones, and dimension 5 operators with dimension 6 ones.

A full list of the independent dimension 5 and dimension 6 operators (containing 2 derivatives) can be found in Appendix 1. The most general dimension 5 Lagrangian with such operators reads

$$\mathcal{L}_{5\partial\partial} = \kappa\sum_i^{14} a_i\mathcal{N}_i \tag{66}$$

Again, the only diffeomorphism invariant combination corresponds to $\sqrt{g}R$, in this case, to the order $O(\kappa^3)$ expansion of it

$$\begin{aligned} &\left(\frac{1}{\kappa^2}\sqrt{g}R\right)_{O(\kappa^3)} \\ &= \kappa\left\{-\frac{1}{4}\mathcal{N}_1 - \frac{1}{2}\mathcal{N}_2 + \frac{1}{4}\mathcal{N}_3 - \frac{1}{2}\mathcal{N}_4 + \frac{1}{2}\mathcal{N}_5 - \frac{1}{4}\mathcal{N}_6 \right. \\ &\quad -\frac{1}{4}\mathcal{N}_8 - \frac{1}{8}\mathcal{N}_9 - \frac{1}{4}\mathcal{N}_{11} + \frac{1}{8}\mathcal{N}_{12} + \frac{3}{16}\mathcal{N}_{13} \\ &\quad \left. -\frac{1}{16}\mathcal{N}_{14}\right\} \end{aligned} \tag{67}$$

This piece then combines with the previous order of the expansion to attain diffeomorphism invariance

$$\left(\sqrt{g}R\right)_{O(\kappa^2)}\Big|_{\delta(\kappa\, h_{\mu\nu})=\partial_\mu\xi_\nu+\partial_\nu\xi_\mu} + \left(\sqrt{g}R\right)_{O(\kappa^3)}\Big|_{\delta(\kappa\, h_{\mu\nu})=\kappa\,\mathcal{L}_\xi h_{\mu\nu}} = 0 \tag{68}$$

We can also look for the most general Lorentz and Weyl invariant Lagrangian built with this kind of operators. Again, we need the dimension 4 operator part that will contribute with the $O(h)$ piece of the Weyl variation, which already has two arbitrary constants appearing in it (22). Taking that piece into account, the most general Weyl invariant Lagrangian up to dimension 5 operators reads

$$\begin{aligned}\mathcal{L}_{W_{5D}} &= \frac{c_1}{4}\partial_\lambda h_{\mu\nu}\partial^\lambda h^{\mu\nu} - \frac{c_2}{2}\partial_\mu h^{\mu\lambda}\partial_\nu h^\nu_{\ \lambda} + \frac{c_2}{4}\partial_\nu h^{\mu\nu}\partial_\mu h \\ &- \frac{2c_1+c_2}{32}\partial_\lambda h\partial^\lambda h \\ &+\kappa\Big\{a_1\mathcal{N}_1 + a_2\mathcal{N}_2 + a_3\mathcal{N}_3 + 4a_1\mathcal{N}_4 + a_5\mathcal{N}_5 \\ &+(a_1 - 2a_2 - 4a_3 + \frac{c_2}{4})\mathcal{N}_6 \\ &+(4a_1 - 4a_2 - a_5)\mathcal{N}_7 + (-a_1 + 2a_2 - \frac{c_2}{4})\mathcal{N}_8 \\ &+a_9\mathcal{N}_9 + \frac{1}{16}(2a_1 - 4a_2 - 4a_9 + c_2)\mathcal{N}_{10} \\ &+a_{11}\mathcal{N}_{11} + \frac{1}{16}(-4a_1 - 8a_{11} + 8a_2 + 16a_3 \\ &+2c_1 - c_2)\mathcal{N}_{12} - \frac{1}{4}(a_{11}+a_9)\mathcal{N}_{13} \\ &+\frac{1}{32}(a_1 + 4a_{11} - 2a_2 - 8a_3 + 2a_9 - c_1)\mathcal{N}_{14}\Big\}\end{aligned} \tag{69}$$

Let us insist on the fact that both pieces are needed so that Weyl invariance is attained, that is, up to certain order in the expansion, the previous order is needed for the computation of the invariance conditions. Thanks to the mixing of the different orders, more freedom is avalaible to attain invariance under the studied symmetries. In particular, there are 8 independent coupling constants $a_1, a_2, a_3, a_5, a_9, a_{11}, c_1$ and $c_2$.

It is straightforward to see that $WTDiff$ (24) is a particular case of this general Weyl invariant Lagrangian, with the constants fixed to

$$\begin{aligned}&a_1 = -\frac{1}{8},\ a_2 = -\frac{1}{4},\ a_3 = \frac{3}{32},\ a_5 = \frac{1}{2},\ a_9 = 0,\\ &a_{11} = -\frac{1}{4},\ c_1 = c_2 = -1\end{aligned} \tag{70}$$

Accordingly, the most general dimension 6 Lagrangian with two derivatives can be written as

$$\mathcal{L}_{6\partial\partial} = \kappa^2\sum_i^{38} b_i\mathcal{K}_i \tag{71}$$

In this case, the expansion of $\left(\sqrt{g}R\right)_{O(\kappa^4)}$ reads

$$\begin{aligned}\left(\frac{1}{\kappa^2}\sqrt{g}R\right)_{O(\kappa^4)} &= \kappa^2\Big\{\frac{1}{4}\mathcal{K}_1 - \frac{1}{8}\mathcal{K}_3 + \frac{1}{2}\mathcal{K}_4 - \frac{1}{4}\mathcal{K}_5 - \frac{3}{16}\mathcal{K}_6 \\ &+\frac{1}{16}\mathcal{K}_7 + \frac{1}{32}\mathcal{K}_8 - \frac{1}{2}\mathcal{K}_9 + \frac{1}{2}\mathcal{K}_{11} \\ &+\frac{1}{8}\mathcal{K}_{12} - \frac{1}{8}\mathcal{K}_{13} - \frac{1}{2}\mathcal{K}_{14} + \frac{1}{4}\mathcal{K}_{16} - \frac{1}{8}\mathcal{K}_{17} \\ &+\frac{1}{4}\mathcal{K}_{20} - \frac{1}{16}\mathcal{K}_{21} + \frac{1}{12}\mathcal{K}_{22} \\ &-\frac{1}{16}\mathcal{K}_{23} + \frac{1}{4}\mathcal{K}_{25} - \frac{1}{8}\mathcal{K}_{26} - \frac{1}{8}\mathcal{K}_{27} + \frac{1}{32}\mathcal{K}_{28} \\ &-\frac{1}{8}\mathcal{K}_{29} + \frac{3}{32}\mathcal{K}_{30} - \frac{1}{96}\mathcal{K}_{31} \\ &+\frac{1}{2}\mathcal{K}_{32} - \frac{1}{2}\mathcal{K}_{33} - \frac{1}{4}\mathcal{K}_{35} - \frac{1}{16}\mathcal{K}_{38}\Big\}\end{aligned} \tag{72}$$

Finally, let us analyze the most general Weyl invariant Lagrangian up to dimension 6 operators. We have different pieces apearing in it. First of all, it contains the pieces up to dimension 5 that were computed in this section (69), together with the dimension 6 piece of two derivative operators, that combine with specific coefficients so that Weyl invariance is attained. Moreover, we have another Weyl invariant combination coming from dimension 6 operators containing four derivatives (45). Taking everything into account, the most general Weyl invariant Lagrangian up to dimension six operators is shown in Appendix 1.

### 2.4 Interaction terms

It would appear quite intuitive to think that there are no potential terms invariant under either *Diff* or Weyl invariance. This is based in our GR intuition, but let us get rid of those prejudices and carry on with our perturbative analysis. It is easy to systematize the perturbative expansion. Up to quartic interactions we have the monomials

$$\begin{aligned}\mathcal{M}_1 &\equiv h_{\alpha\beta}h^{\alpha\beta}\\ \mathcal{M}_2 &\equiv h^2\end{aligned}$$

$$\begin{aligned}\mathcal{J}_1 &\equiv h^{\alpha\beta}h_{\beta\gamma}h^\gamma_\alpha\\ \mathcal{J}_2 &\equiv h^{\alpha\beta}h_{\alpha\beta}h\\ \mathcal{J}_3 &\equiv h^3\end{aligned}$$

$$\begin{aligned}\mathcal{Q}_1 &\equiv h^{\alpha\beta}h_{\beta\gamma}h^{\gamma\delta}h_{\delta\alpha}\\ \mathcal{Q}_2 &\equiv \left(h_{\mu\nu}h^{\mu\nu}\right)^2\end{aligned}$$

$$\begin{aligned}\mathcal{Q}_3 &\equiv h^4\\ \mathcal{Q}_4 &\equiv h^2h_{\alpha\beta}h^{\alpha\beta}\\ \mathcal{Q}_5 &\equiv hh_{\alpha\beta}h^{\beta\gamma}h^\alpha_\gamma\end{aligned}$$

so that the most general potential up to dimension four will read

$$V(h_{\mu\nu}) = mM^2h + \sum_{i=1}^{2}m_i^2\mathcal{M}_i + \sum_{a=1}^{a=3}b_a\mathcal{J}_a + \sum_{a=1}^{a=5}\lambda_a\mathcal{Q}_a + \cdots \tag{73}$$

We want to analyze the invariance under diffeomorphisms as if this $h_{\mu\nu}$ corresponds to the perturbation of the metric around flat spacetime (65), but we take another energy scale $M$ instead of $\kappa$. The crucial point is that owing to the fact that the *Diff* variations contain an order zero piece and an order one piece in the perturbation, each order in the perturbative expansion of the variation of the potential contributes to both the lower and upper orders. Up to total derivatives and dimension four operators, it can be seen that the following interaction Lagrangian is diffeomorphism invariant

$$V^D(h_{\mu\nu}) = mM^2 h + \frac{mM}{4}\left(\mathcal{M}_2 - 2\mathcal{M}_1\right) + m\left(\frac{1}{3}\mathcal{J}_1 - \frac{1}{4}\mathcal{J}_2 + \frac{1}{24}\mathcal{J}_3\right) + \frac{m}{M}\left(-\frac{1}{4}\mathcal{Q}_1 + \frac{1}{16}\mathcal{Q}_2 + \frac{1}{192}\mathcal{Q}_3 - \frac{1}{16}\mathcal{Q}_4 + \frac{1}{6}\mathcal{Q}_5\right) + \ldots \tag{74}$$

In fact this is an iterative process, each term in the expansion determining the following. The final potential contains infinite terms depending on just one arbitrary constant with dimensions of mass, $m$. In fact this is exactly the weak field expansion of $mM^4\left(\sqrt{g} - 1\right)$.

At this point our GR intuition strikes back and asks whether this is not precisely the expansion of the cosmological constant term. (In fact they do not quite fit).

Concentrating in the quadratic terms

$$V_2^D \equiv mM^2 h + \frac{mM}{4}h^2 - \frac{mM}{2}h_{\alpha\beta}^2 = -\frac{Mm}{2}\left(h_{\alpha\beta} + M\eta_{\alpha\beta}\right)^2 + \frac{mM}{4}(h + 4M)^2 - 2mM^3 \tag{75}$$

Not knowing anything on GR we would say that there is spontaneous symmetry breaking in the system and the ground state has shifted from $h_{\mu\nu} = 0$ to $h_{\mu\nu} = -M\eta_{\mu\nu}$, leaving behind a vacuum energy

$$V_0^D \equiv -2mM^3 \tag{76}$$

Fluctuations around the new vacuum state

$$h_{\mu\nu} \equiv -M\eta_{\mu\nu} + H_{\mu\nu} \tag{77}$$

are damped (provided $mM < 0$) with a quadratic term

$$V_2^D + 2mM^3 = -\frac{mM}{2}H_{\alpha\beta}^2 + \frac{mM}{4}H^2 \tag{78}$$

as is not positive semidefinite except for traceless $H^T_{\alpha\beta}$ when $mM < 0$. In order to reach a definite conclusion on positivity, higher order terms should be considered. To the extent that this is related to the weak field expansion of $mM^2\sqrt{g}$, we expect it to have a definite sign however.

Similar reasoning as in the previous paragraph leads to a Lorentz and Weyl invariant potential

$$\begin{aligned} V^W(h_{\mu\nu}) &= mM^2 h + m_1^2\mathcal{M}_1 - \frac{1}{8}\left(mM + 2m_1^2\right)\mathcal{M}_2 \\ &+ b_1\mathcal{J}_1 + \frac{1}{4}\left(-2\frac{m_1^2}{M} - 3b_1\right)\mathcal{J}_2 + \\ &+ \frac{1}{48}\left(m + 6\frac{m_1^2}{M} + 6b_1\right)\mathcal{J}_3 + \kappa\, m\, \lambda_1\mathcal{Q}_1 + \lambda_2\mathcal{Q}_2 \\ &+ \frac{1}{256}\left(-\frac{m}{M} - 12\frac{m_1^2}{M^2} - 24\frac{b_1}{M} - -12\lambda_1 + 16\lambda_2\right)\mathcal{Q}_3 \\ &+ \frac{1}{16}\left(3\frac{m_1^2}{M^2} + 9\frac{b_1}{M} + 6\lambda_1 - 8\lambda_2\right)\mathcal{Q}_4 \\ &- \left(\frac{3}{4}\frac{b_1}{M} + \lambda_1\right)\mathcal{Q}_5 \end{aligned} \tag{79}$$

In this case we have more freedom as more arbitrary constants appear with each order of the perturbative expansion. The quadratic piece can be written as

$$V_2^W = m_1^2\left(h_{\alpha\beta} - M\eta_{\alpha\beta}\right)^2 - \frac{mM + 2m_1^2}{8}(h - 4M)^2 + 2mM^3 \tag{80}$$

Fluctuations around the minimum of the potential

$$h_{\mu\nu} = M\eta_{\mu\nu} + H_{\mu\nu} \tag{81}$$

behave as

$$V_2^W = m_1^2 H_{\mu\nu}^2 - \frac{mM + 2m_1^2}{8}H^2 + 2mM^3 \tag{82}$$

which again is positive semisefinite only for traceless $H^T_{\alpha\beta}$ or else for pure trace when $mM < 0$ as

$$V_2^W = m_1^2 {H^T_{\mu\nu}}^2 - \frac{mM}{8}H^2 + 2mM^3 \tag{83}$$

## 2.5 Global Weyl invariance

There is another symmetry that can be studied in this context, which is global Weyl invariance, that is, when the Weyl scaling factor is just a constant

$$\delta g_{\mu\nu} = \Omega^2 g_{\mu\nu}\,, \quad \partial_\mu\Omega = 0 \tag{84}$$

When we expand the metric around flat spacetime, $g_{\mu\nu} = \eta_{\mu\nu} + \kappa h_{\mu\nu}$, the linearized variation of the quantum fluctuation reads

$$\delta h_{\mu\nu} = 2\omega \left( \frac{1}{\kappa} \eta_{\mu\nu} + h_{\mu\nu} \right) \tag{85}$$

In the case of global (rigid) Weyl invariance where $\omega$ is constant, the variations of the operators quadratic in the fields have to be computed taking into account both terms in the above (that is, the linear order in the quantum field). If we just took the first piece, proportional to the Minkowski metric, all the variations computed in (21) and (44) would just be total derivatives, which have been neglected in this work.

In order to illustrate this point, let us take two simple actions. We know that the Einstein Hilbert action is not globally Weyl invariant in four dimensions,

$$-\frac{1}{2\kappa^2} \int d^4x\, \delta(\sqrt{g}R) = -\frac{1}{2\kappa^2} \left( 2\omega \int d^4x \sqrt{g}R \right) \tag{86}$$

On the other hand, we can take the simplest quadratic action which is invariant in four dimensions

$$\int d^4x\, \delta(\sqrt{g}R^2) = 0 \tag{87}$$

These equalities have to be true order by order in the perturbation of the metric. In this case, the quadratic order in the variation together with the linear order in the Weyl variation, combines with the third order of the perturbation in the action and the lowest order in the Weyl variation. These terms are going to be of order $O(\kappa^2)$ and have to match exactly the $O(\kappa^2)$ part of the rhs of the equation. Namely, The quadratice

$$\int d^4x \left[ (\sqrt{g}R)^{O(\kappa^2)}\Big|_{\delta h_{\mu\nu}=2\omega h_{\mu\nu}} + (\sqrt{g}R)^{O(\kappa^3)}\Big|_{\delta h_{\mu\nu}=2\omega\frac{\eta_{\mu\nu}}{\kappa}} \right] = \left( 2\omega \int d^4x\, (\sqrt{g}R)^{O(\kappa^2)} \right) \tag{88}$$

xpansion of the Einstein Hilbert action reads

$$-\frac{1}{2\kappa^2} \int d^4x\, (\sqrt{g}R)^{O(\kappa^2)} = -\frac{1}{2} \int d^4x \left\{ -\frac{1}{2}\partial_\mu h^{\mu\nu}\partial_\nu h + \frac{1}{2}\partial_\mu h^{\mu\nu}\partial_\rho h^\rho_\nu + \frac{1}{4}\partial_\mu h \partial^\mu h - \frac{1}{4}\partial_\rho h^{\mu\nu}\partial^\rho h_{\mu\nu} \right\} \tag{89}$$

It is straightforward to see that taking the variation $\delta h_{\mu\nu} = 2\omega h_{\mu\nu}$ we get

$$\int d^nx\, (\sqrt{g}R)^{O(\kappa^2)}\Big|_{\delta h_{\mu\nu}=2\omega h_{\mu\nu}} = 4\omega \int d^nx\, (\sqrt{g}R)^{O(\kappa^2)} \tag{90}$$

In order to compute the other piece contributing to $O(\kappa^2)$ we need the third order of the expansion of the Einstein Hilbert action which contains terms with three quantum fields $h_{\mu\nu}$ and two derivatives, which are shown in the Appendix 1. Once we have this expansion, we perform the Weyl variation $\delta h_{\mu\nu} = 2\omega \frac{\eta_{\mu\nu}}{\kappa}$ yielding

$$\int d^nx\, (\sqrt{g}R)^{O(\kappa^2)}\Big|_{\delta h_{\mu\nu}=2\omega\frac{\eta_{\mu\nu}}{\kappa}} = -2\omega \int d^nx\, (\sqrt{g}R)^{O(\kappa^2)} \tag{91}$$

Adding the two contributions

$$\int d^nx \left[ (\sqrt{g}R)^{O(\kappa^2)}\Big|_{\delta h_{\mu\nu}=2\omega h_{\mu\nu}} + (\sqrt{g}R)^{O(\kappa^3)}\Big|_{\delta h_{\mu\nu}=\frac{\eta_{\mu\nu}}{\kappa}} \right] = 2\omega \int d^nx\, (\sqrt{g}R)^{O(\kappa^2)} \tag{92}$$

which precisely yields the right hand side of (88) for $n = 4$. We can see that in $n = 2$ the Einstein Hilbert action is globally Weyl invariant (as well as locally). In fact this is basically the reason all two-dimensional metrics are conformally flat.

In the case of the quadratic action we have

$$\int d^4x \left[ (\sqrt{g}R^2)^{O(\kappa^2)}\Big|_{\delta h_{\mu\nu}=2\omega h_{\mu\nu}} + (\sqrt{g}R^2)^{O(\kappa^3)}\Big|_{\delta h_{\mu\nu}=2\omega\frac{\eta_{\mu\nu}}{\kappa}} \right] = 0 \tag{93}$$

where

$$\int d^4x\, (\sqrt{g}R^2)^{O(\kappa^2)} = \kappa^2 \int d^4x \left\{ \partial_\mu\partial_\nu h^{\mu\nu}\partial_\rho\partial_\sigma h^{\rho\sigma} - 2\partial_\mu\partial_\nu h^{\mu\nu}\Box h + \Box h \Box h \right\} \tag{94}$$

As before, taking the Weyl variation proportional to the quantum field is straightforward and it yields

$$\int d^nx (\sqrt{g}R^2)^{O(\kappa^2)}\Big|_{\delta h_{\mu\nu}=2\omega h_{\mu\nu}} = 4\omega \int d^nx (\sqrt{g}R^2)^{O(\kappa^2)} \tag{95}$$

For the other piece, we need the third order variation of the quadratic action which can be easily computed. After performing the Weyl transformation on the quantum field, $\delta h_{\mu\nu} = 2\omega \frac{\eta_{\mu\nu}}{\kappa}$, we get

$$\int d^nx\, (\sqrt{g}R^2)^{O(\kappa^3)}\Big|_{\delta h_{\mu\nu}=2\omega\frac{\eta_{\mu\nu}}{\kappa}} = -4\omega \int d^nx (\sqrt{g}R^2)^{O(\kappa^2)} \tag{96}$$

Summing both contributions,

$$\int d^4x \left[ (\sqrt{g}R^2)^{O(\kappa^2)}\Big|_{\delta h_{\mu\nu}=2\omega h_{\mu\nu}} + (\sqrt{g}R^2)^{O(\kappa^3)}\Big|_{\delta h_{\mu\nu}=2\omega\frac{\eta_{\mu\nu}}{\kappa}} \right] = 0 \tag{97}$$

## 3 Non-local extensions

There is a permanent temptation to avoid the Källen–Lehman spectral theorem (which states that the price to pay for having propagators that fall off at euclidean infinity faster than $k^{-2}$ is to have negative norm states) by considering non-local theories. For example in [20,21] a non-local generalization of the dimension 4 operators has been considered, namely

$$\begin{aligned}
\mathfrak{O}_1 &\equiv -\frac{1}{4}\, h_{\alpha\beta}\left[c_1\left(\frac{\Box}{M^2}\right)\Box\right] h^{\alpha\beta}\\
\mathfrak{O}_2 &\equiv \frac{1}{2} h_{\alpha\beta}\left[c_2\left(\frac{\Box}{M^2}\right)\partial^\alpha\partial^\gamma\right] h^\beta_\gamma\\
\mathfrak{O}_3 &\equiv -\frac{1}{2}\, h\left[c_3\left(\frac{\Box}{M^2}\right)\partial^\gamma\partial^\delta\right] h_{\gamma\delta}\\
\mathfrak{O}_4 &\equiv \frac{1}{4}\, h\left[c_4\left(\frac{\Box}{M^2}\right)\Box\right] h\\
\mathfrak{O}_5 &\equiv h_{\alpha\beta}\left[c_5\left(\frac{\Box}{M^2}\right)\frac{\partial^\alpha\partial^\beta\partial^\gamma\partial^\delta}{\Box}\right] h_{\gamma\delta}
\end{aligned} \tag{98}$$

so that the general Lagrangian of this type will be

$$L = \sum_{i=1}^{i=5}\mathfrak{O}_i \tag{99}$$

(where $c_i(z)$ are analytic functions with dimensionless argument). The five functions $c_i(z)$, $i = 1\ldots 5$ (which are assumed to include the corresponding coupling constants) characterize the theory. The constants put in front are such that the *LDiff* Fierz–Pauli theory corresponds to

$$\begin{aligned}
&\mathcal{O}_i \leftrightarrow \mathfrak{O}_i \quad (i = 1\ldots 5)\\
&g_1 = g_2 = g_3 = g_4 = 1\\
&g_5 = 0
\end{aligned} \tag{100}$$

The correspondence with the dimension 6 operators in (38) is as follows

$$\begin{aligned}
&c_i(z) = z \quad (i = 1\cdots 5)\\
&\mathcal{O}_1 \leftrightarrow M^2\,\mathfrak{O}_5\\
&\mathcal{O}_2 \leftrightarrow -2M^2\,\mathfrak{O}_3\\
&\mathcal{O}_3 \leftrightarrow 2M^2\,\mathfrak{O}_2\\
&\mathcal{O}_4 \leftrightarrow -4M^2\,\mathfrak{O}_1\\
&\mathcal{O}_5 \leftrightarrow 4M^2\,\mathfrak{O}_4
\end{aligned} \tag{101}$$

i.e. $g_1 = M^2 c_5(z)$, $g_2 = -2M^2 c_3(z)$, $g_3 = 2M^2 c_2(z)$, $g_4 = -4M^2 c_1(z)$ and $g_5 = 4M^2 c_4(z)$, in such a way that the conditions for *LDiff* invariance now translate into

$$\begin{aligned}
&c_2(z) - c_3(z) + c_5(z) = 0\\
&4c_4(z) - c_3(z) = 0\\
&c_2(z) - 4c_1(z) = 0
\end{aligned} \tag{102}$$

It is claimed in [20] that the theory is *ghost-free* provided that

$$\begin{aligned}
&c_1(z) = c_2(z)\\
&c_3(z) = c_4(z)\\
&c_5(z) = 2\,(c_3(z) - c_2(z))
\end{aligned} \tag{103}$$

and the function $c_1(z)$ is chosen as an entire function, such as

$$c_1\,(z) \equiv e^{-z} \tag{104}$$

Note that both constraints, (102) and (103) are different and incompatible.

It is well-known, however, that non-local theories suffer from unitarity and causality problems, some of those can be sometimes hidden uunder the rug of experimental precision of the measurements [22]. However, in order to do that, the theory needs to be *quasi-local*, which means that the corresponding function has got to have bounded support, which seems to contradict other conditions. It is not clear at all that a consistent solution exists.

Outstanding problems in this respect according to [23] are first and foremost, the fact that the presence of the exponential damping factor in the propagator prevents analytic continuation from the riemannian theory to the lorentzian one, owing to the essential singularities in the complex energy plane. It must be stressed, however that such an analytic continuation is problematic in *any* theory involving the gravitational field. Another argument is that none of the theories proposed so far complies with reflexion-positivity, which is believed to be an essential requirement in order to get a consistent quantum field theory.

## 4 Conclusions

In this paper we have presented a complete analysis of operators up to (mass) dimension 6 describing spin 2 theories (e.g. weak field limit of theories linear and quadratic in the curvature), analyzing with some care the conditions for the theory to be (transverse) diffeomorphism invariant, scale invariant, conformal invariant and Weyl invariant. We have also identified a possible non-linear completion of those Lagrangians.

Conformality on shell is attained for any combination of the constants appearing in the dimension 4 and dimension 6 cases. The trace of the energy-momentum tensor is a total derivative, and besides the virial current for specific Lagrangians is also the derivative of a two-index tensor, leading to improved forms of the corresponding energy-momentum tensors.

On the other hand, Weyl invariance instead does impose constraints on the coupling constants. Our main conclusion is to confirm [16–18,24] that Weyl invariance and conformal invariance are independent symmetries: not every Weyl invariant theory is conformal invariant in the weak field limit and conversely, not every conformal invariant theory is Weyl invariant in spite of the fact that it is always invariant under global such Weyl transformations. To illustrate the first part of this statement, let us take for example the following WTDiff invariant theory

$$\int d^4x \left(-\frac{1}{2\kappa^2}R[g^{-1/4}g_{\mu\nu}] + R^2[g^{-1/4}g_{\mu\nu}]\right) \tag{105}$$

where the precise form of these terms after permoning the transformation of the metric can be found in (109) and (120). The weak field expansion of this theory will contain, at quadratic order in the perturbation, dimension 4 operators and dimension 6 operators coming from the linear and quadratic (in curvature) pieces respectively. Theories combining operators of different dimension are not scale invariant, as pointed out in the example in (58).

The analysis of dimension 5 and dimension 6 operators does not bring anything new with respect to diffeomorphism invariant theories, as expected. However, we have given expressions for the most general Lorentz and Weyl invariant Lagrangians up to dimension 5 and dimension 6 operators, and we can clearly see that those theories contain an increasing number of arbitrary constants. We have also discussed global Weyl invariance and it is clear that this symmetry is less restrictive than the local one. An analysis of the interaction terms has been done. It can be seen that potentials with diffeomorphism and weyl invariance can be constructive iteratively, for every orther of the perturbative expansion.

To end up, let us stress that the conditions that are argued to be neccessary for a ghost free non-local theory [20] are not compatible with the ones stemming from diffeomorphism invariance.

We finally point out that our results prove that *any* Lorentz invariant Lagrangian for spin 2 particle up to quadratic order in the field is conformal invariant.

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## A Weak-field limit of geometric scalars

We are interested in the expansion of the geometric invariants when we expand the metric around Minkowski spacetime

$$g_{\mu\nu} = \eta_{\mu\nu} + \kappa h_{\mu\nu} \tag{106}$$

If we take the limit of linear theories of gravity up to quadratic order in the fluctuations we have

$$\begin{aligned}
&-\frac{1}{\kappa^2}\sqrt{g}\,R = \partial^\alpha\partial^\beta h_{\alpha\beta} - \Box h \\
&+ h^{\alpha\beta}\left(\frac{1}{4}\Box\left(\eta_{\alpha\beta}\eta_{\mu\nu} - \eta_{\alpha\mu}\eta_{\beta\nu}\right)\right. \\
&\left.+\frac{1}{2}\eta_{\alpha\mu}\partial_\beta\partial_\nu - \frac{1}{2}\eta_{\mu\nu}\partial_\alpha\partial_\beta\right) h^{\mu\nu} + O(h^3)
\end{aligned} \tag{107}$$

When considering *TDiff* scalars one can also have terms of the type

$$\frac{(\nabla g)^2}{g^2} = \kappa^2(\partial h)^2 + O(h^3) \tag{108}$$

The existence of this operator gives one extra freedom. To build the action which is *WTDiff* invariant we perform a Weyl transformation in the usual Einstein Hilbert action taking $\tilde{g}_{\mu\nu} = g^{-1/4}g_{\mu\nu}$ so that

$$S_{WTDiff} = -\frac{1}{\kappa^2}\int d^4x\, g^{1/4}\left(R + \frac{3}{32}\frac{(\nabla g)^2}{g^2}\right) \tag{109}$$

Expanding it up to quadratic order in the fluctuations and writting it in terms of the four dimensional operators (11) we get

$$S_{WTDiff} = \int d^4x \left(\mathcal{D}_1 + \mathcal{D}_2 + \frac{1}{2}\mathcal{D}_3 + \frac{3}{8}\mathcal{D}_4\right) \tag{110}$$

On the other hand, taking into account that

$$\begin{aligned}
R_{\mu\nu\alpha\beta} = &\frac{\kappa}{2}\left(-\partial_\alpha\partial_\mu h_{\nu\beta} + \partial_\alpha\partial_\nu h_{\mu\beta} + \partial_\beta\partial_\mu h_{\alpha\nu} - \partial_\nu\partial_\beta h_{\mu\alpha}\right) \\
&+ O(h^2)
\end{aligned} \tag{111}$$

we learn that

$$R^2_{\mu\nu\alpha\beta} = \frac{\kappa^2}{4}\left\{4\mathcal{O}_1 - 8\mathcal{O}_3 + 4\mathcal{O}_4\right\} + O(h^3) \tag{112}$$

For the Ricci tensor we have

$$R_{\nu\beta} = \frac{\kappa}{2}\left(-\Box h_{\nu\beta} + \partial_\lambda\partial_\nu h^\lambda_\beta + \partial_\lambda\partial_\beta h^\lambda_\nu - \partial_\beta\partial_\nu h\right) + O(h^2) \tag{113}$$

so that

$$R^2_{\alpha\beta} = \frac{\kappa^2}{4}\left(2\mathcal{O}_1 - 2\mathcal{O}_2 - 2\mathcal{O}_3 + \mathcal{O}_4 + \mathcal{O}_5\right) + O(h^3) \tag{114}$$

Finally the expansion of the Ricci scalar reads

$$R = \kappa\left(\partial^\alpha\partial^\beta h_{\alpha\beta} - \Box h\right) + O(h^2) \tag{115}$$

and it follows that

$$R^2 = \kappa^2\left(\mathcal{O}_1 - 2\mathcal{O}_2 + \mathcal{O}_5\right) + O(h^3) \tag{116}$$

A useful relationship is given by

$$\begin{aligned}\alpha R^2_{\mu\nu\rho\sigma} + \beta R^2_{\mu\nu} + \gamma R^2 = \kappa^2\Bigg[&\left(\alpha + \frac{\beta}{2} + \gamma\right)\mathcal{O}_1 \\ &- \left(\frac{\beta}{2} + 2\gamma\right)\mathcal{O}_2 - \left(2\alpha + \frac{\beta}{2}\right)\mathcal{O}_3 \\ &+ \left(\alpha + \frac{\beta}{4}\right)\mathcal{O}_4 + \left(\frac{\beta}{4} + \gamma\right)\mathcal{O}_5\Bigg] \\ &+ O(h^3)\end{aligned} \tag{117}$$

Using this it can easily be seen that the Euler density vanishes at this level of the expansion, whereas the Weyl squared tensor decomposes into

$$\begin{aligned}W_4 &= R^2_{\mu\nu\rho\sigma} - 2R^2_{\mu\nu} + \frac{1}{3}R^2 \\ &= \frac{\kappa^2}{6}\left\{2\,\mathcal{O}_1 + 2\,\mathcal{O}_2 - 6\,\mathcal{O}_3 + 3\,\mathcal{O}_4 - \mathcal{O}_5\right\}\end{aligned} \tag{118}$$

If we again consider quadratic theories which are *TDiff* invariant, one would have terms of the type must add

$$\frac{(\Box g)^2}{g^2} = (\Box h)^2 \tag{119}$$

again, this yields in this case one extra freedom.

We can make the same analysis for the quadratic invariants but when considering actions that are *WTDiff* invariant. This can be achieved by making a Weyl transformation $\tilde{g}_{\mu\nu} = \Omega^2 g_{\mu\nu}$ on the usual quadratic action (117) and then taking $\Omega^2 = g^{-1/n}$. For a general $\Omega$ we have

$$\begin{aligned}&\alpha\tilde{R}_{\mu\nu\rho\sigma}\tilde{R}^{\mu\nu\rho\sigma} + \beta\tilde{R}_{\mu\nu}\tilde{R}^{\mu\nu} + \gamma\tilde{R}^2 \\ &= \Omega^{-4}\left(\alpha R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma} + \beta R_{\mu\nu}R^{\mu\nu} + \gamma R^2\right) \\ &\quad + \Omega^{-5}\left(-8\alpha - 2(n-2)\beta\right)R_{\mu\nu}\nabla^\mu\nabla^\nu\Omega \\ &\quad + \Omega^{-6}\left(4\alpha + (3n-4)\beta + 4(n-1)^2\gamma\right)(\Box\Omega)^2 \\ &\quad + \Omega^{-6}\left(4(n-2)\alpha + (n-2)^2\beta\right)\nabla_\mu\nabla_\nu\Omega\nabla^\mu\nabla^\nu\Omega \\ &\quad + \Omega^{-6}\left(-4\alpha - 2(n-3)\beta\right. \\ &\quad \left.-2(n-1)(n-4)\gamma\right)R\nabla_\mu\Omega\nabla^\mu\Omega \\ &\quad + \Omega^{-6}\left(16\alpha + 4(n-2)\beta\right)R_{\mu\nu}\nabla^\mu\Omega\nabla^\nu\Omega \\ &\quad \times\Omega^{-7}\left(8(n-3)\alpha + 4(n^2-5n+5)\beta\right. \\ &\quad \left.+4(n-1)^2(n-4)\gamma\right)\Box\Omega\nabla_\mu\Omega\nabla^\mu\Omega \\ &\quad + \Omega^{-7}\left(-16(n-2)\alpha - 4(n-2)^2\beta\right)\nabla_\mu\nabla_\nu\Omega\nabla^\mu\Omega\nabla^\nu\Omega \\ &\quad + \Omega^{-5}\left(-2\beta - 4(n-1)\gamma\right)R\Box\Omega \\ &\quad \times\Omega^{-8}\left(2n(n-1)\alpha + (n-1)(n^2-5n+8)\beta\right. \\ &\quad \left.+(n-1)^2(n-4)^2\gamma\right)(\nabla^\mu\Omega\nabla_\mu\Omega)^2\end{aligned} \tag{120}$$

Using that $\Omega = g^{-1/2n}$ (in order to have *WTDiff*) and keeping dimension six operators with four derivatives and two metric fluctuations, we get for $n = 4$

$$\begin{aligned}&\alpha\tilde{R}_{\mu\nu\rho\sigma}\tilde{R}^{\mu\nu\rho\sigma} + \beta\tilde{R}_{\mu\nu}\tilde{R}^{\mu\nu} + \gamma\tilde{R}^2 \\ &= \kappa^2\Bigg[\left(\alpha + \frac{\beta}{2} + \gamma\right)\mathcal{O}_1 + \frac{1}{2}(\alpha - \gamma)\mathcal{O}_2 \\ &\quad - \left(2\alpha + \frac{\beta}{2}\right)\mathcal{O}_3 + \left(\alpha + \frac{\beta}{4}\right)\mathcal{O}_4 \\ &\quad - \left(\frac{20\alpha + 4\beta - 4\gamma}{32}\right)\mathcal{O}_5\Bigg] + O(h^3)\end{aligned} \tag{121}$$

These are the most general theories that possess *LWTDiff*.

## B Dimension 5 and dimension 6 operators with two-derivatives

This set of operators does not appear in the expansion of terms quadratic in the Riemann tensor, although they appear in the expansion of the Einstein–Hilbert Lagrangian.

For dimension 5, there are 14 independendent operators (up to total derivatives), that form a basis to expand the most general Lorentz invariant Lagrangian containing such opeators

$$
\begin{aligned}
\mathcal{N}_1 &= h^{\mu\nu}\partial_\nu h^\lambda_\mu \partial_\lambda h & \mathcal{N}_8 &= h^{\mu\nu} h \partial_\nu \partial_\lambda h^\lambda_\mu \\
\mathcal{N}_2 &= h^{\mu\nu} h^\lambda_\mu \partial_\lambda \partial_\nu h & \mathcal{N}_9 &= h^{\mu\nu} h_{\mu\nu} \partial_\rho \partial_\lambda h^{\rho\lambda} \\
\mathcal{N}_3 &= h^{\mu\nu} h \partial_\mu \partial_\nu h & \mathcal{N}_{10} &= h^2 \partial_\rho \partial_\lambda h^{\rho\lambda} \\
\mathcal{N}_4 &= h^{\mu\nu} \partial_\lambda h^\lambda_\mu \partial_\rho h^\rho_\nu & \mathcal{N}_{11} &= h^{\mu\nu} h^\lambda_\mu \Box h_{\nu\lambda} \\
\mathcal{N}_5 &= h^{\mu\nu} h^{\rho\lambda} \partial_\rho \partial_\nu h_{\mu\lambda} & \mathcal{N}_{12} &= h^{\mu\nu} h \Box h_{\mu\nu} \\
\mathcal{N}_6 &= h^{\mu\nu} h^{\rho\lambda} \partial_\rho \partial_\lambda h_{\mu\nu} & \mathcal{N}_{13} &= h^{\mu\nu} h_{\mu\nu} \Box h \\
\mathcal{N}_7 &= h^{\mu\nu} h^\lambda_\mu \partial_\rho \partial_\lambda h^\rho_\nu & \mathcal{N}_{14} &= h^2 \Box h
\end{aligned}
$$

For dimension 6, there are 38 independent operators (up to total derivatives), that form a basis to expand the most general Lorentz invariant Lagrangian containing such opeators

$$
\begin{aligned}
\mathcal{K}_1 &= h_{\alpha\beta} h^\lambda_\nu h_{\mu\lambda} \partial^\mu \partial^\nu h^{\alpha\beta} & \mathcal{K}_{20} &= h_{\alpha\beta} h^{\alpha\beta} h^\lambda_\nu \partial^\mu \partial^\nu h_{\mu\lambda} \\
\mathcal{K}_2 &= h h^\lambda_\nu \partial^\nu h \partial^\mu h_{\mu\lambda} & \mathcal{K}_{21} &= h^2 h^\lambda_\nu \partial^\mu \partial^\nu h_{\mu\lambda} \\
\mathcal{K}_3 &= h h^\lambda_\nu \partial^\mu h \partial^\nu h_{\mu\lambda} & \mathcal{K}_{22} &= h_{\alpha\beta} h^{\beta\rho} h^\alpha_\rho \partial^\mu \partial^\nu h_{\mu\nu} \\
\mathcal{K}_4 &= h_{\nu\lambda} h^{\rho\lambda} h_{\mu\rho} \partial^\mu \partial^\nu h & \mathcal{K}_{23} &= h h_{\alpha\beta} h^{\alpha\beta} \partial^\mu \partial^\nu h_{\mu\nu} \\
\mathcal{K}_5 &= h h^\lambda_\nu h_{\mu\lambda} \partial^\mu \partial^\nu h & \mathcal{K}_{24} &= h^3 \partial^\mu \partial^\nu h_{\mu\nu} \\
\mathcal{K}_6 &= h h^{\lambda\alpha} h^\beta_\lambda \Box h_{\alpha\beta} & \mathcal{K}_{25} &= h_{\alpha\lambda} h^{\lambda\rho} h_{\rho\sigma} \Box h^{\sigma\alpha} \\
\mathcal{K}_7 &= h^2 h_{\mu\nu} \partial^\mu \partial^\nu h & \mathcal{K}_{26} &= h_{\mu\nu} h_{\alpha\beta} h^{\alpha\beta} \partial^\mu \partial^\nu h \\
\mathcal{K}_8 &= h_{\alpha\beta} h^{\alpha\beta} \partial_\mu h \partial^\mu h & \mathcal{K}_{27} &= h_{\alpha\beta} h^{\alpha\beta} h_{\rho\sigma} \Box h^{\rho\sigma} \\
\mathcal{K}_9 &= h_{\alpha\beta} h^{\lambda\beta} \partial^\nu h^\alpha_\nu \partial^\mu h_{\mu\lambda} & \mathcal{K}_{28} &= h^2 h_{\alpha\beta} \Box h^{\alpha\beta} \\
\mathcal{K}_{10} &= h h^{\rho\lambda} \partial^\nu h_{\nu\lambda} \partial^\mu h_{\mu\rho} & \mathcal{K}_{29} &= h_{\alpha\beta} h^{\lambda\alpha} h^\beta_\lambda \Box h \\
\mathcal{K}_{11} &= h^\alpha_\nu h^{\lambda\beta} \partial^\mu h_{\mu\lambda} \partial^\nu h_{\alpha\beta} & \mathcal{K}_{30} &= h_{\alpha\beta} h^{\alpha\beta} h \Box h \\
\mathcal{K}_{12} &= h_{\alpha\beta} h^{\alpha\beta} \partial^\nu h^\lambda_\nu \partial^\mu h_{\mu\lambda} & \mathcal{K}_{31} &= h^3 \Box h \\
\mathcal{K}_{13} &= h_{\alpha\beta} h^{\alpha\beta} \partial^\mu h \partial^\nu h_{\mu\nu} & \mathcal{K}_{32} &= h_{\mu\lambda} h^{\lambda\beta} \partial^\mu h^\alpha_\nu \partial^\nu h_{\alpha\beta} \\
\mathcal{K}_{14} &= h^{\lambda\beta} h^\alpha_\nu h_{\mu\lambda} \partial^\mu \partial^\nu h_{\alpha\beta} & \mathcal{K}_{33} &= h h^{\rho\lambda} \partial^\mu h_{\nu\lambda} \partial^\nu h_{\mu\rho} \\
\mathcal{K}_{15} &= h h_{\nu\lambda} h_{\mu\rho} \partial^\mu \partial^\nu h^{\rho\lambda} & \mathcal{K}_{34} &= h_{\alpha\beta} h^{\lambda\beta} \partial^\mu h^\alpha_\nu \partial^\nu h_{\mu\lambda} \\
\mathcal{K}_{16} &= h_{\mu\nu} h^{\beta\rho} h^\alpha_\rho \partial^\mu \partial^\nu h_{\alpha\beta} & \mathcal{K}_{35} &= h h_{\nu\lambda} \partial^\mu h^{\rho\lambda} \partial^\nu h_{\mu\rho} \\
\mathcal{K}_{17} &= h h_{\mu\nu} h^{\alpha\beta} \partial^\mu \partial^\nu h_{\alpha\beta} & \mathcal{K}_{36} &= h_{\alpha\lambda} h^{\lambda\rho} \partial_\mu h_{\rho\sigma} \partial^\mu h^{\sigma\alpha} \\
\mathcal{K}_{18} &= h_{\alpha\beta} h^{\lambda\beta} h^\alpha_\nu \partial^\mu \partial^\nu h_{\mu\lambda} & \mathcal{K}_{37} &= h_{\alpha\beta} h^{\alpha\beta} \partial^\mu h^\lambda_\nu \partial^\nu h_{\mu\lambda} \\
\mathcal{K}_{19} &= h h_{\nu\lambda} h^{\rho\lambda} \partial^\mu \partial^\nu h_{\mu\rho} & \mathcal{K}_{38} &= h_{\alpha\beta} h^{\alpha\beta} \partial_\mu h_{\rho\sigma} \partial^\mu h^{\rho\sigma}
\end{aligned}
$$

Finally, taking all the contributions mentioned in the text, the most general Weyl invariant Lagrangian up to dimension 6 operators reads

$$
\begin{aligned}
\mathcal{L}_{W_{6D}} = \mathcal{L}_{W_{5D}} + \kappa^2 \{ & b_1\mathcal{K}_1 + b_2\mathcal{K}_2 + b_3\mathcal{K}_3 + b_4\mathcal{K}_4 \\
& + b_5\mathcal{K}_5 + b_6\mathcal{K}_6 + b_7\mathcal{K}_7 + b_8\mathcal{K}_8 + b_9\mathcal{K}_9 \\
& + b_{10}\mathcal{K}_{10} + b_{11}\mathcal{K}_{11} + \frac{1}{2}(-3a_2 - b_1 - b_{10} \\
& + 4b_2 - 3b_4 - 4b_5)\,\mathcal{K}_{12} + \frac{1}{8}(2a_1 - 4a_2 - 8a_3 \\
& - 4a_9 - 8b_2 - 8b_3 + c_2)\,\mathcal{K}_{13} + b_{14}\mathcal{K}_{14} \\
& + \frac{1}{4}(-12a_1 - 3a_5 - 4b_{10} - 2b_{14} - 2b_9)\,\mathcal{K}_{15} \\
& + \left(-\frac{3}{2}a_1 + 3a_2 + 12a_3 - b_1 + 4b_5\right. \\
& \left. + 4b_6 + 16b_7 - \frac{3}{8}c_2\right)\mathcal{K}_{16} \\
& + (-3a_3 - 2b_5 - 2b_6 - 8b_7)\,\mathcal{K}_{17} \\
& + (8a_1 + 4a_2 + 2a_5 + 4b_{10} + b_{11} - b_{14} - 4b_4 \\
& + 2b_9)\,\mathcal{K}_{18} \\
& + \frac{1}{4}(20a_1 - 16a_2 + 3a_5 - 8b_1 + 4b_{10} \\
& + 2b_{14} - 12b_4 - 32b_5 + 2b_9 + c_2)\,\mathcal{K}_{19} \\
& + \left(\frac{3}{2}a_1 - 3a_2 - 12a_3 - a_9\right. \\
& \left. + b_1 - 4b_5 - 8b_6 - 16b_7 + \frac{3}{8}c_2\right)\mathcal{K}_{20} \\
& + \frac{1}{16}(-2a_1 + 4a_2 + 48a_3 + 4a_9 \\
& + 8b_2 + 8b_3 + 32b_5 + 32b_6 + 64b_7 - c_2) \\
& \times\mathcal{K}_{21} + b_{22}\mathcal{K}_{22} + \left(-\frac{3}{8}a_1 + \frac{3}{4}a_2 + 3a_3\right. \\
& \left. - \frac{1}{2}a_9 - \frac{1}{2}b_1 - \frac{3}{4}b_{22} + b_5 + 2b_6 + 4b_7 - \frac{3}{32}c_2\right)\mathcal{K}_{23} \\
& + \frac{1}{192}(12a_1 - 24a_2 - 160a_3 + 16a_9 \\
& + 8b_1 - 16b_2 + 24b_{22} - 16b_3 - 64b_5 - 96b_6 \\
& - 192b_7 + 3c_2)\,\mathcal{K}_{24} + b_{25}\mathcal{K}_{25} \\
& + b_{26}\mathcal{K}_{26} + b_{27}\mathcal{K}_{27} \\
& + \frac{1}{256}(16a_1 + 48a_{11} - 32a_2 \\
& + 32a_3 + 16a_9 + 16b_2 - 64b_{26} - 64b_{27} + 16b_3 + 64b_5 \\
& + 64b_6 + 256b_7 - 128b_8 - 10c_1 + 3c_2)\,\mathcal{K}_{28} \\
& + \frac{1}{96}(-12a_1 + 16a_{11} + 24a_2 + 96a_3 - 8a_1 \\
& - 24b_{22} + 32b_{26} + 32b_5 \\
& + 32b_6 + 128b_7 - 3c_2)\,\mathcal{K}_{29} \\
& + \frac{1}{256}(32a_1 + 16a_{11} - 64a_2 - 224a_3 \\
& + 32a_9 + 16b_1 - 16b_2 \\
& + 48b_{22} - 64b_{26} - 64b_{27} - 16b_3 - 64b_5 - 128b_6 \\
& - 256b_7 + 128b_8 - 2c_1 + 9c_2)\,\mathcal{K}_{30} \\
& + \frac{1}{3072}(-104a_1 - 176a_{11} + 208a_2 \\
& + 608a_3 - 128a_9 - 32b_1 + 16b_2 - 96b_{22} + 192b_{26} \\
& + 192b_{27} + 16b_3 + 64b_5 + 192b_6 \\
& + 128b_8 + 38c_1 - 21c_2)\,\mathcal{K}_{31} + (12a_1 + 4a_2 + 3a_5 \\
& + 8b_{10} + b_{11} + 2b_9)\,\mathcal{K}_{32} \\
& + \left(2a_1 - 6a_2 - 2b_1 - 2b_{10} - 6b_4 - 8b_5 + \frac{1}{4}c_2\right)\mathcal{K}_{33}
\end{aligned}
$$

$$
\begin{aligned}
&+ (20a_1 - 8a_2 + 3a_5 - 4b_1 + 4b_{10} + b_{11} \\
&- 12b_{14} - 16b_5 + 3b_9 + \frac{1}{2}c_2\Big) \mathcal{K}_{34} \\
&+ (a_1 + b_{10}) \mathcal{K}_{35} \\
&+ \left(-\frac{3}{2}a_1 + 3a_{11} + 3a_2 + 12a_3 - b_1\right. \\
&\left.+ 3b_{25} + 4b_{26} + 4b_5 + 4b_6 + 16b_7 - \frac{3}{8}c_2\right) \mathcal{K}_{36} \\
&+ \frac{1}{8}(12a_2 + 4b_1 + 4b_{10} + 16b_3 \\
&+ 12b_4 + 16b_5 - c_2) \mathcal{K}_{37} \\
&+ \frac{1}{64}(-8a_1 + 16a_2 + 32a_3 + 16a_9 \\
&+ 16b_2 + 16b_3 - 128b_8 + 2c_1 - 3c_2) \mathcal{K}_{38}\} + \mathcal{L}^W_{6\partial\partial\partial\partial}
\end{aligned} \tag{122}
$$

Let us mention that besides the 8 independent constants of $\mathcal{L}_{W_{5D}}$ (69), the 6 dimensional piece constains 16 new independent constants $b_1, b_2, \ldots, b_{10}, b_{11}, b_{14}, b_{22}, b_{25}, b_{26}, b_{27}$. Moreover, we have the 3 independent constants coming from the six dimensional piece containing 4 derivative operators $\mathcal{L}^W_{6\partial\partial\partial\partial}$ (45).

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# 3

# NON-MINIMAL TINGES OF UNIMODULAR GRAVITY

This chapter contains the article

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# Non-minimal tinges of Unimodular Gravity

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Abstract: Unimodular Gravity is normally assumed to be equivalent to General Relativity for all matters but the character of the Cosmological Constant. Here we discuss this equivalence in the presence of a non-minimally coupled scalar field. We show that when we consider gravitation to be dynamical in a QFT sense, quantum corrections can distinguish both theories if the non-minimal coupling is non-vanishing. In order to show this, we construct a path integral formulation of Unimodular Gravity, fixing the complicated gauge invariance of the theory and computing all one-loop divergences. We find a combination of the couplings in the Lagrangian to which we can assign a physical meaning. It tells whether quantum gravitational phenomena can be ignored or not at a given energy scale. Its renormalization group flow differs depending on if it is computed in General Relativity or Unimodular Gravity.

Keywords: BRST Quantization, Renormalization Group, Classical Theories of Gravity, Models of Quantum Gravity

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## Contents

## 1 Introduction

One of the most everlasting problems in theoretical physics is the Cosmological Constant problem [1, 2] — the question of why our Universe is currently evolving according to the presence of a very small cosmological constant, corresponding to $M_{\mathrm{P}}^2\Lambda \sim 10^{-46}\,\mathrm{GeV}^4$, where $M_{\mathrm{P}} \sim 10^{19}\,\mathrm{GeV}$ is the Planck mass. Being precise, this problem has actually two sides. The first belongs to the realm of model building and aims to describe which concrete physical mechanism leads to the observed value of $\Lambda$. Many attempts have been done in this direction during the last decades (see e.g. [3, 4] and references therein), but so far we do not have any clear experimental signature that favours one or another.

The second facet of the problem is of more fundamental theoretical nature. Even if a sensible mechanism to produce the current value of $\Lambda$ at the classical level is described, it still remains to explain why this value should be stable under radiative corrections. In a Quantum Field Theory (QFT), all dimensionful parameters receive corrections from loops of interacting particles that shift the classical value of parameters. This occurs even if the particles running in the loops do not manifest themselves in the low energy spectrum of the theory. In particular, if we think of an Effective Field Theory (EFT) setting, the cosmological constant receives contributions proportional to the cut-off of the theory, which encodes the ignorance about the UV degrees of freedom [5]. This means that in a gravitational theory described at low energies by General Relativity (GR), we expect corrections of the form $\delta(M_{\mathrm{P}}^2\Lambda) \sim M_{\mathrm{P}}^4$, which are clearly much larger than the observed value of the cosmological constant. Although this *hierarchy problem* can be solved by the inclusion of a very fine-tuned counter-term, it raises a question about the sensitivity of low energy observables to high energy degrees of freedom and thus poses a problem for the viability of the EFT, where separation of scales is critical.

A possible way out of this issue is to modify the infra-red (IR) limit of the gravitational theory, so that the behavior of the cosmological constant gets replaced by a different dynamical avatar. This is the direction of research followed by massive gravity [6, 7], where the graviton mass regulates the IR limit of GR; and of the plethora of (Beyond-) Hordensky/DHOST models [8], where the dynamics of an extra scalar degree of freedom replaces the need for Dark Energy. However, the viability of both approaches has been recently questioned from different directions, and the allowed parameter space is shrinking quickly [9–14].

A particularly simple modification of GR that has attracted scattered attention during the last decades, although it is almost as old as GR itself,[1] is Unimodular Gravity (UG) [16–18], formulated by appending the Einstein-Hilbert action with a condition of constant determinant for the metric tensor. Since the variation of this determinant is proportional to the trace of the equations of motion (eom), this effectively suppresses the trace degree of freedom of the metric. The resulting eom of UG are then the traceless part of Einstein

[1]The equations of motion of UG appear for the first time ever in a 1919 paper by Einstein himself [15]. However, that work was not related to the cosmological constant but instead to the structure of point particles within GR.

equations [19–21]

$$R_{\mu\nu} - \frac{1}{4}g_{\mu\nu}R = G\left(T_{\mu\nu} - \frac{1}{4}Tg_{\mu\nu}\right), \tag{1.1}$$

where $T_{\mu\nu}$ is the energy-momentum tensor of matter coupled to gravity. Any possible cosmological constant in the Lagrangian, or its radiative corrections, would be contained in the trace of Einstein equations and therefore they drop from the eom of UG.

Although this seems to signal a problem to reproduce well-known cosmological physics, it is not the case. The standard classical dynamics for gravity is recovered by the use of Bianchi identities, which are always true for a Riemannian manifold and imply, when taken together with (1.1)

$$\nabla^\mu R_{\mu\nu} = \frac{1}{2}\nabla_\nu R \to R + GT = 4\mathcal{C}, \tag{1.2}$$

after integration in a compact manifold without boundaries, and provided that $\nabla_\mu T^{\mu\nu} = 0$. Here $\mathcal{C}$ is an integration constant. If we now eliminate $T$ from (1.1) by means of (1.2) we recover the full set of Einstein equations

$$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} + \mathcal{C}g_{\mu\nu} = GT_{\mu\nu}, \tag{1.3}$$

where $\mathcal{C}$ takes the role of a cosmological constant. However, here $\mathcal{C}$ is an integration constant instead of a coupling in the Lagrangian and therefore it does not receive radiative corrections [22], effectively solving the second facet of the cosmological constant that we have discussed. The value of the cosmological constant is not given by vacuum energy but instead it is fixed by initial conditions when solving the eom. This mechanism has been explored in the context of inflation in [23], while in [24] it was exploited together with scale invariance to produce the complete thermal history of the Universe.

The fact that UG reproduces Einstein equations has led to a wide discussion of whether it is fully equivalent to GR — apart form the discussed role of the cosmological constant– or if there is some physical phenomenon that can serve to distinguish both theories ([25–27] and references therein). From the previous discussion, it should be clear that (semi-)classically[2] there cannot be any difference between both theories. The equations of motion are the same and the number of degrees of freedom propagated by UG matches those of GR — a single massless graviton [28, 29]. The same must be true for any tree-level computation.

However, things are more subtle when dealing with the quantum nature of the gravitational field. In order to properly formulate a path integral for UG we need to resolve the constraint $|g| = 1$ in an explicit way. As a consequence, and although on-shell states match those of GR, off-shell states are different, owing to a different gauge group. While the graviton fluctuation of GR is traceless only on the mass-shell, the one propagated by UG has a vanishing trace even for off-shell states. This means that loops with running gravitons are potentially different in both theories.

[2]By semi-classically here we mean quantum matter fields, represented by a quantum corrected energy-momentum tensor $\langle T_{\mu\nu}\rangle$, coupled to classical gravity by replacing $T_{\mu\nu}$ by $\langle T_{\mu\nu}\rangle$ in the equations of motion.



JHEP09(2020)041

Quantum phenomena in UG have been previosuly studied from several directions of research [25, 27, 30–34]. Of particular interest are [22, 35], where the one-loop effective action of UG is obtained by using two different approaches. Although the numerical results of both works differ, something which may be a gauge artefact, their physical conclusion is the same — the cosmological constant does not renormalize and UG is one-loop finite. However, since in both works the theory is taken in vacuum, it is not possible to have access to any physical observable in order to compare the dynamics of UG with that of GR. This would require to couple another field to gravitation and account for its backreaction onto the geometry.

In this work we tackle this last point by considering UG together with the action for a non-minimally coupled scalar field. We will thus formulate a perturbative QFT expansion for UG coupled to matter, clarifying the issue of fixing the complicated gauge freedom of the theory and deriving all the elements required to implement perturbation theory around flat space. We will afterwards use these tools to compute the renormalization group (RG) flow of the different coupling constants in the action, at the one-loop level. This will allow us to identify a physically relevant essential coupling and compute its $\beta$-function, that we will be able to compare with the equivalent one as computed in GR.

This paper is organized as follows. First, in section 2 we will describe Unimodular Gravity in more detail, together with the matter action that we consider. In order to quantize the system we will use the Background Field Method, described in section 3 together with the concept of Weyl geometry and the BRST invariance of the gauge fixed action. We will later compute correlation functions at the one-loop order by expanding around flat space, as described in sections 4 and 5, where we compute divergences in the $\overline{MS}$ scheme. Finally, we will derive the $\beta$-functions and anomalous dimensions of all the couplings in the one-loop effective action in section 6, comparing our results with the general relativistic ones in 7. We will draw our conclusions in section 8. For completeness, we add two small appendices describing the computation in GR — appendix A — and the discussion of divergences in UG in vacuum, in appendix B.

## 2 Unimodular Gravity

We *define* UG by adding a condition of constant determinant to the Einstein-Hilbert action

$$S = -\frac{1}{2G}\int d^4x\sqrt{|g|}\; R + S_{\text{matter}}, \quad |g| = \varepsilon, \tag{2.1}$$

where $\varepsilon$ is a constant tensor density. In the following we will be interested on perturbations around flat space, so we fix it to $\varepsilon = 1$ henceforth. Here $G = 8\pi M_{\text{P}}^{-2}$ is the Newton's constant.

As a consequence of the condition $|g| = 1$, UG is not invariant under the full group of diffeomorphisms. Instead, it is invariant only under those that preserve the constraint, corresponding to *volume preserving diffemorphisms, VDiff* [36, 37]. Their action is characterized at the infinitesimal level by a transverse vector

$$\delta g_{\mu\nu} = \mathcal{L}_\xi g_{\mu\nu}, \quad \nabla_\mu \xi^\mu = 0, \tag{2.2}$$

where $\mathcal{L}_\xi$ stands for the Lie derivative along $\xi^\mu$. We will dub the corresponding Lie algebra as *TDiff* for this reason. In the rest of this document we will also use *TDiff* in an sloppy way to refer to the full symmetry group. Effectively, we are replacing the four gauge constraints of GR by three of them — those corresponding to the volume preserving subgroup– plus the unit determinant constraint. The outcome for on-shell states is the same in both theories, four constraints that leave a single transverse and traceless graviton as the only propagating degree of freedom. However, this implies an important difference for off-shell states, since the constraint $|g| = 1$ is also satisfied by them, unlike gauge constraints, which only act on physical degrees of freedom. As a consequence, the metric fluctuations of UG are always exactly traceless

$$\delta g_{\mu\nu} \equiv \delta\tilde{g}_{\mu\nu} - \frac{1}{4}\tilde{g}_{\mu\nu}\tilde{g}^{\alpha\beta}\delta\tilde{g}_{\alpha\beta}, \tag{2.3}$$

with $\tilde{g}_{\alpha\beta}$ an unconstrained metric. Indeed, this is the reason as to why the eom are traceless, since they correspond to a variation with respect to this variable.

Although for classical matters we can use (2.3) to derive the eom, in order to perform a path integral over the gravitational field we need to resolve the constraint $|g| = 1$. It must be included in the integration measure, giving

$$\mathcal{Z}[T_{\mu\nu}] = \int [\mathcal{D}g]\ \delta\left(|g| - 1\right) e^{iS+iT\cdot g}, \tag{2.4}$$

where we have defined the dot product

$$T \cdot g = \int d^4x \sqrt{|g|}\ T^{\mu\nu} g_{\mu\nu}. \tag{2.5}$$

Several ways to resolve this issue have been explored before, including using a Lagrange multiplier [21] and a Stuckelberg field [17, 69, 70]. Here we choose to deal with it by performing a change of variables to a new metric defined by

$$g_{\mu\nu} = \tilde{g}_{\mu\nu}|\tilde{g}|^{\frac{1}{4}}, \tag{2.6}$$

so that $|g| = 1$ is satisfied identically. In terms of the new metric $\tilde{g}_{\mu\nu}$ and after integration by parts, the action of UG reads [38]

$$S_{\mathrm{UG}} = -\frac{1}{2G}\int d^4x\ |\tilde{g}|^{\frac{1}{4}}\ \left(\tilde{R} + \frac{3}{32}\frac{\tilde{\nabla}_\mu|\tilde{g}|\tilde{\nabla}^\mu|\tilde{g}|}{|\tilde{g}|^2}\right), \tag{2.7}$$

where $\tilde{g}_{\mu\nu}$ is now an unconstrained field and variations can be taken freely.

Note that factors of $g$, which behaves as sort of an extra scalar field, have now appeared in the action. In a diffeomorphism invariant theory this is not possible, because the determinant of the metric transforms as a density under a general *Diff* element. However, the fact that here we are dealing only with volume preserving transformations ensure that $g$ will transform as a true scalar and thus the new terms are allowed by symmetry.

The change of variables (2.6) also introduces an extra *fictitious* gauge symmetry in the form of Weyl invariance

$$\tilde{g}_{\mu\nu} \to \Omega(x)^2\tilde{g}_{\mu\nu}, \tag{2.8}$$



JHEP09(2020)041

where $\Omega(x)$ is an arbitrary function of the space-time coordinates. The full gauge symmetry of the theory to be considered is then the direct product of *TDiff* and *Weyl*, a combination that has been dubbed *WTDiff* before [36]. It is precisely Weyl invariance which comes to replace the determinant constraint in this form of the action, giving the extra condition needed to reduce the number of degrees of freedom to a single massless graviton.

Although we will use the action (2.7) in order to evaluate the path integral of UG, we are interested on writing results in terms of the original metric variable, that we choose as our physical metric. This is achieved by simply choosing the gauge $|\tilde{g}| = 1$ for Weyl transformations, thus identifying both metrics in (2.6). This is certainly true at the classical level, but one might be worried by the potential presence of a Weyl anomaly in the effective action, that would then obstruct the identification in the quantum variables. There are no reasons to worry, however, since it can be proven that the identification of the original metric $g_{\mu\nu}$ as the physical one precisely ensures the absence of anomalies [39, 40].

Note that when working with the action (2.7) it is straightforward to understand the main feature of UG. Due to Weyl invariance, a cosmological constant term is forbidden in the action and it cannot be generated by radiative corrections either. Moreover, the Ward identity stemming from (2.8) precisely enforces the tracelessness of the eom and all subsequent variations

$$\tilde{g}^{\mu_1\nu_1}\tilde{g}^{\mu_2\nu_2}\dots\tilde{g}^{\mu_n\nu_n}\frac{\delta}{\delta\tilde{g}_{\mu_1\nu_1}}\frac{\delta}{\delta\tilde{g}_{\mu_2\nu_2}}\dots\frac{\delta S}{\delta\tilde{g}_{\mu_n\nu_n}} = 0. \tag{2.9}$$

This also implies that the graviton excitation $h_{\mu\nu} = \tilde{g}_{\mu\nu} - \eta_{\mu\nu}$ will always be exactly traceless. One can check explicitly that the eom derived from (2.7), in the gauge $|\tilde{g}| = 1$ where we restore the original metric, are indeed the traceless Einstein equations (1.1).

As we discussed before, one of the main conundrums in the formulation of UG is the question of whether it is really a different theory than GR or if otherwise, and barring aside the role of the cosmological constant, they are exactly the same theory. Classically it is obvious that the answer is the latter. Since the eom of both theories are equivalent, the theories are so. However, quantum mechanically there are subtleties due to the different gauge group of UG. This question has been explored in several works from different points of view [22, 25, 27, 30–35], but all of them consider the theory in vacuum, with only gravitation present. Although this is an interesting setting, the simplicity of the theory implies that nothing can be said about the true equivalence of the theories. In particular, if UG is considered alone, there are no physical observables that can be used to establish a comparison with GR, since the one-loop correction in vacuum is finite.

In order to bypass this problem and to be able to define dependable quantities to establish such a comparison, we couple here UG to matter. To keep things simpler — but not trivial — we will consider a toy model comprised of a single massive scalar field with a quartic interaction and non-minimal coupling to gravity

$$S_{\text{matter}} = \int d^4x \, \left(\frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{m^2}{2}\phi^2 - \lambda\phi^4 - \frac{\xi}{2}\phi^2 R\right), \tag{2.10}$$

where we have already fixed $|g| = 1$. Note that, as advertised, the action is written with respect to the original metric $g_{\mu\nu}$ that we consider as physical.

An important difference with respect to GR arises here. Due to the constraint on the metric determinant, neither the mass term nor the quartic interaction couple directly to gravity. This will have a direct effect on the form of the vertices coupling gravity to matter in the perturbative expansion of this Lagrangian. Additionally, and since we will rely on perturbation theory for our later computations, we will always consider $G, \xi, \lambda \ll 1$.

Finally, note that since the Weyl invariance of the action (2.7) appears here as a consequence of the change of variables (2.6), the scalar field *is inert under it*. Unlike standard Weyl transformations, where a scalar would transform with a factor proportional to its energy dimension, here the symmetry is restricted purely to the metric sector. For this reason it has sometimes been dubbed as fake or spurious Weyl Invariance [40].

The total action that we will consider is then the sum of (2.7) and (2.10)

$$S = S_{\mathrm{UG}} + S_{\mathrm{matter}}. \tag{2.11}$$

However, in (2.10) the metric is unimodular. By performing the change of variables to the unconstrained metric $g_{\mu\nu} = |\tilde{g}|^{\frac{1}{4}} \tilde{g}_{\mu\nu}$ we have

$$\begin{aligned} S = \int d^4x \bigg\{ |\tilde{g}|^{\frac{1}{4}} \bigg[ -\frac{1}{2G} \left( \tilde{R} + \frac{3}{32} \frac{\tilde{\nabla}_\mu |\tilde{g}| \tilde{\nabla}^\mu |\tilde{g}|}{|\tilde{g}|^2} \right) + \frac{1}{2} \partial_\mu \phi \partial^\mu \phi \\ - \frac{\xi}{2} \phi^2 \left( \tilde{R} + \frac{3\tilde{\Box}|\tilde{g}|}{4|\tilde{g}|} - \frac{27 \tilde{\nabla}_\mu |\tilde{g}| \tilde{\nabla}^\mu |\tilde{g}|}{32 |\tilde{g}|^2} \right) \bigg] - \frac{m^2}{2} \phi^2 - \lambda \phi^4 \bigg\}, \end{aligned} \tag{2.12}$$

where we have integrated by parts in some terms. All indices in this expression must be contracted by using the unconstrained metric $\tilde{g}_{\mu\nu}$. This is the action that we will use hereinafter.

## 3 The background field expansion

We will formulate the path integral of the theory by using standard tools. In order to be able to preserve explicitly the gauge invariance of gravitational correlation functions we will rely on the use of the background field method [41, 42]. We thus start by defining the complete path integral that we will deal with as

$$\mathcal{Z}[J_{\mu\nu}, j] = \int [\mathcal{D}\tilde{g}][\mathcal{D}\phi] \; e^{i(S + J\cdot\tilde{g} + j\cdot\phi)}, \tag{3.1}$$

where we have introduced two sources $J_{\mu\nu}$ and $j$, which couple to the metric and to the scalar field respectively. We will use those to define correlation functions in the usual way through variational derivatives with respect to them.

Now, following the background field method, we separate the metric into background and fluctuation by

$$\tilde{g}_{\mu\nu} = \bar{g}_{\mu\nu} + h_{\mu\nu}, \tag{3.2}$$

where $h_{\mu\nu}$ is the graviton field. Since this is just a shift of the integration variable, we can set $[\mathcal{D}\tilde{g}] = [\mathcal{D}h]$.

Under this redefinition, the exponent inside the path integral can be expanded in powers of $h_{\mu\nu}$

$$S_J = S_J|_{\tilde{g}=\bar{g}} + \int d^4x \, \frac{\delta S_J}{\delta \tilde{g}(x)_{\mu\nu}}\bigg|_{\tilde{g}=\bar{g}} h(x)_{\mu\nu} \\ + \frac{1}{2}\int d^4x \int d^4y \, h(x)_{\mu\nu} \frac{\delta^2 S_J}{\delta\tilde{g}(x)_{\mu\nu}\delta\tilde{g}(y)_{\alpha\beta}}\bigg|_{\tilde{g}=\bar{g}} h(y)_{\alpha\beta} + \mathcal{O}(h^3), \tag{3.3}$$

where we have defined $S_J = S + J \cdot \tilde{g} + j \cdot \phi$. The first term in the expansion corresponds to the action evaluated in the background field, while the linear term vanishes whenever the background configuration satisfies the classical equations of motion. Since in this work we are interested only in one-loop effects, we cut the expansion at second order, which corresponds to leading order in the $\hbar$ expansion.

Since we have shifted the integration variable to $h_{\mu\nu}$, the background metric can be thought as an extra source, with the path integral depending on it

$$\mathcal{Z}[J_{\mu\nu}, \bar{g}_{\mu\nu}, j] = \int [\mathcal{D}h][\mathcal{D}\phi] \; e^{iS_J}. \tag{3.4}$$

If we now define the Quantum Effective Action in the standard way by a Legendre transform before and after the field redefinition, we find the apparently trivial identity

$$\Gamma[\tilde{g}_{\mu\nu}, \phi] = \Gamma[\bar{g}_{\mu\nu} + h_{\mu\nu}, \phi]. \tag{3.5}$$

However, this is not trivial at all. It means that, due to the appearance of the background metric as a shift of the total one, we can capture any covariant term of the Quantum Effective Action just by computing those correlators in which only $\bar{g}_{\mu\nu}$ and $\phi$ appear on external legs, while $h_{\mu\nu}$ is a pure internal variable over which we integrate. This will clearly make our lives easier and defines our computational strategy.

The other advantage of the background field method is that it allows us to preserve the gauge invariance — *WTDiff* in this case — of the Quantum Effective Action easily, by preserving that of any operator involving the background metric. This is due to the fact that, after the field redefinition, infinitesimal gauge transformations can be split in two

$$\delta_{\rm bg}\bar{g}_{\mu\nu} = \mathcal{L}_\xi \bar{g}_{\mu\nu} + 2\omega\bar{g}_{\mu\nu}, \tag{3.6}$$

$$\delta_{\rm bg}h_{\mu\nu} = \mathcal{L}_\xi h_{\mu\nu} + 2\omega h_{\mu\nu}, \tag{3.7}$$

$$\delta_{\rm q}\bar{g}_{\mu\nu} = 0, \tag{3.8}$$

$$\delta_{\rm q}h_{\mu\nu} = \mathcal{L}_\xi(\bar{g}_{\mu\nu} + h_{\mu\nu}) + 2\omega(\bar{g}_{\mu\nu} + h_{\mu\nu}), \tag{3.9}$$

where $\omega$ is the infinitesimal parameter associated to Weyl transformations (2.8) $\Omega(x) = 1 + \omega + \mathcal{O}(\omega^2)$.

As we see, $\delta_{\rm bg}$ corresponds to the gauge invariance of the background quantities, where $h_{\mu\nu}$ is then regarded as a tensor transforming in the same way as the metric. To this we must append the condition that $\phi$ is a scalar field inert under Weyl transformations.



JHEP09(2020)041

Since the path integral that we need to compute integrates only over $h_{\mu\nu}$ and $\phi$, while $\bar{g}_{\mu\nu}$ is regarded as a source, we will only need to gauge fix the quantum part of the symmetry $\delta_{\text{q}}$. The background symmetry will remain unaltered and therefore gauge invariance of our results is automatically ensured. All correlation functions must then satisfy an analogous expression to (2.9), with the classical action $S$ replaced by the Quantum Effective Action $\Gamma[\tilde{g}_{\mu\nu}, \phi]$.

### 3.1 Weyl Geometry

We thus turn now our attention to the issue of gauge fixing $\delta_{\text{q}}$. In order to do that, we first wish to be able to construct a gauge fixing term which is invariant under the background remaining *WTDiff* symmetry represented by $\delta_{\text{bg}}$. A priori this does not seem like a complicated task, but the complexity of the gauge sector of the theory (cf. later) can make it a cumbersome task. In order to make things easier and more straightforward, we will use here the formalism introduced in [43, 44] and named as *Weyl Geometry*. By defining a full geometric construction which is explicitly Weyl covariant — as well as diffeomorphism covariant — we can construct invariant quantities in a easy way.

The core of the method consists in the introduction of a U(1) gauge field $W_\mu$ which will serve to define Weyl covariant derivatives. However, since this is a Weyl invariant theory, this field *is not* an external ingredient, but instead it can be built out of the fields already in the action. In our case, we define it to be

$$W_\mu = \frac{1}{8}\bar{\nabla}_\mu \log(|\bar{g}|). \tag{3.10}$$

It can be easily checked that under a Weyl transformation (2.8), $W_\mu$ behaves indeed as a U(1) gauge field

$$W_\mu \to W_\mu + \Omega\bar{\nabla}_\mu\Omega. \tag{3.11}$$

Using it we introduce a non-metric connection

$$\Gamma^{(\text{W})\alpha}_{\mu\nu} = \{\ \}^\alpha_{\mu\nu} - \delta^\alpha_\mu W_\nu - \delta^\alpha_\nu W_\mu + \bar{g}_{\mu\nu}W^\alpha, \tag{3.12}$$

where $\{\ \}^\alpha_{\mu\nu}$ is the Levi-Civita connection of the metric $\bar{g}_{\mu\nu}$. As usual, the connection $\Gamma^{(\text{W})\alpha}$ will induce a covariant derivative, that we label $\nabla^{(\text{W})}$.

We complete the construction presented here by introducing the Weyl covariant derivative acting on a generic tensor $\mathcal{T}$

$$D_\mu\mathcal{T} = \nabla^{(W)}_\mu\mathcal{T} - \lambda_{\mathcal{T}}\mathcal{T}, \tag{3.13}$$

where $\lambda_{\mathcal{T}}$ is the scaling dimension of the tensor, defined as the weight of $\Omega$ under a Weyl transformation

$$\mathcal{T} \to \Omega^{\lambda_{\mathcal{T}}}\mathcal{T}. \tag{3.14}$$

Note that, when defined in this way, $D_\mu$ is compatible with the background metric $D_\mu\bar{g}_{\alpha\beta} = 0$.



JHEP09(2020)041

For the future it will be also useful to define a Weyl covariant curvature by using the Ricci identity acting on a generic vector $V^\alpha$

$$[D_\mu, D_\nu]V^\alpha = \mathcal{R}_{\mu\nu}{}^\alpha{}_\beta V^\beta, \tag{3.15}$$

which gives

$$\begin{aligned}\mathcal{R}_{\mu\nu\alpha\beta} = \bar{R}_{\mu\nu\alpha\beta} &+ \bar{g}_{\mu\alpha}\left(\bar{\nabla}_\nu W_\beta + W_\nu W_\beta\right) - \bar{g}_{\mu\beta}\left(\bar{\nabla}_\nu W_\alpha + W_\nu W_\alpha\right) - \bar{g}_{\nu\alpha}\left(\bar{\nabla}_\mu W_\beta + W_\mu W_\beta\right)\\ &+ \bar{g}_{\nu\beta}\left(\bar{\nabla}_\mu W_\alpha + W_\mu W_\alpha\right) - (\bar{g}_{\mu\alpha}\bar{g}_{\nu\beta} - \bar{g}_{\mu\beta}\bar{g}_{\nu\alpha})W^2, \end{aligned}\tag{3.16}$$

and subsequently

$$\mathcal{R}_{\mu\nu} = \bar{R}_{\mu\nu} + 2W_\mu W_\nu + \bar{\nabla}_\mu W_\nu + \bar{\nabla}_\nu W_\mu - 2\bar{g}_{\mu\nu}W^2 + \bar{g}_{\mu\nu}\bar{\nabla}^\alpha W_\alpha, \tag{3.17}$$

$$\mathcal{R} = \bar{R} + 6(\bar{\nabla}^\mu W_\mu - W^2), \tag{3.18}$$

where $\bar{R}_{\mu\nu\alpha\beta}$ is the Riemann tensor of the background metric.

The advantage of using $D_\mu$ now is clear. For any tensor $\mathcal{T}$ with a well-defined scaling dimension — that is, that there are not derivatives of $\Omega$ involved in the transformation of the tensor —, $D_\mu\mathcal{T}$ will transform in the same way as $\mathcal{T}$. Constructing Weyl invariant quantities is just a matter of combining $D_\mu$ with powers of $|\bar{g}|$ — which enjoys scaling dimension $\lambda_{|\bar{g}|} = 8$ — to form scalars under Weyl transformation. A simple example of this is the action (2.7) evaluated in $\bar{g}$, which can be easily written as

$$S|_{\tilde{g}=\bar{g}} = -\frac{1}{2G}\int d^4x\ |\bar{g}|^{\frac{1}{4}}\mathcal{R}. \tag{3.19}$$

### 3.2 Gauge fixing and BRST invariance

We finally turn ourselves to the problem of fixing the *WTDiff* symmetry of the fluctuations. In principle one could think on attempting to fix the symmetry in a standard manner, by introducing a gauge fixing condition

$$F_\mu = 0, \tag{3.20}$$

and appending the action with a gauge fixing term

$$S_{\text{gf}} = \int d^4x\ F_\mu F^\mu, \tag{3.21}$$

and the corresponding action for the ghosts. However, this is not as straightforward as it seems for two reasons. First, we are dealing with the direct product of *TDiff* and *Weyl*. The total number of conditions required to fix the symmetry is still four and thus it seems that choosing a space-time vector $F_\mu$ does the work. However, the resulting gauge fixing term must then satisfy three conditions

1. It must break the quantum part of *TDiff* invariance.

2. It must break the quantum part of *Weyl* invariance.

3. It must preserve the background *WTDiff* symmetry.

As far as we know, it is not possible to choose a function $F_\mu$ such that (3.21) satisfies all three conditions.

The other reason as to why a standard gauge fixing method is cumbersome is related to the structure of the *TDiff* group. Since the generator of transverse diffeomorphisms is constrained to be transverse, the same will be true for the corresponding ghost field $c^\mu$

$$\bar{\nabla}_\mu c^\mu = 0. \tag{3.22}$$

This condition has to be included in the measure in some manner. The easiest way is to follow [22] and use a transverse projector to project an otherwise arbitrary field

$$c^\mu = \left(\delta^\mu_\nu - \bar{\nabla}^\mu \Box^{-1} \bar{\nabla}_\nu\right) d^\mu, \tag{3.23}$$

where the inverse of the Laplace operator $\Box = \bar{\nabla}_\mu \bar{\nabla}^\mu$ is defined by acting on an arbitrary tensor

$$\Box^{-1}\,\Box = \Box\,\Box^{-1}\mathcal{T} = \mathcal{T}. \tag{3.24}$$

By doing this, in a similar way to what happens with the transformation of the metric (2.6), we replace the condition (3.22) by a U(1) gauge symmetry acting on $d^\mu$

$$d^\mu \to \bar{\nabla}^\mu f, \tag{3.25}$$

where $f$ is an arbitrary function of the space-time coordinates. Thus, we will need to introduce a gauge fixing term for this symmetry as well, that will then generate a full new set of ghosts and anti-ghosts, of bosonic character this time. These have been sometimes dubbed in the literature as Nielsen-Kallosh ghosts [45, 46].

In order to circumvent all the complications implied by these properties, we decide here to fix the gauge by using BRST invariance [47]. We thus introduce an operator $\mathfrak{s}$ which, when acting on the graviton fluctuation $h_{\mu\nu}$, implements a gauge transformation with the infinitesimal generator replaced by a ghost

$$\mathfrak{s}h_{\mu\nu} = \mathcal{L}_c(\bar{g}_{\mu\nu} + h_{\mu\nu}) + 2b(\bar{g}_{\mu\nu} + h_{\mu\nu}), \tag{3.26}$$

where we have introduced the ghost field associated to Weyl invariance $b$. Note that since the generators of the transformation are now Grassman variables, the operator $\mathfrak{s}$ is Grassman odd. To this we must append the transformation rules for the ghost fields $c^\mu$ and $b$

$$\mathfrak{s}c^\mu = \mathcal{L}_c c^\mu = c^\rho \nabla_\rho c^\mu, \tag{3.27}$$

$$\mathfrak{s}b = \mathcal{L}_c b = c^\rho \nabla_\rho b, \tag{3.28}$$

which are inert under Weyl transformations.

However, in this work we are only interested in one-loop corrections, that we have already established that correspond to the quadratic approximation in the path integral. Since, as we will see later, the transformation of the ghost will always come in the final gauge-fixed action multiplied by another quantum field, we can just neglect the transformation of both ghost fields and write instead

$$\mathfrak{s}c^\mu = \mathcal{O}(\text{field}^2), \tag{3.29}$$

$$\mathfrak{s}b = \mathcal{O}(\text{field}^2). \tag{3.30}$$

As before, the ghost field $c^\mu$ is forced to satisfy a transversality condition. However, since our ultimate goal is to obtain a gauge fixing term which preserves background *WTDiff* invariance, from now on we define the transverse condition by using the Weyl covariant derivative

$$D_\mu c^\mu = 0. \tag{3.31}$$

We will do the same in any other BRST transformation from now on.

Note that this replacement is always possible, since we can always use whatever derivative we desire to compute the Lie derivative in (3.26). Alternatively, any difference between derivatives can be also absorbed in a redefinition of the ghost field $b$.

Again, we use a transverse projector to satisfy (3.31), which in this case will be given by

$$c^\mu = \left(\delta^\mu_\nu - D^\mu (D^2)^{-1} D_\nu\right) d^\nu, \tag{3.32}$$

and the inherited U(1) invariance will take the form

$$d^\mu \to D^\mu f. \tag{3.33}$$

In general, dealing with this kind of open algebra would require the sophisticated technique of BV quantization [48]. However, in the case of UG things are simple enough so that we can construct the gauge fixing sector by simply including the gauge symmetry of the ghost field in the BRST operator [22, 49, 50]. Consequently, we extend the action of $\mathfrak{s}$ appropriately. We introduce a ghost field $\alpha$ and write

$$\mathfrak{s}d^\mu = D^\mu \alpha, \tag{3.34}$$

where, due to the Grassman parity of $\mathfrak{s}$, we see that $\alpha$ must be a *bosonic* field.

We now append our theory with a complementary set of anti-ghost and auxiliary fields with the goal of closing the algebra of the BRST operator, that we demand to be nilpotent when acting on any field involved in the path integral

$$\mathfrak{s}^2 = 0. \tag{3.35}$$

For symmetries whose associated ghost is Grassman odd, it is enough to add a single anti-ghost and an auxiliary field with even Grassman number to achieve the closure of the algebra. However, for symmetries whose ghost is bosonic, such as $\alpha$, things are more subtle. Since the auxiliary field needs to be Grassman odd, it is impossible to form its square. In that case we are required to introduce two pairs of anti-ghost and auxiliary fields. Following these rules and taking into account that here we have three gauge symmetries — *TDiff*, *Weyl* and the U(1) symmetry of the ghost field — we find that we need the following set of fields and transformations

$$\mathfrak{s}\bar{c}^\mu = \rho^\mu, \qquad \mathfrak{s}\rho^\mu = 0, \tag{3.36}$$

$$\mathfrak{s}\bar{b} = l, \qquad \mathfrak{s}l = 0, \tag{3.37}$$

$$\mathfrak{s}x = m, \qquad \mathfrak{s}m = 0, \tag{3.38}$$

$$\mathfrak{s}\bar{x} = \bar{m}, \qquad \mathfrak{s}\bar{m} = 0. \tag{3.39}$$

Here the first two lines correspond to the auxiliary fields needed to close the algebra for *WTDiff* transformations, while the rest are the two pairs of field required for the U(1). Their Grassman character is

$$\{\bar{c}^\mu, \bar{b}, m, \bar{m}\} \equiv \text{Grassman odd.} \tag{3.40}$$

$$\{\rho^\mu, l, x, \bar{x}\} \equiv \text{Grassman even.} \tag{3.41}$$

This is enough to ensure the nil-potency of the BRST operator when acting on any field of the path integral within the one-loop approximation.

Once the action of $\mathfrak{s}$ onto every field is defined, we introduce the BRST gauge-fixing term, which includes the action of the ghosts, as the result of acting with $\mathfrak{s}$ on a so-called *gauge fermion*

$$S_{\text{BRST}} = -\frac{1}{2G}\int d^4x \; \mathfrak{s}\Psi, \tag{3.42}$$

where $\Psi$ is a term quadratic in the fields and of odd Grassman parity. Thanks to the nil-potency of $\mathfrak{s}$ and once $S_{\text{BRST}}$ is chosen in this way, the total action is invariant under a BRST transformation. The associated Ward-Takahashi identities then become the Slavnov-Taylor identities of the theory, ensuring a successful quantization.

The construction of $\Psi$ now replaces the arbitrary choice of gauge function $F^\mu$. As long as $\Psi$ is Grassman odd and breaks gauge invariance — but not BRST invariance — it is a valid choice. Here we will however follow a conservative approach, still defining a gauge condition

$$F_\mu = D^\nu h_{\mu\nu} + \tau D_\mu h, \tag{3.43}$$

and writing

$$\Psi = |\bar{g}|^{\frac{1}{4}}(\bar{c}_\mu + D_\mu\bar{b})\left(F^\mu - \frac{1}{4\sigma}(\rho^\mu - D^\mu l)\right) + x\left(D_\mu d^\mu + \frac{1}{2\gamma}\bar{m}\right) + y\;\bar{x}\left(\bar{g}^{\frac{1}{4}} D_\mu \bar{c}^\mu - \frac{1}{2\gamma}m\right). \tag{3.44}$$

Here $\sigma$, $\tau$, $y$ and $\gamma$ are gauge parameters whose value we can use either to simplify our computations or to test gauge invariance of our results. The powers of $|g|$ are chosen so that the expression is invariant under background Weyl invariance. The form of this gauge fermion is motivated by the BRST formulation of the usual Faddev-Poppov gauge fixing method. If we were dealing with a simpler symmetry, and in the absence of Weyl invariance, then the first term would be enough to fix it and after integration of the auxiliary field $\rho$ we would have recovered the standard gauge fixing plus ghost action. Here the first term deals with the combined *WTDiff* gauge symmetry while the rest is needed to be able to fix the U(1) symmetry of the ghost sector.

Acting with the BRST operator we then have

$$S_{\text{BRST}} = -\frac{1}{2G}\int d^4x \left\{ |\bar{g}|^{\frac{1}{4}}(\rho_\mu + D_\mu l)\left(F^\mu - \frac{1}{4\sigma}(\rho^\mu - D^\mu l)\right) + |\bar{g}|^{\frac{1}{4}}(\bar{c}_\mu + D_\mu \bar{b})\ \mathfrak{s}F^\mu \right.$$
$$\left. + m\left(D_\mu d^\mu + \frac{1}{2\gamma}\bar{m}\right) + |\bar{g}|^{\frac{1}{4}} x D^2\alpha + y\ \bar{m}\left(|\bar{g}|^{\frac{1}{4}} D_\mu \bar{c}^\mu - \frac{1}{2\gamma}m\right) + y|\bar{g}|^{\frac{1}{4}}\ \bar{x} D_\mu \rho^\mu \right\}. \quad (3.45)$$

Examining this expression we see that $\rho^\mu$, $m$ and $\bar{m}$ are linearly coupled, entering the path integral as sources. We can thus integrate them out by using their equations of motion. This simplifies the BRST term to

$$S_{\text{BRST}} = -\frac{1}{2G}\int d^4x\, |\bar{g}|^{\frac{1}{4}} \left\{ (\bar{c}_\mu + D_\mu \bar{b})\ \mathfrak{s}F^\mu + \frac{2\gamma y}{y+1}\ D_\mu \bar{c}^\mu D_\nu d^\nu + \sigma\,(F_\mu - yD_\mu \bar{x})^2 \right.$$
$$\left. + \frac{1}{4\sigma} D_\mu l D^\mu l + D_\mu l F^\mu + x D^2\alpha \right\}. \quad (3.46)$$

Finally, appending this action to the classical action, we can write the path integral of UG in the background field approach to be

$$\mathcal{Z}[J_{\mu\nu}, \bar{g}_{\mu\nu}, j] = \int [\mathcal{D}h][\mathcal{D}\phi][\mathcal{D}\bar{c}][\mathcal{D}d][\mathcal{D}\bar{b}][\mathcal{D}b][\mathcal{D}\bar{x}][\mathcal{D}l][\mathcal{D}x][\mathcal{D}\alpha]\ e^{i(S_J + S_{\text{BRST}})}. \quad (3.47)$$

## 4 Perturbations around flat space

Once the path integral for the unimodular scalar-tensor theory is properly defined, we come to the task of computing the one-loop correction to the coupling constants. Since background *WTDiff* invariance is ensured by construction, we will perform our computation by expanding the background metric around flat space-time

$$\bar{g}_{\mu\nu} = \eta_{\mu\nu} + H_{\mu\nu}, \quad (4.1)$$

where we will dub $H_{\mu\nu}$ as the *background graviton fluctuation*. This will allow us to use standard techniques to compute Feynman diagrams. Correlation functions of the background metric will become correlators of $H_{\mu\nu}$ and we will capture the renormalization of the coupling constants by computing diagrams with $H_{\mu\nu}$ and $\phi$ in the external legs.

### 4.1 Propagators for bosonic fields

We start by computing the propagators of the fluctuations. In order to do that we take $S_J + S_{\text{BRST}}$ and we set the background metric to be flat, thus retaining only the terms



JHEP09(2020)041

quadratic in the quantum fields. The lagrangian for the bosonic fields then reads

$$\mathcal{L}_2 = -\frac{1}{2G}\Big[h^{\mu\nu}\partial_\alpha\partial_\nu h^\alpha_\mu + h^{\alpha\beta}\partial_\beta\partial_\mu h^\mu_\alpha + \frac{1}{2}\partial^\beta h^{\alpha\mu}\partial_\mu h_{\alpha\beta} - \frac{1}{16}h\partial^2 h - \frac{1}{4}h^{\alpha\beta}\partial^2 h_{\alpha\beta} - \frac{1}{4}h\partial_\alpha\partial_\beta h^{\alpha\beta}$$
$$-h^{\alpha\beta}\partial_\alpha\partial_\beta h + (\sigma-1)\partial_\alpha h^{\alpha\beta}\partial_\mu h^\mu_\alpha - \frac{1+8\sigma\tau}{4}\partial_\alpha h^\beta_\alpha\partial^\alpha h - \frac{1+32\sigma\tau^2}{32}\partial_\alpha h\partial^\alpha h + \frac{1}{4\sigma}\partial_\mu l\partial^\mu l$$
$$+\sigma y^2\partial_\mu\bar{x}\partial^\mu\bar{x} + \partial_\alpha l(\partial_\beta h^{\alpha\beta} + \tau\partial^a h) - 2\sigma y\partial_\alpha\bar{x}(\partial_\beta h^{\alpha\beta} + \tau\partial^a h) + x\partial^2\alpha\Big] + \frac{1}{2}\left(\partial_\alpha\phi\partial^\alpha\phi - m^2\phi^2\right). \tag{4.2}$$

We leave the discussion of the ghost sector involving $d^\mu$, $\bar{c}^\mu$, $b$ and $\bar{b}$ for the next subsection.

Here we find a striking difference between GR and UG. Due to the complicated gauge fixing sector involving bosonic Nielsen-Kallosh ghost fields, we find that the graviton fluctuation $h_{\mu\nu}$ mixes with the bosonic ghosts at the kinetic level, as indicated by the last terms in the second line in (4.2). This means that in order to compute the propagator of the gravitational field, we need to take these fields into account in order to cancel spurious gauge pole contributions. It is not enough to take the $F_\mu F^\mu$ term in the gauge fixing and invert the kinetic term for the graviton by itself, *even for tree-level computations.*

We take the action (4.2), Fourier transforming it to momentum space and we write it in matrix form

$$\mathcal{L}_2 = \frac{1}{2}\Big(h_{\mu\nu}, \bar{x}, l, \phi, x, \alpha\Big)\,\mathcal{M}^{-1}(q)\begin{pmatrix} h_{\alpha\beta} \\ \bar{x} \\ l \\ \phi \\ x \\ \alpha \end{pmatrix} \tag{4.3}$$

where $\mathcal{M}^{-1}(q)$ is the matrix-valued inverse propagator. Inverting it with the following sign convention

$$\mathcal{M}^{-1}(q)\mathcal{M}(q) = i\mathbb{I}, \tag{4.4}$$

gives the following non-vanishing propagators for the fields

$$\begin{aligned}\langle h_{\mu\nu}(-q)h_{\alpha\beta}(q)\rangle = &\frac{2iG}{q^2}\left(\eta_{\mu\alpha}\eta_{\nu\beta} + \eta_{\mu\beta}\eta_{\nu\alpha} - \frac{1+2\sigma(3+8\tau(1+\tau))}{\sigma(1+4\tau)^2}\eta_{\mu\nu}\eta_{\alpha\beta}\right)\\ &+\frac{4iG}{q^6}\frac{1-2\sigma}{\sigma}q_\mu q_\nu q_\alpha q_\beta + \frac{4iG}{q^4}\frac{3+4\tau}{1+4\tau}\left(\eta_{\mu\nu}q_\alpha q_\beta + \eta_{\alpha\beta}q_\mu q_\nu\right)\\ &-\frac{iG}{q^4}\frac{1+2\sigma}{\sigma}\left(\eta_{\mu\alpha}q_\nu q_\beta + \eta_{\nu\alpha}q_\mu q_\beta + \eta_{\mu\beta}q_\nu q_\alpha + \eta_{\nu\beta}q_\mu q_\alpha\right),\end{aligned} \tag{4.5}$$

$$\langle l(-q)l(q)\rangle = -6iG\sigma\frac{1}{q^2}, \tag{4.6}$$

$$\langle l(-q)h_{\mu\nu}(q)\rangle = \frac{2iG}{(1+4\tau)}\frac{\eta_{\mu\nu}}{q^2}, \tag{4.7}$$

$$\langle \bar{x}(-q)\bar{x}(q)\rangle = -\frac{3iG}{2\sigma y^2}\frac{1}{q^2}, \tag{4.8}$$

$$\langle \bar{x}(-q)h_{\mu\nu}(q)\rangle = -\frac{iG}{\sigma y(1+4\tau)}\frac{\eta_{\mu\nu}}{q^2}, \tag{4.9}$$

$$\langle l(-q)\bar{x}(q)\rangle = \frac{iG}{y}\frac{1}{q^2}, \tag{4.10}$$

$$\langle x(-q)\alpha(q)\rangle = \frac{4iG}{q^2}, \tag{4.11}$$

$$\langle \phi(-q)\phi(q)\rangle = \frac{i}{q^2-m^2}. \tag{4.12}$$

In order to simplify our computations we will set the gauge parameter $\tau = -3/4$, for which the graviton propagator reduces to

$$\begin{aligned}\langle h_{\mu\nu}(-q)h_{\alpha\beta}(q)\rangle = {}& \frac{2iG}{q^2}\left(\eta_{\mu\alpha}\eta_{\nu\beta}+\eta_{\mu\beta}\eta_{\nu\alpha}-\frac{1+3\sigma}{4\sigma}\eta_{\mu\nu}\eta_{\alpha\beta}\right)+\frac{4iG}{q^6}\frac{1-2\sigma}{\sigma}q_\mu q_\nu q_\alpha q_\beta \\ & -\frac{iG}{q^4}\frac{1+2\sigma}{\sigma}\left(\eta_{\mu\alpha}q_\nu q_\beta+\eta_{\nu\alpha}q_\mu q_\beta+\eta_{\mu\beta}q_\nu q_\alpha+\eta_{\nu\beta}q_\mu q_\alpha\right).\end{aligned} \tag{4.13}$$

In principle we could further simplify this expression by choosing $\sigma = -1/2$. However, we refrain to do so in order to be able to track the gauge dependence of our results along the computation. We will also leave the parameter $y$ arbitrary.

### 4.2 The ghost propagators

We now focus in the action for the ghost fields

$$S_{\text{gh}} = -\frac{1}{2G}\int d^4x\,|\bar{g}|^{\frac{1}{4}}\left[(\bar{c}_\mu + D_\mu\bar{b})\mathfrak{s}F^\mu + \frac{2\gamma y}{y+1}\,D_\mu\bar{c}^\mu D_\nu d^\nu\right], \tag{4.14}$$

with the goal of computing their propagators.

Acting with the BRST operator on $F^\mu$ gives

$$\mathfrak{s}F^\mu = D^2c^\mu + \mathcal{R}^\mu_\nu c^\nu + (2+8\tau)D^\mu b, \tag{4.15}$$

with $\mathcal{R}_{\mu\nu}$ given by (3.17). However, this is written in terms of the constrained field $c^\mu$. We thus perform the change of variables (3.32) and write

$$\mathfrak{s}F^\mu = D^2d^\mu - D^\mu D_\nu d^\nu + \mathcal{R}^\mu_\nu d^\nu - 2\mathcal{R}^\mu_\nu D^\nu\left(D^2\right)^{-1}D_\alpha d^\alpha + (2+8\tau)D^\mu b. \tag{4.16}$$

Setting the background metric to be flat in order to derive the propagator, we have $D_\mu \equiv \partial_\mu$ and therefore the non-local operator $(\partial^2)^{-1}$ has a well-defined representation in momentum space when acting on an arbitrary tensor, given by

$$(\partial^2)^{-1}\mathcal{T} = \int\frac{d^4q}{2\pi}\left(-\frac{1}{q^2}\right)\mathcal{T}\,e^{iq\cdot x}. \tag{4.17}$$

However, this will never enter into the definition of the propagators, since it comes multiplied by a curvature, which vanishes when $\bar{g}_{\mu\nu} = \eta_{\mu\nu}$. It will be important later when deriving the interaction vertices, but due to the same reason and since we will only need vertices with one external graviton, we will never have to workout the task of inverting this in general, but only take its flat realization.

Around flat space-time, the action that defines the propagator then gives the following Lagrangian

$$\mathcal{L}_2^{(\mathrm{gh})} = \left(\bar{c}_\mu + \partial_\mu \bar{b}\right)\left(\partial^2 d^\mu - \partial^\mu \partial_\nu d^\nu + (2+8\tau)\partial^\mu b\right) - z\, \partial_\mu \bar{c}^\mu \partial_\nu d^\nu, \tag{4.18}$$

where we must note that the different ghost sectors, belonging to *TDiff* and *Weyl*, are mixed at the kinetic level. Here we have defined $z = -2\gamma y(1+y)^{-1}$.

As with the bosonic fields, we now write this in matrix form after integration by parts

$$\mathcal{L}_2^{(\mathrm{gh})} = \left(\bar{c}_\mu, \bar{b}\right) \mathcal{N}^{-1} \begin{pmatrix} d^\nu \\ b \end{pmatrix}, \tag{4.19}$$

and by inverting $\mathcal{N}^{-1}$ we find the following non-vanishing propagators

$$\langle \bar{c}_\nu(-q) d^\mu(q)\rangle = -2G\left(\frac{(1+z)q^\mu q_\nu}{z} - \delta^\mu_\nu\right)\frac{i}{q^2}, \tag{4.20}$$

$$\langle \bar{b}(-q) d^\mu(q)\rangle = -\frac{2Gq^\mu}{zq^4}, \tag{4.21}$$

$$\langle \bar{b}(-q) b(q)\rangle = -\frac{G}{1+4\tau}\frac{i}{q^2}. \tag{4.22}$$

Although we could use $z$ to try to simplify the form of the propagator $\langle d^\mu(-q)\bar{c}_\nu(q)\rangle$ we prefer to keep it arbitrary in order to track gauge independence of our results.

## 5 Computation of correlation functions

Once we have set-up the perturbative expansion of the action and derived the propagators, we can affront the computation of the one-loop RG flow of the different coupling constants in the Lagrangian. In order to understand what we need to compute, let us take a look to the zeroth order action around the background metric

$$S = \int d^4x \Bigg\{ |\bar{g}|^{\frac{1}{4}} \Bigg[ -\frac{1}{2G}\left(\bar{R} + \frac{3}{32}\frac{\bar{\nabla}_\mu|\bar{g}|\bar{\nabla}^\mu|\bar{g}|}{|\bar{g}|^2}\right) + \frac{1}{2}\partial_\mu\phi\partial^\mu\phi \\ -\frac{\xi}{2}\phi^2\left(\bar{R} + \frac{3\bar{\Box}|\bar{g}|}{4|\bar{g}|} - \frac{27\bar{\nabla}_\mu|\bar{g}|\bar{\nabla}^\mu|\bar{g}|}{32|\bar{g}|^2}\right)\Bigg] - \frac{m^2}{2}\phi^2 - \lambda\phi^4 \Bigg\}. \tag{5.1}$$

By expanding this around flat space

$$\bar{g}_{\mu\nu} = \eta_{\mu\nu} + H_{\mu\nu}, \tag{5.2}$$

we see that it is enough to compute the two-point function of $H_{\mu\nu}$ in order to capture the running of $G$, while from the two and four-point functions of the scalar field we derive the running of $m^2$, $\lambda$ and the field strength renormalization of $\phi$, as usual. Finally, from the coupling $\phi^2 H_{\mu\nu}$ we can extract the running of $\xi$. Of course, since the theory is non-renormalizable, we will also find extra divergences corresponding to higher dimension operators — with four derivatives. Therefore, we will adopt an EFT approach to quantization from now on.

We will perform the computation with standard Feynman diagrams, using the propagators (4.5)–(4.12) and (4.20)–(4.22). The interaction vertices are defined in the standard way by variational derivatives of the action $S_J$, after expanding the background metric around flat space and going to momentum space

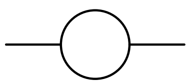
$$\begin{aligned}&\langle H_{\mu_1\nu_1}(q_1)\dots H_{\mu_n\nu_n}(q_n)h_{\alpha_1\beta_1}(p_1)\dots h_{\alpha_m\beta_m}(p_m)\phi(k_1)\phi(k_s)\rangle\\ &=\frac{i}{n!m!s!}\frac{\delta}{\delta H_{\mu_1\nu_1}(q_1)}\cdots\frac{\delta}{\delta H_{\mu_n\nu_n}(q_n)}\frac{\delta}{\delta h_{\alpha_1\beta_1}(p_1)}\cdots\frac{\delta}{\delta h_{\alpha_m\beta_m}(p_m)}\frac{\delta}{\delta\phi(k_1)}\cdots\frac{\delta S_J}{\delta\phi(k_s)}.\end{aligned}\tag{5.3}$$

The explicit formulas for all the vertices are pretty cumbersome and not illuminating at all, so we refrain to show them here explicitly. Let us note however that, due to background Weyl invariance, all vertices and all correlation functions that we will compute must satisfy the Ward identities (2.9).

Regarding loop integrals, we have two possible poles that can enter into the loops from the propagators (4.5)–(4.12) and (4.20)–(4.22). They represent the massless pole of the graviton and ghosts and the massive pole of the scalar field

$$\mathcal{P}_0(q)=\frac{1}{q^2},\quad \mathcal{P}_m(q)=\frac{1}{q^2-m^2}.\tag{5.4}$$

This implies that the denominator in a typical Feynman diagram will be a product of these poles evaluated for the momentum structures running in the loops, that will depend on the external momentum $p^\mu$. For example, a fish diagram will have a typical form



$$\sim\int\frac{d^4k}{(2\pi)^4}\,F(p,k)\,\mathcal{P}_i(k+p)\mathcal{P}_j(k),\tag{5.5}$$

where the form-factor $F(p,k)$ will depend on the particular diagram, and we would have to choose later the pole structures depending if the internal legs are scalars or gravitons.

In the following we will be interested in the computation of divergences, which are the only piece needed to obtain the RG flow of the coupling constants. Therefore we will ignore the finite parts of the diagrams and will capture these divergences by expanding the integrands of the different diagrams in powers of the external momentum and the mass $m$ of the scalar field. After reducing any index structure as usual by using rotational invariance,[3] all divergent integrals in the expansion will have the same form

$$D(n)=\int\frac{d^4q}{(2\pi)^4}\frac{1}{q^n}.\tag{5.6}$$

[3] When expanding the denominators in the Feynman integrals, we will encounter an increasing number of loop momenta $q^\mu$ in the numerators. We will reduce those by Lorentz (rotational) invariance in the standard way, averaging over directions [51],

$$q_{i_1}q_{i_2}\dots q_{i_n}\to|q|^nT_{i_1i_2\dots i_n}\frac{\Gamma\left(\frac{d}{2}\right)\Gamma\left(\frac{n+1}{2}\right)}{\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{d+n}{2}\right)}$$

Once they are taken to this form, we will use dimensional regularization in order to compute them. Since the above integrals have no dimensionful parameter, we can directly see that all of them must vanish — as it is usual in dimensional regularization — unless $n = 4$, so we will only need to retain these integrals

$$\mathcal{I} = D(4) = \int \frac{d^4q}{(2\pi)^4} \frac{1}{q^4}. \tag{5.7}$$

Although we are only interested in the UV divergences of the integral, we must note that (5.7) is however divergent on both ends of the integral. Therefore, it will be convenient for us to regulate the intermediate IR divergences by introducing a soft mass $\eta$ and rewriting $\mathcal{I}$ in $d$ dimensions as

$$\mathcal{I} = \int \frac{d^dq}{(2\pi)^4} \frac{1}{(q^2 - \eta^2)^2}, \tag{5.8}$$

which can now be computed by using standard formulas to give

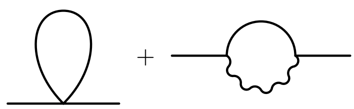
$$\mathcal{I} = \frac{i}{8\pi^2\epsilon} - \frac{i}{16\pi^2} \left(\gamma - \log(4\pi) + \log(\eta^2)\right) + \mathcal{O}(\epsilon), \tag{5.9}$$

where $\gamma$ is the Euler-Mascheroni constant and $\epsilon = 4 - d$. This will be the form that we will later use to regularize the divergences in the Feynman diagrams. From now on we will only focus on those diagrams with non-vanishing divergences under this regularization scheme.

All the computations presented here have been performed with two independent computer codes based on Mathematica, with the help of the package xAct [52, 53]; and FORM [54].

### 5.1 The two-point function of the scalar field

We start by computing the simplest of the correlation functions that we will need to define the RG flow of the coupling constants. That is the two-point function of the scalar field, which will be given by the following Feynman diagrams

$\langle \phi(-p)\phi(p) \rangle_{\text{1-loop}} =$  , (5.10)

where our dictionary for the lines of the diagrams is shown in table 1.

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where $d$ is the space-time dimension and

$$T_{i_1 i_2 \ldots i_n} = \frac{1}{n!} \left[\delta_{i_1 i_2} \ldots \delta_{i_{n-1} i_n} + \text{all permutations of the i's}\right]$$

for even $n$, and $T_{i_1 i_2 \ldots i_n} = 0$ for odd $n$.

Note that the maximum number of free loop momenta that we can find is tied to the number of indices in the external legs of the diagram. Two for every $H_{\mu\nu}$ in a external leg and one for every $p^\mu$. This means that, for example, for the two-point function of the scalar field it is enough to retain terms with up to four free $q^\mu$ (since we can have divergences proportional to $p^4$), while this amount is doubled for the graviton two-point function (four momenta and four indices in the gravitons).



JHEP09(2020)041

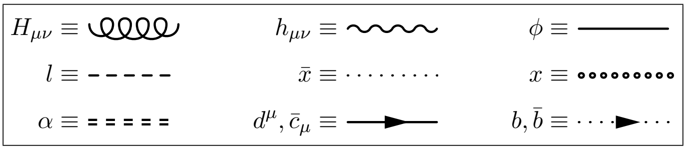


**Table 1**. Dictionary of lines for the Feynman diagrams.

By inspection of the action and the topology of the diagrams, we see that we can expect three types of divergences, proportional to $p^0$, $p^2$ and $p^4$. In principle, we could have added a gravitational tadpole here. However, it is proportional to the integral $D(2)$ and therefore it vanishes in dimensional regularization.

Computing the diagrams as previously discussed, expanding the denominators, and retaining only the UV divergent terms we find

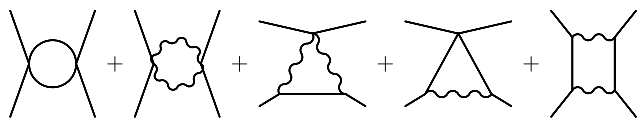
$$\langle\phi(-p)\phi(p)\rangle_{\text{1-loop}} = \left(\frac{Gp^4(3+2\sigma(3+8\xi))}{4\sigma} - \frac{3Gm^2p^2(1+2\sigma(1+8\xi+4\xi^2))}{4\sigma} + 12\lambda m^2 - 6Gm^4\xi^2\right)\mathcal{I}, \tag{5.11}$$

which indeed contains the three possible divergences previously mentioned. From the momentum-independent term we will be able to extract the running of $m^2$, while the term proportional to $p^2$ will give the field strength renormalization of the field. The piece quartic in the external momentum will require the introduction of a higher-derivative operator in order to absorb the divergence, as usual in a non-renormalizable EFT.

### 5.2 The four-point function of the scalar field

We compute now the four-point function of the scalar field. As before, we expect divergences with external momentum up to $p^4$. The corresponding Feynman diagrams contributing to this are

$$\langle\phi(-p)\phi(-p)\phi(p)\phi(p)\rangle_{\text{1-loop}} =$$



, (5.12)

where we are just drawing inequivalent topologies. For all the diagrams considered here, we must sum the contribution of all inequivalent channels once the external momenta are fixed. This amounts to adding the $s$, $t$ and $u$ channels for all the diagrams, plus two permutations of the external vertices for the triangles, which add up to six different channels.

We evaluate the divergences by setting the magnitude of all external momenta to that of $p^\mu$. This is equivalent to the kinematical configuration $s = 4p^2$, $t = u = 0$, which will define our subtraction point. Under this choice, the one-loop contribution to the four-point function becomes

$$\langle\phi(-p)\phi(-p)\phi(p)\phi(p)\rangle_{\text{1-loop}} = \Bigg[36(24\lambda^2-48G\lambda m^2\xi^2+G^2m^4\xi^3(2+9\xi)) + \frac{6Gp^2(-6\lambda(1+\sigma(2+8\xi(4+\xi)))+Gm^2\xi(-3\xi+\sigma(1+2\xi(11+6\xi(7+3\xi\ )))))}{\sigma} + \frac{G^2p^4(117+8\xi(17+40\xi)-4\sigma(27+4\xi(37+45\xi))+4\sigma^2(483+4\xi(259+2\xi(385+3\xi(-28+9\xi)))))}{8\sigma^2}\Bigg]\mathcal{I}. \tag{5.13}$$

As in the previous section, we obtain three kind of divergences. The momentum independent one will dictate the running of $\lambda$, while the other terms will demand higher-derivative operators to be introduced in the EFT expansion.

### 5.3 Corrections to the non-minimal coupling

In order to compute the one-loop contribution to the running of the non-minimal coupling we will need to focus on the three-point function mixing two external scalar fields and a graviton. The tree-level form of this correlator can be obtained by expanding the action to the given order in the background graviton, giving

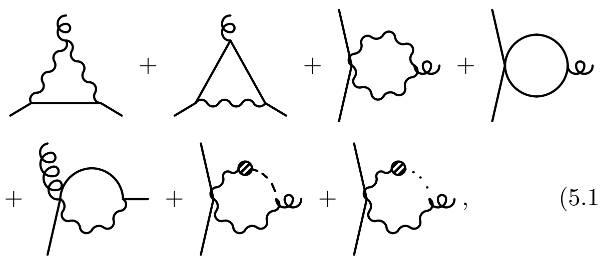
$$\langle\phi(-p)\phi(-p)H_{\mu\nu}(2p)\rangle_{\text{tree}} = -\frac{i}{4}(1+4\xi)(p^2\eta_{\mu\nu}-4p_\mu p_\nu), \tag{5.14}$$

where we have assigned equal incoming momentum for the scalar fields.

Therefore, contributions to $\langle\phi(-p)\phi(-p)H_{\mu\nu}(2p)\rangle$ will renormalize the combination $1+4\xi$, once the effect of the field strength renormalization of $\phi$ is subtracted. Note that, since the theory is Weyl invariant at the background level, the action must satisfy the condition (2.9), which implies that $\langle\phi(-p)\phi(-p)H_{\mu\nu}(p)\rangle$ must be a traceless tensor. This is trivially satisfied by the tree-level contribution (5.14) but it will serve as a strong sanity check of our result for the one-loop computation since in that case the condition is satisfied in a non-trivial way.

The one-loop topologies contributing to this correlator are

$\langle\phi(-p)\phi(-p)H_{\mu\nu}(2p)\rangle_{\text{1-loop}} =$



(5.15)

where the last two diagrams contain a explicit presence of the bosonic ghost fields in the internal lines, with the small shaded blown representing kinetic mixing. Actually, the presence of these bosonic ghost fields is critical, since the sum of all the other diagrams is *not* traceless and therefore violates the Ward identity (2.9). It is only when the last two topologies, which also have a non-vanishing trace, are added, that the whole contribution becomes traceless. This is not surprising, of course. The role of ghosts is precisely to cancel the dynamics of gauge modes, which violate Ward identities, in the internal legs. However, it serves here as a very non-trivial test of the construction of the path integral, the BRST sector and of our computation.

The final result takes the form

$$\begin{aligned}\langle\phi(-p)\phi(-p)H_{\mu\nu}(2p)\rangle_{\text{1-loop}} = \mathcal{I}\left(p_\mu p_\nu - \frac{1}{4}p^2\eta_{\mu\nu}\right)& \\ \times\Bigg[-\frac{64\lambda\sigma(1+6\xi)-Gm^2(-3+2\sigma(1+6\xi)(-3+2\xi(-3+8\xi)))}{4\sigma}& \\ +\frac{Gp^2(-9-40\xi+2\sigma(23+30\xi+\sigma(42+4\xi(53+6(13-6\xi)\xi))))}{12\sigma^2}\Bigg]&. \end{aligned} \quad (5.16)$$

We see that the result is indeed proportional to (5.14). Moreover, no terms independent of the momentum have been generated. Those would require the introduction of counter-terms of the schematic form $|g|^\alpha\phi^2$, with $\alpha$ a constant, that violate Weyl invariance.

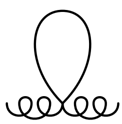
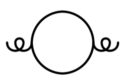
### 5.4 The gravitational two-point function

The last correlation function that we need in order to compute the RG flow of the coupling constants in the action is the two-point function of the gravitational field. Its value is required in order to get the running of the Newton constant G. Additionally, we will also compute the contributions that require the introduction of higher-derivative operators to cancel divergences. This will not only complete our computation but it will also serve as a third additional computation complementary to that of [22, 35]. In the following we will split the computation in three parts — the contribution of the scalar field, that of the rest of bosonic fields, and the one coming from ghost loops.

#### 5.4.1 Contributions from scalar loops

This first contribution in the simplest one of all that we will consider in this subsection. It is equivalent to compute the contribution of a gravitating scalar-field in a background non-dynamical geometry. As such, and by the reasons discussed in this work, its contribution shall be identical to that coming from GR. Indeed, we have checked that it is the case at the level of $\beta$-functions.

There are only two diagrams that need to be taken into account

$\langle H_{\mu\nu}(-p)H_{\alpha\beta}(p)\rangle_\phi =$  $+$  , (5.17) and whose contribution is

$$
\begin{aligned}
\langle H_{\mu\nu}(-p)H_{\alpha\beta}(p)\rangle_\phi =& \Bigg[\frac{p^4}{480}(-1+20\xi+60\xi^2)\eta_{\alpha\beta}\eta_{\mu\nu}-\frac{p^2}{120}(1+20\xi+60\xi^2)\left(\eta_{\mu\nu}p_\alpha p_\beta+\eta_{\alpha\beta}p_\mu p_\nu\right)\\
&-\frac{p^2}{120}\left(\eta_{\beta\nu}p_\alpha p_\mu+\eta_{\alpha\nu}p_\beta p_\mu+\eta_{\beta\mu}p_\alpha p_\nu+\eta_{\alpha\mu}p_\beta p_\nu\right)+\left(\frac{1}{15}+\frac{2\xi}{3}+2\xi^2\right)p_\alpha p_\beta p_\mu p_\nu\\
&+\frac{p^4}{120}\left(\eta_{\alpha\nu}\eta_{\beta\mu}+\eta_{\alpha\mu}\eta_{\beta\nu}\right)-\frac{m^2(1+6\xi)}{48}\Big(4p^2(\eta_{\alpha\nu}\eta_{\beta\mu}+\eta_{\alpha\mu}\eta_{\beta\nu})+4(\eta_{\mu\nu}p_\alpha p_\beta+\eta_{\alpha\beta}p_\mu p_\nu)\\
&-4(\eta_{\beta\nu}p_\alpha p_\mu+\eta_{\alpha\nu}p_\beta p_\mu+\eta_{\beta\mu}p_\alpha p_\nu+\eta_{\alpha\mu}p_\beta p_\nu)-3p^2\eta_{\alpha\beta}\eta_{\mu\nu}\Big)\Bigg]\mathcal{I}. \qquad (5.18)
\end{aligned}
$$

As in the case of the non-minimal coupling, note that there are no terms independent of the external momentum, since those would imply a renormalization of the cosmological constant, violating Weyl invariance of the background. The satisfaction of the Ward identity (2.9) can be seen here from the fact that

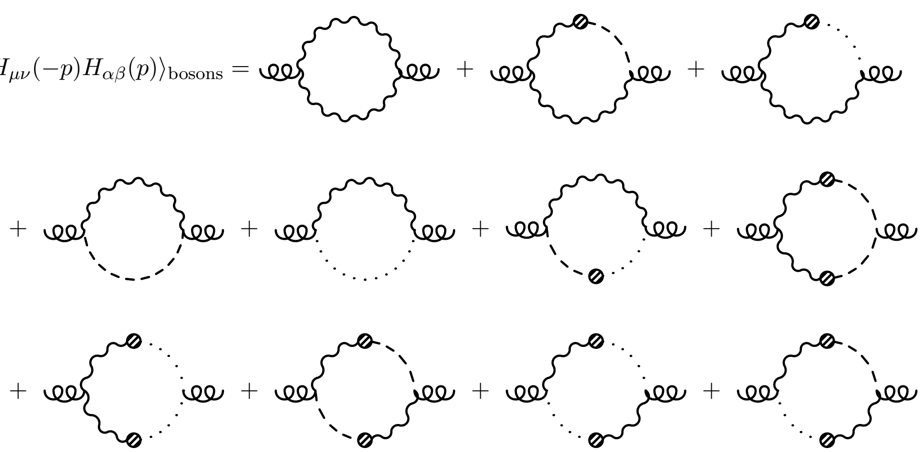
$$
\langle H_{\mu\nu}(-p)H_{\alpha\beta}(p)\rangle_\phi \eta^{\mu\nu}\eta^{\alpha\beta} = 0. \qquad (5.19)
$$

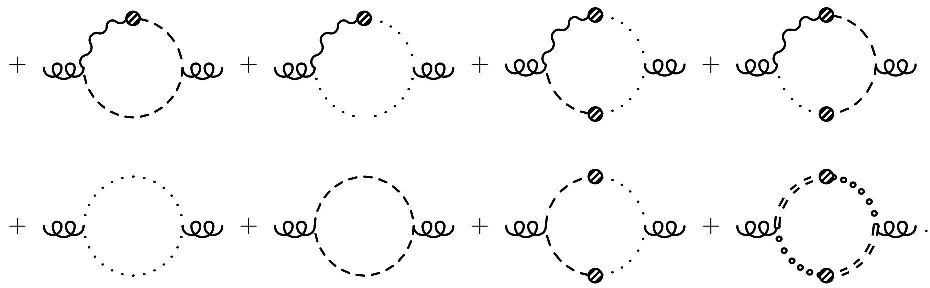
The terms proportional to $p^2$ will renormalize the Newton's constant $G$ — as it can be seen from the fact that they are proportional to the tree-level kinetic term of $H_{\mu\nu}$ —, while the terms with a quartic dependence on $p^4$ will require higher-derivative terms.

### 5.4.2 Contributions from the graviton and bosonic ghost fields

While the contribution from the scalar field to the gravitational two-point function is pretty simple, that of the rest of bosonic fields is pretty cumbersome, due to the kinetic mixing between the graviton fluctuation $h_{\mu\nu}$ and the bosonic ghosts $l$ and $\bar{x}$. This multiplies the number of Feynman diagrams to be considered and leaves the following set of inequivalent topologies

$\langle H_{\mu\nu}(-p)H_{\alpha\beta}(p)\rangle_{\text{bosons}} =$





(5.20)

Computing these Feynman diagrams by following the methods previously described in this paper, we find the following result

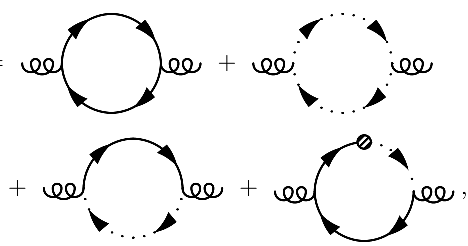
$$\langle H_{\mu\nu}(-p)H_{\alpha\beta}(p)\rangle_{\rm bosons} = \left(\frac{p^4(50{+}100\sigma{+}777\sigma^2)(\eta_{\alpha\mu}\eta_{\beta\nu}{+}\eta_{\alpha\nu}\eta_{\beta\mu})}{480\sigma^2} - \frac{p^4(125{+}250\sigma{+}2167\sigma^2)\eta_{\alpha\beta}\eta_{\mu\nu}}{1920\sigma^2}\right.$$
$$-\frac{p^2(50{+}100\sigma{+}777\sigma^2)(p_\alpha p_\mu\eta_{\beta\nu}{+}p_\alpha p_\nu\eta_{\beta\mu}{+}p_\beta p_\mu\eta_{\alpha\nu}{+}p_\beta p_\nu\eta_{\alpha\mu})}{480\sigma^2} + \frac{(25{+}50\sigma{+}164\sigma^2)p_\alpha p_\beta p_\mu p_\nu}{120\sigma^2}$$
$$\left.+\frac{p^2(25{+}50\sigma{+}613\sigma^2)(\eta_{\mu\nu}p_\alpha p_\beta{+}\eta_{\alpha\beta}p_\mu p_\nu)}{480\sigma^2}\right)\mathcal{I}. \tag{5.21}$$

Note that in this case all divergences are proportional to $p^4$ as a consequence of the absence of any dimensionful parameter in the loops, since all the fields that propagate in these diagrams are massless. As a consequence, this contribution will only renormalize higher-derivative operators.

### 5.4.3 Contributions from fermionic ghosts

The last contribution that we need to compute in order to get the full one-loop divergence contributing to the gravitational two-point functions is that coming from the loops of fermionic ghosts, $\bar{c}_\mu$, $d^\mu$, $\bar{b}$ and $b$. It is given by the following diagrams

$\langle H_{\mu\nu}(-p)H_{\alpha\beta}(p)\rangle_{\rm fermions} =$



(5.22)

where the arrows indicate the fermion flow. Their contribution to the correlation function is

$$\langle H_{\mu\nu}(-p)H_{\alpha\beta}(p)\rangle_{\rm fermions} = \left(\frac{p^4}{16}(\eta_{\alpha\mu}\eta_{\beta\nu}{+}\eta_{\alpha\nu}\eta_{\beta\mu}) - \frac{p^2}{16}(p_\mu p_\alpha\eta_{\nu\beta}{+}p_\nu p_\alpha\eta_{\mu\beta}{+}p_\mu p_\beta\eta_{\nu\alpha}{+}p_\nu p_\beta\eta_{\alpha\mu})\right.$$
$$\left.+\frac{5}{48}p^2(\eta_{\mu\nu}p_\alpha p_\beta{+}\eta_{\alpha\beta}p_\mu p_\nu) - \frac{1}{6}p_\alpha p_\beta p_\mu p_\nu - \frac{11}{192}p^4\eta_{\alpha\beta}\eta_{\mu\nu}\right)\mathcal{I}. \tag{5.23}$$

Again, since there are no dimensionful constant running in the loop propagators, the result is proportional to $p^4$ and will only renormalize higher-derivative operators. Additionally, we see that the dependence on the gauge parameter $z$, which appears explicitly in the ghost propagators (4.20)–(4.22), has cancelled out in the final result. This cancellation is non-trivial, since individual diagrams depend on $z$ and only the total combination is independent of the parameter.

### 5.4.4 The total result

We finally add up all the different contributions computed in the previous sections, finding that the total one-loop correction to the two-point function of the background graviton is

$$\begin{aligned}
&\langle H_{\mu\nu}(-p)H_{\alpha\beta}(p)\rangle_{\text{1-loop}} = \langle H_{\mu\nu}(-p)H_{\alpha\beta}(p)\rangle_{\phi}+\langle H_{\mu\nu}(-p)H_{\alpha\beta}(p)\rangle_{\text{bosons}}+\langle H_{\mu\nu}(-p)H_{\alpha\beta}(p)\rangle_{\text{fermions}}\\
&= \bigg[\frac{p^4(50{+}100\sigma{+}811\sigma^2)(\eta_{\alpha\mu}\eta_{\beta\nu}{+}\eta_{\alpha\nu}\eta_{\beta\mu})}{480\sigma^2}+\frac{p^4(-125{-}250\sigma{+}\sigma^2(-2281{+}80\xi{+}240\xi^2))\eta_{\mu\nu}\eta_{\alpha\beta}}{1920\sigma^2}\\
&+\frac{p^2(25{+}50\sigma{+}\sigma^2(659{-}80\xi{-}240\xi^2))(\eta_{\mu\nu}p_\alpha p_\beta{+}\eta_{\alpha\beta}p_\mu p_\nu)}{480\sigma^2}+\frac{9(25{+}50\sigma{+}8\sigma^2(19{+}10\xi{+}30\xi^2))p_\alpha p_\beta p_\mu p_\nu}{1080\sigma^2}\\
&-\frac{p^2(50{+}100\sigma{+}811\sigma^2)(p_\mu p_\alpha\eta_{\nu\beta}{+}p_\nu p_\alpha\eta_{\mu\beta}{+}p_\mu p_\beta\eta_{\nu\alpha}{+}p_\nu p_\beta\eta_{\alpha\mu})}{480\sigma^2}-\frac{m^2(1{+}6\xi)}{48}\left(4p^2(\eta_{\alpha\nu}\eta_{\beta\mu}{+}\eta_{\alpha\mu}\eta_{\beta\nu})\right.\\
&\left.+4(\eta_{\mu\nu}p_\alpha p_\beta{+}\eta_{\alpha\beta}p_\mu p_\nu){-}4(\eta_{\beta\nu}p_\alpha p_\mu{+}\eta_{\alpha\nu}p_\beta p_\mu{+}\eta_{\beta\mu}p_\alpha p_\nu{+}\eta_{\alpha\mu}p_\beta p_\nu){-}3p^2\eta_{\alpha\beta}\eta_{\mu\nu}\right)\bigg]\mathcal{I}.
\end{aligned} \tag{5.24}$$

### 5.5 Renormalization

Once we have computed the divergent parts of the different correlation functions, we come to the moment of renormalizing the effective action, absorbing the divergences by using a counter-term. For any generic correlation function $\mathcal{G}$, we will compute the value $\mathcal{I}$ by using (5.9) so that we will have

$$\mathcal{G}_{\text{1-loop}} \equiv \bar{\mathcal{G}}\left(\frac{i}{8\pi^2\epsilon} - \frac{i}{16\pi^2}\left(\gamma - \log(4\pi) + \log(\eta^2)\right) + \mathcal{O}(\epsilon)\right), \tag{5.25}$$

where $\bar{\mathcal{G}}$ will be a tensor structure depending on $p^\mu$ and on the coupling constants of the theory. We will then add counterterms to the bare Lagrangian, including also higher-derivative new operators that we will need to absorb the divergences quartic in $p^\mu$. Using the $\overline{MS}$ subtraction scheme then we write

$$\mathcal{G}_{\text{ct}} \propto \delta c\left(\frac{i}{8\pi^2\epsilon} - \frac{i}{16\pi^2}\left(\gamma - \log(4\pi) + \log(\mu^2)\right)\right) = \delta c\,\Re(\mu), \tag{5.26}$$

for a generic coupling $c$. We have defined

$$\Re(\mu) = \left(\frac{i}{8\pi^2\epsilon} - \frac{i}{16\pi^2}\left(\gamma - \log(4\pi) + \log(\mu^2)\right)\right), \tag{5.27}$$

where $\mu$ is the renormalization scale.

We will determine the value of $\delta c$ so that the sum $\mathcal{G}_{\text{1-loop}} + \mathcal{G}_{\text{counter-term}}$ is free of divergences when $\epsilon \to 0$.



JHEP09(2020)041

### 5.5.1 Scalar two-point function

In order to absorb the divergences in the two-point function of the scalar field, we must extend the bare action by including an operator with four derivatives in the kinetic term. The corresponding action for the counter-terms can be written in the frame where the metric is unimodular in the standard way

$$\delta S_{2,\phi} = \int d^4x \left( \frac{\delta Z}{2} \partial\phi\partial^\mu\phi - \frac{\delta m^2}{2}\phi^2 + \frac{\delta a_4}{2}\Box\phi\Box\phi \right), \tag{5.28}$$

where $a_4$ is a dimensionful coupling and $\delta Z$ is the anomalous dimension of the scalar field, related to the field strength renormalization as usual

$$\phi_R = Z^{\frac{1}{2}}\phi, \quad Z = 1 + \delta Z. \tag{5.29}$$

Now, we perform the change of variables to the unconstrained background metric by

$$g_{\mu\nu} = |\bar{g}|^{\frac{1}{4}}\bar{g}_{\mu\nu}, \tag{5.30}$$

for which the action takes the slightly more involved form

$$\delta S_{2,\phi} = \int d^4x \,\left( \frac{\delta Z}{2}|\bar{g}|^{\frac{1}{4}}\partial_\mu\phi\partial^\mu\phi - \frac{\delta m^2}{2}\phi^2 + \frac{\delta a_4}{2}|\bar{g}|^{\frac{1}{2}}D^2\phi D^2\phi \right), \tag{5.31}$$

which is explicitly *WTDiff* invariant.

The contribution from the counter-terms to the correlation function is then

$$\langle\phi(-p)\phi(p)\rangle_{\rm ct} = i\left(\delta Z p^2 - \delta m^2 + \delta a_4 p^4\right)\Re(\mu). \tag{5.32}$$

Adding it to the one-loop result (5.11) and demanding the result to be finite, we find that the value of the counter-terms must be

$$\delta Z = \frac{3Gm^2(1+2\sigma(1+4\xi(2+\xi)))}{4\sigma}, \tag{5.33}$$

$$\delta a_4 = -\frac{G(3+2\sigma(3+8\xi))}{4\sigma}, \tag{5.34}$$

$$\delta m^2 = 12\lambda m^2 - 6Gm^4\xi^2. \tag{5.35}$$

### 5.5.2 Scalar four-point function

In order to renormalize the divergences in the four-point function (5.13) we also need to include higher-derivative operators. As before, we write them in a standard form in the unimodular frame

$$S_{4,\phi} = \int d^4x \left( -\delta\lambda\phi^4 + \frac{\delta b_2}{8}\phi^2(\partial\phi)^2 + \frac{\delta b_4}{24}(\partial\phi)^4 \right). \tag{5.36}$$

Writing it in the unconstrained frame with (5.30), the corresponding action, which is invariant under background *WTDiff* transformations, then reads

$$S_{4,\phi} = \int d^4x \,\left( -\delta\lambda\phi^4 + \frac{\delta b_2}{8}|g|^{\frac{1}{4}}\phi^2(\partial\phi)^2 + \frac{\delta b_4}{24}|\bar{g}|^{\frac{1}{2}}(\partial\phi)^4 \right), \tag{5.37}$$



JHEP09(2020)041

and gives a contribution to the correlator of the form

$$\langle\phi(-p)\phi(-p)\phi(p)\phi(p)\rangle_{\rm ct} = i\left(-24\delta\lambda + \delta b_2 p^2 + \delta b_4 p^4\right)\Re(\mu). \tag{5.38}$$

Adding it to (5.13) and cancelling the divergences in $\epsilon$ we find

$$\delta\lambda = 6\lambda^2 - 72G\lambda m^2\xi^2 + \frac{3}{2}G^2m^4\xi^3(2+9\xi), \tag{5.39}$$

$$\delta b_2 = -\frac{6G(-6\lambda(1+\sigma(2+8\xi(4+\xi)))+Gm^2\xi(-3\xi+\sigma(1+2\xi(11+6\xi(7+3\xi)))))}{\sigma}, \tag{5.40}$$

$$\delta b_4 = -\frac{G^2(117+8\xi(17+40\xi)-4\sigma(27+4\xi(37+45\xi))+4\sigma^2(483+4\xi(259+2\xi(385+3\xi(-28+9\xi)))))}{8\sigma^2}, \tag{5.41}$$

so that the total correlation function in the one-loop approximation is now finite.

### 5.5.3 The non-minimal coupling

Now we come to the renormalization of the corrections to the non-minimal coupling, given by (5.16). In order to do that we will need not only to introduce a counter-term for $\xi$ and a new higher-derivative operator, but also take into account the contribution of two operators that we have already included in a previous section, since they contain the metric and therefore will also contribute to this correlator when expanded around flat space. The full counter-term action that we need is then

$$S_{\phi\phi H} = \int d^4x \left(\frac{\delta Z}{2}\partial_\mu\phi\partial^\mu\phi + \frac{\delta a_4}{2}\Box^2\phi\Box^2\phi - \frac{\delta\xi}{2}\phi^2 R + \frac{\delta\varsigma}{2}\partial_\mu\phi\partial^\mu\phi R\right), \tag{5.42}$$

which in the unconstrained frame reads

$$\begin{aligned} S_{\phi\phi H} = \int d^4x\, |\bar g|^{\frac14}\Bigg[&\frac{\delta Z}{2}\partial_\mu\phi\partial^\mu\phi + \frac{\delta a_4}{2}|\bar g|^{\frac14}D^2\phi D^2\phi - \frac{\delta\xi}{2}\phi^2\left(\bar R + \frac{3\bar\Box|\bar g|}{4|\bar g|} - \frac{27\bar\nabla_\mu|\bar g|\bar\nabla^\mu|\bar g|}{32|\bar g|^2}\right) \\ &+\frac{\delta\varsigma}{2}|\bar g|^{\frac14}\partial_\mu\phi\partial^\mu\phi\left(\bar R + \frac{3\bar\Box|\bar g|}{4|\bar g|} - \frac{27\bar\nabla_\mu|\bar g|\bar\nabla^\mu|\bar g|}{32|\bar g|^2}\right)\Bigg]. \end{aligned} \tag{5.43}$$

The contribution from this action to the corresponding correlation function is then

$$\begin{aligned} \langle\phi(-p)\phi(-p)H_{\mu\nu}(2p)\rangle_{\rm ct} &= -i\left(\frac{p^2(2\delta\varsigma+\delta a_4)}{2} + \frac{\delta Z + 4\delta\xi}{4}\right)\left(\eta_{\mu\nu}p^2 - 4p_\mu p_\nu\right)\Re(\mu) \\ &\equiv -i\left(p^2\delta\varsigma + \delta\xi\right)\left(\eta_{\mu\nu}p^2 - 4p_\mu p_\nu\right)\Re(\mu), \end{aligned} \tag{5.44}$$

where in the last step we have absorbed the value of $\delta a_4$ and $\delta Z$ into the arbitrariness of $\delta\varsigma$ and $\delta\xi$ by redefining them.

Adding this to the divergent result (5.16) and cancelling the divergences we have

$$\delta\xi = \frac{64\lambda\sigma(1+6\xi) + Gm^2(3+\sigma(6+8\xi(6+(5-24\xi)\xi)))}{16\sigma}, \tag{5.45}$$

$$\delta\varsigma = \frac{G(9+40\xi+2\sigma(-23-30\xi+2\sigma(-21+2\xi(-53+6\xi(-13+6\xi)))))}{48\sigma^2}. \tag{5.46}$$

### 5.5.4 Gravitational two-point function

The last correlation function that we need to renormalize is the two-point function of the background graviton $\langle H_{\mu\nu}(-p)H_{\alpha\beta}\rangle$. In order to absorb all the divergences we need to add counter-terms for the Newton's constant $G$ as well as two new standard higher-derivative operators in the form of $R^2$ and $R_{\mu\nu}R^{\mu\nu}$, that we write in the following combination

$$S_{2,H} = \int d^4x \left(-\delta\left(\frac{1}{2G}\right)R + \delta\alpha R^2 + \delta\rho\left(R_{\mu\nu}R^{\mu\nu} - \frac{1}{3}R^2\right)\right). \tag{5.47}$$

In principle we are allowed to add also a term $R_{\mu\nu\alpha\beta}R^{\mu\nu\alpha\beta}$. However its integral corresponds to the Gauss-Bonnet term in four space-time dimensions and therefore its variation — and consequently its expansion around flat space — vanishes.

Of course, the counter-terms must be now written in the unconstrained frame by performing the change of variables $g_{\mu\nu} = |\bar{g}|^{\frac{1}{4}}\bar{g}_{\mu\nu}$ for which we have

$$R_{\mu\nu} = \bar{R}_{\mu\nu} + \frac{\bar{g}_{\mu\nu}\partial_\alpha\partial^\alpha|\bar{g}|}{8|\bar{g}|} - \frac{5g_{\mu\nu}\partial_\alpha|\bar{g}|\partial^\alpha|\bar{g}|}{32|\bar{g}|^2} + \frac{\partial_\mu\partial_\nu|\bar{g}|}{4|\bar{g}|} - \frac{7\partial_\mu|\bar{g}|\partial_\nu|\bar{g}|}{32|\bar{g}|^2}, \tag{5.48}$$

$$R = \bar{R} + \frac{3\Box|\bar{g}|}{4|\bar{g}|} - \frac{27\partial_\mu|\bar{g}|\partial^\mu|\bar{g}|}{32|\bar{g}|^2}. \tag{5.49}$$

We omit the full expression for $S_{2,H}$ since it is very cumbersome. Note that, since $G$ multiplies the kinetic term of the graviton, there is no need to introduce a field strength renormalization for $H_{\mu\nu}$.

The contribution of the counter-terms to the correlator is

$$\begin{aligned}\langle H_{\mu\nu}(-p)H_{\alpha\beta}(p)\rangle_{\rm ct} = \Bigg\{&-\frac{1}{4}\delta\left(\frac{1}{2G}\right)\Bigg(-p^2(\eta_{\mu\alpha}\eta_{\nu\beta}+\eta_{\mu\beta}\eta_{\nu\alpha})+\frac{3p^2}{4}\eta_{\mu\nu}\eta_{\alpha\beta}-(\eta_{\mu\nu}p_\alpha p_\beta+\eta_{\alpha\beta}p_\mu p_\nu)\\
&+(\eta_{\mu\alpha}p_\beta p_\nu+\eta_{\mu\beta}p_\alpha p_\nu+\eta_{\nu\alpha}p_\mu p_\beta+\eta_{\nu\beta}p_\mu p_\alpha)\Bigg)+\delta\alpha\left(\frac{p^4}{8}\eta_{\mu\nu}\eta_{\alpha\beta}-\frac{p^2}{2}(\eta_{\mu\nu}p_\alpha p_\beta+\eta_{\alpha\beta}p_\mu p_\nu)\right.\\
&\left.+2p_\mu p_\nu p_\alpha p_\beta\right)+\delta\rho\Bigg(\frac{p^4}{4}(\eta_{\mu\alpha}\eta_{\beta\nu}+\eta_{\mu\beta}\eta_{\nu\alpha})-\frac{p^4}{6}\eta_{\mu\nu}\eta_{\alpha\beta}+\frac{p^2}{6}(\eta_{\mu\nu}p_\alpha p_\beta+\eta_{\alpha\beta}p_\mu p_\nu)\\
&-\frac{p^2}{4}(\eta_{\mu\alpha}p_\nu p_\beta+\eta_{\nu\alpha}p_\mu p_\beta+\eta_{\mu\beta}p_\nu p_a+\eta_{\nu\beta}p_\mu p_\alpha)+\frac{p^2}{3}p_\alpha p_\beta p_\mu p_\nu\Bigg)\Bigg\}\Re(\mu).\end{aligned} \tag{5.50}$$

Adding this to (5.24) and demanding that the divergences cancel, we find

$$\delta\left(\frac{1}{2G}\right) = \frac{m^2(1+6\xi)}{3}, \tag{5.51}$$

$$\delta\alpha = -\frac{5+10\sigma-\sigma^2(71-48\xi-144\xi^2)}{144\sigma^2}, \tag{5.52}$$

$$\delta\rho = -\frac{50+100\sigma+811\sigma^2}{120\sigma^2}. \tag{5.53}$$

## 6 $\beta$-functions and running couplings

Once we have determined the form of the renormalized correlation functions that we need, we come to the issue of computing the renormalization group flow of the different coupling



JHEP09(2020)041

constants, which is independent of the renormalization and regularization schemes used in the previous sections. We will actually define the running of a given coupling through the Callan-Symanzik (CS) equation for the corresponding correlation function [55, 56]

$$\left(\mu\frac{\partial}{\partial\mu}+\sum_i\beta(c_i)\frac{\partial}{\partial c_i}+\sum_j\gamma(M_j)M_j\frac{\partial}{\partial M_J}+n\gamma_\phi\right)\mathcal{G}(p,\mu)=0, \tag{6.1}$$

which is obtained by demanding independence of the arbitrary scale $\mu$ introduced by renormalization. Here $c_i$ are all possible dimensionless couplings appearing in the correlator, while $M_j$ refers to dimensionful couplings. $\gamma(M_j)$ is then the *anomalous dimension* of the coupling, while $\gamma_\phi$ is the anomalous dimension of the scalar field, being $n$ the number of external scalar legs in the correlator. Here we have already taken into account that the anomalous dimension of $H_{\mu\nu}$ vanishes. Solving the equation (6.1) perturbatively for the couplings in the action will allow us to obtain the running of all of them.

Since we are working at one-loop, we find an important simplification here. The only part of the renormalized correlation function which depends on $\mu$ is the counter-term, so we can make the replacement

$$\mu\frac{\partial\mathcal{G}(p,\mu)}{\partial\mu}\equiv\mu\frac{\partial\mathcal{G}_{\rm ct}(p,\mu)}{\partial\mu}. \tag{6.2}$$

Additionally, since our expansion is polynomial in the couplings, derivatives with respect to them are ordered in the loop expansion, with increasing loops contributing with higher orders. For the one-loop computation at hand, this means that we can also replace

$$\begin{aligned}\left(\sum_i\beta(c_i)\frac{\partial}{\partial c_i}+\sum_j\gamma(M_j)M_j\frac{\partial}{\partial M_J}+n\gamma_\phi\right)\mathcal{G}(p,\mu)\equiv&\\ \left(\sum_i\beta(c_i)\frac{\partial}{\partial c_i}+\sum_j\gamma(M_j)M_j\frac{\partial}{\partial M_J}+n\gamma_\phi\right)&\mathcal{G}_{\rm tree}(p,\mu),\end{aligned} \tag{6.3}$$

since when acting on the corrections we will generate a next-to-leading-order term. These two substitutions simplify the computation greatly.

Let us then start by writing the simplified form of equation (6.1) for the two-point function of the scalar field

$$\left(\mu\frac{\partial}{\partial\mu}+m^2\gamma(m^2)\frac{\partial}{\partial m^2}+a_4\gamma(a_4)\frac{\partial}{\partial a_4}+2\gamma_\phi\right)\langle\phi(-p)\phi(p)\rangle=0. \tag{6.4}$$

Combining the one-loop correction and the counter-term, and ordering this equation by powers of the momentum, it can be easily solved to get

$$\gamma_\phi=\frac{3Gm^2(1+2\sigma(1+4\xi(2+\xi)))}{64\pi^2\sigma}, \tag{6.5}$$

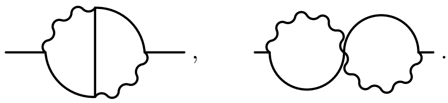
$$\gamma(m^2)=\frac{3\lambda}{2\pi^2}-\frac{3Gm^2(1+2\sigma(1+8\xi(1+\xi)))}{32\pi^2\sigma}, \tag{6.6}$$

$$\gamma(a_4)=-G\left(\frac{3+2\sigma(3+8\xi)}{32a_4\pi^2\sigma}+\frac{3m^2(1+2\sigma(1+8\xi+4\xi^2))}{32\pi^2\sigma}\right). \tag{6.7}$$

From the four-point function of the scalar field we can now compute the running of $\lambda$, $b_2$ and $b_4$. The corresponding CS equation is

$$\left(\mu\frac{\partial}{\partial\mu}+\beta(\lambda)\frac{\partial}{\partial\lambda}+b_2\gamma(b_2)\frac{\partial}{\partial b_2}+b_4\gamma(b_4)\frac{\partial}{\partial b_4}+4\gamma_\phi\right)\langle\phi(-p)\phi(-p)\phi(p)\phi(p)\rangle=0. \tag{6.8}$$

Note however that this imposes a limitation in our computation. While the divergent contribution (5.13) contains pieces proportional to $G^2$, the field strength of the scalar field is only linear in $G$. This means that we should expect two-loop contributions to $\gamma_\phi$ of order $G^2$. Indeed if one notes that the powers of $G$ are brought into the diagrams by gravitational propagators, we can straightforwardly see that the following two diagrams, for instance, will potentially contribute to $\gamma_s$ at order $G^2$



(6.9)

Therefore, if we wanted to solve the CS equation (6.8) at order $G^2$ we would need to add the contribution coming from the two-loop correction to $\gamma_s$. As a consequence, we can only trust our result here up to order $G$ and thus we will cut the perturbative solution to (6.8) at this order. It reads

$$\beta(\lambda)=\frac{9\lambda^2}{2\pi^2}-\frac{3G\lambda m^2(1+2\sigma(1+8\xi+28\xi^2))}{16\pi^2\sigma}, \tag{6.10}$$

$$\gamma(b_2)=G\left(-\frac{3m^2(1+2\sigma(1+8\xi+4\xi^2))}{16\pi^2\sigma}+\frac{9\lambda(1+\sigma(2+32\xi+8\xi^2))}{2b_2\pi^2\sigma}\right), \tag{6.11}$$

$$\gamma(b_4)=-\frac{3Gm^2(1+2\sigma(1+8\xi+4\xi^2))}{16\pi^2\sigma}. \tag{6.12}$$

In the limit of decoupling gravitation $G\to 0$, the running of higher-derivative terms freeze as expected, while the running of $\lambda$ matches the text-book result[4] for $\lambda\phi^4$.

We will find the same issue previously discussed when trying to solve the CS equation for the running of the non-minimal coupling

$$\left(\mu\frac{\partial}{\partial\mu}+\beta(\xi)\frac{\partial}{\partial\xi}+a_4\gamma(a_4)\frac{\partial}{\partial a_4}+\varsigma\gamma(\varsigma)\frac{\partial}{\partial\varsigma}+2\gamma_\phi\right)\langle\phi(-p)\phi(-p)H_{\mu\nu}(2p)\rangle=0, \tag{6.13}$$

since both $\gamma(a_4)$ and $\gamma_\phi$ are of order $G$ and we expect corrections of order $G^2$ coming from two-loop divergences.

We therefore again cut the solution to this equation at order $G$. Taking the form of the divergence (5.16) and the counter-terms (5.45) and ordering the CS equation in powers of $p^\mu$ we find

$$\beta(\xi)=\frac{\lambda(1+6\xi)}{2\pi^2}-\frac{Gm^2\xi(3+\sigma(6+44\xi+72\xi^2))}{32\pi^2\sigma}, \tag{6.14}$$

$$\gamma(\varsigma)=\frac{G(9+40\xi-4\sigma(7+15\xi+9m^2\varsigma)+8\sigma^2(-6-41\xi-78\xi^2+36\xi^3-9m^2(1+4\xi(2+\xi))\varsigma))}{384\pi^2\sigma^2\varsigma}. \tag{6.15}$$

[4]Note however that we are defining our coupling without the standard 4! denominator.

Finally, we write the corresponding equation for the two-point function of the graviton

$$\left(\mu\frac{\partial}{\partial\mu}+G\gamma(G)\frac{\partial}{\partial G}+\beta(\alpha)\frac{\partial}{\partial\alpha}+\beta(\rho)\frac{\partial}{\partial\rho}\right)\langle H_{\mu\nu}(-p)H_{\alpha\beta}(p)\rangle=0. \tag{6.16}$$

Ordering it by powers of the momentum, we complete the computation of the renormalization group flow of the theory with the results

$$\gamma(G)=-\frac{Gm^2(1+6\xi)}{12\pi^2}, \tag{6.17}$$

$$\beta(\alpha)=-\frac{5+10\sigma-\sigma^2(71-48\xi-144\xi^2)}{1152\pi^2\sigma^2}, \tag{6.18}$$

$$\beta(\rho)=-\frac{50+100\sigma+811\sigma^2}{960\pi^2\sigma^2}. \tag{6.19}$$

As a final note in this section, let us note that all the $\beta$ functions and $\gamma$-functions defined here reduce to those of the case of background gravity once the right limit $G\to 0$ is taken. The only subtlety comes from the running of $\alpha$ and $\beta$, which are however irrelevant because in the limit $G\to 0$ they are subleading with respect to the Einstein-Hilbert term in the action.

This completes the computation of the one-loop $\beta$-functions for the unimodular scalar tensor-theory. In the appendix B we discuss the results in the absence of the scalar field, in order to establish a comparison with previous works.

## 7 Unimodular Gravity versus General Relativity

Once the renormalization group flow of the coupling constants is computed, we come to the question that originated this work — is Unimodular Gravity equivalent to General Relativity when coupled to matter?

Although the question is simple enough, the answer is not so. First of all, we note that although the quantization of UG looks much more complicated than the one of GR, no new counter-terms are required in order to absorb all one-loop divergences. Indeed, all required counter-terms — depicted in (5.28), (5.36), (5.42), and (5.47) — are exactly the same ones that would be required in GR, just appended with the condition $|g|=1$.

In order to differentiate both theories we shall then look at a physical observable. However, the running of the couplings that we have just derived does not classify as such, as it can be observed by the explicit dependence on the gauge fixing parameter $\sigma$ of most of them. Moreover, some of our results could, in principle, be modified by a non-linear redefinition of the gravitational field $H_{\mu\nu}\to H_{\mu\nu}(\phi)$, clearly denoting that they lack a physical meaning due to operator mixing after the field redefinition [57, 58]. To determine something which can be thought as physical, we must then find which combinations of the couplings are independent of the gauge choice and blind to field redefinitions. Those, known as *essential couplings*, will be the couplings that control correlation functions of observable quantities. Only the $\beta$-functions of essential couplings have an intrinsic physical meaning. They can be determined by noting that they correspond to the only combinations that do not change when we add to the action a piece proportional to the classical equations of motion [59, 60].

In the following we will focus only on those couplings which are present in the bare Lagrangian, ignoring the higher-derivative operators. Moreover, and for simplicity, we will consider solutions to the eom with unimodular background determinant $|\bar{g}| = 1$. Under this assumption, the background action reads

$$S = \int d^4x \Big( -\frac{1}{2G}\bar{R} + \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{\xi}{2}\phi^2\bar{R} - \frac{m^2}{2}\phi^2 - \lambda\phi^4 \Big). \tag{7.1}$$

In this frame, the equations of motion are the traceless Einstein equations (1.1), which by using Bianchi identities are equivalent to the full set of Einstein equations with an arbitrary cosmological constant $\mathcal{C}$

$$\bar{R}_{\mu\nu} - \frac{1}{2}\bar{R}g_{\mu\nu} + \mathcal{C}\bar{g}_{\mu\nu} = GT_{\mu\nu}. \tag{7.2}$$

The only scalar quantity up to two derivatives that we can form with the eom is then the trace of Einstein equations

$$\mathcal{E} = \bar{R} + GT - 4\mathcal{C} = \bar{R} + G\left[-(1+6\xi)\partial_\mu\phi\partial^\mu\phi - 6\xi\phi\Box\phi + 2m^2\phi^2 + 4\lambda\phi^4 + \xi G\phi^2\bar{R}\right] - 4\mathcal{C}, \tag{7.3}$$

where we have used

$$T_{\mu\nu} = (1+2\xi)\partial_\mu\phi\partial_\nu\phi + 2\xi\phi\bar{\nabla}_\mu\partial_\nu\phi - \xi\phi^2\bar{R}_{\mu\nu} - \bar{g}_{\mu\nu}\left(\frac{1+4\xi}{2}\partial_\alpha\phi\partial^\alpha\phi + 2\xi\phi\Box\phi - \frac{m^2}{2}\phi^2 - \lambda\phi^4 - \frac{\xi}{2}\phi^2\bar{R}\right). \tag{7.4}$$

We thus add a piece proportional to the trace of the eom to the action

$$S \to S + \int d^4x \, \frac{\Sigma\mathcal{E}}{2G}, \tag{7.5}$$

where $\Sigma$ is a constant parameter. Under this addition, ignoring the cosmological constant and integrating by parts, we find that the couplings transform as

$$\delta_\Sigma G = \Sigma G, \tag{7.6}$$

$$\delta_\Sigma m^2 = -2\Sigma m^2, \tag{7.7}$$

$$\delta_\Sigma \lambda = -2\Sigma\lambda, \tag{7.8}$$

$$\delta_\Sigma \xi = -\Sigma\xi. \tag{7.9}$$

Thus, essential couplings will be combinations of these that are invariant under the addition of the evanescent piece proportional to $\Sigma$. Additionally, since we want to avoid the arbitrariness tied to the reference scale for dimensionful quantities, we will demand our essential couplings to be dimensionless as well.

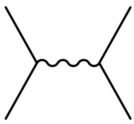
Out of $G$ and $m^2$ we can build the following scale-invariant coupling

$$\mathfrak{G} = Gm^2, \qquad \delta\mathfrak{G} = -\Sigma\mathfrak{G}, \tag{7.10}$$

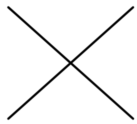
which transforms as $\sqrt{\lambda}$ and therefore we can build a ratio which is an essential coupling, given by

$$\Delta = \frac{\mathfrak{G}^2}{\lambda}. \tag{7.11}$$

It corresponds to the relative strength between interactions dictated by the Einstein-Hilbert term and the self-interaction of the scalar in the Lagrangian, as measured in a $2 \to 2$ scattering of scalar fields. The two channels involved in this process, graviton exchange and contact interactions, are schematically given by



$$\sim \frac{Gm^4}{p^2} = \mathfrak{G}^2 \left(Gp^2\right)^{-1},$$



$$\sim \lambda, \tag{7.12}$$

and we can see that their ratio corresponds to $\Delta$ when the momentum exchanged by the graviton is set at the scale where gravitational interactions dominate $p^2 \sim G^{-1} \sim M_{\mathrm{P}}^2$. Thus, the running of $\Delta$ indicates at which energy scale gravitation becomes important and cannot be ignored when doing QFT with scalar fields. Its $\beta$-function can be easily obtained from the ones of the couplings in the Lagrangian. It reads

$$\beta(\Delta) = \frac{\Delta(-9\lambda + \mathfrak{G}(-1 - 6\xi + 45\xi^2))}{6\pi^2}, \tag{7.13}$$

where as expected, the dependence on the gauge parameter $\sigma$ has cancelled out. In principle, we could also define two other ratios $\Delta_i$ involving $\xi$. However, these enter strong-coupling when either $\mathfrak{G}, \lambda, \xi \to 0$ since their $\beta$-functions are not polynomial in the couplings.[5] We will therefore refrain from discussing them hereinafter.

An unpleasant property about $\beta(\Delta)$ that we must remark here is that although $\Delta$ is an essential coupling, its running, albeit being gauge invariant, depends on the non-physical quantities $\lambda, \mathfrak{G}$ and $\xi$ independently. A similar property has been already noted before in the context of asymptotic safety[6] for the running of essential couplings [61, 62]. It implies that in this situation one cannot disentangle physical contributions from un-physical ones but instead one needs to first compute the latter in order to derive the former. It also poses a conundrum on understanding how the value of $\Delta$ can indeed remain essential along the RG flow and at higher order in perturbation theory. Here we cannot offer any satisfactory explanation beyond hinting that this might be a consequence of the non-renormalizability of the theory.

[5]This can be seen by writing the $\beta$-functions in the form

$$\beta(\Delta_i) = \Delta_i(\dots),$$

where the dots indicate an expansion in the couplings of the theory, which will depend on the particular coupling chosen. For instance, for $\Delta_\xi = \mathfrak{G}/\xi$ the leading term within the parenthesis goes as $\xi^{-1}$ and therefore it is not perturbative in the sense discussed along this work. Note that this is not the case for the coupling (7.13).

[6]We are grateful to R. Percacci for pointing this out.

Following the same reasoning depicted before, we see that the same definition of essential couplings holds for the case of GR, when we restore the $\sqrt{|g|}$ in the integration measure and shift the action accordingly by modifying (7.5). Although the value for the running of the couplings in GR can be found in the literature [63], we prefer here to re-derive the needed ones with the same techniques described for UG. Details of this computation can be found in appendix A. In that case, the value of the $\beta$-function for the essential coupling takes the form

$$\beta_{\text{GR}}(\Delta) = \frac{\Delta(-9\lambda + \mathfrak{G}(-1 + 39\xi + 45\xi^2))}{6\pi^2}, \tag{7.14}$$

which is subtlety but clearly different from (7.13).

Let us remark that the coupling $\Delta$ has a physical meaning. It gives a definite answer to the question of when gravitational interactions can be disregarded. As such, the fact that it does not agree with the UG result is a smoking gun that the theories cannot be considered equivalent once gravitation is dynamical. However, we see that the difference is very minor. The one-loop result $\beta(\Delta)$ agrees in both theories in two very important limiting cases, $\xi \to 0$ and $\xi \gg 1$.

The first limit corresponds to a scalar field minimally coupled. In that case, we see that although the theory is very complicated, the running of the physical parameter $\Delta$ is identical to the more easily computed one in GR. It seems that non-minimal coupling is then an important ingredient to violate the equivalence. One could argue then that the full identification of both theories seems to be connected in a very non-trivial way to the satisfaction of the strong equivalence principle.

The second case is also interesting, since it corresponds to the limit in which several models of inflation — in particular Higgs [64, 65] and Higg-Dilaton inflation [66–68] — are successful. Although strictly speaking we have performed our computations in the limit $\xi \ll 1$ and therefore they would not be valid in the large $\xi$ limit, let us note that in the case $\xi \gg 1$ we can also take $G \gg 1$ and then the role of both couplings is formally exchanged in the action for the case of approximately constant scalar profiles. In that case, the equations for gravity reduce to

$$\xi\phi^2 \left( R_{\mu\nu} - \frac{1}{4} R g_{\mu\nu} + \mathcal{O}\left(\frac{1}{\xi}\right) \right) = 0, \tag{7.15}$$

which corresponds to the vacuum equations, up to sub-leading corrections. This means that in the $\xi \gg 1$ limit, and around flat space, the theory becomes indistinguishable from the case $\xi \ll 1$ and therefore our result should hold.

In any other intermediate value of $\xi$ we find that UG and GR *are not equivalent*. Although this might look minor, since the difference is very subtle, it might have influence in intermediate energy regimes when moving along the flow. For instance, in the thermal history of our Universe. It also poses a question mark on the validity of quantum computations performed in UG without taking into account the very complicated quantization structure and just assuming that, since they are classically equivalent, one can compute in GR instead. This is clearly wrong at the light of our result.



JHEP09(2020)041

Let us finally remark that we strongly believe in the robustness of our result. We have derived it independently by using two different computer codes in different languages. Moreover, the preservation of gauge invariance — both at the background level and from independence of $\sigma$ — is a very non-trivial issue and any minor modification of any ingredient in the computation would produce a result not satisfying it.

## 8 Discussion and conclusions

In this paper we have studied the question of the equivalence between General Relativity and Unimodular Gravity. Although the answer is positive when we look to classical physics, or even classical gravitation in the presence of quantum matter, there are important subtleties when gravitons are dynamical and allowed to run freely in the loops.

In order to discuss this property in a QFT manner, we started by formulating the theory in a frame where the constraint $|g| = 1$ is automatically satisfied, by redefining the metric as $g_{\mu\nu} = |\tilde{g}|^{\frac{1}{4}}\tilde{g}_{\mu\nu}$. In this frame UG becomes a pretty non-standard theory enjoying an extended gauge symmetry, the product of *TDiff* and *Weyl*, that we call *WTDiff*. However, in this form the main properties of UG are explicit. The counting of degrees of freedom is straightfoward and the traceless character of the eom is explicit. In order to compute one-loop corrections we exploited this symmetry by using the background field method in combination with the construction of a Weyl invariant geometry.

The construction of the gauge sector — combining gauge fixing and ghost action — becomes surprisingly much more cumbersome in UG than in GR, mainly due to the fact that *TDiff* generators are not independent but rather constrained to be a transverse vector. Although they can then be represented by using a transverse projector and the full gauge system solved by using BRST symmetry, this generates a tower of new ghost fields of bosonic character. These fields actually couple with the graviton degree of freedom, showing a first non-trivial difference of UG with respect to GR, at least at the technical level. Even for tree-level computations, one cannot just ignore the gauge sector, since the kinetic mixing between $h_{\mu\nu}$ and the bosonic ghosts will have an impact on the propagator of the former.

Nevertheless, once the issue with constructing the gauge sector is solved, then the one-loop corrections to the correlation functions of the theory can be computed in a standard manner by expanding the background around flat space and looking at Feynman diagrams carrying perturbations of the background in the external legs. Although there are plenty of them — in particular due to the mixing of the graviton with the bosonic fields —, this is a task that we were able to carry out with the help of computer codes specialized in tensor algebra.

The result of our computations are the complete set of $\beta$-functions and anomalous dimensions of all the couplings involved in the action, computed in the one-loop approximation and at order $G$, which we reproduce here to collect them together

$$\gamma_\phi = \frac{3Gm^2(1+2\sigma(1+4\xi(2+\xi)))}{64\pi^2\sigma},$$

$$\gamma(m^2) = \frac{3\lambda}{2\pi^2} - \frac{3Gm^2(1+2\sigma(1+8\xi(1+\xi)))}{32\pi^2\sigma},$$

$$
\begin{aligned}
\gamma(a_4) &= -G\left(\frac{3{+}2\sigma(3{+}8\xi)}{32a_4\pi^2\sigma}{+}\frac{3m^2(1{+}2\sigma(1{+}8\xi{+}4\xi^2))}{32\pi^2\sigma}\right),\\
\beta(\lambda) &= \frac{9\lambda^2}{2\pi^2}{-}\frac{3G\lambda m^2(1{+}2\sigma(1{+}8\xi{+}28\xi^2))}{16\pi^2\sigma},\\
\gamma(b_2) &= G\left(-\frac{3m^2(1{+}2\sigma(1{+}8\xi{+}4\xi^2))}{16\pi^2\sigma}{+}\frac{9\lambda(1{+}\sigma(2{+}32\xi{+}8\xi^2))}{2b_2\pi^2\sigma}\right),\\
\gamma(b_4) &= -\frac{3Gm^2(1{+}2\sigma(1{+}8\xi{+}4\xi^2))}{16\pi^2\sigma},\\
\beta(\xi) &= \frac{\lambda(1{+}6\xi)}{2\pi^2}{-}\frac{Gm^2\xi(3{+}\sigma(6{+}44\xi{+}72\xi^2))}{32\pi^2\sigma},\\
\gamma(\varsigma) &= \frac{G(9{+}40\xi{-}4\sigma(7{+}15\xi{+}9m^2\varsigma){+}8\sigma^2({-}6{-}41\xi{-}78\xi^2{+}36\xi^3{-}9m^2(1{+}4\xi(2{+}\xi))\varsigma))}{384\pi^2\sigma^2\varsigma},\\
\gamma(G) &= -\frac{Gm^2(1{+}6\xi)}{12\pi^2},\\
\beta(\alpha) &= -\frac{5{+}10\sigma{-}\sigma^2(71{-}48\xi{-}144\xi^2)}{1152\pi^2\sigma^2},\\
\beta(\rho) &= -\frac{50{+}100\sigma{+}811\sigma^2}{960\pi^2\sigma^2}.
\end{aligned}
\tag{8.1}
$$

They include the couplings present in the classical Lagrangian but also new couplings controlling the strength of higher-derivative terms in the EFT expansion, as required by the non-renormalizability of gravity. The full one-loop EFT action that we obtain is then

$$
\begin{aligned}
S_{\text{1-loop}} = \int d^4x\bigg[&-\frac{1}{2G}R+\frac{1}{2}\partial_\mu\phi\partial^\mu\phi-\frac{\xi}{2}\phi^2R-\frac{m^2}{2}\phi^2-\lambda\phi^4+\frac{a_4}{2}\Box\phi\Box\phi+\frac{b_2}{8}\phi^2(\partial\phi)^2\\
&+\frac{b_4}{24}(\partial\phi)^4+\frac{\varsigma}{2}\partial_\mu\phi\partial^\mu\phi R+\alpha R^2+\rho\left(R_{\mu\nu}R^{\mu\nu}-\frac{1}{3}R^2\right)\bigg],
\end{aligned}
\tag{8.2}
$$

in the frame where the metric is unimodular $|g|=1$.

These runnings are however dependent on the gauge choice used to quantize the theory and therefore they do not correspond to physical quantities. Out of them, we identify the combination $\Delta = G^2m^4\lambda^{-1}$, which controls the relative strength of gravitational interactions with respect to scalar self-interactions and therefore has a physical meaning. It corresponds to an essential coupling of the theory. Its running is then gauge invariant and reads

$$
\beta(\Delta) = \frac{\Delta(-9\lambda+\mathfrak{G}(-1-6\xi+45\xi^2))}{6\pi^2}. \tag{8.3}
$$

We find that this quantity actually differs from the corresponding result in GR, which can be found in (7.14), whenever the non-minimal coupling ranges on intermediate values. Only in the two extremal limits $\xi\to 0$ and $\xi\gg 1$, our result agrees with the general relativistic one. We interpret the first of these agreements as a consequence of the strong equivalence principle, which is then violated by non-minimal coupling. The second coincidence can be traced back to the singular behaviour of the eom in the large $\xi$ limit. However, the difference in intermediate regimes might be important when considering situations in which following the running of physical quantities along an energy history is critical, like the thermal history of the Universe.

Altogether this poses a question on the validity of several approaches found in the literature to computing quantum corrections in the case of UG. One cannot just assume that the theories are equivalent, restore $\sqrt{|g|}$ in the action, and compute quantum corrections in GR by hiding under the carpet the fact that actually one wants to work with UG. In particular, it would be interesting to revisit these results and their effects in models of inflation, which closely resemble the case studied here.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## A Computation of $\beta$-functions in General Relativity

We summarize here the computation of the $\beta$-function of the composite coupling $\Delta$ in GR, following the same techniques as in the case of UG. We will consider the action equivalent to $S_{\rm UG} + S_{\rm matter}$ by restoring $\sqrt{|g|}$

$$S_{\rm GR} = \int d^4x \sqrt{|g|} \left( -\frac{1}{2G} R + \frac{1}{2} \partial_\mu \phi \partial^\mu \phi - \frac{m^2}{2} \phi^2 - \lambda \phi^4 - \frac{\xi}{2} \phi^2 R \right). \tag{A.1}$$

We will also expand this around a background configuration for the gravitational field $g_{\mu\nu} = \bar{g}_{\mu\nu} + h_{\mu\nu}$. However, the absence of Weyl invariance and the independence of the generators of *Diff* allow us to construct a standard gauge fixing *à la Feynman*

$$S_{gf} = -\frac{\sigma}{2G} \int d^4x \sqrt{|\bar{g}|}\, F_\mu F^\mu, \tag{A.2}$$

with $F_\mu$ analogous to (3.43)

$$F_\mu = \bar{\nabla}^\nu h_{\mu\nu} - \frac{1}{2} \bar{\nabla}_\mu h. \tag{A.3}$$

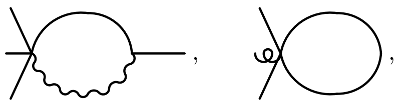
Since we only want to compute the running of $\lambda$, $G$ and $m^2$, we will not need to add the action for the ghost fields in this case.

Expanding now the background metric around flat space $\bar{g}_{\mu\nu} = \eta_{\mu\nu} + H_{\mu\nu}$ and computing correlation functions involving $H_{\mu\nu}$ and $\phi$, we can derive the running of the couplings that we are interested in. The computation is analogous to that of UG with only two



JHEP09(2020)041

important changes — the diagrams containing bosonic ghosts are now absent, and there are two extra diagrams to be considered



(A.4)

whose contributions are actually critical to ensure gauge invariance.

Running our code and computing all the Feynman diagrams, renormalizing, and solving the CS equation, we find

$$\gamma_{\mathrm{GR},\phi} = \frac{Gm^2(-1+6\sigma(-1-\xi+\xi^2))}{16\pi^2\sigma}, \tag{A.5}$$

$$\gamma_{\mathrm{GR}}(m^2) = \frac{6\lambda - 3Gm^2\xi(1+2\xi)}{4\pi^2}, \tag{A.6}$$

$$\beta_{\mathrm{GR}}(\lambda) = \frac{3\lambda(3\lambda - Gm^2\xi(6+7\xi))}{2\pi^2}, \tag{A.7}$$

$$\gamma_{\mathrm{GR}}(G) = -\frac{Gm^2(1+6\xi)}{12\pi^2}, \tag{A.8}$$

from which we can compute the running of $\Delta = Gm^2\lambda^{-1}$ to be

$$\beta_{\mathrm{GR}}(\Delta) = \frac{\Delta(-9\lambda + \mathfrak{G}(-1+39\xi+45\xi^2))}{6\pi^2}, \tag{A.9}$$

where $\mathfrak{G} = Gm^2$. We have also cross-checked the results of $\gamma_{\mathrm{GR},\phi}$ and $\gamma_{\mathrm{GR}}(m^2)$ by using the three-point function mixing scalar fields and a graviton $\langle\phi(-p)\phi(-p)H_{\mu\nu}(2p)\rangle$.

## B Quantum corrections to vacuum Unimodular Gravity

For completeness, we take here a look to the renormalization group flow of the theory in the case of pure UG, when the action is only

$$S_{\mathrm{UG}} = -\frac{1}{2G}\int d^4x\, |\tilde{g}|^{\frac{1}{4}} \left(\tilde{R} + \frac{3}{32}\frac{\tilde{\nabla}_\mu|\tilde{g}|\tilde{\nabla}^\mu|\tilde{g}|}{|\tilde{g}|^2}\right). \tag{B.1}$$

In this case, and using the renormalization scheme previously described in this work, we see that $G$ does not receive divergent one-loop corrections. Only the higher-derivative terms in (5.47) will run in this case. Subtracting the contribution of the scalar loops from our result and repeating the steps in section 5, we find that the running of the higher-derivative terms is controlled by

$$\beta_{\mathrm{vacuum}}(\alpha) = -\frac{5+10\sigma-30\sigma^2}{1152\pi^2\sigma^2}, \tag{B.2}$$

$$\beta_{\mathrm{vacuum}}(\rho) = -\frac{50+100\sigma+807\sigma^2}{960\pi^2\sigma^2}. \tag{B.3}$$

As we expected, these $\beta$-functions are gauge dependent, since they do not correspond to essential couplings. In order to check the robustness of our result we will then reconstruct the full non-linear divergence by noting that due to background field invariance, the divergent part of the one-loop correction — the term proportional to $\epsilon^{-1}$ — can be obtained by expanding the following action around flat space

$$S_{\text{div}} = -\frac{i}{\epsilon} \int d^4x \left( \beta_{\text{vacuum}}(\alpha) R^2 + \beta_{\text{vacuum}}(\rho) \left( R_{\mu\nu} R^{\mu\nu} - \frac{1}{3} R^2 \right) \right). \tag{B.4}$$

Otherwise, we would not be able to absorb it into a counter-term.

Now, gauge independence can be tested with the help of the Kallosh-DeWitt theorem [59, 60]. Since addition of terms proportional to the eom must be able to shift every gauge-dependent quantity, only when we take the previous divergence to be on-shell we must find a gauge-independent result. For the theory in vacuum, the eom of UG are equivalent to the full set of Einstein equations (1.3), which imply

$$R_{\mu\nu} = \mathcal{C} g_{\mu\nu}, \qquad R = 4\mathcal{C}. \tag{B.5}$$

Plugging this into (B.4) we get

$$S_{\text{div}} = -\frac{173 i}{80\pi^2 \epsilon} \int d^4x\, \mathcal{C}^2, \tag{B.6}$$

where the gauge dependence has vanished. Moreover, we find that the divergence is independent of the field, since there is no $\sqrt{|g|}$ term due to the unimodular condition. Therefore we conclude, in the same lines as [22], that UG is one-loop finite *even in the presence of a cosmological constant*.

Note however that our result cannot shed any light on the discrepancy of the results between [22] and [35], since here we only have access to gauge dependent quantities whose on-shell value is not dynamical. Due to the fact that we are working in perturbation theory around flat space, we cannot obtain the value of the topological term, which should be gauge independent by itself.



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JHEP09(2020)041

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# CONCLUSIONS

In this thesis, we have presented various research directions aimed at solving some of the problems of GR from a bottom-up point of view. Let us briefly summarize the key points and the main results of each of the three articles.

*First Order (FO)* quadratic theories of gravity have been the object of study of the first chapter. We started by reviewing the well-known linear and quadratic theories of gravity in the *Second Order* formalism, using the usual spin projectors to compute the propagating degrees of freedom for such theories. After this warm-up, we turned to the FO formalism. When the metric and the connection are treated as independent fields, the connection field plays the main role in these theories, as the graviton propagator vanishes when expanded around flat spacetime. Therefore, the graviton degrees of freedom must come from a spin two piece encoded in the three-index connection field. If this were not the case, these theories would not be able to describe the gravitational interaction. To settle this issue, we carried out a complete analysis of the spin content of the connection field. We focused, in particular, on non-metric connections with vanishing torsion.[1] As a first step, the usual four-index spin projectors were generalized to six-index projectors applicable to the symmetric connection field. The main result of the work was the obtention of a complete basis of 22 projectors, corresponding to one spin three, four spin two, eleven spin one, and six spin zero projectors. All the elements with a different spin content are orthogonal among themselves, although they are not orthogonal to the other projectors of the same spin. This makes it difficult to get to the equations of motion for each of the spin components, and thus, the computation of the asymptotic degrees of freedom is still missing in this preliminary analysis. Nevertheless, they can be used to invert the quadratic operator mediating between two symmetric connection fields and thus, to obtain the propagating degrees of freedom. Generically, a spin three component is present, which is tied to the non-vanishing of the Riemann squared term in the action. This piece needs further investigation, in particular, regarding the possible inconsistencies arising when building an interacting theory for it. Finally, a particular form of the connection perturbations was studied, namely, the Levi Civita form in terms of the derivatives of the metric perturbations. This case resembles the usual *Second Order formalism*, and as expected, the six-index projectors degenerate into the known four-index ones multiplied by a momentum squared term in the

[1] Nevertheless, a study of the torsion-full metric connection was carried out in one of the appendices.

process. Therefore, these extra powers of momentum make the propagators quartic again and lead to the non-unitarity of the theory.

The goal of the second article was to analyze the (in)equivalence of conformal and Weyl invariance for the gravitational field itself. To do that, we studied the most general Lagrangian, up to dimension-six operators, for a spin two field around flat spacetime. We mainly focused on dimension-four and dimension-six operators quadratic in the fields, containing two or four derivatives, and matching the possible low-energy terms of linear and quadratic theories of gravity. We then classified the different possibilities depending on their symmetries. To make contact with gravitational theories, we considered theories invariant under (transverse) diffeomorphism and Weyl transformations, dubbing *LWDiff* and *LWTDiff* to these combinations. We showed that any quadratic (in the field) Lorentz invariant theory constructed purely from dimension four or dimension six operators is automatically conformal invariant. When combining both types of operators, however, no invariant combination is left. On the other hand, Weyl invariance is quite restrictive, in particular, when combined with *Diff* invariance. *WTDiff* invariant theories are more natural in this sense and can be constructed as the low-energy limit of gravitational theories where the metric is transformed as $g_{\mu\nu} \rightarrow g^{-1/n} g_{\mu\nu}$, where $n$ is the spacetime dimension. These findings clarified the central question of the article: Weyl invariance and conformal invariance are not equivalent symmetries for spin two theories. The inequivalence is found to work in both directions. The low-energy limit of a *WTDiff* theory containing linear and quadratic terms in the curvature results in a combination of dimension four and six operators, and as stated above, these are not conformal invariant. Finally, an interesting analysis of the interaction terms was included, where an iterative procedure for the construction of *LWDiff* invariant potentials was developed.

In the final chapter, the quantum (in)equivalence of GR and UG was investigated when a coupling to matter was included. In particular, we considered a non-minimally coupled massive scalar field with a quartic interaction controlled by $\lambda$. The fact that the potential terms such as the mass and the quartic interaction term do not couple to the metric in UG posed a first difference, modifying the vertices involved in the computations. We avoided the use of constrained fields and wrote the theory in terms of an unconstrained metric in the *WTDiff* description of UG. This theory is then invariant under Weyl transformations and transverse diffeomorphisms. The first part of the paper was devoted to dealing with the construction of the gauge sector corresponding to the combined *WTDiff* symmetry. BRST techniques were used, and the needed auxiliary fields and ghost fields were introduced. These fields couple to the graviton modifying the graviton propagator, so that the computations differed for both theories at least at the technical level. After settling the issue of the gauge sector, we carried out the standard computation of all one-loop divergences in the background field approach for both theories, and computed all

$\beta$-functions and anomalous couplings in the Lagrangian. With this in hand, we compared the renormalization group flow of a certain *physical* combination, given by $\Delta = G_N^2 m^4 \lambda^{-1}$. This combination turned out to differ for GR and UG for intermediate values of the non-minimal coupling, $\xi$. There is a caveat here, however. The running of $\Delta$ is a function of non-physical couplings such as $\lambda$ and $G_N^2 m^4$ so that we cannot completely disentangle these contributions from the *physical* ones. Although it could be caused by the non-renormalizability of the theory, this issue requires further investigation. For instance, the same computations could be carried out for renormalizable theories, such as quadratic theories of gravity, to analyze whether this feature is still present.

As a final remark, we have seen that the pillars of modern Theoretical Physics have been built after many small contributions and connections between things that were already known. In the same spirit, the goal of these works has been to explore various subproblems of the gravitational interaction with the hope of adding a modest grain of sand to the construction of a Quantum theory of Gravity. And hopefully, one day, we will sit in the edge of the cube, *adimiring the land where $G_N$, c and $\hbar$ lived happily ever after*.

# CONCLUSIONES

En esta tesis se han presentado varias direcciones de investigación destinadas a resolver algunos de los problemas de la RG tomando como punto de partida la teoría a bajas energías. A continuación resumimos brevemente los puntos clave y los principales resultados de cada uno de los tres artículos.

Las teorías cuadráticas tratadas en el formalismo de *Primer Orden (PO)* han sido el objeto de estudio del primer capítulo. Se ha empezado revisando las conocidas teorías lineales y cuadráticas de la gravedad en el formalismo de *Segundo Orden*, utilizando los proyectores de espín habituales para calcular los grados de libertad de propagación de dichas teorías. Después de este calentamiento, hemos pasado al formalismo de PO. Cuando la métrica y la conexión se tratan como campos independientes, el campo de la conexión juega el papel principal en estas teorías ya que el propagador del gravitón desaparece cuando se expande alrededor del espaciotiempo plano. Por tanto, los grados de libertad del gravitón deben proceder de una pieza de espín dos codificada en el campo de la conexión de tres índices. Si no fuera así, estas teorías no podrían describir la interacción gravitatoria. Para resolver esta cuestión, hemos llevado a cabo un análisis completo del contenido de espín del campo de conexión. Nos hemos centrado, en particular, en las conexiones no métricas con torsión nula.[2] Como primer paso, se han generalizado los proyectores de espín habituales de cuatro índices a proyectores de seis índices aplicables al campo de la conexión simétrica. El principal resultado de este trabajo es la obtención de una base completa de 22 proyectores, correspondientes a un espín tres, cuatro espines dos, once espines uno y seis espines cero. Todos los elementos con diferente contenido de espín son ortogonales entre sí, aunque no lo son respecto a los demás proyectores del mismo espín. Esto dificulta la obtención de las ecuaciones de movimiento para cada uno de las componentes de espín y, por tanto, el cálculo de los grados de libertad asintóticos sigue faltando en este análisis preliminar. No obstante, pueden utilizarse para invertir el operador cuadrático que media entre dos campos de conexión simétricos y, por tanto, obtener los grados de libertad de propagación. En general, hay una componente de espín tres, que está ligada a la presencia del término de Riemann al cuadrado en la acción. Esta pieza requiere de un análisis más detallado, en particular, en lo que respecta a las posibles incoherencias que surgen al construir una teoría interactiva para ella. Por último, se ha estudiado una forma particular de las perturbaciones de la conexión, en particular, la de Levi Civita, construida a par-

[2] No obstante, en uno de los apéndices se lleva a cabo un estudio de la conexión métrica con torsión.

tir de las derivadas de las perturbaciones de la métrica. Este caso se asemeja al formalismo habitual de segundo orden, y como se esperaba, los proyectores de seis índices degeneran en los conocidos proyectores de cuatro índices multiplicados por un término de momento al cuadrado. Por lo tanto, estas potencias extra de momento hacen que los propagadores vuelvan a ser cuárticos y conducen a la no unitariedad de la teoría.

El objetivo del segundo artículo ha sido analizar la (in)equivalencia de la invariancia conforme y de Weyl para el propio campo gravitatorio. Para ello, hemos estudiado el Lagrangiano más general, hasta operadores de dimensión seis, para un campo de espín dos en torno al espaciotiempo plano. Nos hemos centrado principalmente en operadores de dimensión cuatro y dimensión seis cuadráticos en los campos y con dos o cuatro derivadas, que coinciden con los posibles términos de baja energía de las teorías lineales y cuadráticas de la gravedad. A continuación, hemos clasificado las distintas posibilidades en función de sus simetrías. Para hacer contacto con las teorías gravitacionales, hemos considerado teorías invariantes bajo difeomorfismos (transversos) y transformaciones de Weyl, apodando *LWDiff* y *LWTDiff* a estas combinaciones. Hemos demostrado que cualquier teoría invariante de Lorentz cuadrática (en el campo), construida puramente a partir de operadores de dimensión cuatro o seis, es automáticamente invariante conforme. Sin embargo, al combinar ambos tipos de operadores, no sobrevive ninguna combinación invariante. Por otro lado, la invariancia Weyl es bastante restrictiva, en particular, cuando se combina con la invariancia *Diff*. Las teorías invariantes *WTDiff* son más naturales en este sentido y pueden construirse como el límite de baja energía de las teorías gravitacionales donde la métrica se transforma como $g_{\mu\nu} \to g^{-1/n} g_{\mu\nu}$, siendo $n$ la dimensión del espaciotiempo. Estos resultados aclaran la cuestión central del artículo: la invariancia de Weyl y la invariancia conforme no son simetrías equivalentes para las teorías de espín dos. Se encuentra además que esta desigualdad funciona en ambas direcciones. El límite de baja energía de una teoría *WTDiff* que contiene términos lineales y cuadráticos en la curvatura incluye una combinación de operadores de dimensión cuatro y seis, y como se ha dicho anteriormente, esta combinación no es invariante conforme. Por último, se ha incluido un análisis de los términos de interacción, y se ha desarrollado un procedimiento iterativo para la construcción de potenciales invariantes bajo *LWDiff*.

En el último capítulo, se investiga la (in)equivalencia cuántica de la Relatividad General (RG) y la Gravedad Unimodular (GU) cuando se incluye un acoplamiento a la materia. En particular, hemos considerado un campo escalar masivo no mínimamente acoplado con una interacción cuártica controlada por $\lambda$. El hecho de que los términos de potencial como la masa y el término de interacción cuártica no se acoplen a la métrica en GU plantea una primera diferencia que modifica los vértices implicados en los cálculos. Se ha evitado el uso de campos restringidos y se ha escrito la teoría en términos de una métrica no restringida, dando lugar a la versión *WTDiff* de GU. Esta nueva descripción de GU es invariante bajo transformaciones de Weyl y difeomorfismos transversos. La primera parte del artículo se ha dedicado a tratar la construcción del sector gauge correspondiente a la simetría combinada *WTDiff*. Se han utilizado técnicas de BRST y se han introducido los campos auxiliares y los campos fantasma necesarios. Estos campos se acoplan al gravitón modificando su propagador, por lo que los cálculos difieren para ambas teorías al menos a nivel técnico. Una vez resuelta la cuestión del sector gauge, hemos llevado a cabo el cálculo estándar de todas las divergencias a un bucle en la aproximación del campo *background* para ambas teorías, y hemos calculado todas las funciones $\beta$ y los acoplamientos anómalos del Lagrangiano. Con esto en la mano, hemos comparado el flujo del grupo de renormalización de una cierta combinación *física*, dada por $\Delta = G^2 m^4 \lambda^{-1}$. Esta combinación resulta ser diferente para la RG y la GU para valores intermedios del acoplo no mínimo, $\xi$. Sin embargo, hay una sutileza en este resultado. La variacíon de $\Delta$ con la energía es una función de los acoplos no físicos como $\lambda$ y $G^2 m^4$, de modo que no podemos separar completamente estas contribuciones de las contribuciones *físicas*. Aunque podría deberse a que las teorías son no renormalizables, esta cuestión requiere ser investigada más en profundidad. Por ejemplo, los mismos cálculos podrían llevarse a cabo para teorías renormalizables, como las teorías cuadráticas de la gravedad, para analizar si esta característica sigue presente.

Como apunte final, hemos visto que los pilares de la Física Teórica moderna se han construido tras numerosas pequeñas aportaciones y conexiones entre cosas que ya se conocían. En el mismo espíritu, el objetivo de estos trabajos ha sido explorar diversos subproblemas de la interacción gravitatoria con la esperanza de añadir un modesto grano de arena a la construcción de una teoría cuántica de la gravedad. Y con suerte, algún día, nos sentaremos en la arista del cubo, *admirando la tierra donde* $G_N$*, c y* $\hbar$ *vivieron felices y comieron perdices*.

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