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The holographic principle

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ABSTRACT

What does a hologram have in common with our world? According to what is referred to as the holographic principle there seems to be a close connection. The essence of this principle is that our world seems to behave very much like a hologram. In a hologram, information about a three-dimensional object is stored on a two-dimensional surface and basically the same idea appears to be true for our world. The holographic principle implies that all information about a physical system, seems to be gathered on lower-dimensional surfaces, which basically reduces the number of dimensions we have to consider and hence the complexity of our theories. This may have very interesting and intriguing consequences . The holographic principle might turn out to be a cornerstone for a theory of quantum gravity and perhaps even hold the key to its formulation.

This thesis is intended to present the origin, statement and applications of the holographic principle as well as a discussion of possible implications for a theory of quantum gravity. The holographic principle is intimately connected with the concept of entropy. The way towards the holographic principle that I will follow is based on a covariant entropy conjecture which places an upper bound on the entropy in general space-times, following Bousso [(1999)]. This result is besides its connection with the holographic principle interesting by itself, and therefore this conjecture is stated in detail together with evidence for its validity in cosmological solutions and in the interior of black holes. Included is also a brief discussion of some of the predecessors of this covariant entropy conjecture, with a short discussion of their drawbacks.

I have also chosen to include some preliminary basics in Ch.1, to help the reader who encounters this thesis without sufficient knowledge about entropy, general relativity and black hole thermodynamics.

CONTENTS

1. INTRODUCTION

1.1 NOTATION, CONVENTIONS AND UNITS

Throughout this work our stage will be the 3+1-dimensional manifold M, which our space-time consists of. Any three-dimensional subset H of M is called a hypersurface of M. If two of its dimensions are spacelike everywhere and the remaining dimension is timelike (null, spacelike) everywhere, H is called a timelike (null, spacelike) hypersurface. A surface B is always meant to be a two-dimensional spacelike subset of M. A light ray does not refer to an electromagnetic wave or photon, but simply a null geodesic. Unless stated otherwisely, Planck units are used throughout, i.e., units for which h=c=k=G=1.

1.2 A CRASH COURSE IN GENERAL RELATIVITY

Einstein's theory of general relativity (GR), which he introduced in 1915, is a theory of gravitation. According to GR, there are no such thing as a gravitational force. Gravitation is just a manifestation of curved space-time. In GR our world is a 3+1-dimensional space-time. When matter or energy is present, space-time curves, and free particles follow geodesics in the geometry that results from the curvature. What determines the curvature in GR is the energy-momentum tensor, T_{ab}. This tensor contains the matter and energy density. The curvature is given by a metric tensor, g_{ab}, which is calculated from T_{ab} through Einstein's equation:

$$R_{ab} - \frac{1}{2} R g_{ab} = 8\pi T_{ab}\,, \tag{1.1}$$

where

$$R_{ab} = 8\pi (T_{ab} - \frac{1}{2} g_{ab} T) \tag{1.2}$$

$$R = -8\pi T \quad (1.3)$$

and T is the trace of the energy-momentum tensor. As soon as the metric is known, we basically have everything we need to determine the motion of particles. This might sound simple, but the simplicity is deceptive. Actually, Einstein's equation can be solved analytically only in a few simple, although interesting and important, special cases.

This extremely brief and incomplete summary of what GR is all about is actually everything we will need in this thesis. The reader who wants to learn and understand GR and get a firm grip of Einstein's equation, or finds this summary utterly impossible to understand (which by no doubt might be the case), is advised to take a closer look at the large amount of literature written about GR.

1.3 BLACK HOLES AND THERMODYNAMICS

According to the classical theory of general relativity, a gravitationally collapsing star of mass M will shrink, in a short time as measured by an observer on the surface, to a radius of order $2GM/c^2$, at which the gravitational field becomes so strong that no further radiation or anything else can escape. The region of space-time from which it is not possible to escape is called a *black hole*, and its boundary is called the *event horizon*. This is the classical picture of a black hole. However, when quantum effects are taken into account it turns out that a black hole is not completely black. As Hawking was the first to show, radiation tunnels out through the event horizon and escapes at a steady rate.

In the theory of black holes, black-hole physics, there are a number of analogies to thermodynamics. Most striking is the similarity in the behaviours of black-hole area and of entropy: Both quantities tend to increase irreversibly. Hawking has given a general proof that the black-hole area cannot decrease in any process [21], which is analog to the fact that the entropy of a closed thermodynamical system cannot decrease. It might appear that this similarity is of a very superficial nature. It can however be shown that this formal analog for black holes of the second law of thermodynamics extends to the other laws of thermodynamics as well.

If we consider a Kerr black hole, i.e., a charged and rotating black hole, with mass M, charge Q and angular momentum L, it can be shown, see Ref. [14], that the following formula holds:

$$dM = \frac{1}{8\pi}\kappa dA + \Omega dL, \quad (1.4)$$

where

$$\kappa = \frac{(M^2 - a^2 - Q^2)^{1/2}}{2M[M + (M^2 - a^2 - Q^2)^{1/2}] - Q^2}, \quad (1.5)$$

$$\Omega = \frac{a}{r_+^2 + a^2}, \tag{1.6}$$

and

$$a = L/M \tag{1.7}$$

$$r_\pm = M \pm (M^2 - a^2 - Q^2)^{1/2}. \tag{1.8}$$

We noted earlier in this section that the black hole area theorem is analogous to the second law of thermodynamics. We now have a formula for dM, which bears a strong resemblance to the first law of thermodynamics:

$$dE = TdS + \text{work terms}. \tag{1.9}$$

The term ΩdL in Eq. (1.4) is analogous to the ”work term” PdV of the first law, and for an ordinary rotating body a term of the form ΩdL would be present in the thermodynamical formula. The term dA appears in Eq. (1.4) in the same manner as dS appears in the first law of thermodynamics, except being multiplied by $(1/8\pi)\kappa$ rather than T, so κ plays the role of temperature in the black hole laws. It can be shown [14] that κ satisifies an important property analogous to the property of temperature in the zeroth law of thermodynamics: it is uniform over an ”equilibrium” (i.e., stationary) black hole. From Eq. (1.5) we can see that for a charged Kerr black hole, κ vanishes only for the ”extreme” case $M = a^2 + Q^2$. Explicit calculations [22] show that the closer one gets to an ”extreme” black hole, the harder it is to get a further step closer, in a manner similar to the third law of thermodynamics, which in one of its versions states that it is impossible to achieve T=0 by a physical process.

So we have the analogous quantitites E and M, T and ακ, and S and $(1/8\pi\alpha)A$ where α is a constant. A hint that the relation between thermodynamics and black hole physics might be more than just an analogy comes from the fact that E and M are not merely analogs in the formulas, but represent the same physical quantity: total energy. However, the thermodynamical temperature of a black hole in classical general relativity is absolute zero, since a black hole is a perfect absorber, but does not emit anything. So it seems like κ could not physically represent a temperature. But in 1974, Hawking discovered, as discussed earlier in this section, that quantum effects result in an ”emission” of particles from a black hole with a blackbody spectrum at temperature $T = \kappa/2\pi$. Thus, κ does physically represent the thermodynamical temperature of a black hole, which suggests that the relationship between laws of black hole physics and thermodynamics may be much more than an analogy. The black hole laws discussed above may be precisely the ordinary laws of thermodynamics applied to a black hole.

With help of $T = \kappa/2\pi$, and the analogous correspondence between T and ακ, we can set $\alpha=1/2\pi$. This means that the remaining analogous quantities are S and (1/4)A. Does a forth of the horizon area represent the entropy of a black hole? We shall argue that the answer seems to be yes. The ordinary second law of thermodynamics state that the total entropy of matter in the universe never decreases. However, if a black hole is present, one would like to restrict attention to matter outside black holes, since matter that falls in is ”swallowed up” by the singularity within the black hole and, in any case, it cannot be measured by an external observer. However, one easily can make the total entropy, S, of matter outside black holes decrease, by dropping matter into a black hole. On the other hand, the area theorem of classical general relativity states that the surface area, A, of a black hole never decreases.

When we include quantum effects, however, we can violate this. An isolated black hole eventually ”evaporates” completely, decreasing its area to zero. So when black holes are present and we include quantum effects, both the ordinary second law and the area theorem can be violated. However, in the processes where dS > 0 due to loss of matter into a black hole, we increase the black hole area, dA > 0. Similarly, in the evaporation process where dA < 0, we increase the entropy of matter outside black holes, dS > 0, by the emission of thermal radiation. Therefore, we define the *generalized entropy*, S’, by

$$S' = S + \frac{1}{4}A \,, \qquad (1.10)$$

an idea originally due to Bekenstein. The fact that a decrease in S always seems to be compensated by an increase in A and the other way around, suggests that in any process, *the generalized second law* (GSL),

$$dS' \geq 0 \qquad (1.11)$$

may be valid. Actually Eq. (1.10) was proposed by Bekenstein (except for an undetermined numerical factor in front of A) before the discovery of the quantum effects. But it turns out that quantum effects are necessary for it to be satisfied, prohibiting processes who otherwise could violate the generalized law. The generalized second law appears to hold, at least insofar as it can be tested by gedankenexperiments.

The generalized second law has a natural and simple interpretation. It can be viewed as nothing more than the ordinary second law of thermodynamics applied to a system containing a black hole, if we assign one forth of the area A as the physical entropy of the black hole. If this final step is taken, there would no longer be an analogy between black-hole physics and thermodynamics. The laws of black hole thermodynamics would be seen as nothing more than the ordinary laws of thermodynamics applied to a self-gravitating quantum system containing a black hole.

So, the apparent validity of the generalized second law, suggests that (1/4)A is the thermodynamical entropy of a black hole. But the underlying physical basis by which it arises remains unclear. It is a situation in black-hole physics which is very similar to the situation with regard to ordinary thermodynamics prior to the discovery of the underlying basis of these laws arising from statistical physics. We have discovered the laws of black hole thermodynamics, but the underlying basis of these laws is not known and will probably not be fully understood until we have a quantum theory of gravitation. String theory has made some attempts to derive the formula for the entropy of a black hole from more fundamental ideas, but it is still far from obvious that it is the correct road to walk.

1.4 OUTLINE

Ch.1 contains material that is intented to help a reader who encounters this thesis with insufficient knowledge of the key concepts that occur in the text. My reason for including this is that I wanted the thesis to be possible to read and understand for as many people as possible.

In Ch.2 we start by formulating Bekenstein's entropy bound, discussing the ideas that led to this conjecture and its range of validity. Bekenstein was the first to recognize the fact that there might be a bound on the entropy of physical systems, and his conjecture will be an important starting point for the formulation of our covariant entropy conjecture.

Using Bekenstein's conjecture as a starting point, in Ch.3 we try to generalize this conjecture, guided by a wish for general covariance, following the work of Bousso [16-18]. This leads us to a covariant entropy conjecture which we will discuss and state in detail.

In Ch.4 we show that the covariant entropy conjecture implies Bekenstein's bound as a special case, which is an important point if we want to keep our believes in our conjecture. Furthermore, we test our conjecture for different cosmological solutions, showing that the conjecture passes all this test.

The holographic principle is introduced in Ch.5 and is shown to follow from our covariant entropy bound. We introduce the important concept of screens and discuss how to construct them.

Ch.6 is dedicated to applications of the holographic principle to various cosmological situations.

In Ch.7 we discuss the possible implications of the holographic principle for a theory of quantum gravity, as well as the possibility of a general holographic theory.

2. A NON-COVARIANT ENTROPY CONJECTURE

2.1 WHAT BOUNDS THE ENTROPY?

Already in 1981, Bekenstein [1] suggested that there is a surprising limitation for the entropy of a physical system. Let us follow the chain of reasoning which led Bekenstein to conjecturing this limitation.

Classically, we can view entropy S, as a measure of the available phase space for the system in consideration. Let our system have energy E, or alternatively, an energy which does not exceed E. This implies a limitation for the momentum space which is available for the system (on condition that the potential energy is limited from below). If the system is bounded in space as well, its phase space and hence also the entropy is bounded. Clearly the upper bound on the entropy increases when E grows. But *how* does it grow with E? Does it grow linearly or non-linearly? What value does a possible proportionality constant assume? Can we say anything at all about how S grows with E, without knowing how our system is composed in detail?

To approach the answers to these questions we look upon the sky, and for a moment we turn our attention to black holes. For systems with negligible self-gravitation, i.e., systems where gravitation plays a subordinate role and is not the dominating force, it turns out that we can find an upper bound for S/E, the quotient of entropy and energy. According to the generalized form of the second law of thermodynamics (GSL), which we discussed in Sec.1.3, the sum of the entropy outside a black hole and the entropy of the black hole itself (which from Sec.1.3 is a quarter of the area of the horizon of the black hole), can never decrease. A known result regarding black holes, is that when a black hole swallows a body with limited self-gravitation, energy E and effective radius[1] R, the surface area of the black hole has to increase by at least $8\pi ER$ [2,3]. It is possible to arrange this process when the body is swallowed by the black hole in such a way that this least value of the increase is assumed [2].

Where does this $8\pi ER$ come from? Let us calculate the minimum possible increase in black-hole area which must result when a spherical particle with mass μ and proper radius b is captured by a Kerr black hole. We are interested in the increase in area ascribed to the particle itself, and not any increase incidental to the process of bringing the particle to the black-hole horizon. We shall ignore all incidental effects and focus on the increase caused by the particle all by itself.

We assume that the particle is neutral so that it follows a geodesic of the Kerr geometry when falling freely. We use the Kerr metric with Boyer-Lindquist coordinates:

$$ds^2=g_{tt}dt^2+2g_{t\phi}dtd\phi + g_{\phi\phi}d\phi^2+g_{rr}dr^2+g_{\theta\theta}d\theta^2.$$

For g_{rr} we have

$$g_{rr} = (r^2+a^2\cos^2\theta)\Delta^{-1},$$

where

$$\Delta = r^2-2Mr + a^2 + Q^2.$$

The event horizon is located at $r=r_+$, where $r_\pm$ is given by Eq. (X.x). We have

$$\Delta = (r - r_-)(r - r_+).$$

First integrals for geodesic motion in Kerr background have been given by Carter [25]. As a starting point for our analysis we will use the first integral

$$E^2[r^4+a^2(r^2+2Mr-Q^2)] - 2E(2Mr-Q^2)ap_\phi - (r^2-2Mr + Q^2)p_\phi{}^2 - (\mu^2r^2 + q)\,\Delta = (p_r\,\Delta)^2$$

where $E = -p_t$ is the conserved energy, p_ϕ is the conserved component of angular momentum in the direction of the axis of symmetry, q is Carter´s fourth constant of the motion [25], μ is the rest mass of the particle, and p_r is its covariant radial momentum.

We now solve Eq.(X.x) for E, giving us

$$E = Bap_\phi + \{[B^2a^2+A^{-1}(r^2-2Mr+Q^2)]p_\phi{}^2 + A^{-1}[(\mu^2r^2 + q)\,\Delta + (p_r\,\Delta)^2]\}^{1/2},$$

where

$$A = r^4 + a^2(r^2 + 2Mr - Q^2),$$

$$B = (2Mr - Q^2)A^{-1}.$$

At the event horizon, $\Delta=0$ (see Eq. (X:x)) so there we have

$$A = A_+ = (r_+{}^2 + a^2)^2,$$

$$B = B_+ = (r_+{}^2 + a^2)^{-1}.$$

Furthermore, at the horizon $Ba=\Omega$, and the coefficients of $p_\phi{}^2$ and $\mu^2r^2 + q$ in Eq. (X.x) vanish. However,

$$p_r\,\Delta = (r^2 + a^2\cos^2\theta)\,p^r$$

does not vanish at the horizon in general. If the particle's orbit intersects the horizon, we have from Eq. (X.x) that

$$E = \Omega p_\phi + A_+^{-1/2} |p_r \Delta|_+$$

As a result of the capture, the black hole's mass increases by E and its component of angular momentum in the direction of the symmetry axis increases by p_ϕ. Therefore, according to Eq. (X.x), the black hole's rationalized area will increase by $\theta^{-1}A_+^{-1/2} |p_r \Delta|_+$. As pointed out by Christodoulou this increase vanishes only if the particle is captured from a turning point in its orbit in which case $|p_r \Delta|_+ = 0$. In this case we have

$$E = \Omega p_\phi.$$

This analysis shows that it is possible for a black hole to capture a point particle without increasing its area. How is this conclusion changed if the particle has a nonzero proper radius b? We start by noting that no matter how the particle arrives at the horizon, it must clearly aquire its parameters E, p_ϕ and q while every part of it is still outside the horizon, i.e., while it is not yet part of the black hole. Moreover, as the particle is captured, it must already be detached from whatever system brought it to the horizon, so that it may be regarded as falling freely. This means that Eq. (X.x) should always describe the motion of the particle's center of mass at the moment of capture.

To generalize our previous result to the present case, we should evaluate Eq. (X.x) not at $r=r_+$, but at $r = r_+ + \delta$, where δ is determined by

$$\int_{r_+}^{r_+ +\delta} (g_{rr})^{1/2}\, dr = b$$

($r = r_+ + \delta$ is a point a proper distance b outside the horizon). Using Eq. (X.x) we find that

$$b = 2\delta^{1/2}(r_+^2 + a^2\cos^2\theta)^{1/2}(r_+ - r_-)^{-1/2}.$$

We have assumed that $r_+ - r_- >> \delta$ (black hole not nearly extreme). Expanding the argument of the square root in Eq. (X.x) in powers of δ, replacing δ by its value given by Eq. (X.x), and keeping only terms to O(b), we get

$$E = \Omega p_\phi + (1/2)b[(r_+^2 - a^2)(r_+^2 + a^2)^{-1}p_\phi^2 + \mu^2 r_+^2 + q]^{1/2}(r_+ - r_-)(r_+^2 + a^2)^{-1}(r_+^2 + a^2\cos^2\theta)^{-1/2}.$$

We have in this calculation already assumed that the particle reaches a turning point as it is captured since we know that this minimizes the increase in black-hole area.

What is q in Eq. (X.x)? We can obtain a lower bound for it as follows. From the requirement that the θ momentum p_θ be real it follows that [25]

$$q \geq \cos^2\theta[a^2(\mu^2-E^2) + p_\phi^2/\sin^2\theta];$$

where the equality holds when $p_\theta = 0$ at the point in question. If we replace E in Eq. (X.x) by Ωp_ϕ (see Eq. (X.x)) we obtain

$$q \geq \cos^2\theta[a^2\mu^2 + p_\phi^2(1/\sin^2\theta - a^2\,\Omega^2)],$$

which is correct to zeroth order in b. We know that $1/\sin^2\theta \geq 1$ and it is easily shown that $a^2\,\Omega^2 \leq 1/4$ for a charged Kerr black hole. This means that $q \geq a^2\mu^2\cos^2\theta$. Using this in Eq. (X.x) we obtain

$$E \geq \Omega p_\phi + (1/2)\mu b(r_+ - r_-)(r_+^2 + a^2)^{-1}$$

which is correct to O(b). By retracing our steps we can see that the inequality sign in Eq. (X.x) corresponds to the case $p_\phi = p_\theta = p^r = 0$ at the point of capture. The increase in black-hole area, computed by means of Eq. (X.x), Eq. (X.x) and Eq. (X.x), is

$\Delta\alpha \geq 2\mu b.$

This gives the fundamental lower bound on the increase in black-hole area. We note that it is independent of M, Q and L. Using the horizon area A instead of the rationalized area $\alpha = A/4\pi$, we get

$\Delta A \geq 8\pi\mu b.$

But this means that we will violate GSL, unless the entropy of the body cannot exceed $2\pi ER$ (one forth of the increase in the area). This gives us a bound for the entropy of a body with limited self-gravitation:

$$S/E \leq 2\pi R. \tag{2.1}$$

For a system enclosed in a spherical volume, gravitational stability requires that $E \leq R/2$, and the effective radius coincide with the radius of the sphere, so from Eq. (2.1) we then get

$$S \leq A/4 \tag{2.2}$$

where A is the surface of the sphere.

The fascinating thing about this limitation is that we use the generalized form of the second law of thermodynamics (GSL) for its derivation, a law whose significance is strongly coupled to gravitation, but the result is a bound for the entropy of systems with limited self-gravitation, i.e., systems where the gravitation can be neglected! All connections to gravitation has mysteriously vanished. This indicates that the relation might have a more fundamental meaning than what the derivation tells us.

1) The effective radius R is defined as

$$R = \sqrt{\frac{A}{4\pi}}$$

where A is the area of the smallest sphere circumscribing the system

2.2 A CLOSER LOOK AT BEKENSTEIN'S CONJECTURE

Bekenstein's result is truly beautiful and fascinating. Sad to say though, it has its limitations. In Sec.2.1, the conditions Bekenstein specified for the validity of his bound was not stated explicitly. The system under consideration must be of constant, finite size, have limited self-gravity and no matter components with negative energy density can be available. A system which satisfies these conditions will be referred to as a *Bekenstein system*.

The last condition stems from the fact that the bound relies on the gravitational collapse of systems with excessive entropy, and the idea that information requires energy. With matter of negative energy we could add entropy to a system, without an increase in the mass, by adding enthropic matter with positive mass as well as an appropriate amount of negative mass, and this would violate our conjecture.

One of the most serious problems about Eq.(2.2), due to the conditions for its validity, is that it, as stated above, presumes systems where gravitation does not dominate. This automatically excludes systems such as collapsing stars and sufficiently large regions of our cosmological spacetime. In these cases it is a piece of cake to violate Bekenstein's bound. Take a collapsing star for instance. When the star collapses, the surface area becomes arbitrarily small, while the enclosed entropy cannot decrease, which means that Eq.(2.2) no longer holds. Another example is a homogeneous, spacelike hypersurface in a flat Friedmann-Robertson-Walker universe (FRW). In this system, the dynamics is goverened by the cosmological expansion and in this respect gravitation is dominating. For FRW space is infinite so the entropy density is constant, and volumes increase more rapidly than areas. In other words, our bound will be violated [4].

These limitations lead to a number of interesting questions. One of these is which range of validity Bekenstein's formula really have in different cosmological situations. Can we specify this? This question has been examined in numerous situations [4,5-9], where the problems of Bekenstein's bound is demonstrated when we go beyond its range of validity, and I will not touch upon this results in this thesis. We will, however, get an answer to this question in Sec.4.1, where we state the spacelike projection theorem.

Let us instead try a braver project. Can we reformulate our entropy bound, so that we achieve a formulation which holds even in the situations mentioned above, i.e., find a more general formulation? Happily enough, the answer to this question seems to be yes. Some bold leaps in this direction was taken in [4,10-12] and underlies the formulation we are about to state, which is due to Bousso [16]

3. A COVARIANT ENTROPY CONJECTURE

3.1 FOLLOW THE LIGHT!

Bekenstein's bound for the entropy presumed, as discussed in Ch. 2, that the systems we consider are of finite extension and that gravitation is not allowed to dominate within the system. Moreover is Bekenstein's conjecture based on spacelike hypersurfaces, which eliminates any hope of a generally covariant conjecture. What we will try to do is to aim for a similar bound which *is* covariant, i.e., has a universal validity. We are interested in searching for a general formulation of the entropy bound which can be applied to arbitrary space-times. Conservative and fearful as we are, we will try to keep the formula $S \leq A/4$ unchanged, but change the meaning of A in this relation. This is a program which a priori has no guarantees for success, and which has all chances of going completely wrong. We do not have the slightest idea of how things work when Bekenstein's presumptions are not fulfilled. But let us not worry about this for a moment, and see whether such a program is possible and where it might lead us.

As stated above, Bekenstein's formulation has an dissatisfying, inherent non-covariance, which we somehow has to get rid of if we aim for general covariance. The problem is, in a bit more detail than stated above, that given a closed surface B, Bekenstein's entropy bound is concerned with the volume that B encloses, i.e., a specific spacelike hypersurface. But the so chosen hypersurface depends on the choice of time coordinate. How do we get rid of this non-convariance? One possibility is to demand that the bound is valid for *all* spacelike hypersurfaces which are enclosed by B. But the possibility of using any spacelike hypersurface has already been ruled out in our discussions in Sec.2.2. So what to do? A nice idea is that we instead use *null hypersurfaces* which is bounded by B. These are by nature covariant. How do we construct such hypersurfaces?

A natural way of doing this is to start at the surface B and then follow a family of light rays (also called a congruence of null geodesics) which leaves the surface orthogonally to the same on each side. But this leads immediately to a couple of questions. To start with, every two-dimensional surface, regardless size and shape, generates exactly four different null hypersurfaces, four different families of light rays. Our family of light rays can be future-directed and outgoing, future-directed and ingoing, past-directed and outgoing and past-directed and ingoing (Fig. 1). Who or which of these four families should we choose? And moreover, how far should we follow the light rays?

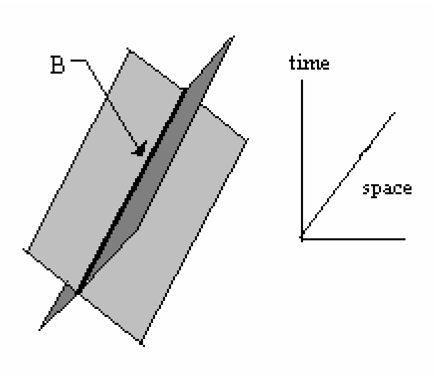

Figure 1: *There are four families of light-rays projecting orthogonally away from a two-dimensional surface B, two future-directed (one to each side of B) and two past-directed families.*

Fischler and Susskind [4] considered spherical surfaces, i.e closed surfaces, and suggested that what should be used are past-directed ingoing null hypersurfaces. The resulting entropy bound works just fine for big regions in flat FRW-universes, for which Bekenstein's bound could not be used. Unfortunately, it turns out that [4,8] Fischler and Susskind's bound does not hold for closed or collapsing spacetimes. Our aim is a generally valid bound, so if we insist on using orthogonal null hypersurfaces we somehow have to find another selection rule for the null hypersurfaces which gives a bound that is satisfied in these situations as well.

Let us in our search for this selection rule once again be inspired by Bekenstein's bound. For a Bekenstein system, the entropy enclosed inside a closed surface B of area A cannot exceed the area A. But our surface B is the boundary of everything outside the enclosed region as well, and the entropy in this region could clearly be anything. The moral of the story is that we somehow should try to consider entropy only on hypersurfaces which do not lie outside the boundary.

But what is meant by ”outside”? There is an ambiguity in the concepts ”inside” and ”outside” in general situations. Is ”inside” the side on which infinity does not reside and ”outside” where infinity exists? This classification works for infinite spaces, but does not lend itself to applications in closed spaces. Fortunately, we can from intuitive concepts generalize ”inside” and ”outside” to a rule which is even covariant.

Consider Euclidean geometry and start with a closed surface B. Construct a new surface B', by moving every point on B an infinitesimal distance away from one of B:s sides along lines orthogonal to B. If this increases the area, we have moved to the outside. If this decreases the area, it was the inside which we moved to (Fig.2a)

We realize from this observation that our light rays which leaves B orthogonally, which we introduced above, is well suited to do the job and provide us with the tools that we need to be able to formulate a rule for which null hypersurfaces of our surface B we should choose. Our intuitive notion of ”going inside” corresponds to the following procedure: Start on B, and follow any of the four families of orthogonal light rays, for which the cross-sectional area of neighbouring rays is constant or decreasing in the neighborhood of B. We can formulate it with more formal accuracy by requiring that the *expansion* of the family of orthogonal light rays should be non-positive near B, in the direction away from B. The expansion θ for a family of light rays is defined as the logarithmic differentiation of the infinitesimal cross section which is spanned by a bunch of neighbouring rays (for a more rigorous definition and additional discussion the reader can consult [13,14]), and measures the *local* rate of change of the cross-sectional area when we move along the light rays. This means that θ is positive if the cross-sectional area increases and negative if it decreases. Worth stressing once more is the fact that the expansion is a local quantity. Using this definition of the expansion θ, our condition for calling a side the ”inside” is simply that we must have

$$\theta \leq 0 \tag{3.1}$$

This gives us a covariant definition of the concepts ”inside” and ”outside”.

If the expansion is positive for the future-directed light rays to one of the sides, it is negative for the past-directed light rays to the other side and the other way around. This follows from continuity across our surface B. Put another way: at least two of the four possible directions are allowed. If the expansion is zero in some directions, three or even all four null hypersurfaces will be allowed.

Perhaps the most interesting property of our covariant definition of ”inside” and ”outside”, is that we have opened the door to a powerful generalization of our attempts to find a covariant entropy bound. We no longer have to demand that B is closed to be able to speak about ”inside”. Only our naive definition of ”inside” as a finite region enclosed inside a surface, needed the surface to be closed. Our selection rule works equally fine for open surfaces, by choosing at least two of the four null hypersurfaces which is bounded by B. Fig. 2b below is an attempt to illustrate the idea.

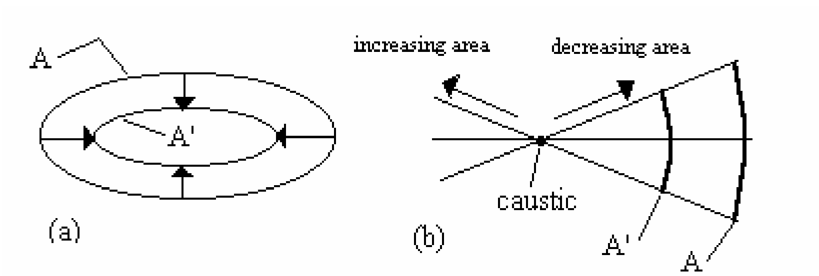

Figure 2. *[We have suppressed time and one spatial dimension.] We define the ”inside” of a two-dimensional surface A to be the light-like direction along which the cross-sectional area decreases. (a): A' ≤ A, or equivalently, θ ≤ 0. We can apply this definition to open surfaces as well (b).*

So from now on we drop the condition that the surface is closed and allow any connected two-dimensional surface. This means that we without loss of generality can assume that the expansion in each of the four null directions does not change sign anywhere on B, and if it does, we simply split B into suitable parts and apply our entropy conjecture to each part separately.

3.2 WHERE TO STOP?

Now we know how to choose orthogonal null hypersurfaces given an arbitrary surface B. But how far should we really follow the light rays which define the null hypersurfaces? We should need some kind of rule that tells us where to stop. For a closed space-time, it is nearly obvious that we should stop if we reach the boundary of the space-time, but for an open space-time, we need another criterion. Otherwise the light rays would continue and generate another cone, which could be arbitrary large, and that is definitely not a situation we want to face if we are to find an entropy bound. Fortunately, we can now use the same idea as we used for choosing null hypersurfaces to find this rule. We simply demand that we should have $\theta \leq 0$ *everywhere* on the null hypersurfaces, not only in the neighbourhood of our surface B.

When neighbouring light rays intersect each other they form a *caustic* or ”focal point” (Fig. 2b). Before the caustic we have $\theta < 0$ and after we have $\theta > 0$. This means that we now know where to stop, using the condition stated above. As soon a light ray reaches a caustic, for instance the vertex of a light cone, we stop. This gives us well-defined null hypersurfaces. Now we are in position to state, in an unambiguous way, how to choose and construct our null hypersurfaces. In what follows, we will call these priviliged null hypersurfaces *light-sheets*. Thus, we define a light-sheet L to a connected spatial two-dimensional surface B as *a null hypersurface which is bounded by B and which is constructed by following a family of light rays orthogonally away from B, in such a way as to keep the expansion non-positive* ($\theta \leq 0$) *all the time.*

Note the fact that the expansion θ and our ”decreasing cross-sectional area” rule enters in two ways: it tells us which null hypersurfaces who are allowed and then it determines the light-sheets, by telling us how far we should proceed along the favoured null hypersurfaces.

3.3 AVOIDING PATHOLOGICAL SCENARIOS

To protect our conjecture about to be stated against pathologies, like for instance superluminal entropy flow, we require the *dominant energy condition* to hold. This condition states that for all timelike v_a, we have

$$T^{ab}v_a v_b \geq 0 \qquad (3.2)$$

and $T^{ab}v_a$ is a non-spacelike vector. This states that to any observer the energy density appears non-negative and the speed of energy flow of matter is less than the speed of light. It implies that a space-time must remain empty if it is empty at one time and no matter is coming in from infinity [13]. It is not a severe restriction, since all ordinary matter is believed to satisfy this condition [13,14].

We also require that the space-time is inextendible and contains no null or timelike (”naked”) singularities. This excludes the possibility of destroying or creating entropy on such boundaries. These conditions are believed to hold in all physical space-times, and hence will not be explicitly stated below. Of course we do not exclude spacelike singularities occuring in cosmology and gravitational collapse.

3.4 A COVARIANT CONJECTURE FOR THE ENTROPY

After so much effort in defining light-sheets, time has come for our reward. With light-sheets well-defined we are now in position to formulate a covariant conjecture for the entropy bound:

(Covariant Entropy Conjecture) *Let M be a four-dimensional space-time, on which Einstein's equation is satisfied with the dominant energy condition holding for matter, A be the area of an arbitrary connected two-dimensional spatial surface B in M, L be a light-sheet bounded by B and let S be the total entropy enclosed on L. Then we have $S \leq A/4$.*

What is remarkable and appealing with this formulation is its generality. Earlier conjectures have all in one way or another suffered from limited presumptions, which led to a restricted range of validity and applicability. In the conjecture above, we associate regions which contain entropy in a very well-defined way to *each* surface area in an *arbitrary* space-time. We dont have to require that gravitation is negligible, and the surface can be closed or open and furthermore assume any shape, which is an enormous strength of our formulation.

Comparing this conjecture, with the conjecture proposed by Fischler and Susskind [4], which also uses null hypersurfaces, the main difference is that our conjecture considers all four light-like directions without prejudice and selects some of them by our selection rule of non-increasing expansion.

Worth stressing though, is the fact that this simple formulation, which our conjecture assumes, is a bit deceptive. The construction of the light sheets can in some situations be everything but trivial. Tavakol and Ellis [15] showed for instance, that light-sheets can be extremely complicated structures, sometimes with fractal properties, in inhomogeneous space-times. But this by no means indicate that we should throw our rule for non-positive expansion, which defines the light-sheets, in the trashcan. Just because a theory suffers from complications when applied in certain situations, it does not mean that the theory has to be discarded. To quote Bousso [16]: " - - - we would not discard the standard model merely because its application becomes impractically complicated when one is describing an elephant".

Another thing worth mentioning is the fact that our covariant bound assumes its strongest form when the light-sheet is made as large as possible, i.e., if we stop only at caustics. Of course it still remains valid, but becomes less powerful, if we choose to stop earlier, since it will decrease the entropy on the light-sheet, but not the boundary area. It has, however, been pointed out by Flanagan, Marolf and Wald [19], that we can strengthen our bound in this case to (A-A')/4, where A' is the surface area spanned by the endpoints of the light-rays (Fig. 2b), which goes to zero as a caustic is approached. This is an appealing expression, since it makes the bound additive over all directions on the light-sheet. In this form the covariant entropy bound implies the generalized second law of thermodynamics [19].

3.5 A RECIPE FOR THE PRACTICALLY-MINDED

After a great deal of formal accuracy and general formulations, it might be a good idea to formulate our conjecture in a more practical way, for instance as a step-by-step-procedure, which is the aim of this section. The recipe to follow is:

1. Pick an arbitrary two-dimensional surface B in the space-time M under consideration.

2. For B there exists four families of light rays which leaves B orthogonally (unless B happen to reside on the boundary of M).
   Let us call these F_1, F_2, F_3 och F_4.

3. If the expansion is positive (in the direction away from B), i.e., the cross-sectional area increases, we must not use that actual family of light rays. If, on the other hand, the expansion is zero or negative, the actual family of light rays can be used.

4. After performing this test for each of the four families of light rays, we have to our disposal at least

two allowed families (if the expansion is zero in some directions we might have as many as three or four allowed families).

5. Choose one of the allowed families, F_i. Now, construct a light-sheet L_i, by following each light ray until anything of the following occurs:

   (a) The light ray reaches the boundary or a singularity of the space-time.
   (b) The expansion becomes positive, i.e., the cross-sectional area which is spanned by the actual family, starts to increase in a neighbourhood of the light ray.

   This gives us a light-sheet for each one of the allowed families.

6. Our conjecture now tells us that the entropy S_i on the light-sheet L_i does not exceed a quarter of the area A of B:

   $S_i \leq A/4$.

   Note that the bound applies to each of the allowed light sheets separately. Since B can have as many as four light sheets, it is possible that the total entropy on all light sheets adds up to as much as A.

A few remarks might be useful at this point. We often refer to $F_1 \ldots F_4$ as future-directed ingoing, future-directed outgoing, past-directed ingoing and past-directed outgoing families respectively. The names ”ingoing” and ”outgoing” are only two arbitrary labels to distinguish between the two sides of B. Only when B is closed there are reasons to be a bit more careful when assigning these names. Actually, nothing in our rule for choosing families makes a distinction between ”ingoing” and ”outgoing”.

If B is closed, we can characterize it as *trapped*, *anti-trapped* or *normal* (se [13,14] for definitions). This gives a simple criterion for the allowed families. If B is trapped (anti-trapped), it has two future-directed (past-directed) light-sheets. If B is normal, it has a future-directed and a past-directed light sheet on the same side, usually called the inside. If B lies on an *apparent horizon* (the boundary between a trapped or anti-trapped region and a normal region), it can have more than two light-sheets.

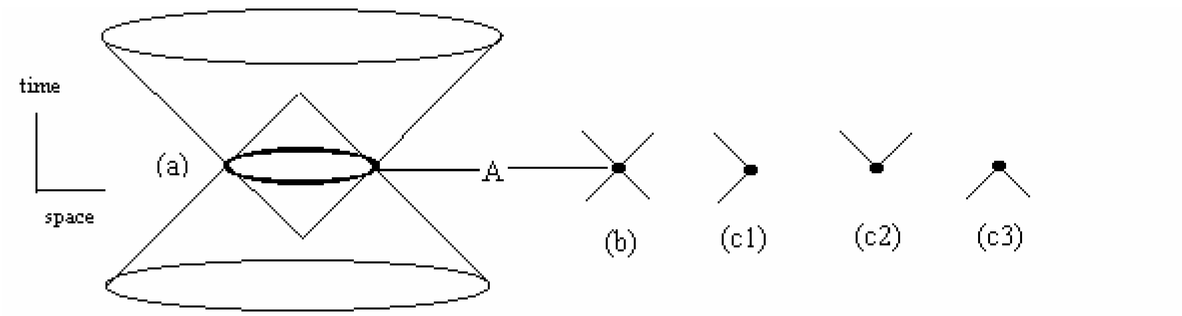

Figure 3. *[In (a) we have suppressed one spatial dimension (a surface becomes a line). In (b,c) we have suppressed two spatial dimensions (a surface becomes a point). A fixed light-like angle separates spacelike and timelike directions.] A is the two-dimensional boundary of four 2+1-dimensional light-like hypersurfaces (a), who are covariantly generated by the past- and future-directed light-rays going to either side of A, so for a normal spherical surface they are given by two cones and two "skirts" (a). In a Penrose diagram, where spheres are represented by points, the null hypersurfaces show up as the three legs of an X (b). Null hypersurfaces with decreasing cross-sectional area, such as the two cones in (a), are called light-sheets. The light sheets for normal (c1), trapped (c2), and anti-trapped (c3) spherical surfaces are shown. If gravity is weak, as in (a), the light-sheet directions agree with our*

intuitive notion of "inside" (c1). For surfaces in a black hole interior, both of the future-directed hypersurfaces collapse (c2). Near the big bang, the cosmological expansion makes the area decrease on both past-directed hypersurfaces (c3).

4. TESTING THE COVARIANT ENTROPY CONJECTURE

4.1 BEKENSTEIN'S CONJECTURE – A LIMITING CASE

After formulating our conjecture for a covariant entropy bound, there remains to see how well it works in various situations, where different more or less intricate situations and cosmologies are used to test the conjecture. Before it is clear that it can withstand these tests without remarks, our conjecture is merely a theoretical construction of limited significance. One failed test is enough to falsify our conjecture. So how does our conjecture cope when put under the magnifying glass?

One first important test, is whether our covariant bound implies Bekenstein's bound in the appropriate limit. How can such a thing be proven? There seems to be an obvious distinction between the two conjectures. In our covariant formulation we use null hypersurfaces, in Bekenstein's formulation spatial surfaces is used, so where is the connection?

It turns out that there is a large class of situations, where an entropy bound on spacelike hypersurfaces is implied by our covariant entropy bound. We will examine under what presumptions this is the case, and derive a theorem for the entropy of spatial regions. We then proceed to show that Bekenstein's bound follows from our covariant bound, as a special case of this theorem.

Let A be the area of a closed surface B, which has at least one future-directed light-sheet L. We furthermore assume that L has no other boundary besides B. We can then call the direction of this light sheet the "inside" of B. Let the spatial region V be the interior of B on some spacelike hypersurface through B. If the region V is enclosed in the causal past of the light-sheet L, the dominant energy condition implies that all matter in V eventually has to pass through the light-sheet L. But then the second law of thermodynamics tells us that the entropy on V, S_V, cannot exceed the entropy on L, S_L. From our covariant entropy bound we know that $S_L \leq A/4$. Thus the entropy of the spatial region V cannot exceed one forth of the boundary area, i.e., $S_V \leq A/4$.

The condition that the future-directed light-sheet L does not possess a boundary ensures us that no entropy in V disappears through holes in L. Neither can any entropy disappear in a black hole singularity, since we have required that the spatial region must reside in the causal past of L. Since we always assumes that the space-time is inextendible and that no naked singularities are present, all entropy on V must go through L.

We are therefore led to the following theorem:

(**Spacelike Projection Theorem**) *Let A be the area of a closed surface B possessing a future-directed light-sheet L, with no boundary other than B. Let the spatial region V be contained in the intersection of the causal past of L with any spacelike hypersurface containing B. Let S be the entropy on V. Then S≤A/4.*

Consider now, in asymptotically flat space, a Bekenstein system in a spatial region V, bounded by a closed surface B of area A. The future-directed ingoing light sheet L of B exists (otherwise B would not have limited self-gravitation), and can be taken to terminate when two light rays meet. We thus have no other boundary besides B. Since the gravitation in a Bekenstein system is not strong enough to form a black hole, V is enclosed in the causal past of L. This means that the presumptions of the theorem are fulfilled, and the entropy of the system must not exceed A/4. This means that we have reached Bekenstein's bound from our covariant formulation.

4.2 COSMOLOGICAL TESTS OF THE COVARIANT ENTROPY CONJECTURE

We have seen in section 4.1 that our covariant entropy conjecture implies Bekenstein's bound in the appropriate limit. Now it is time to see how our conjecture copes with cosmological models. So the question is: Does our conjecture pass the test?

We will apply our covariant bound to the cosmological models that our universe is believed to be described by, and others that certainly not is a good description of our universe. We have, in the formulation of our conjecture, claimed that it is valid for all space-times that satisfy Einstein's equation (with the dominant energy condition), so it must in particular then be valid for cosmological solutions.

We will consider Friedmann-Robertson-Walker (FRW) cosmologies. These have a metric of the form

$$ds^2 = -dt^2 + a^2(t)\left(\frac{dr^2}{1-kr^2} + r^2 d\Omega^2\right) \tag{4.1}$$

or, in comoving coordinates:

$$ds^2 = a^2(\eta)\left[-d\eta^2 + d\chi^2 + f^2(\chi)d\Omega^2\right] \quad (4.2)$$

where as usual, k=-1,0,1 and f(χ)=sinh χ, χ, sin χ, correspond to open, flat and closed universes respectively.

The *Hubble horizon* is the inverse of the expansion rate H:

$$r_{HH} = H^{-1} = a \ / \ da/dt \quad (4.3)$$

The *particle horizon* is the distance travelled by light since the Big Bang:

$$r_{PH} = \eta \quad (4.4)$$

The *apparent horizon* is defined geometrically as a sphere at which at least one pair of orthogonal null congruences have zero expansion, and is given by [7]

$$r_{AH} = \frac{1}{\sqrt{H^2 + \frac{k}{a^2}}}. \quad (4.5)$$

With the help of Friedmann's equation

$$H^2 = \frac{8\pi\rho}{3} - \frac{k}{a^2} \quad (4.6)$$

where ρ is the energy density of matter, we find that

$$r_{AH} = \sqrt{\frac{3}{8\pi\rho}} \quad (4.7)$$

We consider matter described by the energy-momentum tensor T_{ab}=diag(ρ, p, p, p), with p=γρ as the equation of state. Our dominant energy condition then requires that $\rho \geq 0$ and $-1 \leq \gamma \leq 1$. The case with $\gamma = -1$ is special from the others, since it gives a different global structure. It is the so called de Sitter space, which lacks past or future singularities. This is actually an interesting case since it describes an inflationary universe, and we will get back to this later.

What we will do next, is to test our entropy conjecture on spherical surfaces B characterized by some value of r, or (η, χ). As discussed earlier in Sec.3.5 the directions of the light-sheets on a surface depend on its classification as trapped, normal or anti-trapped. Close to χ=0 (and for closed universes, also on the opposite pole, near χ=π), the spherical surfaces will be normal. The larger spheres beyond the apparent horizon(s), will be anti-trapped. Some universes, for instance, most closed universes, or a flat universe with a negative cosmological constant [8], recollapse. Such universes necessarily contain trapped surfaces. Trapped surfaces also can occur by gravitational collapse. Surfaces inside such regions is a serious challenge for our conjecture, since their area shrinks to zero, while the entropy cannot decrease. We will get back to this problem later on, and argue that the conjecture nevertheless holds even in these regions. We will start by considering anti-trapped surfaces and then proceed to normal and trapped surfaces.

4.3 ANTI-TRAPPED SURFACES

An anti-trapped surface B is characterized by two past-directed light-sheets. Unless B lies within the particle horizon, both light-sheets will be ”truncated” at the Planck era near the past singularity. The truncation has the nice property that the volume the light-sheets sweep out grows roughly like the area A, and not like $A^{3/2}$. Actually, the ”ingoing” light-sheet coincides with the ”truncated lightcones” used in the conjecture of Fischler and Susskind [4], and the other light-sheet can be treated similarly. In open universes the bound will be satisfied more comfortably then in flat ones [4]. The bound was checked for flat universes with $0 \leq \gamma \leq 1$ in [4] and for $-1 < \gamma < 0$ in [8]. Here I will present a summary of these results and discuss some connected issues, regarding inflation.

Near a past singularity, it is quite simple to see why our bound is satisfied. The first moment of time we really can talk about without getting into trouble is one Planck time after the singularity. At his time, up to a factor of order one, there cannot have been more than one unit of entropy per Planck volume. This is a mere application of the usual Planck scale cutoff. We cannot continue our light-sheets into regions where we cannot control the physics. A backward light-sheet with area A specified at the time $t=2t_{Pl}$ will be truncated at $t = t_{Pl}$. It will sweep a volume of order Al_{Pl} and the entropy bound will be at most saturated.

We can define σ as the entropy/area ratio at the Planck time. Consider a universe filled with matter allowed by the dominant energy condition, $-1 \leq \gamma \leq 1$. We can exclude $\gamma = -1$, since this case (de Sitter space) contains no past singularity. Our scale factor is

$$a(t) = t^{\frac{2}{3(\gamma+1)}} . \qquad (4.8)$$

If we have adiabatic evolution, we have for the ratio of entropy to area that [8]

$$\frac{S}{A} \leq \sigma t^{\frac{\gamma-1}{\gamma+1}} \qquad (4.9)$$

for any past-directed light-sheet of areas specified at a later time t. We have a non-positive exponent, so the ratio does not increase. Since this bound is satisfied at the Planck time, it will remain satisfied at later times.

If the evolution is non-adiabatic, the entropy bound nevertheless predicts that $S/A \leq 1/4$, which implies that we have a limitation for the rate at which our universe can produce entropy.

In our discussion above, we have used the standard cosmology all the way back to the Planck era. This is not seriously tenable. There are some properties of our universe, such as homogenity and flatness, which usually are explained by assuming that the radiation dominated era was preceded by a vacuum dominated era, where our universe was rapidly blown up like a balloon. This is called *inflation*.

Inflation ends on a spacelike hypersurface V at $t=t_{reheat}$. At this time, all the entropy in the universe is produced through reheating. Both before and after reheating, all spheres will be anti-trapped, except in a small neighbourhood of r=0, of the size of the apparent horizon. So our spacelike projection theorem does not apply to any but the smallest of the surfaces B in V. The other spheres may be exponentially large, but our covariant conjecture does not relate their area to the enclosed entropy. The size and total entropy of the reheating hypersurface V is thus irrelevant.

Outside the apparent horizon, entropy/area comparisons can only be done on the light-sheets specified in the conjecture. The past-directed light-sheets of anti-trapped surfaces of the post-inflationary universe intersect V. Since there are virtually no entropy during inflation, we can consider the light-sheets to be truncated by the reheating surface. Because inflation cannot produce more than one unit of entropy per Planck volume, the bound will be satisifed by the same arguments that were given above for universes with a Big Bang.

4.4 NORMAL SURFACES

Spatial regions enclosed by normal surfaces, will in a sense, turn out to be very much like Bekenstein systems. How can we understand this? First, we have to establish a few properties of Bekenstein systems. We will call the first bound, Eq. (2.1), *Bekenstein's entropy/mass bound*, and the second bound, Eq. (2.2), *Bekenstein's entropy/area bound*. Since a Bekenstein system must be gravitationally stable ($M \leq R/2$), which implies $2\pi RE = \pi R^2 = A/4$, the entropy/mass bound is always at least as tight as the entropy/area bound for a spherical Bekenstein system of a given radius and mass. A spherical Bekenstein system that saturates the entropy/area bound will be called a *saturated Bekenstein system*. This system will also saturate the entropy/mass bound, due to what was said above.

An example of a saturated Bekenstein system is a black hole, viewed from the outside, in semi-classical gravity. In many occasions though, it is simpler to think of an ordinary, maximally enthropic, spherical thermodynamic system just on the verge of gravitational collapse. We also introduce the term *mass-saturated Bekenstein system*, which refers to a system that saturates the entropy/mass bound but not the entropy/area bound.

The normal region contains a past- and future-directed light-sheet. They are both ingoing, i.e., they are directed towards the center of the region at χ=0 (or $\chi=\pi$ for the normal region near the opposite pole in closed universes). This is the important point. If outgoing light-sheets existed even as χ goes to zero, the area would become arbitrarily small while the entropy remained finite. Except for this constraint, the past-directed light-sheet coincides with the light-cones used by Fischler and Susskind [4]. The entropy/area bound has been shown to hold on such surfaces [4,7,8]. Therefore, we focus on the future-directed light-sheet.

The future-directed light-sheet covers the same co-moving volume as the past light-sheet, so the covariant entropy bound will be satisfied on it if the evolution is adiabatic. We though have to allow for the possibility that additional entropy is produced. As an example, consider the outermost surface on which future-directed light-sheets are still allowed, a sphere B on the apparent horizon. Suppose that a group of experimental cosmologists within the apparent horizon are determined to break the entropy bound. They must produce as much entropy as possible before the matter passes through the future-directed light-sheet L of B. What is their best shot?

They cannot collect mass from outside B, since L is a null hypersurface bounded by B and the dominant energy condition holds, which makes spacelike energy flow impossible. The system with most entropy is a saturated Bekenstein system, so they should convert all the matter into such systems.

If all matter is condensed into several small highly enthropic systems, they will be widely separated, i.e., surrounded by empty regions of space which are large, flat and static compared to the length scale of any individual system. Since we are envoking the dominant energy condition, no negative energy is present. So all the conditions for Bekenstein's bound are fulfilled. So we can apply Bekenstein's bounds to the systems, and we will use the entropy/mass bound, Eq. (2.1), since it is tighter. To create the maximum amount of entropy, however, it is best to put the matter into a few large saturated Bekenstein systems, rather than many small ones. So we should take the limit of a Bekenstein system, as large as the apparent horizon and containing the entire mass within it. A question that arises at this point, is whether the calculation remains consistent in this limit, both in its evaluation of the mass and in its treatment of the interior as a Bekenstein system. We will show below, that the interior of the apparent horizon is in fact the largest system for which Bekenstein's conditions can be considered to hold. The same conclusion can be reached within the framework of the covariant entropy conjecture by using the spacelike projection theorem (see Sec.4.1).

The use of an entropy/mass bound in general space-times demands a certain measure of carefulness, since there are no concept of local energy density. In our case, however, the mass in certainly well-defined before we take the limit of a single Bekenstein system, since our multitude of saturated systems will be widely separated and can be treated as residing in asymptotically flat space. In the limit of a single system, we can follow [7], and treat the interior of the apparent horizon as part of an oversized spherical star. We are thus pretending that somewhere beyond the apparent horizon, the space-time may become asymptotically flat and empty. This is not an inconsistent assumption as long as Bekenstein's conditions are satisfied, and we show below that this is indeed the case. Then we can apply the usual mass definition for spherically symmetric systems [21] to the interior region.

The circumferential radius of our system is the apparent horizon radius, and is given by Eq. (4.7):

$$r_{AH} = \sqrt{\frac{3}{8\pi\rho}} \,. \tag{4.10}$$

The mass inside the apparent horizon is given by [21]:

$$M(r_{AH}) = \int_0^{r_{AH}} 4\pi r^2 \rho \, dr = \frac{4\pi}{3} r_{AH}^3 \rho \,. \tag{4.11}$$

This gives us r = $2M(r_{AH})$. From Eq.(1.1), Bekenstein's entropy/mass bound, the entropy cannot exceed $2\pi M(r_{AH}) r_{AH}$. So we find

$$S_{MAX} = \pi r_{AH}^2 \tag{4.12}$$

which is exactly a quarter of the area of the apparent horizon.

Eq. (4.7) follows from the Friedmann equation (which involves only the density but not the pressure), and from Eq. (4.5), which is a geometric property of the FRW metrics, so our calculation holds independently of the equation of state. We have not dropped any factors of order one, and attained exactly the saturation of the bound. This would not be the case for the Hubble horizon or the particle horizon. We thus conclude that the entropy on the future-directed light-sheet L will not exceed a quarter of the area of the boundary B. The covariant entropy bound may be saturated on the apparent horizon, but it is not violated, despite the cosmologists furious attempts to produce entropy.

5. THE HOLOGRAPHIC PRINCIPLE

5.1 THE MYSTERIOUS T-INVARIANCE AND THE HOLOGRAPHIC PRINCIPLE.

An important and surprising property of our covariant entropy bound is that there is no explicit reference to future and past. This means that our bound is invariant under time reversal (T-invariant). This is a bit mysterious in a law about thermodynamic entropy. Our only way out of this mystery is to interpret the covariant bound not only as an entropy bound, but as a bound on the number of degrees of freedom N_{dof} that constitute the statistical origin of entropy. From our conjecture we thus have

$$N_{dof} \leq A/4 \qquad (5.1)$$

where N_{dof} is the number of degrees of freedom present on the light-sheet of A. We have made no assumptions about the microscopic properties of matter, so our limit is fundamental. There cannot be more independent degrees of freedom on L than A/4. This principle is called the *holographic principle*. One might say that logically, since $S \leq N_{dof}$, our entropy bound follows from the holographic principle. It is though intellectually appealing to consider the entropy bound first, since the mystery of its T-invariance naturally leads us to the holographic principle. The analogous step was a lot bolder when it was taken by 't Hooft [10] and Susskind [11], since more conservative interpretations of Bekenstein's bound were available. Given our covariant bound, the necessity for a holographic interpretation is much more obvious.

We formulate our holographic principle rigorously as:

(Holographic Principle) *Let a be the area of a two-dimensional connected spatial surface B. Let L be a light-sheet of B. Then the total number of independent quantum degrees of freedom N_{dof} present on L satisfies $N_{dof} \leq A/4$.*

Another way of expressing the same thing is by formalizing independent quantum degrees of freedom. If N is the number of elements of an orthonormal basis of the quantum Hilbert space that fully describes all physics on the light-sheet L, we then have $N \leq e^{A/4}$.

In the same way as our covariant entropy bound associates the area of a surface B with entropy on null hypersurfaces bounded by B, the holographic principle associates the area of a surface B with degrees of freedom on null hypersurfaces. We saw in Sec.4.1 that we could use our covariant entropy conjecture on spacelike hypersurfaces as well under certain conditions. This derivation can be repeated for the holographic principle, by replacing "entropy" by "N_{dof}". This gives us the holographic version of our spacelike projection theorem:

(Holographic Spacelike Projection Theorem) *Let A be the area of a closed surface B possessing a future-directed light-sheet L with no boundary other than B. Let the spatial region V be contained in the intersection of the causal past of L with any spacelike hypersurface containing B. Let N_{dof} be the total number of independent quantum degrees of freedom present on V. Then $N_{dof} \leq A/4$.*

5.2 SCREENS AND PROJECTIONS

The holographic principle implies that all information contained on the light-sheet L can be stored on the surface B. The construction of light-sheets according to the prescription stated in Sec.3.2 answers the following question: Given a surface B of area A, what is the hypersurface L on which N_{dof} is bounded by A/4? We now consider space-time globally and pose a different question: Which surfaces store the information contained in the entire space-time? To answer this question, we should invert the above prescription. Given a null hypersurface L, we follow the geodesic generators of L in the direction of *non-negative* expansion. We can stop anytime, but *must* stop when the expansion becomes negative. We will call this procedure *projection*. The two-dimensional spatial surface B which is spanned by the points where the projection is terminated will be called a *screen* of the projection. If the expansion vanishes on every point of B, it will be called a *preferred screen*.

Preferred screens are particularly interesting because the expansion of the projection typically changes sign on a preferred screen B, so B will be a preferred screen for projections coming from two directions, e.g., the past-directed outgoing and future-directed ingoing directions. It is thus particularly efficient in encoding global information.

By following all generators of the null hypersurface L in a non-contracting direction to a screen, we obtain a projection of all information on the hypersurface onto one or more screens, which may be embedded in the hypersurface or lie on its boundary.

To project the information in a space-time M, our strategy will be to slice M into a one-parameter family of null hypersurfaces. This turns out to be possible in all the examples we consider. The slicings are usually not unique, but symmetries in the space-times reduce the number of equivalent slicings considerably. For each slice L, we apply the projection rule stated above. This gives us a number of one-parameter families of two-dimensional screens. Each family forms a three-dimensional screen-hypersurface embedded in M or located at the boundary of M. This might sound extremely complicated, but we will see that it is easier than it seems. Our screen-hypersurfaces can be timelike, null or spacelike, and in Ch.6 we will see examples of different types of screen-hypersurfaces.

Usually it is clear from the context, whether we are talking about a screen (a spatial surface) or a three-dimensional hypersurface formed by a one-parameter family of screens. We will therefore a bit sloppy refer to a screen-hypersurface as a ”screen” of our space-time M. A hypersurface consisting of preferred screens will be called a *preferred screen-hypersurface*, or loosely a *preferred screen* of M. If the expansion of both independent pairs of othogonal families of light-rays vanishes on a screen, it is preferred under all four projections that end on it. Such a screen, and hypersurfaces formed by such screens, will be called *optimal*.

We might call the projection discussed so far *null projection***,** i.e., projection along null hyprsurfaces. Sometimes it is possible to project information along spacelike hypersurfaces, that is, *spacelike projection* in a spatial region V onto a screen B. This is allowed if V and B satisfy the conditions of our spacelike projection theorem (see Sec.5.1), and is of significance in de Sitter and Anti-de Sitter space in particular.

5.3 A RECIPE FOR CONSTRUCTING SCREENS

We construct screens by slicing a space-time into null hypersurfaces. In all our examples below, we have spherical symmetry, which makes it natural to slice the space-time into a family of light-cones centered at r=0. As a parameter for our family we can use time. This gives two inequivalent null projections: along past- or future-directed light-cones. The light-cones are often truncated by boundaries of the space-time and will not include r=0, but this does not matter. So when we have spherical symmetry, screens can be constructed using the following recipe:

1. Draw a Penrose diagram. Every point in the diagram represents a 2-sphere. Each diagonal line represents a light-cone. The two inequivalent null slicings can be represented by the ascending and descending families of diagonal lines.

2. Pick one of the two families. In which direction should we now project along the diagonal lines?

3. Identify the apparent horizons, i.e., hypersurfaces on which the expansion of the past or future light-cones vanishes. They will divide the space-time into normal, trapped and anti-trapped regions. In each region, draw a wedge whose legs point in the direction of negative expansion of the cones.

4. On a given diagonal line (i.e., light-cone), project each point towards the tip of the local wedge, onto the nearest point (i.e., sphere) B_i where the direction of the tip flips, or onto the boundary of space-time as the case may be.

5. Repeat for every line in the family. The surfaces B_i will form (preferred) screen-hypersurfaces H_i.

We will in Ch. 6, as we turn to examples, try to make these steps explicit by including a few Penrose diagrams for each example. In the first diagram, we will identify the apparent horizons and mark the wedges. For each inequivalent family of light-cones, we will the draw a diagram and indicate the projection directions by thick arrows. Screens will be denoted by thick points, and preferred screen-hypersurfaces by thick lines. This procedure follows, like most of the material in this and the following chapter, Bousso [17].

6. EXAMPLES OF THE HOLOGRAPHIC PRINCIPLE IN ACTION

6.1 CHAPTER OUTLINE

In this chapter two examples of the holographic principle in action will be discussed. We will show how screens and screen-hypersurfaces are constructed, and illustrate how this is done by diagrams, constructed as described in Sec. 5.3. We will focus on two cosmologies, the Anti-de Sitter space and FRW-cosmologies, which both are of great cosmological importance. In [17], the interested reader can find more examples, including de Sitter space, Minkowski space and a static Einstein universe.

6.2 ANTI-DE SITTER SPACE

Anti-de Sitter space (AdS) can be scaled into the direct product of an infinite time axis with a unit spatial ball [20]. In this form the metric is given by

$$ds^2 = R^2\left[-\frac{1+r^2}{1-r^2}dt^2 + \frac{4}{(1-r^2)^2}(dr^2 + r^2 d\Omega^2)\right] \tag{6.1}$$

The (constant) scale factor R is the radius of curvature. The spacelike hypersurfaces are open balls given by t=constant, $0 \leq r < 1$. The boundary is a two-sphere residing at r=1. The proper area of spheres diverges as r goes to zero.

Consider the past-directed radial light-rays which emanate from a caustic ($\theta = +\infty$) at r=0, $t=t_0$ (see Fig.4). They form a past light-cone L with a spherical boundary. The cone grows with affine time until the light-rays reach the boundary of space at r=1. The expansion, θ, is inversely proportional to the affine time, and decreases monotonically, but remains positive, and as r goes to 1, θ goes to zero.

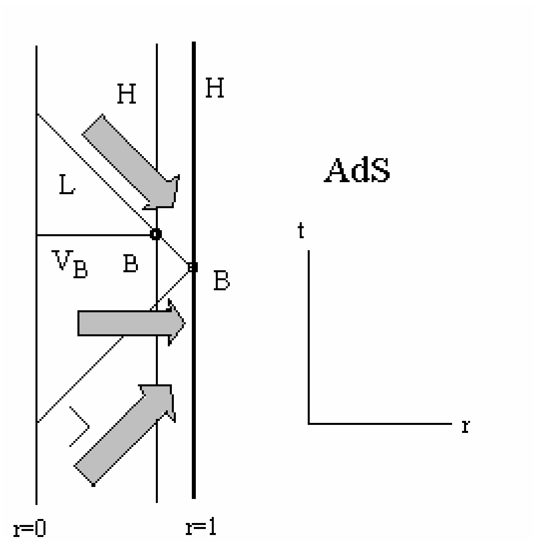

Figure 4: *Conventions and methods used in all diagrams are spelled out in Sec.5.3. Anti-de Sitter space contains no apparent horizons; all spheres are normal. Space-like projection is allowed. All null and spacelike projections are directed away from the center at r=0. Interior information can thus be projected onto a screen-hypersurface H of constant area; H encodes no exterior information. The screen at spatial infinity, H_∞, is optimal and encodes all bulk information.*

Now, consider a sphere B, of area A_B, on the lightcone L. The part of L inside B, L_B, has negative expansion in the direction away from B, and is thus a light-sheet of B. The holographic principle then tells us that the number of degrees of freedom on L_B does not exceed a quarter of the area of B:

$$N_{dof}(L_B) \leq A_B/4. \quad (6.2)$$

Since the cone closes off at r=0, it has no boundary other than B. The spatial interior of B on any spacelike hypersurface through B, V_B, lies entirely in the causal past of L_B. This means that the conditions of our spacelike projection theorem are met, and hence area bounds N_{dof} on spatial regions of AdS:

$$N_{dof}(V_B) \leq A_B/4. \quad (6.3)$$

Since the light-cone expansion is positive for all r, this remains valid in the limit as the sphere B moves to the boundary of space, $B \rightarrow B_\infty$. From the fact that θ goes to zero when r goes to 1, we notice that B_∞ is a preferred screen. As expected, the preferred screen is precisely the one which encodes the entire space. Actually B_∞ is an optimal screen, since the expansion of future-directed radial lightrays arriving at B_∞ also vanish by time reversal invariance of the metric.

Up to this point, we have considered screens bounding N_{dof} on a particular light-cone or spatial hypersurface. A screen-hypersurface encoding the entire space-time is obtained by repeating the construction for every single light-cone in our slicing of AdS. This repetition is trivial by the time-translation invariance of the metric. The family of finite screens B(t) of constant area A_B thus forms a timelike screen-hypersurface H (with topology R x S^2). From the holographic principle, N_{dof} in the enclosed space-time region does not exceed $A_B/4$. Using the spacelike projection theorem we might notice that it does not matter whether we count degrees of freedom on null or on spacelike hypersurfaces intersecting H. After taking the limit $B(t) \rightarrow B_\infty(t)$, we find that the timelike boundary at r=1, H_∞, is an optimal screen of Anti-de Sitter space and encodes all information in the bulk, by spacelike or null projection.

6.3 FRW-COSMOLOGIES

The Friedmann-Robertson-Walker (FRW) universe is described by a metric of the form

$$ds^2 = a^2(\eta)\left[-d\eta^2 + d\chi^2 + f^2(\chi)d\Omega^2\right] \tag{6.4}$$

Here f(χ) = sinh χ, χ, sin χ corresponds to open, flat and closed universes respectively. FRW universes contains homogeneous, isotropic spacelike slices of constant curvature. We will not discuss open universes, since they display no significant features beyond those arising in the treatment of closed or flat universes.

The matter content will be described by T_{ab} = diag(ρ,p,p,p), with pressure $p=\gamma\rho$. We assume that $\rho \geq 0$ and $-1/3 < \gamma \leq 1$. The case $\gamma = -1$ corresponds to de Sitter space, for which the reader is referred to [17]. The apparent horizon is defined geometrically as the spheres on which at least one pair of orthogonal null congruences have zero expansion, and is given by

$$\eta = q\chi \tag{6.5}$$

where

$$q = \frac{2}{1+3\gamma}. \tag{6.6}$$

The solution for a flat universe is given by

$$a(\eta) = \left(\frac{\eta}{q}\right)^q. \tag{6.7}$$

Its causal structure is shown in Fig. 5. The interior of the apparent horizon, $\eta \geq q\chi$, can be projected along past light-cones centered at $\chi=0$, or by space-like projection, onto the apparent horizon. The exterior, $\eta \leq q\chi$, can be projected by the same light-cones, but in the opposite direction, onto the apparent horizon. The apparent horizon is thus a preferred screen encoding the entire space-time. Alternatively, one can use future light-cones to project the entire universe onto future null infinity, another preferred screen.

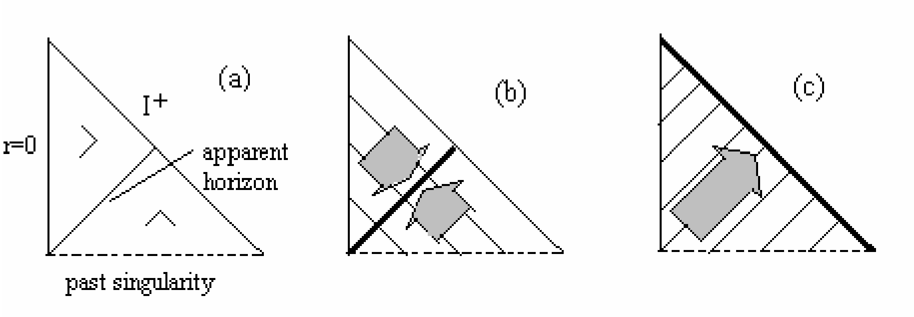

Figure 5: *Penrose diagram for a flat FRW-universe dominated by radiation. The apparent horizon, η=χ, divides the space-time into a normal and an anti-trapped region (a). The information contained*

in the universe can be projected along past light-cones onto the apparent horizon (b), or along future light-cones onto null infinity (c). Both are preferred screen-hypersurfaces.

By Eq. (6.5), the apparent horizon screen is a timelike hypersurface for $-1/3 < \gamma < 1/3$, null for $\gamma = 1/3$, and spacelike for $1/3 < \gamma \leq 1$. In a universe dominated by different types of matter in different eras, the causal character of the apparent horizon hypersurface can change from timelike to spacelike or the other way around.

For a closed universe, the solution is given by

$$a(\eta) = a_{max}\left(\sin\frac{\eta}{q}\right)^{q}. \tag{6.8}$$

In addition to Eq. (6.5), a second apparent horizon emanates from the opposite pole of the spatial S^3, at $\chi=\pi$, and is described by

$$\eta=q(\pi-\chi). \tag{6.9}$$

The two hypersurfaces formed by the apparent horizons divide the space-time into four regions (see Fig. 6). Let us choose the first apparent horizon, Eq. (6.5), as a (preferred) screen-hypersurface. On one side, $\eta \geq q\chi$, lies a normal region and a trapped region. These regions can be projected onto the screen by past-directed radial light-rays moving away from the South pole ($\chi=0$). The other half of the universe, $\eta \leq q\chi$, can be projected onto the same screen by future-directed radial light-rays moving away from the North pole ($\chi=\pi$). Therefore the preferred screen given by Eq. (6.5) encodes the entire closed universe.

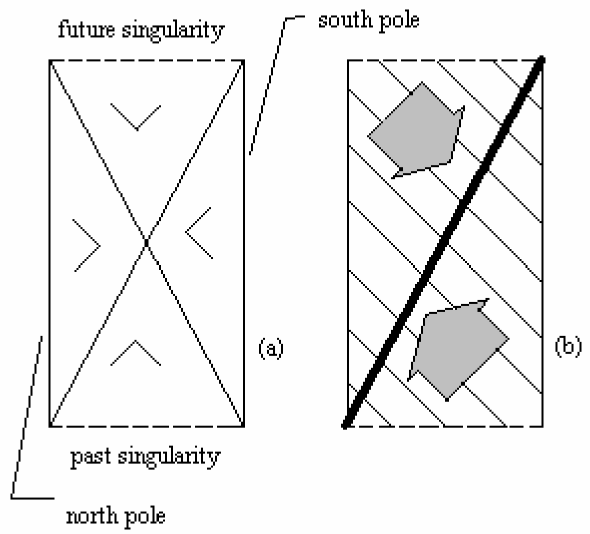

Figure 6: *Penrose diagram for a closed FRW universe dominated by pressureless dust. Two apparent horizons divide the space-time into four regions (a). The information in the universe can be projected onto the embedded screen-hypersurface formed by either horizon (b).*

7. A HOLOGRAPHIC THEORY

7.1 DUAL THEORIES

Our holographic principle and its interpretation has centered on the information needed to describe a state, which is given by the number of degrees of freedom. We can use the holographic principle to project all information in the space-time onto screen-hypersurfaces. This means, in a sense, an enormous reduction of the complexity of nature.

Worth stressing though, is that we did not use the holographic principle to *describe* nature. We have very successful theories that describes our world, such as general relativity and quantum field theory. In these theories the holographic principle is far from manifest. In these theories, one finds a number of non-trivial effects which seem to insure that the entropy bound implied by the holographic principle is always satisfied [16], but the results is not immediately implied by the basic axioms of these theories.

One might be tempted to consider the holographic principle as an external restriction imposed on physical theories. This is, however, a very dissatisfying point of view. If the number of degrees of freedom is limited by the holographic principle, there should be a description of nature in which the principle is manifest. We can call such a theory *the holographic theory*. One would expect the holographic theory to remain valid when semi-classical gravity breaks down, and in this regime it may be the only possible description. So the search for a holographic theory is of great importance.

An idea would be to define a theory on the geometric background given by the screen-hypersurfaces. If the theory was related by a kind of dictionary (”duality”) to the space-time (”bulk”) physics, the holographic principle would be manifest. Let us call this type of theory a *dual theory*. This idea seem to work for certain asymptotically Anti-de Sitter space-times. The screen encoding the entire bulk information is the timelike hypersurface formed by the boundary of space (Sec. 6.2). According to Maldacena’s remarkable conjecture [23], a super-Yang-Mills theory living on this hypersurfaces describs the bulk physics completely, and is a dual theory in the sense of our definition.

Can we find dual theories for other space-times? This would be a bold leap forward in our understanding of holography and quantum gravity. Unfortunately, the dual theory approach will not work in general situations. The theories we usually think of have a fixed number of degrees of freedom built into them. But consider the cosmological solutions studied in Sec.6.3. The area of the screens are time-dependent, so the screen theory would have to be capable of activating degrees of freedom. Moreover, the area can decrease, as seen in the closed universe example, so a screen theory would also have to be able to deactivate degrees of freedom, and eventually the number of degrees of freedom would approach zero and the second law of thermodynamics would be violated.

So what this suggests, is that one should not think of the screen theory as an ”ordinary” theory with a fixed number of degrees of freedom, but a theory with a varying number of degrees of freedom, which means that it would be very different from our ”ordinary” physical theories.

7.2 GEOMETRY FROM ENTROPY?

In Sec. 7.1 we discussed a possible ”screen theory”. One can be more radical than that. Perhaps one should not consider a ”screen theory” at all. A screen theory cannot be fundamental, since it presumes the existence of a space-time background. To use the holographic principle for a full description of nature, Bousso [17] have suggested an interesting idea. It can be summarized as stating that one should not constrain entropy by geometry, but construct geometry from entropy (or more correctly, degrees of freedom), and the construction should be such that the holographic principle is satisfied manifestly.

So it seems like there are (at least) two main problems to solve if we want to find a holographic theory. First, we have to formulate a theory with a varying number of degrees of freedom, as discussed in Sec.7.1. An idea is that the theory can activate och de-activate degrees of freedom from an infinite reservoir. Another more radical resolution would be to treat quantum degrees of freedom not as fundamental ingredients, but as a derived concept. In such models it would be natural for the number of degrees of freedom to vary.

Our second task is to find a way to uniquely reconstruct the space-time geometry from the varying number of degrees of freedom. A part of this will be to equate the number of degrees of freedom with the proper area of an two-dimensional preferred or optimal screen. A more difficult question is how the intrinsic geometry of screen-hypersurfaces can be recovered. But the number and character of the degrees of freedom gives us information about the matter content, so a complete reconstruction of space-time geometry is not inconceivable. It will be of great interest to follow the development in this area in the near future.

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REFERENCES:

[1] J.D. Bekenstein: *A universal upper bound on the entropy to energy ratio for bounded systems.* Phys. Rev. D **23**, 287 (1981)

[2] J.D Bekenstein, Phys. Rev. D **7**, 2333 (1973)

[3] J.D Bekenstein, Phys. Rev. D **12**, 3292 (1974)

[4] W. Fischler and L. Susskind: *Holography and cosmology,* hep-th/9806039

[5] R. Easther and D.A Lowe: *Holography, cosmology and the second law of thermodynamics*, hep-th/9409089

[6] G. Veneziano: *Pre-bangian origin of our entropy and time arrow*, hep-th/9902126

[7] D. Bak and S.-J. Rey: *Cosmic holography*, hep-th/9902173

[8] N. Kaloper and A. Linde: *Cosmology vs. holography*, hep-th/9904120

[9] R. Brustein: *The generalized second law of thermodynamics in cosmology*, gr-qc/9904061

[10] G. 't Hooft: *Dimensional reduction in quantum gravity*, gr-qc/9310026

[11] L. Susskind: *The world as a hologram*, J. Math. Phys. **36**, 6377 (1995), hep-th/9409089

[12] S. Corley and T. Jacobson: *Focusing and the holographic hypothesis*, Phys. Rev. D **53**, 6720 (1996), gr-qc/9602043

[13] S. W. Hawking and G. F. R. Ellis: *The large scale structure of space-time.* Cambridge University Press, Cambridge, England (1973)

[14] R. M. Wald: *General Relativity*. The Universe of Chicago Press, Chicago (1984)

[15] R. Tavakol and G. Ellis: *On holography and cosmology*, hep-th/9908093

[16] R. Bousso: *A Covariant Entropy Conjecture,* hep-th/9905177

[17] R. Bousso: *Holography in General Space-times,* hep-th/9906022

[18] R. Bousso: *The holographic principle for General Backgrounds,* hep-th/9911002

[19] E.E. Flanagan, D.Marolf and R.M. Wald: *Proof of classical versions of the Bousso entropy bound and of the generalized second law*, hep-th/9908070

[20] L. Sussking and E. Witten: *The holographic bound in Anti-de Sitter space,* hep-th/9805114

[21] S. W. Hawking, Phys. Rev. Letters **26**, 1344 (1971); contribution to *Black Holes*, edited by B. DeWitt and C.DeWitt (Gordon and Breach, New York, 1973).

[22] R.M. Wald: *Gedanken Experiments to Destroy a Black Hole*, Ann. Phys., **82**, 548-556

[23] J.Maldacena: *The larger N limit of superconformal field theories and supergravity.*

Adv. Theor. Math. Phys. **2**, 231 (1998), hep-th/9711200